LECTURE I

§0. Introduction

Differential geometry: differential manifolds, additional group structure → Lie groups

Algebraic geometry: algebraic varieties (affine or projective), defined by polynomial equations; additional group structure → algebraic groups

In these lectures, we will restrict to algebraic groups that are affine algebraic varieties, given by polynomial equations $P_i(x_1, \dots x_n) = 0$, $1 \leq i \leq r$ in the affine space. Projective algebraic groups ("abelian varieties") are very interesting, but their theory is quite different.

. We shall also assume that the base field k is algebraically closed for (at least) two reasons:

- Some results are easier to state in this context
- It is very inconvenient to have a nice theory over a non-closed field without using strongly the language (and the general theory) of schemes. The main difficulty is that it is no longer possible to identify a variety with the set of its points in the usual sense, this has to be replaced by the prime ideals of its coordinate ring.

. References:

Borel: Linear algebraic groups } over a closed field
Springer: "

Waterhouse: Introduction to affine group schemes } over an arbitrary fields
Milne: Algebraic groups

Grothendieck and al: SGA 3 (arbitrary group schemes)

§1 Affine varieties

• From now on, k is an algebraically closed field (e.g. $k=\mathbb{C}$, $k=\overline{\mathbb{Q}}$, $k=\overline{\mathbb{F}_p}$).

$\mathbb{A}^n_k = \mathbb{A}^n := \{(a_1, \dots a_n),\ a_i \in k\}$ affine space of dimension n.

The polynomial ring $k[x_1, \dots x_n]$ is noetherian :: every ideal I can be generated by a finite set $\{P_1, \dots P_r\}$. Thus, $P_i(a_1, \dots a_n) = 0$, $\forall i \Leftrightarrow$ $a = (a_1, \dots a_n) \in V(I) = \{a \in (a_1, \dots a_n) \in \mathbb{A}^n,\ f(a) = 0\ \ \forall f \in I\}$, where $I = (P_1 \dots P_r)$

Def: An affine variety is a subset $X \subset \mathbb{A}^n$ of the form $X = V(I)$ for some ideal $I \subset k[x_1, \dots x_n]$.

• Set $I(X) := \{f \in k[x_1, \dots x_n],\ f(a) = 0\ \ \forall a \in X\}$. It is clearly a radical ideal (that is $\sqrt{I(X)} = I(X)$, where $\sqrt{I} = \{x \in k[x_1, \dots x_n],\ x^m \in I$ for some $m\}$)

Th (Hilbert's Nullstellensatz) let I be a radical ideal of $k[x_1, \dots x_n]$. Then $I(V(I)) = I$.

Cor: For each affine variety $X \subset \mathbb{A}^n$, $\exists!$ radical ideal I s.t. $X = V(I)$.

• Recall that the Zariski topology on $\mathbb{A}^n$ is obtained by taking the $V(I)$ as the closed subsets (and equip all affine varieties with the induced topology). Also, the sets $D(f) := \{a \in \mathbb{A}^n,\ f(a) \neq 0\}$ (where $f \in k[x_1, \dots x_n]$) form a basis of open subsets for this topology on $\mathbb{A}^n$.

Rem: a) I is a maximal ideal $\Leftrightarrow X = \{a\}$ is a point and $I = (x_1 - a_1, \dots, x_n - a_n)$

b) I is prime $\Leftrightarrow X$ is an irreducible topological space

$\neq \emptyset$ and (:= if $X = X_1 \cup X_2$ with X_1, X_2 closed, then $X = X_1$ or $X = X_2$; equivalently, every $\neq \emptyset$ open subset is dense)

e.g. $\mathbb{A}^n$, $V(x_1)$ (hyperplane), a lin subspace; stronger than connected in general

c) More generally, if I is a radical ideal of $k[x_1, \dots x_n]$, can write $I = p_1 \cap \dots \cap p_r$ with p_i prime (Lasker-Noether) and this gives the decomposition of the affine variety $V(I)$ into its irreducible components: $V(I) = V(p_1) \cup \dots \cup V(p_r)$ (unique if you demand $V(p_i) \not\subset V(p_j)$ with $i \neq j$).

$V(x_1(x_1-1)) \subset \mathbb{A}^2$

e.g. $V(x_1 x_2) \subset \mathbb{A}^2$

Def: Let $X \subset \mathbb{A}^n$ be an affine variety. The coordinate ring of X is $\mathcal{O}(X) := k[x_1, \dots x_n] / I(X)$ (ring of "regular functions" on X).

It is a reduced ($:=$ no $\neq 0$ nilpotent element) and f.g. k-algebra.

Observe that X irreducible $\iff$ $\mathcal{O}(X)$ is an integral domain.

Def: Let X be an affine variety. A morphism $X \to \mathbb{A}^m$ is an element of $\mathcal{O}(X)^m$. If $Y \subset \mathbb{A}^m$ is an affine variety, a morphism $\Phi : X \to Y$ is the restriction of a morphism $\phi : X \to \mathbb{A}^m$ s.t. $\phi(X) \subset Y$.

