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Linear lattice Boltzmann schemes for acoustic: Parameter choices and isotropy properties

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ABSTRACT

In this paper we investigate the numerous possible parameter choices for linear lattice Boltzmann schemes according to the definition of the order of isotropy given in Augier et al. (in press) [4]. This property – written in a general framework that includes all of the D_dQ_q schemes – can be understood through a group operation. The resulting relationships between the parameters of the scheme (defining the equilibrium states and relaxation times) yield a rigorous methodology that should be followed if one is to ensure isotropy at a given order. For acoustic applications in two spacial dimensions (namely D_2Q_9 and D_2Q_{13} schemes) this methodology is used to propose a full description of the sets of parameters that are isotropic at order m ($m \in \{1, 2, 3, 4, 5\}$ for D_2Q_9 and $m \in \{1, 2\}$ for D_2Q_{13}). We include numerical illustrations for the D_2Q_9 scheme obtained with the code LBMpy developed in the laboratory of Mathematics of the University Paris-Sud.

0. Introduction

The strength of lattice Boltzmann schemes lies in their effectivity. They are intensively used in academic and industrial contexts for numerical simulations of fluid dynamics. However, since the directions of the lattice are biased, these schemes are *a priori* not isotropic even though they are used to solve isotropic phenomena. Furthermore, [1] mentions that isotropy also enhances the stability of lattice Boltzmann schemes. Determining the isotropic properties of D_dQ_q schemes (where d represents the space dimension and q the number of discrete velocities) has been the aim of numerous previous works, although it should be noted that different definitions have often been used (for instance, isotropy of the stress tensor required by fluid equations in [2], isotropy by solving an eigenvalue problem in [3]).

As explained in [4], because of the criss–cross pattern of the scheme, it is natural to look for parameters that give the same behavior on each axis. By extending this argument further, isotropy can be related to the invariance of the spatial frame by special orthogonal transformations. Space transformations are actually considered in [5–7] for discrete cases: isometries of the Bravais lattice in [5], discrete rotations in [6,7]. In [7], an analysis of the moment tensor yields a theoretical procedure to evaluate and systematically construct higher-order BGK models.

In [4], a precise mathematical definition of an isotropic MRT D_dQ_q scheme is given. This definition is based on an invariance of the equivalent equations with regard to all spacial rotations and can be used for every scheme. Moreover, it gives a procedure to specify the parameters (namely equilibrium states and relaxation times) for linear and orthogonalized schemes in order to increase the degree of isotropy.

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In Section 1 we recall first some notations about the linear D_dQ_q scheme defined with orthogonalized moments and then the definition of the equivalent equations that come from a Taylor expansion of the lattice Boltzmann scheme [8].

In Section 2 we briefly recall the definition of isotropy given in [4]. A D_dQ_q scheme is said to be mth-order-isotropic if the equivalent equations of order m are isotropic. This definition is written with respect to a group operation and gives an easy, systematic, procedure to specify some of the parameters of the linear lattice Boltzmann scheme. The end of this section consists in the calculation of the degrees of freedom that we have to take into account for every scheme.

In Section 3 we focus on acoustic applications using the D_2Q_9 and D_2Q_{13} schemes. We first recall the definition of these particular schemes (moments and equilibrium states) and detail how to make use of the systematic procedure described in the previous section. Then we give a complete description of the sets of parameters that involve isotropy of order m (m from 1 to 5 for the D_2Q_9 scheme and to order two for the D_2Q_{13} scheme). Concerning the D_2Q_{13} scheme, we also propose a selected choice of sets of parameters that involve isotropy at third order.

In Section 4, preliminary numerical results are discussed to illustrate the different orders of isotropy for the D_2Q_9 scheme.

1. Lattice Boltzmann method and equivalent equations

In this section we first recall some notation for the lattice Boltzmann method. More precisely, we focus on a linear lattice Boltzmann scheme with orthogonalized moments. We then pay particular attention to the expansion of order m: this is straightforward since it is a partial differential equation of order m. More precisely, these equations are obtained *via* a Taylor expansion and are consistent at order m with the scheme.

1.1. Notations for the lattice Boltzmann method

We use the notation proposed by d'Humières in [9] by considering $\mathcal{L}$, a regular lattice in d dimensions with typical mesh size Δx. The time step Δt is determined after specification of the velocity scale λ by the relation:

$$\Delta t = \frac{\Delta x}{\lambda}.$$

For the scheme denoted by D_dQ_q, we introduce $V = (v_j)_{0\le j\le q-1}$ to denote the set of q velocities and we assume that for each node x of $\mathcal{L}$, and each v_j in V, the point $x + v_j\Delta t$ is also a node of the lattice $\mathcal{L}$.

The aim of the D_dQ_q scheme is to compute a particle distribution $\boldsymbol{f} = (f_j)^{t}_{0\le j\le q-1}$ on the lattice $\mathcal{L}$ at discrete values of time: it is a numerical scheme on a grid to approximately solve (1).

$$\frac{\partial f_j}{\partial t} + v_j \cdot \nabla f_j = -\frac{1}{\tau_j}(f_j - f_j^{\text{eq}}), \quad 0 \le j \le q-1, \tag{1}$$

where f_j^{eq} describes the distribution f_j at the equilibrium and τ_j is the relaxation time (related on f_j).

The interest of the D_dQ_q scheme lies in the method of resolution of (1): we proceed in two steps (as if a splitting method) as explained below.

- First, we are interested in the relaxation $\boldsymbol{f} \to \boldsymbol{f}^*$ step that consists in solving

$$\frac{\partial f_j^*}{\partial t} = -\frac{1}{\tau_j}(f_j - f_j^{\text{eq}}), \quad 0 \le j \le q-1. \tag{2}$$

In order to solve (2), we first consider the moments at equilibrium. These moments are divided into two types: the ones that are conserved at equilibrium are denoted by $\boldsymbol{W} \in \mathbb{R}^N$ and the ones that are not conserved at equilibrium are denoted by $\boldsymbol{Y} \in \mathbb{R}^{q-N}$. More precisely, because of the acoustic applications, the moments $\boldsymbol{W}$ and $\boldsymbol{Y}$ are linear combinations of the distribution functions $\boldsymbol{f}$ (*i.e.* there exists an invertible matrix $M = (M_{ij})_{0\le i,j\le q-1}$ such that $(\boldsymbol{W}^t, \boldsymbol{Y}^t)^t = M\boldsymbol{f}$). The size and the definition of the matrix M depend on the chosen scheme. However, for every D_dQ_q scheme, the N first lines of M are the same. Since our aim is to improve upon the classical D_dQ_q schemes, we only consider the particular case for which the density ρ and the momentum $\boldsymbol{q}^t$ are conserved. Note that the physical energy is not conserved in this scheme. Therefore, $\boldsymbol{W}$ is equal to $(\rho, \boldsymbol{q}^t)^t$ such that:

$$\boldsymbol{W}_0 = \rho = \sum_{j=0}^{q-1} f_j \quad \text{and} \quad \boldsymbol{W}_\alpha = \boldsymbol{q}_\alpha = \sum_{j=0}^{q-1} v_j^\alpha f_j, \quad 1 \le \alpha \le d.$$

Now we define the moments after the relaxation with the help of the relations

$$\begin{cases} \boldsymbol{W}_k^* = \boldsymbol{W}_k, & 0 \le k \le N-1 \\ \boldsymbol{Y}_k^* = \boldsymbol{Y}_k + s_k(\boldsymbol{Y}_k^{\text{eq}} - \boldsymbol{Y}_k), & N \le k \le q-1, \end{cases} \tag{3}$$

where, for $N \leq k \leq q-1$, s_k is related to all the τ_j, $N \leq j \leq q-1$ and $\boldsymbol{Y}_k^{\text{eq}}$ is the moment $\boldsymbol{Y}_k$ at equilibrium. For stability reasons, we choose s_k such that $0 < s_k < 2$, $N \leq k \leq q-1$. In order to simplify the calculations, we introduce the reals σ_k, $N \leq k \leq q-1$, *via*

$$\sigma_k = \frac{1}{s_k} - \frac{1}{2}, \quad \sigma_k > 0. \tag{4}$$

As the scheme is supposed to be linear, the moments at equilibrium $\boldsymbol{Y}^{\text{eq}}$ linearly depend on the conserved moments $\boldsymbol{W}$ through a matrix denoted by E so that Eq. (3) reads as follows

$$\begin{pmatrix} \boldsymbol{W}^* \\ \boldsymbol{Y}^* \end{pmatrix} = J \begin{pmatrix} \boldsymbol{W} \\ \boldsymbol{Y} \end{pmatrix}, \quad \text{with}\, J = \begin{pmatrix} \text{Id}_N & 0 \\ S\,E & \text{Id}_{q-N} - S \end{pmatrix},$$

where S is the diagonal matrix of the relaxation times s_k, $N \leq k \leq q-1$. Finally, we reconstruct the distribution after the relaxation with $f^* = M^{-1}\,(\boldsymbol{W}^{*\mathrm{t}}, \boldsymbol{Y}^{*\mathrm{t}})^{\mathrm{t}}$.

- Second, we solve the transport step

$$\frac{\partial f_j}{\partial t} + v_j \cdot \nabla f_j = 0, \quad 0 \leq j \leq q-1, \tag{5}$$

with the method of characteristics:

$$f_j(x + v_j \Delta t, t + \Delta t) = f_j^*(x, t), \quad \forall x \in \mathcal{L},\ 0 \leq j \leq q-1. \tag{6}$$

Finally a time step of the linear lattice Boltzmann scheme $\mathrm{D}_d\mathrm{Q}_q$ reduces to:

$$\boldsymbol{f}(x + v_j \Delta t, t + \Delta t) = M^{-1} J\, M \boldsymbol{f}(x, t), \quad \forall x \in \mathcal{L},\ 0 \leq j \leq q-1. \tag{7}$$

1.2. Equivalent equations

By convention, we denote by $\cdot^i$ the ith line and $\cdot_j$ the jth column of a tensor $\cdot$. Moreover, a Latin letter is an index related to the moments (of size N) while Greek letters represent an index related to the space dimension (of size d). We then introduce the system of N equivalent equations [8] of order m that is consistent at order m with the linear lattice Boltzmann scheme (*id est* the rest is $\mathcal{O}(\Delta t^m)$):

$$\partial_t \boldsymbol{W}^i + \sum_{n=1}^{m} {}_nA^i \odot \nabla^n \boldsymbol{W} = 0, \quad 1 \leq i \leq N, \tag{8}$$

where ${}_nA^i \in S_{n,d,N}$, $1 \leq n \leq m$, $1 \leq i \leq N$ are tensors of order $n+2$ that take into account the coefficients of E and S. The space $S_{n,d,N}$ is defined as the quotient space $S_{n,d,N} = \mathbb{R}^{N^2 d^n} / \sim$, where two tensors are equivalent for the equivalence relation $\sim$ if the associate partial differential operators are the same (using Schwarz's theorem).

Remark 1. Eqs. (8) could also be seen as a vectorial system: each term of the sum can then be split into terms of type $\widetilde{A}^\alpha \partial_\alpha \boldsymbol{W}$, where $\widetilde{A}^\alpha$ is a N by N matrix and the n-tuple α is the order of derivation.

By convention, the maximal contraction operator $\odot$ is defined by, $1 \leq i \leq N$, $1 \leq n \leq m$:

$$\left({}_nA \odot \nabla^n \boldsymbol{W}\right)^i := {}_nA^i \odot \nabla^n \boldsymbol{W} := \sum_{\substack{1 \leq j \leq N \\ 1 \leq \alpha_1, \ldots, \alpha_n \leq d}} {}_nA_j^{i,\alpha_1,\ldots,\alpha_n} \partial_{\alpha_1} \cdots \partial_{\alpha_n} \boldsymbol{W}^j.$$

These equivalent equations come from a formal calculus explained in [1] and an algorithm that is easy to use is described in [10].

