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The local Cauchy problem for ionized magnetized reactive gas mixtures

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SUMMARY

We investigate a system of partial differential equations modelling ionized magnetized reactive gas mixtures. In this model, dissipative fluxes are anisotropic linear combinations of fluid variable gradients and also include zeroth-order contributions modelling the direct effect of electromagnetic forces. There are also gradient dependent source terms like the conduction current in the Maxwell–Ampere equation. We introduce the notion of partial symmetrizability and that of entropy for such systems of partial differential equations and establish their equivalence. By using entropic variables, we recast the system into a partially normal form, that is, in the form of a quasilinear partially symmetric hyperbolic–parabolic system. Using a result of Vol'Pert and Hudjaev, we prove local existence and uniqueness of a bounded smooth solution. Copyright © 2005 John Wiley & Sons, Ltd.

KEY WORDS: ionized gas mixtures; chemical reactions; entropy; symmetric hyperbolic–parabolic system; normal forms

1. INTRODUCTION

Ionized magnetized reactive gas mixtures have many practical applications such as to laboratory plasmas, high-speed gas flows or atmospheric phenomena. In this paper, we investigate the structure and properties of the corresponding systems of partial differential equations.

The kinetic theory of ionized gas mixtures can be used to obtain the equations governing high density low temperature plasmas. The resulting systems are different according to the various characteristic lengths and times of the phenomena under investigation. Assuming that there is a single temperature in the mixture—this is the case for various practical applications—the corresponding governing equations are derived in Ferziger and Kaper [1] and Giovangigli and Graille [2] for general reactive polyatomic gas mixtures.

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The corresponding equations—governing ionized magnetized reactive gas mixtures—can be split into conservation equations, transport fluxes, thermochemistry, and Maxwell's equations. A remarkable aspect is that the magnetic field yields anisotropic diffusion mass fluxes, heat flux and viscous tensor. Furthermore, diffusion fluxes involve anisotropic linear combinations of fluid variable gradients as well as zeroth-order terms arising from the direct action of electromagnetic forces. There are also gradient-dependent source terms as the conduction current in the Maxwell–Ampere equation. The corresponding structural mathematical assumptions concerning thermoelectrochemistry and transport coefficients are derived from the kinetic theory of gases [2] and they generalize the situation of non-ionized species [3].

The governing equations for reactive ionized magnetized dissipative gas mixtures constitute a second-order quasilinear system of conservation laws with zeroth order terms in dissipative fluxes and gradient-dependent source terms. We introduce the notion of partial symmetrizability as well as that of entropy for such systems of partial differential equations and establish their equivalence. There is in particular an entropic compatibility condition between zeroth order terms of dissipative fluxes and gradient-dependent source terms. We use the terminology 'partial symmetrization' since the resulting quasilinear system contains symmetric as well as antisymmetric contributions in contrast with the non-ionized case [4–7]. The partially symmetric form is the form that reveals most of the structural symmetry properties of the corresponding partial differential operators.

By using entropic variables, we recast the system into a partially normal form, that is, in the form of a quasilinear partially symmetric hyperbolic–parabolic system. We again use the terminology 'partially normal' since the resulting effective first-order differential operators involve non-symmetric matrices in contrast with the non-ionized case [3–5,7]. In particular, global existence results and asymptotic stability of equilibrium states cannot be obtained from the theorems established in [3–5,7]. Nevertheless, we prove local existence of a unique solution to the Cauchy problem with smooth initial conditions. Our method of proof relies on the results of Kawashima [4] or Vol'Pert and Hudjaev [8] concerning the Cauchy problem for symmetric quasilinear hyperbolic–parabolic composite systems of partial differential equations.

The governing equations for ionized magnetized reactive gas mixtures are presented in Section 2. In Section 3, we investigate partial symmetrizability, normal form and existence of solutions for an abstract system. Finally, in Section 4, we apply these results to the system of partial differential equations modelling multicomponent ionized magnetized reactive gas mixtures.

2. EQUATIONS FOR IONIZED MAGNETIZED REACTIVE GAS MIXTURES

The equations governing dissipative plasmas can be split between conservation equations, transport fluxes, thermochemistry, and Maxwell's equations. These equations can be derived from the kinetic theory of gases by using a first-order Enskog expansion [1,2].

2.1. Conservation equations

We denote by $\mathfrak{S}$ the species indexing set $\mathfrak{S}=\{1,\dots,n^s\}$, n^s the number of species, n_k, ρ_k and q_k the number of moles, the mass and the charge per unit volume of the kth species and m_k the molar mass of the kth species.

The species mass conservation equations read

$$\partial_t \rho_k + \boldsymbol{\partial}_{\mathbf{x}}\cdot(\rho_k \mathbf{v}) + \boldsymbol{\partial}_{\mathbf{x}}\cdot\mathscr{F}_k = m_k \omega_k, \quad k \in \mathfrak{S} \tag{1}$$

where $\mathbf{v}$ is the macroscopic velocity of the mixture, $\mathscr{F}_k$ the diffusion flux and ω_k the chemical source term of the kth species.

The momentum conservation equation can be written as

$$\partial_t(\rho \mathbf{v}) + \boldsymbol{\partial}_{\mathbf{x}}\cdot(\rho \mathbf{v}\otimes\mathbf{v} + p\mathbb{I}) + \boldsymbol{\partial}_{\mathbf{x}}\cdot\boldsymbol{\Pi} = \rho\mathbf{g} + q(\mathbf{E} + \mathbf{v}\wedge\mathbf{B}) + \mathbf{j}\wedge\mathbf{B} \tag{2}$$

where ρ denotes the total mass per unit volume, p the pressure, $\mathbb{I}$ the unit tensor, $\boldsymbol{\Pi}$ the viscous tensor, q the total charge per unit volume, $\mathbf{E}$ the electric field, $\mathbf{B}$ the magnetic field, $\mathbf{g}$ a species independent external force, and $\mathbf{j}$ the conduction current density.

Denoting by $e^{\mathrm{t}} = e + \mathbf{v}\cdot\mathbf{v}/2 + \varepsilon_0\mathbf{E}\cdot\mathbf{E}/2 + \mathbf{B}\cdot\mathbf{B}/2\mu_0$ the total energy per unit mass, e the internal energy per unit mass, ε_0 the dielectric constant, μ_0 the magnetic permeability, $\mathbf{P} = (\mathbf{E}\wedge\mathbf{B})/\mu_0$ the Poynting vector, and $\mathbf{Q}$ the heat flux, the energy conservation equation reads

$$\partial_t(\rho e^{\mathrm{t}}) + \boldsymbol{\partial}_{\mathbf{x}}\cdot((\rho e^{\mathrm{t}} + p)\mathbf{v} + \mathbf{P}) + \boldsymbol{\partial}_{\mathbf{x}}\cdot(\mathbf{Q} + \boldsymbol{\Pi}\cdot\mathbf{v}) = \rho\mathbf{g}\cdot\mathbf{v} \tag{3}$$

2.2. *Transport fluxes*

A remarkable aspect of dissipative plasmas is that transport fluxes in strong magnetic fields are anisotropic [1,2]. In order to take into account this anisotropy we define the unitary vector $\mathscr{B} = \mathbf{B}/B$, where B is the norm of the magnetic field $\mathbf{B}$, and for any vector $\mathbf{X}$, we introduce the three vectors

$$\mathbf{X}^{\parallel} = (\mathscr{B}\cdot\mathbf{X})\mathscr{B}, \quad \mathbf{X}^{\perp} = \mathbf{X} - \mathbf{X}^{\parallel} \quad \text{and} \quad \mathbf{X}^{\odot} = \mathscr{B}\wedge\mathbf{X}$$

which are mutually orthogonal. The diffusion flux $\mathscr{F}_k$, $k \in \mathfrak{S}$, is then given by

$$\mathscr{F}_k = \rho_k \mathbf{V}_k, \quad k \in \mathfrak{S} \tag{4}$$

where the diffusion velocity $\mathbf{V}_k$, $k \in \mathfrak{S}$, reads

$$\begin{aligned}\mathbf{V}_k = -\sum_{l\in\mathfrak{S}} (D_{kl}^{\parallel}\mathbf{d}_l^{\parallel} + D_{kl}^{\perp}\mathbf{d}_l^{\perp} + D_{kl}^{\odot}\mathbf{d}_l^{\odot})& \\ -(\theta_k^{\parallel}(\boldsymbol{\partial}_{\mathbf{x}} \log T)^{\parallel} + \theta_k^{\perp}(\boldsymbol{\partial}_{\mathbf{x}} \log T)^{\perp} + \theta_k^{\odot}(\boldsymbol{\partial}_{\mathbf{x}} \log T)^{\odot})&\end{aligned} \tag{5}$$

In these expressions, the species diffusion driving force $\mathbf{d}_k$, $k \in \mathfrak{S}$, is given by

$$\mathbf{d}_k = \frac{1}{p}(\boldsymbol{\partial}_{\mathbf{x}} p_k - \rho_k\mathbf{g} - q_k(\mathbf{E} + \mathbf{v}\wedge\mathbf{B})) \tag{6}$$

and $D_{kl}^{\parallel}$, $D_{kl}^{\perp}$ and $D_{kl}^{\odot}$, $k, l \in \mathfrak{S}$, are the multicomponent diffusion coefficients, $\theta_k^{\parallel}$, $\theta_k^{\perp}$ and $\theta_k^{\odot}$, $k \in \mathfrak{S}$, the thermal diffusion coefficients, T the absolute temperature, and p_k, $k \in \mathfrak{S}$, the species partial pressures. For non-ionized gases, the charges q_k, $k \in \mathfrak{S}$, vanish so that we have $D_{kl}^{\parallel} = D_{kl}^{\perp}$, $D_{kl}^{\odot} = 0$, $k, l \in \mathfrak{S}$, and $\theta_k^{\parallel} = \theta_k^{\perp}$, $\theta_k^{\odot} = 0$, $k \in \mathfrak{S}$, [1,2] and we recover the classical expression [3] of the diffusion velocities $\mathbf{V}_k = -\sum_{l\in\mathfrak{S}} D_{kl}\mathbf{d}_l - \theta_k\boldsymbol{\partial}_{\mathbf{x}} \log T$, $k \in \mathfrak{S}$. For ionized gases, however, the diffusion coefficients are different according to the three spatial directions denoted by $\|$, $\perp$ and $\odot$, as a consequence of anisotropy. We also observe that the species diffusion driving forces $\mathbf{d}_k$, $k \in \mathfrak{S}$, contain additional terms due to the macroscopic electromagnetic forces $q_k(\mathbf{E}+\mathbf{v}\wedge\mathbf{B})$, $k \in \mathfrak{S}$. Although the formalism uses the unitary vector $\mathscr{B}=\mathbf{B}/B$, the fluxes behave smoothly as $\mathbf{B}$ goes to zero thanks to the properties of transport coefficients [9,10]. The corresponding conduction current density $\mathbf{j}$ reads

$$\mathbf{j}=\sum_{k\in\mathfrak{S}} q_k \mathbf{V}_k \tag{7}$$

and the expression of the heat flux is

$$\begin{aligned}\mathbf{Q} = &-\widehat{\lambda}^{\|}(\partial_{\mathbf{x}}T)^{\|} - \widehat{\lambda}^{\perp}(\partial_{\mathbf{x}}T)^{\perp} - \widehat{\lambda}^{\odot}(\partial_{\mathbf{x}}T)^{\odot} \\ &- p\sum_{k\in\mathfrak{S}}(\theta_k^{\|}\mathbf{d}_k^{\|} + \theta_k^{\perp}\mathbf{d}_k^{\perp} + \theta_k^{\odot}\mathbf{d}_k^{\odot}) + \sum_{k\in\mathfrak{S}}\rho_k h_k \mathbf{V}_k\end{aligned} \tag{8}$$

where h_k is the enthalpy per unit mass of the kth species, $\widehat{\lambda}^{\|}$, $\widehat{\lambda}^{\perp}$ and $\widehat{\lambda}^{\odot}$ the partial thermal conductivities. For non-ionized gases, the charges q_k, $k\in\mathfrak{S}$, vanish so that we have $\widehat{\lambda}^{\|}=\widehat{\lambda}^{\perp}$, $\widehat{\lambda}^{\odot}=0$, and $\theta_k^{\|}=\theta_k^{\perp}$, $\theta_k^{\odot}=0$, $k\in\mathfrak{S}$, [1,2], and we also recover the classical expression [3] of the heat flux vector $\mathbf{Q}=-\widehat{\lambda}\partial_{\mathbf{x}}T - p\sum_{k\in\mathfrak{S}}\theta_k\mathbf{d}_k + \sum_{k\in\mathfrak{S}}\rho_k h_k \mathbf{V}_k$. The heat flux is smooth as $\mathbf{B}$ goes to zero thanks to the properties of transport coefficients [9,10].

