

Descent obstructions and Brauer-Manin obstruction in positive characteristic

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Abstract. We prove that the Brauer-Manin obstruction is the only obstruction to the existence of integral points on affine varieties over global fields of positive characteristic p . More precisely, we show that the only obstructions come from étale covers of exponent p or, alternatively, from flat covers coming from torsors under connected group schemes of exponent p .

1. Introduction

Let K be a global field of characteristic $p > 0$, that is: the function field of a geometrically integral curve over a finite field \mathbf{F}_q , where q is a power of p . Let S be a non empty set of primes of K ; we denote by $\mathcal{O}_S \subset K$ the ring of S -integers. Set K^{sep} for a separable closure of K . Let \mathcal{X} be an \mathcal{O}_S -scheme of finite type with generic fibre X over K . For each prime of K , let K_v be the completion of K at v and let \mathcal{O}_v be the ring of integers of K_v . Notation like $\prod_v \dots$ means that the product is taken over all places of K .

The goal of this note is to describe the set $\mathcal{X}(\mathcal{O}_S)$ inside the adelic space $\prod_{v \notin S} \mathcal{X}(\mathcal{O}_v) \times \prod_{v \in S} X(K_v)$ in terms of cohomological obstructions related to certain (fppf or étale) X -torsors and to the Brauer group of X . More precisely we make the following definition :

Definition 1.1 Let G be a K -group scheme. Let $Y \rightarrow X$ be an X -torsor under G . We say that a point $(x_v) \in \prod_v X(K_v)$ is *unobstructed* by Y if the evaluation $[Y]((x_v)) \in \prod_v H^1(K_v, G)$ comes from a global element $a \in H^1(K, G)$ by the diagonal map.

Equivalently this means that there exists a cocycle $a \in Z^1(K, G)$ such that (x_v) lifts to some (y_v) on the twisted torsor Y^a (see [5], section 4.1). Here all cohomology sets are relative to fppf topology (if G is smooth, then étale topology can be used as well). Note that obviously every family $(x_v) \in \prod_v X(K_v)$ coming from a rational point $x \in X(K)$ is unobstructed.

Let $s = q^e$, where e is some positive integer. Consider the finite étale K -group scheme \mathbf{F}_s , which can also be viewed as a Γ_K -module, where $\Gamma_K = \text{Gal}(K^{\text{sep}}/K)$ is the absolute Galois group of K . Let X be a K -variety with ring of regular functions $K[X]$. Let \mathbf{G}_a be the additive group; there is an exact sequence of étale (or fppf) sheaves on $\text{Spec}(K[X])$:

$$0 \rightarrow \mathbf{F}_s \rightarrow \mathbf{G}_a \xrightarrow{\Phi_s} \mathbf{G}_a \rightarrow 0$$

where Φ_s is the additive morphism $x \mapsto x^s - x$. Since $H^1(\text{Spec } R, \mathbf{G}_a) = 0$ (Serre's Theorem) for every affine scheme $\text{Spec } R$, we get a canonical isomorphism

$$K[X]/\Phi_s(K[X]) \simeq H^1(\text{Spec}(K[X]), \mathbf{F}_s),$$

hence a canonical map

$$u_{s,X} : K[X]/\Phi_s(K[X]) \rightarrow H^1(X, \mathbf{F}_s)$$

Definition 1.2 Let X be a K -variety. An *Artin-Schreier torsor* over X is a torsor under the étale group scheme \mathbf{F}_s (for some $s = q^e, e > 0$) given by the equation

$$z^s - z = g$$

for some $g \in K[X]$. Such a torsor corresponds to the cohomology class $u_{s,X}(g) \in H^1(X, \mathbf{F}_s)$, where g is viewed as an element of $K[X]/\Phi_s(K[X])$.

In particular Artin-Schreier torsors are abelian étale torsors.

2. Descent obstructions associated to Artin-Schreier torsors

Consider the affine line \mathbf{A}_K^1 over K and its integral model $\mathbf{A}_{\mathcal{O}_S}^1$. Let e be a positive integer and $s = q^e$. Let $f \in K$. We are interested in Artin-Schreier torsors $Y_{f,s}$ over \mathbf{A}_K^1 given by the equation

$$Y_{f,s} : z^s - z = fx, \quad x \in \mathbf{A}^1 \tag{1}$$

In particular $Y_{f,s}$ has a class $[Y_{f,s}]$ in the étale cohomology group $H^1(\mathbf{A}_K^1, \mathbf{F}_s)$, which is given by $[Y_{f,s}] = u_{s,\mathbf{A}^1}(fx)$.