Such a Φ is continuous because $\Phi^{-1}(D(f)) = D(f \circ \Phi)$.

Prop: a) Let A be a f.g. and reduced k-algebra. Then there exists a (unique up to isomorphism) affine variety X with $A \simeq \mathcal{O}(X)$ as k-algebras.

b) For every $\Phi : X \to Y$ morphism of affine varieties, set $\Phi^* : \mathcal{O}(Y) \to \mathcal{O}(X)$, $\Phi^*(f) = f \circ \Phi$ morphism of k-algebras.

Then $\Phi \mapsto \Phi^*$ is a bijection $\mathrm{Mor}(X, Y) \to \mathrm{Hom}_{k\text{-alg}}(\mathcal{O}(Y), \mathcal{O}(X))$.

Pf) a) A f.g. $\Rightarrow$ $A \simeq k[x_1, \dots x_n]/I$ for some radical ideal I (A reduced)

Take $X = V(I)$ and apply Nullstellensatz.

b) $Y \subset \mathbb{A}^m$, let $e_1, \dots e_m \in \mathcal{O}(Y)$ be the coordinate functions.

An inverse of $\Phi \mapsto \Phi^*$ is $\phi \mapsto (\phi(e_1), \dots \phi(e_m))$ $\quad \phi(e_i) \in \mathcal{O}(X)$. □

To sum up: a) Affine variety $\longleftrightarrow$ f.g. reduced k-algebra

b) morphism $X \xrightarrow{\varphi} Y$ of affine varieties $\longleftrightarrow$ morphism of k-alg $\mathcal{O}(Y) \to \mathcal{O}(X)$

Also, φ is a closed embedding $\Leftrightarrow \varphi^* : \mathcal{O}(Y) \to \mathcal{O}(X)$ is surjective.

" $X \mapsto \mathcal{O}(X)$, $\varphi \mapsto \varphi^*$ is an anti-equivalence of categories between {affine varieties over k} and {f.g. reduced k-algebras}.

Ex: a) Let $U = D(f) \subset \mathbb{A}^n$, then $\mathcal{O}(U) = k[x_1, \dots x_n][\frac{1}{f}] = k[x_1, \dots x_n, u] / (uf - 1)$.

Likewise, if $X \subset \mathbb{A}^n$ is a closed variety, $D(f) \cap X = \mathcal{O}(X)[\frac{1}{f}]$.

b) Let $X \subset \mathbb{A}^n$, $Y \subset \mathbb{A}^m$ be varieties with $X = V(f_1, \dots f_r)$, $Y = V(g_1, \dots g_s)$.

Then $X \times Y \subset \mathbb{A}^{m+n}$ is the variety $V(f_1, \dots f_r, g_1, \dots g_s)$

and its coordinate ring is $\mathcal{O}(X \times Y) = \mathcal{O}(X) \otimes_k \mathcal{O}(Y)$.

In particular, if X and Y are irreducible, so is $X \times Y$ (not obvious!)

Def Let X be an irreducible affine variety. Its dimension $\dim X$ is the Krull dimension of the ring $\mathcal{O}(X)$, e.g. $\dim \mathbb{A}^n = n$, $\dim \{a\} = 0$.

Equivalently, $\dim X$ is the supremum of integers m s.t. there exists a chain $X_m = X \underset{\neq}{\supset} X_{m-1} \underset{\neq}{\supset} \dots \supset X_0$ of irreducible closed subsets.

[Sufficient to every irreducible subset of $\mathbb{A}^n$]

Prop: a) If $X = V(p)$, then $\dim X = \dim (A/p) = \operatorname{tr\,deg}_k (\operatorname{Frac}(A/p))$,

where $A = k[x_1, \dots x_n]$ and $K := \operatorname{Frac}(A/p)$ is the function field of X.

b) If $U \neq \emptyset$ is an (affine) open subset, then $\dim U = \dim X$. (same f.f.)

c) A strict closed subset Y of X has $\dim X < \dim Y$.

d) $\dim (X \times Y) = \dim X + \dim Y$ (not obvious!)

If X is not irreducible, we set $\dim X = \sup_i \dim X_i$, where the X_i are the irreducible components of X.

§2. Affine algebraic groups: first properties

Def: An affine algebraic group is an affine variety G, equipped with two morphisms $m: G\times G \to G$, $(x,y)\mapsto m(x,y) =: xy$ (multiplication) and $i: G\to G$, $x\mapsto i(x) =: x^{-1}$ (inverse) plus a neutral element $e\in G$, satisfying the group axioms.