Finally, the viscous stress tensor can be written in the form

$$\begin{aligned}\mathbf{\Pi} = &-\kappa(\partial_{\mathbf{x}}\cdot\mathbf{v})\mathbb{I} - \eta_1\mathsf{S} - \eta_2(\mathsf{AS}-\mathsf{SA}) - \eta_3(-\mathsf{ASA} + \mathscr{B}\otimes\mathscr{B}\,\mathsf{S}\,\mathscr{B}\otimes\mathscr{B}) \\ &-\eta_4(\mathsf{S}\,\mathscr{B}\otimes\mathscr{B} + \mathscr{B}\otimes\mathscr{B}\,\mathsf{S} - 2\mathscr{B}\otimes\mathscr{B}\,\mathsf{S}\,\mathscr{B}\otimes\mathscr{B}) - \eta_5(\mathscr{B}\otimes\mathscr{B}\,\mathsf{S}\,\mathsf{A} - \mathsf{A}\,\mathsf{S}\,\mathscr{B}\otimes\mathscr{B})\end{aligned} \tag{9}$$

where $\mathscr{B}=\mathbf{B}/B$, κ is the volume viscosity, and η_1, η_2, η_3, η_4, η_5 are the shear viscosities. In this expression, we have denoted by S the symmetric traceless strain rate tensor

$$\mathsf{S} = \partial_{\mathbf{x}}\mathbf{v} + \partial_{\mathbf{x}}\mathbf{v}^{\mathrm{T}} - \tfrac{2}{3}(\partial_{\mathbf{x}}\cdot\mathbf{v})\mathbb{I}$$

where T denotes transposition, and by A the antisymmetric rotation matrix associated with $\mathscr{B}$

$$\mathsf{A} = \begin{pmatrix} 0 & -\mathscr{B}_3 & \mathscr{B}_2 \\ \mathscr{B}_3 & 0 & -\mathscr{B}_1 \\ -\mathscr{B}_2 & \mathscr{B}_1 & 0 \end{pmatrix}$$

The viscous tensor is a linear combination of the identity matrix and of all the symmetric traceless tensors built from S and A, which are linear in S. The viscous tensor behaves smoothly as $\mathbf{B}$ goes to zero thanks to the properties of transport coefficients [9,10]. For non-ionized gases, the viscous tensor reduces to $\mathbf{\Pi}=-\kappa(\partial_{\mathbf{x}}\cdot\mathbf{v})\mathbb{I} - \eta_1\mathsf{S}$ since we then have $\eta_2=\eta_3=\eta_4=\eta_5=0$.

2.3. Chemical source term expression

We consider n^r elementary reversible reactions among the n^s species which can be formally written as

$$\sum_{k\in\mathfrak{S}} v_{kr}^{\mathrm{f}}\mathfrak{M}_k \rightleftarrows \sum_{k\in\mathfrak{S}} v_{kr}^{\mathrm{b}}\mathfrak{M}_k, \quad r\in\mathfrak{R}$$

where $\mathfrak{M}_k$ is the chemical symbol of the kth species, v_{kr}^{f} and v_{kr}^{b} are the forward and the backward stoichiometric coefficients of the kth species in the rth reaction, respectively, and $\mathfrak{R}=\{1,\dots,n^r\}$ is the set of reaction indexes.

The Maxwellian production rates given by the kinetic theory can be written as

$$\omega_k = \sum_{r\in\mathfrak{R}} (v_{kr}^{\mathrm{b}} - v_{kr}^{\mathrm{f}})\tau_r, \quad k\in\mathfrak{S} \tag{10}$$

where τ_r is the rate of progress of the rth reaction. The rates of progress are given by the symmetric expression [3]

$$\tau_r = \mathscr{K}_r^{\mathrm{s}}(\exp\langle v_r^{\mathrm{f}}, \mathsf{M}\mu\rangle - \exp\langle v_r^{\mathrm{b}}, \mathsf{M}\mu\rangle) \tag{11}$$

where $v_r^{\mathrm{f}}=(v_{1r}^{\mathrm{f}},\dots,v_{n^s r}^{\mathrm{f}})^{\mathrm{T}}$, $v_r^{\mathrm{b}}=(v_{1r}^{\mathrm{b}},\dots,v_{n^s r}^{\mathrm{b}})^{\mathrm{T}}$, $\mu=(\mu_1,\dots,\mu_{n^s})^{\mathrm{T}}$, with μ_k, $k\in\mathfrak{S}$, the species reduced chemical potential, M the diagonal matrix defined by $\mathsf{M}=\mathrm{diag}(m_1,\dots,m_{n^s})$ and $\mathscr{K}_r^{\mathrm{s}}$ the symmetric reaction constant. This symmetric formulation of the rates of progress is obtained by using the fundamental reciprocal relation between forward and backward reaction constants that can be deduced from the kinetic theory [3].

2.4. Thermodynamics

Thermodynamics obtained in the framework of the kinetic theory of gases is valid out of equilibrium and has, therefore, a wider range of validity than classical thermodynamics introduced for stationary homogeneous equilibrium states. The formalism obtained from the kinetic theory still coincides with the Gibbs formalism applied to intensive variables.

The total mass per unit volume ρ, the total charge per unit volume q, and the total pressure p can be written in the form

$$\rho = \sum_{k\in\mathfrak{S}} \rho_k, \quad q = \sum_{k\in\mathfrak{S}} q_k, \quad p = \sum_{k\in\mathfrak{S}} p_k$$

where the species partial pressure p_k, $k\in\mathfrak{S}$, is given by $p_k = r_k\rho_k T$ with $r_k = R/m_k$, R the perfect gas constant.

The internal energy e and the entropy per unit mass s can be decomposed into

$$\rho e = \sum_{k\in\mathfrak{S}} \rho_k e_k, \quad \rho s = \sum_{k\in\mathfrak{S}} \rho_k s_k$$

where e_k and s_k are the internal energy and the entropy per unit mass of the kth species and T the temperature. The internal energy is given by

$$e_k(T) = e_k^{\mathrm{st}} + \int_{T^{\mathrm{st}}}^{T} c_{v,k}(\tau)\,\mathrm{d}\tau$$

where $e_k^{\mathrm{st}} = e_k(T^{\mathrm{st}})$ is the formation energy of the kth species at the positive standard temperature T^{st} and $c_{v,k}$ is the constant-volume specific heat of the kth species. The species entropies s_k, $k \in \mathfrak{S}$, are given by

$$s_k(T, \rho_k) = s_k^{\mathrm{st}} + \int_{T^{\mathrm{st}}}^{T} \frac{c_{v,k}(\tau)}{\tau}\, \mathrm{d}\tau - r_k \log\left(\frac{\rho_k}{m_k \gamma^{\mathrm{st}}}\right)$$

where $\gamma^{\mathrm{st}} = p^{\mathrm{st}}/(RT^{\mathrm{st}})$ is the standard concentration, that is, the concentration at the standard state T^{st}, p^{st}. The enthalpy per unit mass h_k and the Gibbs function g_k of the kth species are given by $h_k = e_k + r_k T$ and $g_k = h_k - Ts_k$. We finally define the species reduced chemical potential μ_k by $\mu_k = g_k/(RT)$.

2.5. *Maxwell's equations*

The electric and magnetic fields satisfy the two macroscopic Maxwell's equations

$$\varepsilon_0 \partial_t \mathbf{E} + q\mathbf{v} + \mathbf{j} - \boldsymbol{\partial}_{\mathbf{x}} \wedge \mathbf{B}/\mu_0 = 0 \tag{12}$$

$$\partial_t \mathbf{B} + \boldsymbol{\partial}_{\mathbf{x}} \wedge \mathbf{E} = 0 \tag{13}$$

where ε_0 is the dielectric constant and μ_0 the magnetic permeability. It is well known that the equations $\boldsymbol{\partial}_{\mathbf{x}} \cdot \mathbf{E} = q/\varepsilon_0$ and $\boldsymbol{\partial}_{\mathbf{x}} \cdot \mathbf{B} = 0$ are consequences of (12) and (13) provided that they hold at initial time $t = 0$.

2.6. *Mathematical assumptions*

We describe in this subsection the mathematical assumptions concerning thermoelectrochemistry and transport coefficients for self completeness. These assumptions are obtained from the kinetic theory [2] and are not sufficiently intuitive to be guessed empirically. We assume that these assumptions are satisfied whenever we consider the equations governing reactive ionized magnetized dissipative gas mixtures, that is, in Sections 2 and 4.

The species of the mixture are assumed to be constituted by neutral atoms and electrons. We denote by $\mathfrak{A} = \{1, \dots, n^a\}$ the atoms indexing set, by n^a the number of atoms in the mixture, by $\widetilde{m}_l$, $l \in \mathfrak{A}$, the atom masses and by $\mathfrak{a}_{kl}$ the number of lth atoms in the kth species. We define $\mathfrak{a}_{k0}$ as the number of electrons in the kth species, and for notational convenience, we define $\overline{\mathfrak{A}} = \{0\} \cup \mathfrak{A} = \{0, \dots, n^a\}$. We introduce the atomic vectors $\mathfrak{a}_l$, $l \in \mathfrak{A}$, defined by $\mathfrak{a}_l = (\mathfrak{a}_{1l}, \dots, \mathfrak{a}_{n^s l})^{\mathrm{T}}$, $l \in \mathfrak{A}$, and the electron vector $\mathfrak{a}_0$, by $\mathfrak{a}_0 = (\mathfrak{a}_{10}, \dots, \mathfrak{a}_{n^s 0})^{\mathrm{T}}$. We also define the reaction vectors by $v_r = (v_{1r}, \dots, v_{n^s r})^{\mathrm{T}}$, $r \in \mathfrak{R}$, where $v_{kr} = v_{kr}^{\mathrm{b}} - v_{kr}^{\mathrm{f}}$, $k \in \mathfrak{S}$, so that $v_r = v_r^{\mathrm{b}} - v_r^{\mathrm{f}}$, and we denote by $\mathscr{R}$ the linear space spanned by v_r, $r \in \mathfrak{R}$. We finally define the mass vector per unit volume $\varrho = (\rho_1, \dots, \rho_{n^s})^{\mathrm{T}}$ and the unit vector $\mathsf{u} = (1, \dots, 1)^{\mathrm{T}}$.

2.6.1. *Assumption on thermoelectrochemistry*

(Th_1) The species molar masses m_k, $k \in \mathfrak{S}$, and the gas constant R are positive constants. The formation energies e_k^{st}, $k \in \mathfrak{S}$, and the formation entropies s_k^{st}, $k \in \mathfrak{S}$, are constants. The specific heats $c_{v,k}$, $k \in \mathfrak{S}$, are $\mathscr{C}^\infty$ functions of $T \geqslant 0$. Furthermore, there exist positive constants $\underline{c}_v$ and $\overline{c}_v$ with $0 < \underline{c}_v \leqslant c_{v,k}(T) \leqslant \overline{c}_v$, for $T \geqslant 0$ and $k \in \mathfrak{S}$.

(Th$_2$) The stoichiometric coefficients ν_{kr}^{f} and ν_{kr}^{b}, $k \in \mathfrak{S}$, $r \in \mathfrak{R}$, and the atomic coefficients $\mathfrak{a}_{kl}$, $k \in \mathfrak{S}$, $l \in \mathfrak{A}$, are non-negative integers. The numbers of electrons $\mathfrak{a}_{k0}$, $k \in \mathfrak{S}$, are integers. The atomic vectors $\mathfrak{a}_l$, $l \in \overline{\mathfrak{A}}$, and the reaction vectors ν_r, $r \in \mathfrak{R}$, satisfy the conservation relations $\langle \nu_r, \mathfrak{a}_l \rangle = 0$, $r \in \mathfrak{R}$, $l \in \overline{\mathfrak{A}}$. This relation expresses atom conservation for $l \in \mathfrak{A}$ and charge conservation for $l = 0$.

(Th$_3$) The atom masses $\widetilde{m}_l$, $l \in \mathfrak{A}$, and the electron mass $\widetilde{m}_0$ are positive constants. Moreover, the species molar masses m_k, $k \in \mathfrak{S}$, are given by $m_k = \sum_{l \in \mathfrak{A}} \widetilde{m}_l \mathfrak{a}_{kl} + \widetilde{m}_0 \mathfrak{a}_{k0}$, $k \in \mathfrak{S}$. We also have the proportionality relation between the species charge per unit volume q_k, $k \in \mathfrak{S}$, and the number of electrons in the kth species, $q_k = -\varkappa \mathfrak{a}_{k0} n_k$, $k \in \mathfrak{S}$, where $\varkappa$ is a positive constant which represents the absolute value of charge per unit mole for electrons.

(Th$_4$) The rate constants $\mathscr{K}_r^{\mathrm{s}}$, $r \in \mathfrak{R}$, are $\mathscr{C}^\infty$ positive functions of $T > 0$.

