CUBULATING MALNORMAL AMALGAMS

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Abstract. We examine conditions on a group G splitting as a graph of groups which ensure that G acts properly on a CAT(0) cube complex. The system of conditions include: hyperbolicity of G relative to abelian subgroups, quasiconvexity of the vertex groups, and malnormality and subgroup separability properties of the edge groups. An additional condition involves an extension property for codimension-1 subgroups.

1. Introduction and Main Result

Theorem A (Special case of Main Result)**.** *If G satisfies the following conditions then G acts properly and cocompactly on a CAT(0) cube complex.*

(1) *G splits as $A *_C B$ [or $A*_{C^t}$].*
(2) *G is hyperbolic relative to free-abelian subgroups.*
(3) *A, B [or A] act properly, cocompactly, and virtually specially on CAT(0) cube complexes.*
(4) *C is hyperbolic and C is quasiconvex and malnormal in G.*

The lowest dimension nontrivial case of the main theorem yields the following result:

Corollary B. *Let $G = F_1 *_M F_2$ be an amalgamated product of two finitely generated free groups where M is finitely generated and malnormal. Then G acts properly and cocompactly on a CAT(0) cube complex.*

Corollary C. *Let $G = A *_C B$ [or $G = A*_{C^t}$] be hyperbolic and assume C is infinite cyclic and malnormal. If A, B [or A] act properly and cocompactly on a CAT(0) cube complex then so does G.*

There are many natural classes of relatively hyperbolic groups that can be built up by a sequence of free constructions along cyclic subgroups which are thus cubulated as a consequence of this work. Perhaps the simplest such class, generalizing free groups and surface groups, is the class of limit groups [13, 16].

In [2], Brady and Crisp proved a 2-dimensional rigidity property of a certain group G, and used this to show that most HNN extensions $G*_Z$ along a cyclic subgroup could not act properly and cocompactly on a CAT(0) space. They speculated there that low-dimensional actions were hard to imagine. It follows from [19] that G acts properly and cocompactly on a CAT(0) cube complex. Our results thus imply that their examples $G*_Z$ act properly and cocompactly on CAT(0) cube complexes.

There are several motivations for establishing these results. Firstly, in his original essay on hyperbolicity [5], Gromov raised the possibility of whether every hyperbolic group acts properly and cocompactly on a CAT(0) space. Our results can be interpreted as a very explicit geometrization of a large class of groups. Secondly, nonpositively curved cube complexes serve as a route towards understanding the subgroup structure, residual finiteness properties, and linearity of groups. Together with [8] this work is a fundamental piece in a program to show that many groups arising naturally in geometric group theory are linear and have separable quasiconvex subgroups, because they embed in right-angled Artin groups. For instance, combining [8] and [7] with Corollary B we see that each such group is linear.

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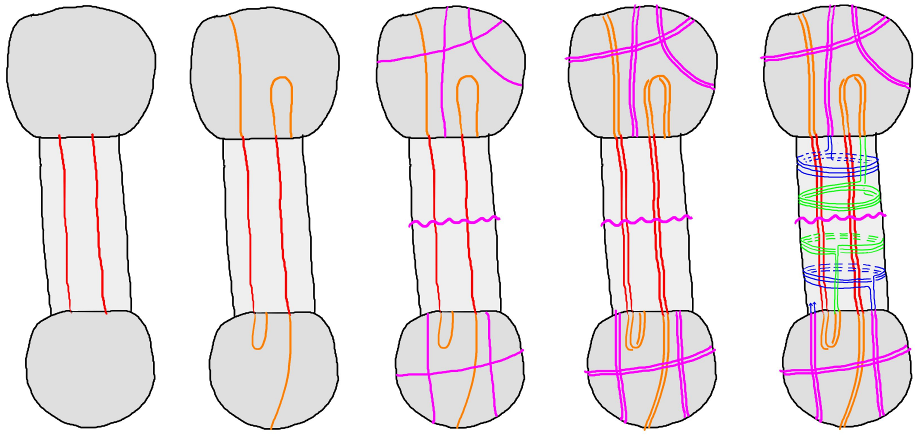


FIGURE 1.1. Producing sufficiently many immersed walls in a graph of spaces for $A *_C B$

There is a growing literature on cubulations of groups and we refer to [11] for a survey. The results markedly generalize our previous results in [12] where we cubulated hyperbolic groups that split as graphs of free groups with cyclic edge groups.

Outline of Paper: In Section 2, we show that piecewise geodesics satisfying a certain divergence properties are quasigeodesics. This can be interpreted as implying that local-geodesics are quasigeodeics – a well-known theme in the hyperbolic case. This enable us to prove that the "walls" we construct later in the paper are quasiisometrically embedded.

In Section 5, we describe a method for "recubulating" a compact nonpositively curved cube complex by amalgamating hyperplanes together along attached collars. This is the most innovative part of the paper and contains a powerful and flexible construction of a new nonpositively curved cube complex $\mathbb{C}'$ from an old nonpositively curved cube complex $\mathbb{C}$ and certain data. In the subsequent application, $\mathbb{C}$ will be the nonpositively curved cube complex corresponding to a vertex group, and the "data" will be prechosen cubulations of one or more edge groups.

In Section 6 we first show how to cubulate a graph of cubulated groups with the property that the vertex groups induce the same system of codimension-1 subgroups on the edge groups. Hyperplanes in the vertex spaces are joined together along the edge spaces, to form walls in the tree of spaces. The main focus here is the linear separation property for these walls.

A suggestive approach towards the main theorem could be: First cubulate A and B so that they induce the same cubulation on C, and then glue the resulting cube complexes together. While it appears the theorem could indeed be proven like this, there are technical reasons to avoid this approach. The main issue is that one would have to redefine in an appropriately coarse way what it means for walls to cross each other, since walls that don't cross in A might cross in B. We have chosen not to follow this route, but to finesse the issue by arranging for compatible wallspaces for A and B, and then using an additional cubulation in a graph of groups. This is done in Section 8 where we prove our main result through a construction combining the results of Sections 5 and 6.

We now heuristically sketch the overall process described in this paper in the case of a graph of spaces $X = X_A \cup X_C \times (-1, 1) \cup X_B$ corresponding to an amalgam $A *_C B$. We describe the collection of immersed walls that are produced in X. Their universal covers provide a system of embedded walls in $\widetilde{X}$, and $\pi_1 X$ acts freely on the resulting dual CAT(0) cube complex. We refer the reader to Figure 1.1. **[1]:** Choose a sufficient collection of immersed walls in X_C and taking their products, we obtain a collection in $X_C \times (-1, 1)$. **[2]:** Extend these immersed walls into X_A and X_B, and note that some of these extensions might return to the edge space. **[3]:** Add a sufficient collection of immersed walls in X_A and X_B as well as a wall $X_C \times \{0\}$ cutting through the edge space. **[4]:** Double the immersed walls constructed thus far. **[5]:** Each immersed wall entering an edge space is "turned" back to its double, by entering and wrapping around the edge space several times.

2. The divergence lemma

2.1. **Background on nonpositively curved spaces with isolated flats.** Let G be hyperbolic relative to a set of virtually abelian subgroups. Suppose G acts properly and cocompactly on the complete CAT(0) space X. Then X has the *isolated flats* property [9, 10], meaning:

(1) there are finitely many *flats* F_i consisting of isometrically embedded copies of $\mathbb{E}^{n_i}$ with $n_i \geq 2$;
(2) there exists $r = r(X)$ such that every flat $F \subset X$ lies in a translated neighborhood $g\mathcal{N}_r(F_i)$ for some i and some $g \in G$;
(3) there is a function $f(s)$ such that for each $s > 0$, we have $\mathcal{N}_s(F_i) \cap \mathcal{N}_s(gF_j)$ has diameter $\leq f(s)$, unless $i = j$ and $g \in \text{Stabilizer}(F_i)$.

Let G act properly and cocompactly on the CAT(0) space with isolated flats X. The following relative thin triangle property is a consequence of [4, Sec 8.1.3].

Proposition 2.1. *There exists $\delta = \delta(X)$ such that the following holds: Let $\Delta(a, b, c)$ be a geodesic triangle in X. Then either Δ is δ-thin or there exists a flat F in X such that Δ is thin relative to F in the sense that there are numbers $r(a), r(b), r(c)$ such that pairs of points at distance $s < r(a)$ from a along $\overline{ab}$ and $\overline{ac}$ lie at distance $\leq \delta$ from each other, and similarly for the b and c corners, and all other points lie in $\mathcal{N}_\delta(F)$.*

Lemma 2.2. *Let X be a CAT(0) space with isolated flats such that geodesic triangles are δ-thin relative to flats. Suppose that for distinct flat representatives F_i, F_j we have $\mathcal{N}_\delta(F_i) \cap F_j$ has diameter $\leq M$.*

Let $\gamma_k = \lambda_1\sigma_2\lambda_2 \cdots \sigma_k\lambda_k$ be a piecewise geodesic. Suppose there exists L with the following property:

(1) *$|\lambda_i| > 4L + 4M$ for each i;*
(2) *$\lambda_i \cap \mathcal{N}_{2\delta}(F)$ has diameter $\leq M$ for each flat F;*
(3) *λ_i, λ_{i+1} subpaths cannot 3δ fellowtravel for a distance of $\geq L$;*
(4) *Consecutive paths $\lambda_i\sigma_{i+1}$ and $\sigma_i\lambda_i$ cannot 2δ-fellowtravel for a distance of $\geq L$.*

Then

(1) *$|\gamma_{k+1}| \geq |\gamma_k| + |\sigma_{k+1}| + |\lambda_{k+1}| - [6L + 6M + 6\delta]$;*
(2) *The terminal subpaths of γ_k, λ_k δ-fellowtravel for a distance of at least $|\lambda_k| - [2L + 2M] > 2L + 2M$.*

Proof. The proof is by induction, where the base case $\gamma_1 = \lambda_1$ is obvious. Let ω_k denote the geodesic from the initial point of γ_k to the endpoint of σ_k. Consider the two geodesic triangles formed by $\gamma_k, \omega_k, \gamma_{k-1}, \lambda_k, \sigma_k$, and consider as well the tail of λ_{k-1} which δ fellowtravels with the tail of γ_{k-1}. Each such geodesic triangle is δ-thin relative to a flat, and its three tails are indicated, and the three complementary segments are δ close to some flat.

There are six possible cases according to the positions of the tails on ω_k, as illustrated in Figures 2.1–2.6. Consideration of the various segments whose lengths are bounded by L, M, δ, because of our hypotheses, as well as copious use of the triangle inequality, allows us to bound the difference between $|\gamma_{k-1}| + |\sigma_k| + |\lambda_k|$ and $|\gamma_k|$. In each picture, the maximum loss can be obtained by summing all letters that appear, so in cases 1,2,3, there is a loss of at most $4L+4M+6\delta$; in cases 4,5, at most $4L+6M+6\delta$; and in case 6, at most $6L + 6M + 6\delta$. Moreover, the tail ends of λ_k and γ_k δ-fellowtravel for all of λ_k, except the initial part of length $L+M$ in cases $1, 2, 3$; length $L+2M$ in cases $4, 5$; and length $2L+2M$ in case 6. Thus statement (2) holds as well. □

Theorem 2.3. *Let X be a CAT(0) space with isolated flats where geodesic triangles are δ-thin relative to flats. Suppose that for distinct flat representatives F_i, F_j we have $\mathcal{N}_\delta(F_i) \cap F_j$ has diameter $\leq M$.*

Let γ be a geodesic with the same endpoints as the piecewise geodesic path $\sigma_1\lambda_1 \ldots \lambda_k\sigma_{k+1}$, and suppose that:

(1) *$\frac{1}{2}|\lambda_i| \geq (6L + 6M + 6\delta)$ for each i.*

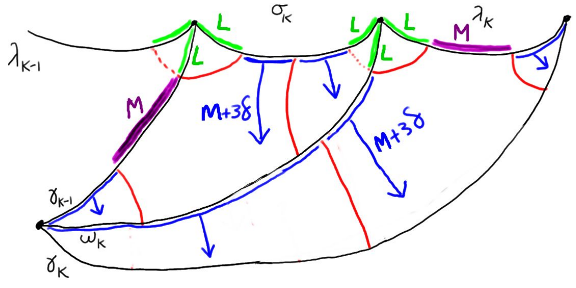


FIGURE 2.1.

