Non-abelian descent

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These notes are the written version of the three one hour talks that I gave for the 2006 Clay summer school in Goettingen.

For any field k of characteristic zero, we fix an algebraic closure $\bar{k}$ of k and we set $\Gamma := \mathrm{Gal}\,(\bar{k}/k)$ (we will sometimes write Γ_k for Γ if several fields are involved). The group Γ is the inverse limit of the groups $\mathrm{Gal}\,(L/k)$ when L runs over all finite Galois extensions of k. If k is a number field, we let Ω_k denote the set of all places of k, and k_v the completion of k at v.

1. Review of non-abelian cohomology

In this section k is any field of characteristic zero. The main reference for the non abelian cohomology of groups is Serre's book [15], chapter I.5.

Let G be an algebraic group over k (all k-groups are assumed to be linear, but not necessarily connected), set $\overline{G} = G \times_k \bar{k}$.

Examples :

- G finite (defining G is the same as giving the abstract finite group $G(\bar{k})$, equipped with a continuous action of Γ for the profinite topology on Γ and the discrete topology on $G(\bar{k})$). E.g. : $\mathbf{Z}/n$ (cyclic group of order n with trivial Galois action), μ_n (group of n-roots of unity in $\bar{k}$ with the natural Galois action).

- G can be a k-torus (this means that $\overline{G}$ is isomorphic to some power of the multiplicative group $\mathbf{G}_m$), e.g. the 1-dimensional torus $R^1_{K/k}\mathbf{G}_m$ defined by the affine equation $x^2 - ay^2 = 1$, where $a \in k^*$ is a constant and $K := k(\sqrt{a})$. More generally, if L is a finite extension of k with k-basis $(\omega_1, ..., \omega_r)$, the $r-1$ dimensional torus $R^1_{L/k}\mathbf{G}_m$ is defined by the affine equation

$$N_{L/k}(x_1\omega_1 + ... + x_r\omega_r) = 1$$

where $x_1, ..., x_r$ are the variables.

- $G = \mathrm{PGL_n}$ (it is semi-simple and *adjoint*, that is the center is trivial), $G = \mathrm{SL_n}$ (it is semi-simple and simply connected).
- $G = O(q)$ (orthogonal group of a quadratic form q); this group is not connected, there is an exact sequence of k-groups

$$1 \to SO(q) \to O(q) \to \mathbf{Z}/2 \to 0$$

  If the rank of q is at least 3, then $SO(q)$ is semi-simple (but not simply connected : its universal covering is $\mathrm{Spin}(q)$); if $q = \langle 1, -a \rangle$ is of rank 2, then $SO(q)$ is just the torus $R^1_{K/k}\mathbf{G}_m$ with $K = k(\sqrt{a})$.

We define the group $H^0(k, G) = H^0(\Gamma, G(\bar{k})) = G(k)$. For example $H^0(\mathbf{Q}, \mu_n)$ is trivial if n is odd. The *Galois cohomology set* $H^1(k, G) = H^1(\Gamma, G(\bar{k}))$ is the quotient of the set of 1-*cocycles* $Z^1(k, G)$ by an equivalence relation defined as follows. The set $Z^1(k, G)$ consists of continuous maps $f : \Gamma \to G(\bar{k})$ satisfying the cocycle condition

$$f(\gamma_1\gamma_2) = f(\gamma_1).^{\gamma_1}f(\gamma_2)$$

for each $\gamma_1, \gamma_2 \in \Gamma$. Two cocycles f, g are equivalent if there exists $b \in G(\bar{k})$ such that $f(\gamma) = b^{-1}g(\gamma)^{\gamma}b$ for every $\gamma \in \Gamma$. There is no canonical group structure on $H^1(k, G)$ if G is not commutative, but there is a distinguished element (denoted 0), namely the class of the trivial cocycle. Therefore $H^1(k, G)$ is a pointed set.

Remark : The continuity assumption implies that

$$H^1(k, G) = \varinjlim_{L} H^1(\mathrm{Gal}\,(L/K), G(L))$$

where L runs over the finite Galois extensions of k.

Other definition of $H^1(k, G)$. It is also possible to define $H^1(k, G)$ as the set of isomorphism classes of *principal homogeneous spaces* (p.h.s.) of G over k. By definition such a p.h.s. is a non empty set A, equipped with a left action of Γ and a simply transitive righ action of $G(\bar{k})$, such that the compatibility formula

$$\gamma(x.g) = \gamma(x).\gamma(g)$$

holds for every $\gamma \in \Gamma$, $x \in A$, $g \in G(\bar{k})$.