Proposition 2.1 *Assume that $(x_v) \in \prod_{v \notin S} \mathbf{A}^1(\mathcal{O}_v) \times \prod_{v \in S} \mathbf{A}^1(K_v)$ is unobstructed by $Y_{f,s}$ for every $s = q^e, e > 0$ and every $f \in K$. Then $(x_v) \in \mathbf{A}^1(\mathcal{O}_S)$.*

Proof Let $\omega \neq 0$ be a differential form of K which we write as $\omega = fdt/t$, where $f \in K$ and t is a separating variable of K (that is: an element of K with $dt \neq 0$). Let s be a power of q such that the places of S and the places in the support of the divisor of ω are \mathbf{F}_s -rational. Let x be a coordinate on \mathbf{A}^1 and consider the étale torsor $Y_{f,s}$. The fact that (x_v) is unobstructed in this cover means that there is an $a \in K$ such that (x_v) lifts to the twist $z^s - z = xf + a$ of this cover. In other words, for all v there exists $y_v \in K_v, y_v^s - y_v = x_v f + a$. It follows that $\text{Res}_v(x_v \omega + adt/t) = \text{Res}_v(y_v^s - y_v)dt/t = 0$. The last equality follows from Theorem 4 of [1] and the fact that the residues are \mathbf{F}_s -rational, by hypothesis. Thus $\sum_v \text{Res}_v(x_v \omega) = -\sum_v \text{Res}_v adt/t = 0$. This being true for all ω implies that (x_v) is global, by Weil's version of Riemann-Roch theorem (see [12], Theorem 2.1.1.). □

We deduce from the previous proposition the following theorem:

Theorem 2.2 *Let \mathcal{X} be an affine \mathcal{O}_S -scheme of finite type with generic fibre X . Let $(x_v) \in \prod_{v \notin S} \mathcal{X}(\mathcal{O}_v) \times \prod_{v \in S} X(K_v)$. Assume that (x_v) is unobstructed by every Artin-Schreier torsor $Y \rightarrow X$. Then $(x_v) \in \mathcal{X}(\mathcal{O}_S)$.*

Proof Embed \mathcal{X} into the affine space $\mathbf{A}_{\mathcal{O}_S}^n$ for some positive integer n . Let $(\gamma_v) \in \prod_v \mathbf{A}^1(K_v)$ be any coordinate of the image of (x_v) in $\prod_v \mathbf{A}^n(K_v)$. By functoriality (x_v) is unobstructed by every Artin-Schreier torsor over \mathbf{A}^1 . By proposition 2.1, the family (γ_v) comes from a global element $\gamma \in \mathbf{A}^1(\mathcal{O}_S)$. Since this is true for each coordinate, we obtain that $(x_v) \in \mathcal{X}(\mathcal{O}_S)$. □

Remarks -We cannot expect such a simple result for projective varieties because in this case $\mathcal{X}(\mathcal{O}_S) = X(K)$, and every adelic point in the closure of $X(K)$ is unobstructed by all torsors (the proof is the same as in the number field case, see [5], Theorem 4.7); in general $X(K)$ is not closed in the adelic space (for example if X is the projective line then the closure of $X(K)$ is the whole adelic space). One can even find examples of projective varieties over K with $X(K) = \emptyset$, but such that X contains adelic points unobstructed by every torsor under a finite étale group scheme (Poonen in [10] for $p > 2$, as well as number fields, and Viray in [11] for $p = 2$).

-In the previous theorem, it is important for the argument to work to consider adelic points in the whole product $\prod_{v \notin S} \mathcal{X}(\mathcal{O}_v) \times \prod_{v \in S} X(K_v)$, and not (as in [6]) in the "truncated" product $\prod_{v \notin S} \mathcal{X}(\mathcal{O}_v)$.