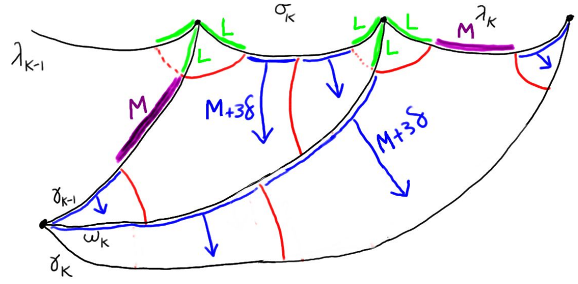
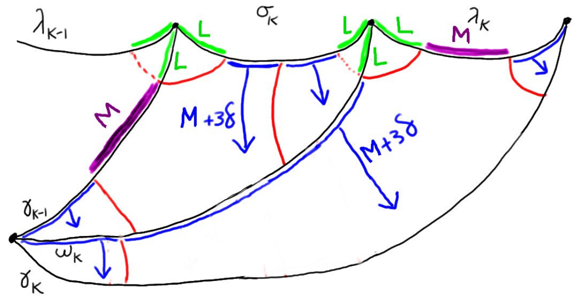


FIGURE 2.2.



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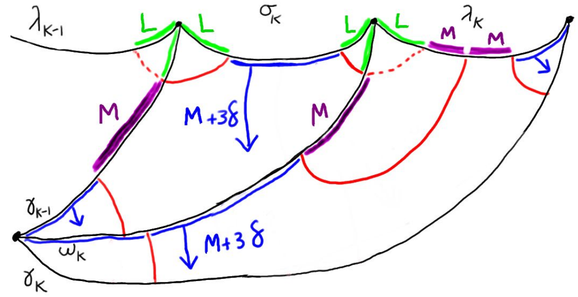

FIGURE 2.4.

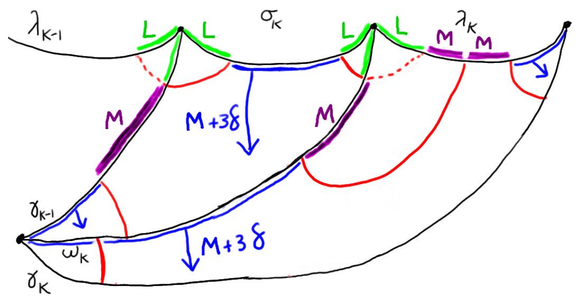

FIGURE 2.5.

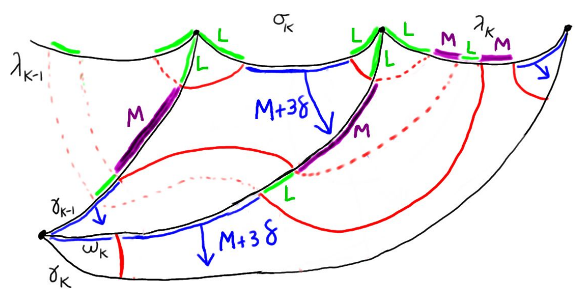

FIGURE 2.6.

(2) *$\lambda_i \cap \mathcal{N}_{3\delta}(F)$ has diameter $\leq M$ for each flat F.*
(3) *λ_i, λ_{i+1} do not have length L subpaths that 3δ-fellowtravel.*
(4) *Consecutive paths $\lambda_i\sigma_{i+1}$ and $\sigma_i\lambda_i$ cannot 3δ-fellowtravel for a distance of $\geq L$.*

Then $|\gamma| \geq \frac{1}{2}\big(\sum_1^k |\lambda_i| + \sum_1^{k+1} |\sigma_i|\big)$.

Proof. Let $\gamma_k = \lambda_1\sigma_2\lambda_2\cdots\sigma_k\lambda_k$, and consider the quadrilateral bounded by $\gamma, \sigma_1, \gamma_k, \sigma_k$, subdivided into two triangles by the addition of a geodesic from the initial point of γ to the terminal point of γ_k. These two triangles are relatively δ-thin. Five of the six cases to consider are illustrated in Figure 2.7. Note that by Lemma 2.2, γ_k δ-fellowtravels λ_k for a distance of greater than $2L+2M$, and so various segments have length bounded by L and M, as shown. But then we see that $|\gamma_k| \leq 2L + 2M$ in all five cases, which makes them impossible. In the sixth case, illustrated in Figure 2.8, we find that

$$|\gamma| \geq |\sigma_1| + |\gamma_k| + |\sigma_{k+1}| - (4L + 2(M + 3\delta) + 2M).$$

By Lemma 2.2, $|\gamma_k| \geq \sum_1^k |\lambda_i| + \sum_2^k |\sigma_i| - (k-1)[6L + 6M + 6\delta]$. Consequently,

$$|\gamma| \geq \sum_1^k |\lambda_i| + \sum_1^{k+1} |\sigma_i| - (k)[6L + 6M + 6\delta].$$

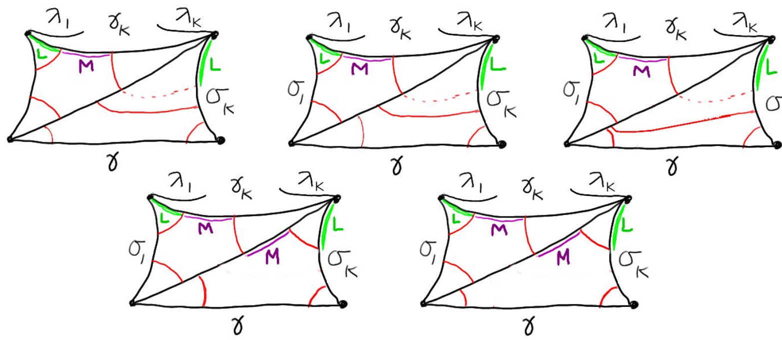

FIGURE 2.7.

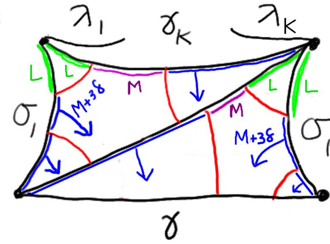

FIGURE 2.8.

Now our hypothesis that $\frac{1}{2}|\lambda_i| \geq (6L + 6M + 6\delta)$ implies that:

$$|\gamma| \geq \sum_1^k |\lambda_i| + \sum_0^{k+1} |\sigma_i| - k(6L + 6M + 6\delta) \geq \sum_1^k [|\lambda_i| - (6L + 6M + 6\delta)] + \sum_0^{k+1} |\sigma_i| \geq \frac{1}{2}[\sum_1^k |\lambda_i| + \sum_0^{k+1} |\sigma_i|]. \qquad \square$$

3. BACKGROUND ON CUBULATING WALLSPACES

This section provides a quick review of results and definitions around Sageev's construction [14]. We will need the following material on the *CAT(0) cube complex of a codimension-1 subgroup*. For proofs, examples, and context, and further references see [11].

Definition 3.1. A wall in X is a connected subspace Λ together with an associated pair of connected subspaces $\{\overleftarrow{\Lambda}, \overrightarrow{\Lambda}\}$ such that $X = \overleftarrow{\Lambda} \cup \overrightarrow{\Lambda}$ and $\Lambda = \overleftarrow{\Lambda} \cap \overrightarrow{\Lambda}$. The subspaces $\overleftarrow{\Lambda} - \Lambda$ and $\overrightarrow{\Lambda} - \Lambda$ are the *open halfspaces* associated to Λ. We will often refer to the wall Λ without mentioning its halfspaces. In particular, if $X - \Lambda$ has exactly two components $\overleftarrow{\Lambda} - \Lambda$ and $\overrightarrow{\Lambda} - \Lambda$, we say that Λ is a *geometric wall*. All of our walls will be geometric until Section 7.

A *wall system for* $\widetilde{X}$ is a set of walls in $\widetilde{X}$. A wall system $\mathcal{W}$ is *locally finite* if every compact subset of $\widetilde{X}$ intersects only finitely many walls of $\mathcal{W}$. Finally, if $\mathcal{W}$ is a locally finite wall system for $\widetilde{X}$, we say that $(\widetilde{X}, \mathcal{W})$ is a *wallspace*.

Definition 3.2. Let $(\widetilde{X}, \mathcal{W})$ be a wallspace. We say $\Lambda \in \mathcal{W}$ *separates* $a, b \in \widetilde{X}$ if a and b lie in distinct open halfspaces of Λ, and for $a, b \in \widetilde{X}$, we let $\#(a, b)$ denote the number of walls in $\mathcal{W}$ that separate a, b. We say $\mathcal{W}$ has the *linear separation* property if there is a constant $K > 0$ such that for all $a, b \in X^0$, we have $\#(a, b) \geq \left\lfloor \frac{1}{k} d(a, b) \right\rfloor$.

Definition 3.3. A group G acts *cosparsely* on a CAT(0) cube complex $\widetilde{C}$ if there is a compact subcomplex $K \subset \widetilde{C}$ and finitely many "quasiflats" (spaces quasi-isometric to Euclidean flats) $\widetilde{F}_i$ in $\widetilde{C}$ such that $\widetilde{C} = GK \cup \left(\bigcup_i G\widetilde{F}_i\right)$, and $g\widetilde{F}_i \cap h\widetilde{F}_i \subset GK$ for $g, h \in G$, unless $i = j$ and $g^{-1}h \in \text{Stabilizer}(\widetilde{F}_i)$.

Proposition 3.4 (Sparse cubulation)**.** *Suppose $(\widetilde{X}, \mathcal{W})$ is a wallspace, $\widetilde{X}$ is a geodesic metric space, G is hyperbolic relative to virtually abelian subgroups, and G acts properly and cocompactly by isometries on X, preserving $\mathcal{W}$. There is a CAT(0) cube complex $\mathbb{C}(\widetilde{X})$, called the* cube complex dual to $\mathcal{W}$*, such that:*

(1) *If every $W \in \mathcal{W}$ is quasi-isometrically embedded in $\widetilde{X}$, then G acts cosparsely on $\mathbb{C}(\widetilde{X})$. Moreover, if in addition, G is actually hyperbolic, then G acts cocompactly on $\mathbb{C}(\widetilde{X})$.*
(2) *If $(\widetilde{X}, \mathcal{W})$ satisfies the linear separation property, then G acts metrically properly on $\mathbb{C}(\widetilde{X})$.*

Proof. This holds by [11, Thm 6.17]. $\square$

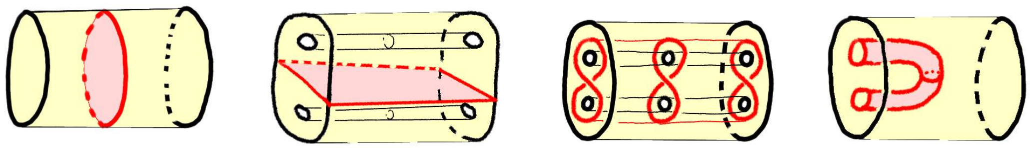

FIGURE 4.1.