Ex: a) The additive group G_a = affine line $\mathbb{A}^1$ with the addition $(x,y)\mapsto x+y$.

b) The multiplicative group $G_m = \mathbb{A}^1 - \{0\}$ with law group $(x,y)\mapsto xy$

c) Identify the vector space of (n,n) matrices M_n to $\mathbb{A}^{n^2}$, let φ be the polynomial in n^2 variables given by the determinant. The the set Gl_n of invertible matrices is the open subset $D(\varphi)\subset \mathbb{A}^{n^2}$. Multiplication of matrices make it a (non commutative) algebraic group.

Observe that $\mathcal{O}(G_a) = k[x]$, $\mathcal{O}(G_m) = k[x^{\pm 1}]$, and Gl_n is an open subset of the irreducible variety $\mathbb{A}^{n^2}$, so in these three examples, the group is an irreducible variety.

d) Every closed subgroup of Gl_n is an affine algebraic group, e.g. D_n (diagonal matrices), T_n (upper triangular), U_n (upper triangular with 1 on the diagonal)

e) In particular, every finite group G is an affine algebraic group (embed G into $\mathcal{S}_n$ for some n, then use $\mathcal{S}_n \hookrightarrow GL_n$). Such a group is not connected if $|G|\geq 2$.

END OF LECTURE I

Rem: Let G be an affine algebraic group. Then all connected components of G are irreducible (hence G connected $\Leftrightarrow$ G irreducible). Also, the connected component G° of e is a normal subgroup of finite index [Hint: decompose $G = X_1 \cup \ldots \cup X_n$ into irreducible components, then choose $x \in X_1 \setminus \bigcup_{j \neq 1} X_j$, then show that every $y \in G$ is in only one X_i. For the second statement, observe that if $g \in G^\circ$, then $gG^\circ = G^\circ$ because gG° is a connected component].

• Using the anti-equivalence $\{\text{affine } k\text{-varieties}\} \longleftrightarrow \{\text{reduced, f.g. } k\text{-algebras}\}$, one gets

Th: Let G be an affine alg. group with coordinate ring $O(G)$. Then the k-algebra $O(G)$ carries the following additional structure, called a Hopf algebra structure:

a) A co-multiplication $\Delta : O(G) \to O(G) \otimes_k O(G)$ induced by $G \times G \to G$

b) A co-unit $\varepsilon : O(G) \to k$ (evaluation at $e \in G$)

c) A co-inverse $j : O(G) \to O(G)$ induced by the inverse map $G \xrightarrow{i} G$.

The three group axioms translate into the following commutative diagrams:

$$\begin{tikzcd}
O(G) \otimes O(G) \otimes O(G) & O(G) \otimes O(G) \arrow[l, "id \otimes \Delta"'] \\
O(G) \otimes O(G) \arrow[u, "\Delta \otimes id"] & O(G) \arrow[l, "\Delta"] \arrow[u, "\Delta"']
\end{tikzcd}$$

(associativity)

$$\begin{tikzcd}
O(G) & O(G) \otimes O(G) \arrow[l, "id \otimes \varepsilon"'] \\
O(G) \otimes O(G) \arrow[u, "\varepsilon \otimes id"] & O(G) \arrow[l, "\Delta"] \arrow[u, "\Delta"'] \arrow[ul, "id"']
\end{tikzcd}$$

(e is a left and right neutral element).

$$\begin{tikzcd}
O(G) & O(G)\otimes O(G) \arrow[l, "id\otimes j"'] \\
O(G)\otimes O(G) \arrow[u, "j\otimes id"] & O(G) \arrow[l, "\Delta"] \arrow[u, "\Delta"'] \arrow[ul, "\gamma"']
\end{tikzcd}$$

where $O(G) \xrightarrow{\varepsilon} k \longrightarrow O(G)$, $\underbrace{\qquad\qquad}_{\gamma}$

($i(g)$ is the left and right inverse of $g \in G$).

Ex: a) $G = G_a$, $O(G) = k[x]$, $\Delta(x) = x\otimes 1 + 1 \otimes x$, $\varepsilon(x) = 0$, $j(x) = -x$.

b) $G = G_m$, $O(G) = k[x^{\pm 1}]$, $\Delta(x) = x \otimes x$, $\varepsilon(x) = 1$, $j(x) = x^{-1}$.

c) $G = GL_n$, $O(X) = k[x_{11}, \dots x_{nn}, \det(x_{rs})]$ $\Delta(x_{rs}) = \sum_{l=1}^{n} x_{rl} \otimes x_{ls}$

$\varepsilon(x_{rs}) = \delta_{rs}$ and $j(x_{rs}) = y_{rs}$, where $[y_{rs}] = [x_{rs}]^{-1}$ as matrices.

Observe that we have an anti-equivalence {affine alg. groups over k} $\longleftrightarrow$ {f.g. and reduced Hopf algebras over k}

Th Let G be an affine alg. group over k. Then G is a linear algebraic group, that is: G is isomorphic to a closed subgroup of GL_n for some n.

Idea: Try to construct a "sufficiently big" finite dimensional G-invariant subspace of $O(G)$.

