Artin–Mazur–Milne duality for fppf cohomology

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Abstract. We provide a complete proof of a duality theorem for the fppf cohomology of either a curve over a finite field or a ring of integers of a number field, which extends the classical Artin–Verdier Theorem in étale cohomology. We also prove some finiteness and vanishing statements.

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1 Introduction

Let K be a number field or the function field of a smooth, projective, geometrically integral curve X over a finite field. In the number field case, set $X := \operatorname{Spec} \mathcal{O}_K$, where $\mathcal{O}_K$ is the ring of integers of K. Let U be a non empty Zariski open subset of X and denote by N a commutative, finite and flat group scheme over U with Cartier dual N^D. Assume that the order of N is invertible on U (in particular N is étale). The classical "étale" *Artin–Verdier Theorem* (cf. [Mi2], Corollary II.3.3.) is a duality statement between étale cohomology $H^{\bullet}_{\text{ét}}(U, N)$ and étale cohomology with compact support $H^{\bullet}_{\text{ét},c}(U, N^D)$. It has been known for a long time that this theorem is especially useful in view of concrete arithmetic applications: for example it yields a very nice method to prove deep results like Cassels–Tate duality for abelian varieties and schemes ([Mi2], section II.5) and their generalizations to 1-motives ([HS], section 4); Artin–Verdier's Theorem also provides a "canonical" path to prove the Poitou–Tate's Theorem and its extension to complex of tori ([Dem1]), which in turn turns out to be very fruitful to deal with local-global questions for (non necessarily commutative) linear algebraic groups ([Dem2]).

It is of course natural to try to remove the condition that the order of N is invertible on U. A good framework to do this is provided by fppf cohomology of finite and flat commutative group schemes over U, as introduced by J.S. Milne in the third part of his book [Mi2]. This includes the case of group schemes of order divisible by $p := \operatorname{Char} K$ in the function field case.

Such a fppf duality theorem has been first announced by B. Mazur[1] ([Maz], Prop. 7.2), relying on work by M. Artin and himself. Special cases have also been proved by M. Artin and Milne ([AM]). The precise statement of the theorem is as follows (see [Mi2], Corollary III.3.2. for the number field case and Theorem III.8.2 for the function field case):

[1]Thanks to A. Schmidt for having pointed this out to us.

Theorem 1.1 *Let $j : U \hookrightarrow X$ be a non empty open subscheme of X. Let N be a finite flat commutative group scheme over U with Cartier dual N^D. For all integers r with $0 \leq r \leq 3$, the canonical pairing*

$$H_c^{3-r}(U, N) \times H^r(U, N^D) \to H_c^3(U, \mathbf{G}_m) \cong \mathbf{Q}/\mathbf{Z}$$

(where $H^r(U, N^D)$ is a fppf cohomology group and $H_c^{3-r}(U, N)$ a fppf cohomology group with compact support) induces a perfect duality between the profinite group $H_c^{3-r}(U, N)$ and the discrete group $H^r(U, N^D)$. Besides, these groups are finite in the number field case, and they are trivial for $r \geq 4$ and $r < 0$ (resp. for $r = 3$ if $U \neq X$) in the function field case.

For example, this extension of the étale Artin–Verdier Theorem is needed to prove the Poitou–Tate exact sequence over global fields of characteristic p ([Gon], Th. 4.8. and 4.11) as well as the Poitou–Tate Theorem over a global field without restriction on the order ([Čes2], Th. 5.1, which in turn is used in [Ros], §5.6 and 5.7). Results of [Mi2], section III.9. (which rely on the fppf duality Theorem) are also a key ingredient in the proof of some cases of the Birch and Swinnerton-Dyer conjecture for abelian varieties over a global field of positive characteristic, in [Bau], §4 and [KT], §2 for instance. Our initial interest in Theorem 1.1 was to try to extend it to complexes of tori in the function field case, following the same method as in the number field case [Dem1]. Such a generalization should then provide results (known in the number field case) about weak and strong approximation for linear algebraic groups defined over a global field of positive characteristic.

However, as K. Česnavičius pointed out to us[2], it seems necessary to add details to the proof in [Mi2], sections III.3. and III.8, for two reasons:

- the functoriality of flat cohomology with compact support and the commutativity of several diagrams are not explained in [Mi2]. Even in the case of an imaginary number field, a definition of $H_c^r(U, \mathcal{F})$ as $H^r(X, j_!\mathcal{F})$ for a fppf sheaf $\mathcal{F}$ (which works for the étale Artin–Verdier Theorem) would not be the right one, because it does not provide the key exact sequence [Mi2] Prop. III.0.4.a) in the fppf setting: indeed the proof of this exact sequence relies on [Mi2], Lemma II.2.4., which in turn uses [Mi2], Prop II.1.1; but the analogue of the latter does not stand anymore with étale cohomology replaced by fppf cohomology, see also Remark 2.2 of the present paper.

  It is therefore necessary to work with an adhoc definition of compact support cohomology as in loc. cit., §III.0. Since this definition involves mapping cones, commutativities of some diagrams have to be checked in the category of complexes and not in the derived category (where there is no good functoriality for the mapping cones). Typically, the isomorphisms that compute $C^\bullet(b)$, $C^\bullet(b \circ a)$ and $C^\bullet(c \circ b \circ a)$ in loc. cit., Prop. III.0.4.c) are not canonical a priori. Hence the required compatibilities in loc. cit., proof of Theorem III.3.1. and Lemma III.8.4. have to be checked carefully.

[2] In particular, he observed that the analogue of [Mi2], Prop. III.0.4.c) is by no means obvious when henselizations are replaced by completions. This analogue is actually false without additional assumptions, as shown by T. Suzuki in [Suz], Rem 2.7.9

- in the positive characteristic case, it is necessary (as explained in [Mi2], §III.8.) to work with a definition of cohomology with compact support involving *completions* of the local rings of points in $X \setminus U$ instead of their henselizations. The reason is that a local duality statement (loc. cit., Th. III.6.10) is needed and this one only works in the context of complete valuation fields, in particular because the H^1 groups involved have to be locally compact (so that Pontryagin duality makes sense). It turns out that some properties of compact support cohomology (in particular loc. cit., Prop. III.0.4.c)) are more difficult to establish in this context: for example the comparison between cohomology of the completion $\widehat{\mathcal{O}_v}$ and of the henselization $\mathcal{O}_v$ is not as straightforward as in the étale case.

The goal of this article is to present a detailed proof of Theorem 1.1 with special regards to the two issues listed above. Section 2 is devoted to general properties of fppf cohomology with compact support (Prop. 2.1), which involves some homological algebra (Lemma 2.3) as well as comparison statements between cohomology of $\mathcal{O}_v$ and $\widehat{\mathcal{O}_v}$ (Lemma 2.6); besides, we make the link to classical étale cohomology with compact support (Lemma 2.10).

We also define a natural topology on the fppf compact support cohomology groups (see section 3) and prove its basic properties. In section 4, we follow the method of [Mi2], §III.8. to prove Theorem 1.1 in the function field case. As a corollary, we get a finiteness statement (Cor. 4.9), which apparently has not been observed before this paper. The case of a number field is simpler once the functorial properties of section 2 have been proved; it is treated in section 5. Finally, we include two useful results in homological algebra in an appendix (section 6).

One week after the first draft of this article was released, Takashi Suzuki kindly informed us that in his preprint [Suz], he obtained (essentially at the same time as us) fppf duality results similar to Theorem 1.1 in a slightly more general context.[3] His methods are somehow more involved than ours, they use the *rational étale site*, which he developed in earlier papers.

Notation. Let X be either a smooth projective curve over a finite field k of characteristic p, or the spectrum of the ring of integers $\mathcal{O}_K$ of a number field K. Let $K := k(X)$ be the function field of X. Throughout the paper, schemes S are endowed with a big fppf site $(\mathrm{Sch}/S)_{\mathrm{fppf}}$ in the sense of [SP, Tag 021R, Tag 021S, Tag 03XB]. By construction, the underlying category in $(\mathrm{Sch}/S)_{\mathrm{fppf}}$ is small and the family of coverings for this site is a set. The corresponding topos is independent of the choices made thanks to [SP, Tag 00VY]. In contrast with [SGA4], the construction of the site $(\mathrm{Sch}/S)_{\mathrm{fppf}}$ in [SP] does not require the existence of universes. The reader who is ready to accept this axiom can replace the site $(\mathrm{Sch}/S)_{\mathrm{fppf}}$ by the big fppf site from [SGA4].

Unless stated otherwise, cohomology is fppf cohomology with respect to this site.

For any closed point $v \in X$, let $\mathcal{O}_v$ (resp. $\widehat{\mathcal{O}_v}$) be the henselization (resp. the completion) of the local ring $\mathcal{O}_{X,v}$ of X at v. Let K_v (resp. $\widehat{K_v}$) be the fraction field of $\mathcal{O}_v$ (resp. $\widehat{\mathcal{O}_v}$). Let U be a non empty Zariski open subset of X and denote by $j : U \to X$ the corresponding open immersion. By [Mat], §34, the local ring $\mathcal{O}_{X,v}$ of

[3]Note, however, that there is still some work to do to obtain our Theorem 1.1 from the very general Th 3.1.3. of [Suz]; compare with section 4.2. of loc. cit., where a similar task is fullfilled for abelian schemes instead of finite group schemes.

X at v is excellent (indeed $\mathcal{O}_{X,v}$ is either of mixed characteristic or the localization of a ring of finite type over a field); hence so are $\mathcal{O}_v$ (by [EGA4], Cor. 18.7.6) as the henselization of an excellent ring, and $\widehat{\mathcal{O}_v}$ as a complete Noetherian local ring ([Mat], §34).

The piece of notation "$v \notin U$" means that we consider all places v corresponding to closed points of $X \setminus U$ *plus the real places in the number field case.* If v is a real place, we set $K_v = \widehat{K_v} = \mathcal{O}_v = \widehat{\mathcal{O}_v}$ for the completion of K at v, and we denote by $H^*(K_v, M)$ the *Tate (or modified)* cohomology groups of a $\mathrm{Gal}(\overline{K_v}/K_v)$-module M.

If $\mathcal{F}$ is a fppf sheaf of abelian groups on U, define the Cartier dual $\mathcal{F}^D$ to be the fppf sheaf $\mathcal{F}^D := \underline{\mathrm{Hom}}(\mathcal{F}, \mathbf{G}_m)$. Notation as $\Gamma(U,\mathcal{F})$ stands for the group of sections of $\mathcal{F}$ over U, and $\Gamma_Z(U,\mathcal{F})$ for the group of sections that vanish over $U \setminus Z$. If E is a field (e.g. $E = K_v$ or $E = \widehat{K_v}$) and $i : \mathrm{Spec}\, E \to U$ is an E-point of U, we will frequently write $H^r(E,\mathcal{F})$ for $H^r(\mathrm{Spec}\, E, i^*\mathcal{F})$. Similarly for an open subset $V \subset U$, the piece of notation $H^r(V,\mathcal{F})$ (resp. $H^r_c(V,\mathcal{F})$) stands for $H^r(V,\mathcal{F}_{|V})$ (resp. $H^r_c(V,\mathcal{F}_{|V})$).

A finite group scheme N over a field E of characteristic $p > 0$ is *local* (or equivalently *infinitesimal*, as in [DG], II.4.7.1) if it is connected (in particular this implies $H^0(E', N) = 0$ for every field extension E' of E). Examples of such group schemes are μ_p (defined by the affine equation $y^p = 1$) and α_p (defined by the equation $y^p = 0$).

Let S be an $\mathbf{F}_p$-scheme. A finite S-group scheme N is of *height 1* if the relative Frobenius map $F_{N/S}$ (cf. [Mi2], §III.0) is trivial.

For any topological abelian group A, let $A^* := \mathrm{Hom}_{\mathrm{cont.}}(A, \mathbf{Q}/\mathbf{Z})$ be the group of continuous homomorphisms from A to $\mathbf{Q}/\mathbf{Z}$ (where $\mathbf{Q}/\mathbf{Z}$ is considered as a discrete group) equipped with the compact–open topology. A morphism $f : A \to B$ of topological groups is *strict* if it is continuous, and the restriction $f : A \to f(A)$ is an open map (where the topology on $f(A)$ is induced by B). This is equivalent to saying that f induces an isomorphism of the topological quotient $A/\ker f$ with the topological subspace $f(A) \subset B$.

Concerning sign conventions in homological algebra, we tried to follow the conventions in [SP] throughout the text.

2 Fppf cohomology with compact support

Define $Z := X \setminus U$ and $Z' := \coprod_{v\in Z} \mathrm{Spec}\,(\widehat{K_v})$ (disjoint union). Then we have a natural morphism $i : Z' \to U$. Let $\mathcal{F}$ be a sheaf of abelian groups on $(\mathrm{Sch}/U)_{\mathrm{fppf}}$. Let $I^\bullet(\mathcal{F})$ be an injective resolution of $\mathcal{F}$ over U. Denote by $\mathcal{F}_v$ and $I^\bullet(\mathcal{F})_v$ their respective pullbacks to $\mathrm{Spec}\, K_v$, for $v \notin U$.

Given a morphism of schemes $f : T \to S$, the fppf pullback functor f^* is exact (see [SP, Tag 021W, Tag 00XL]) and it admits an exact left adjoint $f_!$ (see [SP, Tag 04CC]), hence f^* maps injective (resp. flasque) objects to injective (resp. flasque) objects. Therefore $I^\bullet(\mathcal{F})_v$ is an injective resolution of $\mathcal{F}_v$.

As noticed by A. Schmidt, the definition of the modified fppf cohomology groups in the number field case in [Mi2], III.0.6 (a), has to be written more precisely, because of the non canonicity of the mapping cone in the derived category. We are grateful to him for the following alternative definition.

Let $\Omega_{\mathbf{R}}$ denote the set of real places of K. For $v \in \Omega_{\mathbf{R}}$, let $a^v : (\mathrm{Sch}/\mathrm{Spec}\,(K_v))_{\mathrm{fppf}} \to \mathrm{Spec}\,(K_v)_{\text{ét}}$ be the natural morphism of sites, where $S_{\text{ét}}$ denotes the small étale site on a scheme S. Since K_v is a perfect field, the direct image functor a^v_* associated to

a^v is exact. Hence, by [SGA4], V, Remark 4.6 and Prop. 4.9, the functor a^v_* maps $I^\bullet(\mathcal{F})_v$ to a flasque resolution $a^v_* I^\bullet(\mathcal{F})_v$ of $a^v_*\mathcal{F}_v$. Following [GS] §2, there is a natural acyclic resolution $D^\bullet(a^v_*\mathcal{F}_v) \to a^v_*\mathcal{F}_v$ of the $\mathrm{Gal}(\overline{K_v}/K_v) = \mathbf{Z}/2\mathbf{Z}$-module $a^v_*\mathcal{F}_v$ (identified with $\mathcal{F}_v(\mathrm{Spec}\,(\overline{K_v})))$. Splicing the resolutions $D^\bullet(a^v_*\mathcal{F}_v)$ and $a^v_* I^\bullet(\mathcal{F})_v$ together, one gets a complete acyclic resolution $\widehat{I}^\bullet(\mathcal{F}_v)$ of the $\mathrm{Gal}(\overline{K_v}/K_v)$-module $a^v_*\mathcal{F}_v$, which computes the Tate cohomology of $a^v_*\mathcal{F}_v$. And by construction, there is a natural morphism $\widehat{i}_v : a^v_* I^\bullet(\mathcal{F})_v \to \widehat{I}^\bullet(\mathcal{F}_v)$.

