

Duality for complexes of tori over a global field of positive characteristic

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April 26, 2020

Abstract. If K is a number field, arithmetic duality theorems for tori and complexes of tori over K are crucial to understand local-global principles for linear algebraic groups over K . When K is a global field of positive characteristic, we prove similar arithmetic duality theorems, including a Poitou-Tate exact sequence for Galois hypercohomology of complexes of tori. One of the main ingredients is the Artin-Mazur-Milne duality theorem for fppf cohomology of finite flat commutative group schemes.

AMS codes : 11E72, 11G20, 14F20, 14H25

1 Introduction

Let K be a global field of characteristic $p \geq 0$ and let \mathbf{A}_K denote the ring of adèles of K . Let G be a reductive group over K , and X be a torsor under G . We are interested in rational points on X , and more precisely, on various local-global principles associated to X : does X satisfy the Hasse principle, i.e. does $X(\mathbf{A}_K) \neq \emptyset$ imply $X(K) \neq \emptyset$? If not, can we explain the failure using the so-called Brauer-Manin obstruction to the Hasse principle? Assuming that $X(K) \neq \emptyset$, can we estimate the size of $X(K)$ by studying the so-called weak and strong approximation on X (with a Brauer-Manin obstruction if necessary), i.e. the closure of the set $X(K)$ in the topological space $X(\mathbf{A}_K^S)$, where S is a (not necessarily finite) set of places of K and \mathbf{A}_K^S is the ring of S -adèles (with no components in S)?

The answer to those questions is known in the case where K is a number field, see for instance [San], Corollaries 8.7 and 8.13, for the Hasse principle and weak approximation, and [Dem2], Theorem 3.14 for strong approximation. Note that in the number field case, similar results are known for certain non principal homogeneous spaces of G (see [Bor] or [BD]).

In the case of a global field of positive characteristic, the answer is known for semisimple simply connected groups (thanks to works by Harder, Kneser, Chernousov, Platonov, Prasad), but the general case is essentially open (see [Ros], Theorem 1.9 for some related results). One strategy to attack the remaining local-global questions is similar to one that worked for number fields: arithmetic duality theorems for tori, and abelianization of Galois cohomology (see for instance [Dem1] and [Dem2] for the case of strong approximation over number fields). Indeed, given a reductive group G over a field L (e.g. L is a global or a local field), one can construct a complex of tori of

length two $C := [T_1 \rightarrow T_2]$,¹ together with “abelianization maps” $H^i(L, G) \rightarrow H^i(L, C)$ (cohomology sets here are Galois cohomology or hypercohomology sets), such that the cohomology sets of G can be computed via the abelian cohomology groups of C and the Galois cohomology of a semisimple simply connected group associated to G . The latter is well-understood when L is a local or global field.

Motivated by the discussion above, this paper deals with arithmetic duality theorems for complexes of tori over global fields K of positive characteristic; in characteristic 0, we also get refinements of previously known results.

The aforementioned applications to the arithmetic of reductive groups and homogeneous spaces will be given in a future paper.

The main object is a two-term complex $C := [T_1 \rightarrow T_2]$ of K -tori T_1 and T_2 , and we are particularly interested in its Galois hypercohomology groups $H^i(K, C)$. The main result of the paper can be summarized as follows: we get Poitou-Tate exact sequences relating global Galois cohomology groups $H^i(K, C)$ and local ones $H^i(K_v, C)$ - for any place v of K - via the cohomology of the dual object \widehat{C} of C . To be more precise, let us introduce some notation: K is the function field of a smooth, projective, and geometrically integral curve X over a finite field k . Let $X^{(1)}$ denote the set of closed points in X . If A is a discrete abelian group, then A^* is the Pontryagin dual of homomorphisms from A to \mathbf{Q}/\mathbf{Z} , and A_\wedge denotes the completion $A_\wedge := \varprojlim_{n \in \mathbf{N}^*} A/n$.

We can now state one of the main results in the paper (see Theorem 5.10):

Theorem. *Let $C := [T_1 \rightarrow T_2]$ be a two-term complex of K -tori, and $\widehat{C} := [\widehat{T}_2 \rightarrow \widehat{T}_1]$ be the dual complex, where \widehat{T} is the module of characters of a torus T (T_1 and \widehat{T}_2 are in degree -1). Then there is an exact sequence*

$$\begin{array}{ccccccc}
0 & \longrightarrow & H^{-1}(K, C)_\wedge & \longrightarrow & \left[\prod'_{v \in X^{(1)}} H^{-1}(K_v, C) \right]_\wedge & \longrightarrow & H^2(K, \widehat{C})^* \\
& & & & & & \downarrow \\
& & H^1(K, \widehat{C})^* & \longleftarrow & \left[\prod'_{v \in X^{(1)}} H^0(K_v, C) \right]_\wedge & \longleftarrow & H^0(K, C)_\wedge \\
& & \downarrow & & & & \\
& & H^1(K, C) & \longrightarrow & \bigoplus_{v \in X^{(1)}} H^1(K_v, C) & \longrightarrow & H^0(K, \widehat{C})^* \\
& & & & & & \downarrow \\
0 & \longleftarrow & H^{-1}(K, \widehat{C})^* & \longleftarrow & \bigoplus_{v \in X^{(1)}} H^2(K_v, C) & \longleftarrow & H^2(K, C)
\end{array} \tag{1}$$

In the case where K is a number field, we also recover a generalization of [Dem1], Theorems 6.1 and 6.3. In the function field case, some partial results related to this exact sequence for one single torus can be deduced from section 6 in [Go1].

The main ingredient to prove the Theorem above is the so-called Artin-Mazur-Milne duality theorem for the fppf cohomology of finite flat commutative group schemes over open subsets of X (see [Mi2], Theorem III.8.2 and [DH], Theorem 1.1). It is worth noting that although the complexes that appear in the previous theorem consist of smooth group schemes (hence the result can be stated using only Galois cohomology), it is essential for the proof to involve finite group schemes (which are not smooth in general) over Zariski open subsets of X . That’s why [DH], Theorem 1.1, is required

¹Throughout the paper, the piece of notation $:=$ means that the equality is a definition.

instead of the Artin-Verdier duality Theorem ([Mi2], Cor. II.3.3.) in étale cohomology. Likewise for Theorems 4.11 and 4.9.

Also, since it is necessary at some point to work with the fppf topology, the approach of duality theorems via Ext groups (like [Mi2], Th. II.3.1) does not seem to work, the difficulty being the lack of good notion of constructible sheaf for the fppf topology (see [Mi2], introduction to chapter III).

The structure of the paper is the following: section 2 extends the construction and properties given in [DH] of fppf cohomology with compact support to the case of bounded complexes of finite flat group schemes. General properties of étale cohomology of complexes of tori and of their dual complexes are given in section 3. Section 4 deals with applications of Artin-Mazur-Milne duality theorem to various duality statements for the étale cohomology of complexes of tori over open subsets U of X . In section 5, one deduces several Poitou-Tate exact sequences for Galois cohomology from the results of section 4.

2 Compact support hypercohomology

Let K be the function field of a smooth, projective, and geometrically integral curve X over a finite field k . Let U be a non empty Zariski open subset of X . Denote by $U^{(1)}$ the set of closed points of U .

Let $\mathcal{C} = (C_p)_{p \in \mathbf{Z}}$ be a bounded complex of fppf sheaves over U . In this text, we define the dual of \mathcal{C} to be the Hom-complex $\widehat{\mathcal{C}}$ defined by

$$\widehat{\mathcal{C}} := \underline{\mathbf{Hom}}^\bullet(\mathcal{C}, \mathbf{G}_m[1]),$$

following the sign conventions in [SP], Tag 0A8H or in [AC], X.5.1. Note that there is a functorial morphism of complexes

$$\mathrm{Tot}(\mathcal{C} \otimes \widehat{\mathcal{C}}) \rightarrow \mathbf{G}_m[1]$$

mapping an element $c \otimes \varphi \in C_p \otimes \mathrm{Hom}(C_q, A_n)$ to 0 if $p \neq q$, and to $(-1)^{p(n-1)}\varphi(c) \in A_n$ if $p = q$.

With those conventions, if \mathcal{C} is concentrated in degree 0, i.e. $\mathcal{C} = \mathcal{F}$ with \mathcal{F} an fppf sheaf, then $\widehat{\mathcal{C}}$ is the same as the Cartier dual $\mathcal{F}^D := \underline{\mathbf{Hom}}(\mathcal{F}, \mathbf{G}_m)$ attached in degree -1 , i.e. $\widehat{\mathcal{C}} = \mathcal{F}^D[1]$ and the above pairing coincides with the obvious pairing $\mathcal{F} \otimes \mathcal{F}^D[1] \rightarrow \mathbf{G}_m[1]$ with no extra sign.

Note also that for any bounded complex \mathcal{C} , we have a natural isomorphism of complexes $\widehat{\mathcal{C}[1]} \xrightarrow{\sim} \widehat{\mathcal{C}}[-1]$, given by a sign $(-1)^{n+1}$ in degree n . And given a morphism $f : A \rightarrow B$ of bounded complexes, we have a natural isomorphism of complexes $\widehat{\mathrm{Cone}(f)} \xrightarrow{\sim} \mathrm{Cone}(\widehat{f})[-1]$ such that the following diagram commutes

$$\begin{array}{ccccccc} \widehat{A[1]} & \longrightarrow & \widehat{\mathrm{Cone}(f)} & \longrightarrow & \widehat{B} & \xrightarrow{\widehat{f}} & \widehat{A} \\ \downarrow \sim & & \downarrow \sim & & \downarrow = & & \downarrow = \\ \widehat{A}[-1] & \longrightarrow & \mathrm{Cone}(\widehat{f})[-1] & \longrightarrow & \widehat{B} & \xrightarrow{\widehat{f}} & \widehat{A}. \end{array} \quad (2)$$

If \mathcal{N} is a commutative group scheme (over U or over K), its Cartier dual is denoted \mathcal{N}^D . The Pontryagin dual of a topological abelian group A (consisting of continuous

homomorphism from A to \mathbf{Q}/\mathbf{Z} is denoted A^* . Unless explicitly specified, the topology used for sheaves (resp. complex of sheaves) and cohomology (resp. hypercohomology) is the fppf topology.

For each closed point v of X , the completion of K at v is denoted by K_v : it is a local field of characteristic p with finite residue field \mathbf{F}_v (observe the slight difference of notation with [DH], where K_v stands for the henselization and \widehat{K}_v for the completion). Denote by \mathcal{O}_v the ring of integers of K_v . For every fppf sheaf \mathcal{F} over U with generic fibre F , recall ([DH], Prop 2.1) the long exact sequence (where the piece of notation $v \notin U$ means that we consider all closed points of $X \setminus U$).

$$\dots \rightarrow H_c^i(U, \mathcal{F}) \rightarrow H^i(U, \mathcal{F}) \rightarrow \bigoplus_{v \notin U} H^i(K_v, F) \rightarrow H_c^{i+1}(U, \mathcal{F}) \rightarrow \dots \quad (3)$$

There is also a long exact sequence

$$\dots \rightarrow H_c^i(U, \mathcal{F}') \rightarrow H_c^i(U, \mathcal{F}) \rightarrow H_c^i(U, \mathcal{F}'') \rightarrow H_c^{i+1}(U, \mathcal{F}') \rightarrow \dots \quad (4)$$

associated to every short exact sequence

$$0 \rightarrow \mathcal{F}' \rightarrow \mathcal{F} \rightarrow \mathcal{F}'' \rightarrow 0$$

of fppf sheaves.

Let us now extend the construction of the groups $H_c^i(U, \dots)$ and [DH], Prop 2.1 to the case of bounded complexes. Let $\mathcal{C} := [\dots \rightarrow \mathcal{F}_i \rightarrow \mathcal{F}_{i+1} \rightarrow \dots]$ be a bounded complex of fppf sheaves over U . Let $\mathcal{C} \rightarrow I^\bullet(\mathcal{C})$ be an injective resolution of the complex \mathcal{C} , in the sense of [SP], Tag 013K. Following [DH], section 2, let $Z := X \setminus U$ and $Z' := \coprod_{v \in Z} \text{Spec}(K_v) \xrightarrow{i} U$. Denote by \mathcal{C}_v and $I^\bullet(\mathcal{C})_v$ their respective pullbacks to $\text{Spec} K_v$, for $v \notin U$.

We now define $\Gamma_c(U, I^\bullet(\mathcal{C}))$ to be the following object in the category of complexes of abelian groups:

$$\Gamma_c(U, I^\bullet(\mathcal{C})) := \text{Cone}(\Gamma(U, I^\bullet(\mathcal{C})) \rightarrow \Gamma(Z', i^* I^\bullet(\mathcal{C})))[-1],$$

and $H_c^r(U, \mathcal{C}) := H^r(\Gamma_c(U, I^\bullet(\mathcal{C})))$. We will also denote by $R\Gamma_c(U, \mathcal{C})$ the complex $\Gamma_c(U, I^\bullet(\mathcal{C}))$. Similarly, one can define, for any closed point $v \in X$, complexes $\Gamma_v(\mathcal{O}_v, \mathcal{C})$ computing fppf cohomology groups $H_v^r(\mathcal{O}_v, \mathcal{C})$ over $\text{Spec} \mathcal{O}_v$ with support in the closed point, as in [DH], before Lemma 2.6.

As in [DH], similar definitions could be made when K is a number field (taking into account the real places), but in this article we will focus on the function field case. However, we will make remarks regularly throughout the text explaining similarities and differences appearing in the number field case.

We will need the analogue of [DH], Prop 2.1 and Prop 2.12 for bounded complexes \mathcal{C} : by construction, the first two points of loc. cit. Prop. 2.1 (i.e. exact sequence (3) and (4)) still hold for bounded complexes.

Proposition 2.1. *Let \mathcal{C} be a bounded complex of flat affine commutative group schemes of finite type over U , and let $V \subset U$ be a non empty open subset.*

1. *There is a canonical commutative diagram of abelian groups:*

$$\begin{array}{ccccccc}
& & \bigoplus_{v \notin V} H^{r-1}(K_v, \mathcal{C}) & \xleftarrow{i_2} & \bigoplus_{v \notin U} H^{r-1}(K_v, \mathcal{C}) & & \\
& & \downarrow & & \downarrow & & \\
\cdots \longrightarrow & \bigoplus_{v \in U \setminus V} H^{r-1}(\mathcal{O}_v, \mathcal{C}) & \longrightarrow & H_c^r(V, \mathcal{C}) & \longrightarrow & H_c^r(U, \mathcal{C}) & \longrightarrow \bigoplus_{v \in U \setminus V} H^r(\mathcal{O}_v, \mathcal{C}) \longrightarrow \cdots \\
& & & \downarrow & & \downarrow & \\
& & & H^r(V, \mathcal{C}) & \xleftarrow{\text{Res}} & H^r(U, \mathcal{C}) & \\
& & & \downarrow & & \downarrow & \\
& & \bigoplus_{v \notin V} H^r(K_v, \mathcal{C}) & \xrightarrow{\pi} & \bigoplus_{v \notin U} H^r(K_v, \mathcal{C}), & &
\end{array}$$

where the long row and the columns are exact.

2. Let $V \subset U$ be a non empty open subset. Then there is an exact sequence

$$\cdots \rightarrow \bigoplus_{v \in U \setminus V} H_v^r(\mathcal{O}_v, \mathcal{C}) \rightarrow H^r(U, \mathcal{C}) \rightarrow H^r(V, \mathcal{C}) \rightarrow \bigoplus_{v \in U \setminus V} H_v^{r+1}(\mathcal{O}_v, \mathcal{C}) \rightarrow \cdots$$

Proof: We follow the proofs of [DH], Prop 2.1 3 and of Prop 2.12.

Easy dévissages imply that [DH], Lemmas 2.6 holds with \mathcal{F} replaced by a bounded complex of flat commutative group schemes of finite type and Lemma 2.9 holds for bounded complexes of fppf sheaves. Likewise Lemma 2.10 holds for bounded complexes of étale sheaves or of smooth commutative group schemes. Therefore, one can copy the proofs of [DH], Prop 2.1 (3) and Prop 2.12 to get the required Proposition. \square

Lemma 2.2. Let \mathcal{C} be a bounded complex of flat commutative group schemes of finite type over U with generic fibre C over K . Let i be an integer. For each $v \in U^{(1)}$, denote by $H_{\text{nr}}^i(K_v, \mathcal{C})$ the image of $H^i(\mathcal{O}_v, \mathcal{C})$ in $H^i(K_v, \mathcal{C})$. Let $V \subset U$ be a non empty Zariski open subset. Then there is an exact sequence

$$H^i(U, \mathcal{C}) \rightarrow \prod_{v \notin U} H^i(K_v, \mathcal{C}) \times \prod_{v \in U \setminus V} H_{\text{nr}}^i(K_v, \mathcal{C}) \rightarrow H_c^{i+1}(V, \mathcal{C}).$$

Proof: There is a commutative diagram such that the second line and the left column are exact (by (3) and Prop. 2.1 2.):

$$\begin{array}{ccccc}
H^i(U, \mathcal{C}) & \longrightarrow & \prod_{v \notin U} H^i(K_v, \mathcal{C}) \times \prod_{v \in U \setminus V} H_{\text{nr}}^i(K_v, \mathcal{C}) & & \\
\downarrow & & \downarrow j & & \\
H^i(V, \mathcal{C}) & \longrightarrow & \prod_{v \notin U} H^i(K_v, \mathcal{C}) \times \prod_{v \in U \setminus V} H^i(K_v, \mathcal{C}) & \longrightarrow & H_c^{i+1}(V, \mathcal{C}) \\
\downarrow & & \downarrow & & \\
\prod_{v \in U \setminus V} H_v^{i+1}(\mathcal{O}_v, \mathcal{C}) & \xrightarrow{=} & \prod_{v \in U \setminus V} H_v^{i+1}(\mathcal{O}_v, \mathcal{C}) & &
\end{array}$$

For $v \in U \setminus V$, the localization exact sequence (cf. proof of Proposition 2.1, 2.)

$$H^i(\mathcal{O}_v, \mathcal{C}) \rightarrow H^i(K_v, \mathcal{C}) \rightarrow H_v^{i+1}(\mathcal{O}_v, \mathcal{C})$$

yields that the second column is a complex. Since j is injective by definition, the required exact sequence follows by diagram chasing. \square

For a complex \mathcal{C} of finite flat group schemes, let us now endow the groups $H_c^*(U, \mathcal{C})$ with a natural topology, compatible with the one defined in [DH] in the case where \mathcal{C} is a finite flat group scheme.

Let F be a local field (that is: a field complete for a discrete valuation with finite residue field) and let $\mathcal{C} := [C_r \xrightarrow{f_r} C_{r+1} \rightarrow \cdots \rightarrow C_s]$ be a bounded complex of finite commutative group schemes over $\text{Spec } F$, with C_i in degree i . We assume that F is of positive characteristic p (if F is p -adic, then all groups $H^r(F, \mathcal{C})$ are finite by [Mi2], Cor. I.2.3).

Definition 2.3. A morphism $f : G_1 \rightarrow G_2$ of topological groups is *strict* if it is continuous, and the restriction $f : G_1 \rightarrow f(G_1)$ is an open map (where the topology on $f(G_1)$ is induced by G_2). This is equivalent to saying that f induces an isomorphism of the topological quotient $G_1/\ker f$ with the topological subspace $f(G_1) \subset G_2$.

Let $A := \ker(f_r)$, and let $\bar{\mathcal{C}} := \text{Cone}(A[-r] \xrightarrow{j} \mathcal{C})$. Then there is an exact triangle

$$A[-r] \xrightarrow{j} \mathcal{C} \xrightarrow{i} \bar{\mathcal{C}} \xrightarrow{p} A[1-r]. \quad (5)$$

In addition, we have a natural quasi-isomorphism $\varphi : \bar{\mathcal{C}} \rightarrow \mathcal{C}'$, where $\mathcal{C}' := [C_{r+1}/\text{Im}(f_r) \rightarrow C_{r+2} \rightarrow \cdots \rightarrow C_s]$ has a smaller length than \mathcal{C} .

There is an alternative dévissage for the complex \mathcal{C} , given by the exact triangle:

$$\tilde{\mathcal{C}} \xrightarrow{i'} \mathcal{C} \xrightarrow{p} C_r[-r] \xrightarrow{\partial} \tilde{\mathcal{C}}[1], \quad (6)$$

where $\tilde{\mathcal{C}} := [C_{r+1} \xrightarrow{f_{r+1}} C_{r+2} \rightarrow \cdots \rightarrow C_s]$.

Recall that for a finite and commutative F -group scheme N , the fppf groups $H^i(F, N)$ are finite if $i \neq 1$ ([Mi2], §III.6) and they are equipped with a locally compact topology for $i = 1$ by [Ces1]. By induction on the length of \mathcal{C} , one deduces that if $C_i = 0$, then $H^{i+1}(F, \mathcal{C})$ is finite. In particular, with the previous notation, we get that $H^i(F, \mathcal{C})$ is finite if $i \leq r$ or $i \geq s + 2$.