- Let Y be an X -torsor under a finite group G that extends to a torsor $\mathcal{Y} \rightarrow \mathcal{X}$ under a flat \mathcal{O}_S -group scheme \mathcal{G} . For X -torsors Y under finite

groups G of order prime to p (or arbitrary, in characteristic zero), the Selmer set, consisting of the $a \in H^1(\mathcal{O}_S, \mathcal{G})$ for which the twisted torsor Y^a satisfies $\prod_{v \notin S} \mathcal{Y}^a(\mathcal{O}_v) \times \prod_{v \in S} Y^a(K_v) \neq \emptyset$, is finite (and computable). This is typically not the case for Artin-Schreier torsors and simple examples can be constructed even for \mathbf{A}^1 . If such Selmer sets were finite and computable, our main result would yield an algorithm to decide solubility of diophantine equations over $\mathbf{F}_p[t]$ and it is well-known that such an algorithm cannot exist (see [9]).

-Likewise, one cannot expect, in the number field case, that finite descent obstructions are the only obstructions to the existence of integral points on affine varieties, as this would contradict the negative solution of Hilbert's tenth problem. As was pointed out by Jennifer Park, the affine surface

$$2x^2 + 3y^2 + 4z^2 = 1$$

given by Colliot-Thélène and Wittenberg ([3]) is an explicit counterexample (it is geometrically simply connected, hence has no finite descent obstructions). It has no Brauer-Manin obstruction either (loc. cit., example 5.9.). Another example where finite abelian descent obstructions are not enough is given in [6].

3. Torsors under local group schemes

In this section we consider a different kind of torsors, namely torsors under the local group scheme α_p (given by the equation $x^p = 0$). It turns out that a result similar to Theorem 2.2 holds (and is even simpler):

Proposition 3.1 *Let $(a_v)_v \in \prod_v \mathbf{A}^1(K_v) = \prod_v K_v$. Assume that $(a_v)_v$ is unobstructed by the α_p -torsors $y^p = x$ and $y^p = tx$ of \mathbf{A}_K^1 , where t is a separating variable of K . Then (a_v) is global.*

Proof The assumption means that there are $r, s \in K$ and $(b_v)_v, (c_v)_v \in \prod_v K_v$ with

$$a_v + r = b_v^p, \quad ta_v + s = c_v^p, \quad \forall v$$

Differentiating with respect to t yields

$$a'_v = -r', \quad a_v + ta'_v = -s'$$

and we can eliminate a'_v to get $a_v = -s' + tr' \in K$.

□

The same argument as in the proof of Theorem 2.2 now yields:

Corollary 3.2 *Let X be an affine variety over K . Assume that $(a_v)_v \in \prod_v X(K_v)$ is unobstructed by every α_p -torsor. Then $(a_v) \in X(K)$.*

Observe that in this corollary it is not necessary to assume that (a_v) is an adelic point of X : the hypothesis regarding α_p -torsors is very strong.

4. Relation to the Brauer-Manin obstruction

It has been known for some time ([2]) that (abelian) torsor obstructions to the existence of a rational point are linked to the Brauer-Manin obstruction. In the case of affine varieties over a global field K of positive characteristic, we can use Theorem 2.2 to give a more precise statement. The structure of the Brauer group $\text{Br}(\mathbf{A}_K^1)$ in characteristic p is quite complicated and has been determined by Knus, Ojanguren and Saltman ([7]).

Let $s = q^e$ be a power of q . The Cartier dual G_s of the étale K -group scheme \mathbf{F}_s is a finite K -group scheme of order s ; for example if $s = p$ is a prime number, then G_s is just the K -group scheme μ_q of q -roots of unity. For a K -variety X , the duality pairing

$$G_s \times \mathbf{F}_s \rightarrow \mu_s \hookrightarrow \mathbf{G}_m$$

induces a cup-product map in fppf cohomology

$$H^1(X, G_s) \times H^1(X, \mathbf{F}_s) \rightarrow H^2(X, \mathbf{G}_m) = \text{Br } X$$

For an Artin-Schreier torsor $Y \rightarrow X$ under \mathbf{F}_s , and an element $a \in H^1(K, G_s)$ (fppf cohomology), there is therefore a cup-product $(a \cup [Y]) \in \text{Br } X$, where $[Y]$ is the class of Y in $H^1(X, \mathbf{F}_s)$ and a stands for the image of a in $H^1(X, G_s)$.