As suggested by [Mi2], section III.0, define $\Gamma_c(U, I^\bullet(\mathcal{F}))$ to be the following object in the category of complexes of abelian groups:

$$\Gamma_c(U, I^\bullet(\mathcal{F})) := \mathrm{Cone}\left(\Gamma(U, I^\bullet(\mathcal{F})) \to \Gamma(Z', i^* I^\bullet(\mathcal{F})) \oplus \bigoplus_{v \in \Omega_{\mathbf{R}}} \Gamma(K_v, \widehat{I}^\bullet(\mathcal{F}_v))\right)[-1],$$

and $H^r_c(U, \mathcal{F}) := H^r(\Gamma_c(U, I^\bullet(\mathcal{F})))$. We will also denote by $R\Gamma_c(U, \mathcal{F})$ the complex $\Gamma_c(U, I^\bullet(\mathcal{F}))$ viewed in the derived category of fppf sheaves. Observe that in the number field case the groups $H^r_c(U, \mathcal{F})$ may be non zero even for negative r. In the function field case we have $H^r_c(U, \mathcal{F}) = 0$ for $r < 0$, and also (by Proposition 2.1 below) $H^0_c(U, \mathcal{F}) = 0$ if we assume further $U \neq X$ (the map $H^0(U, \mathcal{F}) \to H^0(\widehat{K}_v, \mathcal{F})$ being injective for each $v \notin U$).

From now on, we will abbreviate Cone(...) by $C(...)$.

Proposition 2.1

1. *Let $\mathcal{F}$ be a sheaf of abelian groups on U_{fppf}. There is a natural exact sequence, for all $r \geq 0$,*

$$\cdots \to H^r_c(U, \mathcal{F}) \to H^r(U, \mathcal{F}) \to \bigoplus_{v \notin U} H^r(\widehat{K}_v, \mathcal{F}) \to H^{r+1}_c(U, \mathcal{F}) \to \ldots .$$

2. *For any short exact sequence*

$$0 \to \mathcal{F}' \to \mathcal{F} \to \mathcal{F}'' \to 0$$

of sheaves of abelian groups on U, there is a long exact sequence

$$\cdots \to H^r_c(U, \mathcal{F}') \to H^r_c(U, \mathcal{F}) \to H^r_c(U, \mathcal{F}'') \to H^{r+1}_c(U, \mathcal{F}') \to \ldots .$$

3. *For any flat affine commutative group scheme $\mathcal{F}$ of finite type over U, and any non empty open subscheme $V \subset U$, there is a canonical exact sequence*

$$\cdots \to H^r_c(V, \mathcal{F}) \to H^r_c(U, \mathcal{F}) \to \bigoplus_{v \in U \setminus V} H^r(\widehat{\mathcal{O}}_v, \mathcal{F}) \to H^{r+1}_c(V, \mathcal{F}) \to \ldots ,$$

and the following natural diagram commutes:

$$
\begin{array}{ccccccc}
 & & \bigoplus_{v \notin V} H^{r-1}(\widehat{K_v}, \mathcal{F}) & \xleftarrow{i_2} & \bigoplus_{v \notin U} H^{r-1}(\widehat{K_v}, \mathcal{F}) & & \\
 & \nearrow^{i_1} & \downarrow & & \downarrow & & \\
\bigoplus_{v \in U \setminus V} H^{r-1}(\widehat{\mathcal{O}_v}, \mathcal{F}) & \longrightarrow & H^r_c(V, \mathcal{F}) & \longrightarrow & H^r_c(U, \mathcal{F}) & \longrightarrow & \bigoplus_{v \in U \setminus V} H^r(\widehat{\mathcal{O}_v}, \mathcal{F}) \\
 & & \downarrow & & \downarrow & \nearrow & \\
 & & H^r(V, \mathcal{F}) & \xleftarrow{\text{Res}} & H^r(U, \mathcal{F}) & & \\
 & & \downarrow & & \downarrow & & \\
 & & \bigoplus_{v \notin V} H^r(\widehat{K_v}, \mathcal{F}) & \xrightarrow{\pi} & \bigoplus_{v \notin U} H^r(\widehat{K_v}, \mathcal{F}), & &
\end{array}
$$

where i_1 (resp. i_2) is obtained by puting 0 at the places $v \notin U$ (resp. $v \in U \setminus V$) and π is the natural projection.

4. *If $\mathcal{F}$ is represented by a smooth group scheme, then $H^r_c(U, \mathcal{F}) \cong H^r_{\text{ét},c}(U, \mathcal{F})$ for $r \neq 1$, where $H^*_{\text{ét},c}$ stands for modified étale cohomology with compact support as defined in [GS], §2. In particular for such $\mathcal{F}$ we have $H^r_c(U, \mathcal{F}) \cong H^r_{\text{ét}}(X, j_!\mathcal{F})$ in the function field case. If in addition the generic fiber $\mathcal{F}_K$ is a finite K-group scheme, then $H^1_c(U, \mathcal{F}) \cong H^1_{\text{ét},c}(U, \mathcal{F})$ (which is identified with $H^1_{\text{ét}}(X, j_!\mathcal{F})$ in the function field case).*

Remark 2.2 Unlike what happens in étale cohomology, the groups $H^1(\mathcal{O}_v, \mathcal{F})$ and $H^1(\widehat{\mathcal{O}_v}, \mathcal{F})$ cannot in general be identified with the group $H^1(k(v), F(v))$, where $k(v)$ is the residue field at v and $F(v)$ the fiber of $\mathcal{F}$ over $k(v)$. For example this already fails for $\mathcal{F} = \mu_p$ and $\widehat{\mathcal{O}_v} = \mathbf{F}_p[[t]]$, because by the Kummer exact sequence

$$0 \to \mu_p \to \mathbf{G}_m \xrightarrow{\cdot p} \mathbf{G}_m \to 0$$

in fppf cohomology, the group $H^1(\widehat{\mathcal{O}_v}, \mathcal{F}) = \widehat{\mathcal{O}_v}^* / \widehat{\mathcal{O}_v}^{*p}$ is an infinite dimensional $\mathbf{F}_p$-vector space, while $H^1(k(v), F(v)) = k(v)^*/k(v)^{*p} = 0$. The situation is better for $r \geq 2$ by [Toe], Cor. 3.4: namely the natural maps from $H^r(\mathcal{O}_v, \mathcal{F})$ and $H^r(\widehat{\mathcal{O}_v}, \mathcal{F})$ to $H^r(k(v), F(v))$ are isomorphisms.

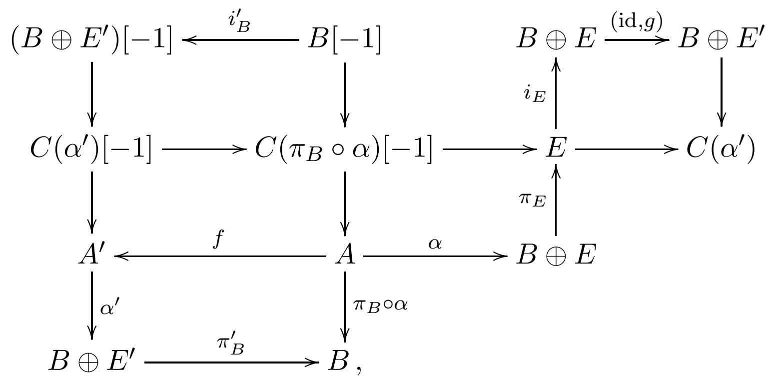
Before proving Proposition 2.1, we need the following lemmas. We start with a lemma in homological algebra:

Lemma 2.3 *Let $\mathcal{A}$ be an abelian category with enough injectives and let $\mathbf{C}(\mathcal{A})$ (resp. $\mathbf{D}(\mathcal{A})$) denote the category (resp. the derived category) of bounded below cochain complexes in $\mathcal{A}$. Consider a commutative diagram in $\mathbf{C}(\mathcal{A})$:*

$$
\begin{array}{ccc}
A & \xrightarrow{\alpha} & B \oplus E \\
{\scriptstyle f}\downarrow & & \downarrow{\scriptstyle (\text{id},g)} \\
A' & \xrightarrow{\alpha'} & B \oplus E',
\end{array}
$$

and denote by π_B (resp. π'_B) the projection $B \oplus E \to B$ (resp. $B \oplus E' \to B$).

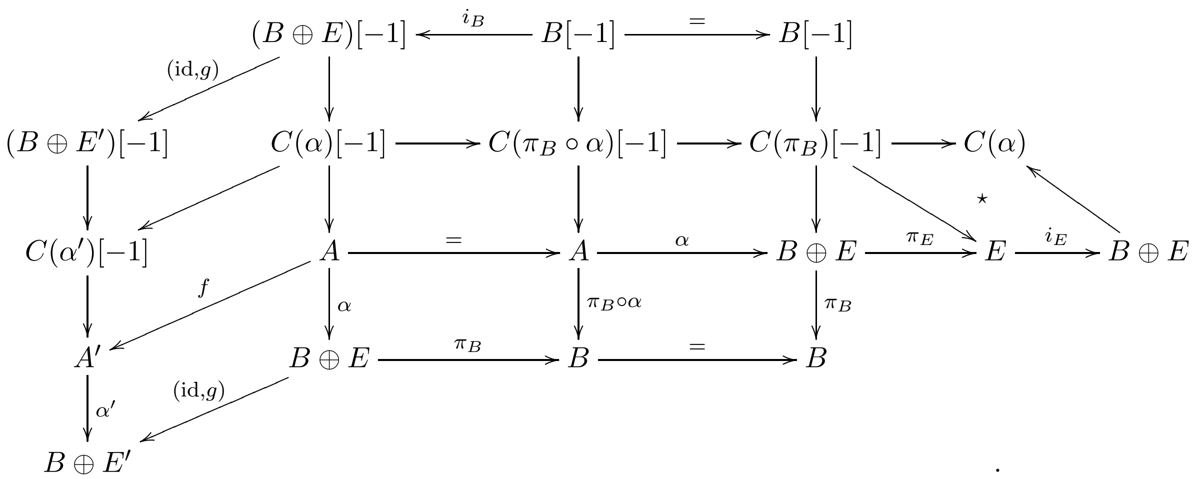
Assume that the natural morphism $C(f) \to C(g)$ in $\mathbf{C}(\mathcal{A})$ is a quasi-isomorphism. Then there exists a canonical commutative diagram in $\mathbf{D}(\mathcal{A})$:



where the second row and the first two columns are exact triangles.

Proof: The assumption that $C(f) \to C(g) \cong C(\mathrm{Id} \oplus g)$ is a quasi-isomorphism implies that $C(\alpha) \to C(\alpha')$ is a quasi-isomorphism (see for instance Proposition 1.1.11 in [BBD] or Corollary A.14 in [PS]).

Functoriality of the mapping cone in the category $\mathbf{C}(\mathcal{A})$ gives the following diagram in $\mathbf{C}(\mathcal{A})$, where the second row (by [Mi2], Prop. II.0.10, or [KS], proof of Theorem 11.2.6) and the columns are exact triangles in the derived category:



As usual, notation as π_B, π_E denotes projections and i_B, i_E are given by puting 0 at the missing piece. Note also that due to our sign conventions, the horizontal map $C(\pi_B)[-1] \to C(\alpha)$ is given by the natural map with a (-1)-sign.

This diagram is commutative in $\mathbf{C}(\mathcal{A})$, except the square $\star$ which is commutative up to homotopy. Indeed, this square defines two maps $f, g : C(\pi_B)[-1] \to C(\alpha)$, which are given in degree n by two maps $f^n, g^n : B^{n-1} \oplus (B^n \oplus E^n) \to (B^n \oplus E^n) \oplus A^{n+1}$, where $f^n(b', b, e) := -(b, e, 0)$ and $g^n(b', b, e) := -(0, e, 0)$. Consider now the maps $s^n : B^{n-1} \oplus (B^n \oplus E^n) \to (B^{n-1} \oplus E^{n-1}) \oplus A^n$ defined by $s^n(b', b, e) := (b', 0, 0)$. Then the collection (s^n) is a homotopy between f and g. Hence the square $\star$ is commutative up to the homotopy (s^n).

Since the map $C(\alpha) \to C(\alpha')$ is a quasi-isomorphism, and since the natural map $C(\pi_B)[-1] \to E$ is a homotopy equivalence, the lemma follows from the commutativity and the exactness of the previous diagram. $\square$

We now need the following result, for which we did not find a suitable reference:

Lemma 2.4 *Let A be a henselian valuation ring with fraction field K. Let $\widehat{A}$ be the completion of A for the valuation topology and $\widehat{K} := \operatorname{Frac} \widehat{A}$. Assume that $\widehat{K}$ is separable over K.*

1. *Let G be a K-group scheme locally of finite type. Then the map $H^1(K, G) \to H^1(\widehat{K}, G)$ has dense image.*
2. *Assume that $\widehat{A}$ is henselian. Let G be a flat A-group scheme locally of finite presentation. Then the map $H^1(A, G) \to H^1(\widehat{A}, G)$ has dense image.*

Here the topology on the pointed sets $H^1(\widehat{A}, G)$ and $H^1(\widehat{K}, G)$ is provided by [Česl], § 3.

Remark 2.5

- The assumption that $\widehat{K}$ is separable over K is satisfied if A is an excellent discrete valuation ring.
- In the second statement, the assumption that $\widehat{A}$ is henselian is satisfied if the valuation on A has height 1 (special case of [Rib], section F, Th. 4). This assumption is used in the proof below to apply [Česl], Theorem B.5. Note also that in general, $\widehat{A}$ is not the same as the completion of A for the $\mathfrak{m}$-adic topology (where $\mathfrak{m}$ denotes the maximal ideal of A).

Proof of Lemma 2.4: We prove both statements at the same time. Let E be either A or K, set $S = \operatorname{Spec} E$. Let $\mathbf{B}G$ denote the classifying E-stack of G-torsors. We need to prove that $\mathbf{B}G(E)$ is dense in $\mathbf{B}G(\widehat{E})$. It is a classical fact that $\mathbf{B}G$ is an algebraic stack ([SP, Tag 0CQJ] and [SP, Tag 06PL]). Let $x \in \mathbf{B}G(\widehat{E})$ and $U \subset \mathbf{B}G(\widehat{E})$ be an open subcategory (in the sense of [Česl], 2.4) containing x. We need to find an object $x' \in \mathbf{B}G(E)$ that maps to $U \subset \mathbf{B}G(\widehat{E})$. Using [Česl], Theorem B.5 and Remark B.6 (applied to the S-scheme $\operatorname{Spec} R := \operatorname{Spec} \widehat{E}$), there exists an affine scheme Y, a smooth S-morphism $\pi : Y \to \mathbf{B}G$ and $y \in Y(\widehat{E})$ such that $\pi_{\widehat{E}}(y) = x$, where $\pi_{\widehat{E}} : Y(\widehat{E}) \to BG(\widehat{E})$ is the map induced by π. In particular, $Y \to S$ is smooth because so are π and $\mathbf{B}G \to S$ (the latter by [Česl], Prop. A.3). Hence Y is locally of finite presentation over S. By assumption, $\pi_{\widehat{E}}^{-1}(U) \subset Y(\widehat{E})$ is an open subset containing y. Hence [MB], Corollary 1.2.1 (in the discrete valuation ring case, it is Greenberg's approximation Theorem) implies that $Y(E) \cap \pi_{\widehat{E}}^{-1}(U) \neq \emptyset$. Applying π_E, we get that the image of $\mathbf{B}G(E)$ meets U, which proves the required result. $\square$

The previous lemma is useful to prove the following crucial (in the function field case) statement. For a local integral domain A with maximal ideal $\mathfrak{m}$, fraction field K and residue field κ, and $\mathcal{F}$ an fppf sheaf on $\operatorname{Spec} A$ with an injective resolution $I^\bullet(\mathcal{F})$, define

$$\Gamma_{\mathfrak{m}}(A, I^\bullet(\mathcal{F})) := \operatorname{Cone}\left(\Gamma(\operatorname{Spec} A, I^\bullet(\mathcal{F})) \to \Gamma(\operatorname{Spec} K, I^\bullet(\mathcal{F}))\right)[-1]$$

and $H^r_{\mathfrak{m}}(A,\mathcal{F}) := H^r(\Gamma_{\mathfrak{m}}(A, I^\bullet(\mathcal{F})))$ (the cohomology with support in $\operatorname{Spec}\kappa$). We have a localization long exact sequence ([Mi2], Prop. III.0.3)

$$... \to H^r_{\mathfrak{m}}(A,\mathcal{F}) \to H^r(A,\mathcal{F}) \to H^r(K,\mathcal{F}) \to H^{r+1}_{\mathfrak{m}}(A,\mathcal{F}) \to ...$$

Lemma 2.6 *Let A be an excellent henselian discrete valuation ring, with maximal ideal $\mathfrak{m}$. Let $\mathcal{F}$ be a flat affine commutative group scheme of finite type over* $\operatorname{Spec} A$. *Then for all $r \geq 0$, the morphism $H^r_{\mathfrak{m}}(A,\mathcal{F}) \to H^r_{\mathfrak{m}}(\widehat{A},\mathcal{F})$ is an isomorphism.*

Remark 2.7 Let $I^\bullet(\mathcal{F})$ be an injective resolution of $\mathcal{F}$ viewed as an fppf sheaf. Another formulation of Lemma 2.6 is that the natural morphism $\Gamma_{\mathfrak{m}}(A, I^\bullet(\mathcal{F})) \to \Gamma_{\mathfrak{m}}(\widehat{A}, I^\bullet(\mathcal{F}))$ is an isomorphism in the derived category. The injective resolution $I^\bullet(\mathcal{F})$ can be replaced by any complex of flasque fppf sheaves that is quasi-isomorphic to $\mathcal{F}$ (indeed the fppf pullback functor f^* associated to $f : \operatorname{Spec}\widehat{A} \to \operatorname{Spec} A$ sends flasque resolutions to flasque resolutions, because f^* is exact and preserves flasque sheaves).