2.6.2. *Assumptions on transport coefficients*

(Tr$_1$) The flux diffusion coefficients $D_{kl}^{\|}$, $D_{kl}^{\perp}$ and $BD_{kl}^{\odot}$, $k, l \in \mathfrak{S}$, the thermal diffusion coefficients $\theta_k^{\|}$, $\theta_k^{\perp}$, $B\theta_k^{\odot}$, $k \in \mathfrak{S}$, the volume viscosity κ, the shear viscosities $\eta_1, B\eta_2, \eta_3, \eta_4$, $B\eta_5$ and the thermal conductivities $\widehat{\lambda}^{\|}$, $\widehat{\lambda}^{\perp}$ and $B\widehat{\lambda}^{\odot}$ are $\mathscr{C}^\infty$ functions of $(T, \varrho, \mathbf{B})$ for $T > 0$, $\varrho > 0$ and $\mathbf{B} \in \mathbb{R}^3$, where B is the norm of the magnetic field $\mathbf{B}$. Moreover, the coefficients $D_{kl}^{\|}$, $k, l \in \mathfrak{S}$, do not depend on the magnetic field $\mathbf{B}$ and we can write $D_{kl}^{\perp} - D_{kl}^{\|} = B^2 \phi_{kl}^{\perp}(B^2)$ and $D_{kl}^{\odot} = B \phi_{kl}^{\odot}(B^2)$, where $\phi_{kl}^{\perp}$ and $\phi_{kl}^{\odot}$, $k, l \in \mathfrak{S}$, are $\mathscr{C}^\infty([0,\infty), \mathbb{R})$ functions. The coefficients $\theta_k^{\|}$, $k \in \mathfrak{S}$, do not depend on the magnetic field $\mathbf{B}$ and we can write $\theta_k^{\perp} - \theta_k^{\|} = B^2 \psi_k^{\perp}(B^2)$, $\theta_k^{\odot} = B \psi_k^{\odot}(B^2)$, where $\psi_k^{\perp}$ and $\psi_k^{\odot}$ are $\mathscr{C}^\infty([0,\infty), \mathbb{R})$ functions. The coefficient $\widehat{\lambda}^{\|}$ does not depend on the magnetic field $\mathbf{B}$ and we can write $\widehat{\lambda}^{\perp} - \widehat{\lambda}^{\|} = B^2 \varsigma^{\perp}(B^2)$ and $\widehat{\lambda}^{\odot} = B \varsigma^{\odot}(B^2)$, where $\varsigma^{\perp}$ and $\varsigma^{\odot}$ are $\mathscr{C}^\infty([0,\infty), \mathbb{R})$ functions. Lastly, we have $\eta_1 = \varphi_1(B^2)$, $\eta_2 = B\varphi_2(B^2)$, $\eta_3 = B^2 \varphi_3(B^2)$, $\eta_4 = B^2 \varphi_4(B^2)$, $\eta_5 = B^3 \varphi_5(B^2)$ and $2\eta_4 - \eta_3 = B^4 \varphi_6(B^2)$, where φ_α, $\alpha \in \{1, \ldots, 6\}$, are $\mathscr{C}^\infty([0,\infty), \mathbb{R})$ functions.

(Tr$_2$) Thermal conductivities $\widehat{\lambda}^{\|}$ and $\widehat{\lambda}^{\perp}$ are positive functions. The volume viscosity κ is a non-negative function and the shear viscosities η_1, η_2, η_3, η_4, η_5 verify $\eta_1 + \eta_4 > 0$, $\eta_1 + \eta_3 > 0$, $\eta_1 - \eta_3 > 0$.

(Tr$_3$) The matrices $A^{\|}$, $A^{\perp}$ and $A^{\odot}$ defined by

$$A^{\diamond} = \begin{pmatrix} \dfrac{T}{p}\widehat{\lambda}^{\diamond} & \theta^{\diamond \mathrm{T}} \\ \theta^{\diamond} & D^{\diamond} \end{pmatrix}, \quad \diamond \in \{\|, \perp, \odot\}$$

are symmetric, $A^{\|}$ and $A^{\perp}$ are positive semidefinite and their nullspace is spanned by the vector $(0, \varrho^{\mathrm{T}})^{\mathrm{T}}$. Moreover, the vector $(0, \varrho^{\mathrm{T}})^{\mathrm{T}}$ is in the nullspace of $A^{\odot}$.

2.7. *Quasilinear form*

We rewrite the system of equations governing reactive ionized magnetized dissipative gas mixtures as a quasilinear system of second-order partial differential equations. We define the conservative variable U by

$$\mathsf{U} = (\varrho^{\mathrm{T}}, \rho\mathbf{v}^{\mathrm{T}}, \mathbf{E}^{\mathrm{T}}, \mathbf{B}^{\mathrm{T}}, \rho e^{\mathrm{t}})^{\mathrm{T}} \tag{14}$$

and the natural variable Z by

$$\mathsf{Z} = (\varrho^{\mathrm{T}}, \mathbf{v}^{\mathrm{T}}, \mathbf{E}^{\mathrm{T}}, \mathbf{B}^{\mathrm{T}}, T)^{\mathrm{T}} \tag{15}$$

The conservation equations can then be written in the compact form

$$\partial_t \mathsf{U} + \sum_{i\in\mathsf{C}} \partial_i \mathsf{F}_i + \sum_{i\in\mathsf{C}} \partial_i \mathsf{F}_i^{\mathrm{diss}} = \Omega \tag{16}$$

where C denotes the set $\{1,2,3\}$, F_i, $i\in\mathsf{C}$, the convective flux in the ith direction, $\mathsf{F}_i^{\mathrm{diss}}$, $i\in\mathsf{C}$, the dissipative flux in the ith direction and Ω the source term. The source term Ω is given by $\Omega = \Omega^{\mathbf{j}} + \Omega_0$, where

$$\Omega^{\mathbf{j}} = (0,\ldots,0,(\mathbf{j}\wedge\mathbf{B})^{\mathrm{T}}, -\mathbf{j}^{\mathrm{T}}/\varepsilon_0, 0_{1,3}, 0)^{\mathrm{T}} \tag{17}$$

$$\Omega_0 = (m_1\omega_1,\ldots,m_{n^s}\omega_{n^s}, (\rho\mathbf{g} + q(\mathbf{E}+\mathbf{v}\wedge\mathbf{B}))^{\mathrm{T}}, -q\mathbf{v}^{\mathrm{T}}/\varepsilon_0, 0_{1,3}, \rho\mathbf{g}\cdot\mathbf{v})^{\mathrm{T}} \tag{18}$$

The convective flux F_i is given by

$$\mathsf{F}_i = (\varrho^{\mathrm{T}} v_i, \rho\mathbf{v}^{\mathrm{T}} v_i + p\mathbf{e}_i^{\mathrm{T}}, -(\mathbf{e}_i\wedge\mathbf{B})^{\mathrm{T}}/\varepsilon_0\mu_0, (\mathbf{e}_i\wedge\mathbf{E})^{\mathrm{T}}, (\rho e^{\mathrm{f}} + p)v_i + P_i)^{\mathrm{T}} \tag{19}$$

and the dissipative flux $\mathsf{F}_i^{\mathrm{diss}}$ can be split into $\mathsf{F}_i^{\mathrm{diss}} = \mathsf{F}_i^{\mathrm{diff}} + \mathsf{F}_i^{\mathrm{visc}}$ where $\mathsf{F}_i^{\mathrm{visc}}$, the viscous flux, and $\mathsf{F}_i^{\mathrm{diff}}$, the diffusion flux, are given by

$$\mathsf{F}_i^{\mathrm{visc}} = (0_{1,n^s}, \mathbf{\Pi}_i, 0_{1,3}, 0_{1,3}, \mathbf{\Pi}_i\cdot\mathbf{v})^{\mathrm{T}} \tag{20}$$

$$\mathsf{F}_i^{\mathrm{diff}} = (\mathscr{F}_{1i},\ldots,\mathscr{F}_{n^s i}, 0_{1,3}, 0_{1,3}, 0_{1,3}, Q_i)^{\mathrm{T}} \tag{21}$$

For notational convenience, we have denoted by $\mathbf{\Pi}_i$ the ith rows extracted from the stress tensor $\mathbf{\Pi}$ and by $\mathbf{e}_1, \mathbf{e}_2, \mathbf{e}_3$, the canonical basis vectors of $\mathbb{R}^3$.

In order to express the natural variable Z in terms of the conservative variable U, we investigate the map $\mathsf{Z}\mapsto\mathsf{U}$ and its range. We introduce the open set $\mathscr{O}_{\mathsf{Z}} = (0,\infty)^{n^s}\times\mathbb{R}^3\times\mathbb{R}^3\times\mathbb{R}^3\times(0,\infty)$ and the open set $\mathscr{O}_{\mathsf{U}}$ defined by

$$\mathscr{O}_{\mathsf{U}} = \{(u_i)\in\mathbb{R}^{n^s+10} : u_1,\ldots,u_{n^s} > 0, u_{n^s+10} > f(u_i)\}$$

where f is the map from $(0,\infty)^{n^s}\times\mathbb{R}^9$ to $\mathbb{R}$ defined by

$$f(u_i) = \frac{1}{2}\frac{\sum_{1\leqslant i\leqslant 3} u_{n^s+i}^2}{\sum_{1\leqslant i\leqslant n^s} u_i} + \frac{\varepsilon_0}{2}\sum_{4\leqslant i\leqslant 6} u_{n^s+i}^2 + \frac{1}{2\mu_0}\sum_{7\leqslant i\leqslant 9} u_{n^s+i}^2 + \sum_{1\leqslant i\leqslant n^s} u_i e_i^0$$

and where e_i^0 is the internal energy of the ith species at $T=0$. The following proposition is easily established as in the non-ionized case [3,7].

Proposition 2.1
The map $\mathsf{Z} \mapsto \mathsf{U}$ is a $\mathscr{C}^\infty$ diffeomorphism from the open set $\mathcal{O}_\mathsf{Z}$ onto the convex open set $\mathcal{O}_\mathsf{U}$.

Thanks to Proposition 2.1, the dissipative fluxes and the source terms, which are naturally expressed in terms of Z, are rewritten in terms of U.

Proposition 2.2
The convective fluxes $\mathsf{F}_i(\mathsf{U})$, $i \in \mathsf{C}$, are $\mathscr{C}^\infty$ functions of the variable $\mathsf{U} \in \mathcal{O}_\mathsf{U}$, the dissipative fluxes $\mathsf{F}_i^{\mathrm{diss}}(\mathsf{U}, \boldsymbol{\partial}_\mathbf{x}\mathsf{U})$, $i \in \mathsf{C}$, can be written in the form

$$\mathsf{F}_i^{\mathrm{diss}}(\mathsf{U}, \boldsymbol{\partial}_\mathbf{x}\mathsf{U}) = -\sum_{j \in \mathsf{C}} \mathsf{B}_{ij}(\mathsf{U})(\partial_j \mathsf{U} + \mathsf{G}_j(\mathsf{U})), \quad i \in \mathsf{C}$$

where the dissipation matrices B_{ij}, $i, j \in \mathsf{C}$, and the zeroth-order contributions G_i, $i \in \mathsf{C}$, are $\mathscr{C}^\infty$ functions of $\mathsf{U} \in \mathcal{O}_\mathsf{U}$. Moreover, the source term $\Omega(\mathsf{U}, \boldsymbol{\partial}_\mathbf{x}\mathsf{U})$ can be written in the form

$$\Omega(\mathsf{U}, \boldsymbol{\partial}_\mathbf{x}\mathsf{U}) = \sum_{i \in \mathsf{C}} \mathsf{M}_i(\mathsf{U})^{\mathsf{T}} \mathsf{F}_i^{\mathrm{diss}}(\mathsf{U}, \boldsymbol{\partial}_\mathbf{x}\mathsf{U}) + \Omega_0(\mathsf{U})$$

where the matrices $\mathsf{M}_i(\mathsf{U})$, $i \in \mathsf{C}$, and the zeroth-order source term $\Omega_0(\mathsf{U})$, are $\mathscr{C}^\infty$ function of $\mathsf{U} \in \mathcal{O}_\mathsf{U}$. Finally, defining the matrices $\mathsf{A}_i(\mathsf{U}) = \partial_\mathsf{U} \mathsf{F}_i$, $i \in \mathsf{C}$, which are $\mathscr{C}^\infty$ functions of $\mathsf{U} \in \mathcal{O}_\mathsf{U}$, the system of partial differential equations (16) can be rewritten in the form

$$\begin{aligned}\partial_t \mathsf{U} + \sum_{i \in \mathsf{C}} \mathsf{A}_i(\mathsf{U}) \partial_i \mathsf{U} = {} & \sum_{i,j \in \mathsf{C}} \partial_i (\mathsf{B}_{ij}(\mathsf{U})(\partial_j \mathsf{U} + \mathsf{G}_j(\mathsf{U}))) \\ & - \sum_{i,j \in \mathsf{C}} \mathsf{M}_i(\mathsf{U})^{\mathsf{T}} \mathsf{B}_{ij}(\mathsf{U})(\partial_j \mathsf{U} + \mathsf{G}_j(\mathsf{U})) + \Omega_0(\mathsf{U}) \end{aligned} \tag{22}$$

Proof
The proof is lengthy and tedious but presents no serious difficulties, and we refer to [9,10] for more details. □

We observe fundamental differences between (22) and the classical case of non-ionized mixtures. For non-ionized mixtures, the dissipative terms $\mathsf{F}_i^{\mathrm{diss}}$, $i \in \mathsf{C}$, are linear combinations of the solution gradients [3–7] whereas for ionized mixtures, they also contain the zeroth-order contributions G_i, $i \in \mathsf{C}$, arising from the direct action of macroscopic electromagnetic forces. A second difference is that for ionized mixtures the source term Ω not only depends on U but also on its gradient $\boldsymbol{\partial}_\mathbf{x}\mathsf{U}$ through the conduction current $\mathbf{j}$ appearing in Maxwell's equations. We will see in the next section that these terms are related through entropy. Finally, for $i, j \in \mathsf{C}$, we denote by $\mathsf{B}_{ij}^{\mathsf{s}}$ and $\mathsf{B}_{ij}^{\mathsf{a}}$ the even and odd parts of the dissipation matrix B_{ij} with respect to the magnetic field $\mathbf{B}$. The odd parts $\mathsf{B}_{ij}^{\mathsf{a}}$, $i, j \in \mathsf{C}$, are due to anisotropy of the species diffusive fluxes, the heat flux and the viscous tensor.