2. Isotropy condition

In this section we briefly recall the definition of isotropy for a $\mathrm{D}_d\mathrm{Q}_q$ scheme (more details are given in [4]). Since the $\mathrm{D}_d\mathrm{Q}_q$ scheme may be characterized through the set of equivalent equations (which are partial differential equations (PDEs)), it seems natural to derive the definition of an isotropic lattice Boltzmann scheme from the definition of an isotropic system of PDEs. Consequently, the isotropy condition is a group operation and involves a set of equations that act on the coefficients of the matrices E and S. The second part of this section consists in counting the number of degrees of freedom given by the isotropy.

2.1. Isotropy and algebra

We first recall the definition of an isotropic system of PDEs:

Definition 2. The system of PDEs of order m

$$\partial_t \boldsymbol{W}^i + \sum_{n=1}^{m} {}_nA^i \odot \nabla^n \boldsymbol{W} = 0, \quad 1 \le i \le N, \tag{9}$$

is said isotropic if it is invariant by special orthogonal transformation of the frame.

For r a special orthogonal transformation of the frame in d spaces dimensions, we define the orthogonal matrix $R(r)$ by

$$R(r) := \begin{pmatrix} 1 & 0 \\ 0 & r \end{pmatrix},$$

such that $R(r)^{-1}\boldsymbol{W}$ is the vector of the conserved moments in the new frame.

Remark 3. In this study, we focus on the group of special orthogonal transformations in two dimensions: that is, the group of rotations $SO_2(\mathbb{R})$. This can be parameterized with just one single real parameter corresponding to the rotation angle. Thus all rotations r are given by

$$\begin{pmatrix} \cos(\theta) & -\sin(\theta) \\ \sin(\theta) & \cos(\theta) \end{pmatrix}, \quad \theta \in \mathbb{R}.$$

As proven in [4], the isotropy property can be understood by a group operation defined in the following definition. Furthermore, this definition gives a set of equations that are very easy to use and give relationships between the parameters of the scheme.

Definition 4. Let n be in $\mathbf{N}^*$. Then the group operation $\Phi_n \; : \; SO_d(\mathbb{R}) \times S_{n,d,N} \to S_{n,d,N}$ is defined for $1 \le i, j \le N$, $1 \le \alpha_1, \ldots, \alpha_n \le d$, $1 \le n \le m$, by the relation:

$$(\Phi_n(r)({}_nA))_j^{i,\alpha_1,\ldots,\alpha_n} := \sum_{1\le\beta_1,\ldots,\beta_n\le d} \; \sum_{1\le k,l\le N} (R(r))_l^i \, {}_nA_k^{l,\beta_1,\ldots,\beta_n} r_{\beta_1}^{\alpha_1} \cdots r_{\beta_n}^{\alpha_n} (R(r)^{-1})_j^k. \tag{10}$$

We then obtain a formal requirement for a PDE to be isotropic (for the proof see [4]):

Proposition 5. *Let (9) be a PDE of order m. It is isotropic if the tensor ${}_nA$ is a fixed point of Φ_n, $1 \le n \le m$, that is $\Phi_n(r)({}_nA) = {}_nA, \forall r \in SO_d(\mathbb{R}), 1 \le n \le m$.*

Finally, we recall the definition for a lattice Boltzmann scheme to be isotropic [4]:

Definition 6. A lattice Boltzmann scheme is said to be m-th order-isotropic if the system of equivalent equations (8) at order m is isotropic.

Furthermore, we denote by $L_N(r) := \sum_{1\le n\le N} (\Phi_n(r)({}_nA) - {}_nA) \triangle t^n$ the lack of isotropy at order N for the special orthogonal transformation r.

2.2. Consequences on the degrees of freedom

In this subsection we count the degrees of freedom that have to be taken into account for studying isotropy. The total number of parameters that can be freely chosen is indeed reduced by considering the relations forced by the physical environment.

Let us start to count the equations and unknowns:

- There are $N^2\,(d+n-1)!/(n!(d-1)!)$ different components of ${}_nA$, each one corresponding to a relation for isotropy thanks to Proposition 5 [4].
- Now we take into account the equilibrium states, that is the coefficients of the matrix E defined in Section 1.1. The matrix E is usually chosen to be sparse and chosen in a way that is consistent with experimental evidence and physical "good sense/intuition". It is remarkable that the isotropy condition for the $\mathrm{D}_d\mathrm{Q}_q$ scheme could give some justifications of some classical choices of coefficients. However, in this paper, E is chosen to be full in order to include as many degrees of freedom as possible. That is, we have $N(q-N)$ additional parameters.

- It remains to consider the relaxation times. Actually, as it is recalled for example in [4], there exist essentially three types of lattice Boltzmann schemes: the Bhatnagar–Gross–Krook (BGK) scheme, the Two Relaxation Times (TRT) scheme and the Multiple Relaxation Times (MRT) scheme. Since our purpose is to choose the relaxation times in order to improve the behavior of the scheme (namely, isotropy is enforced by a good choice of the parameters of the scheme), we consider the third case that is the most general (even if it is not the most often used). Thus, we consider $q - N$ independent unknown parameters on S.

Finally, as we consider the most general $\mathrm{D}_d\mathrm{Q}_q$ scheme in the linear case, there are $(q - N)(N + 1)$ free parameters and $N^2(d + n - 1)!/(n!(d - 1)!)$ equations given by the isotropy conditions. However, some of these parameters are forced by the properties of the physical environment [8]:

- The linear equilibrium state of the energy described by $E_\varepsilon^\rho \rho\lambda^2 + E_\varepsilon^{q_x} q_x\lambda^2 + E_\varepsilon^{q_y} q_y\lambda^2$ is such that the density contribution E_ε^ρ is linked with the sound velocity c_0. For example, as it is recalled in [3,11,12,1], we have $E_\varepsilon^\rho = 6c_0^2 - 4$ for the $\mathrm{D}_2\mathrm{Q}_9$ scheme, $E_\varepsilon^\rho = 26c_0^2 - 28$ for the $\mathrm{D}_2\mathrm{Q}_{13}$ scheme, $E_\varepsilon^\rho = 57c_0^2/\lambda^2 - 30$ for the $\mathrm{D}_3\mathrm{Q}_{19}$ scheme, and $E_\varepsilon^\rho = 3c_0^2 - 2$ for the $\mathrm{D}_3\mathrm{Q}_{27}$ scheme. In fact, these relations come from the physical properties through the relations ${}_1A_1^{21} = {}_1A_1^{32} = c_0^2$, *id est* the contribution of $\partial\rho/\partial x$ (respectively $\partial\rho/\partial y$) in the conservation equation that describes the evolution of the momentum q_x (respectively q_y).
- Furthermore, since there exists a $\mathrm{D}_2\mathrm{Q}_9$ scheme consistent with the Navier–Stokes equations (see [8]), a link shall be made between both the viscosities of the considered fluid and two of the relaxation times (see for example [1,3]). Namely, if the shear viscosity is denoted by μ and the bulk viscosity by ζ, in the case of the $\mathrm{D}_2\mathrm{Q}_9$ scheme, we get

$$\mu = \frac{1}{3}\lambda\,\Delta x\,\sigma_{\varphi_{xx}} \quad \text{and} \quad \zeta = \lambda\,\Delta x\,\sigma_\varepsilon\left(\frac{2}{3} - c_0^2\right), \tag{11}$$

  where c_0^2 is the sound velocity, $\sigma_{\varphi_{xx}}$ (respectively σ_ε) is related to the relaxation time depending on the moment $m_{\varphi_{xx}}$ (respectively m_ε) thanks to (4), where each moment is defined in (12)–(13).
- Since we are considering acoustic applications, we focus our attention on relaxation times that do not link the two viscosities, *i.e.* s_ε and $s_{\varphi_{xx}}$ are linearly independent.
- Moreover, since the moments m_ε and $m_{\varphi_{xx}}$ are of the same order (namely 2), we have to take into account a lattice Boltzmann scheme with multiple relaxation times for acoustic applications (a Two Relaxation Times scheme leads to a link between μ and ζ because s_ε and $s_{\varphi_{xx}}$ are equal).

Finally there remain $(q - N)(N + 1) - 3$ unknown parameters.

3. Isotropy for acoustic applications in two dimensions

In this section we focus on two schemes among the most popular for acoustic applications: $\mathrm{D}_2\mathrm{Q}_9$ and $\mathrm{D}_2\mathrm{Q}_{13}$, and investigate the property of isotropy for each. More precisely, we determine all of the possible choices of coefficients that yield a mth-order-isotropic scheme. We first recall the definition of the $\mathrm{D}_2\mathrm{Q}_9$ and $\mathrm{D}_2\mathrm{Q}_{13}$ schemes through their moments. After specifying the degrees of freedom, we detail the procedure used to determine all of the solutions of these very large systems. In the second part of this section, we explain the results order by order for the $\mathrm{D}_2\mathrm{Q}_9$ scheme. In the third part, as the intricacy of the equations related on the $\mathrm{D}_2\mathrm{Q}_{13}$ scheme increase so much with the order, we have to restrict ourselves to the second order case. Although some sets of parameters that yield isotropy for the $\mathrm{D}_2\mathrm{Q}_{13}$ scheme at third order are given, we are not able to give exhaustible choices of them.

3.1. Generalities and methodology

In this subsection we introduce the notation for the linear $\mathrm{D}_2\mathrm{Q}_9$ and $\mathrm{D}_2\mathrm{Q}_{13}$ schemes and we propose a methodology to solve the very large systems of non-linear equations that appear when the isotropy is enforced at each order. Note that this non-linearity is relative to the parameters of the lattice Boltzmann scheme. These systems are written in terms of the coefficients of the matrices E (describing the equilibrium states) and S (describing the relaxation times) and have to be solved for every rotation angle θ.

For both lattice Boltzmann schemes ($\mathrm{D}_2\mathrm{Q}_9$ and $\mathrm{D}_2\mathrm{Q}_{13}$) considered, three moments are conserved during the collision step: the density ρ and the two coordinates of the macroscopic momentum q_x and q_y where:

$$\rho = \sum_{j=0}^{q-1} f_j, \qquad q_x = \sum_{j=0}^{q-1} v_j^x f_j, \qquad q_y = \sum_{j=0}^{q-1} v_j^y f_j.$$

In other words, we focus on linear Boltzmann schemes that do not conserve the energy. Next, we define six moments that are not conserved during the collision step: the kinetic energy m_ε, the square of the kinetic energy m_{ε_2}, the coordinates of

Table 1
Number of polynomial equations for two dimensions.

Order	1	2	3	4
Polynomial equations	18	27	36	45

the heat flux m_{φ_x} and m_{φ_y}, and two moments of order two: the stress tensor $m_{\varphi_{xx}}$ and $m_{\varphi_{xy}}$. They read as follows:

$$m_\varepsilon = \sum_{j=0}^{q-1} \frac{1}{2}|v_j|^2 f_j, \qquad m_{\varepsilon_2} = \sum_{j=0}^{q-1} \left(\frac{1}{2}|v_j|^2\right)^2 f_j, \qquad m_{\varphi_x} = \sum_{j=0}^{q-1} \frac{1}{2}|v_j|^2 v_j^x f_j, \tag{12}$$

$$m_{\varphi_y} = \sum_{j=0}^{q-1} \frac{1}{2}|v_j|^2 v_j^y f_j, \qquad m_{\varphi_{xx}} = \sum_{j=0}^{q-1} \left(\left(v_j^x\right)^2 - \left(v_j^y\right)^2\right) f_j, \qquad m_{\varphi_{xy}} = \sum_{j=0}^{q-1} v_j^x v_j^y f_j. \tag{13}$$