A subgroup A of a relatively hyperbolic group G is *full* if $P \cap A$ is either finite or finite index in P for each parabolic subgroup P of G. The following is obtained in [6, 15]:

Proposition 3.5 (Sparse core)**.** *Let G be hyperbolic relative to f.g. virtually abelian subgroups and suppose G acts properly and cosparsely on a cube complex C. Let H be a quasi-isometrically embedded subgroup of G. There is a H-cosparse convex subcomplex $D \to C$. Moreover, if H is full, and in particular, if G is hyperbolic, we can assume D is H-compact.*

Proposition 3.6 (Convex cocompact cores)**.** *Let G act properly and cosparsely on the CAT(0) cube complex X. Then there is a convex subspace $Y \subset X$ such that G stabilizes and acts cocompactly on Y.*

Proof. This follows from [11, Cor. 7.4]. □

4. THE NEW IMMERSED WALLS

Given a CAT(0) cube complex $\widetilde{A}$ and a collection of convex subcomplexes $\widetilde{C}_i \hookrightarrow \widetilde{A}$, we form the "collared space" $\widetilde{A}^+$ corresponding to their mapping cylinder (see Figure 4.3). In this section, we construct immersed walls in $\widetilde{A}^+$ by combining hyperplanes of $\widetilde{A}$ with certain codimension-1 subspaces of the "collars" $\widetilde{C}_i \times I$. To this end, after discussing codimension-1 subspaces in Section 4.1, we define "turns" in Section 4.2 and then construct the immersed walls in $\widetilde{A}^+$ in Section 4.3. In Section 5, we show that under favorable circumstances, we may choose turns to obtain sufficiently many genuine walls in $\widetilde{A}^+$.

4.1. **Immersed codimension-1 subspaces.** In this subsection we define certain *(immersed) codimension-1 subspaces* $\phi : Y \hookrightarrow X$ which are (locally) injective maps having the property that Y (locally) separates X in the sense that for each $y \in Y$, there is a neighborhood U of y such that there is a commutative diagram:

$$\begin{array}{ccc} U & \to & Y \\ \downarrow & & \phi\downarrow \\ U \times I & \to & X \end{array}$$

where $I = [-1, 1]$, the top map is the inclusion, the map $U \to U \times I$ is defined by $u \mapsto (u, 0)$, and the bottom map is a topological embedding. We are especially interested in the cases where $Y \to X$ is an embedding or is globally 2-sided.

For instance, $C \times \{0\}$ is a "vertical" codimension-1 subspace of $C \times I$. If B is a codimension-1 subspace of C, then $B \times I$ is a "horizontal" codimension-1 subspace of $C \times I$.

Definition 4.1 (Turn)**.** Let $\hat{C} \to C$ be a connected covering space, and let $B \to \hat{C}$ be an embedded connected 2-sided codimension-1 subspace of $\hat{C}$. We choose $R' \subset \hat{C}$ such that $R' \cong B \times [-2, 2]$ where the commutative diagram below holds. Let $R \subset R'$ correspond to $B \times [-1, 1]$ and let $B_+ = B \times \{+1\}$ and $B_- = B \times \{-1\}$ be the components of ∂R. Note that $B_\pm$ are themselves codimension-1 subspaces of $\hat{C}$.

$$\begin{array}{ccc} B & \to & \hat{C} \\ \downarrow & & \downarrow \\ B \times \{0\} & \to & B \times [-2, 2] \end{array}$$

The *turn* associated to $(\hat{C}, B)$ is the following codimension-1 subspace $D \subset \hat{C} \times I$:

$$D = \Big((\hat{C} \times \{0\}) - (R \times \{0\})\Big) \cup (B_- \times [-1, 0]) \cup (B_+ \times [-1, 0])$$

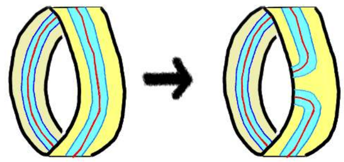

FIGURE 4.2.

The parts $B_{\pm} \times [-1, 0]$ of a turn D are called the "beginning" and "end" of D, and it will often be more natural to consider them as part of the hyperplanes to which they extend in $\widetilde{A}$. The primary part of D is $(\hat{C} \times \{0\})$, so we will refer to paths within $(\hat{C} \times \{0\})$ as *turn segments*. We adopt similar language in the universal cover.

4.2. **High radius turns.**

Definition 4.2. The *radius* of a turn associated to $(\widehat{C}, B)$ is the infimum of lengths of turn segments starting and ending on $B_{\pm} \times [-1, 0]$ that are not path homotopic into $B_{\pm} \times [-1, 0]$.

The following lemma will be used to help produce turns with large radius within the following framework: Let D be an immersed hyperplane in a nonpositively curved cube complex A. Let $C \to A$ be a local isometry with C compact, and let B be an immersed hyperplane of C that maps to D.

Lemma 4.3. *Suppose $\pi_1 C$ is $\pi_1 B$-separable. For each $r \geq 0$, there is a finite cover $\widehat{C} \to C$ such that B lifts to an embedding in $\widehat{C}$, and such that the turn associated to $(\widehat{C}, B)$ has the property that any length $\leq r$ local geodesic turn segment λ with endpoints on $B_{\pm} \times [-1, 0]$ is path-homotopic into $B_{\pm} \times [-1, 0]$.*

Proof. A path with endpoints on $B_{\pm} \times [-1, 0]$ corresponds canonically to a path with endpoints on B by precomposing and postcomposing with a very short path in R. The path homotopy property is preserved mutatis mutandis. Compactness of C and hence B guarantees that for each r there are finitely many double cosets $\pi_1 B g \pi_1 B$ in $\pi_1 C$ with representatives of length $\leq r$. Now, a straightforward use of separability allows us to separate $\pi_1 B$ from the finitely many homotopy classes represented by paths of length $\leq r$. □

4.3. **New walls in $\widetilde{A^+}$.** Let A be a nonpositively curved cube complex. Let $\phi_i : C_i \to A$ be continuous maps such that $\widetilde{\phi}_i : \widetilde{C}_i \to \widetilde{A}$ is an embedding for each i, and consider the mapping cylinder:

$$A^+ = A \cup_i (C_i \times [0, 1]) \, / \, (c_i, 1) \sim \phi_i(c_i) : c_i \in C_i.$$

For each i, we identify a fixed lift of $\widetilde{C}_i$ with its $\widetilde{\phi}_i$ image in $\widetilde{A}$. We also identify $\widetilde{C}_i$ with $\widetilde{C}_i \times \{1\} \subset \widetilde{C}_i \times [0, 1]$.

Remark 4.4. On the level of the base space, we endow A^+ with immersed walls, by extending immersed hyperplanes of A to the $C_i \times [0, 1]$. There are prechosen immersed walls in each $C_i \times [0, 1]$ that arise from turns in $\hat{C}_i$. The process creates new immersed walls by joining various immersed codimension-1 subspaces in the $C_i \times [0, 1]$ together with various immersed hyperplanes in A.

We assume that each $C_i \times [0, 1]$ has a prechosen collection of immersed walls (arising from turns in some $\hat{C} \times I$) that lift to embedded walls $\{W_j\}$ in $\widetilde{C}_i \times [0, 1]$ with the following *matching property*: For hyperplanes $\widetilde{D} \subset \widetilde{A}$, the intersection $\widetilde{D} \cap \widetilde{\phi}_i(\widetilde{C}_i)$ equals a component of $W_j \cap (\widetilde{C}_i \times \{1\})$ for a unique W_j.

This property is an *elementary equivalence* that generates an equivalence relation among the set consisting of the union of all hyperplanes in $\widetilde{A}$ together with the set of walls in the translates of the $\widetilde{C}_i \times [0, 1]$ by $\pi_1 A$. For each equivalence class, we define a *new wall* W in $\widetilde{A}^+$ to consist of the union of all hyperplanes and walls in the equivalence class amalgamated along the intersections between hyperplanes and walls that have an elementary equivalence.

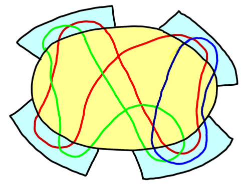

FIGURE 4.3. New walls in a collared space

For each new wall W there is a map $W \to \widetilde{A^+}$, but without further hypotheses this map is unlikely to be an embedding. However, since it is locally codimension-1, and $\widetilde{A^+}$ is simply-connected, we see that W will be a wall if it embeds.

Subdivision: To facilitate the construction, it will be convenient to pass to the first cubical subdivision of the cube complexes in the graph of spaces. This has the effect of replacing each hyperplane with a pair of parallel hyperplanes so that the hyperplanes incident with an edge space C now come in pairs, each corresponding to a single original hyperplane.

Remark 4.5 (Handling Torsion)**.** The construction of new walls in $\widetilde{A^+}$ generalizes as follows: Let $\widetilde{A}$ be a wallspace with a G-equivariant system of walls, and for each i, let $\widetilde{C_i} \times [0,1]$ be a wallspace with a G_i-equivariant system of walls for some $G_i \subset G$. Let $\widetilde{\phi_i} : \widetilde{C_i} \to \widetilde{A}$ be a G_i-equivariant embedding. The space $\widetilde{A^+}$ is the union of $\widetilde{A}$ together with the G-translates of mapping cylinders of the $\widetilde{\phi_i}$. Finally, we assume the walls of $\widetilde{A}$ and $\widetilde{C_i} \times [0,1]$ match.

The construction of turns using an intermediate cover $\widehat{C_i}$ is facilitated by the assumption that each G_i has a finite index torsion-free subgroup G_i', for then we can define $C_i = G_i' \backslash \widetilde{C_i}$, and proceed as before (using the additional separability properties of B).

4.4. **New walls coming from turns.** We now assume that the horizontal walls in $C_i \times I$ are sufficient, in the sense that $\pi_1 C_i$ has linear separation with respect to horizontal walls in $\widetilde{C_i} \times I$. It follows that $\pi_1 C_i$ acts properly on the corresponding dual cube complex.

Moreover, we assume that each non-horizontal wall of $\widetilde{C_i} \times I$ is a high radius turn associated to some $(\widehat{C_i} \to C_i, B)$, where B was an original hyperplane (i.e., before subdividing). The two boundary components $B_\pm$ of the regular neighborhood of B correspond to new hyperplanes of $\widehat{C_i}$, and hence, $\widetilde{C_i}$ (i.e., after subdividing).

Note that each new wall is a nonpositively curved cube complex that is a graph of CAT(0) cube complexes glued together along convex subcomplexes. Thus each new wall is a geodesic metric space.

5. RECUBULATING GROUPS

Lemma 5.1 (Malnormal Divergence)**.** *Let G be hyperbolic relative to abelian subgroups. Let $\{H_i\}$ be a finite collection of quasi-isometrically embedded subgroups of G. Suppose $\{H_i\}$ is an* almost malnormal *collection in the sense that $H_i \cap H_j^g$ is finite unless $i = j$ and $g \in H_i$. Then there exists a function f such that for each r, we have $\mathcal{N}_r(H_i) \cap gH_j$ has diameter $< f(r)$ unless $i = j$ and $g \in H_j$.*

Sketch. An infinite diameter overlap would yield an infinite intersection of corresponding conjugates. □

Theorem 5.2. *Let A be a compact nonpositively curved cube complex such that $\pi_1 A$ is hyperbolic relative to virtually abelian subgroups.*

Let $\phi_i : C_i \to A$ be a collection of maps such that $\{\pi_1 C_i\}$ is almost malnormal, where each C_i is compact, $\pi_1 C_i$ is word-hyperbolic, and $\widetilde{\phi_i} : \widetilde{C_i} \to \widetilde{A}$ is a quasi-isometric embedding.