We now define a natural topology on $H^i(F, \mathcal{C})$ by induction on the length of \mathcal{C} , such that any morphism of such complexes induces a strict map between hypercohomology groups. Using the dévissages given by (5) and (6), one gets the following exact sequences

$$H^{i-1}(F, \mathcal{C}') \rightarrow H^{i-r}(F, A) \xrightarrow{f} H^i(F, \mathcal{C}) \xrightarrow{g} H^i(F, \mathcal{C}') \rightarrow H^{i+1-r}(F, A) \rightarrow \dots \quad (7)$$

and

$$H^{i-r-1}(F, C_r) \rightarrow H^i(F, \tilde{\mathcal{C}}) \xrightarrow{f'} H^i(F, \mathcal{C}) \xrightarrow{g'} H^{i-r}(F, C_r) \rightarrow H^{i+1}(F, \tilde{\mathcal{C}}) \rightarrow \dots, \quad (8)$$

where all the groups except $H^i(F, \mathcal{C})$ are endowed with a natural topology via the induction hypothesis (observe that $C'_r = 0$).

- Assume that $i = r + 1$. Equip $\text{Im } f \cong H^{i-r}(F, A)/\text{Im } H^{i-1}(F, \mathcal{C}')$ with the quotient topology. Then the two rightmost groups in exact sequence (7) are finite, and we can endow $H^{r+1}(F, \mathcal{C})$ with the topology such that $\text{Im } f$ is an open subgroup (see [DH], beginning of section 3). In other words it is the finest topology such that f is continuous. Then f and g are strict.

- Assume that $i \neq r + 1$. Then $H^{i-r}(F, C_r)$ is finite (and discrete), and using exact sequence (8) one can similarly endow $H^i(F, \mathcal{C})$ with the finest topology making f' continuous. Then both maps f' and g' are strict.

By construction, this topology is functorial in \mathcal{C} , i.e. if $\mathcal{C}_1 \rightarrow \mathcal{C}_2$ is a morphism of complexes, then the induced map $H^i(F, \mathcal{C}_1) \rightarrow H^i(F, \mathcal{C}_2)$ is strict. In addition, given a quasi-isomorphism $\mathcal{C}_1 \rightarrow \mathcal{C}_2$, the induced morphism on cohomology groups is a homeomorphism.

Let us now deal with the topology on the groups $H_c^*(U, \mathcal{C})$, where \mathcal{C} is a complex of finite flat commutative group schemes defined over U . Recall that we have an exact sequence analogous to (3):

$$H^{i-1}(U, \mathcal{C}) \rightarrow \bigoplus_{v \notin U} H^{i-1}(K_v, \mathcal{C}) \rightarrow H_c^i(U, \mathcal{C}) \rightarrow H^i(U, \mathcal{C}). \quad (9)$$

We endow the groups $H^i(U, \mathcal{C})$ with the discrete topology, and the groups $H^{i-1}(K_v, \mathcal{C})$ with the topology defined above.

Lemma 2.4. *For all i , the map $H^i(U, \mathcal{C}) \rightarrow \bigoplus_{v \notin U} H^i(K_v, \mathcal{C})$ has discrete image.*

Proof: We prove the result by induction on the length of \mathcal{C} , using the dévissages induced by the exact triangles (5) and (6).

- Assume that $i = r + 1$. Using the exact triangle (6), we get the following commutative diagram of long exact sequences of topological groups (where all maps are strict):

$$\begin{array}{ccccc} H^{r+1}(U, \tilde{\mathcal{C}}) & \longrightarrow & H^{r+1}(U, \mathcal{C}) & \longrightarrow & H^1(U, C_r) \\ \downarrow & & \downarrow & & \downarrow \\ \bigoplus_{v \notin U} H^{r+1}(K_v, \tilde{\mathcal{C}}) & \longrightarrow & \bigoplus_{v \notin U} H^{r+1}(K_v, \mathcal{C}) & \longrightarrow & \bigoplus_{v \notin U} H^1(K_v, C_r). \end{array}$$

The groups on the left hand side are finite, and by [Čes2] Lemma 2.7, the image of the right hand side map is discrete in $\bigoplus_{v \notin U} H^1(K_v, C_r)$. Since $\bigoplus_{v \notin U} H^{r+1}(K_v, \mathcal{C})$ is Hausdorff, an easy topological argument implies that the image of the central vertical map is discrete.

- Assume that $i \neq r + 1$. Using the exact triangle (5), we get the following commutative diagram of long exact sequences of topological groups:

$$\begin{array}{ccccc} H^{i-r}(U, A) & \longrightarrow & H^i(U, \mathcal{C}) & \longrightarrow & H^i(U, \mathcal{C}') \\ \downarrow & & \downarrow & & \downarrow \\ \bigoplus_{v \notin U} H^{i-r}(K_v, A) & \longrightarrow & \bigoplus_{v \notin U} H^i(K_v, \mathcal{C}) & \longrightarrow & \bigoplus_{v \notin U} H^i(K_v, \mathcal{C}'). \end{array}$$

The groups on the left hand side are finite, and by induction on the length of the complex, the image of the right hand side map is discrete in $\bigoplus_{v \notin U} H^i(K_v, \mathcal{C}')$. A similar topological argument as before implies that the central vertical map is discrete.

□

As a consequence of this Lemma, one can endow $H_c^i(U, \mathcal{C})$ with the following topology: we put the quotient topology on the group $\bigoplus_{v \notin U} H^i(K_v, \mathcal{C}) / \text{Im } H^i(U, \mathcal{C})$ (this topology is Hausdorff), and since $H^i(U, \mathcal{C})$ is discrete, there is a unique topology on $H_c^i(U, \mathcal{C})$ such that the maps in the exact sequence (9) are strict.

Lemma 2.5. *The topological group $H_c^i(U, \mathcal{C})$ is profinite.*

Proof: We prove this Lemma by induction on the length of the complex \mathcal{C} . By [DH], Proposition 3.5, the Lemma is proven when \mathcal{C} is a complex of length one, i.e. concentrated in one given degree.

Given a complex \mathcal{C} , consider the previous dévissages:

- Assume that $i = r + 1$. Then the exact sequence (7) implies that the group $H_c^{r+1}(U, \mathcal{C})$ is an extension (the maps being strict) of a (discrete) finite group by a profinite group (which is a quotient of $H_c^1(U, A)$ by a closed subgroup), hence $H_c^{r+1}(U, \mathcal{C})$ is profinite.
- Assume that $i \neq r + 1$. Then the exact sequence (8) implies that $H_c^i(U, \mathcal{C})$ is an extension of a finite (discrete) group by a profinite group (which is a quotient of $H_c^i(U, \tilde{\mathcal{C}})$ by a closed subgroup), hence it is profinite.

□

3 Cohomology of tori and short complexes of tori

Let U be a non empty Zariski open subset of X . Recall that for every U -torus \mathcal{T} (in the sense of [SGA3], IX, Déf. 1.3), there is a finite étale covering (that can be taken to be connected and Galois) V of U such that $\mathcal{T}_V := \mathcal{T} \times_U V$ is *split*, that is: isomorphic to some power \mathbf{G}_m^r ($r \in \mathbf{N}$) of the multiplicative group ([SGA3], X, Th. 5.16). The *group of characters* $\hat{\mathcal{T}}$ of \mathcal{T} is a U -group scheme locally isomorphic to \mathbf{Z}^r for the étale topology, namely it is a torsion-free and finite type $\text{Gal}(V/U)$ -module.

Given a complex of U -tori $\mathcal{C} = [\mathcal{T}_1 \xrightarrow{\rho} \mathcal{T}_2]$ with generic fibre $C = [T_1 \xrightarrow{\rho} T_2]$ over K , where by convention \mathcal{T}_1 is in degree -1 and \mathcal{T}_2 in degree 0 , we can apply the construction of section 2. Namely we have dual complexes $\hat{\mathcal{C}} = [\hat{\mathcal{T}}_2 \rightarrow \hat{\mathcal{T}}_1]$ and $\hat{C} = [\hat{T}_2 \xrightarrow{\hat{\rho}} \hat{T}_1]$ (concentrated in degrees -1 and 0), which are respectively defined over U and over K . Fix a separable closure \bar{K} of K . Denote by S the finite set $X \setminus U$ and by $G_S = \pi_1^{\text{ét}}(U)$ the étale fundamental group of U , which is the Galois group of the maximal field extension $K_S \subset \bar{K}$ of K unramified outside S ; then each $\hat{\mathcal{T}}_i$ ($i = 1, 2$) can be viewed as a discrete G_S -module.

Recall that fppf and étale cohomology coincide for sheaves represented by smooth group schemes ([Mi1], section III.3) like a torus \mathcal{T} , its group of characters $\hat{\mathcal{T}}$, or finite flat group schemes of order prime to p . In particular (by [Mi1], Lemma III.1.16) we have for every integer i :

$$\varinjlim_U H^i(U, \mathcal{C}) \cong H^i(K, C)$$

(where the limit is over all non empty Zariski open subsets U of X), and likewise for the complex $\hat{\mathcal{C}}$.

For such 2-term complexes, the pairings of section 2 can be made explicit (see [Dem1], section 2; note that the sign conventions are slightly different here), and give maps

$$\mathcal{C} \otimes^{\mathbf{L}} \widehat{\mathcal{C}} \rightarrow \mathbf{G}_m[1]; \quad C \otimes^{\mathbf{L}} \widehat{C} \rightarrow \mathbf{G}_m[1]$$

in the bounded derived category $\mathcal{D}^b(U)$ (resp. $\mathcal{D}^b(\text{Spec } K)$) of fppf sheaves over U (resp. over $\text{Spec } K$). In the case $T_1 = 0$ or $T_2 = 0$, we recover (up to shift) the classical pairings $\mathcal{T} \otimes \widehat{\mathcal{T}} \rightarrow \mathbf{G}_m$ and $T \otimes \widehat{T} \rightarrow \mathbf{G}_m$ associated to one single torus \mathcal{T} . We also have for each positive integer n the n -adic realizations

$$T_{\mathbf{Z}/n}(\mathcal{C}) := H^0(\mathcal{C}[-1] \otimes^{\mathbf{L}} \mathbf{Z}/n); \quad T_{\mathbf{Z}/n}(\widehat{\mathcal{C}}) := H^0(\widehat{\mathcal{C}}[-1] \otimes^{\mathbf{L}} \mathbf{Z}/n)$$

and likewise for C and \widehat{C} . The fppf sheaf $T_{\mathbf{Z}/n}(\mathcal{C})$ is representable by a finite group scheme of multiplicative type over U (in the sense of [SGA3], IX, Déf. 1.1) with Cartier dual $T_{\mathbf{Z}/n}(\widehat{\mathcal{C}})$, and similarly for $T_{\mathbf{Z}/n}(C)$ and $T_{\mathbf{Z}/n}(\widehat{C})$ over K . Besides we have exact triangles ([Dem1], Lemme 2.3), where for every abelian group (or group scheme) A , the piece of notation ${}_n A$ stands for the n -torsion subgroup of A :

$${}_n(\ker \rho)[2] \rightarrow \mathcal{C} \otimes^{\mathbf{L}} \mathbf{Z}/n \rightarrow T_{\mathbf{Z}/n}(\mathcal{C})[1] \rightarrow {}_n(\ker \rho)[3] \quad (10)$$

and

$$T_{\mathbf{Z}/n}(\widehat{\mathcal{C}})[1] \rightarrow \widehat{\mathcal{C}} \otimes^{\mathbf{L}} \mathbf{Z}/n \rightarrow \widehat{{}_n(\ker \rho)} \rightarrow T_{\mathbf{Z}/n}(\widehat{\mathcal{C}})[2] \quad (11)$$

in $\mathcal{D}^b(U)$, and similar triangles for C, \widehat{C} in $\mathcal{D}^b(\text{Spec } K)$.

Note also that the objects $\mathcal{C} \otimes^{\mathbf{L}} \mathbf{Z}/n$ and $\widehat{\mathcal{C}} \otimes^{\mathbf{L}} \mathbf{Z}/n$ in the derived category have canonical representatives as complexes of fppf sheaves given by $\mathcal{C} \otimes^{\mathbf{L}} \mathbf{Z}/n = \text{Tot}(\mathcal{C} \otimes [\mathbf{Z} \xrightarrow{n} \mathbf{Z}])$ and $\widehat{\mathcal{C}} \otimes^{\mathbf{L}} \mathbf{Z}/n = \text{Tot}(\widehat{\mathcal{C}} \otimes \mathbf{Z}/n)$.

We also have an exact triangle in $\mathcal{D}^b(U)$:

$$(\ker \rho)[1] \rightarrow \mathcal{C} \rightarrow \text{coker } \rho \rightarrow (\ker \rho)[2], \quad (12)$$

where $\text{coker } \rho$ is a torus and $M := \ker \rho$ is a group of multiplicative type, and the dual exact triangle

$$\widehat{(\text{coker } \rho)}[1] \rightarrow \widehat{\mathcal{C}} \rightarrow \widehat{\ker \rho} \rightarrow \widehat{(\text{coker } \rho)}[2]. \quad (13)$$

For every integer i , there are exact sequences

$$\dots \rightarrow H^i(U, \mathcal{T}_1) \rightarrow H^i(U, \mathcal{T}_2) \rightarrow H^i(U, \mathcal{C}) \rightarrow H^{i+1}(U, \mathcal{T}_1) \rightarrow \dots \quad (14)$$

$$\dots \rightarrow H^i(U, \widehat{\mathcal{T}}_2) \rightarrow H^i(U, \widehat{\mathcal{T}}_1) \rightarrow H^i(U, \widehat{\mathcal{C}}) \rightarrow H^{i+1}(U, \widehat{\mathcal{T}}_2) \rightarrow \dots \quad (15)$$

and we also have similar exact sequences for the compact support fppf cohomology groups $H_c^i(U, \dots)$.

Lemma 3.1. *a) Let \mathcal{T} be a torus over U . Then $H^0(U, \mathcal{T})$ and $H^1(U, \mathcal{T})$ are of finite type. If $U \neq X$, then $H^1(U, \mathcal{T})$ is finite.*

b) Let \mathcal{N} be a finite group scheme of multiplicative type over U . Then $H^1(U, \mathcal{N})$ and $H_c^2(U, \mathcal{N}^D)$ are finite.

Proof: a) Let V be an étale and finite connected Galois covering of U such that $\mathcal{T}_V := \mathcal{T} \times_U V$ is isomorphic to \mathbf{G}_m^r for some non negative integer r . The group $\mathcal{T}(V) \simeq H^0(V, \mathbf{G}_m)^r$ is of finite type by Dirichlet's theorem on units. Therefore $H^0(U, \mathcal{T}) \subset H^0(V, \mathcal{T}) = \mathcal{T}(V)$ is also of finite type.

Set $G = \text{Gal}(V/U)$. Then the Hochschild-Serre spectral sequence provides an exact sequence

$$0 \rightarrow H^1(G, \mathcal{T}(V)) \rightarrow H^1(U, \mathcal{T}) \rightarrow H^1(V, \mathcal{T}_V).$$

Since $\mathcal{T}_V \cong \mathbf{G}_m^r$, the group $H^1(V, \mathcal{T}_V) \cong (\text{Pic } V)^r$ is of finite type (resp. finite if $U \neq X$) by finiteness of the ideal class group of a global field. As $\mathcal{T}(V)$ is of finite type and G is finite, the group $H^1(G, \mathcal{T}(V))$ is finite by [Ser], Chap. VIII, Cor 2. Thus $H^1(U, \mathcal{T})$ is of finite type (resp. finite if $U \neq X$) as well.

b) We can assume (by ([Mi2], Lemma III.8.9 and [DH], Cor. 4.9) that $U \neq X$. Since every finitely generated Galois module is a quotient of a torsion-free and finitely generated Galois module, the assumption that \mathcal{N} is of multiplicative type implies (by [SGA3], X, Prop 1.1) that there is an exact sequence of U -group schemes

$$0 \rightarrow \mathcal{N} \rightarrow \mathcal{T}_1 \rightarrow \mathcal{T}_2 \rightarrow 0,$$

where \mathcal{T}_1 and \mathcal{T}_2 are U -tori. Therefore there is an exact sequence of abelian groups

$$H^0(U, \mathcal{T}_2) \rightarrow H^1(U, \mathcal{N}) \rightarrow H^1(U, \mathcal{T}_1).$$

By a), we know that $H^1(U, \mathcal{T}_1)$ is finite. Let n be the order of \mathcal{N} ; then the map $H^0(U, \mathcal{T}_2) \rightarrow H^1(U, \mathcal{N})$ factorizes through a map $H^0(U, \mathcal{T}_2)/n \rightarrow H^1(U, \mathcal{N})$. But $H^0(U, \mathcal{T}_2)/n$ is finite because $H^0(U, \mathcal{T}_2)$ is of finite type by a). Finally $H^1(U, \mathcal{N})$ is finite. The finiteness of $H_c^2(U, \mathcal{N}^D)$ follows by Artin-Mazur-Milne duality ([DH], Th. 1.1). \square

Remark 3.2. By dévissage, the finiteness of $H^1(U, \mathcal{N})$ holds for a (not necessarily finite) group of multiplicative type \mathcal{N} because such a group is an extension of a finite group by a torus. Recall also ([Mi2], Lemma III.8.9 and [DH], Cor. 4.8) that for every finite and flat commutative group scheme \mathcal{N} over U , the groups $H^i(U, \mathcal{N})$ and $H_c^{3-i}(U, \mathcal{N})$ are finite if $i \neq 1$ or if $U = X$, and also if p does not divide the order of \mathcal{N} (by [DH], Prop 2.1., 4. and [Mi2], Th. II.3.1). Besides these groups are trivial if $i \geq 4$ (this is part of [DH], Th. 1.1).