Finally, we investigate the linear orthogonalized D_2Q_9 scheme (defined thanks to the previous moments), with a full matrix E describing the equilibrium states and a diagonal matrix S describing the relaxation times that read as follows:

$$E := \begin{pmatrix} E_\varepsilon^\rho \lambda^2 & E_\varepsilon^{q_x} \lambda & E_\varepsilon^{q_y} \lambda \\ E_{\varepsilon_2}^\rho \lambda^4 & E_{\varepsilon_2}^{q_x} \lambda^3 & E_{\varepsilon_2}^{q_y} \lambda^3 \\ E_{\varphi_x}^\rho \lambda^3 & E_{\varphi_x}^{q_x} \lambda^2 & E_{\varphi_x}^{q_y} \lambda^2 \\ E_{\varphi_y}^\rho \lambda^3 & E_{\varphi_y}^{q_x} \lambda^2 & E_{\varphi_y}^{q_y} \lambda^2 \\ E_{\varphi_{xx}}^\rho \lambda^2 & E_{\varphi_{xx}}^{q_x} \lambda & E_{\varphi_{xx}}^{q_y} \lambda \\ E_{\varphi_{xy}}^\rho \lambda^2 & E_{\varphi_{xy}}^{q_x} \lambda & E_{\varphi_{xy}}^{q_y} \lambda \end{pmatrix}, \qquad S := \mathrm{Diag}(s_\varepsilon, s_{\varepsilon_2}, s_{\varphi_x}, s_{\varphi_y}, s_{\varphi_{xx}}, s_{\varphi_{xy}}). \tag{14}$$

In order to define the D_2Q_{13} scheme, we introduce four additional moments: the cube of the kinetic energy m_{ε_3}, two moments of order five $m_{x\varepsilon_2}$ and $m_{y\varepsilon_2}$, and finally a moment of order four $m_{xx\varepsilon}$ where:

$$m_{\varepsilon_3} = \sum_{0\le j\le q-1} \left(\frac{1}{2}|v_j|^2\right)^3 f_j, \qquad m_{x\varepsilon_2} = \sum_{0\le j\le q-1} v_j^x \left(\frac{1}{2}|v_j|^2\right)^2 f_j,$$

$$m_{y\varepsilon_2} = \sum_{0\le j\le q-1} v_j^y \left(\frac{1}{2}|v_j|^2\right)^2 f_j, \qquad m_{xx\varepsilon} = \sum_{0\le j\le q-1} \left(\frac{1}{2}|v_j|^2\right) \left((v_j^x)^2 - (v_j^y)^2\right) f_j.$$

More precisely, we take into account the linear and orthogonalized D_2Q_{13} defined from the full matrix E (describing the equilibrium states) and the diagonal matrix S (describing the relaxation times) which read as follows:

$$E := \begin{pmatrix} E_\varepsilon^\rho \lambda^2 & E_\varepsilon^{q_x} \lambda & E_\varepsilon^{q_y} \lambda \\ E_{\varphi_{xx}}^\rho \lambda^2 & E_{\varphi_{xx}}^{q_x} \lambda & E_{\varphi_{xx}}^{q_y} \lambda \\ E_{\varphi_{xy}}^\rho \lambda^2 & E_{\varphi_{xy}}^{q_x} \lambda & E_{\varphi_{xy}}^{q_y} \lambda \\ E_{\varphi_x}^\rho \lambda^3 & E_{\varphi_x}^{q_x} \lambda^2 & E_{\varphi_x}^{q_y} \lambda^2 \\ E_{\varphi_y}^\rho \lambda^3 & E_{\varphi_y}^{q_x} \lambda^2 & E_{\varphi_y}^{q_y} \lambda^2 \\ E_{x\varepsilon_2}^\rho \lambda^5 & E_{x\varepsilon_2}^{q_x} \lambda^4 & E_{x\varepsilon_2}^{q_y} \lambda^4 \\ E_{y\varepsilon_2}^\rho \lambda^5 & E_{y\varepsilon_2}^{q_x} \lambda^4 & E_{y\varepsilon_2}^{q_y} \lambda^4 \\ E_{\varepsilon_2}^\rho \lambda^4 & E_{\varepsilon_2}^{q_x} \lambda^3 & E_{\varepsilon_2}^{q_y} \lambda^3 \\ E_{\varepsilon_3}^\rho \lambda^6 & E_{\varepsilon_3}^{q_x} \lambda^5 & E_{\varepsilon_3}^{q_y} \lambda^5 \\ E_{xx\varepsilon}^\rho \lambda^4 & E_{xx\varepsilon}^{q_x} \lambda^3 & E_{xx\varepsilon}^{q_y} \lambda^3 \end{pmatrix},$$

$$S := \mathrm{Diag}(s_\varepsilon, s_{\varphi_{xx}}, s_{\varphi_{xy}}, s_{\varphi_x}, s_{\varphi_y}, s_{x\varepsilon_2}, s_{y\varepsilon_2}, s_{\varepsilon_2}, s_{\varepsilon_3}, s_{xx\varepsilon}). \tag{15}$$

We then apply Definition 6 on the D_2Q_9 scheme in order to obtain all of the sets of parameters that improve these schemes in terms of isotropy. In two spacial dimensions, the special orthogonal transformations r are the well-known rotation matrices, $N = 3$ and $m \le 5$. We first establish the degrees of freedom for each scheme at each order. Section 2.2 gives the number of equations (this number does not depend on the scheme): $9(n+1)$ where n is the considered order. The explicit calculation of these numbers is given in Table 1. Furthermore, there are 21 unknown parameters for the D_2Q_9 scheme and 37 for the D_2Q_{13} scheme.

We remark that all of these equations are polynomial in $\cos\theta$ and $\sin\theta$, where θ is the angle that parametrizes the rotation. The study of the coefficients yield another set of equations that only depend on the coefficients of S and E. For instance, for the D_2Q_9 scheme, at first order we get 18 polynomial equations thanks to both Proposition 5 and formula:

$${}_1A_j^{i\alpha} = \sum_{1\le\beta\le d} \sum_{1\le k,l\le N} (R(r))_l^i \, {}_1A_k^{l,\beta} r_\beta^\alpha (R(r)^{-1})_j^k, \quad 1 \le i,j \le 3, \ 1 \le \alpha \le 2.$$

Table 2
Number of equations for D_2Q_9.

Order	1	2	3	4
Equations	48	78	116	169
Parameters	21	13	8	6

Table 3
Number of equations for D_2Q_{13}.

Order	1	2	3
Equations	48	78	148
Parameters	37	29	23

All of these equations shall be filed as follows:

- 6 of them are null.
- 8 of them are of type $a\cos^3\theta + b\cos^2\theta\sin\theta + c\cos\theta + d\sin\theta + e = 0$. Since the functions $\theta \mapsto \cos^3\theta, \theta \mapsto \cos^2\theta\sin\theta, \theta \mapsto \cos\theta, \theta \mapsto \sin\theta$, and $\theta \mapsto 1$ are linearly independent, each of these equations implies 5 additional equations: $a = b = c = d = e = 0$.
- 4 of them are of type $a\cos\theta\sin\theta + b\sin^2\theta = 0$. Since the functions $\theta \mapsto \cos\theta\sin\theta$ and $\theta \mapsto \sin^2\theta$ are linearly independent, each of these equations implies 2 additional equations: $a = b = 0$.

The isotropy for the D_2Q_9 scheme at first order is characterized by 48 equations: 8 coefficients of the matrix E are known and it remains to find the $21 - 8 = 13$ parameters that achieve isotropy at second order.

Following this methodology order by order, we are able to list in Tables 2 and 3 for each order the number of equations that we have to take into account (namely, equations that do not depend on the angle θ) and the number of parameters that we have to specify.

Remark 7. Since the tensors ${}_nA$ have the same size (see Table 1), the number of basic polynomial equations for the D_2Q_9 and D_2Q_{13} schemes are equal (the order n being fixed). However, the coefficients of these polynomial equations (in $\sin(\theta)$ and $\cos(\theta)$) depend on the discrete geometry of the mesh and as a result the number of equations we have to solve is not the same for the two schemes (see the second lines of Tables 2 and 3).

The methodology to investigate the nth-order-isotropy is then continued in three steps.

1. Compute the equivalent equations of order n (they could be obtained using a formal code [1,10]) of the lattice Boltzmann scheme described by the matrices E and S taking into account the relations obtained at previous orders.
2. Write the complete set of equations given by Proposition 5, the number of equations being given in Tables 2 and 3.
3. Deduce the relations on the coefficients that have to be fulfilled in order to ensure the desired order of isotropy.

Finally, we introduce a notation that is used to identify the equations:

Notation 8 (*Coeff*$(i, \alpha_1, \ldots, \alpha_n)$)**.** *A natural way to identify the equations consists in specifying the number of the conservation equation and the corresponding coefficient of the tensor* ${}_NA$, $1 \le N \le m$. *Then we denote by* $\mathrm{Coeff}(i, \alpha_1, \ldots, \alpha_n)$ *the equation given thanks to* ${}_nA_j^{i,a_1,\ldots,a_n}$. *We then denote by* $\mathrm{Coeff}(\cos^k \sin^l | i, \alpha_1, \ldots, \alpha_n)$ *the coefficient of* $\cos^k(\theta)\sin^l(\theta)$ *in the equation* $\mathrm{Coeff}(i, \alpha_1, \ldots, \alpha_n)$.

3.2. Results for the D_2Q_9 scheme

In this subsection, the main result on the D_2Q_9 scheme is given in Proposition 9. The proof is then subdivided into five lemmas by proceeding order by order.

Proposition 9. *Let $L_5(r)$ be the lack of isotropy for the D_2Q_9 scheme at fifth order for the rotation r, then we get the following propositions.*

- *The scheme is first-order-isotropic, that is $L_5(r) = \mathcal{O}(\Delta t^2), \forall\, r \in SO_2(\mathbb{R})$, iff $E^{q_x}_{\varepsilon}, E^{q_y}_{\varepsilon}, E^{\rho}_{\varphi_{xx}}, E^{q_x}_{\varphi_{xx}}, E^{q_y}_{\varphi_{xx}}, E^{\rho}_{\varphi_{xy}}, E^{q_x}_{\varphi_{xy}}$, and $E^{q_y}_{\varphi_{xy}}$ are zero.*
- *The scheme is second-order-isotropic, that is $L_5(r) = \mathcal{O}(\Delta t^3), \forall\, r \in SO_2(\mathbb{R})$, iff it is first-order-isotropic, $E^{\rho}_{\varphi_x}, E^{q_y}_{\varphi_x}, E^{\rho}_{\varphi_y}, E^{q_x}_{\varphi_y}$ are zero, and $E^{q_x}_{\varphi_x} = E^{q_y}_{\varphi_y} = (\sigma_{\varphi_{xx}} - 4\sigma_{\varphi_{xy}})/(2\sigma_{\varphi_{xy}} + \sigma_{\varphi_{xx}})$.*
- *The scheme is third-order-isotropic, that is $L_5(r) = \mathcal{O}(\Delta t^4), \forall\, r \in SO_2(\mathbb{R})$, iff the three following properties are satisfied:*
  - *it is second-order-isotropic,*
  - *$\sigma_{\varphi_{xx}} = \sigma_{\varphi_{xy}}$ and $E^{q_x}_{\varphi_x} = -1$,*
  - *at least one of these three properties is satisfied*

* $2E^{\rho}_{\varepsilon_2} + 4 + 3E^{\rho}_{\varepsilon} = 0, E^{q_x}_{\varepsilon_2} = E^{q_y}_{\varepsilon_2} = 0,$
* $\sigma_{\varphi_x} = \sigma_{\varphi_y} = 1/(12\sigma_{\varphi_{xx}}), E^{q_x}_{\varepsilon_2} = E^{q_y}_{\varepsilon_2} = 0,$
* $\sigma_{\varphi_x} = \sigma_{\varphi_y} = 1/(12\sigma_{\varphi_{xx}}), \sigma_{\varphi_{xx}} = \sigma_{\varepsilon}.$

- *The scheme is fourth-order-isotropic, that is $L_5(r) = \mathcal{O}(\Delta t^5), \forall r \in SO_2(\mathbb{R})$, iff the three following properties are satisfied:*
  - *it is third-order-isotropic,*
  - $2E^{\rho}_{\varepsilon_2} + 4 + 3E^{\rho}_{\varepsilon} = 0, \sigma_{\varepsilon} = \sigma_{\varphi_{xx}}, E^{q_x}_{\varepsilon_2} = E^{q_y}_{\varepsilon_2} = 0, \sigma_{\varphi_x} = \sigma_{\varphi_y} = 1/(6\sigma_{\varphi_{xx}}),$
  - $2 + 3E^{\rho}_{\varepsilon} = 0$ *or* $\sigma_{\varepsilon_2} = \sigma_{\varphi_{xx}}$.
- *The scheme is never fifth-order-isotropic, that is to say the property $L_5(r) = \mathcal{O}(\Delta t^6), \forall r \in SO_2(\mathbb{R})$, is never true.*

Remark 10. Some of the properties that ensure the isotropy of the D_2Q_9 scheme are well-known and often used. In particular, the parameters that are taken null in order to obtain the isotropy at first and second orders are also null for isotropic reasons considering the kinetic solution at equilibrium in the continuous environment: namely even (respectively odd) moments only depend on even (respectively odd) conserved moments. However, the results on the third and fourth orders are more surprising even if some usual D_2Q_9 schemes can be seen as peculiar cases of those we propose.