The correspondance between the two definitions goes as follows :

Let $\gamma \mapsto c_\gamma$ be a cocycle in $Z^1(k, G)$. Then define A as the p.h.s. with underlying set $G(\bar{k})$, but the *twisted* action of Γ defined by $\gamma(x) = c_\gamma.^{\gamma}x$ (and

$G(\bar{k})$ acts on the right on A). One checks that cohomologous cocycles give isomorphic p.h.s.

Conversely if A is a p.h.s. of G over k, choose a point $x_0 \in A$; then for each $\gamma \in \Gamma$, there exists a unique $c_\gamma \in G(\bar{k})$ such that $\gamma(x_0) = x_0.c_\gamma$. This defines a cocycle in $Z^1(k, G)$, and the cohomology class of this cocycle does not depend on x_0; moreover isomorphic p.h.s. also give cohomologous cocycles.

Remark : In the case we consider, any p.h.s. A is *representable* by the k-variety X defined as the quotient of $G \times_k \bar{k}$ by the action of Γ corresponding to A (the quotient exists because a group variety is quasi-projective). The k-variety X is a *k-form* of $\overline{G} := G \times_k \bar{k}$ (that is $\overline{X} \simeq \overline{G}$), and the p.h.s. A is trivial iff $X(k) \neq \emptyset$; the latter is also equivalent to the existence of $x_0 \in A$ such that $\gamma(x_0) = x_0$ for all $\gamma \in \Gamma$.

Properties of $H^1(k, G)$.

- The set $H^1(k, G)$ is covariant in G (easy with the cocycle definition), and in k (it is contravariant in $\operatorname{Spec} k$) : if $k \subset L$ is an inclusion of fields, then there is a map $H^1(k, G) \to H^1(L, G)$, induced by the map $X \mapsto X \times_k L$ from isomorphism classes of k-p.h.s. to isomorphism classes of L-p.h.s.

- If
$$1 \to G_1 \to G_2 \to G_3 \to 1$$
is an exact-sequence of k-groups (this means that the sequence of groups $1 \to G_1(\bar{k}) \to G_2(\bar{k}) \to G_3(\bar{k}) \to 1$ is exact), then there is an exact sequence of pointed sets
$$1 \to G_1(k) \to G_2(k) \to G_3(k) \to H^1(k, G_1) \to H^1(k, G_2) \to H^1(k, G_3)$$
In the special case when G_1 is central in G_2, this sequence can be extended with a map $H^1(k, G_3) \to H^2(k, G_1)$, but this map is not a morphism of groups in general, even if G_1 and G_3 are abelian.

Remark : "Exact sequence" of pointed sets means that the image of a map is the kernel of the following; it can happen that a map has trivial kernel but is not injective.

Examples.

- By Hilbert's Theorem 90, we have $H^1(k, \mathrm{GL}_n) = H^1(k, \mathrm{SL}_n) = 0$.

- If T is a non split torus, it can happen hat $H^1(k,T) \neq 0$. For example if $T = R^1_{K/k}\mathbf{G}_m$, we have $H^1(k,T) = k^*/NK^*$; to see this, write T as the kernel of the norm map $R_{K/k}G_m \to \mathbf{G}_m$ (where $R_{K/k}$ stands for Weil's restriction), and use Hilbert's 90 (by Shapiro's lemma, the cohomology group $H^1(k, R_{K/k}\mathbf{G}_m)$ is isomorphic to $H^1(K,\mathbf{G}_m)$ (hence it is zero) because $(R_{K/k}G_m)(\bar{k})$ is the Galois module induced by $\bar{k}^*$ and the inclusion $\Gamma_K \to \Gamma_k$).

- If G is a semi-simple, connected and simply connected group, then $H^1(k,G) = 0$ when G is a p-adic field. For a number field k, the natural map
$$H^1(k,G) \to \bigoplus_{v \in \Omega_{\mathbf{R}}} H^1(k_v,G)$$
is an isomorphism (Kneser/Harder/Chernousov). These are special cases of "Serre's conjecture II" (see [15], III.3).

- The exact sequence $1 \to \mathbf{G}_m \to \mathrm{GL}_n \to \mathrm{PGL}_n \to 1$ is central. It induces an exact sequence
$$1 \to H^1(k,\mathrm{PGL}_n) \to H^2(k,\mathbf{G}_m) = \mathrm{Br}\, k$$
Actually the theory of central simple algebras implies that the map from $H^1(k,\mathrm{PGL}_n)$ to the Brauer group $\mathrm{Br}\, k$ is injective, its image is a subset of the n-torsion $(\mathrm{Br}\, k)[n]$, and the union of the images of $H^1(k,\mathrm{PGL}_n)$ in $\mathrm{Br}\, k$ is the whole $\mathrm{Br}\, k$. By class field theory, the image of $H^1(k,\mathrm{PGL}_n)$ is the whole $(\mathrm{Br}\, k)[n]$ when k is a p-adic field or a number field, but not in general.