Def Let $g \in G$, then $\varphi_g : x \mapsto xg$ is an automorphism of the affine variety G, so it induces an automorphism of k-algebras $\rho_g = \varphi_g^* : O(G) \to O(G)$. Moreover $g \mapsto \rho_g$ is a group homomorphism: $G \to GL(O(G))$ because $\varphi_{gh} = \varphi_h \circ \varphi_g$, hence $\varphi_{gh}^* = \varphi_g^* \circ \varphi_h^*$.

Lemma A: Let $V \subset O(G)$ be a k-linear subspace. Then:

a) $[\forall g \in G,\ \rho_g(V) \subset V] \iff \Delta(V) \subset V \otimes_k O(G)$.

b) Assume further V f.d. Then $\exists$ f.d. $W \subset O(G)$ with $V \subset W$ and $\rho_g(W) \subset W$ for all $g \in G$.

Pf) a) Take a basis $(f_i)_{i\in I}$ of the k-vector space V, complete it with $(g_j)_{j\in J}$ to get a basis of $O(G)$.

Let $f\in V$, write $\Delta(f)=\sum_{i\in I} f_i\otimes u_i + \sum_{j\in J} g_j\otimes v_j$ with $u_i, v_i\in O(G)$ (almost all 0).

Then $\forall h\in G$, $(\rho_g f)(h)=f(hg)=\sum_{i\in I} f_i(h)u_i(g)+\sum_{j\in J} g_j(h)v_j(g)$ by def of co-multiplication.

Thus, $\rho_g f = \sum_{i\in I} u_i(g) f_i + \sum_{j\in J} v_j(g) g_j$ in $O(G)$

which means: $(\forall g\in G,\ \rho_g f\in V) \iff$ all v_j are 0 $\iff \Delta(f)\in V\otimes O(G)$.

b) Since $V=\bigoplus_{i=1}^{n} V_i$ with $\dim V_i=1$, we easily reduce to $\dim V=1$, i.e. $V=\langle f\rangle$.
As seen before, $\rho_g f=\sum_{i=1}^{n} g_i(g) f_i$ with $f_i, g_i\in O(G)$. Set $W'=\langle f_i\rangle$, it is f.d. and contains all $\rho_g f$, hence $W=\langle \rho_g f, g\in G\rangle$ is f.d. and stable by every ρ_g (because $\rho_{gh}=\rho_g\circ\rho_h$, $\forall g,h\in G$) □

Pf of th: Take a finite system of generators of the k-algebra $O(G)$, they span a f.d. linear subspace. Applying lemma A(b) to this subspace yields a f.d. W s.t.: W is stable by all ρ_g and W generates $O(G)$ as a k-algebra

Let $\mathcal{B}=(f_1,\dots,f_n)$ a basis of W, by lemma A(a):

$\Delta(f_i)=\sum_{j=1}^{n} f_j\otimes a_{ij}$ with $a_{ij}\in O(G)$, hence $\rho_g(f_i)=\sum_{j=1}^{n} a_{ij}(g) f_j$

This means $Mat_{\mathcal{B}}(\rho_g)=[a_{ij}(g)]$. Let Φ be the group homomorphism $G\to GL_n$, $g\mapsto [a_{ij}(g)]$

$O(GL_n)=k[x_{ij}, \det(x_{ij})^{-1}]$, then $\Phi^*(x_{ij})=a_{ij}\in O(G)$

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Since $\rho_g(f_i) = \sum_{j=1}^{n} a_{ij}(g) f_j$, we have

$$f_i(g) = (\rho_g(f_i))(1) = \sum_{j=1}^{n} a_{ij}(g) f_j(1), \text{ thus}$$

$$f_i = \sum_{j=1}^{n} f_j(1)\, a_{ij} \in \mathrm{Im}\,\Phi^* \quad (\text{recall that } a_{ij} = \Phi^*(x_{ij}) \in \mathrm{Im}\,\Phi^*).$$

Since $(f_1, \dots, f_n)$ generate the k-algebra $\mathcal{O}(G)$, Φ^* is surjective and Φ is a closed embedding. □

§4. Jordan decomposition, unipotent groups

Let E be a f.d. vector space over an alg. closed field k. Recall that for every $v \in \mathrm{End}(E)$, $\exists!$ pair $(s,n) \in \mathrm{End}(E)$ s.t.: $v = n+s = s+n$, s is <u>semi-simple</u> ($:=$ diagonalizable) and n is <u>nilpotent</u> ($\exists k \in \mathbb{N}^*$, $n^k = 0$). Another form of this decomposition is that if $g \in GL(E)$, $\exists!$ $(s,u) \in GL(E)$ s.t. $g = su = us$, s is semi-simple and u is <u>unipotent</u> ($:=$ $(u - \mathrm{id})$ is nilpotent). Moreover, s and u are polynomials in g.