The proof of Proposition 9 is detailed below using the five lemmas: one for each order. In fact, this proof follows the procedure explained previously and makes use of the positivity of the coefficients $\sigma_{\varepsilon}, \sigma_{\varepsilon_2}, \sigma_{\varphi_x}, \sigma_{\varphi_y}, \sigma_{\varphi_{xx}}$, and $\sigma_{\varphi_{xy}}$.

3.2.1. First order

The isotropy at first order is detailed in the following lemma.

Lemma 11. *The D_2Q_9 scheme is first-order-isotropic if, and only if the properties* (I$_1$)–(I$_2$) *are satisfied.*

(I$_1$) *At equilibrium, the energy is proportional to the density (the proportionality factor is relative to the sound velocity).*
(I$_2$) *Both moments $m_{\varphi_{xx}}$ and $m_{\varphi_{xy}}$ vanish at equilibrium.*

Properties (I$_1$)–(I$_2$) *involve the following structure for the matrix E (there is no constraint on the matrix S at first order):*

$$E = \begin{pmatrix} E^{\rho}_{\varepsilon}\lambda^2 & 0 & 0 \\ E^{\rho}_{\varepsilon_2}\lambda^4 & E^{q_x}_{\varepsilon_2}\lambda^3 & E^{q_y}_{\varepsilon_2}\lambda^3 \\ E^{\rho}_{\varphi_x}\lambda^3 & E^{q_x}_{\varphi_x}\lambda^2 & E^{q_y}_{\varphi_x}\lambda^2 \\ E^{\rho}_{\varphi_y}\lambda^3 & E^{q_x}_{\varphi_y}\lambda^2 & E^{q_y}_{\varphi_y}\lambda^2 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}.$$

Remark 12. This result gives a rigorous proof for usual physical assumptions.

Proof. Considering equations $\mathrm{Coeff}(\cos^1\sin^1|2,1,1)$ and $\mathrm{Coeff}(\cos^0\sin^1|2,1,1)$ yields to $E^{\rho}_{\varphi_{xx}} = E^{\rho}_{\varphi_{xy}} = 0$. Then the system of four equations

$$\begin{cases} \mathrm{Coeff}(\cos^2\sin^1|2,2,2) = 0, \\ \mathrm{Coeff}(\cos^1\sin^0|2,2,2) = 0, \\ \mathrm{Coeff}(\cos^0\sin^1|2,2,2) = 0, \\ \mathrm{Coeff}(\cos^0\sin^0|2,2,2) = 0, \end{cases}$$

eliminates the four unknowns $E^{q_x}_{\varphi_{xx}}, E^{q_y}_{\varphi_{xx}}, E^{q_x}_{\varphi_{xy}}$ and $E^{q_y}_{\varphi_{xy}}$. In order to solve all of the remained equations, it is sufficient to consider equations $\mathrm{Coeff}(\cos^0\sin^0|2,1,2)$ and $\mathrm{Coeff}(\cos^0\sin|3,2,2)$ in the unknowns $E^{q_x}_{\varepsilon}$ and $E^{q_y}_{\varepsilon}$: more precisely the equilibrium energy is characterized by $E^{q_x}_{\varepsilon} = E^{q_y}_{\varepsilon} = 0$. □

3.2.2. Second order

For the D_2Q_9 scheme, the isotropy at second order can be characterized by the following lemma:

Lemma 13. *The D_2Q_9 scheme is second-order-isotropic if and only if the properties* (I$_1$)–(I$_4$) *are satisfied.*

(I$_3$) *At equilibrium, the heat flux is proportional to the momentum, that is $m_{\varphi_x} = E^{q_x}_{\varphi_x}\lambda^2 q_x$ and $m_{\varphi_y} = E^{q_x}_{\varphi_x}\lambda^2 q_y$.*
(I$_4$) *The proportionality factor $E^{q_x}_{\varphi_x}$ is related to the relaxation times by the relation $E^{q_x}_{\varphi_x} = -(4\sigma_{\varphi_{xy}} - \sigma_{\varphi_{xx}})/(\sigma_{\varphi_{xx}} + 2\sigma_{\varphi_{xy}})$.*

Properties (I_1)–(I_4) *involve the following structure for the matrix E (there is no constraint on the matrix S at second order):*

$$E = \begin{pmatrix} E^{\rho}_{\varepsilon}\lambda^2 & 0 & 0 \\ E^{\rho}_{\varepsilon_2}\lambda^4 & E^{q_x}_{\varepsilon_2}\lambda^3 & E^{q_y}_{\varepsilon_2}\lambda^3 \\ 0 & c\lambda^2 & 0 \\ 0 & 0 & c\lambda^2 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix},$$

with $c = -(4\sigma_{\varphi_{xy}} - \sigma_{\varphi_{xx}})/(\sigma_{\varphi_{xx}} + 2\sigma_{\varphi_{xy}})$.

Proof. Considering equation Coeff(2, 2, 2, 1) and more precisely the system

$$\begin{cases} \mathrm{Coeff}(\cos^0 \sin^0 |2, 2, 2, 1) = 0, \\ \mathrm{Coeff}(\cos^0 \sin^0 |3, 1, 1, 1) = 0, \end{cases}$$

we find that $E^{\rho}_{\varphi_x}$, $E^{\rho}_{\varphi_y}$ vanish. Then the system

$$\begin{cases} \mathrm{Coeff}(\cos^0 \sin^2 |2, 2, 2, 2) = 0, \\ \mathrm{Coeff}(\cos^0 \sin^2 |3, 1, 1, 2) = 0, \end{cases}$$

implies both results $E^{q_x}_{\varphi_x} = E^{q_y}_{\varphi_y}$ and $E^{q_x}_{\varphi_y} = -E^{q_y}_{\varphi_x}$. Using equation Coeff($\cos^0 \sin^2$ |3, 1, 1, 3) yields $E^{q_y}_{\varphi_x} = 0$ and equation Coeff($\cos^2 \sin^2$ |2, 2, 3, 3) gives the characterization on $E^{q_x}_{\varphi_x}$. Finally, we prove that the previous relations are sufficient to enforce the isotropy at second order. □

Remark 14. The relation $E^{q_x}_{\varphi_x} = -(4\sigma_{\varphi_{xy}} - \sigma_{\varphi_{xx}})/(\sigma_{\varphi_{xx}} + 2\sigma_{\varphi_{xy}})$ is exactly the relation (41) of [3] except for the roles of the relaxation times $\sigma_{\varphi_{xx}}$ and $\sigma_{\varphi_{xy}}$ that have to be exchanged.

3.2.3. Third order

For the D_2Q_9 scheme, the isotropy at third order can be characterized by the following lemma:

Lemma 15. *The* D_2Q_9 *scheme is third-order-isotropic if and only if all of the properties* (I_1)–(I_5) *and one of the properties* (I_6)–(I_8) *are satisfied.*

(I_5) *The relaxation times* $\sigma_{\varphi_{xx}}$ *and* $\sigma_{\varphi_{xy}}$ *(relating to second order moments) are the same, so that* $c = -1$.

(I_6) *At equilibrium, the square of the energy is proportional to the density: that is* $E^{q_x}_{\varepsilon_2} = 0$ *and* $E^{q_y}_{\varepsilon_2} = 0$, *and is linked with the energy through the relation* $2E^{\rho}_{\varepsilon_2} + 4 + 3E^{\rho}_{\varepsilon} = 0$.

(I_7) *At equilibrium, the square of the energy is proportional to the density: that is* $E^{q_x}_{\varepsilon_2} = 0$ *and* $E^{q_y}_{\varepsilon_2} = 0$, *and both relaxation times related to the heat flux satisfy* $\sigma_{\varphi_x} = \sigma_{\varphi_y} = 1/(12\sigma_{\varphi_{xx}})$.

(I_8) *Both relaxation times related to the heat flux satisfy* $\sigma_{\varphi_x} = \sigma_{\varphi_y} = 1/(12\sigma_{\varphi_{xx}})$, *and both viscosities are linked by* $\sigma_{\varepsilon} = \sigma_{\varphi_{xx}}$.

Properties (I_1)–(I_5) + (I_6) *involve the following structure of the matrices E and S:*

$$E = \begin{pmatrix} E^{\rho}_{\varepsilon}\lambda^2 & 0 & 0 \\ \dfrac{-4 - 3E^{\rho}_{\varepsilon}}{2}\lambda^4 & 0 & 0 \\ 0 & -\lambda^2 & 0 \\ 0 & 0 & -\lambda^2 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \qquad S = \mathrm{Diag}(s_{\varepsilon}, s_{\varepsilon_2}, s_{\varphi_x}, s_{\varphi_y}, s_{\varphi_{xx}}, s_{\varphi_{xx}})^{\mathtt{t}}.$$

Properties (I_1)–(I_5) + (I_7) *involve the following structure of the matrices E and S:*

$$E = \begin{pmatrix} E^{\rho}_{\varepsilon}\lambda^2 & 0 & 0 \\ E^{\rho}_{\varepsilon_2}\lambda^4 & 0 & 0 \\ 0 & -\lambda^2 & 0 \\ 0 & 0 & -\lambda^2 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \qquad S = \mathrm{Diag}\left(s_{\varepsilon}, s_{\varepsilon_2}, 3\frac{2 - s_{\varphi_{xx}}}{3 - s_{\varphi_{xx}}}, 3\frac{2 - s_{\varphi_{xx}}}{3 - s_{\varphi_{xx}}}, s_{\varphi_{xx}}, s_{\varphi_{xx}}\right)^{\mathtt{t}}.$$

Properties (I_1)–(I_5) + (I_8) *involve the following structure of the matrices E and S:*

$$E = \begin{pmatrix} E_\varepsilon^\rho \lambda^2 & 0 & 0 \\ E_{\varepsilon_2}^\rho \lambda^4 & E_{\varepsilon_2}^{q_x} \lambda^3 & E_{\varepsilon_2}^{q_y} \lambda^3 \\ 0 & -\lambda^2 & 0 \\ 0 & 0 & -\lambda^2 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix},$$

$$S = \mathrm{Diag}\left(s_{\varphi_{xx}}, s_{\varepsilon_2}, 3\frac{2 - s_{\varphi_{xx}}}{3 - s_{\varphi_{xx}}}, 3\frac{2 - s_{\varphi_{xx}}}{3 - s_{\varphi_{xx}}}, s_{\varphi_{xx}}, s_{\varphi_{xx}} \right)^{\mathrm{t}}.$$

Proof. We first consider equation Coeff($\cos^2 \sin^2$ |1, 1, 1, 1, 2) and we establish property (I_6). Then, multiple choices appear considering equations Coeff(3, 2, 2, 2, 3), Coeff($\cos^2 \sin^1$ |2, 2, 2, 2, 1) and Coeff($\cos^2 \sin^1$ |2, 1, 1, 1, 2). In order to give more details about these choices we have to define seven relations:

(I_a) $E_{\varepsilon_2}^{q_y} = 0$, (I_d) $-1 + 6\sigma_{\varphi_{xx}}(\sigma_{\varphi_x} + \sigma_{\varphi_y}) = 0$,
(I_b) $\sigma_\varepsilon = \sigma_{\varphi_{xx}}$, (I_e) $4 + 2E_{\varepsilon_2}^\rho + 3E_\varepsilon^\rho = 0$,
(I_c) $E_{\varepsilon_2}^{q_x} = 0$, (I_f) $-1 + 4\sigma_{\varphi_{xx}}(2\sigma_{\varphi_x} + \sigma_{\varphi_y}) = 0$,
(I_g) $\sigma_{\varphi_x} = \sigma_{\varphi_y}$.