3. LOCAL EXISTENCE FOR AN ABSTRACT SYSTEM

In this section, we investigate partial symmetrization and entropy for an abstract second-order quasilinear system with zeroth-order terms in dissipative fluxes and gradient dependent source terms. Partially normal forms are next obtained by using the nullspace invariance condition introduced by Kawashima and Shizuta [5]. Local existence of solutions are finally obtained by using theorems of Kawashima [4] or Vol'Pert and Hudjaev [8].

3.1. Quasilinear abstract system

We consider an abstract second-order quasilinear system in the form

$$\partial_t \mathsf{U}^* + \sum_{i\in \mathsf{C}^*} \partial_i \mathsf{F}_i^* + \sum_{i\in \mathsf{C}^*} \partial_i \mathscr{F}_i^* = \Omega^* \tag{23}$$

where $\mathsf{U}^* \in \mathcal{O}_{\mathsf{U}^*}$, $\mathcal{O}_{\mathsf{U}^*}$ is an open convex set of $\mathbb{R}^{n^*}$, $\mathsf{C}^* = \{1,\dots,d\}$ the set of direction indexes of $\mathbb{R}^d$, F_i^*, $i\in\mathsf{C}^*$, the convective fluxes, $\mathscr{F}_i^*$, $i\in\mathsf{C}^*$, the dissipative fluxes, and Ω^* the source term. The superscript $*$ is used to distinguish between the abstract second-order system (23) of size n^* in $\mathbb{R}^d$ and the particular multicomponent reactive magnetized flows system (16) of size n^s+10 in $\mathbb{R}^3$. The convective fluxes are functions of U^*

$$\mathsf{F}_i^* = \mathsf{F}_i^*(\mathsf{U}^*), \quad i\in\mathsf{C}^* \tag{24}$$

and the dissipative fluxes are assumed to be in the form

$$\mathscr{F}_i^*(\mathsf{U}^*, \boldsymbol{\partial}_\mathbf{x}\mathsf{U}^*) = -\sum_{j\in\mathsf{C}^*} \mathsf{B}_{ij}^*(\mathsf{U}^*)(\partial_j \mathsf{U}^* + \mathsf{G}_j^*(\mathsf{U}^*)), \quad i\in\mathsf{C}^* \tag{25}$$

where B_{ij}^*, $i,j\in\mathsf{C}^*$, are the dissipation matrices and G_i^*, $i\in\mathsf{C}^*$ are the zeroth-order contributions. We assume that the dissipation matrices B_{ij}^*, $i,j\in\mathsf{C}^*$, can be split into

$$\mathsf{B}_{ij}^* = \mathsf{B}_{ij}^{\mathsf{s}*} + \mathsf{B}_{ij}^{\mathsf{a}*}, \quad i,j\in\mathsf{C}^* \tag{26}$$

where the partial dissipation matrices $\mathsf{B}_{ij}^{\mathsf{s}*}$ and $\mathsf{B}_{ij}^{\mathsf{a}*}$, $i,j\in\mathsf{C}^*$, will have different symmetry properties. We also assume that the source term can be split into

$$\Omega^*(\mathsf{U}^*, \boldsymbol{\partial}_\mathbf{x}\mathsf{U}^*) = \sum_{i\in\mathsf{C}^*} \mathsf{M}_i^*(\mathsf{U}^*)^{\mathrm{T}} \mathscr{F}_i^*(\mathsf{U}^*, \boldsymbol{\partial}_\mathbf{x}\mathsf{U}^*) + \Omega_0^*(\mathsf{U}^*) \tag{27}$$

where $\mathsf{M}_i^*(\mathsf{U}^*)$, $i\in\mathsf{C}^*$, are matrices and $\Omega_0^*(\mathsf{U}^*)$ is a vector. Defining the convective Jacobian matrices by $\mathsf{A}_i^* = \partial_{\mathsf{U}^*}\mathsf{F}_i^*$, $i\in\mathsf{C}^*$, we finally obtain

$$\begin{aligned}\partial_t \mathsf{U}^* + \sum_{i\in\mathsf{C}^*} \mathsf{A}_i^*(\mathsf{U}^*)\partial_i \mathsf{U}^* = {} & \sum_{i,j\in\mathsf{C}^*} \partial_i(\mathsf{B}_{ij}^*(\mathsf{U}^*)(\partial_j \mathsf{U}^* + \mathsf{G}_j^*(\mathsf{U}^*))) \\ & - \sum_{i,j\in\mathsf{C}^*} \mathsf{M}_i^*(\mathsf{U}^*)^{\mathrm{T}} \mathsf{B}_{ij}^*(\mathsf{U}^*)(\partial_j \mathsf{U}^* + \mathsf{G}_j^*(\mathsf{U}^*)) + \Omega_0^*(\mathsf{U}^*)\end{aligned} \tag{28}$$

and we assume that the following properties hold for system (28).

(Edp_1) The convective fluxes F_i^*, $i\in\mathsf{C}^*$, the dissipation matrices B_{ij}^*, $\mathsf{B}_{ij}^{\mathsf{s}*}$, $\mathsf{B}_{ij}^{\mathsf{a}*}$, $i,j\in\mathsf{C}^*$, the zeroth-order terms G_i^*, $i\in\mathsf{C}^*$, the matrices M_i^*, $i\in\mathsf{C}^*$, and the source term Ω_0^* are smooth functions of the variable $\mathsf{U}^*\in\mathcal{O}_{\mathsf{U}^*}$, where $\mathcal{O}_{\mathsf{U}^*}$ is a convex open set of $\mathbb{R}^{n^*}$.

3.2. Partial symmetrization and entropy

Symmetric forms are a fundamental step towards existence results for systems of partial differential equations of hyperbolic–parabolic type [3–8]. In the framework of isotropic hyperbolic–parabolic systems, existence of a conservative symmetric formulation has been shown to be equivalent to the existence of a mathematical entropy [5]. We generalize in this section the notion of symmetrization as well as that of entropy to the situation of hyperbolic–parabolic systems with zeroth-order contributions in dissipative fluxes and gradient dependent source terms. We then use the terminology 'partial symmetrization' since the resulting quasilinear systems contain symmetric as well as antisymmetric contributions. However, it is the form that reveals most of the structural symmetry properties of the partial differential operators under investigation. Finally, it is the structure that corresponds to the situation of ionized magnetized dissipative gaz mixtures.

Definition 3.1

Assume that $\mathsf{U}^* \mapsto \mathsf{V}^*$ is a diffeomorphism from $\mathcal{O}_{\mathsf{U}^*}$ onto $\mathcal{O}_{\mathsf{V}^*}$, and consider the system in the V^* variable

$$\begin{aligned}\widetilde{\mathsf{A}}_0^*(\mathsf{V}^*)\partial_t \mathsf{V}^* + \sum_{i\in \mathsf{C}^*} \widetilde{\mathsf{A}}_i^*(\mathsf{V}^*)\partial_i \mathsf{V}^* &= \sum_{i,j\in \mathsf{C}^*} \partial_i(\widetilde{\mathsf{B}}_{ij}^*(\mathsf{V}^*)(\partial_j \mathsf{V}^* + \widetilde{\mathsf{G}}_j^*(\mathsf{V}^*)))\\ &\quad - \sum_{i,j\in \mathsf{C}^*} \widetilde{\mathsf{M}}_i^*(\mathsf{V}^*)^{\mathsf{T}}\widetilde{\mathsf{B}}_{ij}^*(\mathsf{V}^*)(\partial_j \mathsf{V}^* + \widetilde{\mathsf{G}}_j^*(\mathsf{V}^*)) + \widetilde{\Omega}_0^*(\mathsf{V}^*) \end{aligned} \tag{29}$$

where $\widetilde{\mathsf{A}}_0^* = \partial_{\mathsf{V}^*}\mathsf{U}^*$, $\widetilde{\mathsf{A}}_i^* = \mathsf{A}_i^*\partial_{\mathsf{V}^*}\mathsf{U}^*$, $\widetilde{\mathsf{B}}_{ij}^* = \mathsf{B}_{ij}^*\partial_{\mathsf{V}^*}\mathsf{U}^*$, $\widetilde{\mathsf{B}}_{ij}^{\mathsf{s}*} = \mathsf{B}_{ij}^{\mathsf{s}*}\partial_{\mathsf{V}^*}\mathsf{U}^*$, $\widetilde{\mathsf{B}}_{ij}^{\mathsf{a}*} = \mathsf{B}_{ij}^{\mathsf{a}*}\partial_{\mathsf{V}^*}\mathsf{U}^*$, $\widetilde{\mathsf{G}}_i^* = (\partial_{\mathsf{V}^*}\mathsf{U}^*)^{-1}\mathsf{G}_i^*$, $\widetilde{\mathsf{M}}_i^* = \mathsf{M}_i^*$, and $\widetilde{\Omega}_0^* = \Omega_0^*$. System (29) is said to be of the partially symmetric form if the vector and matrix coefficients satisfy the following properties:

- (S_1) The matrix $\widetilde{\mathsf{A}}_0^*(\mathsf{V}^*)$ is symmetric positive definite for $\mathsf{V}^* \in \mathcal{O}_{\mathsf{V}^*}$.
- (S_2) The convective matrices $\widetilde{\mathsf{A}}_i^*(\mathsf{V}^*)$, $i \in \mathsf{C}^*$, are symmetric for $\mathsf{V}^* \in \mathcal{O}_{\mathsf{V}^*}$.
- (S_3) The dissipation matrices satisfy the reciprocity relations $\widetilde{\mathsf{B}}_{ij}^{\mathsf{s}*}(\mathsf{V}^*)^{\mathsf{T}} = \widetilde{\mathsf{B}}_{ji}^{\mathsf{s}*}(\mathsf{V}^*)$, and $\widetilde{\mathsf{B}}_{ij}^{\mathsf{a}*}(\mathsf{V}^*)^{\mathsf{T}} = -\widetilde{\mathsf{B}}_{ji}^{\mathsf{a}*}(\mathsf{V}^*)$, for $i,j \in \mathsf{C}^*$, $\mathsf{V}^* \in \mathcal{O}_{\mathsf{V}^*}$.
- (S_4) The matrix $\widetilde{\mathsf{B}}^*(\mathsf{V}^*, \boldsymbol{\xi}) = \sum_{i,j\in \mathsf{C}^*} \widetilde{\mathsf{B}}_{ij}^{\mathsf{s}*}(\mathsf{V}^*)\xi_i\xi_j$ is symmetric positive semidefinite for $\mathsf{V}^* \in \mathcal{O}_{\mathsf{V}^*}$ and $\boldsymbol{\xi} \in \Sigma^{d-1}$ where Σ^{d-1} is the unit sphere in d dimensions, and for any x in $N(\widetilde{\mathsf{B}}^*)$ we have $\widetilde{\mathsf{B}}_{ij}^{\mathsf{s}*}x = 0$ and $\widetilde{\mathsf{B}}_{ij}^{\mathsf{a}*}x = 0$.
- (S_5) We have the compatibility conditions $\widetilde{\mathsf{G}}_i^*(\mathsf{V}^*) = \widetilde{\mathsf{M}}_i^*(\mathsf{V}^*)\,\mathsf{V}^*$, $i \in \mathsf{C}^*$.

Properties (S_1)–(S_2) are the same as those of non-ionized mixtures whereas (S_3) takes into account the symmetric and antisymmetric parts of the dissipation matrices $\widetilde{\mathsf{B}}_{ij}^*$, $i,j \in \mathsf{C}^*$. In practical applications, the antisymmetric contributions $\widetilde{\mathsf{B}}_{ij}^{\mathsf{a}*}$, $i,j \in \mathsf{C}^*$ are a consequence of the Onsager reciprocal relations $\widetilde{\mathsf{B}}_{ij}{}^{\mathsf{T}}(\mathbf{B}) = \widetilde{\mathsf{B}}_{ji}(-\mathbf{B})$, $i,j \in \mathsf{C}$, where we have emphasized the dependence on the magnetic field $\mathbf{B}$. The condition (S_4) express as usual that the system is degenerate strongly parabolic. The necessary conditions that $\widetilde{\mathsf{B}}_{ij}^{\mathsf{s}*}$ and $\widetilde{\mathsf{B}}_{ij}^{\mathsf{a}*}$ vanish over $N(\widetilde{\mathsf{B}}^*)$ have been included in (S_4) rather than in the nullspace invariance condition [3]. Finally, the condition (S_5) is a fundamental entropic compatibility condition between the zeroth-order contributions of dissipatives fluxes and gradient dependent source terms. We now introduce the concept of entropy for the system of partial differential equations (28).