In infinite dimension, you have such a <u>Jordan decomposition</u> if you assume further g <u>locally finite</u> ($:=$ E is a union of g-stable f.d. subspaces). Then $\exists!$ decomposition $g = g_s g_u = g_u g_s$ with g_s <u>semi-simple</u> and g_u <u>locally unipotent</u> (meaning: for every g-stable f.d. subspace V, $(g_s)_{|V}$ and $(g_u)_{|V}$ are resp. semi-simple, unipotent).

Moreover: the maps $g \mapsto g_s$ and $g \mapsto g_u$ are compatible to: restriction to a g-stable subspace W, $\oplus$ and $\otimes$ of vector spaces. Finally if $a \in \mathrm{End}(E)$, $b \in \mathrm{End}(F)$ et $\varphi : E \to F$ satisfies $b \circ \varphi = \varphi \circ a$, then

$\boxed{b_s \circ \varphi = \varphi \circ a_s \text{ and } b_u \circ \varphi = \varphi \circ b_u}$ $(*)$ $\boxed{\text{END OF LECTURE II}}$

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LECTURE III

Let G be an affine alg group. For each $g \in G$, $\rho_g \in GL(O(G))$ is locally finite (Lemma A1(b)), whence a Jordan decomposition $(\rho g) = (\rho g)_s (\rho g)_u = (\rho g)_u (\rho g)_s$

Th a) $\exists!$ pair $g_s, g_u \in G$ s.t. $(\rho g)_s = \rho_{g_s}$, $(\rho g)_u = \rho_{g_u}$ and $g = g_s g_u = g_u g_s$.
This is called Jordan decomposition in the affine alg. group G.
b) In the case $G = GL_n$, g_u and g_s are given by the classical Jordan decomposition

Using b), we see that for every closed imbedding $\Phi : G \to GL_n$, we have
$\Phi(g_s) = \Phi(g)_s$ and $\Phi(g_u) = \Phi(g)_u$.

Def: $g \in G$ is semi-simple if $g = g_s$, unipotent if $g = g_u$.
An affine alg. group G is unipotent if all its elements are unipotent. Equivalently, if G is imbedded into GL_n, ~~this~~ means that all $g \in G$ are unipotent in the usual sense.
In particular, every $g \in G$ is conjugate in GL_n to an element of $T_n = \left\{ \begin{pmatrix} 1 & & * \\ & \ddots & \\ 0 & & 1 \end{pmatrix} \right\}$. Actually, the classical Kolchin theorem says that a unipotent subgroup of GL_n is conjugate to a subgroup of T_n.

Pf of th a) Let $g \in G$. The fact that $\rho_g \in \mathrm{Aut}_{k\text{-alg}}(O(G))$ means: $m \circ (\rho_g \otimes \rho_g) = \rho_g \circ m$, where $m : O(G) \otimes O(G) \to O(G)$, $x \otimes y \mapsto xy$

Using (*) and compatibility of $v \mapsto v_s$ with $\otimes$ in a vector space, we get $m \circ ((\rho g)_s \otimes (\rho g)_s) = (\rho g)_s \circ m$

This means that $(\rho_g)_s$ is not only an automorphism of the k-vector space $O(G)$, but also an element of $\mathrm{Aut}_{k\text{-alg}}(O(G))$.

- We can therefore define a homomorphism of k-algebras $\varphi : O(G) \to k$ by $\varphi(f) := ((\rho_g)_s f)(e)$. This corresponds to a point $g_s \in G$ s.t. $\varphi(f) = f(g_s)$ for every $f \in O(G)$

Now it is sufficient to prove the formula: $((\rho_g)_s f)(h) = f(h g_s)$ (1) for every $f \in O(G)$, $g \in G$, $h \in G$. Indeed, this yields $((\rho_g)_s f)(h) = \rho_{g_s} f)(h)$, hence $(\rho_g)_s = \rho_{g_s}$. The same argument gives $(\rho_g)_u = \rho_{g_u}$, the unicity of g_s, g_u and the equality $g = g_s \cdot g_u = g_u g_s$ come from the fact that $\rho : G \to GL(O(G))$, $g \mapsto \rho_g$ is clearly injective.

- <u>Pf of (1)</u>: For $h \in G$, define $(\lambda_h f)(x) = f(h^{-1}x)$ $\forall x \in G$. Then λ_h commutes with ρ_g, and by $(*)$, λ_h commutes also with $(\rho_g)_s$ and $(\rho_g)_u$. Thus:

$$((\rho g)_s f)(h) = [(\lambda_{h^{-1}} \circ (\rho_g)_s)(f)](e) = ((\rho_g)_s(\lambda_{h^{-1}} f)](e)$$

$$= \varphi(\lambda_{h^{-1}} f) = (\lambda_{h^{-1}} f)(g_s) = f(h g_s), \text{ as expected.} \quad \square$$