Also note that Lemma 2.6 is a variant of [Suz], Prop. 2.6.2: our result is slightly more general in the affine case, while the notion of cohomological approximation in [Suz] is a priori a little stronger than the conclusion of Lemma 2.6. In addition, this lemma answers a variant of a question raised after Prop 2.6.2. of loc. cit. (under a flatness assumption).

Proof of Lemma 2.6:

- $r = 0$:

  Since $\mathcal{F}$ is separated (as an affine scheme), the morphisms $H^0(A,\mathcal{F}) \to H^0(K,\mathcal{F})$ and $H^0(\widehat{A},\mathcal{F}) \to H^0(\widehat{K},\mathcal{F})$ are injective, which implies that

  $$H^0_{\mathfrak{m}}(A,\mathcal{F}) = H^0_{\mathfrak{m}}(\widehat{A},\mathcal{F}) = 0\,.$$

- $r = 1$:

  Consider the following commutative diagram with exact rows:

  $$\begin{array}{ccccccccc} H^0(A,\mathcal{F}) & \longrightarrow & H^0(K,\mathcal{F}) & \longrightarrow & H^1_{\mathfrak{m}}(A,\mathcal{F}) & \longrightarrow & H^1(A,\mathcal{F}) & \longrightarrow & H^1(K,\mathcal{F}) \\ \downarrow & & \downarrow & & \downarrow & & \downarrow & & \downarrow \\ H^0(\widehat{A},\mathcal{F}) & \longrightarrow & H^0(\widehat{K},\mathcal{F}) & \longrightarrow & H^1_{\mathfrak{m}}(\widehat{A},\mathcal{F}) & \longrightarrow & H^1(\widehat{A},\mathcal{F}) & \longrightarrow & H^1(\widehat{K},\mathcal{F})\,. \end{array} \qquad (1)$$

  Since A is excellent, Artin approximation (see [Art], Theorem 1.12) implies that the morphism $H^1(A,\mathcal{F}) \to H^1(\widehat{A},\mathcal{F})$ is injective: indeed, given a $(\operatorname{Spec} A)$-torsor $\mathcal{P}$ under $\mathcal{F}$, $\mathcal{P}$ is locally of finite presentation, and Artin approximation ensures that $\mathcal{P}(\widehat{A}) \neq \emptyset$ implies that $\mathcal{P}(A) \neq \emptyset$.

  The affine A-scheme of finite type $\mathcal{F}$ is of the form $\operatorname{Spec}(A[x_1,...,x_n]/(f_1,...,f_r))$, where $f_1,...,f_r$ are polynomials. Since the discrete valuation ring A satisfies $A = K \cap \widehat{A} \subset \widehat{K}$, the square on the left hand side in (1) is cartesian.

  Hence an easy diagram chase implies that $H^1_{\mathfrak{m}}(A,\mathcal{F}) \to H^1_{\mathfrak{m}}(\widehat{A},\mathcal{F})$ is injective.

  By Proposition A.6 in [GP], the right hand side square in (1) is cartesian. In addition, $H^0(\widehat{A},\mathcal{F}) \subset H^0(\widehat{K},\mathcal{F})$ is open ([GGM], Prop. 3.3.4), and $H^0(K,\mathcal{F}) \subset H^0(\widehat{K},\mathcal{F})$ is dense by [GGM], Proposition 3.5.2 (weak approximation for $\mathcal{F}$).

Therefore, an easy diagram chase implies that the map $H^1_{\mathfrak{m}}(A, \mathcal{F}) \to H^1_{\mathfrak{m}}(\widehat{A}, \mathcal{F})$ is surjective.

- $r = 2$:

  Consider the commutative diagram with exact rows:

$$\begin{array}{ccccccccc} H^1(A,\mathcal{F}) & \longrightarrow & H^1(K,\mathcal{F}) & \longrightarrow & H^2_{\mathfrak{m}}(A,\mathcal{F}) & \longrightarrow & H^2(A,\mathcal{F}) & \longrightarrow & H^2(K,\mathcal{F}) \\ \downarrow & & \downarrow & & \downarrow & & \downarrow & & \downarrow \\ H^1(\widehat{A},\mathcal{F}) & \longrightarrow & H^1(\widehat{K},\mathcal{F}) & \longrightarrow & H^2_{\mathfrak{m}}(\widehat{A},\mathcal{F}) & \longrightarrow & H^2(\widehat{A},\mathcal{F}) & \longrightarrow & H^2(\widehat{K},\mathcal{F}) . \end{array} \qquad (2)$$

  By [Toe], Corollary 3.4, the map $H^2(A, \mathcal{F}) \to H^2(\widehat{A}, \mathcal{F})$ is an isomorphism. And we already explained (in the case $r = 1$) that the left hand side square in (2) is cartesian. Hence a diagram chase proves that the map $H^2_{\mathfrak{m}}(A, \mathcal{F}) \to H^2_{\mathfrak{m}}(\widehat{A}, \mathcal{F})$ is injective.

  Using [GGM], Proposition 3.5.3.(3), the map $H^2(K, \mathcal{F}) \to H^2(\widehat{K}, \mathcal{F})$ is also an isomorphism. By [Česl], Proposition 2.9 (e), the map $H^1(\widehat{A}, \mathcal{F}) \to H^1(\widehat{K}, \mathcal{F})$ is open. Lemma 2.4 implies that the map $H^1(K, \mathcal{F}) \to H^1(\widehat{K}, \mathcal{F})$ has dense image. By diagram chase, we get that the map $H^2_{\mathfrak{m}}(A, \mathcal{F}) \to H^2_{\mathfrak{m}}(\widehat{A}, \mathcal{F})$ is surjective.

- $r \geq 3$:

  Corollary 3.4 in [Toe] implies that the morphisms $H^{r-1}(A, \mathcal{F}) \to H^{r-1}(\widehat{A}, \mathcal{F})$ and $H^r(A, \mathcal{F}) \to H^r(\widehat{A}, \mathcal{F})$ are isomorphisms. Proposition 3.5.3.(3) in [GGM] implies that the maps $H^{r-1}(K, \mathcal{F}) \to H^{r-1}(\widehat{K}, \mathcal{F})$ and $H^r(K, \mathcal{F}) \to H^r(\widehat{K}, \mathcal{F})$ are isomorphisms. Therefore, the five-lemma proves that $H^r_{\mathfrak{m}}(A, \mathcal{F}) \to H^r_{\mathfrak{m}}(\widehat{A}, \mathcal{F})$ is an isomorphism. □

Remark 2.8 We will apply the previous lemma to a finite and flat commutative group scheme N. As was pointed out to us by K. Česnavičius, it is then possible to argue without using Corollary 3.4 in [Toe] (whose proof is quite involved): indeed there exists (cf. [Mi2], Th. III.A.5) an exact sequence

$$0 \to N \to G_1 \to G_2 \to 0$$

of affine A-group schemes such that G_1 and G_2 are smooth. Now for $i > 0$ we have $H^i(A, G_j) \cong H^i(\widehat{A}, G_j)$ $(j = 1, 2)$ by [Mi1], Rem. III.3.11 because A and $\widehat{A}$ are henselian, and fppf cohomology coincides with étale cohomology for smooth group schemes. It remains to apply the five-lemma to get $H^i(A, N) \cong H^i(\widehat{A}, N)$ for $i \geq 2$, which is the input from [Toe] that we used in the proof.

The following lemma is a version of the excision property for fppf cohomology with respect to étale morphisms:

Lemma 2.9 *Let X, X' be schemes, $Z \hookrightarrow X$ (resp. $Z' \hookrightarrow X'$) be closed subschemes, $\pi : X' \to X$ be an étale morphism. Assume that π restricted to Z' is an isomorphism from Z' to Z and that $\pi(X' \setminus Z') \subset X \setminus Z$. Let $\mathcal{F}$ be a sheaf on $(\mathrm{Sch}/X)_{\mathrm{fppf}}$. Then for all $r \geq 0$, the natural morphism $H^r_Z(X, \mathcal{F}) \to H^r_{Z'}(X', \pi^*\mathcal{F})$ is an isomorphism.*

Proof: Since π^* is exact and maps injective objects to injective objects, the proof is exactly the same as the proof of [Mi1], Proposition III.1.27. □

We continue with a lemma comparing the definition of modified étale cohomology with compact support in [GS] and our definition of modified fppf cohomology with compact support. For any scheme T, consider the morphisms of sites

$$(\mathrm{S}ch/T)_{\mathrm{fppf}} \xrightarrow{\varepsilon_T} (\mathrm{S}ch/T)_{\text{ét}} \xrightarrow{\pi_T} T_{\text{ét}}, \qquad a_T = \pi_T \circ \varepsilon_T : (\mathrm{S}ch/T)_{\mathrm{fppf}} \to T_{\text{ét}}$$

where $(\mathrm{S}ch/T)_{\text{ét}}$ denotes the big étale site of T. Recall that $Z := X \setminus U$, $Z' := \coprod_{v \in Z} \operatorname{Spec}(\widehat{K}_v)$, $j : U \to X$ is the open immersion and $i : Z' \to U$ is the natural morphism. Set $a := a_U$ and $\varepsilon := \varepsilon_U$.

Let $\mathcal{F}$ be a sheaf on $U_{\text{ét}}$, and let $\pi_X^* j_! \mathcal{F} \to J^\bullet(\mathcal{F})$ be an injective resolution in the big étale topos of X. By [SP, Tag 0758] and [SP, Tag 04BT], the restriction $J^\bullet(\mathcal{F})_{\text{ét}}$ of $J^\bullet(\mathcal{F})$ to the small étale site of X is an injective resolution of $j_!\mathcal{F}$. For every place $v \notin U$ of K, let $\mathcal{F}_v$ be the pull-back of $\mathcal{F}$ to $(\operatorname{Spec} K_v)_{\text{ét}}$. As in the fppf case (explained in the beginning of section 2), we have for v real a complete resolution $\widehat{J}^\bullet(\mathcal{F}_v)$ of the $\mathrm{Gal}(\overline{K}_v/K_v)$-module $\mathcal{F}_v$, which computes its Tate cohomology. Following [GS], section 2, we define

$$\Gamma_{\text{ét},c}(U, J^\bullet(\mathcal{F})) := \operatorname{Cone}\left(\Gamma(X, J^\bullet(\mathcal{F})_{\text{ét}}) \to \bigoplus_{v \in \Omega_{\mathbf{R}}} \Gamma(K_v, \widehat{J}^\bullet(\mathcal{F}_v)) \right) [-1],$$

and $H^r_{\text{ét},c}(U, \mathcal{F}) := H^r(\Gamma_{\text{ét},c}(U, J^\bullet(\mathcal{F})))$.

Denote by $R\Gamma_{\text{ét},c}(U, \mathcal{F})$ the complex $\Gamma_{\text{ét},c}(U, J^\bullet(\mathcal{F}))$ (viewed in the derived category of abelian groups); similarly for v real, set $\widehat{R\Gamma_{\text{ét}}}(K_v, \mathcal{F})$ (resp. $\widehat{R\Gamma}(K_v, a^*\mathcal{F})$) for the complex $\Gamma(K_v, \widehat{J}^\bullet(\mathcal{F}_v))$ (resp. $\Gamma(K_v, \widehat{I}^\bullet((a^*\mathcal{F})_v))$, where $I^\bullet(a^*\mathcal{F})$ is a flasque resolution of $a^*\mathcal{F}$, cf. beginning of section 2) in the derived category of étale sheaves (resp. fppf sheaves) over $\operatorname{Spec} K_v$. Finally, let $R\Gamma_{\text{ét},Z}(X, j_!\mathcal{F})$ denote the complex

$$\Gamma_{\text{ét},Z}(X, J^\bullet(\mathcal{F})) := \operatorname{Cone}(\Gamma(X, J^\bullet(\mathcal{F})_{\text{ét}}) \to \Gamma(U, J^\bullet(\mathcal{F})_{\text{ét}}))[-1].$$

Lemma 2.10

1. *Let $\mathcal{F}$ be a sheaf of abelian groups over $U_{\text{ét}}$. Then there is a canonical commutative diagram in the derived category of abelian groups, where the rows are exact triangles:*

$$\begin{array}{ccccccc}
R\Gamma_{\text{ét},c}(U,\mathcal{F}) & \longrightarrow & R\Gamma_{\text{ét}}(U,\mathcal{F}) & \longrightarrow & R\Gamma_{\text{ét},Z}(X, j_!\mathcal{F})[1] \oplus \bigoplus_{v\in\Omega_{\mathbf{R}}} \widehat{R\Gamma_{\text{ét}}}(K_v,\mathcal{F}) & \longrightarrow & R\Gamma_{\text{ét},c}(U,\mathcal{F})[1] \\
\downarrow & & \downarrow \wr & & \downarrow & & \downarrow \\
R\Gamma_c(U, a^*\mathcal{F}) & \longrightarrow & R\Gamma(U, a^*\mathcal{F}) & \longrightarrow & R\Gamma(Z', i^*a^*\mathcal{F}) \oplus \bigoplus_{v\in\Omega_{\mathbf{R}}} \widehat{R\Gamma}(K_v, a^*\mathcal{F}) & \longrightarrow & R\Gamma_c(U, a^*\mathcal{F})[1].
\end{array}$$

Besides, the complex $R\Gamma_{\text{ét},Z}(X, j_!\mathcal{F})[1]$ is quasi-isomorphic to $\bigoplus_{v\in Z} R\Gamma_{\text{ét}}(K_v, \mathcal{F})$.

2. *Let G be a smooth commutative group scheme over U. Let $\underline{G}$ denote the fppf sheaf associated to G and $G_{\text{ét}} := a_* \underline{G}$. Then there is a canonical commutative diagram in the derived category of abelian groups, where the rows are exact triangles:*

$$\begin{array}{ccccccc}
R\Gamma_{\text{ét},c}(U,G_{\text{ét}}) & \longrightarrow & R\Gamma_{\text{ét}}(U,G_{\text{ét}}) & \longrightarrow & R\Gamma_{\text{ét},Z}(X,j_!G_{\text{ét}})[1]\oplus\bigoplus_{v\in\Omega_{\mathbf{R}}}\widehat{R\Gamma_{\text{ét}}}(K_v,G_{\text{ét}}) & \longrightarrow & R\Gamma_{\text{ét},c}(U,G_{\text{ét}})[1] \\
\downarrow & & \downarrow\sim & & \downarrow & & \downarrow \\
R\Gamma_c(U,\underline{G}) & \longrightarrow & R\Gamma(U,\underline{G}) & \longrightarrow & R\Gamma(Z',i^*\underline{G})\oplus\bigoplus_{v\in\Omega_{\mathbf{R}}}\widehat{R\Gamma}(K_v,\underline{G}) & \longrightarrow & R\Gamma_c(U,\underline{G})[1]\,.
\end{array}$$

Besides, the complex $R\Gamma_{\text{ét},Z}(X,j_!G_{\text{ét}})[1]$ is quasi-isomorphic to $\bigoplus_{v\in Z} R\Gamma_{\text{ét}}(K_v,G_{\text{ét}})$.