Definition 3.2
A smooth function $\sigma^*(\mathsf{U}^*)$ defined on a convex set $\mathcal{O}_{\mathsf{U}^*}$ is said to be an entropy function for system (28) if the following properties hold:

- (E_1) The function σ^* is a strictly convex function on $\mathcal{O}_{\mathsf{U}^*}$ in the sense that the Hessian matrix is positive definite on $\mathcal{O}_{\mathsf{U}^*}$.
- (E_2) There exists smooth functions $\mathsf{q}_i^*(\mathsf{U}^*)$, $i \in \mathsf{C}^*$, such that on $\mathcal{O}_{\mathsf{U}^*}$

$$(\partial_{\mathsf{U}^*}\sigma^*)\mathsf{A}_i^* = \partial_{\mathsf{U}^*}\mathsf{q}_i^*, \quad i \in \mathsf{C}^*, \quad \mathsf{U}^* \in \mathcal{O}_{\mathsf{U}^*}$$

- (E_3) The matrices $\mathsf{B}_{ij}^{\mathsf{s}*}$ and $\mathsf{B}_{ij}^{\mathsf{a}*}$, $i,j \in \mathsf{C}^*$, satisfy the reciprocity relations

$$(\partial^2_{\mathsf{U}^*}\sigma^*(\mathsf{U}^*))^{-1}\mathsf{B}_{ij}^{\mathsf{s}*}(\mathsf{U}^*)^{\mathrm{T}} = \mathsf{B}_{ji}^{\mathsf{s}*}(\mathsf{U}^*)(\partial^2_{\mathsf{U}^*}\sigma^*(\mathsf{U}^*))^{-1}$$

$$(\partial^2_{\mathsf{U}^*}\sigma^*(\mathsf{U}^*))^{-1}\mathsf{B}_{ij}^{\mathsf{a}*}(\mathsf{U}^*)^{\mathrm{T}} = -\mathsf{B}_{ji}^{\mathsf{a}*}(\mathsf{U}^*)(\partial^2_{\mathsf{U}^*}\sigma^*(\mathsf{U}^*))^{-1}$$

- (E_4) The matrix $\mathsf{B}^*(\mathsf{U}^*, \boldsymbol{\xi}) = \sum_{i,j \in \mathsf{C}^*} \mathsf{B}_{ij}^{\mathsf{s}*}(\mathsf{U}^*)(\partial^2_{\mathsf{U}^*}\sigma^*(\mathsf{U}^*))^{-1}\xi_i\xi_j$ is symmetric positive semidefinite for $\mathsf{U}^* \in \mathcal{O}_{\mathsf{U}^*}$ and $\boldsymbol{\xi} \in \Sigma^{d-1}$. Furthermore, for any $x \in N(\mathsf{B}^*)$ we have $\mathsf{B}_{ij}^{\mathsf{s}*}(\partial^2_{\mathsf{U}^*}\sigma^*)^{-1}x = 0$ and $\mathsf{B}_{ij}^{\mathsf{a}*}(\partial^2_{\mathsf{U}^*}\sigma^*)^{-1}x = 0$.
- (E_5) We have the entropic compatibility conditions

$$\mathsf{G}_i^*(\mathsf{V}^*) = (\partial^2_{\mathsf{U}^*}\sigma^*(\mathsf{U}^*))^{-1}\mathsf{M}_i^*(\mathsf{U}^*)\,\partial_{\mathsf{U}^*}\sigma^*(\mathsf{U}^*)^{\mathrm{T}}, \quad i \in \mathsf{C}^*$$

We now establish the equivalence between partial symmetrizability and the existence of an entropy function for system (28).

Theorem 3.3
System (28) can be partially symmetrized on the open convex set $\mathcal{O}_{\mathsf{U}^*}$ if and only if the system admits an entropy function σ^* on $\mathcal{O}_{\mathsf{U}^*}$. In this situation, the symmetrizing variable V^* and the entropy σ^* satisfy the relation $\mathsf{V}^* = (\partial_{\mathsf{U}^*}\sigma^*)^{\mathrm{T}}$.

Proof
Assume first that there exists an entropy σ^*, and let $\mathsf{V}^* = (\partial_{\mathsf{U}^*}\sigma^*)^{\mathrm{T}}$ be the symmetrizing variable. The map $\mathsf{U}^* \mapsto \mathsf{V}^*$ is then a diffeomorphism since $\mathcal{O}_{\mathsf{U}^*}$ is convex and $\partial_{\mathsf{U}^*}\mathsf{V}^* = \partial^2_{\mathsf{U}^*}\sigma^*$ is positive definite. We can thus define the smooth functions

$$\widehat{\sigma}^*(\mathsf{V}^*) = \mathsf{U}^{*\mathrm{T}}\mathsf{V}^* - \sigma^*(\mathsf{U}^*) \quad \text{and} \quad \widehat{\mathsf{q}}_i^*(\mathsf{V}^*) = \mathsf{F}_i^{*\mathrm{T}}\mathsf{V}^* - \mathsf{q}_i^*(\mathsf{U}^*), \quad i \in \mathsf{C}^*$$

Differentiating these equalities then yields the relations $(\partial_{\mathsf{V}^*}\widehat{\sigma}^*)^{\mathrm{T}} = \mathsf{U}^*$ and $(\partial_{\mathsf{V}^*}\widehat{\mathsf{q}}_i^*)^{\mathrm{T}} = \mathsf{F}_i^*$, making use of property (E_2). We then obtain that $\widetilde{\mathsf{A}}_0^* = \partial_{\mathsf{V}^*}\mathsf{U}^* = (\partial_{\mathsf{U}^*}\mathsf{V}^*)^{-1} = (\partial^2_{\mathsf{U}^*}\sigma^*)^{-1}$ and $\widetilde{\mathsf{A}}_i^* = \partial_{\mathsf{V}^*}\mathsf{F}_i^* = \partial^2_{\mathsf{V}^*}\widehat{\mathsf{q}}_i^*$, $i \in \mathsf{C}^*$, so that the matrix $\widetilde{\mathsf{A}}_0^*$ is symmetric definite positive and the matrices $\widetilde{\mathsf{A}}_i^*$, $i \in \mathsf{C}^*$, are symmetric. Moreover, we directly get from properties (E_3)–(E_5) that the matrices $\widetilde{\mathsf{B}}_{ij}^* = \mathsf{B}_{ij}^*(\partial^2_{\mathsf{U}^*}\sigma^*)^{-1}$, $i,j \in \mathsf{C}^*$, and the vectors $\widetilde{\mathsf{G}}_i^* = (\partial^2_{\mathsf{U}^*}\sigma^*)\mathsf{G}_i^*$, $i,j \in \mathsf{C}^*$, are such that property (S_3)–(S_5) holds.

Conversely, assume that the system can be partially symmetrized in the sense of Definition 3.1. Since $\partial_{\mathsf{V}^*}\mathsf{U}^*$ and $\partial_{\mathsf{V}^*}\mathsf{F}_i^*$, $i\in\mathsf{C}^*$, are symmetric and $\mathcal{O}_{\mathsf{V}^*}$ is simply connected, there exists $\widehat{\sigma}^*$ and $\widehat{\mathsf{q}}_i^*$, $i\in\mathsf{C}^*$, defined over $\mathcal{O}_{\mathsf{V}^*}$, such that $(\partial_{\mathsf{V}^*}\widehat{\sigma}^*)^{\mathrm{T}}=\mathsf{U}^*$ and $(\partial_{\mathsf{V}^*}\widehat{\mathsf{q}}_i^*)^{\mathrm{T}}=\mathsf{F}_i^*$, $i\in\mathsf{C}^*$. We can thus define the functions

$$\sigma^*(\mathsf{U}^*)=\mathsf{U}^{*\mathrm{T}}\mathsf{V}^*-\widehat{\sigma}^*(\mathsf{V}^*)\quad\text{and}\quad \mathsf{q}_i^*(\mathsf{U}^*)=\mathsf{F}_i^{*\mathrm{T}}\mathsf{V}^*-\widehat{\mathsf{q}}_i^*(\mathsf{V}^*),\quad i\in\mathsf{C}^*$$

Differentiating these identities, and using properties (S_1)–(S_3), it is then straightforward to establish that σ^* is an entropy with fluxes q_i^*, $i\in\mathsf{C}^*$, such that $\mathsf{V}^*=(\partial_{\mathsf{U}^*}\sigma^*)^{\mathrm{T}}$. Properties (S_3)–(S_5) are then easily shown to be equivalent to (E_3)–(E_5). □

Corollary 3.4

Assume that system (28) can be partially symmetrized into (29) and introduce the dissipative entropy fluxes $\mathfrak{p}_i^*$, $i\in\mathsf{C}^*$, defined by

$$\mathfrak{p}_i^*=\langle\mathsf{V}^*,\mathscr{F}_i^*\rangle,\quad i\in\mathsf{C}^*$$

The entropy balance equation can then be obtained upon multiplying the symmetrized system by the entropic variable V^* and reads

$$\partial_t\sigma^*+\sum_{i\in\mathsf{C}^*}\partial_i\mathsf{q}_i^*+\sum_{i\in\mathsf{C}^*}\partial_i\mathfrak{p}_i^*=-\sum_{i,j\in\mathsf{C}^*}\langle\partial_i\mathsf{V}^*+\widetilde{\mathsf{M}}_i^*\mathsf{V}^*,\widetilde{\mathsf{B}}_{ij}^{\mathsf{s}*}(\partial_j\mathsf{V}^*+\widetilde{\mathsf{M}}_j^*\mathsf{V}^*)\rangle+\langle\widetilde{\Omega}_0,\mathsf{V}^*\rangle \tag{30}$$

Proof

This results from straightforward calculations. □

The physical meaning of the entropy conservation equation (30) is that when $\langle x,\widetilde{\mathsf{B}}_{ij}^{\mathsf{s}*}(\mathsf{V}^*)x\rangle\geqslant 0$ and $\langle\widetilde{\Omega}_0(\mathsf{V}^*),\mathsf{V}^*\rangle\leqslant 0$, for any $\mathsf{V}^*\in\mathcal{O}_{\mathsf{V}^*}$ and any $x\in\mathbb{R}^{d\times n^*}$, then the integral $\int_{\mathbb{R}^d}\sigma^*\,\mathrm{d}x$ is decreasing in time, which corresponds to the second principle of thermodynamics. This reveals the close links between the second principle of thermodynamics and the parabolic nature of systems of conservation laws. Note also that we have $\langle x,\widetilde{\mathsf{B}}_{ij}^{\mathsf{s}*}x\rangle=\langle x,\widetilde{\mathsf{B}}_{ij}^{*}x\rangle$ for any $x\in\mathbb{R}^{d\times n^*}$ so that the matrices $\widetilde{\mathsf{B}}_{ij}^{\mathsf{a}*}$, $i,j\in\mathsf{C}^*$, do not contribute to entropy production. Finally, it is fundamental to note that the zeroth-order contributions are included in the entropy production term associated with dissipative processes, and this is notably the case for magnetized dissipative mixtures [2,9,10].

3.3. Partially normal form

The purpose of this section is to introduce partially normal forms. We first assume that the system admits an entropy according to Definition 3.2.

(Edp_2) The system of partial differential equations (28) admits an entropy function σ^* on the open convex set $\mathcal{O}_{\mathsf{U}^*}$.

From Theorem 3.3 the system can be partially symmetrized in the form (29). We now want to rewrite this system by regrouping with the convective terms all first-order derivatives arising from the zeroth-order contributions of dissipative fluxes $\widetilde{\mathsf{G}}_i^*$, $i\in\mathsf{C}^*$, and from the gradient-dependent source terms $\sum_{j\in\mathsf{C}^*}\widetilde{\mathsf{M}}_j^{*\mathrm{T}}\widetilde{\mathsf{B}}_{ji}^*$. To this purpose, we define the matrices $\widetilde{\mathsf{A}}_i^{*\mathsf{a}}$, $i\in\mathsf{C}^*$, by

$$\widetilde{\mathsf{A}}_i^{*\mathsf{a}}(\mathsf{V}^*)=\sum_{j\in\mathsf{C}^*}(\widetilde{\mathsf{M}}_j^{*\mathrm{T}}\widetilde{\mathsf{B}}_{ji}^*-\widetilde{\mathsf{B}}_{ij}^*\widetilde{\mathsf{M}}_j^*-\partial_{\mathsf{V}^*}(\widetilde{\mathsf{B}}_{ij}^*\widetilde{\mathsf{M}}_j^*)\mathsf{V}^*) \tag{31}$$

where, with a small abuse of notation, the quantity $\partial_{\mathsf{V}^*}(\widetilde{\mathsf{B}}_{ij}^*\widetilde{\mathsf{M}}_j^*)\mathsf{V}^*$ has the following meaning:

$$\{\partial_{\mathsf{V}^*}(\widetilde{\mathsf{B}}_{ij}^*\widetilde{\mathsf{M}}_j^*)\mathsf{V}^*\}_{rs}=\sum_{1\leqslant u,v\leqslant n^*}\partial_{\mathsf{V}^{*s}}(\widetilde{\mathsf{B}}_{ij}^{*ru}\widetilde{\mathsf{M}}_j^{*uv})\mathsf{V}^{*v}$$

and we also define

$$\widetilde{\mathsf{L}}^*(\mathsf{V}^*)=\sum_{i,j\in\mathsf{C}^*}\widetilde{\mathsf{M}}_i^{*\mathrm{T}}\widetilde{\mathsf{B}}_{ij}^*\widetilde{\mathsf{M}}_j^*,\quad \widetilde{\Omega}^*(\mathsf{V}^*)=-\widetilde{\mathsf{L}}^*(\mathsf{V}^*)\,\mathsf{V}^*+\widetilde{\Omega}_0^*(\mathsf{V}^*)$$

Then, after a little algebra, the partially symmetrized system (29) is easily rewritten in the effective form

$$\widetilde{\mathsf{A}}_0^*(\mathsf{V}^*)\partial_t\mathsf{V}^*+\sum_{i\in\mathsf{C}^*}(\widetilde{\mathsf{A}}_i^*(\mathsf{V}^*)+\widetilde{\mathsf{A}}_i^{*\mathsf{a}}(\mathsf{V}^*))\partial_i\mathsf{V}^*=\sum_{i,j\in\mathsf{C}^*}\partial_i(\widetilde{\mathsf{B}}_{ij}^*(\mathsf{V}^*)\partial_j\mathsf{V}^*)+\widetilde{\Omega}^*(\mathsf{V}^*) \tag{32}$$

In order to rewrite this system as a composite hyperbolic–parabolic system, we introduce the following definition of partially normal forms.