Suppose each $\widetilde{C_i} \times I$ is a wallspace, each of whose walls is either a horizontal wall (i.e., genuine hyperplane) or a turn. Suppose the walls of $\widetilde{A}$ and $\widetilde{C_i} \times [0,1]$ match along $\widetilde{C_i} \times \{1\}$. Suppose the horizontal walls are sufficient in the sense that $\pi_1 C_i$ acts properly on the associated dual cube complex. Suppose each turn has sufficiently large radius.

Let $\widetilde{A}^+ = \widetilde{A} \cup \bigcup_{i,g} g\widetilde{C}_i \times I$ *with new walls defined as in the construction of Section 4.3.*
Then the new walls in $\widetilde{A^+}$*:*

1. *quasi-isometrically embed*
2. *embed*
3. *are sufficient, in the sense that the linear separation property holds with respect to* G.

Remark 5.3 (Torsion)**.** Theorem 5.2 holds in the following context: G acts properly and cocompactly on a CAT(0) cube complex $\widetilde{A}$ whose walls are hyperplanes and where each $\widetilde{C}_i$ is a convex subspace whose stabilizer G_i acts cocompactly on it. In this case, the virtual torsion-freeness of each G_i together with separability of hyperplane stabilizers will be used to apply Lemma 4.3. Indeed, letting G'_i denote a finite index subgroup of G_i, we let $C_i = G'_i\backslash\widetilde{C}_i$ and proceed from there using the separability of hyperplanes.

Remark 5.4 (Sparseness)**.** Theorem 5.2 will be applicable when G is hyperbolic relative to virtually abelian subgroups and G acts properly on a CAT(0) cube complex with finitely many orbits of hyperplanes each of which has a quasi-isometrically embedded stabilizer. Indeed, in this case, G acts properly and cosparsely on a CAT(0) cube complex $\widetilde{X}$ (possibly different from the original action). Moreover, by Proposition 3.6, $\widetilde{X}$ contains a convex G-invariant cocompact subspace $\widetilde{A}$.

For a finite collection of convex subcomplexes $\widetilde{D}_i$ in $\widetilde{X}$ whose stabilizers are quasi-isometrically embedded, $\widetilde{A}$ can be chosen so that $\widetilde{C}_i = \widetilde{A} \cap \widetilde{D}_i$ have the same stabilizers, but they now act cocompactly.

Finally, the hyperplanes of $\widetilde{X}$ determine hyperplanes of $\widetilde{A}$, and in a suitable sense $\widetilde{A}$ can be treated as if it were a cocompact cube cube complex for our purposes. One minor discrepancy is that the walls in $\widetilde{A}^*$ are only CAT(0) spaces instead of cubical complexes. For the application of Lemma 5.9 in the proof of Theorem 5.2, one notes that $\widetilde{C}$ lies in $\widetilde{A}$ by construction, and hence $p\widetilde{C}$ lies in $\widetilde{A}$, and thus lies in the "hyperplane" $H \cap \widetilde{A}$.

Proof of Theorem 5.2. Letting G_i denote the stabilizer of $\widetilde{C}_i$ in G we have: $\widetilde{A}^+ = \widetilde{A} \cup \bigcup_{i,g\in G/G_i} g\widetilde{C}_i \times [0,1]$. New walls are unions of hyperplanes of $\widetilde{A}$ together with turns and horizontal walls in copies of $g\widetilde{C}_i \times [0,1]$, and we shall examine the map from a new wall to $\widetilde{A^+}$ without presupposing that it embeds.

Consider a geodesic path in a new wall expressed as an alternating concatenation $\sigma_1\lambda_1\sigma_2\cdots\lambda_k\sigma_{k+1}$, where each hyperplane segment σ_i lies in a hyperplane H_i of $\widetilde{A^+}$ used to build the new wall and each turn segment λ_i lies in a turn in $\widetilde{C}_i \times [0,1]$ that starts on H_i and ends on H_{i+1}. (Note that horizontal walls in $\widetilde{C}_i \times [0,1]$ are parts of hyperplanes of $\widetilde{A^+}$.)

By choosing the decomposition of this path so that k is minimal, we can assume that $H_i \neq H_{i+1}$, for otherwise λ_i "backtracks", in the sense that it is path-homotopic into $H_i = H_{i+1}$ and we may reduce k by replacing $\sigma_i\lambda_i\sigma_{i+1}$ with a single geodesic σ'_i lying in $H_i = H_{i+1}$. Since each λ_i does not backtrack, Lemma 4.3 allows us to construct the turns so that each λ_i is long enough to satisfy the hypothesis of Theorem 2.3.

Claim: λ_i and λ_{i+1} must lie in distinct collars $\widetilde{C}_i$ and $\widetilde{C}_{i+1}$. Indeed, if $\widetilde{C}_i = \widetilde{C}_{i+1}$, then since the turn begins at $\widetilde{C}_i \cap H_{i+1} = \widetilde{C}_{i+1} \cap H_{i+1}$, we see that $\lambda_i\sigma_{i+1}\lambda_{i+1}$ can be replaced with a single turn segment λ'_i with $|\lambda'_i| \le |\lambda_i\sigma_{i+1}\lambda_{i+1}|$.

There is thus an upper bound K_1 on $\operatorname{diam}\big(\mathcal{N}_\delta(\lambda_i) \cap \lambda_{i+1}\big)$ since Lemma 5.1 and the almost malnormality of $\{\pi_1 C_j\}$ imply an upper bound K_1 on the diameters of $\mathcal{N}_\delta(\widetilde{C}_j) \cap g\widetilde{C}_k$ when $j \neq k$ or $j = k, g \notin \pi_1 C_j$.

There is an upper bound $K_2 = \frac{1}{2} + \frac{9}{2}\delta$ on the 3δ-fellowtraveling between a hyperplane segment and a turn segment σ_i, λ_i or λ_i, σ_{i+1}. To see this, let $\sigma = ab$ and $\lambda = bc$, so λ and σ meet at the point b. Let p be a point on σ that is 3δ-close to its comparison point p' on λ. See Figure 5.1. Let $\overline{p}$ denote the projection of p onto $\widetilde{C}$. Since H is a hyperplane, and $\widetilde{C}$ is convex, by Lemma 5.9, we see that the geodesic $p\overline{p}$ lies in H. Of course $d(p', \overline{p}) \le d(p', p) + d(p, \overline{p}) \le 2d(p, p') \le 6\delta$.

Let $\sigma' = a\overline{p}$ and let $\lambda' = \overline{p}c$. If λ, σ would 3δ-fellowtravel for $\ge K_2$ then these replacements for λ, σ will yield a shorter minimal path. Indeed, the following two equations hold by the triangle inequality:

$$|\sigma'| = d(a, \overline{p}) \le d(a, p) + d(p, \overline{p}) = |\sigma| - d(p, b) + d(p, \overline{p})$$
$$|\lambda'| = d(c, \overline{p}) \le d(c, p') + d(p', \overline{p}) = |\lambda| - d(p', b) + d(p', \overline{p})$$

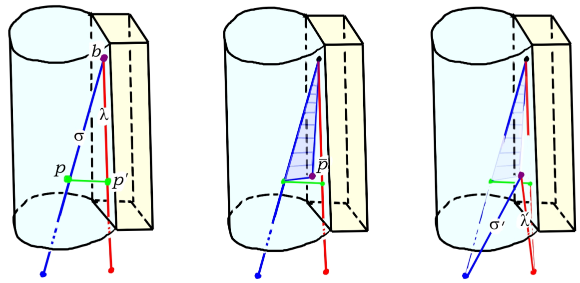

FIGURE 5.1.

Since $d(p, b) = d(p', b)$ and $d(p, \overline{p}) \leq d(p, p')$ and $d(p', \overline{p}) \leq 2d(p', p)$ we have:

$$|\lambda'| + |\sigma'| \leq |\lambda| + |\sigma| - 2d(p, b) + 3d(p, p').$$

So the λ', σ' replacement shows that minimality is violated when $d(p, b) \geq K_2 = \frac{1}{2} + \frac{9}{2}\delta \geq \frac{1}{2} + \frac{3}{2}d(p, p')$.

By Lemma 5.1, the almost malnormality of $\{\pi_1 C_i\}$ implies that there is an upper bound K_1 on the diameters of $\mathcal{N}_\delta(\widetilde{C}_i) \cap g\widetilde{C}_j$ when $i \neq j$ or $i = j, g \notin \pi_1 C_i$. By Lemma 5.1, there is an M upper bound on the diameter of a 3δ overlap between a flat, and a path λ in a turn.

The three conclusions of the theorem are proven in Theorem 5.5, 5.6, and 5.12. □

We now use that each new wall W is a nonpositively curved cube complex, and we let d_W be the associated metric. Note that the inclusion $W \subset \widetilde{A^+}$ is an embedding of nonpositively curved cube complexes (after we subdivide $\widetilde{A^+}$) but W is not a convex subcomplex. We now use the geodesic metric d_W to understand the geometry of $W \to \widetilde{A^+}$.

Theorem 5.5 (Quasi-isometry). *Let $R_o = 2(6L + 6M + 6\delta)$. If each turn in $\widetilde{A}^+$ has radius $\geq R_o$, then $d_{\widetilde{A}}(\psi(p), \psi(q)) \geq \frac{1}{2}d_W(p, q) - 4R_o$ and the map $\psi : W \to \widetilde{A^+}$ is a $(\frac{1}{2}, 4R_o)$-quasi-isometric embedding.*

Proof. Since each turn has radius $\geq R_o$, each $|\lambda_i| \geq R_o$, and by Theorem 2.3, the distance between the endpoints of $\sigma_1\lambda_1 \cdots \lambda_k\sigma_{k+1}$ in $\widetilde{A}$ is $\geq \frac{1}{2}(\sum |\sigma_i| + |\lambda_i|)$. In particular, this holds when σ_1 or σ_{k+1} is trivial.

Next, consider a path $\lambda_1'\sigma_2 \cdots \sigma_k\lambda_k'$ between p, q. If both $|\lambda_1'| > R_o$ and $|\lambda_k'| > R_o$, Theorem 2.3 still applies, and $d_{\widetilde{A}}(\psi(p), \psi(q)) \geq \frac{1}{2}d_W(p, q)$. If both $|\lambda_1'| \leq R_o$ and $|\lambda_2'| \leq R_o$, then the endpoints $\psi(p), \psi(q)$ of the image path are within R_o of the endpoints $\psi(p'), \psi(q')$ of the image subpath $\sigma_2\lambda_2 \cdots \sigma_{k-1}$, and

$$\begin{aligned} d_{\widetilde{A}}(\psi(p), \psi(q)) &\geq d_{\widetilde{A}}(\psi(p'), \psi(q')) - 2R_o \geq \frac{1}{2}(|\sigma_2| + |\lambda_2| + \cdots + |\lambda_{k-1}| + |\sigma_k|) - 2R_o \\ &\geq \frac{1}{2}d_W(p, q) - |\lambda_1'| - |\lambda_k'| - 2R_o \geq \frac{1}{2}d_W(p, q) - 4R_o. \end{aligned}$$

If only one of $|\lambda_1'|, |\lambda_k'| \leq R_o$, a stronger statement holds for similar reasons. Finally, by the definition of d_W and the triangle inequality, $d_W(p, q) \geq d_{\widetilde{A}}(\psi(p), \psi(q))$. □

Theorem 5.6 (Embed). *If all turns have radius $R > 8R_o$ then the new walls embed in $\widetilde{A}^+$.*

Proof. A geodesic γ in W from p to q decomposes as a concatenation of hyperplane segments and turn segments, and $|\gamma|$ is the sum of their lengths, so if γ contains a complete turn segment, then $|\gamma| > 8R_o$, and by Theorem 5.5, $d_{\widetilde{A}}(\psi(p), \psi(q)) \geq \frac{1}{2}d_W(p, q) - 4R_o > 0$. It therfore suffices to consider the case where γ contains no complete turn segments.