Proof:

1. Set $J := J^\bullet(\mathcal{F})$. Since $j_!\mathcal{F} \to J_{\text{ét}} := J^\bullet(\mathcal{F})_{\text{ét}}$ is an injective resolution, we get an injective resolution $\mathcal{F} = j^*j_!\mathcal{F} \to j^*J_{\text{ét}}$ in $U_{\text{ét}}$. The functor ε^* is an exact functor that maps flasque étale sheaves to flasque fppf sheaves (see [SP, Tag 0DDU]), we get a flasque resolution $a^*\mathcal{F} \to I := \varepsilon^* j^* J_{\text{ét}}$. Let $\widehat{J}_v := \widehat{J^\bullet}(\mathcal{F}_v)$; define $\widehat{I}_v = \widehat{I^\bullet}((\varepsilon^*\mathcal{F})_v)$ (associated to the flasque resolution I of $\varepsilon^*\mathcal{F}$) as in the beginning of section 2.

   Consider now the following commutative diagram of complexes, where $\widetilde{\Gamma}_{\text{ét},Z}(U,J)$ and $\widetilde{\Gamma}_{\text{ét},c}(U,J)$ are mapping cones defined such that the third and fourth rows are exact triangles:

$$\begin{array}{ccccccc}
\Gamma_{\text{ét},c}(U,J_{\text{ét}}) & \longrightarrow & \Gamma_{\text{ét}}(U,j^*J_{\text{ét}}) & \longrightarrow & \Gamma_{\text{ét},Z}(X,J_{\text{ét}})[1]\oplus\bigoplus_{v\in\Omega_{\mathbf{R}}}\Gamma_{\text{ét}}(K_v,\widehat{J_{\text{ét}}}_v) & \longrightarrow & \Gamma_{\text{ét},c}(U,J_{\text{ét}})[1] \\
\uparrow\varphi_c & & \uparrow\varphi & & \uparrow\varphi' & & \uparrow \\
\Gamma_{\text{ét},c}(U,J) & \longrightarrow & \Gamma_{\text{ét}}(U,j^*J) & \longrightarrow & \Gamma_{\text{ét},Z}(X,J)[1]\oplus\bigoplus_{v\in\Omega_{\mathbf{R}}}\Gamma_{\text{ét}}(K_v,\widehat{J}_v) & \longrightarrow & \Gamma_{\text{ét},c}(U,J)[1] \\
\downarrow d' & & \downarrow = & & \downarrow d & & \downarrow \\
\widetilde{\Gamma}_{\text{ét},Z}(U,J) & \longrightarrow & \Gamma_{\text{ét}}(U,j^*J) & \longrightarrow & \bigoplus_{v\in Z}\Gamma_{\text{ét},v}(\mathcal{O}_v,J)[1]\oplus\bigoplus_{v\in\Omega_{\mathbf{R}}}\Gamma_{\text{ét}}(K_v,\widehat{J}_v) & \longrightarrow & \widetilde{\Gamma}_{\text{ét},Z}(U,J)[1] \\
\uparrow b' & & \uparrow = & & \uparrow b & & \uparrow \\
\widetilde{\Gamma}_{\text{ét},c}(U,J) & \longrightarrow & \Gamma_{\text{ét}}(U,j^*J) & \longrightarrow & \bigoplus_{v\in Z}\Gamma_{\text{ét}}(K_v,J)\oplus\bigoplus_{v\in\Omega_{\mathbf{R}}}\Gamma_{\text{ét}}(K_v,\widehat{J}_v) & \longrightarrow & \widetilde{\Gamma}_{\text{ét},c}(U,J)[1] \\
\downarrow & & \downarrow c & & \downarrow & & \downarrow \\
\Gamma_c(U,I) & \longrightarrow & \Gamma(U,I) & \longrightarrow & \Gamma(Z',i^*I)\oplus\bigoplus_{v\in\Omega_{\mathbf{R}}}\Gamma(K_v,\widehat{I}_v) & \longrightarrow & \Gamma_c(U,I)[1]\,.
\end{array}$$

   In this diagram, the rows are exact triangles (by definition for the last three rows, using the proof of Lemma 2.7 in [GS] for the first ones). The maps φ, φ' and φ_c are quasi-isomorphisms by [SP, Tag 0DDH]. In addition, the maps d and b (hence also d' and b') are quasi-isomorphisms: for the map d, this is the excision property for étale cohomology (see [Mi1], Proposition III.1.27); for the map b, this is exactly [Mi2], Proposition II.1.1.(a). In addition, the map c is a quasi-isomorphism, using [SP, Tag 0DDU]. This proves the lemma.

2. Consider the following commutative diagram of exact triangles in the derived category

$$
\begin{array}{ccccccc}
R\Gamma_{\text{ét},c}(U,G_{\text{ét}}) & \longrightarrow & R\Gamma_{\text{ét}}(U,G_{\text{ét}}) & \longrightarrow & R\Gamma_{\text{ét},Z}(X,j_!G_{\text{ét}})[1] \oplus \bigoplus_{v\in\Omega_{\mathbf{R}}} \widehat{R\Gamma_{\text{ét}}}(K_v,G_{\text{ét}}) & \longrightarrow & R\Gamma_{\text{ét},c}(U,G_{\text{ét}})[1] \\
\downarrow & & \downarrow & & \downarrow & & \downarrow \\
R\Gamma_c(U,a^*G_{\text{ét}}) & \longrightarrow & R\Gamma(U,a^*G_{\text{ét}}) & \longrightarrow & R\Gamma(Z',i^*a^*G_{\text{ét}}) \oplus \bigoplus_{v\in\Omega_{\mathbf{R}}} \widehat{R\Gamma}(K_v,a^*G_{\text{ét}}) & \longrightarrow & R\Gamma_c(U,a^*G_{\text{ét}})[1] \\
\downarrow & & \downarrow & & \downarrow & & \downarrow \\
R\Gamma_c(U,\underline{G}) & \longrightarrow & R\Gamma(U,\underline{G}) & \longrightarrow & R\Gamma(Z',i^*\underline{G}) \oplus \bigoplus_{v\in\Omega_{\mathbf{R}}} \widehat{R\Gamma}(K_v,\underline{G}) & \longrightarrow & R\Gamma_c(U,\underline{G})[1]\,,
\end{array}
$$

where the vertical maps between the first two rows come from the first point of this Lemma, and the ones between the last two rows come from the adjunction morphism $a^*G_{\text{ét}} = a^*a_*\underline{G} \to \underline{G}$ and from the functoriality of the triangle defining the complexes $R\Gamma_c(U,\cdot)$. Now [Gro], Theorem 11.7, ensures that the composed vertical morphism $R\Gamma_{\text{ét}}(U,G_{\text{ét}}) \to R\Gamma(U,\underline{G})$ is an isomorphism. Whence the required result.

□

Proof of Proposition 2.1:

1. This is immediate from the definitions, cf. [Mi2], III, Proposition 0.4.a) and Remark 0.6. b).

2. The claim follows from the definitions, from the exactness of the functors i^*, a^v_* and $D^\bullet(\cdot)$ at the beginning of section 2, and from the exactness of the cone functor on the category of complexes of abelian groups (see also [Mi2], III, Proposition 0.4.b) and Remark 0.6. b)).

3. As in the proof of [Mi2], III, Proposition 0.4.c), let $I^\bullet(\mathcal{F})$ be an injective resolution of $\mathcal{F}$. In the number field case, *the piece of notation* $\Gamma(\widehat{K_v},I^\bullet(\mathcal{F}))$ *will stand for* $\Gamma(K_v,\widehat{I}^\bullet(\mathcal{F}_v))$ *when v is a real place of K*, where $\widehat{I}^\bullet(\mathcal{F}_v)$ is the modified resolution constructed in the beginning of section 2.

   Consider the following commutative diagram of complexes in the category of bounded below complexes of abelian groups:

$$
\begin{array}{ccccc}
\Gamma(U,I^\bullet(\mathcal{F})) & \xrightarrow{\alpha} & \bigoplus_{v\notin U}\Gamma(\widehat{K_v},I^\bullet(\mathcal{F})) \oplus \bigoplus_{v\in U\setminus V}\Gamma(\widehat{\mathcal{O}_v},I^\bullet(\mathcal{F})) & \xrightarrow{\pi_{\mathcal{O}}} & \bigoplus_{v\notin U}\Gamma(\widehat{K_v},I^\bullet(\mathcal{F})) \\
\downarrow f & & \downarrow (\mathrm{id},g) & \nearrow \pi_K & \\
\Gamma(V,I^\bullet(\mathcal{F})) & \xrightarrow{\alpha'} & \bigoplus_{v\notin U}\Gamma(\widehat{K_v},I^\bullet(\mathcal{F})) \oplus \bigoplus_{v\in U\setminus V}\Gamma(\widehat{K_v},I^\bullet(\mathcal{F}))\,, & &
\end{array}
$$

   where the maps are the natural ones.

   Functoriality of the mapping cone in the category of complexes gives morphisms

$$
\Gamma_{U\setminus V}(U,I^\bullet(\mathcal{F})) \to \bigoplus_{v\in U\setminus V}\Gamma_v(\mathcal{O}_v,I^\bullet(\mathcal{F})) \to \bigoplus_{v\in U\setminus V}\Gamma_v(\widehat{\mathcal{O}_v},I^\bullet(\mathcal{F}))\,,
$$

where

$$\Gamma_{U\setminus V}(U, I^\bullet(\mathcal{F})) := C(f)[-1], \quad \Gamma_v(\mathcal{O}_v, I^\bullet(\mathcal{F})) := \Gamma_{\mathfrak{m}_v}(\mathcal{O}_v, I^\bullet(\mathcal{F}))$$

and

$$\Gamma_v(\widehat{\mathcal{O}_v}, I^\bullet(\mathcal{F})) := \Gamma_{\mathfrak{m}_v}(\widehat{\mathcal{O}_v}, I^\bullet(\mathcal{F})).$$

The excision property (Lemma 2.9) implies that the first morphism $\Gamma_{U\setminus V}(U, I^\bullet(\mathcal{F})) \to \bigoplus_{v\in U\setminus V} \Gamma_v(\mathcal{O}_v, I^\bullet(\mathcal{F}))$ is a quasi-isomorphism.

Since for all $v \in X$, the ring $\mathcal{O}_v$ is an excellent henselian discrete valuation ring, Lemma 2.6 ensures that the second map

$$\bigoplus_{v\in U\setminus V} \Gamma_v(\mathcal{O}_v, I^\bullet(\mathcal{F})) \to \bigoplus_{v\in U\setminus V} \Gamma_v(\widehat{\mathcal{O}_v}, I^\bullet(\mathcal{F}))$$

is a quasi-isomorphism. Therefore, the natural morphism $C(f) \to C(g)$ is a quasi-isomorphism.

Apply now Lemma 2.3 to get a commutative diagram in the derived category of abelian groups:

$$\begin{array}{ccccccc}
\left(\bigoplus_{v\notin V} \Gamma(\widehat{K_v}, I^\bullet(\mathcal{F}))\right)[-1] & \xleftarrow{i_K} & \left(\bigoplus_{v\notin U} \Gamma(\widehat{K_v}, I^\bullet(\mathcal{F}))\right)[-1] & & \bigoplus_{v\notin U} \Gamma(\widehat{K_v}, I^\bullet(\mathcal{F})) \oplus \bigoplus_{v\in U\setminus V} \Gamma(\widehat{\mathcal{O}_v}, I^\bullet(\mathcal{F})) & \xrightarrow{(\mathrm{id},g)} & \bigoplus_{v\notin V} \Gamma(\widehat{K_v}, I^\bullet(\mathcal{F})) \\
\downarrow & & \downarrow & & \uparrow{\scriptstyle i'_{\mathcal{O}}} & & \downarrow \\
\Gamma_c(V, I^\bullet(\mathcal{F})) & \longrightarrow & \Gamma_c(U, I^\bullet(\mathcal{F})) & \longrightarrow & \bigoplus_{v\in U\setminus V} \Gamma(\widehat{\mathcal{O}_v}, I^\bullet(\mathcal{F})) & \longrightarrow & \Gamma_c(V, I^\bullet(\mathcal{F}))[1] \\
\downarrow & & \downarrow & & \uparrow{\scriptstyle \pi'_{\mathcal{O}}} & & \\
\Gamma(V, I^\bullet(\mathcal{F})) & \xleftarrow{f} & \Gamma(U, I^\bullet(\mathcal{F})) & \xrightarrow{\alpha} & \bigoplus_{v\notin U} \Gamma(\widehat{K_v}, I^\bullet(\mathcal{F})) \oplus \bigoplus_{v\in U\setminus V} \Gamma(\widehat{\mathcal{O}_v}, I^\bullet(\mathcal{F})) & & \\
\downarrow{\scriptstyle \alpha'} & & \downarrow{\scriptstyle \pi_{\mathcal{O}}\circ\alpha} & & & & \\
\bigoplus_{v\notin V} \Gamma(\widehat{K_v}, I^\bullet(\mathcal{F})) & \xrightarrow{\pi_K} & \bigoplus_{v\notin U} \Gamma(\widehat{K_v}, I^\bullet(\mathcal{F})), & & & &
\end{array} \tag{3}$$

where the second row and the first two columns are exact triangles.

Now the cohomology of this diagram gives the following canonical commutative diagram, with an exact second row (and the two first columns exact):

$$\begin{array}{ccccccccc}
 & & \bigoplus_{v\notin V} H^{r-1}(\widehat{K_v}, \mathcal{F}) & \longleftarrow & \bigoplus_{v\notin U} H^{r-1}(\widehat{K_v}, \mathcal{F}) & & \bigoplus_{v\notin U} H^r(\widehat{K_v}, \mathcal{F}) \oplus \bigoplus_{v\in U\setminus V} H^r(\widehat{\mathcal{O}_v}, \mathcal{F}) & \longrightarrow & \bigoplus_{v\notin V} H^r(\widehat{K_v}, \mathcal{F}) \\
 & & \downarrow & & \downarrow & & \uparrow & & \downarrow \\
\cdots & \longrightarrow & H^r_c(V, \mathcal{F}) & \longrightarrow & H^r_c(U, \mathcal{F}) & \longrightarrow & \bigoplus_{v\in U\setminus V} H^r(\widehat{\mathcal{O}_v}, \mathcal{F}) & \longrightarrow & H^{r+1}_c(V, \mathcal{F}) \longrightarrow \cdots \\
 & & \downarrow & & \downarrow & & \uparrow & & \\
 & & H^r(V, \mathcal{F}) & \xleftarrow{\mathrm{Res}} & H^r(U, \mathcal{F}) & \longrightarrow & \bigoplus_{v\notin U} H^r(\widehat{K_v}, \mathcal{F}) \oplus \bigoplus_{v\in U\setminus V} H^r(\widehat{\mathcal{O}_v}, \mathcal{F}) & & \\
 & & \downarrow & & \downarrow & & & & \\
 & & \bigoplus_{v\notin V} H^r(\widehat{K_v}, \mathcal{F}) & \longrightarrow & \bigoplus_{v\notin U} H^r(\widehat{K_v}, \mathcal{F}), & & & &
\end{array}$$

which proves the required exactness and commutativity.

4. Lemma 2.10 gives a commutative diagram with exact rows:

$$\begin{array}{ccccccccc} H^{r-1}_{\text{ét}}(U,\mathcal{F}) & \longrightarrow & \bigoplus_{v\notin U} H^{r-1}_{\text{ét}}(K_v,\mathcal{F}) & \longrightarrow & H^r_{\text{ét},c}(U,\mathcal{F}) & \longrightarrow & H^r_{\text{ét}}(U,\mathcal{F}) & \longrightarrow & \bigoplus_{v\notin U} H^r_{\text{ét}}(K_v,\mathcal{F}) \\ \downarrow\sim & & \downarrow & & \downarrow & & \downarrow\sim & & \downarrow \\ H^{r-1}(U,\mathcal{F}) & \longrightarrow & \bigoplus_{v\notin U} H^{r-1}(\widehat{K_v},\mathcal{F}) & \longrightarrow & H^r_c(U,\mathcal{F}) & \longrightarrow & H^r(U,\mathcal{F}) & \longrightarrow & \bigoplus_{v\notin U} H^r(\widehat{K_v},\mathcal{F}). \end{array}$$

Here $H_{\text{ét}}$ stands for étale cohomology (modified over K_v for v real) and $H_{\text{ét},c}$ for (modified) étale cohomology with compact support (as defined in [GS], §2, or before Lemma 2.10; recall also that in the number field case, the piece of notation $v \notin U$ means that we consider the places corresponding to closed points of $\operatorname{Spec}(\mathcal{O}_K) \setminus U$ *and* the real places).