For an fppf sheaf (or a bounded complex of fppf sheaves) \mathcal{F} on U with generic fibre F , we set (cf. exact sequence (3))

$$D^i(U, \mathcal{F}) = \text{Ker} [H^i(U, \mathcal{F}) \rightarrow \bigoplus_{v \notin U} H^i(K_v, F)] = \text{Im} [H_c^i(U, \mathcal{F}) \rightarrow H^i(U, \mathcal{F})].$$

Lemma 3.3. *We have $D^2(U, \mathbf{G}_m) = 0$.*

Proof: Let \mathbf{F}_v be the residue field of K_v . By [Mi2], Prop II.1.1 b), we have

$$H^2(\mathcal{O}_v, \mathbf{G}_m) = \text{Br } \mathcal{O}_v \cong \text{Br } \mathbf{F}_v = 0$$

because \mathbf{F}_v is finite. The Brauer group $\text{Br } U$ of U injects into $\text{Br } K$ ([Mi1], Cor. IV.2.6). Now every element of $D^2(U, \mathbf{G}_m) \subset \text{Br } U \subset \text{Br } K$ has trivial restriction to $\text{Br } K_v$

for all places v of K , hence it is trivial by Brauer-Hasse-Noether Theorem ([NSW], Th. VIII.1.17). □

Lemma 3.4. *Let \mathcal{T} be a U -torus with generic fibre T .*

- a) *The group $H^1(U, \widehat{\mathcal{T}})$ is finite; the groups $H^0(U, \widehat{\mathcal{T}})$ and $H_c^0(U, \widehat{\mathcal{T}})$ are of finite type and torsion-free. The group $H_c^1(U, \widehat{\mathcal{T}})$ is of finite type.*
- b) *The group $H_c^2(U, \mathcal{T})$ is finite. In particular, if $U = X$, then $H^2(X, \mathcal{T})$ is finite.*
- c) *Assume $U \neq X$. Then $H_c^2(U, \widehat{\mathcal{T}})$ is finite.*
- d) *Assume $i \geq 4$. Then $H^i(U, \widehat{\mathcal{T}}) = H_c^i(U, \widehat{\mathcal{T}}) = 0$. If $U \neq X$, then $H^3(U, \widehat{\mathcal{T}}) = 0$.*
- e) *If $U = X$, then $H^3(U, \widehat{\mathcal{T}}) = H^3(X, \widehat{\mathcal{T}})$ is finite.*

Proof: a) Let V be a finite connected Galois étale covering of U such that $\mathcal{T} \times_U V$ is split. Let L be the function field of V and set $G = \text{Gal}(L/K)$. We have $H^1(V, \mathbf{Z}) = 0$ because $H^1(V, \mathbf{Z})$ injects into $H^1(L, \mathbf{Z})$ by Leray spectral sequence. Therefore we have $H^1(V, \widehat{\mathcal{T}}) = 0$, hence the group $H^1(U, \widehat{\mathcal{T}})$ identifies (by the Hochschild-Serre spectral sequence) to a subgroup of $H^1(G, \widehat{\mathcal{T}})$, which is finite because $\widehat{\mathcal{T}}$ is a G -module of finite type. The assertion about $H^0(U, \widehat{\mathcal{T}})$ (which is a subgroup of $H^0(K, \widehat{\mathcal{T}})$) and $H_c^0(U, \widehat{\mathcal{T}}) \subset H^0(U, \widehat{\mathcal{T}})$ is obvious (we even have $H_c^0(U, \widehat{\mathcal{T}}) = 0$ if $U \neq X$). Also the exact sequence

$$\bigoplus_{v \notin U} H^0(K_v, \widehat{\mathcal{T}}) \rightarrow H_c^1(U, \widehat{\mathcal{T}}) \rightarrow H^1(U, \widehat{\mathcal{T}})$$

shows that $H_c^1(U, \widehat{\mathcal{T}})$ is of finite type.

b) By (3), there is an exact sequence

$$\bigoplus_{v \notin U} H^1(K_v, T) \rightarrow H_c^2(U, \mathcal{T}) \rightarrow D^2(U, \mathcal{T}) \rightarrow 0.$$

The groups $H^1(K_v, T)$ are finite ([Mi2], Cor. I.2.3). Since we have $D^2(V, \mathbf{G}_m) = 0$ by Lemma 3.3, a restriction-corestriction argument shows that $D^2(U, \mathcal{T})$ is a subgroup of ${}_n H^2(U, \mathcal{T})$, where $n = \#\text{Gal}(V/U)$. By the Kummer sequence in the fppf topology

$$0 \rightarrow {}_n \mathcal{T} \rightarrow \mathcal{T} \xrightarrow{\cdot n} \mathcal{T} \rightarrow 0,$$

the group ${}_n H^2(U, \mathcal{T})$ is a quotient of $H^2(U, {}_n \mathcal{T})$, which is finite (even if p divides n , cf. Remark 3.2). Thus $H_c^2(U, \mathcal{T})$ is finite.

c) Let $n > 0$. Using the exact sequence in the fppf topology

$$0 \rightarrow \widehat{\mathcal{T}} \xrightarrow{\cdot n} \widehat{\mathcal{T}} \rightarrow \widehat{\mathcal{T}}/n \rightarrow 0, \tag{16}$$

we see that ${}_n H_c^2(U, \widehat{\mathcal{T}})$ is a quotient of $H_c^1(U, \widehat{\mathcal{T}}/n)$, which is finite (see Remark 3.2). It is therefore sufficient to show that $H_c^2(U, \widehat{\mathcal{T}})$ is of finite exponent, and by a restriction-corestriction argument, we reduce to the case $\widehat{\mathcal{T}} = \mathbf{Z}$. As $H^1(K_v, \mathbf{Z}) = 0$, we have

$$H_c^2(U, \mathbf{Z}) = \text{Ker}[H^2(U, \mathbf{Z}) \rightarrow \bigoplus_{v \notin U} H^2(K_v, \mathbf{Z})].$$

By [Mi2], Lemma II.2.10, this yields

$$H_c^2(U, \mathbf{Z}) \cong \text{Ker}[H^1(U, \mathbf{Q}/\mathbf{Z}) \rightarrow \bigoplus_{v \notin U} H^1(K_v, \mathbf{Q}/\mathbf{Z})],$$

hence

$$H_c^2(U, \mathbf{Z}) \cong \text{Ker}[H^1(G_S, \mathbf{Q}/\mathbf{Z}) \rightarrow \bigoplus_{v \notin U} H^1(K_v, \mathbf{Q}/\mathbf{Z})].$$

Therefore $H_c^2(U, \mathbf{Z})$ is a subgroup of $H^1(\text{Gal}(L/K), \mathbf{Q}/\mathbf{Z})$, where $L \subset K_S$ is the maximal abelian extension of K that is unramified outside S and totally decomposed at every $v \in S = X \setminus U$. The group $\text{Gal}(L/K)$ is isomorphic to $\text{Pic } U$ by class field theory, which implies that it is finite because $U \neq X$. Hence $H_c^2(U, \mathbf{Z})$ is finite, which proves the lemma.

d) For $i \geq 4$, the groups $H_c^i(U, \widehat{\mathcal{T}})$ and $H^i(U, \widehat{\mathcal{T}})$ coincide thanks to exact sequence (3) because the local field K_v is of strict cohomological dimension 2 ([NSW], Cor. 7.2.5). Assume $U \neq X$ and $i \geq 3$. Then, by [Mi2], Prop II.2.9., we have $H^i(U, \widehat{\mathcal{T}}) = H^i(G_S, \widehat{\mathcal{T}})$, which is zero: indeed G_S is of strict cohomological dimension 2 by [NSW], Th. 8.3.17. It remains to deal with the case $U = X$ (now we assume $i \geq 4$). By [Mi2], Lemma II.2.10, the group $H^i(X, \mathbf{Z})$ is torsion; by a restriction-corestriction argument, the same holds for $H^i(X, \widehat{\mathcal{T}})$. Since \mathbf{Q} is uniquely divisible, this yields

$$H^i(X, \widehat{\mathcal{T}}) = H^{i-1}(X, \widehat{\mathcal{T}} \otimes \mathbf{Q}/\mathbf{Z}) = \varprojlim_n H^{i-1}(X, \widehat{\mathcal{T}}/n).$$

For $i \geq 5$, the group $H^{i-1}(X, \widehat{\mathcal{T}}/n)$ is zero (cf. Remark 3.2), so we are done. Assume $i = 4$. We observe that the finite group $H^3(X, \widehat{\mathcal{T}}/n)$ is dual to $H^0(X, {}_n\mathcal{T})$ by Artin-Mazur-Milne duality ([DH], Th. 1.1), so the dual of the discrete torsion group $H^4(X, \widehat{\mathcal{T}})$ is the profinite group

$$\varprojlim_n H^0(X, {}_n\mathcal{T}) = \varprojlim_n (T(K))$$

(the equality holds because the X -group scheme ${}_n\mathcal{T}$ is finite and X is connected). But K is a global field, hence $T(K)_{\text{tors}}$ is finite: indeed if L is a finite extension of K such that T splits over L , then $T(K) \subset T(L)$ with $T(L) \simeq (L^*)^r$ for some r , and L^* contains only finitely many roots of unity. Therefore $T(K)$ has trivial Tate module, which yields the result.

e) Using exact sequence (16), we get a surjection $H^2(X, \widehat{\mathcal{T}}/n) \rightarrow {}_nH^3(X, \widehat{\mathcal{T}})$, so it is sufficient (by Remark 3.2) to show that $H^3(X, \widehat{\mathcal{T}})$ is of finite exponent. By restriction-corestriction, we can therefore assume that $\widehat{\mathcal{T}} = \mathbf{Z}$. By the same method as in d), we get that the dual of $H^3(X, \mathbf{Z})$ is $\varprojlim_n H^1(X, \mu_n)$. As $H^0(X, \mathbf{G}_m) = k^*$ because X is a proper and geometrically integral curve, we get an exact sequence of finite groups

$$0 \rightarrow k^*/k^{*n} \rightarrow H^1(X, \mu_n) \rightarrow {}_n\text{Pic } X \rightarrow 0.$$

Since $\text{Pic } X$ is of finite type, we have $\varprojlim_n ({}_n\text{Pic } X) = 0$, hence $\varprojlim_n H^1(X, \mu_n)$ is the inverse limit of the k^*/k^{*n} , which is k^* itself because k is finite. Thus $H^3(X, \mathbf{Z})$ is the dual of k^* , which is finite (but not zero). □

Remark 3.5. Assume $U = X$. Then the group $H_c^2(U, \widehat{\mathcal{T}}) = H^2(X, \widehat{\mathcal{T}})$ is in general infinite: for example $H^2(X, \mathbf{Z}) \cong H^1(X, \mathbf{Q}/\mathbf{Z})$ is the dual of the étale fundamental group $\pi_1^{\text{ét}}(X)$ and the latter is an extension of $\text{Gal}(\bar{k}/k) = \widehat{\mathbf{Z}}$; therefore $H^2(X, \mathbf{Z})$ contains a copy of \mathbf{Q}/\mathbf{Z} .

Proposition 3.6. Let $\mathcal{C} = [\mathcal{T}_1 \xrightarrow{\rho} \mathcal{T}_2]$ be a complex of U -tori with generic fibre $C = [T_1 \rightarrow T_2]$.

a) Let $i \in \{-1, 0\}$. Then the groups $H^i(U, \widehat{\mathcal{C}})$ and $H_c^i(U, \widehat{\mathcal{C}})$ are of finite type, and the restriction map $H^i(U, \widehat{\mathcal{C}}) \rightarrow H^i(K, \widehat{\mathcal{C}})$ is an isomorphism. The restriction map $H^1(U, \widehat{\mathcal{C}}) \rightarrow H^1(K, \widehat{\mathcal{C}})$ is injective. If $U \neq X$, then $H_c^1(U, \widehat{\mathcal{C}})$ is of finite type.

b) The groups $H^{-1}(U, \mathcal{C})$ and $H_c^{-1}(U, \mathcal{C})$ are of finite type, and so is $H^0(U, \mathcal{C})$. If $U = X$, then $H^1(U, \mathcal{C}) = H^1(X, \mathcal{C})$ is of finite type.

c) Assume $U \neq X$. Then $D^1(U, \mathcal{C})$ and $D^1(U, \widehat{\mathcal{C}})$ are finite.

Proof: a) The fact that $H^i(U, \widehat{\mathcal{C}})$ and $H_c^i(U, \widehat{\mathcal{C}})$ are of finite type for $i \in \{-1, 0\}$ follows immediately by dévissage (cf. exact sequences (15) and (3)) from Lemma 3.4 a), and we even have $H_c^{-1}(U, \widehat{\mathcal{C}}) = 0$ if $U \neq X$. For a U -torus \mathcal{T} , the restriction map $H^0(U, \widehat{\mathcal{T}}) \rightarrow H^0(K, \widehat{\mathcal{T}})$ obviously is an isomorphism. Let V be a connected Galois covering of U with function field L and group G , such that \mathcal{T} splits over V . As seen before, we have $H^1(V, \mathbf{Z}) = H^1(L, \mathbf{Z}) = 0$, hence $H^1(V, \widehat{\mathcal{T}}) = H^1(L, \widehat{\mathcal{T}}) = 0$. By the Hochschild-Serre spectral sequence we get $H^1(U, \widehat{\mathcal{T}}) \cong H^1(K, \widehat{\mathcal{T}})$ because both groups identify to $H^1(G, \widehat{\mathcal{T}})$. By [Mi2], Lemma II.2.10, we have

$$H^2(U, \widehat{\mathcal{T}}) \cong H^1(U, \widehat{\mathcal{T}} \otimes \mathbf{Q}/\mathbf{Z}) = \varinjlim_{n>0} H^1(U, \widehat{\mathcal{T}}/n),$$

and $H^1(U, \widehat{\mathcal{T}}/n) \hookrightarrow H^1(K, \widehat{\mathcal{T}}/n)$ because $\widehat{\mathcal{T}}/n$ is a finite U -group scheme, which implies $H^2(U, \widehat{\mathcal{T}}) \hookrightarrow H^2(K, \widehat{\mathcal{T}})$.

The commutative diagram with exact lines

$$\begin{array}{ccccccccc} 0 & \longrightarrow & H^i(U, \widehat{\mathcal{T}}_1) & \longrightarrow & H^i(U, \widehat{\mathcal{T}}_2) & \longrightarrow & H^i(U, \widehat{\mathcal{C}}) & \longrightarrow & H^{i+1}(U, \widehat{\mathcal{T}}_1) & \longrightarrow & H^{i+1}(U, \widehat{\mathcal{T}}_2) \\ & & \downarrow & & \downarrow & & \downarrow & & \downarrow & & \downarrow \\ 0 & \longrightarrow & H^i(K, \widehat{\mathcal{T}}_1) & \longrightarrow & H^i(K, \widehat{\mathcal{T}}_2) & \longrightarrow & H^i(K, \widehat{\mathcal{C}}) & \longrightarrow & H^{i+1}(K, \widehat{\mathcal{T}}_1) & \longrightarrow & H^{i+1}(K, \widehat{\mathcal{T}}_2) \end{array}$$

and the five lemma now give that the restriction map $H^i(U, \widehat{\mathcal{C}}) \rightarrow H^i(K, \widehat{\mathcal{C}})$ is an isomorphism for $i \in \{-1, 0\}$, and is injective for $i = 1$. The fact that $H_c^1(U, \widehat{\mathcal{C}})$ is of finite type if $U \neq X$ is immediate by dévissage thanks to Lemma 3.4 c).

b) The first two assertions follow from Lemma 3.1, using exact sequence (14). For $U = X$, every X -torus T satisfies that $H^1(X, \mathcal{T})$ is of finite type (Lemma 3.1) and $H^2(X, \mathcal{T})$ is finite (Lemma 3.4, b), whence the result.

c) By functoriality, the image of $D^1(U, \mathcal{C})$ by the map $u : H^1(U, \mathcal{C}) \rightarrow H^2(U, \mathcal{T}_1)$ is a subgroup of $D^2(U, \mathcal{T}_1)$. The latter is finite by Lemma 3.4 b), because it is a quotient of $H_c^2(U, \mathcal{T}_1)$. As the kernel of u is a quotient of $H^1(U, \mathcal{T}_2)$ (which is finite by Lemma 3.1 a), this means that $D^1(U, \mathcal{C})$ is finite.

The group $H_c^2(U, \widehat{\mathcal{T}}_2)$ is finite by Lemma 3.4, c). Hence $D^2(U, \widehat{\mathcal{T}}_2)$ is finite. On the other hand, the kernel of the map $H^1(U, \widehat{\mathcal{C}}) \rightarrow H^2(U, \widehat{\mathcal{T}}_2)$ is a quotient of the group $H^1(U, \widehat{\mathcal{T}}_1)$, which is finite by Lemma 3.4 a). Thus $D^1(U, \widehat{\mathcal{C}})$ is finite. \square

Remark 3.7. The same argument as in Proposition 3.6 a) shows that for $v \in U$, the restriction map $H^i(\mathcal{O}_v, \widehat{\mathcal{C}}) \rightarrow H^i(K_v, \widehat{\mathcal{C}})$ is an isomorphism for $i \in \{-1, 0\}$, and is injective for $i = 1$.

Recall ([Dem1], section 3) that for $v \in X^{(1)}$, given a complex C of K_v -tori, the groups $H^{-1}(K_v, C)$ and $H^0(K_v, C)$ are equipped with a natural Hausdorff topology (and the groups $H^i(K_v, C)$ are endowed with the discrete topology for $i \geq 1$, as are all groups $H^r(K_v, \widehat{\mathcal{C}})$ for $-1 \leq r \leq 2$).

Lemma 3.8. *The image I of the group $H^0(U, \mathcal{C})$ into $\bigoplus_{v \notin U} H^0(K_v, C)$ is a discrete (hence closed) subgroup, and so is the image of $H^{-1}(U, \mathcal{C})$ into $\bigoplus_{v \notin U} H^{-1}(K_v, C)$.*

Proof: We can assume $U \neq X$. Let us start with the case when $\mathcal{C} = \mathbf{G}_m$. Then $\mathcal{O}_U^* := H^0(U, \mathbf{G}_m)$ is a discrete subgroup of $\bigoplus_{v \notin U} K_v^*$, because its intersection with the open subgroup $\bigoplus_{v \notin U} \mathcal{O}_v^*$ is $H^0(X, \mathbf{G}_m) = k^*$, which is finite. Consider now a U -torus \mathcal{T} . Let W be a connected Galois finite covering of U (with function field $L \supset K$) that splits \mathcal{T} . Let $G := \text{Gal}(L/K)$. By the case $\mathcal{T} = \mathbf{G}_m$, the subgroup $H^0(W, \mathcal{T})$ is discrete in $\bigoplus_{w \notin W} H^0(L_w, T)$, so $H^0(U, \mathcal{T})$ is discrete in $\bigoplus_{v \notin U} H^0(K_v, T)$ because it is the intersection of $H^0(W, \mathcal{T}) \subset \bigoplus_{w \notin W} H^0(L_w, T)$ with $\bigoplus_{v \notin U} H^0(K_v, T) = (\bigoplus_{w \notin W} H^0(L_w, T))^G$. Thus I is discrete when $\mathcal{C} = \mathcal{T}$ is one single torus.

In the general case, exact triangle (12) yields a commutative diagram with exact lines

$$\begin{array}{ccccc} H^1(U, \mathcal{M}) & \longrightarrow & H^0(U, \mathcal{C}) & \xrightarrow{u} & H^0(U, \mathcal{T}) \\ \downarrow & & \downarrow j & & \downarrow \\ \bigoplus_{v \notin U} H^1(K_v, M) & \longrightarrow & \bigoplus_{v \notin U} H^0(K_v, C) & \longrightarrow & \bigoplus_{v \notin U} H^0(K_v, T), \end{array}$$

where \mathcal{M} is a U -group of multiplicative type and \mathcal{T} is a U -torus. Since $U \neq X$, the right vertical map is injective. As the lemma holds for $\mathcal{C} = \mathcal{T}$, the image J of $H^0(U, \mathcal{T})$ into $\bigoplus_{v \notin U} H^0(K_v, T)$ is discrete, hence there is an open subgroup H of $\bigoplus_{v \notin U} H^0(K_v, T)$ such that $J \cap H = \{0\}$. Let H_1 be the inverse image of H in $\bigoplus_{v \notin U} H^0(K_v, C)$, it is an open subgroup of $\bigoplus_{v \notin U} H^0(K_v, C)$ such that $j^{-1}(H_1)$ is a subgroup of $\ker u$. As $H^1(U, \mathcal{M})$ is finite (Remark 3.2), we also have that $\ker u$ is finite and so is $j^{-1}(H_1)$. Therefore $I \cap H_1 = j(j^{-1}(H_1))$ is finite, which implies that I is discrete.

The same result for the image of $H^{-1}(U, \mathcal{C})$ into $\bigoplus_{v \notin U} H^{-1}(K_v, C)$ follows immediately because $H^{-1}(U, \mathcal{C})$ is a subgroup of $H^0(U, \mathcal{T}_1)$ (which has just been shown to be a discrete subgroup of $\bigoplus_{v \notin U} H^0(K_v, T_1)$), and $\bigoplus_{v \notin U} H^{-1}(K_v, C)$ is a topological subspace of $\bigoplus_{v \notin U} H^0(K_v, T_1)$. □

Remark 3.9. The analogue of Lemma 3.8 does not hold over a number field as soon as at least one finite place of K is not in U and K has at least two non archimedean places: indeed for the exact sequence

$$0 \rightarrow H^0(U, \mathbf{G}_m) \rightarrow \bigoplus_{v \notin U} H^0(K_v, \mathbf{G}_m) \rightarrow H_c^1(U, \mathbf{G}_m) \rightarrow H^1(U, \mathbf{G}_m) \rightarrow 0$$

to hold (cf. [DH], beginning of section 2), the groups $H^0(K_v, \mathbf{G}_m)$ at the archimedean places must be understood as the *modified* Tate group $\widehat{H}^0(K_v, \mathbf{G}_m)$. The intersection I of $H^0(U, \mathbf{G}_m)$ with the compact subgroup $\bigoplus_{v \notin U} \mathcal{O}_v^* \subset \bigoplus_{v \notin U} H^0(K_v, \mathbf{G}_m)$ (where by convention \mathcal{O}_v^* means $\widehat{H}^0(K_v, \mathbf{G}_m)$ at the archimedean places) is countable and infinite (it is isomorphic to \mathcal{O}_K^*), hence I is not compact by Baire's Theorem. Therefore the image of $H^0(U, \mathbf{G}_m)$ in $\bigoplus_{v \notin U} H^0(K_v, \mathbf{G}_m)$ is not closed.

Equip the finitely generated group $H^0(U, \mathcal{C})$ with the discrete topology. We give $H_c^0(U, \mathcal{C})$ the unique topology such that all maps in the exact sequence

$$H^{-1}(U, \mathcal{C}) \rightarrow \bigoplus_{v \notin U} H^{-1}(K_v, \mathcal{C}) \rightarrow H_c^0(U, \mathcal{C}) \rightarrow H^0(U, \mathcal{C}) \quad (17)$$

are strict (by Lemma 3.8, the left map is strict and the quotient of $\bigoplus_{v \notin U} H^{-1}(K_v, \mathcal{C})$ by the image of $H^{-1}(U, \mathcal{C})$ is a locally compact Hausdorff group). We also give the finite group (cf. Proposition 3.6 c) $D^1(U, \mathcal{C})$ the discrete topology, and topologize $H_c^1(U, \mathcal{C})$ such that all maps in the exact sequence

$$H^0(U, \mathcal{C}) \rightarrow \bigoplus_{v \notin U} H^0(K_v, \mathcal{C}) \rightarrow H_c^1(U, \mathcal{C}) \rightarrow D^1(U, \mathcal{C}) \rightarrow 0 \quad (18)$$

are strict.