(b) is more subtle, we just provide a few hints about the method

Set $V = k^n$, $G = GL(V)$. Let $g = g_u \cdot g_s$ be the classical Jordan decomposition of $g \in G$. We have to show that $\rho_{g_s} = (\rho_g)_s$ (it's similar for ρ_{g_u}). The first observation is that ρ_{g_s} and $(\rho_g)_s$ act not only on $O(G) \simeq k[x_{ij}, \det(x_{ij})^{-1}]$, but also on the sub-algebra $O(\mathrm{End}\, V) \simeq k[x_{ij}]$. A computation (using $(*)$ again) shows that ρ_{g_s} and $(\rho_g)_s$ coincide on $(\mathrm{End}\, V)^\vee \subset O(\mathrm{End}(V))$ (linear forms = polynomial of degree 1 on $\mathrm{End}(V)$). A similar argument (taking $\otimes^m$ for every $m \in \mathbb{N}$) shows that ρ_{g_s} and $(\rho_g)_s$ coincide on $\mathrm{Sym}((\mathrm{End}(V))^\vee) = O(\mathrm{End}(V))$. Finally, this implies that they coincide also on $O(GL(V)) = O(\mathrm{End}\, V)[\det(x_{ij})^{-1}]$ because they are automorphisms of the k-algebra $O(GL(V))$.

§5. Commutative algebraic groups

Rem: Let G be an affine algebraic group over k. Then the set G_u of unipotent elements is always closed (indeed, if $G \subset GL_n$, then G_u is given by the algebraic equation $(g - id)^n = 0$). In general, G_s is not closed (e.g. in GL_n, G_s is dense). The situation is better for commutative groups.

Th: Let G be a commutative linear algebraic group. Then G_u and G_s are closed subgroups of G, and
$$\varphi : G_s \times G_u \to G, \quad (s, u) \mapsto su$$
is an isomorphism of algebraic groups.

(Pf) Embed $G \hookrightarrow GL(V)$. Since the elements of G commute pairwise, a classical result in linear algebra yields that there exists a basis $\mathcal{B}$ of the f.d. vector space V s.t. $\forall g \in G$, $\mathrm{Mat}_{\mathcal{B}}\, g$ is upper triangular and $\forall g \in G_s$, $\mathrm{Mat}_{\mathcal{B}}\, g$ is diagonal (this can be proved by induction on $\dim V$, applying the fact that if two endomorphisms v, u commute, then the eigenspaces of v are u-stable). We can therefore assume that $G \subset T_n$ and $G_s = D_n \cap G$, where $D_n = \begin{pmatrix} * & & 0 \\ & \ddots & \\ 0 & & * \end{pmatrix}$ and $T_n = \begin{pmatrix} * & & * \\ & * & \\ & & \ddots \\ & & & * \end{pmatrix}$. This shows that G_s is a closed subgroup of G, and $G_u = G \cap U_n$ (where $U_n = \begin{pmatrix} 1 & & * \\ & \ddots & \\ & & 1 \end{pmatrix}$) likewise. Since $G_s \cap G_u = \{1\}$, ϕ is injective. It is surjective by Jordan decomposition. Finally, $g \mapsto g_s$ is the morphism $\begin{pmatrix} d_1 & & * \\ & \ddots & \\ 0 & & d_n \end{pmatrix} \mapsto \begin{pmatrix} d_1 & & 0 \\ & \ddots & \\ 0 & & d_n \end{pmatrix}$, hence ϕ^{-1} is also algebraic. $\square$.

<u>Def</u>: A <u>diagonalisable group</u> is a commutative algebraic group G s.t. $G = G_s$. Equivalently, G is isomorphic to a subgroup of $D_n \simeq \mathbb{G}_m^n$. G is a <u>torus</u> if it is isomorphic to D_n for some $n \geq 0$.

<u>Prop</u>: Let G be a diagonalisable group. Then the <u>group of characters</u> $\hat{G} = \mathrm{Hom}_{\text{alg. groups}}(G, \mathbb{G}_m)$ is a f.g. abelian group, without elements of order p if $\mathrm{char}\, k = p > 0$

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Pf) • Recall that $\mathcal{O}(\mathbb{G}_m) = k[t^{\pm 1}]$ with $\Delta(t) = t \otimes t$. Therefore there is a bijection between $\hat{G}$ and $\{x \in \mathcal{O}(G),\ \Delta(x) = x \otimes x\}$ (set of group-like elements in $\mathcal{O}(G)$). By Dedekind's independence of morphisms, group-like elements are linearly independent in $\mathcal{O}(G)$.

• Now $\hat{\mathbb{G}}_m = \mathbb{Z}$, $\widehat{\mathbb{G}_m^n} = \mathbb{Z}^n$ and $\mathcal{O}(\mathbb{G}_m^n) = k[t_1^{\pm 1}, \dots, t_n^{\pm 1}]$.
Let $\Phi : G \hookrightarrow \mathbb{G}_m^n$ be a closed embedding, then $\Phi^* : k[t_1^{\pm 1}, \dots, t_n^{\pm 1}] \to \mathcal{O}(G)$ is a surjective morphism of Hopf algebras. Since the $t_i^{m_i}$, $m_i \in \mathbb{Z}$ are group-like and span linearly $\mathcal{O}(\mathbb{G}_m^n)$, the same holds for the $\Phi^*(t_i^{m_i})$, but since group-like elements are linearly independent, the only possibility is that: the $\Phi^*(t_i^{m_i})$ are a basis of the vector space $\mathcal{O}(G)$, and they are exactly <u>the</u> group-like elements in $\mathcal{O}(G)$.