By [GGM], Lemma 3.5.3, and [Mi1] III.3, we have

$$H^r_{\text{ét}}(K_v,\mathcal{F}) \cong H^r_{\text{ét}}(\widehat{K}_v,\mathcal{F}) \xrightarrow{\sim} H^r(\widehat{K_v},\mathcal{F})$$

for all $r \geq 1$ (resp. for all integers r if $\mathcal{F}_K$ is finite; indeed K_v and $\widehat{K_v}$ have the same absolute Galois group via [AC], §8, Corollary 4 to Theorem 2 and [Rib], section F, Cor. 2 to Th. 2) and all places v of K. Therefore the five-lemma gives the result. □

Remark 2.11 The definition of fppf compact support cohomology and its related properties are specific to schemes of dimension 1. To the best of our knowledge, there is no good analogue in higher dimension, unlike what happens for étale cohomology.

We will need the following complement to Proposition 2.1:

Proposition 2.12 *Let $\mathcal{F}$ be a flat affine commutative group scheme of finite type over U. Let $V \subset U$ be a non empty open subset. Then there is a long exact sequence*

$$\cdots \to \bigoplus_{v\in U\setminus V} H^r_v(\widehat{\mathcal{O}_v},\mathcal{F}) \to H^r(U,\mathcal{F}) \to H^r(V,\mathcal{F}) \to \bigoplus_{v\in U\setminus V} H^{r+1}_v(\widehat{\mathcal{O}_v},\mathcal{F}) \to \dots \quad (4)$$

Proof: The map $\bigoplus_{v\in U\setminus V} H^r_v(\widehat{\mathcal{O}_v},\mathcal{F}) \to H^r(U,\mathcal{F})$ is given by the identification of the first group with $H^r_Z(U,\mathcal{F})$, where $Z = U \setminus V$, via Lemma 2.6 and Lemma 2.9. By the localization exact sequence ([Mi2], Prop. III.0.3. c), this identification yields the required long exact sequence. □

3 Topology on cohomology groups with compact support

With the previous notation, let us define a natural topology on the groups $H^*_c(U,N)$, where N is a finite flat commutative U-group scheme. Th. 1.1 actually immediately implies that $H^2_c(U,N)$ is profinite, but this duality theorem will not be used in this paragraph. The "a priori" approach we adopt in this section answers a question raised by Milne ([Mi2], Problem III.8.8.).

We restrict ourselves to the function field case, because when K is a number field the groups involved are finite (cf. [Mi2], Th. III.3.2; see also section 5 of this article).

Recall that as usual (cf. for example [Mi2], §III.8), the groups $H^*(U,N)$ are endowed with the discrete topology. Our first goal in this section is to define a natural topology on the groups $H_c^*(U,N)$.

Given an exact sequence of abelian groups

$$0 \to A \to B \to C \to 0\,,$$

such that A is a topological group, there exists a unique topology on B such that B is a topological group, A is an open subgroup of B, and C is discrete when endowed with the quotient topology. Indeed, the topology on B is generated by the subsets $b+U$, where $b \in B$ and U is an open subset of A. In addition, given another abelian group B' with a subgroup $A' \subset B'$ that is a topological group, and a commutative diagram of abelian groups

$$\begin{array}{ccc} A & \hookrightarrow & B \\ \downarrow f & & \downarrow g \\ A' & \hookrightarrow & B'\,, \end{array}$$

then f is continuous if and only if g is continuous, for the aforementioned topologies. And f is open if and only if g is.

We can therefore topologize the groups $H_c^i(U,N)$ for $i \neq 2$, using the exact sequence (see Proposition 2.1, 1.)

$$\bigoplus_{v \notin U} H^{i-1}(\widehat{K_v}, N) \to H_c^i(U,N) \to H^i(U,N).$$

Since the groups $H^{i-1}(\widehat{K_v}, N)$ are finite for $i \neq 2$ ([Mi2], §III.6) and $H^i(U,N)$ is discrete, all groups $H_c^i(U,N)$ are discrete if $i \neq 2$.

Let us now focus on the case $i = 2$. Consider the exact sequence (Proposition 2.1, 1.)

$$H^1(U,N) \to \bigoplus_{v \notin U} H^1(\widehat{K}_v, N) \to H_c^2(U,N) \to H^2(U,N)\,. \tag{5}$$

and for $i = 1, 2$, set

$$D^i(U,N) = \mathrm{Im}\,[H_c^i(U,N) \to H^i(U,N)] = \mathrm{Ker}\,[H^i(U,N) \to \bigoplus_{v \notin U} H^i(\widehat{K_v}, N)].$$

By Proposition 2.1, 1., there is an exact sequence

$$\bigoplus_{v \notin U} H^{i-1}(\widehat{K_v}, N) \to H_c^i(U,N) \to D^i(U,N) \to 0. \tag{6}$$

The following result has been proved by Česnavičius ([Čes3], Th. 2.9).[4]

Theorem 3.1 (Česnavičius) *The map $H^1(U,N) \to \bigoplus_{v \notin U} H^1(\widehat{K}_v, N)$ is a strict morphism of topological groups, that is: the image of $H^1(U,N)$ is a discrete subgroup of $\bigoplus_{v \notin U} H^1(\widehat{K}_v, N)$. Besides, the group $D^1(U,N)$ is finite.*

Corollary 3.2 *The group $H_c^1(U,N)$ is finite.*

[4]Proposition 2.3 of loc. cit. uses the fppf duality Theorem 1.1, but this proposition is actually not needed to prove Theorem 3.1 because a discrete subgroup of a Hausdorff topological group is automatically closed by [TG], §2, Prop. 5.

Proof: The group $\bigoplus_{v \notin U} H^0(\widehat{K_v}, N)$ is finite (N being a finite U-group scheme). Thus the finiteness of $H^1_c(U,N)$ is equivalent to the finiteness of $D^1(U,N)$ by (6). □

Put the quotient topology on $(\bigoplus_{v \notin U} H^1(\widehat{K}_v, N))/\operatorname{Im} H^1(U,N)$. Using Th. 3.1, the previous facts define a natural topology on $H^2_c(U,N)$, so that morphisms in the exact sequence (5) are continuous (and even strict). This topology makes $H^2_c(U,N)$ a Hausdorff and locally compact group (cf. [TG], §2, Prop. 18, a).

To say more about the topology of $H^2_c(U,N)$, we need a lemma:

Lemma 3.3

1. *Let $r : N \to N'$ be a morphism of finite flat commutative U-group schemes. Then the corresponding map $s : H^2_c(U,N) \to H^2_c(U,N')$ is continuous. If we assume further that r is surjective, then s is open. If*

$$0 \to N' \to N \to N'' \to 0$$

   is an exact sequence of finite flat commutative U-group schemes, then the connecting map $H^2_c(U,N'') \to H^3_c(U,N')$ is continuous.

2. *Let $V \subset U$ be a non empty open subset. Then the natural map $u : H^2_c(V,N) \to H^2_c(U,N)$ is continuous.*

Proof:

1. By definition of the topology on the groups H^2_c, it is sufficient to prove that for $v \notin U$, the map $H^1(\widehat{K_v}, N) \to H^1(\widehat{K_v}, N')$ is continuous (resp. open if r is surjective). Continuity follows from [Česl], Prop. 4.2 and the openness statement from loc. cit., Prop 4.3 d). Similarly, the last assertion follows from the continuity of the connecting map $H^1(K_v, N'') \to H^2(K_v, N')$ (loc. cit., Prop. 4.2).

2. Since (by definition of the topology) the image I of $A := \bigoplus_{v \notin V} H^1(\widehat{K_v}, N)$ is an open subgroup of $H^2_c(V,N)$, it is sufficient to show that the restriction of u to I is continuous. As I is equipped with the quotient topology (induced by the topology of A), this is equivalent to showing that the natural map $s : A \to H^2_c(U,N)$ is continuous. Now we observe that A is the direct sum of $A_1 := \bigoplus_{v \notin U} H^1(\widehat{K_v}, N)$ and $A_2 := \bigoplus_{v \in U \setminus V} H^1(\widehat{K_v}, N)$. The restriction of s to A_1 is continuous by the commutative diagram of Prop. 2.1, 3. Therefore it only remains to show that the restriction s_2 of s to A_2 is continuous. By loc. cit., the restriction of s_2 to $\bigoplus_{v \in U \setminus V} H^1(\widehat{\mathcal{O}_v}, N)$ is zero. Since $\bigoplus_{v \in U \setminus V} H^1(\widehat{\mathcal{O}_v}, N)$ is an open subgroup of $\bigoplus_{v \in U \setminus V} H^1(\widehat{K_v}, N)$ ([Česl], Prop. 3.10), the result follows.

□

Recall also the following (probably well-known) lemma:

Lemma 3.4 *Let $f : A \to B$ be a continuous morphism of topological groups, with B Hausdorff.*

1. *Assume that A is profinite. Then f is strict.*

2. *Assume that f is injective and A is compact. Then f is strict.*

3. *Let*
$$0 \to A \overset{i}{\to} B \overset{\pi}{\to} C \to 0$$
be an exact sequence of topological groups with i strict and π continuous. If A and C are completely disconnected, then so is B.

Proof:

1. Since f is continuous and B Hausdorff, the image of f is a compact subspace of B, so we can assume that B is compact and f is onto. The topology of A has a basis consisting of open subgroups, so it is sufficient to show that the image of such a subgroup U is open. As U is closed (hence compact) and of finite index in A, its image $f(U)$ is also compact and of finite index in B, hence it is an open subgroup of B.

2. Since A is compact and B is Hausdorff, we get that i is a closed map (because the image of a compact subspace of A is compact), hence it induces a homeomorphism from i onto the subspace $i(A) \subset B$. This means that i is strict.

3. Let D be a connected subset of B. Then $\pi(D)$ is connected, hence is a singleton. Thus, by translating, one can assume that $D \subset i(A)$; as i is strict, the subset $i^{-1}(D) \subset A$ is connected, so it is reduced to a point, hence D is a singleton. This proves the statement.

□

Proposition 3.5 *For every integer i with $0 \leq i \leq 3$, the topology on $H^i_c(U, N)$ is profinite.*

Proof: The only non trivial case is $i = 2$. We first observe that if there is an exact sequence of finite flat commutative U-group schemes

$$0 \to N' \to N \to N'' \to 0,$$

then it is sufficient to prove that $H^2_c(U, N')$ and $H^2_c(U, N'')$ are profinite to get the same result for $H^2_c(U, N)$. Indeed by Proposition 2.1, 3., there is an exact sequence

$$H^1_c(U, N'') \to H^2_c(U, N') \to H^2_c(U, N) \to H^2_c(U, N'').$$

The group $H^1_c(U, N'')$ is finite by Corollary 3.2; besides, all maps are continuous and the map $H^2_c(U, N) \to H^2_c(U, N'')$ is open (in particular it is strict, and its image is profinite as soon as $H^2_c(U, N'')$ is) by Lemma 3.3, 1. Therefore if $H^2_c(U, N')$ and $H^2_c(U, N'')$ are profinite, then $H^2_c(U, N)$ is profinite as an extension

$$0 \overset{i}{\to} A \to H^2_c(U, N) \overset{\pi}{\to} B \to 0$$

of two profinite groups A, B such that the map π is open (the map i is strict by Lemma 3.4, 2.; the group $H^2_c(U, N)$ is completely disconnected by Lemma 3.4 3., and its compactness follows from the fact that π is a proper map by [TG], §4, Cor. 2 to Prop. 2).

This being said, note now that Proposition 2.1, 4. implies the result when the order of N is prime to p by [Mi2], Corollary II.3.3 (in this case $H^2_c(U,N)$ is even finite). One can therefore assume by devissage that the order of N is a power of p. The generic fiber N_K of N is a finite commutative group scheme over K. By [DG], IV, §3.5, N_K admits a composition series whose quotients are étale (with a dual of height one), local (of height one) with étale dual, or α_p. The schematic closure in N of this composition series provides a composition series defined over U. Thus, using the same devissage argument as above, one reduces to the case where the generic fiber N_K or its dual N_K^D has height one.

Proposition III.B.4 and Corollary III.B.5 in [Mi2] now imply that there exists a non empty open subset $V \subset U$ such that $N_{|V}$ extends to a finite flat commutative group scheme $\widetilde{N}$ over the proper k-curve X.

Then Proposition 2.1, 3. gives an exact sequence

$$H^1_c(X,\widetilde{N}) \to \bigoplus_{v\in X\setminus V} H^1(\widehat{\mathcal{O}}_v,\widetilde{N}) \to H^2_c(V,N) \to H^2_c(X,\widetilde{N}) \qquad (7)$$

and since we are in the function field case with X proper over k, we have $H^i_c(X,\widetilde{N}) = H^i(X,\widetilde{N})$ for every positive integer i.

By Proposition 2.1, 3., the map $\bigoplus_{v\in X\setminus V} H^1(\widehat{\mathcal{O}}_v,\widetilde{N}) \to H^2_c(V,N)$ factors through $\bigoplus_{v\in X\setminus V} H^1(\widehat{K}_v,N)$, hence it is continuous. By Lemma 3.3, all maps in (7) are continuous. In addition, the groups $H^1_c(X,\widetilde{N}) = H^1(X,\widetilde{N})$ and $H^2_c(X,\widetilde{N}) = H^2(X,\widetilde{N})$ are finite by [Mi2], Lemma III.8.9. Besides, $\bigoplus_{v\in X\setminus V} H^1(\widehat{\mathcal{O}}_v,\widetilde{N})$ is profinite by loc. cit., §III.7; hence $H^2_c(V,N)$ is profinite as an extension (the maps being strict by Lemma 3.4, 2.) of a finite group by a profinite group.

Since $H^2(\widehat{\mathcal{O}}_v,N) = 0$ for every $v \in U$ ([Mi2], §III.7), Prop. 2.1, 3. gives an exact sequence of groups

$$\bigoplus_{v\in U\setminus V} H^1(\widehat{\mathcal{O}}_v,N) \to H^2_c(V,N) \to H^2_c(U,N) \to 0,$$

which implies that $H^2_c(U,N)$ is profinite, the map $H^2_c(V,N) \to H^2_c(U,N)$ being continuous by Lemma 3.3 2., hence strict by Lemma 3.4 1., because $H^2_c(V,N)$ is profinite and $H^2_c(U,N)$ is Hausdorff. □

The following statement will be useful in the next section:

Proposition 3.6 *Assume that $\mathcal{F} = N$, $\mathcal{F}' = N'$ and $\mathcal{F}'' = N''$ are finite and flat commutative group schemes over U. Then all the maps in Proposition 2.1 are strict (in particular continuous).*

Proof: For the maps in assertion 1. of Prop. 2.1, this follows from the definition of the topology and Th. 3.1.

Let us consider the maps in assertion 2. The finiteness of the H^1_c groups (Cor. 3.2) implies that it only remains to deal with the maps between H^2_c's and the connecting map $H^2_c(U,\mathcal{F}'') \to H^3_c(U,\mathcal{F}')$. All these maps are continuous by Lemma 3.3, hence strict by Lemma 3.4 1. and Prop. 3.5.