The remaining cases of γ have the form λ_1, σ_1, $\sigma_1\lambda_1$, $\lambda_1\sigma_2$, and $\lambda_1\sigma_1\lambda_2$. Individual hyperplanes and turns of $\widetilde{A}^+$ clearly embed. The intersection of a hyperplane and a turn is a convex subset, and so the "concatenation" of a hyperplane segment and a turn segment embeds. Finally, a path of the form $\lambda_1\sigma_1\lambda_2$ embeds by the Claim in the proof of Theorem 5.2. □

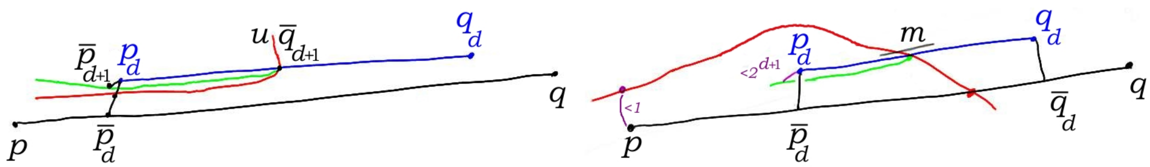

FIGURE 5.2.

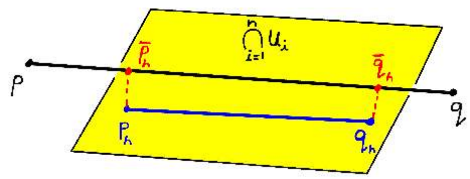

FIGURE 5.3.

5.1. **Preliminaries to support linear separation.** We employ the following statements in the proof of the linear separation property for the recubulation. By [10, Prop 4.1.6] we have:

Proposition 5.7 (Fellow Travelling Relative to Flats)**.** *Let X be a cocompact CAT(0) space with isolated flats. For each ϵ, there exists $\mu = \mu(X, \epsilon)$ with the following property: Let ω be an ϵ-quasigeodesic and let γ be a geodesic with the same endpoints. Then there exist flats $F_1, \ldots, F_m$ such that ω is a concatenation $\omega_1\omega_1'\omega_2\omega_2' \cdots \omega_m\omega_m'$ and γ is a concatenation $\gamma_1\gamma_1'\gamma_2\gamma_2' \cdots \gamma_m\gamma_m'$, and for each i, we have ω_i, γ_i lie in a μ neighborhood of each other, and ω_i', γ_i' lie in a μ neighborhood of F_i.*

The *angle* between a geodesic γ and a hyperplane Λ in a CAT(0) cube complex, is the CAT(0) angle [3] between γ and $\overline{\gamma}$ where $\overline{\gamma}$ is the projection of γ on Λ. It follows from the CAT(0) inequality, that a lower bound on the angle θ between Λ and γ gives a lower bound on the divergence between rays in the directions of Λ and γ. Indeed, the divergence is bounded by that of rays in $\mathbb{E}^2$ meeting at θ.

We refer to [15] for a proof of the following:

Lemma 5.8. *Let X be a finite-dimensional CAT(0) cube complex. There exists a constant $M = M(dim(X))$ and a real number $\theta = \theta(dim(X)) > 0$ such that for any length M geodesic γ, there exists a hyperplane Λ crossed by γ such that $\sphericalangle(\Lambda, \gamma) \geq \theta$.*

Lemma 5.10 will support the proof of linear separation. Note that Lemma 5.8 affirms (both the hypothesis and) the conclusion of Lemma 5.10 when U_i is the full set of hyperplanes. We first require the following easily verified property of hyperplanes in CAT(0) cube complexes.

Lemma 5.9. *Let H be a hyperplane and let B be a convex subcomplex of the CAT(0) cube complex $\widetilde{X}$. Suppose that $H \cap B \neq \emptyset$. Let $p \in H$ and let pB denote the geodesic from p to B. Then pB lies in H.* □

We will also need the following monstrous variation of Lemma 5.8.

Lemma 5.10. *Let X be a finite dimensional CAT(0) cube complex. Suppose there is a collection $\mathcal{U}$ of hyperplanes and a number $M > 0$ such that for any length M geodesic γ in X, there exists $U \in \mathcal{U}$ such that γ crosses U. Then there exists L such that, for each geodesic γ of length at least L and endpoints p, q, there exists $U \in \mathcal{U}$ such that γ crosses U and $p, q \notin \mathcal{N}_1(U)$.*

Proof. Let $N = \dim(X)$. We define a sequence of constants $L_N, L_{N-1}, \ldots, L_2, L_1, L_0$ recursively by declaring that $L_N = 1$ and $L_n = 2L_{n+1} + 4^{n+1} + M$ for $0 \leq n < N$.

Let γ be a geodesic with endpoints p, q such that $|\gamma| \geq L = L_0$, and consider the following condition indexed by n ($0 \leq n \leq N$), as illustrated in Figure 5.3.

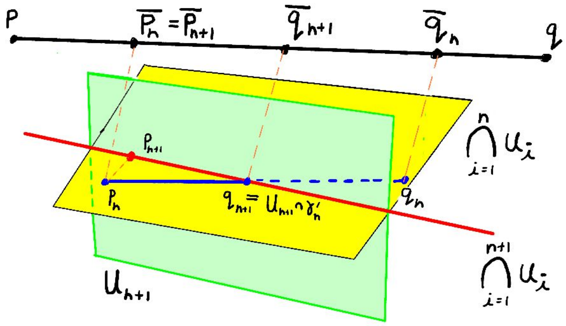

FIGURE 5.4.

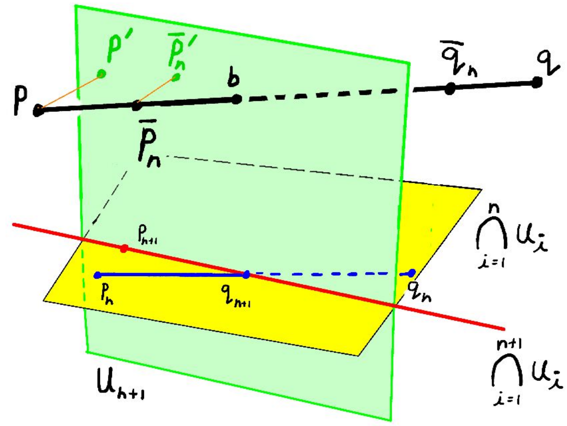

FIGURE 5.5.

Condition(n)**:** There exist hyperplanes $U_1, \ldots, U_n \in \mathcal{U}$, a geodesic γ_n with endpoints p_n, q_n, and points $\overline{p_n}, \overline{q_n}$ on γ such that:

(1) $\cap_{i=1}^{n} U_i$ is a codimension-n hyperplane; (a codimension-0 hyperplane is X itself)
(2) γ_n lies inside $\cap_{i=1}^{n} U_i$;
(3) $d(p_n, \overline{p_n}) \leq 3^n$ and $d(q_n, \overline{q_n}) \leq 3^n$;
(4) $|\gamma_n| \geq L_n$.

Let n be the largest integer such that Condition(n) holds. Note that Condition(0) holds by taking $\gamma_0 = \gamma$, $p_0 = \overline{p_0} = p$, and $q_0 = \overline{q_0} = q$, and note that Condition(N) cannot hold, as a geodesic of length at least $L_N = 1$ cannot be contained in a codimension-N hyperplane (a point). Therefore, $0 \leq n < N$.

Let c denote the midpoint of γ_n and, since $L_n \geq M$, consider the length M subgeodesic γ_n' centered at c. By hypothesis, there is a hyperplane U_{n+1} crossing γ_n'. Note that since U_{n+1} crosses a nontrivial geodesic in $\cap_{i=1}^{n} U_i$, the hyperplane $\cap_{i=1}^{n+1} U_i$ is codimension-$(n+1)$. We will show that U_{n+1} crosses γ separating p, q and that $p, q \notin \mathcal{N}_1(U_{n+1})$.

We first show that U_{n+1} crosses γ in the subgeodesic $\overline{p_n}\,\overline{q_n}$. This follows by showing that it crosses neither the geodesic $p_n\overline{p_n}$ nor $q_n\overline{q_n}$. We exclude the first case, as the arguments are identical. Suppose U_{n+1} crosses $p_n\overline{p_n}$. This will contradict the maximality of n by a construction illustrated in Figure 5.4. Observe that p_n lies at a distance at most 3^n from a point in U_{n+1}, which means that if we let p_{n+1} be the projection of p_n to U_{n+1}, then $d(p_{n+1}, p_n) \leq 3^n$. Furthermore, Lemma 5.9 implies that the geodesic $p_n p_{n+1}$ lies in $\cap_{i=1}^{n} U_i$ so p_{n+1} lies in $\cap_{i=1}^{n+1} U_i$. If we then let $\overline{p_{n+1}} = \overline{p_n}$, by Condition($n$) we see that:

$$d(p_{n+1}, \overline{p_{n+1}}) = d(p_{n+1}, \overline{p_n}) \leq d(p_{n+1}, p_n) + d(p_n, \overline{p_n}) \leq 3^n + 3^n < 3^{n+1}.$$

Letting $q_{n+1} = U_{n+1} \cap \gamma_n'$, and noting that q_{n+1} lies within 3^n from a point $\overline{q_{n+1}}$ in γ by the CAT(0) inequality, we now claim that the hyperplanes $U_1, \ldots, U_n, U_{n+1}$, the geodesic $\gamma_{n+1} = p_{n+1}q_{n+1}$, and the points $\overline{p_{n+1}}, \overline{q_{n+1}}$ (Figure 5.4) satisfy Condition($n+1$), contradicting the maximality of n. Indeed, the last thing to check is that $|\gamma_{n+1}| \geq L_{n+1}$, but

$$\begin{aligned} |\gamma_{n+1}| = d(p_{n+1}, q_{n+1}) &\geq d(p_n, c) - d(c, q_{n+1}) - d(p_{n+1}, p_n) \\ &\geq \frac{1}{2}L_n - \frac{1}{2}M - 3^n \geq \frac{1}{2}([2L_{n+1} + 4^{n+1} + M] - M) - 3^n \geq L_{n+1}. \end{aligned}$$

We now show that $p, q \notin \mathcal{N}_1(U_{n+1})$ by means of another construction that would violate the maximality of n (Figure 5.4). Since the argument for q is identical, we will focus on showing that $p \notin \mathcal{N}_1(U_{n+1})$. Let $b = \overline{p_n}\,\overline{q_n} \cap U_{n+1}$, and suppose there exists a point p' in U_{n+1} with $d(p, p') \leq 1$. Since $\overline{p_n}$ lies on the geodesic

pb and the geodesic $p'b$ lies in U_{n+1}, applying the CAT(0) inequality, we see that there is a point $\overline{p_n}'$ in U_{n+1} such that $d(\overline{p_n}, \overline{p_n}') < 1$.