Definition 4.1 Let X be a K -variety. Set $\text{Br}_0 X := \text{Im} [\text{Br } K \rightarrow \text{Br } X]$. For each $e > 0$, we define a subgroup $B_{AS,e}(X)$ of the Brauer group $\text{Br } X$ as the subgroup generated by $\text{Br}_0 X$ and the cup-products $(a \cup [Y])$, where Y runs over all Artin-Schreier X -torsors under \mathbf{F}_{q^e} and a runs over all elements of $H^1(K, G_{q^e})$. Then we set

$$B_{AS}(X) := \bigcup_{e>0} B_{AS,e}(X)$$

Remark In the special case $q = p, e = 1$, we have the identification $H^1(K, G_p) = H^1(K, \mu_p) = K^*/K^{*p}$. In particular elements of $B_{AS,1}(\mathbf{A}_{\mathbf{F}_p}^1)$ are the p -torsion elements of $\text{Br}(\mathbf{A}_{\mathbf{F}_p}^1)$ given by symbols (a, b) , where $a \in K^*$ and $b \in K[T]$ (this corresponds to a cup-product, where the element a is viewed in $H^1(\mathbf{F}_p, \mu_p)$ and the element b is in $H^1(\text{Spec}(\mathbf{F}_p[T]), \mathbf{F}_p)$). More generally the p -torsion subgroup of $\text{Br}(\mathbf{A}_{\mathbf{F}_p}^1)$ is generated by the symbols (a, b) , where a, b are arbitrary in $K[T]$ and are considered in $H^1(\text{Spec}(K[T], \alpha_p) = K[T]^1/K[T]^p$ to make the cup-product ([7], Theorem 6.7).

Theorem 4.2 *Let X be an affine K -variety. Let $(a_v) \in \prod_v X(K_v)$ be an adelic point on X . Assume that for every element $\theta \in B_{AS}(X)$, the evaluation $\theta((a_v))$ is global (that is: comes from an element of $\text{Br} K$ by the diagonal embedding). Then $(a_v) \in X(K)$.*

In other words: the Brauer-Manin obstruction related to the subgroup $B_{AS}(X)$ is the only one. Recall that the condition that $\theta((a_v))$ is global is equivalent to

$$\sum_v j_v(\theta(a_v)) = 0$$

where $j_v : \text{Br} K_v \rightarrow \mathbf{Q}/\mathbf{Z}$ is the local invariant.

Proof Taking integral models and using Theorem 2.2, it is sufficient to check that for every Artin-Schreier torsor $Y \rightarrow X$ under \mathbf{F}_s (with $s = q^e, e > 0$), the adelic point (a_v) is unobstructed by Y . By assumption the family $(b_v) := ([Y](a_v))$ is orthogonal to $H^1(K, G_s)$ for the Poitou-Tate pairing (which is obtained via the local Tate pairings):

$$H^1(K, G_s) \times \prod_v H^1(K_v, \mathbf{F}_s) \rightarrow \mathbf{Q}/\mathbf{Z}, \quad (a, (b_v)) \mapsto \sum_v j_v(a \cup b_v)$$

Now by Poitou-Tate exact sequence for finite group schemes over global fields of characteristic p ([4], Theorem 4.11, applied to $N = \mathbf{F}_s$) this implies that (b_v) comes from a global element $b \in H^1(K, \mathbf{F}_s)$ by the diagonal map, which means exactly that (a_v) is unobstructed by Y . □

There is a similar statement if we take into account more general p -torsion elements of the Brauer group:

Proposition 4.3 *Let X be an affine K -variety. Let (a_v) be an adelic point on X . Assume that (a_v) is orthogonal to the p -torsion subgroup of $\text{Br} X$ in*

the Brauer-Manin pairing

$$((a_v), \theta) \mapsto \sum_v j_v(\theta(a_v))$$

Then $(a_v) \in X(K)$.

