ON EXACT SEQUENCES OF ABELIAN TOPOLOGICAL GROUPS

BY DAVID HARARI

In this short note (which is a complement to the Appendix of [1]), we collect some (probably well known) results about morphisms of abelian topological groups. The two (independent) statements that we prove can be useful to deal with Galois or étale cohomology of number fields.

Proposition 1. *Let $(f_n : A_n \to B_n)_{n\geq 1}$ be a projective system of strict morphisms of abelian topological groups. Assume that each A_n is Hausdorff, locally compact, and completely disconnected. Assume further that for every n, the kernel of f_n is finite. Then the morphism $f : \varprojlim A_n \to \varprojlim B_n$ induced by the system (f_n) is strict.*

Here the topology of a projective limit is by definition the topology induced by the product topology. To simplify the notation, the transition maps $A_{n+1} \to A_n$ and $B_{n+1} \to B_n$ will both be denoted π_n.

Proof. Set $I_n = \operatorname{Im} f_n$ (equipped with the topology induced by B_n) and $K_n = \ker f_n$ (it is a topological subgroup of A_n). Since K_n is finite, Mittag-Leffler condition is satisfied, hence $\operatorname{Im} f = \varprojlim I_n \subset \varprojlim B_n$; replacing B_n with I_n if necessary, we can assume that each f_n is surjective and is an open map (because f_n is strict).

As A_n is locally compact and completely disconnected, it has a basis of neighborhoods of zero consisting of open subgroups. Therefore (by definition of the projective limit topology) $\varprojlim A_n$ has a basis of open neighborhoods of zero consisting of subgroups $U = \varprojlim U_n$ that satisfy the two following conditions: U_n is an open subgroup of A_n for every n, and there exists $n_0 \geq 1$ such that for all $n \geq n_0$ we have $U_{n+1} = \pi_n^{-1}(U_n)$. Let $V_n = f_n(U_n)$; then V_n is an open subgroup of B_n and $f(U) = \varprojlim V_n$ thanks to the exact sequence

$$0 \to (K_n \cap U_n) \to U_n \xrightarrow{f_n} V_n \to 0$$

and the finiteness of $K_n \cap U_n$. Thus it is sufficient to check the following condition : there exists $r > 0$ such that for every $n \geq r$, we have $\pi_n^{-1}(V_n) = V_{n+1}$.

For each K_n, denote $K'_n \subset K_n$ the image of $K_m \to K_n$ for m sufficiently large (here we use again Mittag-Leffler condition). Then the transition maps $K'_{n+1} \to K'_n$ are surjective and there exists $r > n_0$ such that the image of the transition map $K_r \to K_{n_0}$ is K'_{n_0}. This implies that if $n \geq r$, then for every $x_n \in K_n$ there exists $x'_n \in K'_n$ with same image as x_n in A_{n_0}; namely $K_n \subset (K'_n + \ker[A_n \to A_{n_0}])$ for $n \geq r$, so $K_n \subset (K'_n + U_n)$ for $n \geq r$: indeed the inverse image of U_{n_0} in A_n is U_n thanks to the condition on U.

Consider now the exact commutative diagram

$$\begin{tikzcd}
0 \arrow[r] & K_{n+1} \arrow[r] \arrow[d] & A_{n+1} \arrow[r, "f_{n+1}"] \arrow[d, "\pi_n"] & B_{n+1} \arrow[r] \arrow[d, "\pi_n"] & 0 \\
0 \arrow[r] & K_n \arrow[r] & A_n \arrow[r, "f_n"] & B_n \arrow[r] & 0
\end{tikzcd}$$

We have $\pi_n^{-1}(U_n) = U_{n+1}$ and $K_n \subset (K'_n + U_n)$ for $n \geq r$. Also f_{n+1} and the map $K'_{n+1} \to K'_n$ are both surjective. Now an easy diagram chasing yields $V_{n+1} = \pi_n^{-1}(V_n)$ for $n \geq r$.

□

If A is an abelian topological group, we let $A^\wedge$ denote the completion of A for the topology induced by the family (Ω_A) of open subgroups of finite index of A. Namely $A^\wedge = \varprojlim_{U \in \Omega_A} (A/U)$.

Proposition 2. *Let $f : A \to B$ be a morphism of abelian topological groups. Then the following conditions are equivalent :*

a) The map $f^\wedge : A^\wedge \to B^\wedge$ induced by f is injective.

b) For every $U \in \Omega_A$, there exists $V \in \Omega_B$ such that $f^{-1}(V) \subset U$.

These conditions are in particular satisfied when the following assumption holds:

c) The completion morphism $B \to B^\wedge$ is injective. and the morphism f is strict and injective.

Proof. Assume b). Let $x = (x_U)_{U \in \Omega_A}$ be an element of $\ker f^\wedge$ (with $x_U \in A/U$ for each U). For every $U \in \Omega_A$, choose $V \in \Omega_B$ such that $f^{-1}(V) \subset U$. Set $U' = f^{-1}(V)$, then $x_{U'} = 0$ because $x \in \ker f^\wedge$ and the map $A/U' \to B/V$ induced by f is injective; hence $x_U = 0$ (indeed x_U is the image of $x_{U'}$ by the transition map $A/U \to A/U'$).

Assume a). Let $U \in \Omega_A$, then U is the inverse image of an open subgroup W of finite index of $A^\wedge$ by the completion map $A \to A^\wedge$ (take $W = \ker[A^\wedge \to A/U]$), and conversely the inverse image of an open subgroup of finite index by the completion map is of finite index. It is therefore sufficient to prove b) when A and B are already profinite and f is injective. In this case this is clear because A/U is a finite (hence discrete) subgroup of the profinite group B/U.

If c) holds, then $A^\wedge$ identifies with the closure of A in $B^\wedge$, so b) holds as well.

□

Remark : The map $B \to B^\wedge$ is in particular injective when B is Hausdorff, locally compact, completely disconnected, and compactly generated (cf. [1], Appendix).

References

[1] D. Harari, T. Szamuely, Arithmetic duality theorems for 1-motives, *J. reine angew. Math.* **578** (2005), 93–128.