Applying Lemma 5.9 again, as above, we see that the projection p_{n+1} of p_n in U_{n+1} lies in $\cap_{i=1}^{n+1} U_i$. Furthermore, as p_{n+1} is the point of U_{n+1} closest to p_n, we see that $d(p_n, p_{n+1}) \le d(p_n, \overline{p_n}') \le d(p_n, \overline{p_n}) + d(\overline{p_n}, \overline{p_n}') \le 3^n + 1$, by Condition($n$). Let $\overline{p_{n+1}} = \overline{p_n}$, and applying Condition(n) again we have:

$$d(p_{n+1}, \overline{p_{n+1}}) = d(p_{n+1}, \overline{p_n}) \le d(p_{n+1}, p_n) + d(p_n, \overline{p_n}) \le (3^n + 1) + 3^n \le 3^{n+1}.$$

Finally, letting $q_{n+1} = U_{n+1} \cap \gamma'_n$, and noting that q_{n+1} lies within 3^n from a point $\overline{q_{n+1}}$ in γ by the CAT(0) inequality, we again claim that the hyperplanes $U_1, \ldots, U_n, U_{n+1}$, the geodesic $\gamma_{n+1} = p_{n+1}q_{n+1}$, and the points $\overline{p_{n+1}}, \overline{q_{n+1}}$ (Figure 5.5) satisfy Condition($n+1$), contradicting the maximality of n. Again, the last thing to check is that $|\gamma_{n+1}| \ge L_{n+1}$, but

$$\begin{aligned} |\gamma_{n+1}| &= d(p_{n+1}, q_{n+1}) \ge d(p_n, c) - d(c, q_{n+1}) - d(p_{n+1}, p_n) \\ &\ge \frac{1}{2}L_n - \frac{1}{2}M - (3^n + 1) \ge \frac{1}{2}([2L_{n+1} + 4^{n+1} + M] - M) - 3^n - 1 \ge L_{n+1}. \end{aligned}$$

□

5.2. **Linear Separation.**

Lemma 5.11. *Let γ be a geodesic and let $\omega = \sigma_1\lambda_1\sigma_2\lambda_2\ldots\sigma_m$ be an ϵ-quasigeodesic with the same endpoints such that each σ_i and λ_i is a geodesic. By Proposition 5.7, γ and ω must μ-fellowtravel relative to flats; let V be a uniform upper bound on* $\operatorname{diam}(\lambda_i \cap \mathcal{N}_\mu(F))$ *for any flat F, and suppose each $|\lambda_i| > V$. Then there exists κ such that γ and ω lie in κ-neighborhoods of each other.*

Proof. Let $\gamma = \gamma_1\gamma'_1\gamma_2\gamma'_2\cdots$ and $\omega = \omega_1\omega'_1\omega_2\omega'_2\cdots$ be the decompositions from Proposition 5.7. Note that we may assume that no λ_j contains an entire ω'_i. Indeed, if $\omega'_i \subseteq \lambda_j$, CAT(0) convexity implies that the entire geodesic ω'_i lies within μ of γ, as its endpoints are within μ of γ. We may then shorten the decomposition by treating $\gamma_i\gamma'_i\gamma_{i+1}$ and $\omega_i\omega'_i\omega_{i+1}$ as single corresponding subpaths within μ of each other.

Each λ_j is thus a subpath of a $\omega'_{i-1}\omega_i\omega'_i$ that intersects ω_i (since $|\lambda_j| > V$). The part of λ_j that is contained in each of ω'_{i-1} and ω'_i is bounded by V, so each λ_j lies in $\mathcal{N}_{\mu+V}(\gamma)$. Then, since each σ_j begins and ends with a λ_i segment or endpoint, CAT(0) convexity implies that $\sigma_j \subset \mathcal{N}_{\mu+V}(\gamma)$. □

Let $\#(p,q)$ denote the number of walls in $\widetilde{A}^+$ separating p and q.

Theorem 5.12 (Linear Separation)**.** *If all turns are sufficiently large, there exist K_1, K_2 s.t. $\#(p,q) \ge K_1 d_{\widetilde{A}^+}(p,q) - K_2$ for all $p, q \in \widetilde{A}^+$.*

Proof. Since $\widetilde{A}^+ \subseteq \mathcal{N}_1(\widetilde{A})$, it suffices to prove the theorem for $p, q \in \widetilde{A}$. We will do so by verifying the following claim: There exists a constant J such that for each length J subsegment γ' of a geodesic γ in $\widetilde{A}$, there exists a hyperplane Λ of $\widetilde{A}$ such that Λ separates the endpoints of γ' and Λ extends to a new wall Λ^+ that crosses γ at a single point p of γ'.

Let M be the constant from Lemma 5.8. If $J \ge M + 2S$ (where S is declared below), we have a hyperplane Λ_1 crossed by γ' at a point p_1 with $\sphericalangle(\Lambda_1, \gamma) \ge \theta$. Moreover we can choose Λ_1 such that p_1 is within the central length M part of γ'. Let Λ_1^+ denote the wall of $\widetilde{A}^+$ containing Λ_1.

If $\gamma \cap \Lambda_1^+ = \{p_1\}$ then we are done. So let us suppose that $\gamma \cap \Lambda_1^+$ contains a second point q_1. Let ω_1 be a geodesic in Λ_1^+ between p_1, q_1. As usual, we express ω_1 as $\sigma_1\lambda_1\cdots$, where σ_1 is a geodesic hyperplane segment contained in Λ_1, and λ_1 is a geodesic turn segment in a collar $\widetilde{C}_1$, and so forth.

Note that ω_1 cannot equal σ_1 since $\sphericalangle \ge \theta > 0$. By Theorem 5.5, ω_1 is a $(\frac{1}{2}, 4R_o)$-quasigeodesic in $\widetilde{A}$. By Proposition 5.7, ω_1 and the geodesic p_1q_1 must μ-fellowtravel relative to flats. By cocompactness and aparabolicity, let V be a uniform upper bound on $\operatorname{diam}(\widetilde{C}_i \cap \mathcal{N}_\mu(F))$ for any flat F; we assume all turns have radius $> V$. By Lemma 5.11, the endpoints of λ_1 lie within κ of the endpoints s_1, t_1 of a subpath γ_1 of γ. Since it is the initial point of λ_1, the terminal point of σ_1 is within κ of γ_1, and hence, since $\sphericalangle \ge \theta$, we find that $|\sigma_1|$ is uniformly bounded, and therefore, the initial point s_1 of γ_1 is a uniformly bounded distance ξ_1 from p_1. We refer the reader to Figure 5.6.

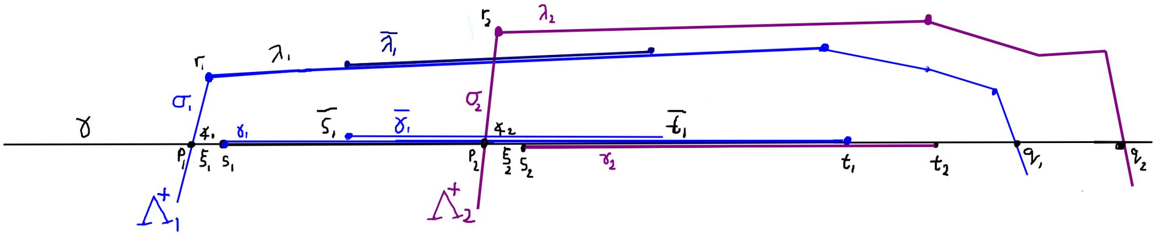

Figure 5.6.

Assume all turns have radius $\geq 2\kappa L + 2S$ (where L is the constant from Lemma 5.10, $S = [Q + \xi_2 + 2\kappa]$ and Q, ξ_2 are identified below). Let $\overline{\lambda}_1$ denote the length $2\kappa L$ part of λ_1 appearing after the initial length S part. Let $\overline{\lambda}_1$ denote the subsegment of γ_1 whose endpoints $\overline{s}_1, \overline{t}_1$ are within κ of the endpoints s_1, t_1 of $\overline{\lambda}_1$. Note that $d(s_1, \overline{s}_1) \geq Q + \xi_2$ and $d(t_1, \overline{t}_1) \geq Q + \xi_2$.

Consider the finitely many G-orbits of collars attached along convex subcomplexes $\widetilde{C}_i$. By hypothesis, each has a sufficient collection of horizontal hyperplanes to separate geodesics, and each is finite dimensional. By Lemma 5.10, for any geodesic of length $\geq L$ in $\widetilde{C}_i$, there is a horizontal hyperplane cutting through this geodesic but not coming within 1 of either of its endpoints. From the CAT(0) inequality, we see that for any geodesic of length $2\kappa L$ there exists a horizontal hyperplane passing through it such that neither endpoint of the geodesic comes within 2κ of this hyperplane. We thus have a horizontal hyperplane Λ_2 of $\widetilde{C}_1$ that crosses $\overline{\lambda}_1$ but does not come within 2κ of its endpoints. Observe that Λ_2 must also cross γ_1 at a point p_2 in $\overline{\gamma}_1$, as it is impossible to "slip out" since the distances between the endpoints of $\overline{\gamma}_1$ and $\overline{\lambda}_1$ are bounded by κ. We note that p_2 is within κ of a point s_2 of $\overline{\lambda}_1$. Since $\overline{\lambda}_1$ and $\overline{\gamma}_1$ have length $\leq 2\kappa L + 2\kappa$, and κ-fellow travel, there is a uniform lower bound θ_2 on the angle between Λ_2 and γ at p_2.

The hyperplane Λ_2 extends to a wall Λ_2^+ that only crosses γ at p_2. Indeed, if Λ_2^+ crosses γ at a second point q_2, then as before, let $\omega_2 = \sigma_2\lambda_2 \cdots$ be a geodesic in Λ_2^+ from p_2 to q_2, and note that ω_2 must μ-fellowtravel γ relative to flats. We show this is impossible. By Lemma 5.11, the endpoints of λ_2 lie within κ of a subpath γ_2 of γ. The initial point s_2 of γ_2 lies within κ of the endpoint r_2 of σ_2, and the lower bound θ_2 on $\angle(\sigma_2, \gamma)$ ensures that $d(p_2, r_2)$ is uniformly bounded, and hence $d(p_2, s_2)$ is bounded by a uniform constant ξ_2. By Lemma 5.1, almost malnormality implies that there is a uniform bound $Q = Q(2\kappa)$ on diam $(\mathcal{N}_\kappa(\widetilde{C}_1) \cap \mathcal{N}_\kappa(\widetilde{C}_2))$. However, since $\gamma_1 \cap \gamma_2$ lies within κ of both $\widetilde{C}_1$ and $\widetilde{C}_2$, and since $d(s_2, t_2) \geq 2\kappa L + 2[Q + \xi_2 + 2\kappa] - 2\kappa \geq Q$, we see that diam$(\gamma_1 \cap \gamma_2) \geq Q$, which is impossible.

Since $d(p_1, p_2) \leq \xi_1 + S + 2\kappa$, the result follows for $J = M + 2[\xi_1 + S + 2\kappa]$. □

The following will be used later in the proof of the main theorem, but it is of independent interest.

Theorem 5.13. *Let $\complement(\widetilde{A}^+)$ denote the cube complex dual to the system of walls in $\widetilde{A}^+$. The action of G on $\complement$ is metrically proper. Furthermore, $\complement$ is cocompact when G is hyperbolic, and cosparse when G is hyperbolic relative to abelian subgroups.*

Proof. This follows from Proposition 3.4, since G is relatively hyperbolic and acts cocompactly, the system of walls is locally finite, the walls are quasi-isometrically embedded by Theorem 5.5, and the linear separation property holds by Theorem 5.12. □

5.3. **No Backtracking.** We will need the following lemma in the proof of Theorem 8.1.