The investigation of these equations involves some of these seven relations. More precisely, we have:

- equation Coeff($\cos^1 \sin^0$ |3, 2, 2, 2, 3) yields to a dichotomy between (I_a) and (I_c),
- equation Coeff($\cos^0 \sin^1$ |3, 2, 2, 2, 3) yields to a dichotomy between (I_b) and (I_c),
- equation Coeff($\cos^5 \sin^0$ |3, 2, 2, 2, 3) yields to a dichotomy between (I_a) and (I_d),
- equation Coeff($\cos^4 \sin^1$ |3, 2, 2, 2, 3) yields to a dichotomy between (I_c) and (I_d),
- equation Coeff($\cos^1 \sin^1$ |2, 2, 2, 2, 1) yields to a dichotomy between (I_e) and (I_f),
- equation Coeff($\cos^0 \sin^2$ |2, 1, 1, 1, 2) yields to a dichotomy between (I_e) and (I_g).

This gives *a priori* $2^8 = 256$ possibilities. However, it is straightforward to see that only three cases remain:

- properties (I_a), (I_c) and (I_e) are true,
- properties (I_a), (I_c), (I_f) and (I_g) are true,
- properties (I_b), (I_d) and (I_f) are true.

Finally, each of these three cases solves all of the remaining equations and we get isotropy at the third order. □

Remark 16. – Since the properties (I_5)–(I_6) are well-known (the equality $2E_{\varepsilon_2}^\rho + 4 + 3E_\varepsilon^\rho = 0$ is a generalization of the classical choice $E_\varepsilon^\rho = -2$ and $E_{\varepsilon_2}^\rho = 1$), it is really interesting to obtain a rigorous justification based on isotropic properties.
- The equality $s_{\varphi_x} = 3(2 - s_{\varphi_{xx}})/(3 - s_{\varphi_{xx}})$ is also given in [3].
- To our knowledge, the sets of parameters achieved by properties (I_1)–(I_5)-(I_7) and (I_1)–(I_5)-(I_8) are new, though the second one is less interesting for acoustic applications because of the link between both viscosities.

3.2.4. Fourth order

For the D_2Q_9 scheme, the isotropy at fourth order can be characterized by the following lemma:

Lemma 17. *The* D_2Q_9 *scheme is fourth-order-isotropic if and only if all of the properties* (I_1)–(I_6), (I_9)–(I_{11}) *and one of the two properties* (I_{12})–(I_{13}) *are satisfied.*

(I_9) *The relaxation times related to odd moments satisfy* $\sigma_{\varphi_x} = \sigma_{\varphi_y} = 1/(6\sigma_{\varphi_{xx}})$.
(I_{10}) *At equilibrium, the square of the kinetic energy is proportional to the density* $\varepsilon_2 = E_{\varepsilon_2}^\rho \rho \lambda^4$.
(I_{11}) *At equilibrium, the kinetic energy and its square are linked by* $2E_{\varepsilon_2}^\rho + 4 + 3E_\varepsilon^\rho = 0$.
(I_{12}) *The relaxation times related to even moments satisfy* $\sigma_\varepsilon = \sigma_{\varepsilon_2} = \sigma_{\varphi_{xx}}$.
(I_{13}) *The sound velocity is imposed through the equality* $2 + 3E_\varepsilon^\rho = 0$.

These conditions involve the following structure for the matrices E and S:

$$E = \begin{pmatrix} E_\varepsilon^\rho \lambda^2 & 0 & 0 \\ \dfrac{-4 - 3E_\varepsilon^\rho}{2}\lambda^4 & 0 & 0 \\ 0 & -\lambda^2 & 0 \\ 0 & 0 & -\lambda^2 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \qquad S = \mathrm{Diag}\left(s_\varepsilon, s_{\varepsilon_2}, 6\frac{2 - s_{\varphi_{xx}}}{6 - s_{\varphi_{xx}}}, 6\frac{2 - s_{\varphi_{xx}}}{6 - s_{\varphi_{xx}}}, s_\varepsilon, s_\varepsilon \right)^{\mathrm{t}},$$

where either $s_{\varepsilon_2} = s_\varepsilon$ *or* $E_\varepsilon^\rho = -2/3$.

Proof. Since there are three sets of coefficients giving third order isotropy, the proof is divided into three cases.

The cases with (I_1)–(I_5) and either (I_7) or (I_8) are forbidden because they both imply the condition $\sigma_{\varphi_{xx}} = 0$, using equation Coeff$(\cos^2 \sin^2 |1, 1, 1, 1, 1, 1)$.

So we have to assume (I_1)–(I_6) in order to enforce isotropy at third order. These assumptions imply $E^{q_x}_{\varepsilon_2} = E^{q_y}_{\varepsilon_2} = 0$ using equations Coeff$(\cos^5 \sin^0 |1, 1, 1, 2, 2, 2)$ and Coeff$(\cos^4 \sin^1 |1, 1, 1, 2, 2, 2)$ and $\sigma_{\varphi_y} = \sigma_{\varphi_x}$ thanks to equation Coeff$(\cos^5 \sin^1 |2, 1, 2, 2, 2, 2)$. Then equation Coeff$(\cos^2 \sin^2 |3, 1, 1, 1, 1, 3)$ involves property (I_9) and equation Coeff$(\cos^1 \sin^1 |1, 1, 2, 2, 2, 1)$ involves property (I_{11}). Finally the dichotomy between (I_{12}) and (I_{13}) comes from equation Coeff$(\cos^2 \sin^2 |2, 1, 1, 1, 1, 2)$. □

Remark 18. – Since fourth-order-isotropy imposes the equality between both relaxation times s_{φ_x} and s_{φ_y}, this lemma is the last step that yields the linear dependence between the heat flux and the momentum.
- The property (I_{11}) is involved in a particular case in [3] ($E^\rho_\varepsilon = -2, E^\rho_{\varepsilon_2} = 1$) but without the condition on the relaxation times given in (I_9).
- The case (I_1)–(I_5)-(I_9)–(I_{11})-(I_{13}) ensures isotropy at fourth order.
- The isotropy at fourth order is quite restrictive: both relaxation times linked to the viscosities (s_ε and $s_{\varphi_{xx}}$) must be equal.
- Note that these results generalize those obtained in [12] for quartic parameters. More precisely, the values $\sigma_{\varphi_x} = \sqrt{3}/3$ and $\sigma_{\varphi_{xx}} = \sqrt{3}/6$ given in [12] are consistent with property (I_9).

3.2.5. Fifth order

For the D_2Q_9 scheme, we have the following lemma:

Lemma 19. *The D_2Q_9 scheme is never fifth-order-isotropic.*

Proof. In both cases (I_1)–(I_5)-(I_9)–(I_{12}) and (I_1)–(I_5)-(I_9)–(I_{11})-(I_{13}), Proposition 5 gives equations that cannot be solved for every rotation of the frame: for example Coeff$(1, 1, 1, 1, 1, 1, 2)$ is of type $c \cos\theta \sin\theta$, where c is a given real constant independent of the parameters (namely, c do not vanish). □

3.3. D_2Q_{13} scheme

In this subsection, we give the main result on the D_2Q_{13} scheme in Proposition 20. The proof of this proposition is then subdivided into two lemmas in order to explain the methodology.

Proposition 20. *Let $L_4(r)$ be the lack of isotropy for the D_2Q_{13} scheme at fourth order for the rotation r. Then we have the following propositions.*

- *The scheme is first-order-isotropic, that is $L_4(r) = \mathcal{O}(\Delta t^2)$, $\forall r \in SO_2(\mathbb{R})$, iff $E^{q_x}_\varepsilon$, $E^{q_y}_\varepsilon$, $E^\rho_{\varphi_{xx}}$, $E^{q_x}_{\varphi_{xx}}$, $E^{q_y}_{\varphi_{xx}}$, $E^\rho_{\varphi_{xy}}$, $E^{q_x}_{\varphi_{xy}}$, and $E^{q_y}_{\varphi_{xy}}$ are zero.*
- *The scheme is second-order-isotropic, that is $L_4(r) = \mathcal{O}(\Delta t^3)$, $\forall r \in SO_2(\mathbb{R})$, iff the four following properties are satisfied:*
  - *it is first-order-isotropic,*
  - *$E^\rho_{\varphi_x}$, $E^\rho_{\varphi_y}$, $E^\rho_{x\varepsilon_2}$, and $E^\rho_{y\varepsilon_2}$ are zero,*
  - *$E^{q_x}_{\varphi_x}$ and $E^{q_y}_{\varphi_y}$ are equal to $(\sigma_{\varphi_{xx}} - 4\sigma_{\varphi_{xy}})/(3\sigma_{\varphi_{xy}} + \sigma_{\varphi_{xx}})$,*
  - *$E^{q_x}_{x\varepsilon_2} = E^{q_y}_{y\varepsilon_2}$, $E^{q_y}_{\varphi_x} = -E^{q_x}_{\varphi_y}$, and $E^{q_y}_{x\varepsilon_2} = -E^{q_x}_{y\varepsilon_2}$.*

The proof of Proposition 20 is detailed below with the aid of two lemmas: one for each order. The methodology is the same as the one for the D_2Q_9 scheme.

3.3.1. First order

For the D_2Q_{13} scheme, the isotropy at first order is characterized by the following lemma:

Lemma 21. *The D_2Q_{13} scheme is first-order-isotropic if and only if the properties (I_1')–(I_2') are satisfied.*

(I_1') *At equilibrium, the energy is proportional to the density (the proportionality factor is relative to the sound velocity).*

(I_2') *At equilibrium, both moments $m_{\varphi_{xx}}$ and $m_{\varphi_{xy}}$ are zero.*

Both properties (I_1')–(I_2') *imply the following structure of the matrix E (there is no constraint on the matrix S at first order):*

$$E = \begin{pmatrix} E_{\varepsilon}^{\rho}\lambda^2 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \\ E_{\varphi_x}^{\rho}\lambda^3 & E_{\varphi_x}^{q_x}\lambda^2 & E_{\varphi_x}^{q_y}\lambda^2 \\ E_{\varphi_y}^{\rho}\lambda^3 & E_{\varphi_y}^{q_x}\lambda^2 & E_{\varphi_y}^{q_y}\lambda^2 \\ E_{x\varepsilon_2}^{\rho}\lambda^5 & E_{x\varepsilon_2}^{q_x}\lambda^4 & E_{x\varepsilon_2}^{q_y}\lambda^4 \\ E_{y\varepsilon_2}^{\rho}\lambda^5 & E_{y\varepsilon_2}^{q_x}\lambda^4 & E_{y\varepsilon_2}^{q_y}\lambda^4 \\ E_{\varepsilon_2}^{\rho}\lambda^4 & E_{\varepsilon_2}^{q_x}\lambda^3 & E_{\varepsilon_2}^{q_y}\lambda^3 \\ E_{\varepsilon_3}^{\rho}\lambda^6 & E_{\varepsilon_3}^{q_x}\lambda^5 & E_{\varepsilon_3}^{q_y}\lambda^5 \\ E_{xx\varepsilon}^{\rho}\lambda^4 & E_{xx\varepsilon}^{q_x}\lambda^3 & E_{xx\varepsilon}^{q_y}\lambda^3 \end{pmatrix}.$$

Remark 22. The first order isotropy conditions are the same for the D_2Q_9 and D_2Q_{13} schemes.

3.3.2. Second order

For the D_2Q_{13} scheme, the isotropy at second order is described in the following lemma:

Lemma 23. *The* D_2Q_{13} *scheme is second-order-isotropic if and only if the properties* (I_1')–(I_5') *are satisfied.*

(I_3') *At equilibrium the heat flux is a rotation–dilatation of the momentum, more precisely* $m_{\varphi_x} = E_{\varphi_x}^{q_x}\lambda^2 q_x + E_{\varphi_x}^{q_y}\lambda^2 q_y$ *and* $m_{\varphi_y} = -E_{\varphi_x}^{q_y}\lambda^2 q_x + E_{\varphi_x}^{q_x}\lambda^2 q_y$.