Definition 3.10. Define \mathcal{E} as the class of those abelian topological groups A that are an extension

$$0 \rightarrow P \rightarrow A \xrightarrow{\pi} F \rightarrow 0 \quad (19)$$

(the maps being continuous) of a finitely generated group F (equipped with the discrete topology) by a profinite group P (this implies that all maps in this exact sequence are strict by [DH], Lemma 3.4).

It is easy to check that for every group A in \mathcal{E} , a closed subgroup of A and the quotient of A by any closed subgroup of A are still in \mathcal{E} . Also a topological extension of a (discrete) finitely generated group by A stays in \mathcal{E} . Finally, every group A in \mathcal{E} is isomorphic to the direct product of a finitely generated group (equipped with discrete topology) by a profinite group: indeed up to replacing F by F/F_{tors} and P by $\pi^{-1}(F_{\text{tors}})$ in the extension (19), we can assume that $F = \mathbf{Z}^r$ for some $r \geq 0$. Since F is free, the morphism π has a section $s : F \rightarrow A$, which is automatically continuous because F is discrete. Setting $B = s(F)$, we get a topological isomorphism $A \cong P \times B$.

Definition 3.11. Let A be an abelian group. For every prime number ℓ , the ℓ -adic completion of A is

$$A^{(\ell)} := \varprojlim_{m \in \mathbf{N}} (A/\ell^m).$$

We also set

$$A_{\wedge} := \varprojlim_{n \in \mathbf{N}^*} A/n = \prod_{\ell \text{ prime}} A^{(\ell)}.$$

The piece of notation $A\{\ell\}$ stands for the ℓ -primary torsion subgroup of A .

For A finitely generated, we have $A^{(\ell)} = A \otimes_{\mathbf{Z}} \mathbf{Z}_{\ell}$; since \mathbf{Z}_{ℓ} is a torsion-free (hence flat) \mathbf{Z} -module, the functors $A \mapsto A^{(\ell)}$ and $A \mapsto A_{\wedge}$ are exact in the category of finitely generated abelian groups.

Lemma 3.12. *Let $A \rightarrow B \rightarrow E \rightarrow 0$ be an exact sequence of abelian groups.*

a) *The induced map $B^{(\ell)} \rightarrow E^{(\ell)}$ is surjective.*

b) *If ${}_{\ell}E$ is finite, then the induced sequence*

$$A^{(\ell)} \rightarrow B^{(\ell)} \rightarrow E^{(\ell)} \rightarrow 0$$

is exact. Likewise if A/ℓ is finite.

Proof: a) Since the functor $\cdot \otimes_{\mathbf{Z}} \mathbf{Z}/\ell^m$ is right exact, the sequence

$$A/\ell^m \rightarrow B/\ell^m \rightarrow E/\ell^m \rightarrow 0$$

is exact. Therefore the projective system $(\ker[B/\ell^m \rightarrow E/\ell^m])_{m \geq 1}$ has surjective transition maps, which implies that the map $\varprojlim_m (B/\ell^m) \rightarrow \varprojlim_m (E/\ell^m)$ remains surjective.

b) Assume that ${}_{\ell}E$ is finite. Then (by induction on m) we also have that ${}_{\ell^m}E$ is finite for every positive integer m thanks to the exact sequence

$${}_{\ell}E \rightarrow {}_{\ell^{m+1}}E \xrightarrow{\cdot \ell} {}_{\ell^m}E.$$

Let $I \subset B$ be the image of A by the map $A \rightarrow B$. By a), the map $A^{(\ell)} \rightarrow I^{(\ell)}$ is surjective, so it is sufficient to prove that the sequence

$$I^{(\ell)} \rightarrow B^{(\ell)} \rightarrow E^{(\ell)}$$

is exact. By the snake lemma, we have an exact sequence

$${}_{\ell^m}E \rightarrow I/\ell^m \rightarrow B/\ell^m \rightarrow E/\ell^m.$$

Taking projective limit over m yields the required exact sequence because the kernel of the map $I/\ell^m \rightarrow B/\ell^m$ is finite (it is a quotient of ${}_{\ell^m}E$). Similarly, if A/ℓ is finite, then A/ℓ^m (hence also I/ℓ^m) is finite for every positive m and the same argument works. \square

If we assume further that A is a topological abelian group, its *profinite completion* is $A^{\wedge} := \varprojlim_H (A/H)$, where H runs over all open subgroups of finite index in A . If A is profinite, then $A = A_{\wedge} = A^{\wedge}$. If A is in the class \mathcal{E} , then $A \hookrightarrow A_{\wedge} = A^{\wedge}$.

Proposition 3.13. *Let $\mathcal{C} = [\mathcal{T}_1 \rightarrow \mathcal{T}_2]$ be a complex of U -tori with generic fiber $\mathcal{C} = [T_1 \rightarrow T_2]$. Let $v \in X^{(1)}$. The topological groups $H^{-1}(K_v, \mathcal{C})$ and $H^0(K_v, \mathcal{C})$ are in \mathcal{E} , as are the groups $H_c^0(U, \mathcal{C})$ and $H_c^1(U, \mathcal{C})$. In particular, for $i \in \{-1, 0\}$, we have $H^i(K_v, \mathcal{C})_{\wedge} = H^i(K_v, \mathcal{C})^{\wedge}$ and for $i \in \{0, 1\}$, we have $H_c^i(U, \mathcal{C}) \hookrightarrow H_c^i(U, \mathcal{C})_{\wedge}$.*

Proof: Let \mathcal{T} be a U -torus with generic fibre T . Let v be a closed point of X and let L be a finite Galois extension of K_v such that \mathcal{T} splits over L . As $L^* \simeq \mathbf{Z} \times \mathcal{O}_L^*$ is in \mathcal{E} , so is $T(L)$. Then $H^0(K_v, \mathcal{T})$ is in \mathcal{E} as a closed subgroup of $T(L)$ (the subgroup of $\text{Gal}(L/K_v)$ -invariants). The exact sequence

$$H^0(K_v, T_1) \rightarrow H^0(K_v, T_2) \rightarrow H^0(K_v, \mathcal{C}) \rightarrow H^1(K_v, T_1)$$

and the definition of the topology on $H^0(K_v, \mathcal{C})$ now imply that $H^0(K_v, \mathcal{C})$ is in \mathcal{E} . The exact sequence (18) and Lemma 3.8 yield that $H_c^1(U, \mathcal{C})$ is in \mathcal{E} because $D^1(U, \mathcal{C})$ is finite.

Similarly the group $H^{-1}(K_v, \mathcal{C}) = \ker[H^0(K_v, \mathcal{T}_1) \rightarrow H^0(K_v, \mathcal{T}_2)]$ is a closed subgroup of $H^0(K_v, \mathcal{T}_1)$, hence is in \mathcal{E} . This implies that $H_c^0(U, \mathcal{C})$ is in \mathcal{E} thanks to the exact sequence (17), the group $H^0(U, \mathcal{C})$ being of finite type by Proposition 3.6. \square

Lemma 3.14. *Let $v \in U$ and let \mathcal{C} be a complex of \mathcal{O}_v -tori. Then $H^i(\mathcal{O}_v, \mathcal{C}) = 0$ for $i \geq 1$.*

Proof: Using the exact sequence (14) with U replaced by \mathcal{O}_v , we can assume that $\mathcal{C} = \mathcal{T}$ is one single torus. Since \mathcal{T} is smooth over \mathcal{O}_v , the fppf cohomology group $H^i(\mathcal{O}_v, \mathcal{T})$ coincides with the étale group, and it is isomorphic ([Mi2], Prop. II.1.1. b) to the Galois cohomology group $H^i(\mathbf{F}_v, \tilde{T})$, where \mathbf{F}_v is the residue field of \mathcal{O}_v and \tilde{T} the reduction of \mathcal{T} mod. v , which is a torus over the finite field \mathbf{F}_v . Now $H^1(\mathbf{F}_v, \tilde{T}) = 0$ by Lang's theorem ([Lan]). For $i \geq 2$, the Galois cohomology group $H^i(\mathbf{F}_v, \tilde{T})$ is torsion. Let $n > 0$. By the Kummer sequence applied to the torus \tilde{T} over the perfect field \mathbf{F}_v , the n -torsion subgroup ${}_n H^i(\mathbf{F}_v, \tilde{T})$ is a quotient of $H^i(\mathbf{F}_v, {}_n \tilde{T})$, which is zero because \mathbf{F}_v is of cohomological dimension 1 ([Ser], Chap. XIII, Prop. 2). This proves the lemma. \square

Remark 3.15. Using the definition of fppf compact support cohomology given in [DH] (which, in particular, takes care of the set $\Omega_{\mathbf{R}}$ of real places; see loc. cit., Prop. 2.1), most results of this section hold (with the same proof) if we replace K by a number field with ring of integers \mathcal{O}_K , X by $\text{Spec } \mathcal{O}_K$, and U by a non empty Zariski open subset of X . Also the piece of notation $v \notin U$ means that we consider the closed points of $X \setminus U$ and the real places of K ; for $v \in \Omega_{\mathbf{R}}$ and $i \leq 0$, the groups $H^i(K_v, \dots)$ must be understood as the modified Tate groups (cf. Remark 3.9). More precisely:

- Lemma 3.1 a) hold without the restriction $U \neq X$; b) and Remark 3.2 are useless because for every finite flat commutative U -group scheme \mathcal{N} , all groups $H^i(U, \mathcal{N})$ and $H_c^i(U, \mathcal{N})$ are finite (cf. [DH], Th.1.1).
- Lemma 3.4 a) and b) are unchanged; c) holds without the restriction $U \neq X$. In d), the vanishing of $H_c^i(U, \widehat{\mathcal{T}})$ for $i \geq 4$ still holds but the proof uses a different argument (namely that the dual of this group is the inverse limit of the $H^{4-i}(U, {}_n \mathcal{T})$, which is zero even in the case $i = 4$ because the finitely generated group $H^0(U, \mathcal{T})$ has trivial Tate module). The vanishing of $H^i(U, \widehat{\mathcal{T}})$ for $i \geq 4$ must be replaced by its finiteness if $\Omega_{\mathbf{R}} \neq \emptyset$. Finally, *the vanishing of $H^3(U, \widehat{\mathcal{T}})$ does not hold any more*, even if $\Omega_{\mathbf{R}} = \emptyset$ (if Leopoldt's conjecture is assumed, then $H^3(U, \widehat{\mathcal{T}})\{\ell\} = 0$ for ℓ invertible on U , but not in general for other primes; [Mi2], Th. II.4.6. b) is wrong for $r = 3$, the problem in the proof being that the second line of the diagram needs not remain exact after taking profinite completions); neither does e) hold as soon as there are at least two archimedean places. Also, there is no more counterexample as in Remark 3.5.
- Proposition 3.6 a) and c) hold (without the condition $U \neq X$), b) is also true (and when $U = X$, the group $H^1(X, \mathcal{C})$ is even finite).

4 Duality theorems in fppf cohomology

In order to state and prove duality results for the cohomology of complexes of fppf sheaves, we need to extend some constructions from [DH] to the context of bounded complexes. Let A and B be two bounded complexes of fppf sheaves over U . Following [SGA4], XVII, 4.2.9 or [FS], appendix A, one can consider the Godement resolutions $G(A)$ and $G(B)$ of A and B . As in [God], II.6.6 or in [FS], appendix A, there is a natural commutative diagram of complexes

$$\begin{array}{ccc} A \otimes B & & \\ \downarrow & \searrow & \\ \text{Tot}(G(A) \otimes G(B)) & \longrightarrow & G(A \otimes B). \end{array}$$

Following [DH], proof of Lemma 4.1, one gets a functorial morphism of complexes

$$\text{Tot}(\Gamma_c(U, G(A)) \otimes \Gamma(U, G(B))) \rightarrow \Gamma_c(U, G(A \otimes B)) \quad (20)$$

and a functorial pairing

$$R\Gamma_c(U, A) \otimes^{\mathbf{L}} R\Gamma(U, B) \rightarrow R\Gamma_c(U, A \otimes B).$$

In particular, if C is a bounded complex and $\widehat{C} := \underline{\text{Hom}}^{\bullet}(C, \mathbf{G}_m[1])$ its dual, using the morphism $\text{Tot}(C \otimes \widehat{C}) \rightarrow \mathbf{G}_m[1]$ from section 2, we get functorial pairings

$$\text{Tot}(\Gamma_c(U, G(C)) \otimes \Gamma(U, G(\widehat{C}))) \rightarrow \Gamma_c(U, G(\mathbf{G}_m[1]))$$

and

$$R\Gamma_c(U, C) \otimes^{\mathbf{L}} R\Gamma(U, \widehat{C}) \rightarrow R\Gamma_c(U, \mathbf{G}_m[1]). \quad (21)$$

Following section 2, given a bounded complex \mathcal{C} of finite flat commutative group schemes over U , there is a natural topology on the abelian groups $H_c^i(U, \mathcal{C})$. This topology is profinite via Lemma 2.5, and considering $H^j(U, \widehat{\mathcal{C}})$ as discrete torsion groups, the pairings

$$H_c^i(U, \mathcal{C}) \times H^j(U, \widehat{\mathcal{C}}) \rightarrow H_c^{i+j+1}(U, \mathbf{G}_m)$$

are continuous by the same argument as [DH], Lemma 4.4.

Proposition 4.1. *Let \mathcal{C} be a bounded complex of finite flat commutative group schemes over U . For all $i \in \mathbf{Z}$, there is a perfect pairing between profinite and discrete torsion groups*

$$H_c^{2-i}(U, \mathcal{C}) \times H^i(U, \widehat{\mathcal{C}}) \rightarrow H_c^3(U, \mathbf{G}_m) \cong \mathbf{Q}/\mathbf{Z}.$$

Proof: The isomorphism $H_c^3(U, \mathbf{G}_m) \cong \mathbf{Q}/\mathbf{Z}$ follows from [Mi2], Prop. II.2.6 and [DH], Prop. 2.1., 4).

We now prove the Proposition by induction on the length of the complex \mathcal{C} .

- if \mathcal{C} is concentrated in a given degree n , then the Proposition is a direct consequence of [DH], Theorem 1.1.

- assume that $\mathcal{C} := [C_r \xrightarrow{f_r} C_{r+1} \rightarrow \cdots \rightarrow C_s]$, with C_i in degree i , has length $s - r \geq 1$. Let $A := \ker(f_r)$, and let $\overline{\mathcal{C}} := \text{Cone}(A[-r] \xrightarrow{j} \mathcal{C})$. Then there is an exact triangle

$$A[-r] \xrightarrow{j} \mathcal{C} \xrightarrow{i} \overline{\mathcal{C}} \xrightarrow{p} A[1-r].$$

We apply the functor $\widehat{}$ to this exact triangle. Then, using (2), we get that the natural triangle

$$\widehat{A[1-r]} \xrightarrow{\widehat{p}} \widehat{\overline{\mathcal{C}}} \xrightarrow{\widehat{i}} \widehat{\mathcal{C}} \xrightarrow{\widehat{j}} \widehat{A[-r]}$$

is exact.

Since the pairing between a complex and its dual is functorial, we get from the previous triangles a commutative diagram of topological groups with exact rows, where the vertical maps comes from the pairings (21):

$$\begin{array}{ccccccccc} H_c^{i-1}(U, \overline{\mathcal{C}}) & \longrightarrow & H_c^{i-r}(U, A) & \longrightarrow & H_c^i(U, \mathcal{C}) & \longrightarrow & H_c^i(U, \overline{\mathcal{C}}) & \longrightarrow & H_c^{i+1-r}(U, A) \\ \downarrow & & \downarrow & & \downarrow & & \downarrow & & \downarrow \\ H^{3-i}(U, \widehat{\overline{\mathcal{C}}})^* & \longrightarrow & H^{3+r-i}(U, A^D)^* & \longrightarrow & H^{2-i}(U, \widehat{\mathcal{C}})^* & \longrightarrow & H^{2-i}(U, \widehat{\overline{\mathcal{C}}})^* & \longrightarrow & H^{2+r-i}(U, A^D)^* \end{array} \quad (22)$$

The second line remains exact as the dual of an exact sequence of discrete groups.

Note that we have a quasi-isomorphism $\varphi : \overline{\mathcal{C}} \rightarrow \mathcal{C}'$, where $\mathcal{C}' := [C_{r+1}/\text{Im}(f_r) \rightarrow C_{r+2} \rightarrow \cdots \rightarrow C_s]$ has a smaller length than \mathcal{C} , hence by induction, we can assume that the natural maps $H_c^r(U, \mathcal{C}') \rightarrow H^{2-r}(U, \widehat{\mathcal{C}'})^*$ are isomorphisms. Since all the C_i 's are finite flat group schemes, we see that the dual morphism $\widehat{\varphi} : \widehat{\mathcal{C}'} \rightarrow \widehat{\overline{\mathcal{C}}}$ is a quasi-isomorphism, hence by functoriality of the pairings, we deduce that the maps $H_c^r(U, \overline{\mathcal{C}}) \rightarrow H^{2-r}(U, \widehat{\overline{\mathcal{C}}})^*$ are isomorphisms too.

Hence in diagram (22), all vertical morphisms, except perhaps the central one, are isomorphisms. Then the five Lemma implies that the central morphism is an isomorphism.

By induction on the length of \mathcal{C} , the Proposition is proven. \square

From now on, we denote by $\mathcal{C} = [\mathcal{T}_1 \rightarrow \mathcal{T}_2]$ a complex of U -tori with generic fibre $C = [T_1 \rightarrow T_2]$ and dual $\widehat{\mathcal{C}}$ (cf. section 3). As a consequence of the previous proposition, we can get the global duality results for the cohomology of the complex $\mathcal{C} \otimes^{\mathbf{L}} \mathbf{Z}/n$:

Proposition 4.2. *Let n be a positive integer (not necessarily prime to p). Let i be an integer with $-2 \leq i \leq 2$.*

1. *There is a perfect pairing of finite groups*

$$H^i(U, \mathcal{C} \otimes^{\mathbf{L}} \mathbf{Z}/n) \times H_c^{1-i}(U, \widehat{\mathcal{C}} \otimes^{\mathbf{L}} \mathbf{Z}/n) \rightarrow \mathbf{Q}/\mathbf{Z}.$$

If $U \neq X$, the groups $H^2(U, \mathcal{C} \otimes^{\mathbf{L}} \mathbf{Z}/n)$ and $H_c^{-1}(U, \widehat{\mathcal{C}} \otimes^{\mathbf{L}} \mathbf{Z}/n)$ are zero.

2. *There is a perfect pairing*

$$H_c^i(U, \mathcal{C} \otimes^{\mathbf{L}} \mathbf{Z}/n) \times H^{1-i}(U, \widehat{\mathcal{C}} \otimes^{\mathbf{L}} \mathbf{Z}/n) \rightarrow \mathbf{Q}/\mathbf{Z}$$

between the profinite group $H_c^i(U, \mathcal{C} \otimes^{\mathbf{L}} \mathbf{Z}/n)$ and the discrete group $H^{1-i}(U, \widehat{\mathcal{C}} \otimes^{\mathbf{L}} \mathbf{Z}/n)$. These groups are finite if $i \notin \{0, 1\}$ or if p and n are coprime. The groups $H_c^{-2}(U, \mathcal{C} \otimes^{\mathbf{L}} \mathbf{Z}/n)$ and $H^3(U, \widehat{\mathcal{C}} \otimes^{\mathbf{L}} \mathbf{Z}/n)$ are zero if $U \neq X$.

Moreover, all the groups involved are zero if $|i| > 2$.

Proof: Recall that there is a quasi-isomorphism of complexes $\psi = C' \rightarrow \mathcal{C} \otimes^{\mathbf{L}} \mathbf{Z}/n$, where $C' := [{}_n T_1 \xrightarrow{\rho} {}_n T_2]$, with ${}_n T_1$ in degree -2 , and that the dual morphism $\widehat{\psi} : \widehat{\mathcal{C}} \otimes^{\mathbf{L}} \mathbf{Z}/n \rightarrow \widehat{C}' = [\widehat{T}_2/n \xrightarrow{-\widehat{\rho}} \widehat{T}_1/n] = (\widehat{\mathcal{C}} \otimes^{\mathbf{L}} \mathbf{Z}/n)[-1]$ (with \widehat{T}_2/n in degree 0) is also a quasi-isomorphism.

Since C' is a bounded complex of finite flat commutative group schemes, then Proposition 4.1 implies that the pairings in the statement of the Proposition are perfect pairings of topological groups.

Let us now check the finiteness and vanishing results. Using the exact triangle (10), we get an exact sequence:

$$H^{i+2}(U, {}_n \ker \rho) \rightarrow H^i(U, \mathcal{C} \otimes^{\mathbf{L}} \mathbf{Z}/n) \rightarrow H^{i+1}(U, T_{\mathbf{Z}/n}(\mathcal{C})).$$

As the finite U -group schemes $T_{\mathbf{Z}/n}(\mathcal{C})$ and ${}_n \ker \rho$ are of multiplicative type, Lemma 3.1 and Remark 3.2 imply that the groups $H^r(U, T_{\mathbf{Z}/n}(\mathcal{C}))$ and $H^r(U, {}_n \ker \rho)$ are finite for every integer r . Thus $H^i(U, \mathcal{C} \otimes^{\mathbf{L}} \mathbf{Z}/n)$ is finite.