Finally, it has already been proven (cf. proof of Prop. 3.5) that the maps in the exact sequence of assertion 3. are continuous. They are strict via Lemma 3.4 1. because $H^1_c(U,\mathcal{F})$ is finite, $H^2_c(U,\mathcal{F})$ (resp. $\bigoplus_{v\in U\setminus V} H^1(\widehat{\mathcal{O}_v},\mathcal{F})$) is profinite, and the other groups are discrete. □

4 Proof of Theorem 1.1 in the function field case

In this section K is the function field of a projective, smooth and geometrically integral curve X defined over a finite field k of characteristic p. The proof follows the same lines as the proof of [Mi2], Theorem III.8.2, replacing Proposition III.0.4 in [Mi2] by Proposition 2.1 and using the results of section 2.

For every non empty open subset $V\subset U$, the natural map $H^3_c(V,\mathbf{G}_m)\xrightarrow{s} H^3_c(U,\mathbf{G}_m)$ is an isomorphism, and the trace map identifies $H^3_c(U,\mathbf{G}_m)$ with $\mathbf{Q}/\mathbf{Z}$ (this identification being compatible with s). Indeed since $\mathbf{G}_m$ is a smooth group scheme we can apply Prop 2.1, 4. and [Mi2], §II.3.

For a fppf sheaf $\mathcal{F}$ on U, let us first define the pairing of abelian groups

$$H^{3-r}_c(U,\mathcal{F})\times H^r(U,\mathcal{F}^D)\to H^3_c(U,\mathbf{G}_m)\cong\mathbf{Q}/\mathbf{Z}\,.$$

Since the cohomology groups with compact support are defined via a mapping cone construction, we need to construct this pairing carefully at the level of complexes in order to be able to prove the compatibilities that follow (see Lemmas 4.3 and 4.7 for instance).

Lemma 4.1 *Let A and B be two fppf sheaves of abelian groups on U. Then there exists a canonical pairing in the derived category of abelian groups:*

$$R\Gamma_c(U,A)\otimes^{\mathbf{L}} R\Gamma(U,B)\to R\Gamma_c(U,A\otimes B)\,.$$

Moreover, this pairing is functorial in A and B.

Proof: For any complex C of fppf sheaves, let $G(C)$ denote the Godement resolution of C (see for instance [SGA4], XVII, 4.2.9; Godement resolutions exist on the big fppf site because this site has enough points, see Remark 1.6. of [GK] or [SP, Tag 06VX]).

Then there is a commutative diagram of complexes of sheaves (see [God], II.6.6 or [FS], Appendix A)

$$\begin{array}{ccc} A\otimes B & & \\ \downarrow & \searrow & \\ \mathrm{Tot}(G(A)\otimes G(B)) & \longrightarrow & G(A\otimes B)\,. \end{array}$$

The horizontal morphism induces a morphism of complexes of abelian groups

$$\mathrm{Tot}(\Gamma(U,G(A))\otimes\Gamma(U,G(B)))\to\Gamma(U,G(A\otimes B))$$

hence a canonical morphism in the derived category of abelian groups

$$\Gamma(U,G(A))\otimes^{\mathbf{L}}\Gamma(U,G(B))\to\Gamma(U,G(A\otimes B))\,.$$

Considering the local versions of the previous pairings, one gets a commutative diagram of complexes of abelian groups

$$\begin{array}{ccc} \mathrm{Tot}(\Gamma(U,G(A))\otimes\Gamma(U,G(B))) & \longrightarrow & \Gamma(U,G(A\otimes B)) \\ \downarrow & & \downarrow \\ \prod_{v\notin U}\mathrm{Tot}(\Gamma(\widehat{K}_v,G(A))\otimes\Gamma(U,G(B))) & \longrightarrow & \prod_{v\notin U}\Gamma(\widehat{K_v},G(A\otimes B)) \,, \end{array}$$

and functoriality of cones gives a canonical morphism of complexes (via Proposition 6.1 in the Appendix)

$$\mathrm{Tot}(\Gamma_c(U,G(A))\otimes\Gamma(U,G(B)))\to\Gamma_c(U,G(A\otimes B))\,. \tag{8}$$

Since Godement resolutions are acyclic (see [SGA4], XVII, Proposition 4.2.3), we know that $R\Gamma(U,C)\cong\Gamma(U,G(C))$ in the derived category, for any fppf sheaf C. Hence the pairing (8) gives the required morphism in the derived category

$$R\Gamma_c(U,A)\otimes^{\mathbf{L}}R\Gamma(U,B)\to R\Gamma_c(U,A\otimes B)\,.$$

The functoriality of Godement resolutions implies the functoriality of the pairing in A and B. □

Using Lemma 4.1, [SP, Tag 068G] gives a natural pairing

$$H^r_c(U,A)\times H^s(U,B)\to H^{r+s}_c(U,A\otimes B)\,,$$

whence we deduce the required canonical pairings, for any sheaf $\mathcal{F}$ on $(\mathrm{Sch}/U)_{\mathrm{fppf}}$

$$H^r_c(U,\mathcal{F})\times H^s(U,\mathcal{F}^D)\to H^{r+s}_c(U,\mathbf{G}_m)\,, \tag{9}$$

using the canonical map $\mathcal{F}\otimes\mathcal{F}^D=\mathcal{F}\otimes\underline{\mathrm{Hom}}(\mathcal{F},\mathbf{G}_m)\to\mathbf{G}_m$.

Let us describe explicitely the pairing above: the map

$$\mathrm{Tot}(\Gamma_c(U,G(A))\otimes\Gamma(U,G(B)))\to\Gamma_c(U,G(A\otimes B))$$

is given by maps

$$\left(\prod_{v\notin U}\Gamma(\widehat{K}_v,G_{r-1}(A))\oplus\Gamma(U,G_r(A))\right)\otimes\Gamma(U,G_s(B))\to\prod_{v\notin U}\Gamma(\widehat{K}_v,G_{r+s-1}(A\otimes B))\oplus\Gamma(U,G_{r+s}(A\otimes B))$$

$$(a_{r-1},a_r)\otimes b_s\mapsto(a_{r-1}\cup\beta(b_s),a_r\cup b_s)\,,$$

where the maps denoted by $\cup$ are the natural pairings, and $\beta:\Gamma(U,G_s(B))\to\prod_{v\notin U}\Gamma(\widehat{K}_v,G_s(B))$ is the localization map.

In the following, we will need an alternative version of the above pairing: with the same notation as above, one defines a pairing in the derived category

$$R\Gamma(U,A)\otimes^{\mathbf{L}}R\Gamma_c(U,B)\to R\Gamma_c(U,A\otimes B)\,.$$

The definition is similar to the one in Lemma 4.1: the commutative diagram of complexes

$$\begin{array}{ccc} \mathrm{Tot}(\Gamma(U,G(A))\otimes\Gamma(U,G(B))) & \longrightarrow & \Gamma(U,G(A\otimes B)) \\ \downarrow & & \downarrow \\ \prod_{v\notin U}\mathrm{Tot}(\Gamma(U,G(A))\otimes\Gamma(\widehat{K}_v,G(B))) & \longrightarrow & \prod_{v\notin U}\Gamma(\widehat{K_v},G(A\otimes B))\,, \end{array}$$

and Proposition 6.1 in the Appendix gives a morphism of complexes

$$\mathrm{Tot}(\Gamma(U,G(A))\otimes\Gamma_c(U,G(B)))\to\Gamma_c(U,G(A\otimes B))\,. \tag{10}$$

Taking into account the signs in Proposition 6.1, one can describe the pairing $\mathrm{Tot}(\Gamma(U,G(A))\otimes\Gamma_c(U,G(B)))\to\Gamma_c(U,G(A\otimes B))$ explicitely as follows:

$$\Gamma(U,G_r(A))\otimes\left(\prod_{v\notin U}\Gamma(\widehat{K}_v,G_{s-1}(B))\oplus\Gamma(U,G_s(B))\right)\to\prod_{v\notin U}\Gamma(\widehat{K}_v,G_{r+s-1}(A\otimes B))\oplus\Gamma(U,G_{r+s}(A\otimes B))$$
$$a_r\otimes(b_{s-1},b_s)\mapsto((-1)^r\alpha(a_r)\cup b_{s-1},a_r\cup b_s)\,,$$

where $\alpha:\Gamma(U,G_r(A))\to\prod_{v\notin U}\Gamma(\widehat{K}_v,G_r(A))$ is the localization map.

We now compare the two pairings defined above:

Lemma 4.2 *The following diagram of complexes*

$$\begin{array}{ccc} \mathrm{Tot}(\Gamma_c(U,G(A))\otimes\Gamma(U,G(B))) & \longrightarrow & \Gamma_c(U,G(A\otimes B)) \\ =\uparrow \quad\quad \uparrow & & \\ \mathrm{Tot}(\Gamma_c(U,G(A))\otimes\Gamma_c(U,G(B))) & & \Big\downarrow = \\ \downarrow \quad\quad \downarrow = & & \\ \mathrm{Tot}(\Gamma(U,G(A))\otimes\Gamma_c(U,G(B))) & \longrightarrow & \Gamma_c(U,G(A\otimes B)) \end{array}$$

commutes up to homotopy.

Proof: Using the explicit descriptions above, one needs to prove that the map $\varphi_{r,s}$ from $\left(\prod_{v\notin U}\Gamma(\widehat{K}_v,G_{r-1}(A))\oplus\Gamma(U,G_r(A))\right)\otimes\left(\prod_{v\notin U}\Gamma(\widehat{K}_v,G_{s-1}(B))\oplus\Gamma(U,G_s(B))\right)$ to $\prod_{v\notin U}\Gamma(\widehat{K}_v,G_{r+s-1}(A\otimes B))\oplus\Gamma(U,G_{r+s}(A\otimes B))$ given by

$$(a_{r-1},a_r)\otimes(b_{s-1},b_s)\mapsto((-1)^r\alpha(a_r)\cup b_{s-1}-a_{r-1}\cup\beta(b_s),0)$$

is homotopically trivial. To prove this, consider the maps

$$\left(\prod_{v\notin U}\Gamma(\widehat{K}_v,G_{r-1}(A))\oplus\Gamma(U,G_r(A))\right)\otimes\left(\prod_{v\notin U}\Gamma(\widehat{K}_v,G_{s-1}(B))\oplus\Gamma(U,G_s(B))\right)\xrightarrow{h_{r,s}}\prod_{v\notin U}\Gamma(\widehat{K}_v,G_{r+s-2}(A\otimes B))\oplus\Gamma(U,G_{r+s-1}(A\otimes B))$$

given by $h_{r,s} : (a_{r-1}, a_r) \otimes (b_{s-1}, b_s) \mapsto (0, (-1)^r a_{r-1} \cup b_{s-1})$. Then these maps define an homotopy equivalence between the map $\bigoplus_{r+s=n} \varphi_{r,s}$ and the zero map, proving the lemma. □

We now prove that the pairing is compatible with coboundary maps in cohomology coming from short exact sequences:

Lemma 4.3 *Let $0 \to A \to B \to C \to 0$ and $0 \to C' \to B' \to A' \to 0$ be two exact sequences of fppf sheaves on U, and let $B \otimes B' \to D$ be a morphism of fppf sheaves. Assume that the induced morphism $A \otimes C' \to D$ is trivial.*

Consider the following diagram

$$\begin{array}{ccc} H^r_c(U,C) \times H^{s+1}(U,C') & \xrightarrow{\cup} & H^{r+s+1}_c(U,D) \\ \downarrow \partial_r \qquad\qquad \uparrow \partial'_s & & \downarrow = \\ H^{r+1}_c(U,A) \times H^s(U,A') & \xrightarrow{\cup} & H^{r+s+1}_c(U,D)\,, \end{array}$$

where the horizontal morphisms are induced by the pairings in Lemma 4.1 and by the morphism $B \otimes B' \to D$, and the vertical maps are the coboundary morphisms.

Then for all $c \in H^r_c(U,C)$ and $a' \in H^s(U,A')$, we have

$$\partial_r(c) \cup a' + (-1)^r c \cup \partial'_s(a') = 0\,.$$

Proof: For all fppf sheaves E, let $\partial_i^E : G_i(E) \to G_{i+1}(E)$ denote the coboundary map in the Godement complex $G(E)$.

Consider the diagram induced by $B \otimes B' \to D$:

$$\begin{array}{ccc} \Gamma(U,G_r(B)) \otimes \Gamma(U,G_{s+1}(B')) & \xrightarrow{\cup} & \Gamma(U,G_{r+s+1}(D)) \\ \downarrow \partial_r^B \qquad\qquad \uparrow \partial_s^{B'} & & \downarrow = \\ \Gamma(U,G_{r+1}(B)) \otimes \Gamma(U,G_s(B')) & \xrightarrow{\cup} & \Gamma(U,G_{r+s+1}(D))\,, \end{array}$$

together with the similar diagrams over $\operatorname{Spec} \widehat{K}_v$, for all $v \notin S$.

By compatibility of the Godement resolution with tensor product (cf. [FS], Appendix A), the pairing $\operatorname{Tot}(G(B) \otimes G(B')) \to G(D)$ is a morphism of complexes. Hence for all $b \in \Gamma(U, G_r(B))$ and $b' \in \Gamma(U, G_s(B'))$, we have

$$\partial_r^B(b) \cup b' + (-1)^r b \cup \partial_s^{B'}(b') = \partial_{r+s}^D(b \cup b')\,.$$

This formula, its analogue over $\operatorname{Spec} \widehat{K}_v$ for $v \notin S$, together with the definition of the connecting maps in cohomology via Godement resolutions (recall that for all n, the functor $\mathcal{F} \mapsto G_n(\mathcal{F})$ is exact, see [SGA4], XVII, Proposition 4.2.3), implies Lemma 4.3. □

Lemma 4.4 *Let N be a finite flat commutative U-group scheme of order n, then the pairings* (9)

$$H^r_c(U,N) \times H^s(U,N^D) \to H^{r+s}_c(U,\mu_n)$$

are continuous.

Proof: The pairings (9) are defined via the cup-product on U and the local duality pairings $H^a(\widehat{K}_v, N) \times H^b(\widehat{K}_v, N^D) \to H^{a+b}(\widehat{K}_v, \mu_n)$. These local pairings are continuous (see [Čes1], Theorems 5.11 and 6.5). Hence the lemma follows from the definition of the topologies on the cohomology groups (see section 3). □

Remark 4.5 In [Mi2] (see for example Th. III.3.1), the pairings are defined via the Ext groups, which is quite convenient for the definition itself but makes the required commutativities of diagrams more difficult to check. Nevertheless, Proposition V.1.20 in [Mi1] provides a similar comparison between both definitions: see the details in the Appendix, Proposition 6.2.

In order to prove Theorem 1.1, we now need to show that the induced map $H^{3-r}_c(U,N) \to H^r(U,N^D)^*$ is an isomorphism (of topological groups) for every finite flat commutative group scheme N over U and every $r \in \{0,1,2,3\}$ (recall that the groups $H^r(U,N^D)$ are equipped with the discrete topology).

We first recall the following lemma ([Mi2], Lemma III.8.3):

Lemma 4.6 *Let*

$$0 \to N' \to N \to N'' \to 0$$

be an exact sequence of finite flat commutative group schemes over U. If Theorem 1.1 is true for N' and N'', then it is true for N.

Proof: Using Proposition 2.1, 2., the exactness of Pontryagin duality for discrete groups and the pairing in Lemma 4.1, one gets a diagram of long exact sequences:

$$\begin{array}{ccccccccc} \dots \longrightarrow & H^{3-r}_c(U,N') & \longrightarrow & H^{3-r}_c(U,N) & \longrightarrow & H^{3-r}_c(U,N'') & \longrightarrow \dots \\ \star & \downarrow & & \downarrow & & \downarrow & \star \\ \dots \longrightarrow & H^r(U,N'^D)^* & \longrightarrow & H^r(U,N^D)^* & \longrightarrow & H^r(U,N''^D)^* & \longrightarrow \dots . \end{array}$$

The functoriality of the pairing (see Lemma 4.1) implies that both central squares in the diagram are commutative. Lemma 4.3 implies that both extreme squares (denoted $\star$) are commutative up to sign. By Lemma 3.4 1., Prop. 3.6 and Lemma 4.4, all the maps in this diagram are continuous. Hence the lemma follows from the five-lemma. □

We now want to show that it is sufficient to prove Theorem 1.1 for a smaller open subset $V \subset U$. To do this, we need to check the compatibility of the pairing in Theorem 1.1 with restriction to an open subset of U and with the local duality pairing (see Lemma 4.7 below).