2. Extension to étale cohomology

A reference for this section is Skorobogatov's book [17], I.5. See also [11], section 4.

Let X be an algebraic k-variety. The cohomology set $H^1(X,G)$ is defined using Cech cocycles for the étale topology. As in the case $X = \mathrm{Spec}\, k$, the pointed set $H^1(X,G)$ classifies isomorphism classes of (right) X-torsors (i.e. p.h.s.) under G. Namely such a torsor is a k-variety Y equipped with a faithfully flat morphism $f : Y \to X$ and a right action of G on Y, such that $G(\bar{k})$ acts simply transitively on $Y_{\bar{x}} := f^{-1}(\bar{x})$ for each geometric point $\bar{x} \in X(\bar{k})$.

The functorial properties of $H^1(X,G)$ are as in the case $X = \mathrm{Spec}\, k$, and there is also the same behaviour relatively to short exact sequences of

k-groups (simply replacing k by X). In particular the class $[Y]$ of a torsor Y in $H^1(X,G)$ is zero iff Y is isomorphic to the trivial torsor $X \times_k G$ iff the morphism $f: Y \to X$ has an X-section. If $X' \to X$ is a morphism of k-varieties, it induces a map $H^1(X,G) \to H^1(X',G)$, which maps $[Y]$ to $[Y \times_X X']$. A morphism of k-groups $G \to H$ induces a map $H^1(X,G) \to H^1(X,H)$, such that the image of $[Y]$ is the class of the *contracted product* $Y \times^G H$, which is defined as the quotient of $Y \times G$ by the diagonal action

$$(y,g).h := (y.h, h^{-1}g)$$

of H.

Let $m \in X(k)$ and $[Y] \in H^1(X,G)$. The k-morphism $\operatorname{Spec} k \to X$ corresponding to m induces an *evaluation map* $[Y] \mapsto [Y](m) \in H^1(k,G)$, and we have $[Y](m) = 0$ iff the fibre Y_m of the torsor $Y \to X$ has a k-point. More generally, for every cocycle $c \in Z^1(k,G)$, the equality $[Y](m) = [c]$ holds iff $[Y^c](m) = 0$, where Y^c is the *twisted torsor* of Y by c : it is an X-torsor under the *twisted group* G^c. The group G^c is an inner form of G : namely $\overline{G}^c = \overline{G}$ and the new Galois action on G^c is given by $\gamma(g) = c_\gamma {}^\gamma g c_\gamma^{-1}$ for every $\gamma \in \Gamma$ and $g \in G^c(\bar{k}) = G(\bar{k})$; the torsor Y^c is isomorphic to Y over $\bar{k}$, but the Galois action on Y^c is twisted via the formula

$$\gamma(y) = {}^\gamma y.c_\gamma^{-1}$$

If G is abelian, then $G^c = G$ and $[Y^c] = [Y] - [c]$ in the abelian group $H^1(X,G)$. We obtain the obvious (albeit important) *descent statement* :

Proposition 2.1 *Let $f: Y \to X$ be a torsor under a k-group G. For each $c \in Z^1(k,G)$, let $f^c: Y^c \to X$ be the corresponding twisted torsor. Then*

$$X(k) = \bigcup_{[c] \in H^1(k,G)} f^c(Y^c(k))$$

From now on we asume that k is a number field. Let X be a smooth variety such that $X(k_v) \neq \emptyset$ for every completion k_v of k. Let $X(\mathbf{A}_k)$ be the set of adelic points of X; if X is projective this set is just $\prod_{v \in \Omega_k} X(k_v)$. Let $f: Y \to X$ be a torsor under a k-algebraic group G, define

$$X(\mathbf{A}_k))^f = \bigcup_{[c] \in H^1(k,G)} f^c(Y^c(\mathbf{A}_k))$$

In other words $X(\mathbf{A}_k)^f$ is the subset of $X(\mathbf{A}_k)$ consisting of those points (P_v) such that the evaluation $[Y](P_v) \in \prod_{v \in \Omega_k} H^1(k_v,G)$ belongs to the diagonal image of $H^1(k,G)$. In particular $X(k) \subset X(\mathbf{A}_k)^f$, hence the condition $X(\mathbf{A}_k)^f = \emptyset$ is an obstruction to the Hasse principle, the *descent obstruction* associated to the torsor $f: Y \to X$ (or to the cohomology class $[Y] \in H^1(X,G)$).

Remark : This construction is not interesting if G is semi-simple and simply connected, or if G is a split torus. Indeed in these cases, we have $H^1(k, G) = 0$ for every field k, hence $X(\mathbf{A}_k)^f = X(\mathbf{A}_k)$.