Lemma 5.14 (No Backtracking into Collars)**.** *If each turn has radius* $> \max(8R_o, Q)$*, where* $Q = Q(\kappa)$ *bounds the diameter of the κ-overlap between distinct collars, then the following holds:*

Let W be a wall of $\widetilde{A}^+$, and let $\widetilde{C} \times [0,1]$ be a collar. Then $W \cap (\widetilde{C} \times [0,1])$ is either empty, or consists of a single horizontal hyperplane of $\widetilde{C} \times [0,1]$, or consists of a single turn of $\widetilde{C} \times [0,1]$.

Proof. If this is false, then there is a wall W of $\widetilde{A}^+$ containing two hyperplanes H_1, H_{m+1} that intersect $\widetilde{C} \times \{0\}$ but are not the beginning and end of a turn of $\widetilde{C} \times [0,1]$ in W.

Let ω denote a geodesic in W that starts on $H_1 \cap (\widetilde{C} \times \{0\})$ and ends on $H_{m+1} \cap (\widetilde{C} \times \{0\})$. As usual, we express $\omega = \sigma_1 \lambda_1 \cdots \lambda_m \sigma_{m+1}$ as an alternating concatenation of hyperplane segments and turn segments, where σ_1, σ_{m+1} lie in H_1, H_{m+1}. We also assume that m is minimal.

Let γ be a geodesic in $\widetilde{A}$ with the same endpoints as ω and note that γ lies in the convex subspace $\widetilde{C} \times \{0\}$. By Lemma 5.11, the paths γ and ω must κ-fellow travel in $\widetilde{A}^+$. Since the turn radii exceed the diameter of κ-overlap of distinct collars, we see that each λ_i actually lies in $\widetilde{C} \times 0$. Therefore, as claimed in the proof of Theorem 5.2, any subpath $\lambda_1 \sigma_2 \lambda_2$ could be subsumed by a single turn segment λ', contradicting the minimality of m. We conclude that $\omega = \sigma_1 \lambda_1 \sigma_2$, which is contained in a single turn. □

6. A wallspace structure on a tree of geometric wallspaces

We now create and examine a wallspace structure on a tree of wallspaces satisfying certain properties.

Definition 6.1 (Embedding of geometric wallspace)**.** We say $\phi : A \to B$ is an *embedding of geometric wallspaces* if ϕ is a topological embedding, and there is a chosen inclusion $\Phi : \mathcal{W}_A \to \mathcal{W}_B$ such that $\phi^{-1}(\Phi(W)) = W$ for each $W \in \mathcal{W}_A$. When there are group actions involved, we insist that the relevant maps are equivariant. For simplicity we assume that the walls of A are the nonempty intersections $A \cap W$ where $W \in \mathcal{W}_B$, and that distinct walls of B cannot intersect A in the same nonempty subset, and that $A - W$ consists of exactly two components and these map to the corresponding two components of $B - \Phi(W)$.

Definition 6.2 (Tree of geometric wallspaces)**.** Let T be a directed tree, with a geometric wallspace X_v associated to each vertex, and two embeddings of geometric wallspaces $\phi_{-e} : X_e \to X_{\iota(v)}$ and $\phi_{+e} : X_e \to X_{\tau(v)}$ associated to an edge e directed from $\iota(e)$ to $\tau(e)$. The associated *tree of wallspaces* is the union

$$X = \bigcup X_v \,\cup\, \bigcup (X_e \times [-1,1]) \;/\; \{(x, \pm 1) \sim \phi_{\pm e}(x)\}.$$

We will be interested in the setting where there is a G action on X and on T, and the natural map $X \to T$ is G-equivariant, and that for each v the stabilizer G_v acts on the wallspace X_v, and likewise G_e acts on the wallspace X_e, and the attaching maps $\phi_{\pm e}$ are G_e equivariant.

Definition 6.3 (The walls of X)**.** There are two types of walls in a tree of wallspaces X. Each edge e of T is dual to a wall $X_e \times \{0\}$. The other walls are unions of a collection of walls in vertex spaces and walls in $X_e \times [-1,1]$ of the form $W \times [-1,1]$ where $W \in \mathcal{W}_{X_e}$. Declare a wall $W \times [-1,1]$ in $X_e \times [-1,1]$ to be *simply equivalent* to the image walls of W in $X_{\iota(e)}$ and $X_{\tau(e)}$. This generates an equivalence relation on walls in the spaces X_v and $X_e \times [-1,1]$. We define a *wall of* X to be the union of the subspaces corresponding to walls in an equivalence class.

New-walls embed in X since by hypothesis each is embedded within each vertex space and they do not "backtrack" because each $\Phi : \mathcal{W}_{X_e} \to \mathcal{W}_{X_v}$ is a monomorphism. New-walls are 2-sided in X since they are locally 2-sided (in vertex and edge spaces) and globally embed. The collection of walls in X is locally finite because of the local finiteness of walls in vertex spaces and that finitely many vertex spaces intersect any finite ball in X.

We supplement the condition in Proposition 3.4.(2) for properness of the G action on $\mathbb{C}(\widetilde{X})$ with Lemma 6.4, which is a variant of a properness criterion examined in [11].

Let g be an automorphism of a wallspace X. A wall W *cuts* the element g provided that there is a g-invariant subspace $\mathbb{R}_g$ that is a copy of $\mathbb{R}$ such that $g^n W \cap \mathbb{R}_g = \{n\}$ for each $n \in \mathbb{Z}$.

Lemma 6.4. *Suppose each torsion subgroup of G is finite. If each infinite order element $g \in G$ is cut by a wall $W \in \mathcal{W}$, then G acts with torsion-stabilizers on the cube complex C dual to $(X, \mathcal{W})$.*

Walls cutting elements exist when X is a CAT(0) cube complex whose walls are the hyperplanes:

Lemma 6.5. *Let g be a fixed-point free automorphism of a finite dimensional CAT(0) cube complex X. Then g is cut by some hyperplane.*

Proof. Let $\mathbb{R}_g$ be a geodesic axis for g. Then each hyperplane H passing through $\mathbb{R}_g$ provides a cut, since if H intersects $\mathbb{R}_g$ in more than one point, then $\mathbb{R}_g$ is not a geodesic. □

The following is used to show that new walls are quasiconvex.

Proposition 6.6 (Quasiconvexity Criterion [1])**.** *Let G be a relatively hyperbolic group that splits as a finite graph of groups with each edge group almost malnormal and quasiconvex. Let T be the Bass-Serre tree. Let H be a f.g. subgroup with finitely many H-orbits of nontrivial H-stabilizers in T, and such that each of these is a quasiconvex subgroup of its vertex group. Then H is quasiconvex in G.*

The following target will guide us in arranging the proof of our main theorem.

Theorem 6.7. *Suppose X is a tree of geometric wallspaces with a cocompact G-action satisfying:*

(1.1) *Walls in vertex spaces do not backtrack into edge spaces;*
(1.2) *The stabilizer G_v of each vertex space X_v acts properly on its dual cube complex $\mathsf{C}(X_v)$.*
(2.1) *G is hyperbolic relative to f.g. virtually abelian subgroups;*
(2.2) *Each edge group G_e is almost malnormal in G;*
(2.3) *Each edge group G_e is quasiconvex in G;*
(2.4) *Each vertex space X_v is a wallspace with κ-quasiconvex walls.*

Then X has an induced G-invariant wallspace structure with the following properties:

(1) *Every infinite order element $g \in G$ is cut by a wall of X;*
(2) *Each wall of X is κ'-quasiisometrically embedded;*
(3) *And so G acts properly and cosparsely on the cube complex $\mathsf{C}(X)$ dual to this wallspace.*

Proof. Claim (3) follows from (2.1) and Claims (1) and (2) by Lemma 6.4 and Theorem 3.4.

To prove Claim (1), first note that g is either hyperbolic or elliptic with respect to the tree of spaces. In the hyperbolic case, a horizontal wall dual to an edge of the underlying tree of X cuts g. In the elliptic case, a vertex space X_v is stabilized by g. Since g has infinite order and G_v acts properly on X_v, Lemma 6.5 ensures that X_v has a wall W_v that cuts g, and hence W_v extends to a new wall W that still cuts v.

Each new wall is quasi-isometrically embedded in X by Proposition 6.6. (Only this last point requires relative hyperbolicity.) □

The following construction will facilitate building our tree of wallspaces in the proof of Theorem 8.1:

Definition 6.8 (Hybrids)**.** Let $\phi : C \to D$ be a J-equivariant map, and form the *hybrid space*:

$$\overrightarrow{CD} = C \times [-1,0] \sqcup D \times [0,1] \ / \ \{(c,0) \sim (\phi(c),0) : c \in C\}$$

Note that $\overrightarrow{CD}$ has an induced J-action, and that it is J-cocompact if C, D are both J-cocompact, and the action on $\overrightarrow{CD}$ is proper if the actions on the factors are proper. (The motivating case arises from the universal cover of the analogous space formed from a homotopy equivalence $\overline{C} \to \overline{D}$ where $J = \pi_1\overline{C}$, but our more general framework allows torsion.)

Suppose there are geometric walls $U \subset C$ and $V \subset D$ that are stabilized by a subgroup $H \subset J$, and let $N = \mathcal{N}_r(V)$ be a neighborhood so that $\phi(U) \subset N$. The *hybrid wall* associated to this data is the subspace:

$$\overrightarrow{UV} = U \times [-1,0] \ \cup \ N \times \{0\} \ \cup \ V \times [0,1].$$

See Figure 6.1 for an example of a hybrid space with a single illustrated hybrid wall. Assume that the positive and negative sides (i.e. halfspaces) $U^\pm$ of U in C have been declared, and likewise the sides $V^\pm$ of V in D, and the sides $N^\pm$ of N in D have been declared. Moreover assume these declarations are compatible in the sense that $N^\pm \subset V^\pm$ and $\phi^{-1}(N^\pm) \subset U^\pm$. There are then induced positive and negative sides of $U \times [-1,0)$ in

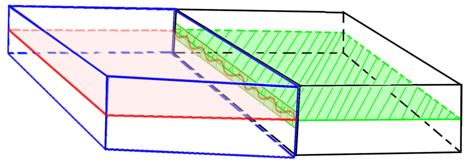

FIGURE 6.1. The wall $\overrightarrow{UV}$ within a hybrid $\overrightarrow{CD}$

$C \times [-1, 0)$ and of $V \times [0, -1]$ in $D \times [0, 1]$. We accordingly declare the positive and negative sides of $\overrightarrow{UV}$ in $\overrightarrow{CD}$ to be $U^{\pm} \times [-1, 0] \ \cup \ N^{\pm} \times \{0\} \ \cup \ V^{\pm} \times [0, 1]$.

Using that the complements of U, N, V consist of exactly two path components, one verifies that $\overrightarrow{UV}$ has the same property. We imagine a map $\overrightarrow{CD} \to \mathbb{R}$ such that $\overrightarrow{UV}$ maps to 0 and its positive and negative sides in $\overrightarrow{CD}$ are path connected and map to the positive and negative real numbers.

Assuming that C and D are geometric wallspaces with proper cocompact J-actions and a one-to-one correspondence between geometric walls (with agreeing stabilizers), we make $\overrightarrow{CD}$ into a wallspace with a J-action by equivariantly producing a hybrid wall $\overrightarrow{UV}$ for each associated pair of walls U, V.

7. The extension property

Definition 7.1. An A-*wall* in a group B is a finite neighborhood $\mathcal{N}_r(A)$ of A in $\Gamma = \Gamma(B)$ and a decomposition of Γ as the union of two A-invariant subspaces $\Gamma = \overleftarrow{A} \cup \overrightarrow{A}$ with $\overleftarrow{A} \cap \overrightarrow{A} = \mathcal{N}_r(A)$. An A-wall in B *extends* to an H-wall in G if $A = B \cap H$ and $\overleftarrow{A} \subset \mathcal{N}_s(\overleftarrow{H})$ and $\overrightarrow{A} \subset \mathcal{N}_s(\overrightarrow{H})$ for some s. We use the term *quasiconvex A-wall* to additionally indicate that A is a quasiconvex subgroup. G has the *extension property for quasiconvex walls* if it has a finite index subgroup G' such that for any quasiconvex $A \subset G'$, each A-wall in a quasiconvex subgroup B extends to a quasiconvex H-wall in G.