We first define the maps that appear in this lemma. Let $\mathcal{F}$ be a flat affine commutative U-group scheme of finite type and let $V \subset U$ be a non empty open subset. Let W denote $U \setminus V$. In diagram (11) below, the first column is the long exact sequence of Proposition 2.1, 3., and the second column is the localization exact sequence from Prop. 2.12. The horizontal pairings are either the local duality pairings from [Mi2], Theorem III.7.1 (first and last rows), using the same sign convention as in the pairing (10), or the global pairings from Lemma 4.1 (second and third rows). The proof of Proposition 2.12 provides an isomorphism $H^3_W(U,\mathbf{G}_m) \cong \bigoplus_{v\in W} H^3_v(\widehat{\mathcal{O}_v}, \mathbf{G}_m)$, and the natural morphism of complexes $\Gamma_W(U, I^\bullet(\mathbf{G}_m)) \to \Gamma_c(V, I^\bullet(\mathbf{G}_m))$ gives a morphism $H^3_W(U,\mathbf{G}_m) \to H^3_c(V,\mathbf{G}_m)$, whence natural morphisms $\bigoplus_{v\in W} H^3_v(\widehat{\mathcal{O}_v}, \mathbf{G}_m) \to H^3_c(V,\mathbf{G}_m) \to H^3_c(U,\mathbf{G}_m)$.

Lemma 4.7 *Let $\mathcal{F}$, $\mathcal{G}$ be flat affine commutative group schemes of finite type on U, together with a pairing $\mathcal{F} \otimes \mathcal{G} \to \mathbf{G}_m$. Let $V \subset U$ be a non empty open subscheme and $W := U \setminus V$. Then the following diagram*

$$\begin{array}{ccc}
\bigoplus_{v\in W} H^{2-r}(\widehat{\mathcal{O}_v},\mathcal{F}) \times \bigoplus_{v\in W} H^{r+1}_v(\widehat{\mathcal{O}_v},\mathcal{G}) & \longrightarrow & \bigoplus_{v\in W} H^3_v(\widehat{\mathcal{O}_v},\mathbf{G}_m) \\
\downarrow \qquad \uparrow & & \downarrow \\
H^{3-r}_c(V,\mathcal{F}) \times H^r(V,\mathcal{G}) & \longrightarrow & H^3_c(V,\mathbf{G}_m) \\
\downarrow \qquad \uparrow & & \downarrow \wr \\
H^{3-r}_c(U,\mathcal{F}) \times H^r(U,\mathcal{G}) & \longrightarrow & H^3_c(U,\mathbf{G}_m) \\
\downarrow \qquad \uparrow & & \uparrow \\
\bigoplus_{v\in W} H^{3-r}(\widehat{\mathcal{O}_v},\mathcal{F}) \times \bigoplus_{v\in W} H^{r}_v(\widehat{\mathcal{O}_v},\mathcal{G}) & \longrightarrow & \bigoplus_{v\in W} H^3_v(\widehat{\mathcal{O}_v},\mathbf{G}_m)
\end{array} \tag{11}$$

is commutative.

In addition, if $\mathcal{F}$ and $\mathcal{G}$ are finite and flat group schemes, then all the maps in the diagram are continuous.

Proof:

1. We first prove the commutativity of the top rectangle. It is sufficient to prove that the following diagram commutes:

$$\begin{array}{ccc}
\mathrm{Tot}\left(\bigoplus_{v\in W}\Gamma(\widehat{\mathcal{O}_v},G(\mathcal{F}))[-1]\otimes\bigoplus_{v\in W}\Gamma_v(\widehat{\mathcal{O}_v},G(\mathcal{G}))[1]\right) & \longrightarrow & \bigoplus_{v\in W}\Gamma_v(\widehat{\mathcal{O}_v},G(\mathbf{G}_m)) \\
\downarrow \qquad \uparrow & & \uparrow \\
\mathrm{Tot}\left(\bigoplus_{v\in W}\Gamma(\widehat{K}_v,G(\mathcal{F}))[-1]\otimes\bigoplus_{v\in W}\Gamma(\widehat{K}_v,G(\mathcal{G}))\right) & \longrightarrow & \bigoplus_{v\in W}\Gamma(\widehat{K}_v,G(\mathbf{G}_m))[-1] \\
\downarrow \qquad \uparrow & & \downarrow \\
\mathrm{Tot}\left(\Gamma_c(V,G(\mathcal{F}))\otimes\Gamma(V,G(\mathcal{G}))\right) & \longrightarrow & \Gamma_c(V,G(\mathbf{G}_m))
\end{array}$$

   where the vertical maps are the natural ones and the horizontal pairings are defined earlier. The top rectangle is commutative because of the definition of the pairing involving cohomology with support in a closed subscheme, taking into account the sign conventions in Proposition 6.1 in the Appendix. The bottom one is commutative by definition of the pairing involving compact support cohomology.

   Assume now that $\mathcal{F}$ and $\mathcal{G}$ are finite flat group schemes. Then the following maps are continuous: the pairing $H^2_c(V,\mathcal{F}) \times H^1(V,\mathcal{G}) \to H^3_c(V,\mathbf{G}_m)$ (see Lemma 4.4), the pairing $H^1(\widehat{\mathcal{O}_v},\mathcal{F}) \times H^2_v(\widehat{\mathcal{O}_v},\mathcal{G}) \to H^3_v(\widehat{\mathcal{O}_v},\mathbf{G}_m)$ (see [Mi2], Theorem III.7.1) and the map $\bigoplus_{v\in W} H^1(\widehat{\mathcal{O}_v},\mathcal{F}) \to H^2_c(V,\mathcal{F})$ (see Proposition 3.6).

2. We now prove the commutativity of the rectangle in the middle. Let $\widetilde{\Gamma}(U,G(\mathcal{F})) := \mathrm{Cone}(\Gamma(U,G(\mathcal{F})) \to \bigoplus_{v\notin U}\Gamma(\widehat{K_v},G(\mathcal{F}))\oplus\bigoplus_{v\in U\setminus V}\Gamma(\widehat{\mathcal{O}_v},G(\mathcal{F})))[-1]$. Then functoriality of the cone gives a commutative diagram (similar to (3), where $I^\bullet(\mathcal{F})$ is

replaced by $G(\mathcal{F})$ and by $G(\mathbf{G}_m)$) of complexes of abelian groups:

$$\begin{array}{ccc}
\mathrm{Tot}(\Gamma_c(V, G(\mathcal{F})) \otimes \Gamma(V, G(\mathcal{G}))) & \longrightarrow & \Gamma_c(V, G(\mathbf{G}_m)) \\
{\scriptstyle q}\uparrow \qquad\qquad \uparrow & & \uparrow{\scriptstyle q} \\
\mathrm{Tot}(\widetilde{\Gamma}(U, G(\mathcal{F})) \otimes \Gamma(U, G(\mathcal{G}))) & \longrightarrow & \widetilde{\Gamma}(U, G(\mathbf{G}_m)) \\
\downarrow \qquad\qquad \downarrow{\scriptstyle =} & & \downarrow \\
\mathrm{Tot}(\Gamma_c(U, G(\mathcal{F})) \otimes \Gamma(U, G(\mathcal{G}))) & \longrightarrow & \Gamma_c(U, G(\mathbf{G}_m)) .
\end{array}$$

Here the maps denoted by q are quasi-isomorphisms (see Remark 2.7 and the proof of the third point in Proposition 2.1, which uses Lemma 2.4). This diagram gives a commutative diagram in the derived category of abelian groups (where all the maps are either the natural ones or the ones constructed above):

$$\begin{array}{ccc}
\Gamma_c(V, G(\mathcal{F})) \otimes^{\mathbf{L}} \Gamma(V, G(\mathcal{G})) & \longrightarrow & \Gamma_c(V, G(\mathbf{G}_m)) \\
\downarrow \qquad\qquad \uparrow & & \downarrow \\
\Gamma_c(U, G(\mathcal{F})) \otimes^{\mathbf{L}} \Gamma(U, G(\mathcal{G})) & \longrightarrow & \Gamma_c(U, G(\mathbf{G}_m)) .
\end{array}$$

Taking cohomology of this diagram gives a commutative diagram of abelian groups:

$$\begin{array}{ccc}
H^r_c(V, \mathcal{F}) \times H^s(V, \mathcal{G}) & \longrightarrow & H^{r+s}_c(V, \mathbf{G}_m) \\
\downarrow \qquad\qquad \uparrow & & \downarrow \\
H^r_c(U, \mathcal{F}) \times H^s(U, \mathcal{G}) & \longrightarrow & H^{r+s}_c(U, \mathbf{G}_m) ,
\end{array}$$

which implies the required commutativity.

The continuity of the maps in the case where $\mathcal{F}$ and $\mathcal{G}$ are finite flat group schemes is a consequence of Lemma 4.4 and of Lemma 3.3.

3. We now need to prove the commutativity of the bottom rectangle. By Lemma 4.2, the following diagram

$$\begin{array}{ccc}
\Gamma_c(U, G(\mathcal{F})) \otimes^{\mathbf{L}} \Gamma(U, G(\mathcal{G})) & \longrightarrow & \Gamma_c(U, G(\mathbf{G}_m)) \\
\downarrow \qquad\qquad \uparrow & & \downarrow{\scriptstyle =} \\
\Gamma(U, G(\mathcal{F})) \otimes^{\mathbf{L}} \Gamma_c(U, G(\mathcal{G})) & \longrightarrow & \Gamma_c(U, G(\mathbf{G}_m))
\end{array}$$

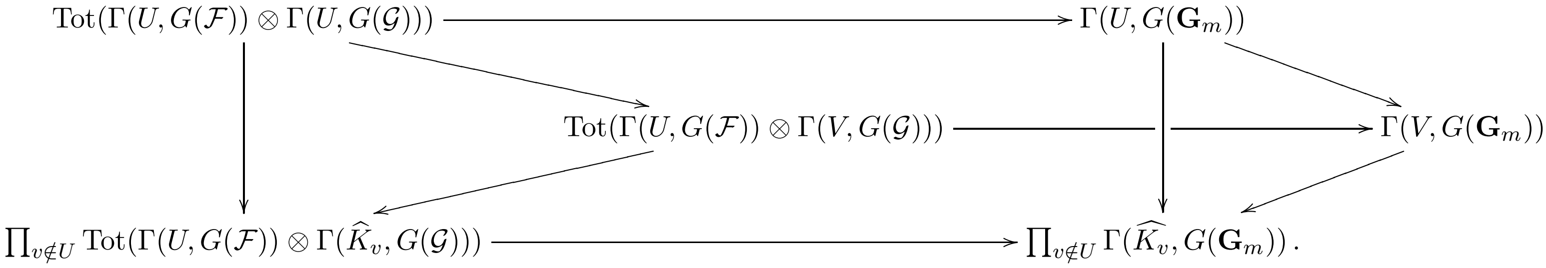
commutes in the derived category. Computing cohomology gives a commutative diagram of abelian groups:

$$\begin{array}{ccc}
H^r_c(U, \mathcal{F}) \times H^s(U, \mathcal{G}) & \longrightarrow & H^{r+s}_c(U, \mathbf{G}_m) \\
\downarrow \qquad\qquad \uparrow & & \downarrow{\scriptstyle =} \\
H^r(U, \mathcal{F}) \times H^s_c(U, \mathcal{G}) & \longrightarrow & H^{r+s}_c(U, \mathbf{G}_m) .
\end{array}$$

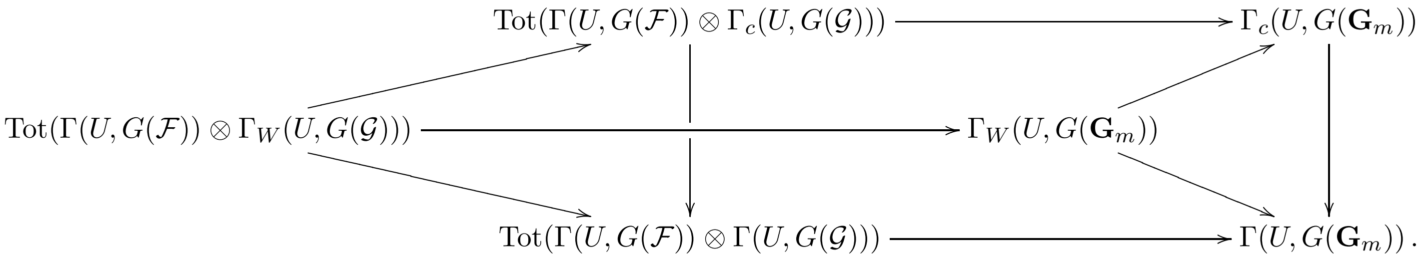
Let $\Gamma_W(U, G(\mathcal{G})) := \mathrm{Cone}(\Gamma(U, G(\mathcal{G})) \to \Gamma(V, G(\mathcal{G})))[-1]$. In order to prove the required commutativity, it is enough to prove that the natural diagram

$$\begin{array}{ccc} \Gamma(U, G(\mathcal{F})) \otimes^{\mathbf{L}} \Gamma_c(U, G(\mathcal{G})) & \longrightarrow & \Gamma_c(U, G(\mathbf{G}_m)) \\ {\scriptstyle =}\uparrow \qquad\quad \uparrow & & \uparrow \\ \Gamma(U, G(\mathcal{F})) \otimes^{\mathbf{L}} \Gamma_W(U, G(\mathcal{G})) & \longrightarrow & \Gamma_W(U, G(\mathbf{G}_m)) \end{array}$$

commutes in the derived category, where the pairing on the bottom row is defined in a similar way as the pairing (10). Consider the following diagram in the category of complexes:



This diagram is commutative, hence, using Proposition 6.1, it induces a commutative diagram of complexes at the level of cones:



The commutativity of the upper face of this last diagram concludes the proof.

Assume now that $\mathcal{F}$ and $\mathcal{G}$ are finite flat group schemes. The only possibly non discrete groups in the diagram are $H^2_c(U, \mathcal{F})$ (in the case $r = 1$) and $H^1(\widehat{\mathcal{O}_v}, \mathcal{F})$ (in the case $r = 2$). If $r = 1$, the pairing $H^2_c(U, \mathcal{F}) \times H^1(U, \mathcal{G}) \to H^3_c(U, \mathbf{G}_m)$ is continuous by Lemma 4.4 and $H^2(\widehat{\mathcal{O}_v}, \mathcal{F}) = 0$ for all $v \in W$ (see for instance [Mi2], Lemma 1.1), hence all maps are continuous in this case. If $r = 2$, then the local pairings $H^1(\widehat{\mathcal{O}_v}, \mathcal{F}) \times H^2_v(\widehat{\mathcal{O}_v}, \mathcal{G}) \to H^3_v(\widehat{\mathcal{O}_v}, \mathbf{G}_m)$ are continuous by [Mi2], Theorem III.7.1. All the other maps are obviously continuous.

This finishes the proof of Lemma 4.7. ◻

We can now prove the following lemma (see [Mi2], Lemma III.8.4):

Lemma 4.8 *Let $V \subset U$ be a non empty open subscheme. Let N be a finite flat commutative group scheme over U. Then Theorem 1.1 holds for N over U if and only if it holds for $N_{|V}$ over V.*

Proof: Proposition 2.1,3., Proposition 2.12, Proposition 3.6 and Lemma 4.7 give a commutative diagram of long exact sequences of topological groups:

$$\begin{array}{ccccccccc}
\dots & \longrightarrow & H^{3-r}_c(V,N) & \longrightarrow & H^{3-r}_c(U,N) & \longrightarrow & \bigoplus_{v\in U\setminus V} H^{3-r}(\widehat{\mathcal{O}_v},N) & \longrightarrow & \dots \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
\dots & \longrightarrow & H^r(V,N^D)^* & \longrightarrow & H^r(U,N^D)^* & \longrightarrow & \bigoplus_{v\in U\setminus V} H^r_v(\widehat{\mathcal{O}_v},N^D)^* & \longrightarrow & \dots,
\end{array}$$

where the vertical maps are defined via the pairings (9) and the local duality pairings of [Mi2], III.7.1. In particular, the maps $H^{3-r}(\widehat{\mathcal{O}_v},N) \to H^r_v(\widehat{\mathcal{O}_v},N^D)^*$ are isomorphisms by [Mi2], Theorem III.7.1. The middle vertical map is strict by Lemma 4.4 and Lemma 3.4. Therefore the five-lemma gives the result. □

The end of the proof of Theorem 1.1 (which implies in particular that by duality the groups $H^r(U,N^D)$ are zero for $r \geq 4$, resp. for $r = 3$ if $U \neq X$) is exactly the same as the end of the proof of Theorem III.8.2 in [Mi2]: let $U \subset X$ be a non empty open subset and N be a finite flat commutative group scheme over U.