Theorem 2.2 ([11], Th. 4.7) *Assume further that X is projective. Then $X(\mathbf{A}_k)^f$ contains the closure $\overline{X(k)}$ of $X(k)$ in $X(\mathbf{A}_k)$.*

This theorem is a consequence of the so-called Borel-Serre's theorem in Galois cohomology ([15], III.4). If X is projective and $X(\mathbf{A}_k)^f \neq X(\mathbf{A}_k)$, we obtain a *descent obstruction to weak approximation.*

A natural question consists of comparing these descent obstructions to the so-called *Brauer-Manin obstruction.* Let X be a smooth and geometrically integral k-variety and $\operatorname{Br} X = H^2(X, \mathbf{G}_m)$ its Brauer group (if X is the spectrum of a field F, then $\operatorname{Br} X$ is just the classical Brauer group $\operatorname{Br} F$ of the field F). Reciprocity law in global class field theory yields an exact sequence

$$0 \to \operatorname{Br} k \to \bigoplus_{v \in \Omega_k} \operatorname{Br} k_v \xrightarrow{\sum_v j_v} \mathbf{Q}/\mathbf{Z} \to 0$$

where $j_v : \operatorname{Br} k_v \to \mathbf{Q}/\mathbf{Z}$ is the local invariant. Therefore, the set $X(k)$ is a subset of the subset

$$X(\mathbf{A}_k)^{\mathrm{Br}} := \{(P_v) \in X(\mathbf{A}_k), \forall \alpha \in \operatorname{Br} X, \sum_{v \in \Omega_k} j_v(\alpha(P_v)) = 0\}$$

In particular the condition $X(\mathbf{A}_k)^{\mathrm{Br}} = \emptyset$ implies that $X(k) = \emptyset$. This is the Brauer-Manin obstruction to the Hasse principle. If X is assumed to be projective, then the set $X(\mathbf{A}_k)^{\mathrm{Br}}$ contains $\overline{X(k)}$ ([3], III.1) and the condition $X(\mathbf{A}_k)^{\mathrm{Br}} \neq X(\mathbf{A}_k)$ is the Brauer-Manin obstruction to weak approximation.

Special cases. a) The theory of descent of Colliot-Thélène and Sansuc [3] (refined by Skorobogatov) implies that the Brauer-Manin obstruction associated to $\operatorname{Br}_1 X := \ker[\operatorname{Br} X \to \operatorname{Br} \overline{X}]$ corresponds to considering all descent obstructions associated to groups G *of multiplicative type* (i.e. commutative linear groups whose connected component of 1 is a torus), see [11], Theorem 4.9.

b) There are examples of Brauer-Manin obstructions associated to "transcendental" elements (that is: elements that do not vanish in $\operatorname{Br} \overline{X}$) of $\operatorname{Br} X$ ([8], [19]); they correspond to descent obstructions related to $G = \mathrm{PGL}_n$ ([11], Th. 4.10). This uses the exact sequence $H^1(X, \mathrm{GL}_n) \to H^1(X, \mathrm{PGL}_n) \to \operatorname{Br} X$, and a theorem of Gabber (cf. [6]) saying that $\operatorname{Br} X$ is the union of the images of $H^1(X, \mathrm{PGL}_n)$ in $\operatorname{Br} X$.

c) For G finite and non commutative, the descent obstruction can refine the Brauer-Manin obstruction, that is: the set $X(\mathbf{A}_k)^{\mathrm{Br}}$ can be strictly bigger than $X(\mathbf{A}_k)^f$. An example of this situation will be explained in the next section.

3. Bielliptic surfaces

3.1. First properties of bielliptic surfaces

Geometrically (that is: over $\bar{k}$), a *bielliptic surface* is the quotient of the product $E_1 \times E_2$ of two elliptic curves by the free action of a finite group F (there are 7 possibilities for F, see for example [1], VI.20). We shall say that a k-variety X is a bielliptic surface if $\overline{X} := X \times_k \bar{k}$ is a bielliptic surface. Then the geometric invariants of X are $H^2(X, \mathcal{O}_X) = 0$ and $\dim H^1(X, \mathcal{O}_X) = 1$. In particular the geometric Brauer group $\mathrm{Br}\,\overline{X}$ is finite by Grothendieck's results ([7]).