Definition 3.5
Consider a system in partially symmetric form (29) rewritten as (32), a diffeomorphism $\mathsf{V}^*\mapsto\mathsf{W}^*$ from $\mathcal{O}_{\mathsf{V}^*}$ onto $\mathcal{O}_{\mathsf{W}^*}$ and the system in the new variable W^*

$$\overline{\mathsf{A}}_0^*\partial_t\mathsf{W}^*+\sum_{i\in\mathsf{C}^*}(\overline{\mathsf{A}}_i^*+\overline{\mathsf{A}}_i^{*\mathsf{a}})\partial_i\mathsf{W}^*=\sum_{i,j\in\mathsf{C}^*}\partial_i(\overline{\mathsf{B}}_{ij}^*\partial_j\mathsf{W}^*)+\overline{\mathscr{T}}^*+\overline{\Omega}^* \tag{33}$$

where we define

$$\begin{aligned}
\overline{\mathsf{A}}_0^*&=(\partial_{\mathsf{W}^*}\mathsf{V}^*)^{\mathrm{T}}\,\widetilde{\mathsf{A}}_0^*\,(\partial_{\mathsf{W}^*}\mathsf{V}^*),\quad \overline{\mathsf{B}}_{ij}^*=(\partial_{\mathsf{W}^*}\mathsf{V}^*)^{\mathrm{T}}\,\widetilde{\mathsf{B}}_{ij}^*\,(\partial_{\mathsf{W}^*}\mathsf{V}^*)\\
\overline{\mathsf{B}}_{ij}^{\mathsf{s}*}&=(\partial_{\mathsf{W}^*}\mathsf{V}^*)^{\mathrm{T}}\,\widetilde{\mathsf{B}}_{ij}^{\mathsf{s}*}\,(\partial_{\mathsf{W}^*}\mathsf{V}^*),\quad \overline{\mathsf{B}}_{ij}^{\mathsf{a}*}=(\partial_{\mathsf{W}^*}\mathsf{V}^*)^{\mathrm{T}}\,\widetilde{\mathsf{B}}_{ij}^{\mathsf{a}*}\,(\partial_{\mathsf{W}^*}\mathsf{V}^*)\\
\overline{\mathsf{A}}_i^*&=(\partial_{\mathsf{W}^*}\mathsf{V}^*)^{\mathrm{T}}\,\widetilde{\mathsf{A}}_i^*\,(\partial_{\mathsf{W}^*}\mathsf{V}^*),\quad \overline{\Omega}^*=(\partial_{\mathsf{W}^*}\mathsf{V}^*)^{\mathrm{T}}\,\widetilde{\Omega}^*\\
\overline{\mathsf{M}}_i^*&=(\partial_{\mathsf{W}^*}\mathsf{V}^*)^{-1}\,\widetilde{\mathsf{M}}_i^*\,(\partial_{\mathsf{W}^*}\mathsf{V}^*),\quad \overline{\mathsf{G}}_i^*=(\partial_{\mathsf{W}^*}\mathsf{V}^*)^{-1}\,\widetilde{\mathsf{G}}_i^*\\
\overline{\mathsf{A}}_i^{*\mathsf{a}}&=(\partial_{\mathsf{W}^*}\mathsf{V}^*)^{\mathrm{T}}\,\widetilde{\mathsf{A}}_i^{*\mathsf{a}}\,(\partial_{\mathsf{W}^*}\mathsf{V}^*),\quad \overline{\mathscr{T}}^*=-\sum_{i,j\in\mathsf{C}^*}\partial_i(\partial_{\mathsf{W}^*}\mathsf{V}^*)^{\mathrm{T}}\,\widetilde{\mathsf{B}}_{ij}^*\,(\partial_{\mathsf{W}^*}\mathsf{V}^*)\,\partial_j\mathsf{W}^*
\end{aligned}$$

Then we have

$$\overline{\mathsf{A}}_i^{*\mathsf{a}}=\sum_{j\in\mathsf{C}^*}(\overline{\mathsf{M}}_j^{*\mathrm{T}}\,\overline{\mathsf{B}}_{ji}^*-\overline{\mathsf{B}}_{ij}^*\overline{\mathsf{M}}_j^*-(\partial_{\mathsf{W}^*}\mathsf{V}^*)^{\mathrm{T}}\partial_{\mathsf{W}^*}(\widetilde{\mathsf{B}}_{ij}^*\widetilde{\mathsf{M}}_j^*)\mathsf{V}^*) \tag{34}$$

and the matrix and vector coefficients satisfy $(\overline{\mathsf{S}}_1)$–$(\overline{\mathsf{S}}_5)$:

- $(\overline{\mathsf{S}}_1)$ The matrix $\overline{\mathsf{A}}_0^*(\mathsf{W}^*)$ is symmetric positive definite for $\mathsf{W}^*\in\mathcal{O}_{\mathsf{W}^*}$.
- $(\overline{\mathsf{S}}_2)$ The matrices $\overline{\mathsf{A}}_i^*(\mathsf{W}^*)$, $i\in\mathsf{C}^*$, are symmetric for $\mathsf{W}^*\in\mathcal{O}_{\mathsf{W}^*}$.

($\overline{\mathsf{S}}_3$) The dissipation matrices satisfy the reciprocity relations $\overline{\mathsf{B}}_{ij}^{\mathsf{s}*\mathrm{T}} = \overline{\mathsf{B}}_{ji}^{\mathsf{s}*}$ and $\overline{\mathsf{B}}_{ij}^{\mathsf{a}*\mathrm{T}} = -\overline{\mathsf{B}}_{ji}^{\mathsf{a}*}$, for $\mathsf{W}^* \in \mathcal{O}_{\mathsf{W}^*}$, $i,j \in \mathsf{C}^*$.

($\overline{\mathsf{S}}_4$) The matrix $\overline{\mathsf{B}}^*(\mathsf{W}^*, \boldsymbol{\xi}) = \sum_{i,j\in\mathsf{C}^*} \overline{\mathsf{B}}_{ij}^{\mathsf{s}*}(\mathsf{W}^*)\xi_i\xi_j$ is symmetric positive semidefinite for $\mathsf{W}^* \in \mathcal{O}_{\mathsf{W}^*}$ and $\boldsymbol{\xi} \in \Sigma^{d-1}$, and for any x in $N(\overline{\mathsf{B}}^*)$ we have $\overline{\mathsf{B}}_{ij}^{\mathsf{s}*} x = 0$ and $\overline{\mathsf{B}}_{ij}^{\mathsf{a}*} x = 0$.

($\overline{\mathsf{S}}_5$) We have the compatibility conditions $\overline{\mathsf{G}}_i^* = \overline{\mathsf{M}}_i^* (\partial_{\mathsf{V}^*}\mathsf{W}^*)\mathsf{V}^*$, $i \in \mathsf{C}^*$.

This system is said to be of the partially normal form if there exists a partition of $\{1,\dots,n^*\}$ into $\mathrm{I} = \{1,\dots,n_0^*\}$ and $\mathrm{II} = \{n_0^*+1,\dots,n^*\}$, such that the following properties hold:

($\overline{\mathsf{Nor}}_1$) The matrices $\overline{\mathsf{A}}_0^*$, $\overline{\mathsf{A}}_i^{*\mathsf{a}}$, $i \in \mathsf{C}^*$, and $\overline{\mathsf{B}}_{ij}^*$, $i,j \in \mathsf{C}^*$, have the bloc structure

$$\overline{\mathsf{A}}_0^* = \begin{pmatrix} \overline{\mathsf{A}}_0^{*\mathrm{I},\mathrm{I}} & 0 \\ 0 & \overline{\mathsf{A}}_0^{*\mathrm{II},\mathrm{II}} \end{pmatrix}, \quad \overline{\mathsf{A}}_i^{*\mathsf{a}} = \begin{pmatrix} 0 & \overline{\mathsf{A}}_i^{*\mathsf{a}\mathrm{I},\mathrm{II}} \\ \overline{\mathsf{A}}_i^{*\mathsf{a}\mathrm{II},\mathrm{I}} & \overline{\mathsf{A}}_i^{*\mathsf{a}\mathrm{II},\mathrm{II}} \end{pmatrix}, \quad \overline{\mathsf{B}}_{ij}^* = \begin{pmatrix} 0 & 0 \\ 0 & \overline{\mathsf{B}}_{ij}^{*\mathrm{II},\mathrm{II}} \end{pmatrix}$$

($\overline{\mathsf{Nor}}_2$) The matrix $\overline{\mathsf{B}}^{*\mathrm{II},\mathrm{II}}(\mathsf{W}^*, \boldsymbol{\xi}) = \sum_{i,j\in\mathsf{C}} \overline{\mathsf{B}}_{ij}^{\mathsf{s}*\mathrm{II},\mathrm{II}}(\mathsf{W}^*)\xi_i\xi_j$ is positive definite for $\mathsf{W}^* \in \mathcal{O}_{\mathsf{W}^*}$ and $\boldsymbol{\xi} \in \Sigma^{d-1}$.

($\overline{\mathsf{Nor}}_3$) We have $\overline{\mathcal{T}}^*(\mathsf{W}^*, \boldsymbol{\partial}_{\mathbf{x}}\mathsf{W}^*) = (\overline{\mathcal{T}}_{\mathrm{I}}^*(\mathsf{W}^*, \boldsymbol{\partial}_{\mathbf{x}}\mathsf{W}_{\mathrm{II}}^*)^{\mathrm{T}}, \overline{\mathcal{T}}_{\mathrm{II}}^*(\mathsf{W}^*, \boldsymbol{\partial}_{\mathbf{x}}\mathsf{W}^*)^{\mathrm{T}})^{\mathrm{T}}$

In these properties, we have used the vector and the matrix block structure induced by the partitioning of $\{1,\dots,n^*\}$, so that we have $\mathsf{W}^* = (\mathsf{W}_{\mathrm{I}}^{*\mathrm{T}}, \mathsf{W}_{\mathrm{II}}^{*\mathrm{T}})^{\mathrm{T}}$, for instance.

The main interest of partially normal forms is that, for any $i \in \mathsf{C}^*$, the symmetric hyperbolic bloc of the convective matrix $\overline{\mathsf{A}}_i^*$ is not perturbed by the matrix $\overline{\mathsf{A}}_i^{*\mathsf{a}}$. Using partially normal forms, it is then possible to apply local existence theorem. Note, however, that the matrices $\overline{\mathsf{A}}_i^{*\mathsf{a}}$, $i \in \mathsf{C}^*$, which contain antisymmetric factors, prevent application of global existence theorems as in References [3,4]. We now introduce the nullspace invariance property [3,5] which is a sufficient condition for system (29) to be recast into a partially normal form.

($\overline{\mathsf{Edp}}_3$) The nullspace of the symmetric matrix $\widetilde{\mathsf{B}}^*(\mathsf{V}^*, \boldsymbol{\xi}) = \sum_{i,j\in\mathsf{C}^*} \widetilde{\mathsf{B}}_{ij}^{\mathsf{s}*}(\mathsf{V}^*)\xi_i\xi_j$ does not depend on $\mathsf{V}^* \in \mathcal{O}_{\mathsf{V}^*}$ and $\boldsymbol{\xi} \in \Sigma^{d-1}$. We denote by N this nullspace and by n_0^* its dimension.

An important consequence of partial symmetrization and nullspace invariance is that for any x and y in $N(\widetilde{\mathsf{B}}^*)$, we have $\langle x, \widetilde{\mathsf{A}}_i^{*\mathsf{a}} y\rangle = 0$, $i \in \mathsf{C}^*$. This is a direct consequence of the definition (31) of the matrices $\widetilde{\mathsf{A}}_i^{*\mathsf{a}}$, $i \in \mathsf{C}^*$ This property will imply that there are normal forms such that the hyperbolic blocs of the symmetric convective Jacobians $\widetilde{\mathsf{A}}_i^*$, $i \in \mathsf{C}^*$, are not perturbed by the matrices $\widetilde{\mathsf{A}}_i^{*\mathsf{a}}$, $i \in \mathsf{C}^*$. Note that the necessary conditions $\widetilde{\mathsf{B}}_{ij}^{\mathsf{s}*} x = 0$, and $\widetilde{\mathsf{B}}_{ij}^{\mathsf{a}*} x = 0$, $i,j \in \mathsf{C}^*$, for any $x \in N$ [3], have been included in the definition of partially symmetric form for the sake of clarity.