For instance, when G acts properly on a CAT(0) cube complex, each infinite cyclic subgroup H of G has the extension property. If G is hyperbolic and contains a finite index subgroup G' that is the fundamental group of a compact special cube complex, then G has the quasiconvex wall extension property. The following slightly more general statement is obtained in [18]:

Proposition 7.2. *Let G act cosparsely on a CAT(0) cube complex X, and suppose there is a finite index subgroup G' such that $G'\backslash X$ is special. Let H be a quasiconvex subgroup of G, and let $A \subset H$ be a quasiconvex full subgroup of G'. Then any A-wall of H extends to a quasiconvex B-wall of G.*

8. Main Theorem

Main Theorem 8.1. *G acts properly and cosparsely on a CAT(0) cube complex provided:*

1. *G is hyperbolic relative to virtually abelian subgroups;*
2. *G splits as a finite directed graph Γ of groups with quasiconvex, almost malnormal, edge groups;*
3. *The collection of edge groups $\{G_e\}$ at G_v is almost malnormal, and each G_e is hyperbolic;*
4. *Each vertex group acts properly and cosparsely on a CAT(0) cube complex;*
5. *Each vertex group has the quasiconvex extension property with respect to its edge subgroups;*
6. *For each embedding of an (incoming) edge group $G_e \subset G_v$, and each quasiconvex subgroup H of G_v, the subgroup $G_e \cap H$ is separable in G_e.*

A collection of walls is *sufficient* if the resulting action on the dual cube complex is proper. The proof works if we can extend sufficiently many quasiconvex walls of each edge group into its vertex groups.

Proof. For each e, let $G'_e = \phi_{-e}^{-1}(G'_{\iota(e)}) \cap \phi_{+e}^{-1}(G'_{\tau(e)})$ be the intersection of the extension-assuring finite index subgroups of its vertex groups. Choose a sufficient collection of quasiconvex walls for G'_e, and extend each wall into $G_{\iota(e)}$ and $G_{\tau(e)}$. Thus for each G_v, we obtain a collection of *immigrant* quasiconvex walls from its edge groups. By hypothesis, each G_v acts properly and cosparsely on a CAT(0) cube complex, from which we obtain a sufficient collection of *native* quasiconvex walls for G_v. Combining native and immigrant quasiconvex walls, we obtain a proper cosparse action of G_v on a CAT(0) cube complex Z_v.

By Proposition 3.5, for each e at v, let C_e denote a G_e-cocompact convex subcomplex of Z_v. By Proposition 3.6, let A_v denote a G_v-cocompact convex subspace of Z_v that contains each C_e. At this point, for each G_e, there are systems of walls coming from $G_{\iota(e)}$ and $G_{\tau(e)}$, and they may not agree with each other. We now resolve this by turning them.

Consider the collared space A_v^+ associated to the mapping cylinder of the union of local isometries $\{C_e \to A_v\}$ representing edge groups. We apply Theorem 5.2 to obtain a wallspace structure on A_v^+, which we denote by $(X_v, \mathcal{W}_v)$. That construction produces quasiconvex walls on A_v^+, and duplicates each wall of each C_e. The inclusion $C_e = C_e \times \{1\} \subset A_v^+$ is an embedding of geometric wallspaces by Lemma 5.14.

For each edge e, we have produced above two G_e-wallspaces associated to the distinct inclusions of $G_e \to G_\iota(e)$ and $G_e \to G_\tau(e)$. To indicate the orientation of the edge e, we denote these by C_e and D_e. Let $C_e \to D_e$ be a G_e-equivariant map, and following Definition 6.8, let $\overrightarrow{C_eD_e}$ denote the hybrid wallspace.

Let T be the barycentric subdivision of the Bass-Serre tree T_o of Γ. The original vertices of T are *primary* and the new vertices are *hybrid*. We now produce a tree X of geometric wallspaces over T. Its *primary* vertex spaces are appropriate copies of the X_v for primary vertices $v \in T$. Its *hybrid* vertex spaces are appropriate copies of $\overrightarrow{C_eD_e}$ for hybrid vertices of $v \in T$. The edge spaces X_e are copies of $C_e \times I$ and $D_e \times I$ equivariantly attached to hybrid and primary vertex spaces by geometric embeddings of wallspaces.

Finally, G acts cosparsely on the cube complex dual to the wallspace X by Theorem 6.7. □

Remark 8.2 (Ensuring Cocompactness)**.** When G is hyperbolic, the dual cube complex $\mathbb{C}(X)$ of Theorem 8.1 is G-cocompact. More generally, Proposition 3.4 yields a cocompact cubulation precisely when each parabolic subgroup P of G is not "overcubulated". This means that the number of *slopes* i.e. distinct parallelism classes of walls crossing a maximal parabolic subgroup P, equals d where P is virtually $\mathbb{Z}^d$. Let us therefore assume that each G_v acts cocompactly on a CAT(0) cube complex, and examine the steps in the proof of Theorem 8.1 to see when extra slopes are added.

Doubling of hyperplanes in the proof of Theorem 5.2 does not produce new slopes, and turns do not produce new slopes. Taken together, we note that the recubulation procedure of Theorem 5.2 has cocompact output $\mathbb{C}(A^+)$ when it is given a cocompact input $\mathbb{C}(A)$.

The extension result in [18] is obtained using a combinatorial retraction map, and therefore does not produce any new slopes. The only way that a new slope can arise for a parabolic subgroup P in a vertex group G_v, is when an immigrant wall arrives in G_v through some edge group. One way to exclude this possibility is to assume that the edge groups are hyperbolic.

We conclude that $\mathbb{C}(X)$ is G-cocompact when each G_v acts cocompactly on a CAT(0) cube complex and each edge group is malnormal quasiconvex and hyperbolic.

Remark 8.3. There is a G-equivariant quotient $\mathbb{C}(X) \to T$ where T is the Bass-Serre tree of Γ, and cubes are quotiented to vertices or projectively to 1-cubes of T. In particular, for each edge group G_e there is a hyperplane H of $\mathbb{C}(X)$ such that $G_e = \text{Stabilizer}(H)$ and the action of G_e on H preserves its halfspaces, and $gH \cap H = \emptyset$ unless $g \in \text{Stabilizer}(H)$. This holds because of our choice of walls for the edge groups of G. The analogous remark holds for Theorem 8.4 below.

For an amalgamated product, we relax almost malnormality and aparabolicity on one side as follows:

Theorem 8.4. *Suppose $G = L *_E R$ is hyperbolic relative to virtually abelian subgroups, where L, R are quasiconvex and act properly and cosparsely on CAT(0) cube complexes, $E \subset L$ is almost malnormal, each quasiconvex subgroup of E is separable in L, and finally, there exists a finite index normal subgroup $R' \subset R$ such that each quasiconvex wall of $E \cap R'$ extends to a quasiconvex wall of L. Then G acts properly on a cosparse CAT(0) cube complex.*

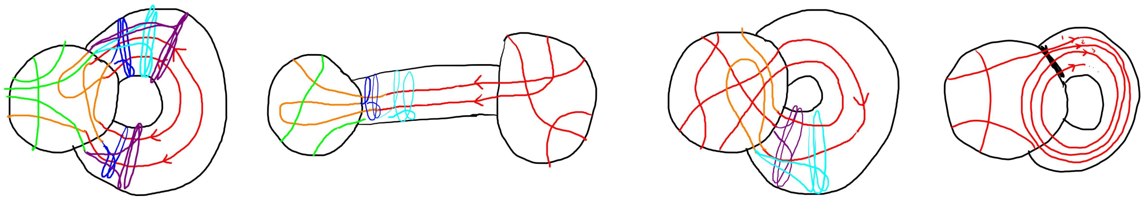

FIGURE 8.1. The 1st figure illustrates the immersed walls for an HNN extension $A*_{C^t=D}$ with $\{C, D\}$ malnormal in A. (We do not illustrated the doubling step or the wall cutting through the edge space.) The 2nd figure illustrates the case of $A *_C B$ where C is malnormal and has the extension property in A. For $A*_{C^t=D}$, when only $C \subset A$ is malnormal and has the extension property in A, we need to be sure that extensions of walls in C do not produce new walls in D as in the 3rd figure. Otherwise a feedback loop could develop as in the 4th figure.

Remark 8.5. If we only assume E is quasiconvex in L, and we assume that all walls in E are extendible into L, then we see that G is properly cubulated, but cannot draw any cocompactness conclusions.

Proof of Theorem 8.4. The cubulation of R' provides a sufficient collection of quasiconvex walls in E. Each of these extends to a quasiconvex wall in L. These immigrant walls are combined together with a hypothesized native collection, and all excess walls are turned.

R acts properly and cosparsely on a CAT(0) cube complex, and we apply Proposition 3.6 to choose a R-cocompact convex subspace A_R whose walls are the intersections with hyperplanes. Section 5 provides a wallspace structure on a cocompact CAT(0) spaces A_L and an E-cocompact subspace $C \subset A_L$ representing $E \to L$. While C_L is actually a subcomplex of the original CAT(0) cube complex containing A_L, since E is not full on the R-side, we can only obtain a convex subspace $D \subset A_R$ arising from an application of Propositions 3.6 and 3.5 to D and A_R. We then form the hybrid space $\overrightarrow{CD}$ and proceed as before.

Quasiconvexity of the new walls follows from a variant of Proposition 6.6 in [1] that only requires almost malnormality on one side in the amalgamated free product case. □

Remark 8.6 (Avoiding feedback loops)**.** Lack of malnormality of the edge groups is harder to compensate for in the HNN case. This is because the extension property can generate a "feedback loop". Consider $G = A*_{C^t=D}$. If we assume that G has the extension property with respect to C, and that $\{C, D\}$ is almost malnormal, then the proof proceeds as follows: A cubulation of A yields native walls of A that are sufficient in D. These extend into A through C to produce a further collection of immigrant walls. Now the excess immigrant walls and native walls of A are turned in C and D. However, if we assume that only one of C or D is malnormal, then we need further control of the extension property to avoid a feedback loop.

In the case where the extension property holds for C and where C is almost malnormal, we need to know that no further (induced) walls in D are created when a wall in C is extended into A. For we can start with a cubulation of A, and then extend the walls in the edge group that are induced by $C \subset D$, and finally turn all excess on the almost malnormal C side.

Example 8.7. The simplest examples where "feedback loops" prevent cubulation are the infamous Baumslag-Solitar HNN extensions: $\langle a, t \mid (a^n)^t = a^m \rangle$ where $n, m \neq 0$ and $n \neq \pm m$. But there are more subtle examples. For a group G splitting as a graph of $\mathbb{Z}^2$ groups with $\mathbb{Z}$ vertex groups, it is determined exactly when G can be cubulated in [17]. One group described there that cannot be cubulated is: $\langle a, b, s, t \mid [a, b], (ab)^s = a^3b, (ab)^t = ab^3 \rangle$. Using $c = ab, d = a^3b$, it can be re-presented as: $\langle c, d, s, t \mid [c, d], csds^{-1}, c^t = c^4d^{-1} \rangle$. This is an HNN extension whose stable letter is t and where the presentation of the base group provides a npc square complex with a special double cover.

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