Similarly, using the exact triangle (10), the group $H_c^i(U, \mathcal{C} \otimes^{\mathbf{L}} \mathbf{Z}/n)$ is finite for $i \notin \{0, 1\}$ (or if p and n are coprime) by Remark 3.2.

Recall that for a finite and flat group scheme \mathcal{N} over U , we have $H^r(U, \mathcal{N}) = 0$ for $r < 0$ (obvious), for $r \geq 4$, and also for $r = 3$ if $U \neq X$: indeed by loc. cit. $H^r(U, \mathcal{N})$ is dual to $H_c^{3-r}(U, \mathcal{N}^D)$; the latter is clearly zero if $r \geq 4$ and if $U \neq X$, we also have $H_c^0(U, \mathcal{N}^D) = 0$ thanks to the exact sequence

$$0 \rightarrow H_c^0(U, \mathcal{N}^D) \rightarrow H^0(U, \mathcal{N}^D) \rightarrow \bigoplus_{v \notin U} H^0(K_v, \mathcal{N}^D),$$

the last map being injective by the assumption $U \neq X$. The previous dévissages now yield the vanishing assertions of the proposition. \square

Proposition 4.3. *Let i be an integer. Let n be a positive integer (not necessarily prime to p).*

a) *The groups ${}_n H^i(U, \mathcal{C})$ and $H^i(U, \mathcal{C})/n$ are finite. The group $H^i(U, \mathcal{C})$ is torsion if $i \geq 2$ (resp. if $i \geq 1$ and $U \neq X$). Besides $H^i(U, \mathcal{C}) = 0$ in the following cases: $i \geq 4$; $i \leq -2$; $i = 3$ and $U \neq X$.*

b) *The groups ${}_n H_c^i(U, \widehat{\mathcal{C}})$ and $H_c^i(U, \widehat{\mathcal{C}})/n$ are finite. The group $H_c^i(U, \widehat{\mathcal{C}})$ is torsion if $i \geq 2$, and it is zero if $i \geq 4$ or $i \leq -2$. Assume further $U \neq X$; then $H_c^i(U, \widehat{\mathcal{C}}) = 0$ for $i = -1$.*

Proof: a) Using the exact triangle in $\mathcal{D}^b(U)$:

$$\mathcal{C} \xrightarrow{n} \mathcal{C} \rightarrow \mathcal{C} \otimes^{\mathbf{L}} \mathbf{Z}/n \rightarrow \mathcal{C}[1],$$

we get an exact sequence of abelian groups

$$0 \rightarrow H^i(U, \mathcal{C})/n \rightarrow H^i(U, \mathcal{C} \otimes^{\mathbf{L}} \mathbf{Z}/n) \rightarrow {}_n H^{i+1}(U, \mathcal{C}) \rightarrow 0. \quad (23)$$

since $H^i(U, \mathcal{C} \otimes^{\mathbf{L}} \mathbf{Z}/n)$ is finite (Proposition 4.2 1.), the finiteness of the groups ${}_n H^{i+1}(U, \mathcal{C})$ and $H^i(U, \mathcal{C})/n$ follows for all i .

To prove that $H^i(U, \mathcal{C})$ is torsion if $i \geq 2$ (resp. if $U \neq X$ and $i = 1$), we can restrict by dévissage (using exact sequence (14)) to the case when $\mathcal{C} = \mathcal{T}$ is one single torus. If $U \neq X$ and $i = 1$, this follows from Lemma 3.1, a), so assume $i \geq 2$. We can also assume by a restriction-corestriction argument that $\mathcal{T} = \mathbf{G}_m$ because the torus \mathcal{T} is split by some finite étale covering of U . Now $H^2(U, \mathbf{G}_m) = \text{Br } U$ is torsion because it injects into $\text{Br } K$; also $H^3(U, \mathbf{G}_m)$ is torsion (it is even 0 if $U \neq X$) and $H^i(U, \mathbf{G}_m) = 0$ for $i \geq 4$ by [Mi2], Prop. II.2.1., the group scheme \mathbf{G}_m being smooth (hence étale and fppf cohomology coincide).

For every U -torus \mathcal{T} , we have $H^i(U, \mathcal{T}) = 0$ for negative i (obvious), hence by dévissage $H^i(U, \mathcal{C}) = 0$ for $i < -1$. Let $i \geq 3$; as seen before $H^i(U, \mathcal{C})$ is torsion and ${}_n H^i(U, \mathcal{C})$ is a quotient of $H^{i-1}(U, \mathcal{C} \otimes^{\mathbf{L}} \mathbf{Z}/n)$ by exact sequence (23). The latter is zero if $i \geq 4$, and also if $i = 3$ when $U \neq X$ by the vanishing assertions in Proposition 4.2 1.. Thus $H^i(U, \mathcal{C})$ is zero if $i \geq 4$, and also if $i = 3$ if we assume further $U \neq X$.

b) Similarly, the finiteness statements follow from the exact sequence

$$0 \rightarrow H_c^i(U, \widehat{\mathcal{C}})/n \rightarrow H_c^i(U, \widehat{\mathcal{C}} \otimes^{\mathbf{L}} \mathbf{Z}/n) \rightarrow {}_n H_c^{i+1}(U, \widehat{\mathcal{C}}) \rightarrow 0 \quad (24)$$

combined to Proposition 4.2 1.. Let $i \geq 2$. To prove that $H_c^i(U, \widehat{\mathcal{C}})$ is torsion we can assume that $\widehat{\mathcal{C}}$ is the dual of a torus (via exact sequence (15)), then that $\widehat{\mathcal{C}} = \mathbf{Z}$ (by a restriction-corestriction argument). Using the exact sequence

$$\bigoplus_{v \notin U} H^{i-1}(K_v, \mathbf{Z}) \rightarrow H_c^i(U, \mathbf{Z}) \rightarrow H^i(U, \mathbf{Z}),$$

it is sufficient to prove that $H^i(U, \mathbf{Z})$ is torsion because the Galois cohomology groups $H^{i-1}(K_v, \mathbf{Z})$ are torsion for $i - 1 > 0$. This holds by [Mi2], Lemma II.2.10.

Let \mathcal{T} be a U -torus. For each integer i , there is an exact sequence

$$\bigoplus_{v \notin U} H^{i-1}(K_v, \widehat{\mathcal{T}}) \rightarrow H_c^i(U, \widehat{\mathcal{T}}) \rightarrow H^i(U, \widehat{\mathcal{T}}) \rightarrow \bigoplus_{v \notin U} H^i(K_v, \widehat{\mathcal{T}}). \quad (25)$$

Therefore $H_c^i(U, \widehat{\mathcal{T}}) = 0$ for $i < 0$, hence $H_c^i(U, \widehat{\mathcal{C}}) = 0$ (by dévissage) for $i < -1$. For $i \geq 4$, we have $H_c^i(U, \widehat{\mathcal{T}}) = 0$ by Lemma 3.4 d), and $H_c^i(U, \widehat{\mathcal{C}}) = 0$ by dévissage.

Assume now $U \neq X$. Then $H_c^0(U, \widehat{\mathcal{T}}) = 0$ by exact sequence (25) applied to $i = 0$: indeed the map $H^0(U, \widehat{\mathcal{T}}) \rightarrow \bigoplus_{v \notin U} H^0(K_v, \widehat{\mathcal{T}})$ is injective (choose a closed point v of $X \setminus U$; then the restriction maps $H^0(U, \widehat{\mathcal{T}}) \rightarrow H^0(K, \widehat{\mathcal{T}})$ and $H^0(K, \widehat{\mathcal{T}}) \rightarrow H^0(K_v, \widehat{\mathcal{T}})$ are injective). Therefore $H_c^{-1}(U, \widehat{\mathcal{C}}) = 0$ by dévissage. \square

Proposition 4.4. *Let i be an integer. Let n be a positive integer (not necessarily prime to p).*

a) *The group ${}_n H^i(U, \widehat{\mathcal{C}})$ is finite if $i \notin \{1, 2\}$. The group $H^i(U, \widehat{\mathcal{C}})/n$ is finite if $i \neq 1$. The group $H^i(U, \widehat{\mathcal{C}})$ is torsion if $i \geq 1$, and it is zero if $i \geq 4$ or $i \leq -2$. If we assume further $U \neq X$, then $H^3(U, \widehat{\mathcal{C}}) = 0$.*

b) *The group ${}_n H_c^i(U, \mathcal{C})$ is finite if $i \neq 1$. The group $H_c^i(U, \mathcal{C})/n$ is finite if $i \notin \{0, 1\}$. The group $H_c^i(U, \mathcal{C})$ is torsion for $i \geq 2$, and it is zero if $i \geq 4$ or $i \leq -2$ (resp. $i = -1$ if $U \neq X$).*

Proof: a) The exact sequence

$$0 \rightarrow H^i(U, \widehat{\mathcal{C}})/n \rightarrow H^i(U, \widehat{\mathcal{C}} \otimes^{\mathbf{L}} \mathbf{Z}/n) \rightarrow_n H^{i+1}(U, \widehat{\mathcal{C}}) \rightarrow 0 \quad (26)$$

and Proposition 4.2 2. yield the finiteness of ${}_n H^i(U, \widehat{\mathcal{C}})$ for $i \notin \{1, 2\}$ and of $H^i(U, \widehat{\mathcal{C}})/n$ for $i \notin \{0, 1\}$. Besides the abelian group $H^0(U, \widehat{\mathcal{C}})$ is of finite type by Proposition 3.6. In particular $H^0(U, \widehat{\mathcal{C}})/n$ is finite.

Let $i \geq 1$. To prove that $H^i(U, \widehat{\mathcal{C}})$ is torsion, we can assume (by dévissage) that $\widehat{\mathcal{C}} = \widehat{\mathcal{T}}$, where \mathcal{T} is one single torus, then that $\widehat{\mathcal{C}} = \mathbf{Z}$ (by restriction-corestriction); then the result holds by [Mi2], Lemma II.2.10. For $i \leq -2$ or $i \geq 4$ (resp. $i = 3$ if $U \neq X$), the group $H^i(U, \widehat{\mathcal{C}})$ is zero by dévissage (using Lemma 3.4, d) for the latter).

b) There is an exact sequence

$$0 \rightarrow H_c^i(U, \mathcal{C})/n \rightarrow H_c^i(U, \mathcal{C} \otimes^{\mathbf{L}} \mathbf{Z}/n) \rightarrow_n H_c^{i+1}(U, \mathcal{C}) \rightarrow 0 \quad (27)$$

By Proposition 4.2 2., the group ${}_n H_c^i(U, \mathcal{C})$ is finite for $i \notin \{1, 2\}$ and the group $H_c^i(U, \mathcal{C})/n$ is finite for $i \notin \{0, 1\}$. To prove that the groups $H_c^i(U, \mathcal{C})$ are torsion for $i \geq 2$, we can assume as usual that $\mathcal{C} = \mathcal{T}$ is a torus. Then we apply Proposition 4.3 a) and the exact sequence

$$\bigoplus_{v \notin U} H^{i-1}(K_v, T) \rightarrow H_c^i(U, \mathcal{T}) \rightarrow H^i(U, \mathcal{T}).$$

Besides $H_c^2(U, \mathcal{T})$ is finite by Lemma 3.4 b), so $H_c^2(U, \mathcal{C})$ is also of cofinite type by dévissage because we already know that $H_c^3(U, \mathcal{T}_1)$ is torsion of cofinite type.

Obviously we have $H_c^i(U, \mathcal{T}) = 0$ for every negative i , hence $H_c^i(U, \mathcal{C}) = 0$ by dévissage if $i \leq -2$. Assume $i \geq 4$. then $H_c^i(U, \mathcal{C}) = H^i(U, \mathcal{C})$ (apply exact sequence (3) and use the fact that K_v is of strict cohomological dimension 2), so $H_c^i(U, \mathcal{C}) = 0$ by Proposition 4.3 a). If we assume further $U \neq X$, then $H_c^0(U, \mathcal{T}) = 0$ (same argument as in Proposition 4.3 b), so $H_c^{-1}(U, \mathcal{C}) = 0$ by dévissage. \square

Remark 4.5. Using Remark 3.2, it is easy to see that the finiteness assertions of Proposition 4.4 hold for every i if we assume further that p does not divide n , but this is no longer true in general if $U \neq X$. Indeed the group $H^1(U, \mathbf{Z}/p)$ and its dual $H_c^2(U, \mu_p)$ can be infinite (cf. [Mi2], Lemma III.8.9). Since $H^1(U, \mathbf{Z}) = 0$ and ${}_p H_c^2(U, \mathbf{G}_m)$ is finite, this implies that ${}_p H^2(U, \mathbf{Z})$ and $H_c^1(U, \mathbf{G}_m)/p$ are infinite, which gives examples of ${}_p H^i(U, \widehat{\mathcal{C}})$ infinite for $i = 1, 2$ and of $H_c^i(U, \mathcal{C})/p$ infinite for $i = 0, 1$. The complex $\mathcal{C} = [\mathbf{G}_m \xrightarrow{\cdot p} \mathbf{G}_m]$ is an example with ${}_p H_c^1(U, \mathcal{C})$ and $H^1(U, \widehat{\mathcal{C}})/p$ infinite (indeed \mathcal{C} is quasi-isomorphic to $\mu_p[1]$ and $\widehat{\mathcal{C}}$ is quasi-isomorphic to \mathbf{Z}/p).

Remark 4.6. For every integer r and every positive integer n , the groups $H^r(U, \mathcal{C})/n$ and $H_c^r(U, \widehat{\mathcal{C}})/n$ are finite by Proposition 4.3 b), so for each prime number ℓ (including $\ell = p$), the ℓ -adic completions

$$H^r(U, \mathcal{C})^{(\ell)} := \varprojlim_m H^r(U, \mathcal{C})/\ell^m; \quad H_c^r(U, \widehat{\mathcal{C}})^{(\ell)} := \varprojlim_m H_c^r(U, \widehat{\mathcal{C}})/\ell^m$$

are profinite. Exact sequence (27) shows that $H_c^r(U, \mathcal{C})/n$ is a closed subgroup of the profinite group $H_c^r(U, \mathcal{C} \otimes^{\mathbf{L}} \mathbf{Z}/n)$, hence $H_c^r(U, \mathcal{C})/n$ is profinite and so is the ℓ -adic completion $H_c^r(U, \mathcal{C})^{(\ell)}$.

The map $\mathcal{C} \otimes^{\mathbf{L}} \widehat{\mathcal{C}} \rightarrow \mathbf{G}_m[1]$ induces for every integer r pairings

$$H^r(U, \mathcal{C}) \times H_c^{2-r}(U, \widehat{\mathcal{C}}) \rightarrow H_c^3(U, \mathbf{G}_m) \cong \mathbf{Q}/\mathbf{Z}. \quad (28)$$

$$H_c^r(U, \mathcal{C}) \times H^{2-r}(U, \widehat{\mathcal{C}}) \rightarrow H_c^3(U, \mathbf{G}_m) \cong \mathbf{Q}/\mathbf{Z}. \quad (29)$$

We now prove a key lemma.

Lemma 4.7. *Let ℓ be a prime number (possibly equal to p). Let i be an integer.*

a) *The maps*

$$\begin{aligned} \psi &: H^{i+1}(U, \mathcal{C})\{\ell\} \rightarrow (H_c^{1-i}(U, \widehat{\mathcal{C}})^{(\ell)})^* \\ \psi' &: H_c^{1-i}(U, \widehat{\mathcal{C}})\{\ell\} \rightarrow (H^{i+1}(U, \mathcal{C})^{(\ell)})^* \end{aligned}$$

induced by the pairing (28) are surjective and have divisible kernel. Besides ψ' is an isomorphism if $i = -2$ and ψ is an isomorphism if we have both $i = 1$ and $U \neq X$.

b) *The map*

$$\varphi : H^{i+1}(U, \widehat{\mathcal{C}})\{\ell\} \rightarrow (H_c^{1-i}(U, \mathcal{C})^{(\ell)})^*$$

induced by the pairing (29) is surjective, and has divisible kernel (resp. is an isomorphism if we assume both $i = 1$ and $U = X$). Assume $i \notin \{-1, 0\}$. Then the map

$$\varphi' : H_c^{1-i}(U, \mathcal{C})\{\ell\} \rightarrow (H^{i+1}(U, \widehat{\mathcal{C}})^{(\ell)})^*$$

is surjective and has divisible kernel (resp. is an isomorphism if $i = -2$).

Observe that for $U \neq X$, the groups involved can be non zero only if $-2 \leq i \leq 1$.

Proof: a) For each positive integer m , there is an exact commutative diagram of finite abelian groups,

$$\begin{array}{ccccccc} 0 & \longrightarrow & H^i(U, \mathcal{C})/\ell^m & \longrightarrow & H^i(U, \mathcal{C} \otimes^{\mathbf{L}} \mathbf{Z}/\ell^m) & \longrightarrow & {}_{\ell^m}H^{i+1}(U, \mathcal{C}) \longrightarrow 0 \\ & & \downarrow & & \downarrow & & \downarrow \psi_m \\ 0 & \longrightarrow & ({}_{\ell^m}H_c^{2-i}(U, \widehat{\mathcal{C}}))^* & \longrightarrow & H_c^{1-i}(U, \widehat{\mathcal{C}} \otimes^{\mathbf{L}} \mathbf{Z}/\ell^m)^* & \longrightarrow & (H_c^{1-i}(U, \widehat{\mathcal{C}})/\ell^m)^* \longrightarrow 0. \end{array} \quad (30)$$

By Proposition 4.2 1., the middle vertical map is an isomorphism. Taking direct limit over m and applying the snake lemma, we get that $\psi = \varinjlim_m \psi_m$ is surjective and $\text{Ker } \psi$ is a quotient of $(T_{\ell}H_c^{2-i}(U, \widehat{\mathcal{C}}))^*$. Since each ${}_{\ell^m}H_c^{2-i}(U, \widehat{\mathcal{C}})$ is finite, the ℓ -adic Tate module $T_{\ell}H_c^{2-i}(U, \widehat{\mathcal{C}})$ is profinite and torsion-free, which implies that the dual $(T_{\ell}H_c^{2-i}(U, \widehat{\mathcal{C}}))^*$ is divisible. For $U \neq X$ and $i = 1$, the group $H_c^1(U, \widehat{\mathcal{C}})$ is of finite type by Proposition 3.6 a), so its ℓ -adic Tate-module is zero and ψ has trivial kernel.

The argument for ψ' is similar, using the exact commutative diagram

$$\begin{array}{ccccccc} 0 & \longrightarrow & H_c^{-i}(U, \widehat{\mathcal{C}})/\ell^m & \longrightarrow & H_c^{-i}(U, \widehat{\mathcal{C}} \otimes^{\mathbf{L}} \mathbf{Z}/\ell^m) & \longrightarrow & {}_{\ell^m}H_c^{1-i}(U, \widehat{\mathcal{C}}) \longrightarrow 0 \\ & & \downarrow & & \downarrow & & \downarrow \psi'_m \\ 0 & \longrightarrow & ({}_{\ell^m}H^{2+i}(U, \mathcal{C}))^* & \longrightarrow & H^{2+i}(U, \mathcal{C} \otimes^{\mathbf{L}} \mathbf{Z}/\ell^m)^* & \longrightarrow & (H^{1+i}(U, \mathcal{C})/\ell^m)^* \longrightarrow 0. \end{array} \quad (31)$$

Besides, for $i = -2$, the Tate module of $H^{2+i}(U, \mathcal{C}) = H^0(U, \mathcal{C})$ is trivial because $H^0(U, \mathcal{C})$ is a finitely generated abelian group (Proposition 3.6, b), which gives that ψ' has trivial kernel.

b) There is an exact commutative diagram of discrete abelian groups (observe that the second line is obtained by dualizing an exact sequence of profinite groups):

$$\begin{array}{ccccccc}
0 & \longrightarrow & H^i(U, \widehat{\mathcal{C}})/\ell^m & \longrightarrow & H^i(U, \widehat{\mathcal{C}} \otimes^{\mathbf{L}} \mathbf{Z}/\ell^m) & \longrightarrow & \ell^m H^{i+1}(U, \widehat{\mathcal{C}}) \longrightarrow 0 \\
& & \downarrow & & \downarrow & & \downarrow \varphi_m \\
0 & \longrightarrow & (\ell^m H_c^{2-i}(U, \mathcal{C}))^* & \longrightarrow & H_c^{1-i}(U, \mathcal{C} \otimes^{\mathbf{L}} \mathbf{Z}/\ell^m)^* & \longrightarrow & (H_c^{1-i}(U, \mathcal{C})/\ell^m)^* \longrightarrow 0.
\end{array} \tag{32}$$

Since the middle vertical is an isomorphism by Proposition 4.2 2. and $\ell^m(H_c^{2-i}(U, \mathcal{C}))$ is profinite for each m (hence $H_c^{2-i}(U, \mathcal{C})$ has profinite ℓ -adic Tate module), the same argument as in a) yields that φ is surjective with divisible kernel. If $U = X$ and $i = 1$, then $H_c^{2-i}(U, \mathcal{C}) = H^1(X, \mathcal{C})$ is of finite type by Proposition 3.6 b), so it has trivial ℓ -adic Tate module and φ is an isomorphism.