(I_4') *At equilibrium the moment of order five* $(m_{x\varepsilon_2}, m_{y\varepsilon_2})$ *is a rotation–dilatation of the momentum, more precisely* $m_{x\varepsilon_2} = E_{x\varepsilon_2}^{q_x}\lambda^4 q_x + E_{x\varepsilon_2}^{q_y}\lambda^4 q_y$ *and* $m_{y\varepsilon_2} = -E_{x\varepsilon_2}^{q_y}\lambda^4 q_x + E_{x\varepsilon_2}^{q_x}\lambda^4 q_y$.

(I_5') *The equilibrium states are related to the relaxation times by the relations* $E_{x\varepsilon_2}^{q_x} = a$ *and* $E_{x\varepsilon_2}^{q_y} = b$*, with*

$$a := -\frac{1}{12}\frac{7(7\sigma_{\varphi_{xx}} + 2\sigma_{\varphi_{xy}})E_{\varphi_x}^{q_x} + 5(17\sigma_{\varphi_{xx}} - 4\sigma_{\varphi_{xy}})}{\sigma_{\varphi_{xx}} + \sigma_{\varphi_{xy}}}, \qquad b := \frac{7}{12}E_{\varphi_y}^{q_x}\frac{7\sigma_{\varphi_{xx}} + 2\sigma_{\varphi_{xy}}}{\sigma_{\varphi_{xx}} + \sigma_{\varphi_{xy}}}. \tag{16}$$

Properties (I_1')–(I_5') *imply the following structure of the matrix E (there is no condition on the matrix S at second order):*

$$E = \begin{pmatrix} E_{\varepsilon}^{\rho}\lambda^2 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & E_{\varphi_x}^{q_x}\lambda^2 & E_{\varphi_x}^{q_y}\lambda^2 \\ 0 & -E_{\varphi_x}^{q_y}\lambda^2 & E_{\varphi_x}^{q_x}\lambda^2 \\ 0 & a\lambda^4 & b\lambda^4 \\ 0 & -b\lambda^4 & a\lambda^4 \\ E_{\varepsilon_2}^{\rho}\lambda^4 & E_{\varepsilon_2}^{q_x}\lambda^3 & E_{\varepsilon_2}^{q_y}\lambda^3 \\ E_{\varepsilon_3}^{\rho}\lambda^6 & E_{\varepsilon_3}^{q_x}\lambda^5 & E_{\varepsilon_3}^{q_y}\lambda^5 \\ E_{xx\varepsilon}^{\rho}\lambda^4 & E_{xx\varepsilon}^{q_x}\lambda^3 & E_{xx\varepsilon}^{q_y}\lambda^3 \end{pmatrix},$$

where a and b are given in (16).

Remark 24. The second-order-isotropic property for both D_2Q_9 and D_2Q_{13} schemes are of same type, and more precisely the D_2Q_{13} scheme is a generalization of the D_2Q_9 scheme (see for example the condition on $E_{\varphi_x}^{q_x}$).

3.3.3. Third order

Because of the very high number of cases offered to enforce the isotropy at third order (to our knowledge, not less than 17 different cases have to be investigated), only some sufficient conditions are given in this paper. First of all, we already know that the isotropy at third order implies properties (I_1')–(I_9').

(I_7') *At equilibrium, the heat flux is proportional to the momentum: that is* $m_{\varphi_x} = E_{\varphi_x}^{q_x}\lambda^2 qx$ *and* $m_{\varphi_y} = E_{\varphi_x}^{q_x}\lambda^2 qy$.

(I_8') *At equilibrium the moment of order five* $(m_{x\varepsilon_2}, m_{y\varepsilon_2})$ *is proportional to the momentum: that is* $m_{x\varepsilon_2} = E_{x\varepsilon_2}^{q_x}\lambda^4 qx$ *and* $m_{\varphi_y} = E_{x\varepsilon_2}^{q_x}\lambda^4 qy$.

(I_9') *The proportional coefficients* $E_{\varphi_x}^{q_x}$ *and* $E_{x\varepsilon_2}^{q_x}$ *are linked by* $E_{\varepsilon_2}^{q_x} = -(21/8)E_{\varphi_x}^{q_x} - 65/24$.

Assuming properties (I_1')–(I_9'), we give some example of sets of coefficients that make the D_2Q_{13} scheme isotropic in the Appendix.

4. Numerical results for the D_2Q_9 scheme

In this section we present some preliminary numerical results in order to appreciate the lack of isotropy order by order for the D_2Q_9 scheme. Since it is not easy to observe the lack of isotropy in the sense of Definition 6, we have to investigate in another way to explicitly determine the isotropy error.

We propose to investigate the evolution of a radial function after a few time steps, and in particular to compare the profiles for several angles. We use a D_2Q_9 scheme with periodic boundary conditions on $[-1, 1] \times [-1, 1]$, initialized with the equilibrium state corresponding to the Gaussian function for the first moment $\rho(x, y) = \exp(-50x^2 - 50y^2)$, and to the zero function for the second and third moments $q_x(x, y) = q_y(x, y) = 0$. Note that the first-order correction for non-conserved momenta (through Chapman–Enskog expansion for instance) is not used here: first, the purpose of this paper is not to give numerical results that are in good agreement with the physics but rather to investigate a particular property of numerical schemes; second, the results do not depend on the choice of the initial condition provided that it remains analytically isotropic.

In order to determine the error of isotropy, we first plot the profile of the density $r \mapsto \rho(\theta, r)$ for several angles θ at $t = 0.25$ and then the difference $r \mapsto \rho(\theta, r) - \rho(0, r)$. The space step Δx is fixed equal to 0.0025 in this first analysis. For visualization convenience, we restrict the choice of the angles to $\theta \in \{\theta_i,\ 0 \leq i \leq 3\}$.

Name	Value	Corresponding direction	Symbol
θ_0	0	[1, 0]	solid line '-'
θ_1	$\pi/4$	[1, 1]	dashed line '- -'
θ_2	$\arctan(1/2)$	[1, 2]	dash-dot line '-.'
θ_3	$\arctan(1/3)$	[1, 3]	dotted line ':'

Because of the symmetries of the lattice, a lot of other angles could be included in this analysis without changing the figures, for instance angles of the form $\pm\theta_i + k\pi/2$.

For each result we only have the values on the nodes of the mesh: namely for each angle θ, the discrete values of the abscissa r are not equal. Since we have to compare these densities, we interpolate the value with a spline method of order 5 so that the interpolation error does not interfere.

Remark 25. By plotting the density ρ, we illustrate (albeit imperfectly) the theoretical lack of isotropy $L_5(r)$: the following figures only show the error of isotropy on the first moment of a particular function and not the error of the discrete operator. Strictly speaking, $L_5(r)$ is the norm of an operator and we would have to take a maximum over all initial conditions. However, for practical use, it is more important to investigate the isotropy for a given initial condition.

We first consider the following parameters for which the scheme is first-order-isotropic but not second order:

$$E = \begin{pmatrix} -2 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & -2 & 0 \\ 0 & 0 & -2 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} \quad \text{and} \quad S = \operatorname{diag}\begin{pmatrix} s_\varepsilon \\ s_{\varepsilon_2} \\ s_{\varphi_x} \\ s_{\varphi_x} \\ s_{\varphi_{xx}} \\ s_{\varphi_{xy}} \end{pmatrix},$$

where S is given thanks to (4) and $\sigma = \{0.5, 0.4, 0.6, 0.3, 0.45, 0.55\}$.

We then consider the following parameters, with no change on the relaxation times, for which the scheme is second-order-isotropic but not third-order-isotropic:

$$E = \begin{pmatrix} -2 & 0 & 0 \\ 6 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & -1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}.$$

We next consider the following parameters, with no change on the relaxation times, for which the scheme is third-order-isotropic but not fourth-order-isotropic:

$$E = \begin{pmatrix} -2 & 0 & 0 \\ 1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & -1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} \quad \text{and} \quad S = \operatorname{diag}\begin{pmatrix} s_\varepsilon \\ s_{\varepsilon_2} \\ s_{\varphi_x} \\ s_{\varphi_x} \\ s_{\varphi_{xx}} \\ s_{\varphi_{xy}} \end{pmatrix}.$$

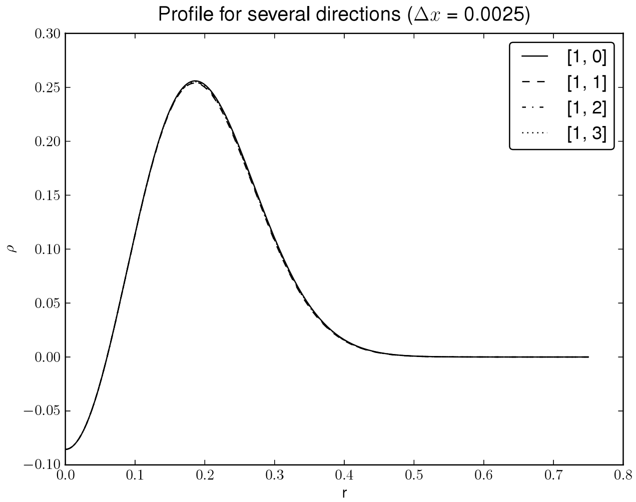

Fig. 1. Isotropy at first order.

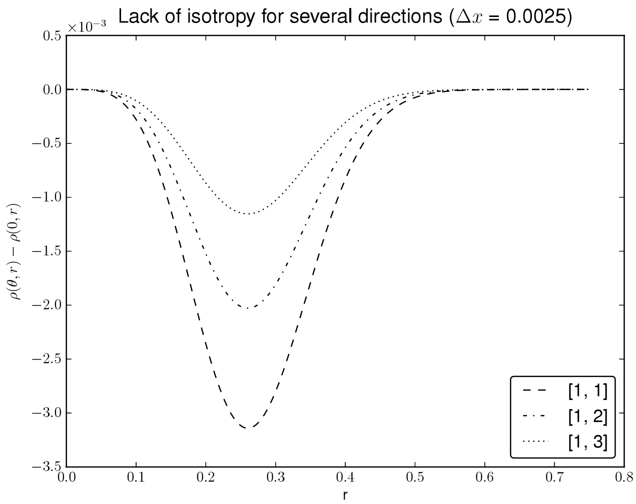

Fig. 2. Isotropy error at first order.

And finally, we consider the following parameters for which the scheme is fourth-order-isotropic:

$$E = \begin{pmatrix} -2 & 0 & 0 \\ 1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & -1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} \quad \text{and} \quad S = \mathrm{diag} \begin{pmatrix} s_\varepsilon \\ s_{\varepsilon_2} \\ s_{\varphi_x} \\ s_{\varphi_x} \\ s_{\varphi_{xx}} \\ s_{\varphi_{xy}} \end{pmatrix},$$

where S is given thanks to (4) and $\sigma = \{0.5, 0.5, 1/3, 1/3, 0.5, 0.5\}$.

The different curves ρ_θ are plotted in Figs. 1, 3, 5 and 7 for each order of isotropy: the lack of isotropy is not high enough to be seen in these figures. However, from Figs. 2, 4, 6 and 8, the reader can clearly observe the falling of the isotropy error.

In order to visualize the orders of isotropy, we investigate the link between the error of isotropy and the space step. More precisely, for a given Δx and a given angle θ_i, we define the function $E_i(\theta_i, \Delta x)$ as the difference between the interpolates of $\rho(\theta_i)$ and of $\rho(0)$. In Fig. 9, we can read the maximum over all the angles θ_i of the L^2-norm of E_i, and that for the four different linear D_2Q_9 schemes. The space step Δx is taken from 0.025 (80×80 mesh) to 0.00025 (8000×8000 mesh). We observe that asymptotically the errors are consistent and that the numerical slopes are in good agreement with the theory.