• This means that $\mathbb{Z}^n = \widehat{\mathbb{G}_m^n} \to \hat{G}$ is surjective, hence $\hat{G}$ is a f.g. abelian group. Finally, if $\mathrm{char}\, k = p > 0$, $\chi^p = 1 \Rightarrow \forall g \in G,\ \chi(g)^p - 1 = 0$
$\Rightarrow (\chi(g) - 1)^p = 0$ because $\mathrm{char}\, k = p$, thus $\chi = 1$. $\square$

Actually, $G \to \hat{G}$ is an anti-equivalence of categories between diagonalizable groups and f.g. abelian groups without elements of order $\mathrm{char}\, k$ (and tori correspond to free abelian groups).

Indeed, if A is such an abelian group, the group algebra

$k[A] = \{\sum_{a\in A} n_a a, n_a \in \mathbb{Z}, (n_a)$ almost all $0\}$ carries a Hopf algebra

structure given by $\Delta(a) = a \otimes a$, the corresponding affine group is the diagonalizable with character group A. Conversely, if G is a diagonalisable group, then $\mathcal{O}(G) = k[\hat{G}]$ because group-like elements are a basis of the k-vector space $\mathcal{O}(G)$.

• Thus, every diagonalisable group is $\simeq \mathbb{G}_m^n \times \mu_{m_1} \times \dots \times \mu_{m_r}$, where $\mu_m :=$ group of m-roots of unity.

Rk: If k is an arbitrary field of char 0, there is a similar correspondence between <u>affine groups of multiplicative type</u> (:= groups that become diagonalisable over the algebraic closure $\bar{k}$) and f.g. abelian groups equipped with an action of $Gal(\bar{k}/k)$

$\boxed{\text{END OF LECTURE III}}$

§6. Quotients of algebraic groups, Borel fixed point theorem

· Let G be an affine alg. group, $H \subset G$ a closed subgroup. It turns out that if $H \trianglelefteq G$, G/H can be defined as an affine alg. group, but in general G/H is only a (quasi-projective) variety.

· The projective space $\mathbb{P}^n$ is $(\mathbb{A}^{n+1} - \{0\})/\sim$, where $\sim$ is colinearity relation.

Def: A projective variety is a subset of $\mathbb{P}^n$ of the form $V(f_1, \dots f_r) = \{a \in \mathbb{P}^n, f_i(a) = 0 \ \forall i\}$, where $f_1 \dots f_r$ are homogeneous polynomials in $k[x_0, x_1, \dots x_n]$.

The Zariski topology on $\mathbb{P}^n$ is defined by taking the $V(f_1, \dots f_r)$ as closed subsets: $\mathbb{P}^n$ has an open covering by $D_+(x_i) := \mathbb{P}^n - V(x_i)$, and each of these is $\simeq \mathbb{A}^n$.

Def: A (quasi-projective) variety is a Zariski open subset of a projective variety (e.g.: projective varieties, affine varieties).

Such a variety X has a basis of open subsets consisting of affine varieties.

Def: A morphism of varieties is a continuous map $\varphi: X \to Y$ s.t. for every pair of open affine subsets U, V with $\varphi(U) \subset V$, the restriction $\varphi|_U^V$ is a morphism of affine varieties.

Ex: a) morphism of affine varieties

b) Inclusion $U \subset X$ of an open subset

c) If $X \subset \mathbb{P}^n$ is projective and $F_0, \dots F_m \in k[x_0, \dots x_n]$ are homogeneous with $V(F_0, \dots F_m) \cap X = \phi$, then $a \mapsto (F_0(a), \dots F_m(a))$ is a morphism from X to $\mathbb{P}^m$

d) If $X \subset \mathbb{P}^n$ and $Y \subset \mathbb{P}^m$ are varieties, then $X \times Y := S^{n,m}(X \times Y)$ is also one (and it is projective if X, Y are), where $S^{n,m} : \mathbb{P}^n \times \mathbb{P}^m \to \mathbb{P}^N$ is the Segre embedding $((a_0, \dots a_n), (b_0, \dots b_m)) \longmapsto (a_0 b_0, a_0 b_1, \dots a_n b_m)$.

• The notion of dimension, function field... extend to any irreducible variety X ($\dim X = \dim U$ and $k(X) = k(U)$, where U is any $\neq \phi$ affine open subset).