The argument for φ' is similar, except that we use the exact commutative diagram

$$\begin{array}{ccccccc}
0 & \longrightarrow & H_c^{-i}(U, \mathcal{C})/\ell^m & \longrightarrow & H_c^{-i}(U, \mathcal{C} \otimes^{\mathbf{L}} \mathbf{Z}/\ell^m) & \longrightarrow & \ell^m H_c^{1-i}(U, \mathcal{C}) \longrightarrow 0 \\
& & \downarrow & & \downarrow & & \downarrow \varphi'_m \\
0 & \longrightarrow & (\ell^m H^{i+2}(U, \widehat{\mathcal{C}}))^* & \longrightarrow & H^{i+1}(U, \widehat{\mathcal{C}} \otimes^{\mathbf{L}} \mathbf{Z}/\ell^m)^* & \longrightarrow & (H^{i+1}(U, \widehat{\mathcal{C}})/\ell^m)^* \longrightarrow 0.
\end{array} \tag{33}$$

only for $i \notin \{-1, 0\}$ (for $i \in \{-1, 0\}$ and $U \neq X$, the diagram would consist of profinite but possibly infinite groups if $\ell = p$, so direct limits would not necessarily behave well; in particular, ℓ -adic completions involved would not necessarily be profinite). The same argument as in a) shows that φ' is surjective with divisible kernel, and this kernel is trivial for $i = -2$ because the finitely generated abelian group $H^0(U, \widehat{\mathcal{C}})$ (cf. Proposition 3.6, a) has trivial ℓ -adic Tate module. \square

Remark 4.8. For abelian groups A, B , assertions like “ $A\{\ell\} \rightarrow (B^{(\ell)})^*$ is surjective with divisible kernel” can be rephrased as follows : the pairing $A\{\ell\} \times B^{(\ell)} \rightarrow \mathbf{Q}/\mathbf{Z}$ has trivial right kernel and divisible left kernel.

The following theorem extends the function field case of [Mi2], Th. II.4.6. (which corresponds to $\mathcal{C} = \mathcal{T}$ or $\mathcal{C} = \mathcal{T}[1]$, where \mathcal{T} is a torus).

Theorem 4.9. a) *The pairing (28) induces a perfect duality between the discrete torsion group $H_c^3(U, \widehat{\mathcal{C}})$ and the finite-type $\widehat{\mathbf{Z}}$ -module $H^{-1}(U, \mathcal{C})_{\wedge}$, resp. between the discrete torsion group $H_c^2(U, \widehat{\mathcal{C}})$ and the finite-type $\widehat{\mathbf{Z}}$ -module $H^0(U, \mathcal{C})_{\wedge}$.*

b) *Assume $U \neq X$. The pairing (28) induces a perfect duality between the discrete torsion group $H^1(U, \mathcal{C})$ and the finite-type $\widehat{\mathbf{Z}}$ -module $H_c^1(U, \widehat{\mathcal{C}})_{\wedge}$, resp. between the discrete torsion group $H^2(U, \mathcal{C})$ and the finite-type $\widehat{\mathbf{Z}}$ -module $H_c^0(U, \widehat{\mathcal{C}})_{\wedge}$.*

Proof: a) Let ℓ be a prime number. Then the map ψ' of Lemma 4.7 a) is an isomorphism for $i = -2$, which yields the first point (recall that $H_c^3(U, \widehat{\mathcal{C}})$ and $H_c^2(U, \widehat{\mathcal{C}})$ are torsion of cofinite type by Proposition 4.3; also $H^{-1}(U, \mathcal{C})$ and $H^0(U, \mathcal{C})$ are finitely generated by Proposition 3.6).

In the case $U = X$, the second point is a duality between $H^2(X, \widehat{\mathcal{C}})$ and $H^0(X, \mathcal{C})_\wedge$, which follows from Lemma 4.7 b) in the case $i = 1$. Now assume $U \neq X$ and let \mathcal{T} be a U -torus. By the first point applied to $\mathcal{C} = \mathcal{T}[1]$, the group $H_c^3(U, \widehat{\mathcal{T}})$ is dual to $H^0(U, \mathcal{T})_\wedge$. By Lemma 4.7 a) with $i = -1$ and $\mathcal{C} = \mathcal{T}[1]$, the finite group $H_c^2(U, \widehat{\mathcal{T}})$ (cf. Lemma 3.4, c) is dual to the finite group $H^1(U, \mathcal{T})$ (cf. Lemma 3.1, a): indeed a finite group coincides with its ℓ -adic completion and doesn't contain a non trivial divisible subgroup. There is a commutative diagram with exact lines (observe that the second line is obtained by applying the ℓ -completion functor to an exact sequence of finitely generated group, then dualizing an exact sequence of profinite groups):

$$\begin{array}{ccccccccc}
H_c^2(U, \widehat{\mathcal{T}}_2)\{\ell\} & \longrightarrow & H_c^2(U, \widehat{\mathcal{T}}_2)\{\ell\} & \longrightarrow & H_c^2(U, \widehat{\mathcal{C}})\{\ell\} & \longrightarrow & H_c^3(U, \widehat{\mathcal{T}}_2)\{\ell\} & \longrightarrow & H_c^3(U, \widehat{\mathcal{T}}_1)\{\ell\} \\
\downarrow f_1 & & \downarrow f_2 & & \downarrow h & & \downarrow g_1 & & \downarrow g_2 \\
(H^1(U, \mathcal{T}_2)^{(\ell)})^* & \longrightarrow & (H^1(U, \mathcal{T}_1)^{(\ell)})^* & \longrightarrow & (H^0(U, \mathcal{C})^{(\ell)})^* & \longrightarrow & (H^0(U, \mathcal{T}_2)^{(\ell)})^* & \longrightarrow & (H^0(U, \mathcal{T}_1)^{(\ell)})^*.
\end{array}$$

Now h is an isomorphism by the five-lemma, whence the result.

b) Consider diagram (30) for $i = 0$. By a), the left vertical map is an isomorphism and by Proposition 4.2 1., the middle vertical map is an isomorphism, hence ψ_m is an isomorphism from ${}_{\ell^m}H^1(U, \mathcal{C})$ to $(H_c^1(U, \widehat{\mathcal{C}})/\ell^m)^*$. Taking direct limit over m , then direct sum over all prime ℓ , yields the duality between the torsion group of cofinite type (cf. Proposition 4.3) $H^1(U, \mathcal{C})$ and the finite type $\widehat{\mathbf{Z}}$ -module (cf. Proposition 3.6) $H_c^1(U, \widehat{\mathcal{C}})_\wedge$.

Lemma 4.7 a) for $i = 1$ yields that for $U \neq X$, the map ψ an isomorphism, which immediately gives the duality between the torsion group of finite type (cf. Proposition 4.3) $H^2(U, \mathcal{C})$ and the finite type $\widehat{\mathbf{Z}}$ -module (cf. Proposition 3.6) $H_c^0(U, \widehat{\mathcal{C}})_\wedge$. \square

Remark 4.10. In the case $U = X$, the first assertion of Theorem 4.9 b) should be replaced by a duality between $H^1(X, \widehat{\mathcal{C}})$ and $H^1(X, \mathcal{C})_\wedge$ (see Theorem 4.11 b) below in the case $U = X$). The second assertion (duality between $H^2(X, \mathcal{C})$ and $H^0(X, \widehat{\mathcal{C}})_\wedge$) actually still holds, cf. Theorem 4.11 a).

The following duality theorem has the same flavour as [Mi2], Th II.4.6. b) (but one should be careful that in the number field case, the case $r = 3$ of the latter does not hold in general, see also Remark 4.18).

Theorem 4.11. a) *The pairing (29) induces a perfect duality between the discrete torsion group $H_c^3(U, \mathcal{C})$ and the finite-type $\widehat{\mathbf{Z}}$ -module $H^{-1}(U, \widehat{\mathcal{C}})_\wedge$, resp. between the discrete torsion group $H_c^2(U, \mathcal{C})$ and the finite-type $\widehat{\mathbf{Z}}$ -module $H^0(U, \widehat{\mathcal{C}})_\wedge$.*

b) *The pairing (29) induces a perfect duality between the discrete torsion group $H^1(U, \widehat{\mathcal{C}})$ and the profinite group $H_c^1(U, \mathcal{C})_\wedge$, resp. between the discrete torsion group $H^2(U, \widehat{\mathcal{C}})$ and the profinite group $H_c^0(U, \mathcal{C})_\wedge$.*

Proof: a) Let ℓ be any prime number. The map φ' of Lemma 4.7 b) is an isomorphism for $i = -2$, which yields the first point (Proposition 4.2 2. yields that $H_c^3(U, \mathcal{C})$ and $H_c^2(U, \mathcal{C})$ are torsion groups of cofinite type; Proposition 3.6 gives that $H^{-1}(U, \widehat{\mathcal{C}})$ and $H^0(U, \widehat{\mathcal{C}})$ are finitely generated).

There is a commutative diagram with exact lines

$$\begin{array}{ccccccccc}
H_c^2(U, \mathcal{T}_1)\{\ell\} & \longrightarrow & H_c^2(U, \mathcal{T}_2)\{\ell\} & \longrightarrow & H_c^2(U, \mathcal{C})\{\ell\} & \longrightarrow & H_c^3(U, \mathcal{T}_1)\{\ell\} & \longrightarrow & H_c^3(U, \mathcal{T}_2)\{\ell\} \\
\downarrow f_1 & & \downarrow f_2 & & \downarrow h & & \downarrow g_1 & & \downarrow g_2 \\
(H^1(U, \widehat{\mathcal{T}}_1)^{\ell})^* & \longrightarrow & (H^1(U, \widehat{\mathcal{T}}_2)^{\ell})^* & \longrightarrow & (H^0(U, \widehat{\mathcal{C}})^{\ell})^* & \longrightarrow & (H^0(U, \widehat{\mathcal{T}}_1)^{\ell})^* & \longrightarrow & (H^0(U, \widehat{\mathcal{T}}_2)^{\ell})^*.
\end{array}$$

The maps g_1 and g_2 are isomorphisms by the first point applied to $\mathcal{C} = \mathcal{T}_1$, $\mathcal{C} = \mathcal{T}_2$. The maps f_1 and f_2 are isomorphisms by Lemma 4.7 b) applied to the same complexes (map φ in the case $i = -1$): indeed for a U -torus \mathcal{T} , the groups $H^1(U, \widehat{\mathcal{T}})$ and $H_c^2(U, \mathcal{T})$ are finite (Lemma 3.4 a) and b), hence they coincide with their ℓ -adic completions and do not contain a non trivial divisible subgroup. Therefore h is an isomorphism by the five-lemma, whence the second point.

b) Consider diagram (32) for $i = 0$. By a), the left vertical map is an isomorphism and the middle vertical map is an isomorphism by Proposition 4.2 1., hence φ_m is an isomorphism from $\ell^m H^1(U, \widehat{\mathcal{C}})$ to $(H_c^1(U, \mathcal{C})/\ell^m)^*$. Taking direct limit over m , then direct sum over all prime ℓ , yields the duality between $H^1(U, \widehat{\mathcal{C}})$ (which is torsion by Proposition 4.4, but not necessarily of cofinite type, cf. Remark 4.5) and $H_c^1(U, \mathcal{C})_{\wedge}$.

Now consider diagram (32) for $i = 1$. By the previous duality, the left vertical map induces an isomorphism between $H^1(U, \widehat{\mathcal{C}})/\ell^m$ and $(\ell^m(H_c^1(U, \mathcal{C})_{\wedge}))^*$. Since $H_c^1(U, \mathcal{C})$ is in the class \mathcal{E} (that is: it is the product of a finite type group by a profinite group) by Proposition 3.13, the ℓ^m -torsion of $H_c^1(U, \mathcal{C})$ and of $H_c^1(U, \mathcal{C})_{\wedge}$ coincide, hence the left vertical map is actually an isomorphism and the right vertical map ψ_m is an isomorphism as well (the middle vertical map is an isomorphism by Proposition 4.2 2.). Taking direct limit and direct sum over all prime ℓ , we get the duality between the torsion group (cf. Proposition 4.4) $H^2(U, \widehat{\mathcal{C}})$ and the profinite group $H_c^0(U, \mathcal{C})_{\wedge}$. \square

Proposition 4.12. *The pairing (28) for $r = 1$ induces a perfect pairing of finite groups*

$$D^1(U, \mathcal{C}) \times D^1(U, \widehat{\mathcal{C}}) \rightarrow \mathbf{Q}/\mathbf{Z}. \quad (34)$$

Proof: Fix a prime number ℓ . There is a commutative diagram :

$$\begin{array}{ccccccc}
0 & \longrightarrow & D^1(U, \mathcal{C})\{\ell\} & \longrightarrow & H^1(U, \mathcal{C})\{\ell\} & \longrightarrow & \bigoplus_{v \notin U} H^1(K_v, \mathcal{C})\{\ell\} \\
& & & & \downarrow \psi & & \downarrow \beta \\
0 & \longrightarrow & (D^1(U, \widehat{\mathcal{C}})^{\ell})^* & \longrightarrow & (H_c^1(U, \widehat{\mathcal{C}})^{\ell})^* & \longrightarrow & (\bigoplus_{v \notin U} H^0(K_v, \widehat{\mathcal{C}})^{\ell})^*.
\end{array} \quad (35)$$

The first line is exact by definition of $D^1(U, \mathcal{C})$. The sequence

$$\bigoplus_{v \notin U} H^0(K_v, \widehat{\mathcal{C}}) \rightarrow H_c^1(U, \widehat{\mathcal{C}}) \rightarrow D^1(U, \widehat{\mathcal{C}}) \rightarrow 0$$

is also exact by definition of $D^1(U, \widehat{\mathcal{C}})$. Using Lemma 3.12 and the fact that the ℓ -adic completion functor (ℓ) commutes with finite direct sums, the sequence

$$\bigoplus_{v \notin U} H^0(K_v, \widehat{\mathcal{C}})^{\ell} \rightarrow H_c^1(U, \widehat{\mathcal{C}})^{\ell} \rightarrow D^1(U, \widehat{\mathcal{C}})^{\ell} \rightarrow 0$$

of profinite groups is exact as well, and its dual sequence (which is the second line of the diagram) remains exact.

The commutative diagram (35) defines a map

$$\theta : D^1(U, \mathcal{C})\{\ell\} \rightarrow (D^1(U, \widehat{\mathcal{C}})^{(\ell)})^*.$$

We observe that by [Dem1] Th. 3.1 (which is the local duality theorem), the map β is an isomorphism, and we also know by Theorem 4.9 that ψ is an isomorphism. By diagram chasing θ is an isomorphism. Since this holds for every prime ℓ (including $\ell = p$), the proposition is proven, the finiteness of $D^1(U, \mathcal{C})$ and $D^1(U, \widehat{\mathcal{C}})$ being known by Proposition 3.6 c). □

Remark 4.13. Of course, the pairing (34) can also be defined via the pairing (29).

Lemma 4.14. *Assume $U \neq X$. Then $D^2(U, \mathcal{C})$ and $D^0(U, \widehat{\mathcal{C}})$ are finite.*

Proof: Using the exact triangle (12) and the fact that $\text{coker } \rho := \mathcal{T}$ is a torus, we know that $D^2(U, \mathcal{T})$ is finite and is sufficient to show that $H^3(U, \ker \rho)$ is finite to get the finiteness of $D^2(U, \mathcal{C})$. But $\ker \rho$ is a group of multiplicative type, so there is an exact sequence

$$0 \rightarrow \mathcal{T}_1 \rightarrow \ker \rho \rightarrow \mathcal{F} \rightarrow 0,$$

where \mathcal{F} is a finite group of multiplicative type and \mathcal{T}_1 is a torus. Since $H^3(U, \mathcal{T}_1) = 0$ by Proposition 4.3 a) and $H^3(U, \mathcal{F}) = 0$ (cf. Remark 3.2; it is dual to $H_c^0(U, \widehat{\mathcal{F}})$, which is zero because $U \neq X$), the group $H^3(U, \ker \rho)$ is actually zero.

The group $D^0(U, \widehat{\ker \rho})$ is trivial thanks to the assumption $U \neq X$. Thus the exact triangle (13) shows that $D^0(U, \widehat{\mathcal{C}})$ is finite because so is $H^1(U, \widehat{\mathcal{T}})$ (Lemma 3.4 a). □

Lemma 4.15. *Assume $U \neq X$. Then the groups $D^0(U, \mathcal{C})$ and $D^2(U, \widehat{\mathcal{C}})$ are finite.*

Proof: For a U -torus \mathcal{T} , we have $D^0(U, \mathcal{T}) = 0$ because $U \neq X$. For a U -group of multiplicative type \mathcal{M} , we also know (Remark 3.2) that $H^1(U, \mathcal{M})$ is finite, whence the finiteness of $D^0(U, \mathcal{C})$ via the exact triangle (12).

Exact triangle (13) and the vanishing of $H^3(U, \widehat{\mathcal{T}})$ for a U -torus \mathcal{T} (Lemma 3.4 d) imply that $D^2(U, \widehat{\mathcal{T}})$ injects into $D^2(U, \widehat{\mathcal{M}})$, so it only remains to show that the latter is finite. We show that $H_c^2(U, \widehat{\mathcal{M}})$ is finite. By dévissage it is sufficient to prove this when M is a finite group of multiplicative type and when M is a torus. The first case follows from Lemma 3.1 b) and the second one from Lemma 3.4 c). □

Proposition 4.16. *The pairing (28) for $r = 2$ induces a perfect pairing of finite groups*

$$D^2(U, \mathcal{C}) \times D^0(U, \widehat{\mathcal{C}}) \rightarrow \mathbf{Q}/\mathbf{Z}. \tag{36}$$

Proof: The argument is exactly the same as in the proof of Proposition 4.12, using now the commutative diagram with exact lines:

$$\begin{array}{ccccccc}
0 & \longrightarrow & D^2(U, \mathcal{C})\{\ell\} & \longrightarrow & H^2(U, \mathcal{C})\{\ell\} & \longrightarrow & \bigoplus_{v \notin U} H^2(K_v, \mathcal{C})\{\ell\} \\
& & & & \downarrow & & \downarrow \\
0 & \longrightarrow & (D^0(U, \widehat{\mathcal{C}})^{\ell})^* & \longrightarrow & (H_c^0(U, \widehat{\mathcal{C}})^{\ell})^* & \longrightarrow & (\bigoplus_{v \notin U} H^{-1}(K_v, \widehat{\mathcal{C}})^{\ell})^*.
\end{array} \tag{37}$$

Indeed the right vertical map is an isomorphism by [Dem1], Th. 3.1 and the middle vertical map is an isomorphism as well by Theorem 4.9, b). It remains to apply Lemma 4.14. \square

Proposition 4.17. *The pairing (29) for $r = 0$ induces a perfect pairing of finite groups*

$$D^0(U, \mathcal{C}) \times D^2(U, \widehat{\mathcal{C}}) \rightarrow \mathbf{Q}/\mathbf{Z}. \tag{38}$$

Proof: Again the argument is the same as in Proposition 4.12, using this time the commutative diagram with exact lines:

$$\begin{array}{ccccccc}
0 & \longrightarrow & D^2(U, \widehat{\mathcal{C}})\{\ell\} & \longrightarrow & H^2(U, \widehat{\mathcal{C}})\{\ell\} & \longrightarrow & \bigoplus_{v \notin U} H^2(K_v, \widehat{\mathcal{C}})\{\ell\} \\
& & & & \downarrow & & \downarrow \\
0 & \longrightarrow & (D^0(U, \mathcal{C})^{\ell})^* & \longrightarrow & (H_c^0(U, \mathcal{C})^{\ell})^* & \longrightarrow & (\bigoplus_{v \notin U} H^{-1}(K_v, \mathcal{C})^{\ell})^*.
\end{array} \tag{39}$$

The right vertical map is an isomorphism by [Dem1], Th. 3.1 (recall that by Proposition 3.13, the groups $H^{-1}(K_v, \mathcal{C})_{\wedge}$ and $H^{-1}(K_v, \mathcal{C})^{\wedge}$ coincide), and the middle vertical map is an isomorphism as well by Theorem 4.11, b); Lemma 4.15 then yields the result. \square

Remark 4.18. Again there are analogous results over a number field :

- Proposition 4.2 1. holds except that for n even, the vanishing statements for $i \geq 2$ do not hold any more if $\Omega_{\mathbf{R}} \neq \emptyset$. In Proposition 4.2 2., all groups involved are finite, but the vanishing statements for $i \leq -2$ are in general false if n is even and $\Omega_{\mathbf{R}} \neq \emptyset$.
- In Proposition 4.3 and Proposition 4.4, the vanishing statements must be replaced by finiteness statements if $\Omega_{\mathbf{R}} \neq \emptyset$ for the following groups: $H^i(U, \mathcal{C})$ for $i \geq 4$, $H_c^i(U, \widehat{\mathcal{C}})$ for $i \leq -2$, $H^i(U, \widehat{\mathcal{C}})$ for $i \geq 4$, $H_c^i(U, \mathcal{C})$ for $i \leq -2$. The groups $H^3(U, \mathcal{C})$, $H_c^{-1}(U, \widehat{\mathcal{C}})$, and $H_c^{-1}(U, \mathcal{C})$ are still finite if $U \neq X$ (resp. zero if $U = X$ and $\Omega_{\mathbf{R}} = \emptyset$). Also, the group $H^1(U, \mathcal{C})$ is finite even if $U = X$ and the finiteness assertions in Proposition 4.4 a) hold without any condition on i . Finally, *the vanishing of $H^3(U, \widehat{\mathcal{C}})$ does not hold any more* (see Remark 3.15 about $H^3(U, \widehat{\mathcal{T}})$) even for $\Omega_{\mathbf{R}} = \emptyset$.
- Lemma 4.7 is unchanged, except that the restriction $U \neq X$ can be removed in a) for $i = 1$.