In these notes, we will restrict ourselves to the case $F = \mathbf{Z}/2$. We consider a bielliptic surface X over k, equipped with an étale covering Y with group $\mathbf{Z}/2$, such that $\overline{Y}$ is the product of two elliptic curves. In particular there is an exact sequence associated to the geometric étale fundamental groups

$$1 \to \pi_1(\overline{Y}) \to \pi_1(\overline{X}) \to \mathbf{Z}/2 \to 1.$$

Unlike $\pi_1(\overline{Y})$, $\pi_1(\overline{X})$ is not abelian. Indeed $\pi_1(\overline{Y})$ is isomorphic to $\widehat{\mathbf{Z}}^4$ and $\pi_1(\overline{X})^{\mathrm{ab}}$ is of rank 2 because $\dim H^1(X, \mathcal{O}_X) = 1$.

Bielliptic surfaces were first used by Colliot-Thélène, Skorobogatov and Swinnerton-Dyer ([5]) to disprove a conjecture of Mazur. Then Skorobogatov exploited the properties of these surfaces to give the first counter-example to the Hasse principle not accounted for by the Brauer-Manin obstruction. In the next paragraph, we will summarize his construction.

3.2. Skorobogatov's construction

The reference for this paragraph is the paper [16].

Theorem 3.1 (Skorobogatov, 1997) *There exists a bielliptic surface X over $k = \mathbf{Q}$ such that: $X(\mathbf{Q}) = \emptyset$ but $X(\mathbf{A}_{\mathbf{Q}})^{\mathrm{Br}} \neq \emptyset$.*

The idea is as follows. Skorobogatov constructs a tower of coverings

$$Y' = C' \times D \to Y = C \times D \xrightarrow{f} X$$

where C and D are curves of genus one with $D(\mathbf{Q}) \neq \emptyset$ (but $C(\mathbf{Q}) = \emptyset$), with the following properties. The map $C' \to C$ makes C' into a torsor under the finite k-group $E[2]$ consisting of 2-torsion points of an elliptic curve E, such that C' itself is a k-torsor under E. The class $[C'] \in H^1(k, E)$ is an element of order exactly 4 in the Tate-Shafarevitch group $Ш(E)$. Recall that by definition $Ш(E)$ is the subgroup of $H^1(k, E)$ corresponding to elements whose restriction to $H^1(k_v, E)$ is zero for every place v of k. In particular C' has points in every completion of k but $C'(k) = \emptyset$.

Now the proof of Theorem 3.1 essentially breaks into two steps.

a) Under some assumptions (mainly the fact that $E(k)$ has no points of order exactly 2), prove that $(f')^*(\operatorname{Br} X) \subset \pi^*(\operatorname{Br} D)$, where f' is the map $Y' \to X$ and π the projection $Y' \to D$. This relies on careful computations of $\operatorname{Br} \overline{X} \simeq E[2]$ (hence $(\operatorname{Br} \overline{X})^\Gamma = 0$) and of $\operatorname{NS} \overline{X}$. Then it is very easy to construct points in $X(\mathbf{A}_k)^{\operatorname{Br}}$: it is sufficient to take the projection (Q_v) of $((P_v), R) \in Y'(\mathbf{A}_k)$, where $R \in D(k)$ and $(P_v) \in C'(\mathbf{A}_k)$); indeed for $\alpha \in \operatorname{Br} X$ such that $(f')^*(\alpha) = \pi^*(\beta)$ with $\beta \in \operatorname{Br} D$, we have

$$\sum_{v \in \Omega_k} j_v(\alpha(Q_v)) = \sum_{v \in \Omega_k} \beta(D)$$

(by functoriality) and $\beta(D) = 0$ because D is a rational point.

b) Prove that $X(k) \neq \emptyset$. This uses a descent argument. Only Y and the twist Y^- of Y by $(-1) \in H^1(\mathbf{Q}, \mathbf{Z}/2) = \mathbf{Q}^*/\mathbf{Q}^{*^2}$ have points everywhere locally. Then one shows by a direct computation that $Y(\mathbf{Q}) = Y^-(\mathbf{Q}) = \emptyset$.

3.3. Interpretation in terms of non-abelian torsors

In his paper [16], Skorobogatov explains his counterexample by an "iterated version" of the Brauer-Manin obstruction. Namely he shows that all twisted torsors Y^c of $Y \to X$ satisfy: $Y^c(\mathbf{A}_k)^{\operatorname{Br}} = \emptyset$. This implies $Y^c(k) = \emptyset$, hence $X(k) = \emptyset$ by Proposition 2.1.

Actually (see [11], subsection 5.1 for a complete description of the situation) the emptyness of $Y(\mathbf{A}_k)^{\operatorname{Br}}$ corresponds to a descent obstruction associated to a torsor $g : Z \to Y$ under a finite abelian k-group (which is a k-form of $E[4]$). The composite map $h = f \circ g$ makes Z a torsor over X, but its structural group G is not abelian ($\overline{G}$ is a semi-direct product $E[4] \rtimes \mathbf{Z}/2$). We have $X(\mathbf{A}_k)^h = \emptyset$, which shows that the descent obstruction associated to a finite and non-abelian group can refine the Brauer-Manin obstruction. The situation is different for commutative groups or linear connected groups (see [10], Th. 2).