Th B (difficult, uses the notion of flatness) Let $\varphi : X \to Y$ be a morphism of irreducible varieties with $\operatorname{Im} \varphi$ dense. Then $\exists$ non empty open subset U of X s.t. $\varphi : U \to Y$ is an open mapping. Also, in char 0, φ injective $\Rightarrow$ φ^* induces an isomorphism $k(Y) \xrightarrow{\sim} k(X)$

• Some (not obvious!) consequences of Th B:

a) Let $\varphi : G \to G'$ morphism of affine alg. groups. Then $\varphi(G) \subset G'$ is a closed subset [reduce to G irreducible, use Th B to show that $\varphi(G)$ is open in $H = \overline{\varphi(G)}$, then prove that $\forall h \in H$, $h\varphi(G) \cap \varphi(G) \neq \phi$]

b) The same kind of argument can be used to show that the <u>derived subgroup</u> $[G,G]$ is a closed subgroup of G if G is a connected alg. group.

c) Let's say G <u>acts</u> on a variety X if a morphism of varieties

$$\begin{aligned} G \times X &\longrightarrow X \\ (g, x) &\longmapsto g\cdot x \end{aligned}$$

is given and satisfies the axioms of an action.

Then every orbit is an ~~open~~ open subset of its closure and every orbit of minimal dimension is closed.

All these tools can be used to prove:

<u>Th</u>: Let G be an affine algebraic group. Let H be a closed subgroup of G. Then:

a) There exists a variety X, equipped with a morphism $\rho : G \to X$ which is surjective and constant on the left cosets of H, s.t. the pair (X, ρ) is "universal" for this property (that is: if (X', ρ') is another such pair, $\exists\, \varphi : X \to X'$ s.t. $\rho' = \varphi \circ \rho$).
X is the <u>quotient</u> of G by H, denoted G/H.

b) Moreover, if $H \trianglelefteq G$, G/H is an affine alg. group.

When H is not normal in G, G/H is not affine in general.

X is constructed as a homogeneous space of G := a variety equipped with a transitive action of G s.t. $H = \mathrm{Stab}\, x_0$ (if $x_0 \in X$) and the fibres of $\begin{cases} G \to X \\ g \mapsto g \cdot x_0 \end{cases}$ are the left cosets gH of H in G.

__Ex__: a) $G = GL_n$, $H = \{\lambda \cdot \mathrm{id}, \lambda \in k^*\}$, $PGL_n := G/H$, more generally, the quotient of an affine alg. group G by its center is an affine alg. group. Similarly for $G^{ab} = G/[G,G]$ if G is connected.

b) Let V be a f.d. vector space. A __complete flag__ in V is a strictly increasing chain $\{0\} = V_0 \subsetneq V_1 \subsetneq \cdots \subsetneq V_n = V$ of linear subspaces, where $n = \dim V$. The set $Fl(V)$ of flags is made a projective variety as follows: let $Gr_d(V)$ be the set of d-dim. subspaces, and take:

$$Gr_d(V) \xrightarrow{P_d} \mathbb{P}(\Lambda^d(V)) \simeq \mathbb{P}^{\binom{n}{d}-1}$$
$$S \longmapsto \Lambda^d(S) \to \text{space of dim } 1 \text{ in } \Lambda^d(V)$$

(in terms of matrix, send an (n,d) matrix to all (d,d) minors of this matrix). It turns out P_d is injective with closed image.

Now $p_V : Fl(V) \longrightarrow Gr_0(V) \times \cdots \times Gr_n(V)$
$$(V_0, \dots V_n) \longmapsto (p_i(V_i))$$
identifies $Fl(V)$ with a projective variety! ↦ (we show that $\mathrm{Im}(Fl(V))$ is closed) $G = GL(V)$ ($\simeq GL_n$) acts transitively on $Fl(V)$, so the stabilizer B of a flag is a closed subgroup s.t. the variety $G/B \simeq Fl(V)$ is projective.

Such a B is called a <u>Borel subgroup</u> in $GL(V)$.

Typically $T_n = \begin{pmatrix} * & & * \\ & \ddots & \\ & & * \end{pmatrix}$ is a Borel subgroup (stabilizes the canonical flag).

<u>Th</u> (Borel fixed point theorem) Let G be a connected and <u>solvable</u> affine alg. group acting on a projective variety X.

Then $\exists x \in X$, $g.x = x$ for every $g \in G$

<u>Idea</u>: induction of length of commutator series, reduce to $\forall g \in (G,G)$, $g.x = x$. $\forall x \in X$ Then an orbit Z of p of minimal dim is closed, hence projective. Since $\mathrm{Stab}_p \supset (G,G)$, $\mathrm{Stab}_p \triangleleft G$, hence G/G_p is affine but also projective because it is $\simeq Z$. Finally $Z = \{p\}$ $\square$

<u>Cor</u>: (Kolchin) A connected solvable affine group $\subset GL(V)$ has a fixed complete flag (hence a basis of trigonalisation)

$\boxed{\text{END OF LECTURE IV}}$