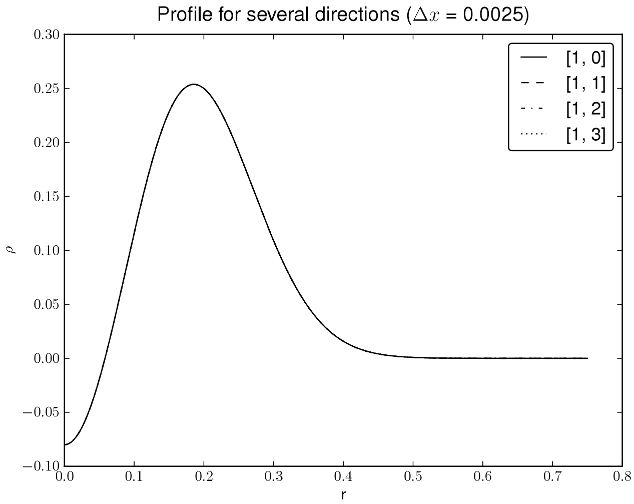

Fig. 3. Isotropy at second order.

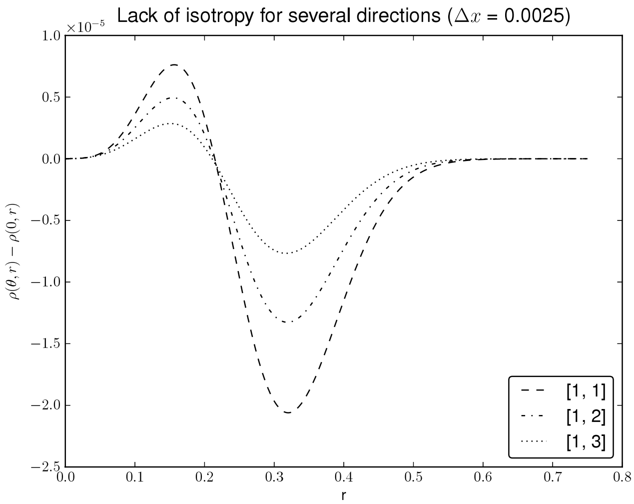

Fig. 4. Isotropy error at second order.

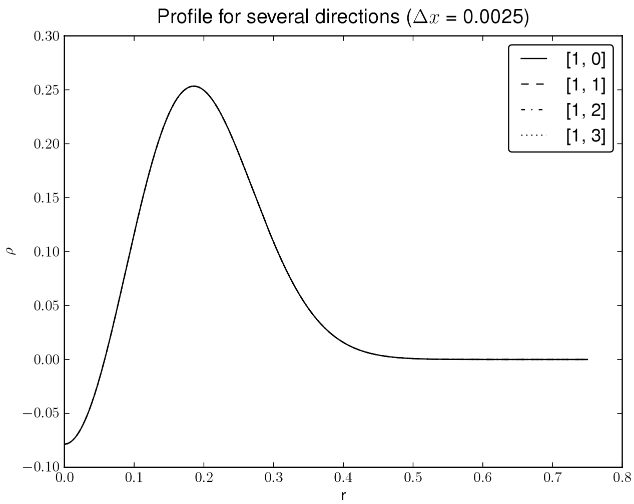

Fig. 5. Isotropy at third order.

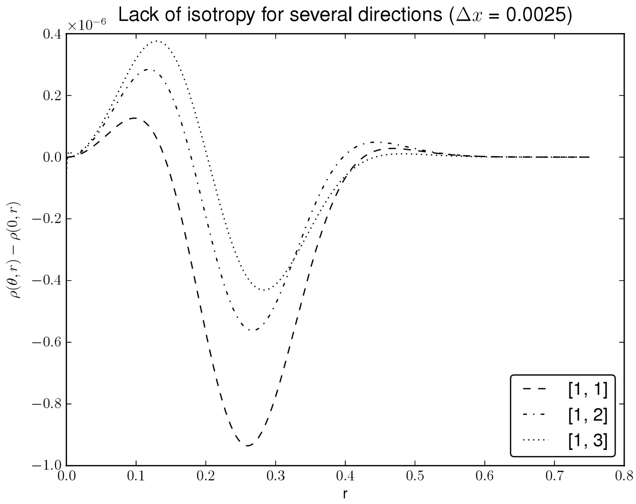

Fig. 6. Isotropy error at third order.

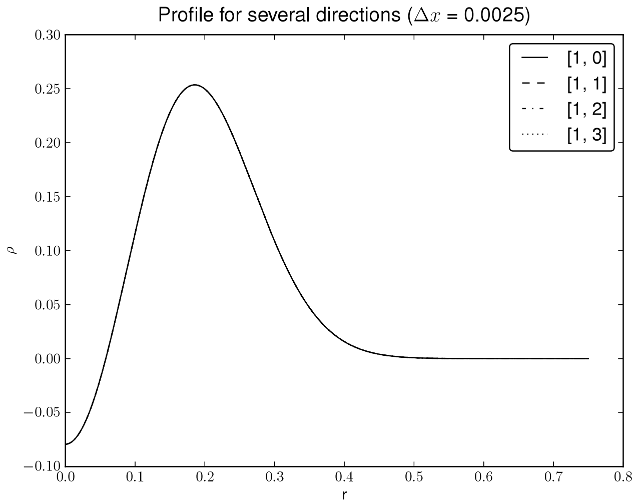

Fig. 7. Isotropy at fourth order.

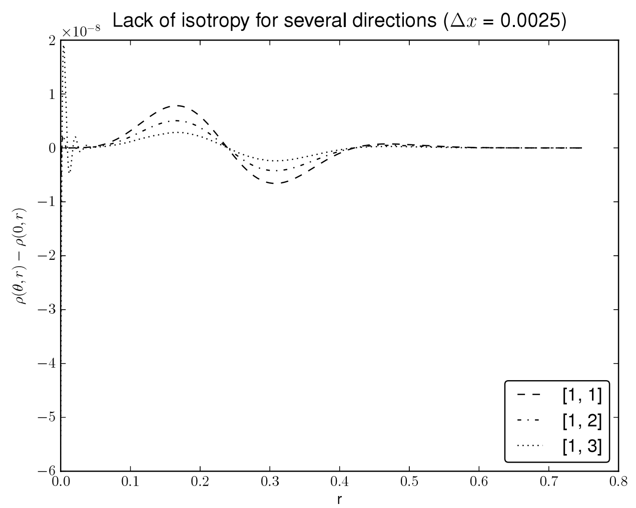

Fig. 8. Isotropy at fourth order.

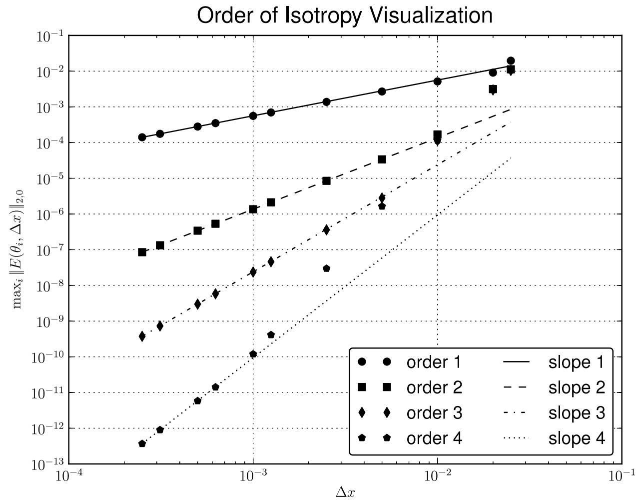

Fig. 9. Order of isotropy.

Indeed, the asymptotic slopes are given in the following tabular form.

Theoretical order	Numerical order
1	0.9974
2	1.9988
3	2.9948
4	3.9912

5. Conclusion

We used a general methodology that guarantees isotropy of a lattice Boltzmann scheme at a given order. This approach is based on the expansion of the equivalent PDEs at an arbitrary order and the invariance by the group operation Φ_n. We have given details of all the possible cases for the basic scheme D_2Q_9 applied to linearized fluid mechanics. Results (up to the second order) have also been proposed for the D_2Q_{13} scheme. Using "inappropriate" choices of parameters in the D_2Q_9 scheme, elementary test cases highlight the lack of isotropy at various orders. The error of isotropy has been quantified with a precise numerical experiment. This work can be extended without conceptional difficulties for 3D lattice Boltzmann schemes and will be done so in a forthcoming contribution.

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Appendix

In this appendix, we propose some details concerning the isotropy at third order for the D_2Q_{13} scheme. This work is in progress: we express the different cases that have to be studied.

In fact, knowing properties (I_1')–(I_9'), equations Coeff(1, 1, 1, 1, 3) and Coeff(2, 2, 2, 3, 3) give multiple choices between the thirteen following relations:

(I_a') $E_{\varphi_x}^{q_x} = -3$,
(I_b') $\sigma_{\varphi_{xy}} = \sigma_{\varphi_{xx}}$,
(I_{c1}') First value for $E_{\varepsilon_3}^{q_y}$,
(I_d') $\sigma_{y\varepsilon_2} = 1/12\sigma_{\varphi_{xx}}$,
(I_e') $\sigma_{x\varepsilon_2} = 1/12\sigma_{\varphi_{xx}}$,
(I_{f1}') First value for $E_{\varepsilon_3}^{q_x}$,
(I_{c2}') Second value for $E_{\varepsilon_3}^{q_y}$,
(I_{f2}') Second value for $E_{\varepsilon_3}^{q_x}$,
(I_g') $3(\sigma_{\varphi_{xx}} + \sigma_{\varphi_{xy}})(\sigma_{x\varepsilon_2} + \sigma_{y\varepsilon_2}) = 1$,
(I_{f3}') Third value for $E_{\varepsilon_3}^{q_x}$,
(I_{g2}') $6\sigma_{\varphi_{xy}}(\sigma_{x\varepsilon_2} + \sigma_{y\varepsilon_2}) + 3\sigma_{\varphi_{xx}}(\sigma_{y\varepsilon_2} + 3\sigma_{x\varepsilon_2})$,
(I_{c3}') Third value for $E_{\varepsilon_3}^{q_y}$,
(I_{c4}') Fourth value for $E_{\varepsilon_3}^{q_y}$.

The investigation of these equations involves some of these thirteen relations. More precisely, we have:

- eq. Coeff($\cos^1 \sin^1$ |1, 1, 1, 1, 3) yields to a dichotomy between $(\mathrm{I}'_{\mathrm{a}})$ and $(\mathrm{I}'_{\mathrm{b}})$,
- eq. Coeff($\cos^0 \sin^0$ |3, 2, 2, 2, 3) yields to a dichotomy between $(\mathrm{I}'_{\mathrm{c1}})$ and $(\mathrm{I}'_{\mathrm{d}})$,
- eq. Coeff($\cos^0 \sin^1$ |3, 2, 2, 2, 3) yields to a dichotomy between $(\mathrm{I}'_{\mathrm{e}})$ and $(\mathrm{I}'_{\mathrm{f1}})$,
- eq. Coeff($\cos^1 \sin^0$ |3, 2, 2, 2, 3) yields to a dichotomy between $(\mathrm{I}'_{\mathrm{c2}})$ and $(\mathrm{I}'_{\mathrm{e}})$,
- eq. Coeff($\cos^4 \sin^1$ |3, 2, 2, 2, 3) yields to a dichotomy between $(\mathrm{I}'_{\mathrm{f2}})$ and $(\mathrm{I}'_{\mathrm{g1}})$,
- eq. Coeff($\cos^2 \sin^1$ |3, 2, 2, 2, 3) yields to a dichotomy between $(\mathrm{I}'_{\mathrm{f3}})$ and $(\mathrm{I}'_{\mathrm{g2}})$,
- eq. Coeff($\cos^3 \sin^0$ |3, 2, 2, 2, 3) yields to a dichotomy between $(\mathrm{I}'_{\mathrm{c3}})$ and $(\mathrm{I}'_{\mathrm{g2}})$,
- eq. Coeff($\cos^5 \sin^0$ |3, 2, 2, 2, 3) yields to a dichotomy between $(\mathrm{I}'_{\mathrm{g1}})$ and $(\mathrm{I}'_{\mathrm{c4}})$.

That gives *a priori* $2^8 = 256$ possibilities. However, preliminary calculations yield that only seventeen cases remain: in order to have isotropy of third order, it is necessary to satisfy at least one of these sets of properties.