In order to characterize more easily, partially normal forms for partially symmetrized systems satisfying the nullspace invariance property, we introduce the auxiliary variables $\mathsf{U}^{*\prime}$ and $\mathsf{V}^{*\prime}$, depending linearly on U^* and V^*, respectively. The dissipation matrices corresponding to these auxiliary variables have non-zero coefficients only in the lower right bloc of size $n^*-n_0^*$, and moreover, the Jacobian matrices $\widetilde{\mathsf{A}}_i^{*\mathsf{a}\prime}$, $i \in \mathsf{C}$, have zero coefficients in the upper left bloc of size n_0^*. Partial normal forms are then equivalently—and more easily—obtained from the $\mathsf{V}^{*\prime}$ symmetric equation.

Lemma 3.6
Consider a system (29) that is partially symmetric in the sense of Definition 3.1 and assume that the nullspace invariance property holds. Denote by σ^* the associated entropy function and by $\mathsf{V}^* = (\partial_{\mathsf{U}^*}\sigma^*)^{\mathrm{T}}$ the partial symmetrizing variable, and assume that the nullspace invariance property is satisfied over $\mathcal{O}_{\mathsf{V}^*}$. Further consider any constant non-singular matrix P of dimension n^*, such that its first n_0^* columns span the nullspace N. Then the auxiliary variable $\mathsf{U}^{*\prime} = \mathsf{P}^{\mathrm{T}}\mathsf{U}^*$ satisfies the equation

$$\partial_t \mathsf{U}^{*\prime} + \sum_{i\in\mathsf{C}^*}(\mathsf{A}_i^{*\prime}+\mathsf{A}_i^{*\mathsf{a}\prime})\partial_i\mathsf{U}^{*\prime} = \sum_{i,j\in\mathsf{C}^*}\partial_i(\mathsf{B}_{ij}^{*\prime}\partial_j\mathsf{U}^{*\prime}) + \Omega^{*\prime} \tag{35}$$

where $\mathsf{A}_i^{*\prime} = \mathsf{P}^{\mathrm{T}}\mathsf{A}_i^*(\mathsf{P}^{\mathrm{T}})^{-1}$, $\mathsf{A}_i^{*\mathsf{a}\prime} = \mathsf{P}^{\mathrm{T}}\mathsf{A}_i^{*\mathsf{a}}(\mathsf{P}^{\mathrm{T}})^{-1}$, $\mathsf{B}_{ij}^{*\prime} = \mathsf{P}^{\mathrm{T}}\mathsf{B}_{ij}^*(\mathsf{P}^{\mathrm{T}})^{-1}$, and $\Omega^{*\prime} = \mathsf{P}^{\mathrm{T}}\Omega^*$. The corresponding entropy is the functional $\sigma^{*\prime}(\mathsf{U}^{*\prime}) = \sigma^*((\mathsf{P}^{\mathrm{T}})^{-1}\mathsf{U}^{*\prime})$, and the associated entropic variable $\mathsf{V}^{*\prime} = (\partial_{\mathsf{U}^{*\prime}}\sigma^{*\prime})^{\mathrm{T}}$ is given by $\mathsf{V}^{*\prime} = \mathsf{P}^{-1}\mathsf{V}^*$ and satisfies the equation

$$\partial_t \mathsf{V}^{*\prime} + \sum_{i\in\mathsf{C}^*}(\widetilde{\mathsf{A}}_i^{*\prime}+\widetilde{\mathsf{A}}_i^{*\mathsf{a}\prime})\partial_i\mathsf{V}^{*\prime} = \sum_{i,j\in\mathsf{C}^*}\partial_i(\widetilde{\mathsf{B}}_{ij}^{*\prime}\partial_j\mathsf{V}^{*\prime}) + \widetilde{\Omega}^{*\prime} \tag{36}$$

where $\widetilde{\mathsf{A}}_0^{*\prime} = \mathsf{P}^{\mathrm{T}}\widetilde{\mathsf{A}}_0^*\mathsf{P}$, $\widetilde{\mathsf{A}}_i^{*\prime} = \mathsf{P}^{\mathrm{T}}\widetilde{\mathsf{A}}_i^*\mathsf{P}$, $\widetilde{\mathsf{A}}_i^{*\mathsf{a}\prime} = \mathsf{P}^{\mathrm{T}}\widetilde{\mathsf{A}}_i^{*\mathsf{a}}\mathsf{P}$, $\widetilde{\mathsf{B}}_{ij}^{*\prime} = \mathsf{P}^{\mathrm{T}}\widetilde{\mathsf{B}}_{ij}^*\mathsf{P}$, and $\widetilde{\Omega}^{*\prime} = \mathsf{P}^{\mathrm{T}}\widetilde{\Omega}^*$. In particular, the matrices $\widetilde{\mathsf{A}}_i^{*\mathsf{a}\prime}$, $i\in\mathsf{C}^*$, and $\widetilde{\mathsf{B}}_{ij}^{*\prime}$, $i,j\in\mathsf{C}^*$, are in the form

$$\widetilde{\mathsf{A}}_i^{*\mathsf{a}\prime} = \begin{pmatrix} 0 & \widetilde{\mathsf{A}}_i^{*\mathsf{a}\prime_{\mathrm{I,II}}} \\ \widetilde{\mathsf{A}}_i^{*\mathsf{a}\prime_{\mathrm{II,I}}} & \widetilde{\mathsf{A}}_i^{*\mathsf{a}\prime_{\mathrm{II,II}}} \end{pmatrix}, \quad \widetilde{\mathsf{B}}_{ij}^{*\prime} = \begin{pmatrix} 0 & 0 \\ 0 & \widetilde{\mathsf{B}}_{ij}^{*\prime_{\mathrm{II,II}}} \end{pmatrix} \tag{37}$$

and the matrix $\widetilde{\mathsf{B}}^{*\prime_{\mathrm{II,II}}}(\mathsf{V}^{*\prime},\boldsymbol{\xi}) = \sum_{i,j\in\mathsf{C}^*}\widetilde{\mathsf{B}}_{ij}^{\mathsf{s}*\prime_{\mathrm{II,II}}}(\mathsf{V}^{*\prime})\xi_i\xi_j$, is positive definite for $\mathsf{V}^{*\prime}\in\mathcal{O}_{\mathsf{V}^{*\prime}}$ and $\boldsymbol{\xi}\in\Sigma^{d-1}$. Finally, the partially normal form (33) is equivalently obtained by multiplying the V^* equation (29) by $(\partial_{\mathsf{W}^*}\mathsf{V}^*)^{\mathrm{T}}$ or the $\mathsf{V}^{*\prime}$ equation (36) by $(\partial_{\mathsf{W}^*}\mathsf{V}^{*\prime})^{\mathrm{T}}$.

Proof
Equation (35) is easily established by multiplying (28) on the left by P^{T}. This also yields the matrix relations $\mathsf{A}_i^{*\prime} = \mathsf{P}^{\mathrm{T}}\mathsf{A}_i^*(\mathsf{P}^{\mathrm{T}})^{-1}$, $\mathsf{A}_i^{*\mathsf{a}\prime} = \mathsf{P}^{\mathrm{T}}\mathsf{A}_i^{*\mathsf{a}}(\mathsf{P}^{\mathrm{T}})^{-1}$, $\mathsf{B}_{ij}^{*\prime} = \mathsf{P}^{\mathrm{T}}\mathsf{B}_{ij}^*(\mathsf{P}^{\mathrm{T}})^{-1}$, $\mathsf{B}_{ij}^{\mathsf{s}*\prime} = \mathsf{P}^{\mathrm{T}}\mathsf{B}_{ij}^{\mathsf{s}*}(\mathsf{P}^{\mathrm{T}})^{-1}$, $\mathsf{B}_{ij}^{\mathsf{a}*\prime} = \mathsf{P}^{\mathrm{T}}\mathsf{B}_{ij}^{\mathsf{a}*}(\mathsf{P}^{\mathrm{T}})^{-1}$, and $\Omega^{*\prime} = \mathsf{P}^{\mathrm{T}}\Omega^*$. It is also easily checked that the functional $\sigma^{*\prime}(\mathsf{U}^{*\prime}) = \sigma^*((\mathsf{P}^{\mathrm{T}})^{-1}\mathsf{U}^{*\prime})$ is the corresponding entropy. From the definition $\mathsf{V}^{*\prime} = (\partial_{\mathsf{U}^{*\prime}}\sigma^{*\prime})^{\mathrm{T}}$ and the chain rule, we then get that $\mathsf{V}^{*\prime} = \mathsf{P}^{-1}\mathsf{V}^*$ and (36) is obtained as in (28)–(29). Since $\widetilde{\mathsf{B}}^{*\prime} = \mathsf{P}^{\mathrm{T}}\widetilde{\mathsf{B}}^*\mathsf{P}$ and the first n_0^* columns of P span N, we next deduce that $\widetilde{\mathsf{B}}^{*\prime}$ is in the form

$$\widetilde{\mathsf{B}}^{*\prime} = \begin{pmatrix} 0 & 0 \\ 0 & \widetilde{\mathsf{B}}^{*\prime_{\mathrm{II,II}}} \end{pmatrix}$$

and similarly, all matrices $\widetilde{\mathsf{B}}_{ij}^{*\prime}$, $i,j\in\mathsf{C}^*$, and $\widetilde{\mathsf{A}}_i^{*\mathsf{a}\prime}$, $i\in\mathsf{C}^*$, are also in the form (37). Moreover, the matrix $\widetilde{\mathsf{B}}^{*\prime_{\mathrm{II,II}}}(\mathsf{V}^{*\prime},\boldsymbol{\xi})$, is symmetric positive definite, for $\mathsf{V}^{*\prime}\in\mathcal{O}_{\mathsf{V}^{*\prime}}$ and $\boldsymbol{\xi}\in\Sigma^{d-1}$, since the $n^*-n_0^*$ last columns of P span a subspace complementary to N. □

Partially normal forms for partially symmetrizable systems satisfying the nullspace consistency property are now completely characterized in the following theorem, in terms of the auxiliary variables $\mathsf{U}^{*\prime}$ and $\mathsf{V}^{*\prime}$.

Theorem 3.7
Keeping the assumptions and notations of Lemma 3.6, any partially normal form of system (29) is given by a change of variables in the form

$$\mathsf{W}^* = (\psi_{\mathrm{I}}(\mathsf{U}_{\mathrm{I}}^{*\prime}), \phi_{\mathrm{II}}(\mathsf{V}_{\mathrm{II}}^{*\prime}))^{\mathrm{T}} \tag{38}$$

where ψ_{I} and ϕ_{II} are two diffeomorphisms of $\mathbb{R}^{n_0^*}$ and $\mathbb{R}^{n^*-n_0^*}$, respectively. Furthermore, we have

$$\overline{\mathscr{T}}^*(\mathsf{W}^*, \partial_{\mathbf{x}}\mathsf{W}^*) = (0, \overline{\mathscr{T}}_{\mathrm{II}}^*(\mathsf{W}^*, \partial_{\mathbf{x}}\mathsf{W}_{\mathrm{II}}^*))^{\mathrm{T}}$$

Proof
The proof is exactly the same as in Reference [3] for non-ionized gases. The only difference concerns the matrices $\overline{\mathsf{A}}_i^{*\mathrm{a}}$ which can be treated as $\overline{\mathsf{B}}_{ij}^*$ □

3.4. An existence theorem in $V_l(\mathbb{R}^d)$

The system in partially normal form (33) can be split into a hyperbolic subsystem and a parabolic subsystem

$$\begin{cases} \overline{\mathsf{A}}_0^{*\mathrm{I},\mathrm{I}}\ \partial_t \mathsf{W}_{\mathrm{I}}^* = -\sum_{i\in\mathsf{C}^*} \overline{\mathsf{A}}_i^{*\mathrm{I},\mathrm{I}}\ \partial_i \mathsf{W}_{\mathrm{I}}^* + \overline{\Gamma}_{\mathrm{I}}^* \\ \overline{\mathsf{A}}_0^{*\mathrm{II},\mathrm{II}}\ \partial_t \mathsf{W}_{\mathrm{II}}^* = -\sum_{i\in\mathsf{C}^*} (\overline{\mathsf{A}}_i^{*\mathrm{II},\mathrm{I}} + \overline{\mathsf{A}}_i^{*\mathrm{aII},\mathrm{I}})\partial_i \mathsf{W}_{\mathrm{I}}^* + \sum_{i,j\in\mathsf{C}^*} \partial_i(\overline{\mathsf{B}}_{ij}^{*\mathrm{II},\mathrm{II}}\partial_j \mathsf{W}_{\mathrm{II}}^*) + \overline{\Gamma}_{\mathrm{II}}^* \end{cases} \tag{39}$$

where

$$\overline{\Gamma}_{\mathrm{I}}^* = \overline{\Omega}_{\mathrm{I}}^* - \sum_{i\in\mathsf{C}^*} (\overline{\mathsf{A}}_i^{*\mathrm{aI},\mathrm{II}} + \overline{\mathsf{A}}_i^{*\mathrm{I},\mathrm{II}})\partial_i \mathsf{W}_{\mathrm{II}}^*, \quad \overline{\Gamma}_{\mathrm{II}}^* = \overline{\Omega}_{\mathrm{II}}^* + \overline{\mathscr{T}}_{\mathrm{II}}^* - \sum_{i\in\mathsf{C}^*} (\overline{\mathsf{A}}_i^{*\mathrm{aII},\mathrm{II}} + \overline{\mathsf{A}}_i^{*\mathrm{II},\mathrm{II}})\partial_i \mathsf{W}_{\mathrm{II}}^*$$

We consider the Cauchy problem for this system (39) with initial conditions

$$\mathsf{W}_{\mathrm{I}}^*(0,\mathbf{x}) = \mathsf{W}_{\mathrm{I}}^{*0}(\mathbf{x}), \quad \mathsf{W}_{\mathrm{II}}^*(0,\mathbf{x}) = \mathsf{W}_{\mathrm{II}}^{*0}(\mathbf{x}) \tag{40}$$

These equations are considered in the strip $\bar{Q}_\Theta$ where Θ is positive and $Q_t = (0,t)\times\mathbb{R}^d$, for $t>0$. The unknown vectors $\mathsf{W}_{\mathrm{I}}^*$ and $\mathsf{W}_{\mathrm{II}}^*$ are assumed to be in the convex open sets $\mathcal{O}_{\mathsf{W}_{\mathrm{I}}^*} \subset \mathbb{R}^{n_0^*}$ and $\mathcal{O}_{\mathsf{W}_{\mathrm{II}}^*} \subset \mathbb{R}^{n^*-n_0^*}$.