More generally, the fact that the geometric étale fundamental group $\pi^1(\overline{X})$ is not abelian is often crucial to construct counterexamples as above. Here is a general statement about weak approximation:

Theorem 3.2 ([9]) *Let X be a smooth, projective and geometrically integral k-variety with $X(k) \neq \emptyset$. Assume that $H^2(X, \mathcal{O}_X) = 0$ and that $\pi^1(\overline{X})$ is not abelian. Assume further that the Albanese map (over $\bar{k}$) is flat with connected and reduced fibres. Then the closure $\overline{X(k)}$ of $X(k)$ in $X(\mathbf{A}_k)$ is strictly smaller than $X(\mathbf{A}_k)^{\mathrm{Br}}$.*

The condition on the Albanese map is technical (anyway it holds as soon as $H^1(X, \mathcal{O}_X) = 0$, or $\dim H^1(X, \mathcal{O}_X) = 1$ and $\dim X \geq 2$), the important point here being $\dim X > \dim H^1(X, \mathcal{O}_X)$.

For example the theorem applies to any bielliptic surface. It works also for some étale quotients of abelian varieties (in higher dimension), and for some elliptic surfaces, as well as for certain general type surfaces. Nevertheless, constructing a similar counterexample to the Hasse principle for a variety of general type remains an open problem.

The idea to prove Theorem 3.2 is that the conditions on H^1 and H^2 mean that the set $X(\mathbf{A}_k)^{\mathrm{Br}}$ is sufficiently big. Then the condition on $\pi^1(\overline{X})$ yields a descent obstruction (associated to a finite and non-abelian group) for some points in $X(\mathbf{A}_k)^{\mathrm{Br}}$.

The theorem does not apply to Enriques surfaces (the geometric fundamental group is $\mathbf{Z}/2$). However we will see in the next sections that using torsors under an extension of $\mathbf{Z}/2$ by a torus, it is still possible to refine the Brauer-Manin obstruction for such surfaces.

4. Composition of two torsors

From now on we follow the paper [12]. Our goal is to construct an Enriques surface X over k and a torsor $f : Z \to X$ under a linear algebraic group G such that $X(\mathbf{A}_k)^{\mathrm{Br}}$ is not a subset of $X(\mathbf{A}_k)^f$ (in particular the Brauer-Manin obstruction to weak approximation is not the only one). As mentionned before, the group G has to be non connected and non commutative. Since we are going to define G as an extension, it is necessary to know that under certain conditions, the composition of two torsors is still a torsor. That is the aim of this section.

Let $Z \to Y$ be a torsor under a k-torus T. Colliot-Thélène and Sansuc defined the notion of *type* of the torsor Y: it is an element of $\mathrm{Hom}_\Gamma(\widehat{T}, \mathrm{Pic}\,\overline{Y})$, where $\widehat{T}$ is the Galois module of characters of $\overline{T} = T \times_k \bar{k}$. To define the type, observe that each element χ of $\widehat{T} = \mathrm{Hom}(\overline{T}, \mathbf{G}_m)$ induces a pushout $\chi_*([Z]) \in H^1(\overline{Y}, \mathbf{G}_m) = \mathrm{Pic}\,\overline{Y}$ of the class $[\overline{Z}] \in H^1(\overline{Y}, \overline{T})$; we obtain a homomorphism $\widehat{T} \to \mathrm{Pic}\,\overline{Y}$, which is clearly Γ-equivariant: this is the type of the torsor Z. When $\mathrm{Pic}\,\overline{Y}$ is torsion-free and T is the *Néron-Severi torus* of Y (that is: $\widehat{T}$ is isomorphic to $\mathrm{Pic}\,\overline{Y}$), Colliot-Thélène and Sansuc also defined *universal torsors* as torsors whose type λ is an isomorphism $\mathrm{Pic}\,\overline{Y} \to \mathrm{Pic}\,\overline{Y}$ (see for example [17], (2.22) for more details).