- If the order of N is prime to p, then Theorem 1.1 is a consequence of Proposition 2.1, 4. and étale Artin–Verdier duality (Corollary II.3.3 in [Mi2] or Theorem 4.6 in [GS]). Note that it requires to compare the pairing defined in Lemma 4.1 with the Artin–Verdier pairing using Ext groups as defined in [Mi2] or [GS] : this is explained for instance in Proposition 6.2 of the Appendix. Hence by Lemma 4.6, it is sufficient to prove Theorem 1.1 when the order of N is a power of p.

- If the order of N is a power of p, the proof of Lemma 3.5 implies that N admits a composition series such that the generic fiber of each quotient is either of height one or the dual of a group of height one. By Lemma 4.6, it is therefore sufficient to prove Theorem 1.1 in the case N_K or N_K^D have height one.

- If N_K or N_K^D have height one, Proposition B.4 and Corollary B.5 in [Mi2] imply that there exists a non empty open subset $V \subset U$ such that $N_{|V}$ extends to a finite flat commutative group scheme $\widetilde{N}$ over the proper k-curve X, such that $\widetilde{N}$ or $\widetilde{N}^D$ have height one. Using Lemma 4.8 twice, it is enough to prove Theorem 1.1 when $U = X$ and N or N^D have height one.

- Lemma III.8.5 in [Mi2] proves Theorem 1.1 for $U = X$ and N (resp. N^D) of height one, by reduction to the classical Serre duality for vector bundles over the smooth projective curve X. Indeed, Proposition V.1.20 in [Mi1] proves that the pairings $R\Gamma(X,\mathcal{F}^D) \otimes^{\mathbf{L}} R\Gamma(X,\mathcal{F}) \to R\Gamma(X,\mathbf{G}_m)$ defined via Godement resolutions in the proof of Lemma 4.1 are compatible with the classical pairings using Ext groups that appear in Serre duality. □

As observed in [Mi2], §III.8 (remark before Lemma 8.9), the group $H^1(U,N)$ is in general infinite if $U \neq X$ and by duality, the same is true for $H^2_c(U,N)$. However, the situation is better for H^2 and H^1_c :

Corollary 4.9 *Let N be a finite and flat commutative group scheme over a non empty Zariski open subset U of X. Then the groups $H^2(U,N)$ and $H^1_c(U,N)$ are finite.*

Proof: The statement about $H^1_c(U,N)$ is Corollary 3.2. The finiteness of $H^2(U,N)$ follows by the duality Theorem 1.1. □

The previous corollary can be refined in some cases:

Proposition 4.10 *Let N be a finite and flat commutative group scheme over a non empty affine open subset $U \subset X$, such that the generic fiber N_K is local. Then $H^1_c(U,N) = 0$.*

Proof: By the valuative criterion of properness, the restriction map $H^1(U,N) \to H^1(K,N)$ is injective. It is sufficient to show that if we choose $v \notin U$, the restriction map $H^1(K,N) \to H^1(\widehat{K}_v,N)$ is injective when N_K is local. Indeed this implies that $D^1(U,N) = 0$, hence $H^1_c(U,N) = 0$ by exact sequence (6) because $H^0(\widehat{K}_v,N) = 0$ for every completion $\widehat{K}_v$ of K.

We also reduce to showing that for every finite subextension L/K of $\widehat{K}_v/K$, the restriction map $r : H^1(K,N) \to H^1(L,N)$ is injective (indeed a K-torsor under the finite K-group scheme N_K is of finite type over K, hence it has a point over an extension K' of K if and only if it has a point over a finite subextension of K'). To do this, we argue as in [Česl], Lemma 5.7 a). Since by [Rib], section F, Th. 2, L is a separable extension of K, the K-algebra $E := L \otimes_K L$ is reduced. As N_K is finite and connected, the group $N(E)$ is trivial. Let $C^1 := \mathbf{R}_{E/K}(N \times_K E)$ (where $\mathbf{R}$ denotes Weil's restriction of scalars) be the scheme of 1-cochains with respect to L/K, we obtain that $C^1(K)$ is trivial, which in turn implies that $\ker r$ is trivial by [Česl], §5.1. □

Remark 4.11 The finiteness of $H^1_c(U,N)$ (Cor. 3.2) relies on the finiteness of $D^1(U,N)$ proven in [Čes3], Th. 2.9. An alternative argument is actually available. By [Mi2], Lemma III.8.9., we can assume that $U \neq X$, namely that U is affine. By loc. cit., Th. II.3.1. and Prop. 2.1, 4., we can also assume that the order of N is a power of p. Let N_K be the generic fiber of N, it is a finite group scheme over K. By [DG], IV, §3.5, and Prop. 2.1, 2., it is sufficient to prove the required finiteness in the following cases : N_K is étale, N_K is local with étale dual, $N_K = \alpha_p$. The last two cases are taken care of by Prop. 4.10, so we can suppose that N_K is étale. Let $V \subset U$ be a non empty open subset. By Prop. 2.1, we have an exact sequence

$$H^1_c(V,N) \to H^1_c(U,N) \to \bigoplus_{v \in U \setminus V} H^1(\widehat{\mathcal{O}_v},N).$$

Since the generic fiber of N is étale, the group $H^1(\widehat{\mathcal{O}_v},N)$ is finite by [Mi2], Rem. III.7.6. (this follows from the fact that $H^1(\widehat{\mathcal{O}_v},N)$ is a compact subgroup of the discrete group $H^1(K_v,N)$), hence the finiteness of $H^1_c(U,N)$ is equivalent to the finiteness of $H^1_c(V,N)$, which in turn is equivalent to the finiteness of $D^1(V,N)$. The latter holds for V sufficiently small: either apply [Gon], Lemma 4.3. (which relies on an embedding of N_K into an abelian variety) or reduce (as in [Mi2], Lemma III.8.9.) to the case when N^D is of height one. Indeed by loc. cit., Cor. III.B.5., the assumption that N^D is of height one implies that for V sufficiently small, the restriction of N to V extends to a finite and flat commutative group scheme $\widetilde{N}$ over X. Then the finiteness of $H^1_c(X,\widetilde{N})$ implies the finiteness of $H^1_c(V,\widetilde{N}) = H^1_c(V,N)$ by Prop 2.1, 3., because the groups $H^0(\widehat{\mathcal{O}_v},\widetilde{N})$ are finite.

5 The number field case

Assume now that K is a number field and set $X = \operatorname{Spec}\mathcal{O}_K$. Let U be a non empty Zariski open subset of X. Let n be the order of the finite and flat commutative group scheme N. To prove Theorem 1.1 in this case, one follows exactly the same method as in [Mi2], Th. III.3.1. and Cor. III.3.2. once Proposition 2.1 has been proved. Namely Proposition 2.1, 4., shows that on $U[1/n]$, Theorem 1.1 reduces to the étale Artin–Verdier Theorem ([Mi2], II.3.3 or Theorem 4.6 in [GS]). Here we can use a definition of the pairings similar to Lemma 4.1, or a definition via the Ext pairings as in loc. cit. (the two definitions coincide, the argument being the same as in Proposition 6.2 of the Appendix). Now Proposition 2.1, 3., gives a commutative diagram as in the end of the proof of [Mi2], Th. III.3.1. (with completions $\widehat{\mathcal{O}}_v$ instead of henselizations $\mathcal{O}_v$). Theorem 1.1 follows by the five-lemma, using the result on $U[1/n]$ and the local duality statement [Mi2], Th. III.1.3.

Remark 5.1 In the number field case, one can as well (as in [Mi2], §III.3) work from the very beginning with henselizations $\mathcal{O}_v$ and not with completions $\widehat{\mathcal{O}}_v$ to define cohomology with compact support. Indeed the local theorem (loc. cit., Th. III.1.3) still holds with henselian (not necessarily complete) discrete valuation ring with finite residue field when the fraction field is of characteristic zero. Hence the only issue here is commutativity of diagrams. Nevertheless, we felt that it was more convenient to have a uniform statement (Proposition 2.1) in both characteristic 0 and characteristic p situations.

6 Appendix

6.1 Cone and tensor products

Proposition 6.1 *Let $\mathcal{A}$ be the category of fppf sheaves over a scheme T. Let A, B and C be three complexes in $\mathcal{A}$. Let $f : A \to B$ be a morphism of complexes. Then there are commutative diagrams (where $\otimes$ denotes the total tensor product of complexes) such that the vertical maps are isomorphisms of complexes:*

$$\begin{array}{ccccccc}
A\otimes C & \xrightarrow{f\otimes 1} & B\otimes C & \xrightarrow{i\otimes 1} & \operatorname{Cone}(f)\otimes C & \xrightarrow{-\pi\otimes 1} & A[1]\otimes C \\
\downarrow{=} & & \downarrow{=} & & \downarrow{\sim} & & \downarrow{\sim} \\
A\otimes C & \xrightarrow{f\otimes 1} & B\otimes C & \xrightarrow{i'} & \operatorname{Cone}(f\otimes 1) & \xrightarrow{-\pi} & (A\otimes C)[1]\,,
\end{array}$$

where the vertical isomorphisms involve no signs, and

$$\begin{array}{ccccccc}
C\otimes A & \xrightarrow{1\otimes f} & C\otimes B & \xrightarrow{1\otimes i} & C\otimes \operatorname{Cone}(f) & \xrightarrow{-1\otimes\pi} & C\otimes A[1] \\
\downarrow{=} & & \downarrow{=} & & \downarrow{\sim} & & \downarrow{\sim} \\
C\otimes A & \xrightarrow{1\otimes f} & C\otimes B & \xrightarrow{i'} & \operatorname{Cone}(1\otimes f) & \xrightarrow{-\pi} & (C\otimes A)[1]\,,
\end{array}$$

where the two last vertical maps involve a sign $(-1)^r$ on the factor $C_r \otimes A_s$.

Proof: In the first diagram, define the non obvious map $\mathrm{Cone}(f) \otimes C \to \mathrm{Cone}(f \otimes 1)$ (resp. $A[1] \otimes C \to (A \otimes C)[1]$) by the isomorphism $(B_r \oplus A_{r+1}) \otimes C_s \to (B_r \otimes C_s) \oplus (A_{r+1} \otimes C_s)$ (resp. by the identity of $A_{r+1} \otimes C_s$). In the second diagram, the non obvious map $C \otimes \mathrm{Cone}(f) \to \mathrm{Cone}(1 \otimes f)$ (resp. $C \otimes A[1] \to (C \otimes A)[1]$) is given by the isomorphism $C_r \otimes (B_s \oplus A_{s+1}) \to (C_r \otimes B_s) \oplus (C_r \otimes A_{s+1})$ that maps $c \otimes (b, a)$ to $(c \otimes b, (-1)^r c \otimes a)$ (resp. by the automorphism of $C_r \otimes A_{s+1}$ given by $c \otimes a \mapsto (-1)^r c \otimes a$). The proposition is then straightforward. □

6.2 Comparison of two pairings

Let U be a non empty Zariski open subset of a smooth, projective, geometrically integral curve defined over a finite field.

Proposition 6.2 *Let A, B and C be three fppf sheaves on U, endowed with a pairing $A \otimes B \to C$. Then there is a commutative diagram*

$$\begin{array}{ccc} H^r(U, A) \otimes H^s_c(U, B) & \longrightarrow & H^{r+s}_c(U, C) \\ \downarrow & & \downarrow = \\ \mathrm{Ext}^r_U(B, C) \otimes H^s_c(U, B) & \longrightarrow & H^{r+s}_c(U, C), \end{array}$$

where the top pairing is the one from (10) *and the bottom one is the pairing from [Mi2], Proposition III.0.4.e. The same holds for étale sheaves instead of fppf sheaves if we replace fppf cohomology (resp. compact support fppf cohomology) by étale cohomology (resp. compact support étale cohomology); in the étale case the bottom pairing is the one from loc. cit., Proposition II.2.5. (or [GS]).*

Proof: We prove the statement for fppf sheaves (the étale case is similar). Consider the natural morphisms of complexes:

$$\mathrm{Tot}(G(A) \otimes G(B)) \to G(A \otimes B) \to G(C).$$

Using [SP, Tag 0A90], one gets a natural morphism of complexes $G(A) \to \mathcal{H}om^\bullet(G(B), G(C))$ and a commutative diagram of complexes:

$$\begin{array}{ccccc} \mathrm{Tot}(G(A) \otimes G(B)) & \longrightarrow & G(A \otimes B) & \longrightarrow & G(C) \\ \downarrow & & & & \downarrow = \\ \mathrm{Tot}(\mathcal{H}om^\bullet(G(B), G(C)) \otimes G(B)) & & \longrightarrow & & G(C), \end{array}$$

where the second pairing is the natural one. All morphisms in this diagram involve no extra-sign.

Let $G(C) \to I$ be an injective resolution. Then one gets a commutative diagram

$$\begin{array}{ccccc} \mathrm{Tot}(G(A) \otimes G(B)) & \longrightarrow & G(A \otimes B) & \longrightarrow & G(C) \\ \downarrow & & & & \downarrow \\ \mathrm{Tot}(\mathcal{H}om^\bullet(G(B), I) \otimes G(B)) & & \longrightarrow & & I. \end{array}$$

Taking global sections, one gets a commutative diagram:

$$\begin{array}{ccc} \operatorname{Tot}(\Gamma(U,G(A))\otimes\Gamma(U,G(B))) & \longrightarrow & \Gamma(U,G(C)) \\ \downarrow & & \downarrow\wr \\ \operatorname{Tot}(\operatorname{Hom}_U^\bullet(G(B),I)\otimes\Gamma(U,G(B))) & \longrightarrow & \Gamma(U,I)\,. \end{array} \tag{12}$$

Taking cohomology, one gets a commutative diagram comparing the pairing from the beginning of section 4 to the classical Ext-pairing:

$$\begin{array}{ccc} H^r(U,A)\otimes H^s(U,B) & \longrightarrow & H^{r+s}(U,C) \\ \downarrow & & \downarrow = \\ \operatorname{Ext}_U^r(B,C)\otimes H^s(U,B) & \longrightarrow & H^{r+s}(U,C)\,. \end{array}$$

Applying functoriality of cone to (12) and to the similar pairing over completions of K, one gets a commutative diagram of complexes

$$\begin{array}{ccc} \operatorname{Tot}(\Gamma(U,G(A))\otimes\Gamma_c(U,G(B))) & \longrightarrow & \Gamma_c(U,G(C)) \\ \downarrow & & \downarrow\wr \\ \operatorname{Tot}(\operatorname{Hom}_U^\bullet(G(B),I)\otimes\Gamma_c(U,G(B))) & \longrightarrow & \Gamma_c(U,I)\,. \end{array}$$

Taking cohomology, we get the required commutative diagram. □

Remark 6.3

1. A similar diagram holds with compact support cohomology groups on the left and Ext-groups on the right. In this case, one gets a commutative diagram

$$\begin{array}{ccc} H_c^r(U,A)\otimes H^s(U,B) & \longrightarrow & H_c^{r+s}(U,C) \\ \downarrow & & \downarrow = \\ H_c^r(U,A)\otimes \operatorname{Ext}_U^s(A,C) & \longrightarrow & H_c^{r+s}(U,C)\,, \end{array}$$

where the first pairing is the one from Lemma 4.1, while the vertical map and the bottom pairing both involve a $(-1)^{rs}$ sign.

2. Similar commutative diagrams hold over an arbitrary basis, with compact support cohomology replaced by cohomology with support in a closed subscheme (with a similar proof).

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