Proof Arguing as in Corollary 3.2, it is sufficient to treat the case $X = \mathbf{A}_K^1$. Consider p -torsion elements of $\text{Br } X$ of the form $(a \cup [Y])$ with $a \in H^1(K, \alpha_p)$ and $[Y] \in H^1(X, \alpha_p)$ (the finite group scheme α_p is its own Cartier dual). The assumption implies that for a given α_p -torsor $Y \rightarrow X$, the family $([Y](a_v))_v$ is orthogonal to every $a \in H^1(K, \alpha_p)$ in the Poitou-Tate pairing. By Poitou-Tate exact sequence, this means that $([Y](a_v))_v$ is global. In other words $(a_v)_v$ is unobstructed by every α_p -torsor $Y \rightarrow X$, which implies that it comes from $X(K)$ by Proposition 3.1. □

5. Counterexamples to the Hasse principle

In this section we construct an example of affine scheme \mathcal{X} over $\mathcal{O}_S = \mathbf{F}_p[t]$ (here $K := \text{Frac } \mathcal{O}_S = \mathbf{F}_p(t)$ and the finite set S consists of the prime at infinity) such that \mathcal{X} has points over every completion of \mathcal{O}_S but $\mathcal{X}(\mathcal{O}_S) = \emptyset$. In particular every point of $\prod_{v \neq \infty} \mathcal{X}(\mathcal{O}_v) \times X(K_\infty)$ is obstructed by some torsor. Hence we get a counterexample to the integral Hasse principle in characteristic p .

Proposition 5.1 *Let $p > 2$ and $A, B \in \mathbf{F}_p[t]$ with $\deg A = 3, \deg B = 1$. Consider the affine scheme over $\mathcal{O}_S = \mathbf{F}_p[t]$*

$$\mathcal{X} : x^2 + Ay^2 = B$$

Assume that A, B are chosen so that the generic fibre X of \mathcal{X} (defined over $K = \mathbf{F}_p(t)$) has a K -point. Then every point in $\prod_{v \neq \infty} X(\mathcal{O}_v) \times X(K_\infty)$ is obstructed by the torsor $Y : z^p - z = y$. In particular \mathcal{X} has no $\mathbf{F}_p[t]$ -point.

Proof Suppose that there exists a global twist

$$z^p - z = y + c$$

with local points everywhere. The conditions at $v \neq \infty$ give, in particular, that c is a polynomial. Looking at the equation for X , we see that the component y_∞ has valuation equal to 1 at infinity so is in $\Phi_p(K_\infty)$. It follows

that c is in $\Phi_p(K_\infty)$ also and, being a polynomial, is in $\Phi_p(\mathbf{F}_p[t])$, so without loss of generality we can assume $c = 0$. Now, as $p > 2$ and B is linear, there exists $\alpha \in \mathbf{F}_p$, with $B(\alpha)$ not a square. The condition that $z^p - z = y$ has a local point in the place $v = t - \alpha$ gives that y_v vanishes at α and so $x_v(\alpha)^2 = B(\alpha)$, contradiction. □

Remark Of course in this example one can check directly that \mathcal{X} has no $\mathbf{F}_p[t]$ -point by degree considerations. For a specific example one can take $p = 3$, $A = t^2(t + 1)$, $B = t + 1$ which has the rational point $(0, 1/t)$.

We conclude with an example of finite étale group scheme M over $K = \mathbf{F}_p(t)$, such that the kernel

$$\text{III}^1(K, M) := \ker[H^1(K, M) \rightarrow \prod_v H^1(K_v, M)]$$

of the diagonal map is not trivial; in other words, we get a principal homogeneous space of M that is a counterexample to the Hasse principle.

Proposition 5.2 *Let $K = \mathbf{F}_2(t)$. Then there exists a Γ_K -module M such that $\text{III}^1(K, M) \neq 0$.*

Proof Consider the two quadratic extensions K_0, K_∞ of K associated to Artin-Schreier torsors

$$y^2 - y = t \quad z^2 - z = 1/t$$

Let $F = K_1K_2$ be the composite biquadratic extension. Then F/K is unramified outside 0 and ∞ , hence the corresponding decomposition subgroups are cyclic. Moreover at 0 and ∞ one of the two extensions K_0 and K_∞ is split, which implies that the decompositions subgroups at 0 and at ∞ are of order ≤ 2 , hence cyclic. Now we apply the same construction as in [8], example 9.1.: we define M as the kernel of the augmentation map $(\mathbf{Z}/4\mathbf{Z})[G] \rightarrow \mathbf{Z}/4\mathbf{Z}$ where $G := \text{Gal}(F/K)$. The cohomology sequence identifies $\mathbf{Z}/4\mathbf{Z}$ with $H^1(G, M)$; now if c is a generator of $H^1(G, M)$, the image of $2c$ in $H^1(K, M)$ is a non zero element of $\text{III}^1(K, M)$. □

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