- Theorem 4.9 is unchanged (which gives a more precise statement than [Dem1], Th. 4.3) except that the assumption $U \neq X$ can be removed in b). Theorem 4.11 a) is still true, as is the first assertion of Theorem 4.11 b), but *not the second assertion of Theorem 4.11 b)*: the pairing $H^2(U, \widehat{\mathcal{C}})\{\ell\} \times H_c^0(U, \mathcal{C})^{(\ell)}$ has trivial right kernel and divisible left kernel, but for triviality of the left kernel we need ℓ invertible on U and Leopoldt's conjecture.
- Proposition 4.12 is unchanged (this removes the condition $\ell \in \mathcal{O}_U^*$ in [Dem1], Cor 4.7). Lemma 4.14 also holds (in the proof, the groups $H^3(U, \mathcal{T}_1)$ and $H_c^0(U, \widehat{\mathcal{F}})$ might be only finite if $\Omega_{\mathbf{R}} \neq \emptyset$, but this does not affect the result), as does Proposition 4.16 (the assumption $\ker \rho$ finite made in [Dem1], Lemma 5.13, is not necessary). The first part of Lemma 4.15 still holds, but *not its second part* because in general the ℓ -primary part of $H^3(U, \widehat{\mathcal{T}})$ is infinite if ℓ is not invertible on U (and even for $\ell \in \mathcal{O}_U^*$, the finiteness of $H^3(U, \widehat{\mathcal{T}})\{\ell\}$ relies on Leopoldt's conjecture). Similarly *Proposition 4.17 does not hold any more in general*, we only get that the pairing (38) has trivial left kernel and divisible right kernel (see also [Dem1], paragraph 5.4. for a variant).

5 Poitou-Tate exact sequences

Let $C = [T_1 \rightarrow T_2]$ be a complex of K -tori with dual $\widehat{C} = [\widehat{T}_2 \rightarrow \widehat{T}_1]$. We can choose a non empty Zariski open subset U_0 of X such that C extends to a complex $\mathcal{C} = [\mathcal{T}_1 \xrightarrow{\rho} \mathcal{T}_2]$ of U_0 -tori with dual $\widehat{\mathcal{C}}$. For every integer i and every K -group scheme (or bounded complex of K -group schemes) M (e.g. $M = T$, $M = \widehat{T}$), define

$$\text{III}^i(M) := \text{Ker} [H^i(K, M) \rightarrow \prod_{v \in X^{(1)}} H^i(K_v, M)].$$

Lemma 5.1. *There exists a non empty Zariski open subset $U_1 \subset U_0$ such that for every Zariski open subset $V \subset U_1$:*

a) *For $i \in \{1, 2\}$, the restriction map $r_{U_1, V} : H^i(U_1, \mathcal{C}) \rightarrow H^i(V, \mathcal{C})$ induces isomorphisms*

$$D^i(U_1, \mathcal{C}) \cong D^i(V, \mathcal{C}) \cong \text{III}^i(\mathcal{C}).$$

b) *For $r \in \{0, 1\}$, the canonical map $H^r(V, \widehat{\mathcal{C}}) \rightarrow H^r(K, \widehat{\mathcal{C}})$ is injective and identifies $D^r(V, \widehat{\mathcal{C}})$ with $\text{III}^r(\widehat{\mathcal{C}})$.*

Proof: We can deal with the two properties a) and b) separately (up to taking the intersections of the various provided U_1).

a) Let's start with arbitrary non empty Zariski open subsets $V \subset U \subset U_0$. Take $i \in \{1, 2\}$. For all $v \in U$, we have $H^i(\mathcal{O}_v, \mathcal{C}) = 0$ by Lemma 3.14, which implies that the image of $D^i(U, \mathcal{C})$ by $r_{U, V}$ is contained in $D^i(V, \mathcal{C})$. The induced map $D^i(U, \mathcal{C}) \rightarrow D^i(V, \mathcal{C})$ is surjective thanks to the compatibility of the covariant map $H_c^i(V, \mathcal{C}) \rightarrow H_c^i(U, \mathcal{C})$ with $r_{U, V}$ ([DH] Prop 2.1 (3)). Since all $D^i(U, \mathcal{C})$ are finite by Proposition 3.6 c) and Lemma 4.14, the decreasing sequence of positive integers $\#D^i(U, \mathcal{C})$ (when U becomes smaller and smaller) must stabilize for some $U = U_1$. We get an isomorphism from $D^i(U_1, \mathcal{C})$ to $D^i(V, \mathcal{C})$ for all $V \subset U_1$. Since $H^i(K, \mathcal{C})$ is the direct limit over V of the $H^i(V, \mathcal{C})$, we get an injective map $u : D^i(U_1, \mathcal{C}) \rightarrow H^i(K, \mathcal{C})$. As $D^i(U_1, \mathcal{C})$ is the

same as $D^i(V, \mathcal{C})$ for every $V \subset U_1$, the image of u is contained in $\text{III}^i(C)$ (because its restriction to $H^i(K_v, C)$ is zero for all $v \notin V$ and V can be taken arbitrarily small). Conversely, every element of $\text{III}^i(C)$ can be lifted to an $a \in H^i(V, \mathcal{C})$ for some V , and by definition $a \in D^i(V, \mathcal{C}) = D^i(U_1, \mathcal{C})$, so the image of u contains $\text{III}^i(C)$.

b) Let $V \subset U_0$ be an arbitrary non empty Zariski open subset. Let $r \in \{0, 1\}$. The injectivity of $H^r(V, \widehat{\mathcal{C}}) \rightarrow H^r(K, \widehat{\mathcal{C}})$ has been proven in Proposition 3.6 a). Identifying now $D^r(V, \widehat{\mathcal{C}})$ with a subgroup of $H^r(K, \widehat{\mathcal{C}})$, we get (again using the maps $H^r(V, \widehat{\mathcal{C}}) \rightarrow H^r(U, \widehat{\mathcal{C}})$ for $V \subset U \subset U_0$) a decreasing sequence of finite subgroups (when V becomes smaller and smaller), which stabilizes for some U_1 . Since $D^r(U_1, \widehat{\mathcal{C}})$ is also $D^r(V, \widehat{\mathcal{C}})$ for every $V \subset U_1$, we have $D^r(U_1, \widehat{\mathcal{C}}) \subset \text{III}^r(\widehat{\mathcal{C}})$. On the other hand, every element of $\text{III}^r(\widehat{\mathcal{C}})$ comes from $H^r(V, \widehat{\mathcal{C}})$ for some $V \subset U_1$, and it is then automatically in $D^r(V, \widehat{\mathcal{C}}) = D^r(U_1, \widehat{\mathcal{C}})$ because it is everywhere locally trivial. \square

Theorem 5.2. *There are perfect pairing of finite groups*

$$\text{III}^1(C) \times \text{III}^1(\widehat{\mathcal{C}}) \rightarrow \mathbf{Q}/\mathbf{Z}.$$

$$\text{III}^2(C) \times \text{III}^0(\widehat{\mathcal{C}}) \rightarrow \mathbf{Q}/\mathbf{Z}.$$

Proof: This follows immediately from Lemma 5.1 and Proposition 4.12 (resp. Proposition 4.16) applied to U_1 . \square

Lemma 5.3. *There exists a non empty Zariski open subset U_1 of U_0 such that for every non empty Zariski open subset V of U_1 :*

-the restriction map $H^0(V, \mathcal{C}) \rightarrow H^0(K, C)$ is injective.

-For all non empty Zariski open subsets $W \subset V$, the canonical map

$$j_{W,V} : H_c^2(W, \widehat{\mathcal{C}}) \rightarrow H_c^2(V, \widehat{\mathcal{C}})$$

is surjective and the image of $D^2(V, \widehat{\mathcal{C}})$ by the restriction map

$$r_{V,W} : H^2(V, \widehat{\mathcal{C}}) \rightarrow H^2(W, \widehat{\mathcal{C}})$$

is a subgroup of $D^2(W, \widehat{\mathcal{C}})$.

Proof: Let $U \subset U_0$ be a non empty Zariski open subset. By the exact triangle (12), there is a commutative diagram with exact lines

$$\begin{array}{ccccccc} 0 & \longrightarrow & H^1(U, \mathcal{M}) & \longrightarrow & H^0(U, \mathcal{C}) & \longrightarrow & H^0(U, \mathcal{T}_0) \\ & & \downarrow & & \downarrow & & \downarrow \\ 0 & \longrightarrow & H^1(K, \mathcal{M}) & \longrightarrow & H^0(K, C) & \longrightarrow & H^0(K, \mathcal{T}_0), \end{array}$$

where \mathcal{M} is a U -group of multiplicative type with generic fibre M and \mathcal{T}_0 is a U -torus. Since the right vertical map is clearly injective, it is sufficient to prove the injectivity of the left vertical map for U small enough. We can write \mathcal{M} as an extension

$$0 \rightarrow \mathcal{T} \rightarrow \mathcal{M} \rightarrow \mathcal{F} \rightarrow 0$$

of a finite U -group of multiplicative type \mathcal{F} by a U -torus \mathcal{T} . This yields a commutative diagram with exact lines

$$\begin{array}{ccccccc} 0 & \longrightarrow & H^0(U, \mathcal{F}) & \longrightarrow & H^1(U, \mathcal{T}) & \longrightarrow & H^1(U, \mathcal{M}) & \longrightarrow & H^1(U, \mathcal{F}) \\ & & \downarrow & & \downarrow & & \downarrow & & \downarrow \\ 0 & \longrightarrow & H^0(K, \mathcal{F}) & \longrightarrow & H^1(K, \mathcal{T}) & \longrightarrow & H^1(K, \mathcal{M}) & \longrightarrow & H^1(K, \mathcal{F}). \end{array}$$

Since \mathcal{F} is finite (hence proper) over U , the left vertical map is an isomorphism and the right vertical map is injective. It is therefore sufficient to prove that for a U -torus \mathcal{T} , the restriction map $H^1(U, \mathcal{T}) \rightarrow H^1(K, \mathcal{T})$ is injective for U sufficiently small. Set $N_U = \ker[H^1(U, \mathcal{T}) \rightarrow H^1(K, \mathcal{T})]$. For every Zariski open subset $V \rightarrow U$, the restriction map $H^1(U, \mathcal{T}) \rightarrow H^1(V, \mathcal{T})$ induces a homomorphism $i_{U,V} : N_U \rightarrow N_V$. By Lemma 2.2, this homomorphism is surjective. Lemma 3.1 implies that the group N_V is finite, and the decreasing sequence of positive integers $(\#N_V), V \subset U_0$ must stabilize for some $V = U_1 \subset U_0$. Then the maps $i_{U_1, V}$ for $V \subset U_1$ are isomorphisms, which implies (passing to the limit) that the restriction map $N_{U_1} \rightarrow H^1(K, \mathcal{T})$ is injective. By definition of N_{U_1} , this means that $N_{U_1} = 0$, hence $N_V = 0$ for every $V \subset U_1$. This gives the first point.

For $W \subset V \subset U_1$, the restriction map $H^0(V, \mathcal{C}) \rightarrow H^0(W, \mathcal{C})$ is injective because so is its composition with $H^0(W, \mathcal{C}) \rightarrow H^0(K, \mathcal{C})$. As $H^0(V, \mathcal{C})$ and $H^0(W, \mathcal{C})$ are finitely generated by Proposition 3.6 b), the induced map $H^0(V, \mathcal{C})_{\wedge} \rightarrow H^0(W, \mathcal{C})_{\wedge}$ is still injective. By Theorem 4.9, the dual map $H_c^2(W, \widehat{\mathcal{C}}) \rightarrow H_c^2(V, \widehat{\mathcal{C}})$ is surjective. Now the compatibility of $r_{V,W}$ with $j_{W,V}$ ([DH], Prop 2.1 c) gives that $r_{V,W}(D^2(V, \widehat{\mathcal{C}})) \subset D^2(W, \widehat{\mathcal{C}})$. □

Theorem 5.4. *There is a perfect pairing of finite groups*

$$\text{III}^0(\mathcal{C}) \times \text{III}^2(\widehat{\mathcal{C}}) \rightarrow \mathbf{Q}/\mathbf{Z}.$$

Proof: As in the proof of Lemma 5.1, Lemma 5.3 and Lemma 4.15 imply that for a sufficiently small Zariski open subset $U \subset U_0$, we have $\text{III}^0(\mathcal{C}) = D^0(U, \mathcal{C})$ and $\text{III}^2(\widehat{\mathcal{C}}) \cong D^2(U, \widehat{\mathcal{C}})$. Now apply Proposition 4.17. □

For each integer i , denote by $\prod'_{v \in X^{(1)}} H^i(K_v, \mathcal{C})$ (resp. $\prod'_{v \in X^{(1)}} H^i(K_v, \mathcal{C})_{\wedge}$) the restricted product of the $H^i(K_v, \mathcal{C})$ (resp. of the $H^i(K_v, \mathcal{C})_{\wedge}$) with respect to the $H_{\text{nr}}^i(K_v, \mathcal{C})$ (resp. to the image of $H^1(\mathcal{O}_v, \mathcal{C})$ in $H^i(K_v, \mathcal{C})_{\wedge}$). The same notation stands for $\widehat{\mathcal{C}}$. The groups $\prod'_{v \in X^{(1)}} H^i(K_v, \mathcal{C})$ and $\prod'_{v \in X^{(1)}} H^i(K_v, \widehat{\mathcal{C}})$ are equipped with their restricted product topology (associated to the topology previously defined on the $H^i(K_v, \mathcal{C})$ and $H^i(K_v, \widehat{\mathcal{C}})$). All groups $H^i(K, \mathcal{C})$ (resp. $H^i(K, \widehat{\mathcal{C}})$) are equipped with the discrete topology.

Lemma 5.5. *Let i be an integer. Then the image of $H^i(K, \mathcal{C})$ in $\prod'_{v \in X^{(1)}} H^i(K_v, \mathcal{C})$ is discrete for the subspace topology. The same holds if \mathcal{C} is replaced by $\widehat{\mathcal{C}}$.*

Proof: As the local fields K_v are of strict cohomological dimension 2, the statement is obvious except for $-1 \leq i \leq 2$. Fix a Zariski open subset $U \subset U_0$ with $U \neq X$.

All groups $H^i(K_v, \widehat{C})$ are discrete, so the subgroup $E := \prod_{v \notin U} \{0\} \times \prod_{v \in U} H_{\text{nr}}^i(K_v, \widehat{C})$ is open in $\prod'_{v \in X^{(1)}} H^i(K_v, \widehat{C})$. Let I be the image of $H^i(K, \widehat{C})$ in $\prod'_{v \in X^{(1)}} H^i(K_v, \widehat{C})$. Every element of $H^1(K, \widehat{C})$ comes from $H^1(V, \widehat{C})$ for some $V \subset U$, hence by Lemma 2.2, there is a surjection $D^i(U, \widehat{C}) \rightarrow I \cap E$. Since all groups $D^i(U, \widehat{C})$ are finite by Proposition 3.6 c), Lemma 4.14 and Lemma 4.15, this implies that $I \cap E$ is finite, hence I is discrete.

The same argument shows that the image J of $H^i(K, C)$ in $\prod'_{v \in X^{(1)}} H^i(K_v, C)$ is discrete for $i \geq 1$. For $i \in \{-1, 0\}$, this is an immediate consequence of Lemma 3.8 (again combined with Lemma 2.2). \square

Lemma 5.6. *Let $U \subset U_0$ be a non empty Zariski open subset with $U \neq X$.*

a) *There are exact sequences*

$$\begin{aligned} H^0(U, C) &\rightarrow \prod_{v \notin U} H^0(K_v, C) \times \prod_{v \in U} H_{\text{nr}}^0(K_v, C) \rightarrow H^1(K, \widehat{C})^*. \\ 0 &\rightarrow H^{-1}(U, C) \rightarrow \prod_{v \notin U} H^{-1}(K_v, C) \times \prod_{v \in U} H_{\text{nr}}^{-1}(K_v, C) \rightarrow H^2(K, \widehat{C})^*. \end{aligned}$$

b) *There are exact sequences*

$$\begin{aligned} H^2(U, \widehat{C}) &\rightarrow \prod_{v \notin U} H^2(K_v, \widehat{C}) \times \prod_{v \in U} H_{\text{nr}}^2(K_v, \widehat{C}) \rightarrow H^{-1}(K, C)^* \rightarrow 0. \\ H^1(U, \widehat{C}) &\rightarrow \prod_{v \notin U} H^1(K_v, \widehat{C}) \times \prod_{v \in U} H_{\text{nr}}^1(K_v, \widehat{C}) \rightarrow H^0(K, C)^* \rightarrow D^2(U, \widehat{C}). \end{aligned}$$

Proof: a) Let $V \subset U$ be a non empty Zariski open subset. Let $i \in \{-1, 0\}$. By Lemma 2.2, we have an exact sequence

$$H^i(U, C) \rightarrow \prod_{v \notin U} H^i(K_v, C) \times \prod_{v \in U \setminus V} H_{\text{nr}}^i(K_v, C) \rightarrow H_c^{i+1}(V, C).$$

By Proposition 3.13, the map $H_c^{i+1}(V, C) \rightarrow H_c^{i+1}(V, C)_\wedge$ is injective, thus by Theorem 4.11 we get an exact sequence

$$H^i(U, C) \rightarrow \prod_{v \notin U} H^i(K_v, C) \times \prod_{v \in U \setminus V} H_{\text{nr}}^i(K_v, C) \rightarrow H^{1-i}(V, \widehat{C})^*,$$

where $H^{1-i}(V, \widehat{C})$ is a discrete torsion group. Besides, the kernel of the first map is a subgroup of $D^i(U, C)$, hence it is finite for $i = 0$ by Lemma 4.15. This kernel is also obviously zero for $i = -1$ as soon as $V \neq U$. This implies that the inverse limit of this exact sequence (when V runs over all non empty Zariski open subsets of U) remains exact, which yields the result.

b) We apply again Lemma 2.2 and observe that for $i \in \{1, 2\}$:

-we have $H_c^{i+1}(V, \widehat{C}) \simeq (H^{1-i}(V, C)_\wedge)^* \simeq H^{1-i}(V, C)^*$ by Theorem 4.9, because the discrete finitely generated (cf. Proposition 3.6, b) group $H^{1-i}(V, C)$ and its completion have same dual.

-the groups $D^i(U, \widehat{C})$ are finite (Lemma 4.15 and Proposition 3.6 c).