Proposition 4.1 ([12]) *Let X be a smooth, projective, geometrically integral k-variety. Let $f : Y \to X$ be a torsor under a finite k-group H, and let $p : Z \to Y$ be a torsor under a k-torus T. Assume that the image $\mathrm{Im}\,\lambda \subset \mathrm{Pic}\,\overline{Y}$ of the type λ of Z is $H(\bar{k})$-invariant (e.g. Z universal). Then there exist a k-group G (extension of H by T) such that $f \circ p : Z \to X$ makes Z an X-torsor under G.*

The special case of this proposition we are interested in is when X is an Enriques surface. In this case (assuming $X(k) \neq \emptyset$), we have a $\mathbf{Z}/2$-torsor $f : Y \to X$, where Y is a $K3$-surface, and a universal torsor $Z \to Y$ under the Néron-Severi torus of Y. We obtain a torsor $g : Z \to X$ under a linear k-group G and an exact sequence

$$1 \to T \to G \to \mathbf{Z}/2 \to 1$$

It can be shown ([12], page 9, example 3) that the group G is commutative if and only if the map $f^* : \mathrm{Pic}\,\overline{X} \to \mathrm{Pic}\,\overline{Y}$ is surjective; this is the "generic" situation, but not the one we are going to consider for our construction.

5. A family of Enriques surfaces of Kummer type

The main theorem is the following.

Theorem 5.1 ([12]) *There exist an Enriques surface X over $k = \mathbf{Q}$, a torsor $g : Z \to X$ under a linear group G, and an adelic point $(P_v) \in X(\mathbf{A}_k)$ such that: $(P_v) \in X(\mathbf{A}_k)^{\mathrm{Br}}$ but $(P_v) \notin X(\mathbf{A}_k)^g$. In particular the Brauer-Manin obstruction to weak approximation is not the only one for X.*

It is likely that there exists an Enriques surface X such that $X(\mathbf{A}_k)^{\mathrm{Br}} \neq \emptyset$ and $X(k) = \emptyset$ (via a descent obstruction associated to a torsor as in Theorem 5.1), but no such example is known.

Let us explain briefly the construction leading to Theorem 5.1. We start with genus one projective curves D_1, D_2 given by affine equations

$$y_1^2 = d_1(x^2 - a)(x^2 - ab^2)$$

$$y_2^2 = d_2(t^2 - a)(t^2 - ac^2)$$

where b, c, d_1, d_2 are constant elements of k^*, and a is a non-square element of k^*. We also demand that b, c are not ± 1. Note that the Jacobian varieties E_1, E_2 of D_1, D_2 have all 2-torsion points defined over k. Let Y be the *Kummer surface* defined as the minmal desingularization of $(D_1 \times D_2)/(-1)$, where (-1) is the involution induced by multiplication by -1 on D_1 and D_2. Namely the $K3$ surface Y is a minimal smooth and projective model of the affine variety

$$y^2 = d(x^2 - a)(x^2 - ab^2)(t^2 - a)(t^2 - ac^2)$$

where $d = d_1 d_2$. It is equipped with the fixed-point free involution $\sigma : (x, t, y) \mapsto (-x, -t, -y)$, and the quotient $X = Y/\sigma$ is an Enriques surface (the associated morphism $Y \to X$ will be denoted f).

Under very mild conditions on the constants a, b, c, d_1, d_2, we obtain that the elliptic curves $\overline{E}_1$ and $\overline{E}_2$ are not $\bar{k}$-isogeneous (one just has to check that the modular invariant j_1 of E_1 is not integral over $\mathbf{Z}[j_2]$, where j_2 is the modular invariant of E_2). From this we deduce the following important fact:

Proposition 5.2 *There exist 24 lines on $\overline{Y}$, defined over $L = k(\sqrt{a})$, such that:*

a) $\operatorname{Pic}\overline{Y}$ is generated by the classes of these 24 lines.

b) The action of the Enriques involution σ on the 24 lines coincides with the action of $\operatorname{Gal}(L/k)$.

The property b) is especially interesting, because it simplifies computations of group cohomology related to X. For example we can now show the following result:

Proposition 5.3 *Let $\operatorname{Br}_1 X = \ker[\operatorname{Br} X \to \operatorname{Br} \overline{X}]$. Then $f^*(\operatorname{Br}_1 X)$ consists of constants (i.e. elements of $\operatorname{Im}[\operatorname{Br} k \to \operatorname{Br} Y]$).*

Proof : We have $\operatorname{Br}_1 X/\operatorname{Br} k = H^1(k, \operatorname{Pic}\overline{X})$ (cf. [17], Corollary 2.3.9). The image of this group in $H^1(k, \operatorname{Pic}\overline{Y}) = \operatorname{Br}_1 Y/\operatorname{Br} k$ factorizes through $H^1(k, \operatorname{Pic}\overline{X}/\mathrm{tors})$ because $\operatorname{Pic}\overline{Y}$ is torsion-free. Thus it is sufficient to prove that $H^1(k, \mathrm{Pic}\,\overline{X}/\mathrm{tors}) = 0$. Hochschild-Serre's spectral sequence associated to $\bar{f} : \overline{Y} \to \overline{X}$ yields an exact sequence

$$0 \to \mathbf{Z}/2 \to \mathrm{Pic}\,\overline{X} \to (\mathrm{Pic}\,\overline{Y})^\sigma \to 0$$

(here we are using $H^2(\mathbf{Z}/2, \bar{k}^*) = \widehat{H}^0(\mathbf{Z}/2, \bar{k}^*) = 0$). Therefore $\mathrm{Pic}\,\overline{X}/\mathrm{tors} = (\mathrm{Pic}\,\overline{Y})^\sigma$ is a lattice with trivial Galois action (because the Galois action on $\mathrm{Pic}\,\overline{Y}$ coincides with the action of σ thanks to Proposition 5.2), which implies $H^1(k, \mathrm{Pic}\,\overline{X}/\mathrm{tors}) = 0$.