(1) $(\mathrm{I}'_{\mathrm{a}})$, $(\mathrm{I}'_{\mathrm{c3}})$, $(\mathrm{I}'_{\mathrm{c4}})$, $(\mathrm{I}'_{\mathrm{d}})$, $(\mathrm{I}'_{\mathrm{e}})$, $(\mathrm{I}'_{\mathrm{f2}})$ and $(\mathrm{I}'_{\mathrm{f3}})$
(2) $(\mathrm{I}'_{\mathrm{b}})$, $(\mathrm{I}'_{\mathrm{d}})$, $(\mathrm{I}'_{\mathrm{e}})$, $(\mathrm{I}'_{\mathrm{g1}})$ and $(\mathrm{I}'_{\mathrm{g2}})$
(3) $(\mathrm{I}'_{\mathrm{a}})$, $(\mathrm{I}'_{\mathrm{c2}})$, $(\mathrm{I}'_{\mathrm{c3}})$, $(\mathrm{I}'_{\mathrm{c4}})$, $(\mathrm{I}'_{\mathrm{d}})$, $(\mathrm{I}'_{\mathrm{f1}})$, $(\mathrm{I}'_{\mathrm{f2}})$ and $(\mathrm{I}'_{\mathrm{f3}})$
(4) $(\mathrm{I}'_{\mathrm{a}})$, $(\mathrm{I}'_{\mathrm{c2}})$, $(\mathrm{I}'_{\mathrm{c4}})$, $(\mathrm{I}'_{\mathrm{d}})$, $(\mathrm{I}'_{\mathrm{f1}})$, $(\mathrm{I}'_{\mathrm{f2}})$ and $(\mathrm{I}'_{\mathrm{g2}})$
(5) $(\mathrm{I}'_{\mathrm{a}})$, $(\mathrm{I}'_{\mathrm{c2}})$, $(\mathrm{I}'_{\mathrm{c3}})$, $(\mathrm{I}'_{\mathrm{d}})$, $(\mathrm{I}'_{\mathrm{f1}})$, $(\mathrm{I}'_{\mathrm{f3}})$ and $(\mathrm{I}'_{\mathrm{g1}})$
(6) $(\mathrm{I}'_{\mathrm{a}})$, $(\mathrm{I}'_{\mathrm{c1}})$, $(\mathrm{I}'_{\mathrm{c3}})$, $(\mathrm{I}'_{\mathrm{c4}})$, $(\mathrm{I}'_{\mathrm{e}})$, $(\mathrm{I}'_{\mathrm{f2}})$ and $(\mathrm{I}'_{\mathrm{f3}})$
(7) $(\mathrm{I}'_{\mathrm{a}})$, $(\mathrm{I}'_{\mathrm{c1}})$, $(\mathrm{I}'_{\mathrm{c4}})$, $(\mathrm{I}'_{\mathrm{e}})$, $(\mathrm{I}'_{\mathrm{f2}})$ and $(\mathrm{I}'_{\mathrm{g2}})$
(8) $(\mathrm{I}'_{\mathrm{a}})$, $(\mathrm{I}'_{\mathrm{c1}})$, $(\mathrm{I}'_{\mathrm{c3}})$, $(\mathrm{I}'_{\mathrm{e}})$, $(\mathrm{I}'_{\mathrm{f3}})$ and $(\mathrm{I}'_{\mathrm{g1}})$
(9) $(\mathrm{I}'_{\mathrm{a}})$, $(\mathrm{I}'_{\mathrm{c1}})$, $(\mathrm{I}'_{\mathrm{c2}})$, $(\mathrm{I}'_{\mathrm{c3}})$, $(\mathrm{I}'_{\mathrm{c4}})$, $(\mathrm{I}'_{\mathrm{f1}})$, $(\mathrm{I}'_{\mathrm{f2}})$ and $(\mathrm{I}'_{\mathrm{f3}})$
(10) $(\mathrm{I}'_{\mathrm{a}})$, $(\mathrm{I}'_{\mathrm{c1}})$, $(\mathrm{I}'_{\mathrm{c2}})$, $(\mathrm{I}'_{\mathrm{c4}})$, $(\mathrm{I}'_{\mathrm{f1}})$, $(\mathrm{I}'_{\mathrm{f2}})$ and $(\mathrm{I}'_{\mathrm{g2}})$
(11) $(\mathrm{I}'_{\mathrm{a}})$, $(\mathrm{I}'_{\mathrm{c1}})$, $(\mathrm{I}'_{\mathrm{c2}})$, $(\mathrm{I}'_{\mathrm{c3}})$, $(\mathrm{I}'_{\mathrm{f1}})$, $(\mathrm{I}'_{\mathrm{f3}})$ and $(\mathrm{I}'_{\mathrm{g1}})$
(12) $(\mathrm{I}'_{\mathrm{a}})$, $(\mathrm{I}'_{\mathrm{c1}})$, $(\mathrm{I}'_{\mathrm{c2}})$, $(\mathrm{I}'_{\mathrm{f1}})$, $(\mathrm{I}'_{\mathrm{g1}})$ and $(\mathrm{I}'_{\mathrm{g2}})$
(13) $(\mathrm{I}'_{\mathrm{b}})$, $(\mathrm{I}'_{\mathrm{c1}})$, $(\mathrm{I}'_{\mathrm{c2}})$, $(\mathrm{I}'_{\mathrm{c3}})$, $(\mathrm{I}'_{\mathrm{c4}})$, $(\mathrm{I}'_{\mathrm{f1}})$, $(\mathrm{I}'_{\mathrm{f2}})$ and $(\mathrm{I}'_{\mathrm{f3}})$
(14) $(\mathrm{I}'_{\mathrm{b}})$, $(\mathrm{I}'_{\mathrm{c1}})$, $(\mathrm{I}'_{\mathrm{c2}})$, $(\mathrm{I}'_{\mathrm{c4}})$, $(\mathrm{I}'_{\mathrm{f1}})$, $(\mathrm{I}'_{\mathrm{f2}})$ and $(\mathrm{I}'_{\mathrm{g2}})$
(15) $(\mathrm{I}'_{\mathrm{b}})$, $(\mathrm{I}'_{\mathrm{c1}})$, $(\mathrm{I}'_{\mathrm{c2}})$, $(\mathrm{I}'_{\mathrm{c3}})$, $(\mathrm{I}'_{\mathrm{f1}})$, $(\mathrm{I}'_{\mathrm{f3}})$ and $(\mathrm{I}'_{\mathrm{g1}})$
(16) $(\mathrm{I}'_{\mathrm{b}})$, $(\mathrm{I}'_{\mathrm{c2}})$, $(\mathrm{I}'_{\mathrm{c3}})$, $(\mathrm{I}'_{\mathrm{c4}})$, $(\mathrm{I}'_{\mathrm{d}})$, $(\mathrm{I}'_{\mathrm{f1}})$, $(\mathrm{I}'_{\mathrm{f2}})$ and $(\mathrm{I}'_{\mathrm{f3}})$
(17) $(\mathrm{I}'_{\mathrm{b}})$, $(\mathrm{I}'_{\mathrm{c1}})$, $(\mathrm{I}'_{\mathrm{c3}})$, $(\mathrm{I}'_{\mathrm{c4}})$, $(\mathrm{I}'_{\mathrm{e}})$, $(\mathrm{I}'_{\mathrm{f2}})$ and $(\mathrm{I}'_{\mathrm{f3}})$

The study of these cases is in progress and we cannot then give a full characterization of it. However, we know some examples of matrices E and S that involve isotropy at third order. We propose here two cases given by properties $(\mathrm{I}'_{\mathrm{a}})$, $(\mathrm{I}'_{\mathrm{c3}})$, $(\mathrm{I}'_{\mathrm{c4}})$, $(\mathrm{I}'_{\mathrm{d}})$, $(\mathrm{I}'_{\mathrm{e}})$, $(\mathrm{I}'_{\mathrm{f2}})$ and $(\mathrm{I}'_{\mathrm{f3}})$.

$$
E = \begin{pmatrix}
E^{\rho}_{\varepsilon}\lambda^2 & 0 & 0 \\
0 & 0 & 0 \\
0 & 0 & 0 \\
0 & -3\lambda^2 & 0 \\
0 & 0 & -3\lambda^2 \\
0 & \dfrac{31}{6}\lambda^4 & 0 \\
0 & 0 & \dfrac{31}{6}\lambda^4 \\
E^{\rho}_{\varepsilon_2}\lambda^4 & E^{q_x}_{\varepsilon_2}\lambda^3 & E^{q_y}_{\varepsilon_2}\lambda^3 \\
\left(\dfrac{274}{39} - \dfrac{67}{462}E^{\rho}_{\varepsilon_2} - \dfrac{137}{3003}E^{\rho}_{\varepsilon}\right)\lambda^6 & -\dfrac{67}{462}E^{q_x}_{\varepsilon_2}\lambda^5 & -\dfrac{67}{462}E^{q_y}_{\varepsilon_2}\lambda^5 \\
E^{\rho}_{xx\varepsilon}\lambda^4 & E^{q_x}_{xx\varepsilon}\lambda^3 & E^{q_y}_{xx\varepsilon}\lambda^3
\end{pmatrix},
$$

$$
S = \operatorname{diag}\left(s_{\varphi_{xx}}, s_{\varphi_{xx}}, s_{\varphi_{xy}}, s_{\varphi_x}, s_{\varphi_x}, s_{\varphi_x}, s_{\varphi_x}, s_{\varepsilon_2}, s_{\varepsilon_3}, s_{xx\varepsilon}\right)^{\mathrm{t}}, \quad \text{where } s_{\varphi_x} = 3\frac{2 - s_{\varphi_{xx}}}{3 - s_{\varphi_{xx}}},
$$

or for instance

$$E = \begin{pmatrix} E_\varepsilon^\rho \lambda^2 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & -3\lambda^2 & 0 \\ 0 & 0 & -3\lambda^2 \\ 0 & \frac{31}{6}\lambda^4 & 0 \\ 0 & 0 & \frac{31}{6}\lambda^4 \\ \left(-\frac{3234}{13} - \frac{361}{26}E_\varepsilon^\rho\right)\lambda^4 & 0 & 0 \\ \left(\frac{1681}{39} + \frac{307}{156}E_\varepsilon^\rho\right)\lambda^6 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix},$$

$$S = \text{diag}\left(s_\varepsilon, s_{\varphi_{xx}}, s_{\varphi_{xy}}, 2\frac{7s_\varepsilon + 5s_{\varphi_{xx}} - 6s_{\varphi_{xx}}s_\varepsilon}{7s_\varepsilon + 5s_{\varphi_{xx}} - 4s_{\varphi_{xx}}s_\varepsilon}, s_{\varphi_y}, s_{x\varepsilon_2}, s_{x\varepsilon_2}, s_{\varepsilon_2}, s_{\varepsilon_3}, s_{xx\varepsilon}\right)^t, \quad \text{where } s_{x\varepsilon_2} = 3\frac{2 - s_{\varphi_{xx}}}{3 - s_{\varphi_{xx}}}.$$

Remark 26. Since the D_2Q_{13} scheme takes into account the velocities of the D_2Q_9 and 4 additional ones, it can be read as a generalization of the D_2Q_9 scheme. In order to illustrate this remark, we propose here a set of coefficients that gives isotropy at third order for both D_2Q_9 and D_2Q_{13} schemes:

$$E = \begin{pmatrix} E_\varepsilon^\rho \lambda^2 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & -\lambda^2 & 0 \\ 0 & 0 & -\lambda^2 \\ 0 & -\frac{1}{12}\lambda^4 & 0 \\ 0 & 0 & -\frac{1}{12}\lambda^4 \\ \left(-\frac{3234}{13} - \frac{361}{26}E_\varepsilon^\rho\right)\lambda^4 & 0 & 0 \\ \left(\frac{1681}{39} + \frac{307}{156}E_\varepsilon^\rho\right)\lambda^6 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix},$$

$$S = \text{diag}\left(s_{\varphi_{xx}}, s_{\varphi_{xx}}, s_{\varphi_{xx}}, \varphi_x, s_{\varphi_x}, s_{\varphi_x}, s_{\varphi_x}, s_{\varepsilon_2}, s_{\varepsilon_3}, s_{xx\varepsilon}\right)^t, \quad \text{where } s_{\varphi_x} = 3\frac{2 - s_{\varphi_{xx}}}{3 - s_{\varphi_{xx}}}.$$

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