Now the same method as in a) gives the exactness of

$$H^i(U, \widehat{C}) \rightarrow \prod_{v \notin U} H^i(K_v, \widehat{C}) \times \prod_{v \in U} H_{\text{nr}}^i(K_v, \widehat{C}) \rightarrow H^{1-i}(K, C)^*.$$

Besides, by [DH], Prop 2.1, there is a commutative diagram with exact lines:

$$\begin{array}{ccccc} & & & & D^{i+1}(U, \widehat{C}) \\ & & & & \uparrow \\ \prod_{v \in U \setminus V} H_{\text{nr}}^i(K_v, \widehat{C}) & \longrightarrow & H_c^{i+1}(V, \widehat{C}) & \longrightarrow & H_c^{i+1}(U, \widehat{C}) \\ & & \uparrow & & \uparrow \\ & & \prod_{v \notin V} H^i(K_v, \widehat{C}) & \longleftarrow & \prod_{v \notin U} H^i(K_v, \widehat{C}). \end{array}$$

The right column is also exact by definition of $D^{i+1}(U, \widehat{C})$. By diagram chasing, this yields an exact sequence

$$\prod_{v \notin U} H^i(K_v, \widehat{C}) \times \prod_{v \in U \setminus V} H_{\text{nr}}^i(K_v, \widehat{C}) \xrightarrow{s_V} H_c^{i+1}(V, \widehat{C}) \rightarrow D^{i+1}(U, \widehat{C}). \quad (40)$$

As seen before, the kernel of s_V is the image of $H^i(U, \widehat{C})$, which implies that for $W \subset V$, the transition map $\ker s_W \rightarrow \ker s_V$ is surjective. The map

$$\prod_{v \notin U} H^i(K_v, \widehat{C}) \times \prod_{v \in U \setminus W} H_{\text{nr}}^i(K_v, \widehat{C}) \rightarrow \prod_{v \notin U} H^i(K_v, \widehat{C}) \times \prod_{v \in U \setminus V} H_{\text{nr}}^i(K_v, \widehat{C})$$

is also obviously surjective. Thus taking projective limit over V in (40) gives an exact sequence

$$\prod_{v \notin U} H^i(K_v, \widehat{C}) \times \prod_{v \in U} H_{\text{nr}}^i(K_v, \widehat{C}) \rightarrow H^{1-i}(K, C)^* \rightarrow D^{i+1}(U, \widehat{C})$$

(indeed recall that $H_c^{i+1}(V, \widehat{C}) \cong H^{1-i}(V, C)^*$). It remains to observe that for $i = 2$, we have $D^3(U, \widehat{C}) \subset H^3(U, \widehat{C}) = 0$ (Proposition 4.4). □

Theorem 5.7 (Poitou-Tate I). *In the sequence*

$$\begin{array}{ccccccc} 0 & \longrightarrow & H^{-1}(K, C) & \longrightarrow & \prod'_{v \in X^{(1)}} H^{-1}(K_v, C) & \longrightarrow & H^2(K, \widehat{C})^* \\ & & & & & & \downarrow \\ & & H^1(K, \widehat{C})^* & \longleftarrow & \prod'_{v \in X^{(1)}} H^0(K_v, C) & \longleftarrow & H^0(K, C) \\ & & \downarrow & & & & \\ & & H^1(K, C) & \longrightarrow & \bigoplus_{v \in X^{(1)}} H^1(K_v, C) & \longrightarrow & H^0(K, \widehat{C})^* \\ & & & & & & \downarrow \\ 0 & \longleftarrow & H^{-1}(K, \widehat{C})^* & \longleftarrow & \bigoplus_{v \in X^{(1)}} H^2(K_v, C) & \longleftarrow & H^2(K, C) \end{array} \quad (41)$$

every sequence of three consecutive terms is exact, except the two ones respectively finishing with $H^0(K, C)$ and $H^1(K, C)$, which must be replaced with the following “completed” exact sequences:

$$\prod'_{v \in X^{(1)}} H^{-1}(K_v, C)_\wedge \rightarrow H^2(K, \widehat{C})^* \rightarrow H^0(K, C).$$

$$\prod'_{v \in X^{(1)}} H^0(K_v, C)_\wedge \rightarrow H^1(K, \widehat{C})^* \rightarrow H^1(K, C).$$

Proof: Let $U \subset U_0$ be a non empty Zariski open subset with $U \neq X$.

First take $i \in \{1, 2\}$. By (3), there is an exact sequence

$$H^i(U, C) \rightarrow \bigoplus_{v \notin U} H^i(K_v, C) \rightarrow H_c^{i+1}(U, C)$$

and for $i = 2$ the last map is surjective because $H^3(U, C) = 0$ by Proposition 4.3 a). By Theorem 4.9 and Proposition 3.13, we have

$$H_c^{i+1}(U, C) \cong H^{1-i}(U, \widehat{C})^* \cong H^{1-i}(K, \widehat{C})^*,$$

whence (by Lemma 3.14) for every non empty Zariski open subset $V \subset U$, a commutative diagram with exact lines

$$\begin{array}{ccccc} H^i(U, C) & \longrightarrow & \bigoplus_{v \notin U} H^i(K_v, C) & \longrightarrow & H^{1-i}(K, \widehat{C})^* \\ \downarrow & & \downarrow j & & \downarrow = \\ H^i(V, C) & \longrightarrow & \bigoplus_{v \notin V} H^i(K_v, C) & \longrightarrow & H^{1-i}(K, \widehat{C})^*, \end{array}$$

where j is obtained by putting 0 at the missing places (and the right horizontal maps are surjective for $i = 2$). Therefore taking direct limit over U in the first line of this diagram gives that the last two lines of (41) are exact. The exactness of the first two lines of (41) comes from Lemma 5.6 after taking again direct limit over U .

It remains to prove the exactness of the following three sequences:

$$\prod'_{v \in X^{(1)}} H^{-1}(K_v, C)_\wedge \rightarrow H^2(K, \widehat{C})^* \rightarrow \mathbb{H}^0(C) \rightarrow 0.$$

$$\prod'_{v \in X^{(1)}} H^0(K_v, C)_\wedge \rightarrow H^1(K, \widehat{C})^* \rightarrow \mathbb{H}^1(C) \rightarrow 0.$$

$$\bigoplus_{v \in X^{(1)}} H^1(K_v, C) \rightarrow H^0(K, \widehat{C})^* \rightarrow \mathbb{H}^2(C) \rightarrow 0.$$

We observe that for $0 \leq i \leq 2$, the following sequence is exact:

$$0 \rightarrow \mathbb{H}^i(\widehat{C}) \rightarrow H^i(K, \widehat{C}) \xrightarrow{p_i} \prod'_{v \in X^{(1)}} H^i(K_v, \widehat{C}).$$

Set $A_i := \text{Imp}_i$; we get exact sequences (the maps being strict by Lemma 5.5) of Hausdorff, totally disconnected groups

$$0 \rightarrow \mathbb{H}^i(\widehat{C}) \rightarrow H^i(K, \widehat{C}) \rightarrow A_i \rightarrow 0.$$

$$0 \rightarrow A_i \rightarrow \prod'_{v \in X^{(1)}} H^i(K_v, \widehat{C}),$$

where A_i is equipped with the discrete topology. By [HSS], Lemma 2.4 (where the groups are assumed to be locally compact, but the proof shows that it sufficient to assume that they have a basis of neighborhoods of zero consisting of open subgroups; this is the case for all groups considered here), the duals of these exact sequences are also exact. Recall that for $i \in \{-1, 0\}$, the group $H^i(K_v, C)_\wedge = \varprojlim_{n>0} (H^i(K_v, C)/n)$ is also the completion $H^i(K_v, C)^\wedge$ of $H^i(K_v, C)$ with respect to the open subgroups of finite index. By [Dem1], Th 3.1. and 3.3., the dual of $\prod'_{v \in X^{(1)}} H^i(K_v, \widehat{C})$ is $\prod'_{v \in X^{(1)}} H^{1-i}(K_v, C)_\wedge$ for $1 \leq i \leq 2$, and the dual of the group $\prod'_{v \in X^{(1)}} H^0(K_v, \widehat{C}) = \prod_{v \in X^{(1)}} H^0(K_v, \widehat{C})$ (cf. Remark 3.7) is $\bigoplus_{v \in X^{(1)}} H^1(K_v, C)$. By Theorems 5.4 and 5.2, the dual of the finite group $\text{III}^i(\widehat{C})$ is $\text{III}^{2-i}(C)$. This proves the result. \square

Theorem 5.8 (Poitou-Tate II). *In the sequence*

$$\begin{array}{ccccccc}
0 & \longrightarrow & H^{-1}(K, \widehat{C}) & \longrightarrow & \prod_{v \in X^{(1)}} H^{-1}(K_v, \widehat{C}) & \longrightarrow & H^2(K, C)^* \\
& & & & & & \downarrow \\
& & H^1(K, C)^* & \longleftarrow & \prod_{v \in X^{(1)}} H^0(K_v, \widehat{C}) & \longleftarrow & H^0(K, \widehat{C}) \\
& & \downarrow & & & & \\
& & H^1(K, \widehat{C}) & \longrightarrow & \prod'_{v \in X^{(1)}} H^1(K_v, \widehat{C}) & \longrightarrow & H^0(K, C)^* \\
& & & & & & \downarrow \\
0 & \longleftarrow & H^{-1}(K, C)^* & \longleftarrow & \prod'_{v \in X^{(1)}} H^2(K_v, \widehat{C}) & \longleftarrow & H^2(K, \widehat{C})
\end{array} \tag{42}$$

every sequence of three consecutive terms is exact, except the two ones respectively finishing with $H^0(K, \widehat{C})$ and $H^1(K, \widehat{C})$, which must be replaced with the following completed exact sequences:

$$\begin{array}{l}
\prod_{v \in X^{(1)}} H^{-1}(K_v, \widehat{C})_\wedge \rightarrow H^2(K, C)^* \rightarrow H^0(K, \widehat{C}). \\
\prod_{v \in X^{(1)}} H^0(K_v, \widehat{C})_\wedge \rightarrow H^1(K, C)^* \rightarrow H^1(K, \widehat{C}).
\end{array}$$

Proof: This is very similar to the proof of Theorem 5.7. Let $U \subset U_0$ be a non empty open subset with $U \neq X$. For $i \in \{-1, 0\}$, (3) and Proposition 3.6 a) give an exact sequence

$$H^i(K, \widehat{C}) \rightarrow \prod_{v \notin U} H^i(K_v, \widehat{C}) \rightarrow H_c^{i+1}(U, \widehat{C}),$$

such that the kernel of the first map is zero for $i = -1$, and this kernel is finite (by Lemma 4.14) for $i = 0$. Applying Theorem 4.9 and taking projective limit over U , we obtain that the first two lines of (42) are exact.

Taking direct limit over U in the exact sequences of Lemma 5.6 b) yields the exactness of the last line and of the sequence

$$H^1(K, \widehat{C}) \rightarrow \prod'_{v \in X^{(1)}} H^1(K_v, \widehat{C}) \rightarrow H^0(K, C)^* \rightarrow \text{III}^2(\widehat{C}).$$

because $\mathbb{H}^2(\widehat{C}) \cong \varinjlim_U D^2(U, \widehat{C})$ (cf. Theorem 5.4).

Finally, dualizing the exact sequence of discrete groups

$$0 \rightarrow \mathbb{H}^i(C) \rightarrow H^i(K, C) \rightarrow \bigoplus_{v \in X^{(1)}} H^i(K_v, C)$$

for $i \in \{1, 2\}$ gives the missing pieces of Theorem 5.8, thanks to Theorem 5.2 and [Dem1], Th. 3.1. and 3.3. □

Remark 5.9. As the groups $H^1(K, \widehat{C})$ and $H^2(K, \widehat{C})$ are torsion, it is also possible to replace the last two lines of (42) by the following exact sequences

$$0 \rightarrow \mathbb{H}^1(\widehat{C}) \rightarrow H^1(K, \widehat{C}) \rightarrow \left(\prod'_{v \in X^{(1)}} H^1(K_v, \widehat{C}) \right)_{\text{tors}} \rightarrow (H^0(K, C)_\wedge)^* \rightarrow \mathbb{H}^2(\widehat{C}) \rightarrow 0. \quad (43)$$

$$0 \rightarrow \mathbb{H}^2(\widehat{C}) \rightarrow H^2(K, \widehat{C}) \rightarrow \left(\prod'_{v \in X^{(1)}} H^2(K_v, \widehat{C}) \right)_{\text{tors}} \rightarrow (H^{-1}(K, C)_\wedge)^* \rightarrow 0. \quad (44)$$

Indeed for $i \in \{-1, 0\}$, the dual of $H^i(K, C)_\wedge = \varprojlim_{n>0} (H^i(K, C)/n)$ (equipped with the inverse limit topology) is

$$\varinjlim_{n>0} (H^i(K, C)/n)^* = \varinjlim_{n>0} {}_n H^i(K, C)^* = (H^i(K, C)^*)_{\text{tors}},$$

which gives that (43) and (44) are exact, except that we don't have the surjectivity of $(H^0(K, C)_\wedge)^* \rightarrow \mathbb{H}^2(\widehat{C})$ yet. To see the latter, we first observe that for a K -torus T , the subgroup of divisible elements in $H^0(K, T)$ is trivial (indeed we may assume that T is split and this is so for K^* because K is a global field); then the same property holds by dévissage (using (12)) for $H^0(K, C)$ because for a group of multiplicative type M , the group $H^1(K, M)$ is of finite exponent via Hilbert 90. Therefore the canonical map $H^0(K, C) \rightarrow H^0(K, C)_\wedge$ is injective, whence an injection $\mathbb{H}^0(C) \hookrightarrow H^0(K, C)_\wedge$, whose dual $(H^0(K, C)_\wedge)^* \rightarrow \mathbb{H}^2(\widehat{C})$ (cf. Theorem 5.4) is surjective, the group $\mathbb{H}^0(C)$ being finite and $H^0(K, C)_\wedge$ having a basis of neighborhoods of zero consisting of open subgroups (cf. [HSS], Lemma 2.4).

We can now prove the following variant of Theorem 5.7:

Theorem 5.10 (Poitou-Tate, Γ). *There is an exact sequence*

$$\begin{array}{ccccccc} 0 & \longrightarrow & H^{-1}(K, C)_\wedge & \longrightarrow & [\prod'_{v \in X^{(1)}} H^{-1}(K_v, C)]_\wedge & \longrightarrow & H^2(K, \widehat{C})^* \\ & & & & & & \downarrow \\ & & H^1(K, \widehat{C})^* & \longleftarrow & [\prod'_{v \in X^{(1)}} H^0(K_v, C)]_\wedge & \longleftarrow & H^0(K, C)_\wedge \\ & & \downarrow & & & & \\ & & H^1(K, C) & \longrightarrow & \bigoplus_{v \in X^{(1)}} H^1(K_v, C) & \longrightarrow & H^0(K, \widehat{C})^* \\ & & & & & & \downarrow \\ 0 & \longleftarrow & H^{-1}(K, \widehat{C})^* & \longleftarrow & \bigoplus_{v \in X^{(1)}} H^2(K_v, C) & \longleftarrow & H^2(K, C) \end{array} \quad (45)$$

Proof: Dualizing (43) and (44) yields the two exact sequences

$$0 \rightarrow \text{III}^0(C) \rightarrow H^0(K, C)_\wedge \rightarrow [\prod'_{v \in X^{(1)}} H^0(K_v, C)]_\wedge \rightarrow H^1(K, \widehat{C})^* \rightarrow \text{III}^1(C) \rightarrow 0.$$

$$0 \rightarrow H^{-1}(K, C)_\wedge \rightarrow [\prod'_{v \in X^{(1)}} H^{-1}(K_v, C)]_\wedge \rightarrow H^2(K, \widehat{C})^* \rightarrow \text{III}^0(C) \rightarrow 0.$$

The other parts of the sequence follow from Theorem 5.7. □

Remark 5.11. A subtle point here is that for $i \in \{-1, 0\}$, there is a canonical injective map (which is induced by the isomorphism $\prod_v H^i(K_v, C)_\wedge \cong [\prod_v H^i(K_v, C)]_\wedge$):

$$\prod'_{v \in X^{(1)}} H^i(K_v, C)_\wedge \hookrightarrow [\prod'_{v \in X^{(1)}} H^i(K_v, C)]_\wedge,$$

but this map is not surjective in general. For instance if $i = 0$ and $C = \mathbf{G}_m$, we have $H^0(K_v, C) \cong \mathcal{O}_v^* \times \mathbf{Z}$; as $\bigoplus_{v \in X^{(1)}} \widehat{\mathbf{Z}}$ is smaller than $[\bigoplus_{v \in X^{(1)}} \mathbf{Z}]_\wedge$, the aforementioned map is not surjective. Observe that the natural map $H^i(K, C) \rightarrow \prod'_{v \in X^{(1)}} H^i(K_v, C)_\wedge$ does not in general extend to $H^i(K, C)_\wedge$, it is only defined on the bidual $H^i(K, C)^{**}$, which is smaller than $H^i(K, C)_\wedge$.

Remark 5.12. Dualizing the exact sequence

$$0 \rightarrow \text{III}^i(C) \rightarrow H^i(K, C) \rightarrow \bigoplus_{v \in X^{(1)}} H^i(K_v, C) \rightarrow H^{1-i}(K, \widehat{C})^* \rightarrow \text{III}^{i+1}(K, C) \rightarrow 0$$

for $i \in \{1, 2\}$ also yields an exact Poitou-Tate sequence II' , which is the same as (42) except that for $r \in \{-1, 0\}$, the group $H^r(K, \widehat{C})$ (resp. $\prod_{v \in X^{(1)}} H^r(K_v, \widehat{C})$) has to be replaced by $H^r(K, \widehat{C})_\wedge$ (resp. by $(\prod_{v \in X^{(1)}} H^r(K_v, \widehat{C}))_\wedge = \prod_{v \in X^{(1)}} H^r(K_v, \widehat{C})_\wedge$).

Remark 5.13. If we replace the function field K by a number field, some results of this section still hold and some of them have to be modified. Namely:

- Theorem 5.2 is unchanged (with the same proof), see also Theorems 5.7 and 5.12 in [Dem1] (in the latter the assumption $\ker \rho$ finite is unnecessary).
- Lemma 5.3 still holds. Therefore Theorem 5.4 is also true: indeed since the pairing (38) has divisible right kernel and trivial left kernel, taking the direct limit over U (and using the facts that the sequence of finite groups $D^0(U, C)$ stabilizes for U sufficiently small) yields a pairing $\text{III}^0(C) \times \text{III}^2(\widehat{C})$ with divisible right kernel and trivial left kernel. But it is known that $\text{III}^2(\widehat{C})$ is finite (see [Dem1], proof of Th. 5.14), whence the result (which extends [Dem1], Th. 5.23). Actually the image of $H_c^2(U, \widehat{C})$ into $H^2(K, \widehat{C})$ is finite by dévissage thanks to exact triangle (13): indeed $H^3(K, \widehat{T}) \cong \bigoplus_{v \in \Omega_{\mathbf{R}}} H^3(K_v, \widehat{T})$ is finite for a torus T , and $H_c^2(U, \widehat{\mathcal{M}})$ is also finite for a group of multiplicative type \mathcal{M} because we already saw (cf. Remark 3.15 and Lemma 3.4, c) that this holds when \mathcal{M} is a torus or a finite group.
- In Lemma 5.5, one has to restrict to $i \geq 1$ for the assertion about C . The result about \widehat{C} holds for an arbitrary i (although the group $D^2(U, \widehat{C})$ might be infinite, we just saw that its image in $H^2(K, \widehat{C})$ is finite, which is sufficient).

- Lemma 5.6 a) is not valid any more (one has to complete the first two terms in the exact sequences); the second exact sequence of b) still holds (same proof), as does the first one except for the surjectivity of the last map, which must be replaced by the exact sequence

$$\prod_{v \notin U} H^2(K_v, \widehat{\mathcal{C}}) \times \prod_{v \in U} H_{\text{nr}}^2(K_v, \widehat{\mathcal{C}}) \rightarrow H^{-1}(K, C)^* \rightarrow D^3(U, \widehat{\mathcal{C}}) \rightarrow 0$$

because we lack the vanishing of $D^3(U, \widehat{\mathcal{C}})$. Also, since $D^2(U, \widehat{\mathcal{C}})$ is in general infinite, the proof of the exactness of

$$H^2(U, \widehat{\mathcal{C}}) \rightarrow \prod_{v \notin U} H^2(K_v, \widehat{\mathcal{C}}) \times \prod_{v \in U} H_{\text{nr}}^2(K_v, \widehat{\mathcal{C}}) \rightarrow H^{-1}(K, C)^*$$

is a little bit more complicated (using exact triangle (13) one reduces to the case when \mathcal{C} is quasi-isomorphic to $\mathcal{M}[1]$, where \mathcal{M} is a group of multiplicative type; then one proceeds as in Lemma 5.6 b), the group $D^2(U, \widehat{\mathcal{M}})$ being finite because $H_c^2(U, \widehat{\mathcal{M}})$ is finite).

- By the previous observations, the end of sequence (41) starting with $H^1(K, \widehat{\mathcal{C}})^* \rightarrow H^1(K, C) \rightarrow \dots$ remains exact. Theorem 5.8 is valid with one single slight complication in the proof: we do not know in general that $D^3(U, \widehat{\mathcal{C}}) = 0$, but the direct limit over U of the $D^3(U, \widehat{\mathcal{C}})$ is $\text{III}^3(\widehat{\mathcal{C}})$, which is zero. Theorem 5.10 is therefore also unchanged, which extends [Dem1], Theorems 6.1 and 6.3.

Remark 5.14. In the case of one single torus T with module of characters \widehat{T} , some of our results of section 4 and 5 can be deduced from similar theorems on 1-motives proven by González-Avilés ([Go1], Th. 6.6) and González-Avilés/Tan ([Go2], Th. 3.11).

Acknowledgements. We thank J.L. Colliot-Thélène and the anonymous referee for helpful comments.

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