□

Since X is a projective surface satisfying $H^2(X, \mathcal{O}_X) = 0$ and $\mathrm{NS}\,\overline{X} = \mathbf{Z}/2$, Grothendieck's results [7] imply that $\mathrm{Br}\,\overline{X} = \mathbf{Z}/2$. The most difficult part in [12] consists of proving that the non trivial element of $\mathrm{Br}\,\overline{X}$ does not come from an element of $\mathrm{Br}\,X$, which means $\mathrm{Br}\,X = \mathrm{Br}_1 X$. This holds as soon as neither $-d$ nor $-ad$ is a square in k^*. Using Proposition 5.3 and functoriality, we obtain

Proposition 5.4 *Assume that neither $-d$ nor $-ad$ is a square in k^*. Then the projection on X of every adelic point $(N_v) \in Y(\mathbf{A}_k)$ belongs to $X(\mathbf{A}_k)^{\mathrm{Br}}$.*

The end of the proof of Theorem 5.1 consists of finding an adelic point (N_v) on Y such that $(N_v) \notin Y(\mathbf{A}_k)^p$, where $p : Z \to Y$ is a universal torsor. This is possible for example for $k = \mathbf{Q}$, $a = 5$, $b = 13$, $c = 2$, $d = 1$. Using Prop 4.1, we obtain a torsor $g : Z \to X$ under a group G by composing p with $f : Y \to X$. The group G is an extension of $\mathbf{Z}/2$ by a torus, but it is not commutative. Finally a Galois cohomology computation (sort of non commutative "diagram-chasing") shows that the property $(N_v) \notin Y(\mathbf{A}_k)^p$ implies that $(M_v) := f(N_v)$ does not belong to $X(\mathbf{A}_k)^g$, although it is an element of $X(\mathbf{A}_k)^{\mathrm{Br}}$ by Proposition 5.4.

Remark : Actually instead of working with a universal torsor it is easier to work with a torsor of another type (satisfying the assumptions of Proposition 4.1), which is associated to the 1-dimensional torus $R^1_{L/k}\mathbf{G}_m$. Then G is a k-form of an orthogonal group O_2.

6. A summary of results, conjectures, and questions

The following summarizes what is known, what should be true, and what is completely unknown about Hasse principle and weak approximation on surfaces. Notice that for geometrically simply connected varieties, descent obstructions associated to linear groups cannot refine the Brauer-Manin obstruction because of [10], Th.2.

- Rational surfaces : it has been conjectured by Colliot-Thélène and Sansuc that the Brauer-Manin obstruction to the Hasse principle and weak approximation is the only one. Several significant cases are known (Châtelet surfaces, conic bundles with at most 5 degenerate fibres [4], [2], [14]).
- Abelian surfaces (with finite Tate-Shafarevitch group): The Brauer-Manin obstruction to the Hasse principle is the only one, and the same results holds for weak approximation if archimedean places are not taken into account ([13], [18]).
- Bielliptic surfaces : The Brauer-Manin obstruction to the Hasse principle is not the only one ([16]), and similarly for weak approximation ([9]). Descent obstruction (associated to a finite non-commutative group) can refine Brauer-Manin obstruction.
- $K3$ surfaces : since a $K3$ surface is geometrically simply connected, descent obstructions do not refine Brauer-Manin obstruction according to [10], Th. 2. (but "transcendental" obstructions can play a role, see [19]). I have no clear idea whether the Brauer-Manin obstruction should be the only one (neither for Hasse principle nor for weak approximation).
- Enriques surfaces : Descent obstruction (associated to a non-connected linear group) can refine Brauer-Manin obstruction, which is not the only one for weak approximation ([12]). It is likely (but not known) that the same should hold for the Hasse principle.
- Elliptic surfaces with Kodaira dimension 1 : the Brauer-Manin obstruction is not the only one for weak approximation, because of descent obstructions associated to finite non-commutative groups ([9]). The same should hold for the Hasse principle.
- General type surfaces : the situation is the same as for elliptic surfaces with Kodaira dimension 1.

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