Local existence theorems for the system (39) can be obtained from the results of Kawashima [4] or of Vol'Pert and Hudjaev [8]. The results of Kawashima are valid for initial conditions near equilibrium states and directly applies to the system in partially normal form [4]. The results of Vol'Pert and Hudjaev [8], which are summarized in this section, require a slightly stronger parabolicity condition easily established in the situation of magnetized mixtures.

We will use the classical functional spaces $L_p(\mathbb{R}^d)$ with norm

$$\|\phi\|_{0,p} = \left(\int_{\mathbb{R}^d} |\phi(\mathbf{x})|^p \,\mathrm{d}\mathbf{x}\right)^{1/p} \quad \text{if } 1\leqslant p<\infty \quad \text{and} \quad \|\phi\|_{0,\infty} = \sup_{\mathbb{R}^d} |\phi(\mathbf{x})|$$

the Sobolev spaces $W_p^l(\mathbb{R}^d)$, $1\leqslant p\leqslant\infty$, with norm

$$\|\phi\|_{l,p} = \sum_{k\in[\![0,l]\!]} |\phi|_{k,p}, \quad |\phi|_{k,p} = \sum_{|\beta|=k} \|\partial^\beta\phi\|_{0,p}$$

and the Vol'Pert spaces $V_l(\mathbb{R}^d)$ with norm [8]

$$\|\phi\|_l = |\phi|_{0,\infty} + \sum_{k\in[\![1,l]\!]} |\phi|_{k,2}$$

These definitions are extended to vector functions by using the Euclidean norm of $\mathbb{R}^d$. According to the Sobolev inequalities, there is an embedding of $W_2^l(\mathbb{R}^d)$ into $W_\infty^k(\mathbb{R}^d)$ for $l>k+d/2$ and an embedding of $W_2^l(\mathbb{R}^d)$ into $V_l(\mathbb{R}^d)$ for $l>d/2$.

In the following, $\mathscr{L}$ denotes an arbitrary fixed positive continuous convex function, on the open convex set $\mathcal{O}_{\mathsf{W}^*} = \mathcal{O}_{\mathsf{W}_{\mathrm{I}}^*}\times\mathcal{O}_{\mathsf{W}_{\mathrm{II}}^*}$, which grows without bound as any finite point of the boundary of $\mathcal{O}_{\mathsf{W}^*}$ is approached. The following theorem of Vol'Pert and Hudjaev [8] shows that, in a certain strip, there exists a solution which preserves the smoothness of the initial condition.

Theorem 3.8
Suppose that system (39)–(40) satisfies the following assumptions where $l>d/2+3$ denotes an integer:

- (Ex_1) The initial conditions $\mathsf{W}_{\mathrm{I}}^{*0}$, $\mathsf{W}_{\mathrm{II}}^{*0}$ satisfy $\sup_{x\in\mathbb{R}^d}\mathscr{L}(\mathsf{W}_{\mathrm{I}}^{*0}(\mathbf{x}),\mathsf{W}_{\mathrm{II}}^{*0}(\mathbf{x}))<+\infty$ and $\mathsf{W}_{\mathrm{I}}^{*0}$ and $\mathsf{W}_{\mathrm{II}}^{*0}$ are in the space $V_l(\mathbb{R}^d)$.
- (Ex_2) The matrix coefficients $\overline{\mathsf{A}}_0^{*\mathrm{I},\mathrm{I}}(w_{\mathrm{I}},w_{\mathrm{II}})$, $\overline{\mathsf{A}}_0^{*\mathrm{II},\mathrm{II}}(w_{\mathrm{I}},w_{\mathrm{II}})$ and $\overline{\mathsf{B}}_{ij}^{*\mathrm{II},\mathrm{II}}(w_{\mathrm{I}},w_{\mathrm{II}})$, $i,j\in\mathsf{C}^*$, have continuous derivative of order l with respect to $w_{\mathrm{I}}\in\mathcal{O}_{\mathsf{W}_{\mathrm{I}}^*}$ and $w_{\mathrm{II}}\in\mathcal{O}_{\mathsf{W}_{\mathrm{II}}^*}$.
- (Ex_3) The matrix coefficients $\overline{\mathsf{A}}_i^{*\mathrm{I},\mathrm{I}}(w_{\mathrm{I}},w_{\mathrm{II}},\xi)$, $\overline{\mathsf{A}}_i^{*\mathrm{II},\mathrm{I}}(w_{\mathrm{I}},w_{\mathrm{II}},\xi)$, $\overline{\mathsf{A}}_i^{*a\mathrm{II},\mathrm{I}}(w_{\mathrm{I}},w_{\mathrm{II}},\xi)$, $i\in\mathsf{C}^*$, and the vector coefficients $\overline{\Gamma}_{\mathrm{I}}^*(w_{\mathrm{I}},w_{\mathrm{II}},\xi)$ and $\overline{\Gamma}_{\mathrm{II}}^*(w_{\mathrm{I}},w_{\mathrm{II}},\xi)$ have continuous derivative of order l with respect to $w_{\mathrm{I}}\in\mathcal{O}_{\mathsf{W}_{\mathrm{I}}^*}$, $w_{\mathrm{II}}\in\mathcal{O}_{\mathsf{W}_{\mathrm{II}}^*}$ and $\boldsymbol{\xi}\in\mathbb{R}^{d\times(n^*-n_0^*)}$.
- (Ex_4) The matrix coefficients $\overline{\mathsf{A}}_0^{*\mathrm{I},\mathrm{I}}(w_{\mathrm{I}},w_{\mathrm{II}})$ and $\overline{\mathsf{A}}_0^{*\mathrm{II},\mathrm{II}}(w_{\mathrm{I}},w_{\mathrm{II}})$ are symmetric and positive definite for $w_{\mathrm{I}}\in\mathcal{O}_{\mathsf{W}_{\mathrm{I}}^*}$ and $w_{\mathrm{II}}\in\mathcal{O}_{\mathsf{W}_{\mathrm{II}}^*}$.
- (Ex_5) The matrix coefficients $\overline{\mathsf{A}}_i^{*\mathrm{I},\mathrm{I}}(w_{\mathrm{I}},w_{\mathrm{II}},\xi)$, $i\in\mathsf{C}^*$, are symmetric for $w_{\mathrm{I}}\in\mathcal{O}_{\mathsf{W}_{\mathrm{I}}^*}$, $w_{\mathrm{II}}\in\mathcal{O}_{\mathsf{W}_{\mathrm{II}}^*}$ and $\xi\in\mathbb{R}^{d\times(n^*-n_0^*)}$.
- (Ex_6) The matrices $\overline{\mathsf{A}}_0^{*\mathrm{I},\mathrm{I}}(w_{\mathrm{I}},w_{\mathrm{II}})$, $\overline{\mathsf{A}}_0^{*\mathrm{II},\mathrm{II}}(w_{\mathrm{I}},w_{\mathrm{II}})$ and the vectors $\overline{\Gamma}_{\mathrm{I}}^*(w_{\mathrm{I}},w_{\mathrm{II}},0)$ and $\overline{\Gamma}_{\mathrm{II}}^*(w_{\mathrm{I}},w_{\mathrm{II}},0)$ have continuous derivatives to order $l+3$ in $w_{\mathrm{I}}\in\mathcal{O}_{\mathsf{W}_{\mathrm{I}}^*}$ and $w_{\mathrm{II}}\in\mathcal{O}_{\mathsf{W}_{\mathrm{II}}^*}$.
- (Ex_7) For any compact subset K of $\mathcal{O}_{\mathsf{W}^*}=\mathcal{O}_{\mathsf{W}_{\mathrm{I}}^*}\times\mathcal{O}_{\mathsf{W}_{\mathrm{II}}^*}$, there exists $\alpha>0$ such that for any smooth function $w=(w_{\mathrm{I}},w_{\mathrm{II}})$ from $\mathbb{R}^d$ to $\mathbb{R}^{n^*}$ with value in K we have

$$\int_{\mathbb{R}^d}\sum_{i,j\in\mathsf{C}^*}(\partial_i\phi_{\mathrm{II}})^{\mathsf{T}}\overline{\mathsf{B}}_{ij}^{*\mathrm{II},\mathrm{II}}(w(\mathbf{x}))(\partial_j\phi_{\mathrm{II}})\,\mathrm{d}\mathbf{x}\geqslant\alpha\int_{\mathbb{R}^d}\sum_{i\in\mathsf{C}^*}(\partial_i\phi_{\mathrm{II}})^{\mathsf{T}}(\partial_i\phi_{\mathrm{II}})\,\mathrm{d}\mathbf{x} \tag{41}$$

  where ϕ_{II} is any function in $W_2^1(\mathbb{R}^d)$ with $n^*-n_0^*$ components.

Then there exists t_0, $0<t_0\leqslant\Theta$, such that the Cauchy problem (39), (40), admits a unique solution $(\mathsf{W}_{\mathrm{I}}^{*\mathsf{T}},\mathsf{W}_{\mathrm{II}}^{*\mathsf{T}})^{\mathsf{T}}$ defined on $\bar{Q}_{t_0}=[0,t_0]\times\mathbb{R}^d$, which is continuous with its derivatives of first-order in t and second-order in $\mathbf{x}$, and for which the following quantities are finite:

$$\sup_{0\leqslant t\leqslant t_0}\|(\mathsf{W}_{\mathrm{I}}^*(t),\mathsf{W}_{\mathrm{II}}^*(t))\|_l,\quad \sup_{\bar{Q}_{t_0}}\mathscr{L}(\mathsf{W}_{\mathrm{I}}^*,\mathsf{W}_{\mathrm{II}}^*) \tag{42}$$

$$\sup_{0\leqslant t\leqslant t_0}\|\partial_t\mathsf{W}_{\mathrm{I}}^*(t)\|_{l-1},\quad \int_0^{t_0}(\|\partial_t\mathsf{W}_{\mathrm{II}}^*(\tau)\|_{l-1}^2+\|\mathsf{W}_{\mathrm{II}}^*(\tau)\|_{l+1}^2)\,\mathrm{d}\tau \tag{43}$$

Moreover, either $t_0 = \Theta$, or there exists t_1 such that the theorem is true for any $t_0 < t_1$ and such that for $t_0 \to t_1^-$, at least one of the quantities

$$\|\mathsf{W}_{\mathrm{I}}^*(t_0)\|_{1,\infty} + \|\mathsf{W}_{\mathrm{II}}^*(t_0)\|_{2,\infty}, \quad \sup_{\bar{Q}_{t_0}} \mathscr{L}(\mathsf{W}_{\mathrm{I}}^*, \mathsf{W}_{\mathrm{II}}^*) \tag{44}$$

grows without bound, that is to say, the solution can be extended as long as quantities (44) remain finite.

4. EXISTENCE THEOREM FOR MULTICOMPONENT MAGNETIZED DISSIPATIVE MIXTURES

We now apply the general results of Section 3 to the system of equations governing multicomponent ionized magnetized reactive flows (22).

4.1. Partial symmetrization

We establish in this section that the equations governing ionized magnetized dissipative mixtures admit an entropy function. We define the mathematical entropy function σ as the opposite of the physical entropy per unit volume

$$\sigma = -\sum_{k\in\mathfrak{S}} \rho_k s_k = -\frac{1}{T}\sum_{k\in\mathfrak{S}} \rho_k (h_k - g_k) \tag{45}$$

and the corresponding entropic variable V is easily obtained

$$\mathsf{V} = \frac{1}{T}(g_1 - \tfrac{1}{2}\mathbf{v}\cdot\mathbf{v}, \ldots, g_{n^s} - \tfrac{1}{2}\mathbf{v}\cdot\mathbf{v}, \mathbf{v}^{\mathrm{T}}, \varepsilon_0\mathbf{E}^{\mathrm{T}}, \mathbf{B}^{\mathrm{T}}/\mu_0, -1)^{\mathrm{T}} \tag{46}$$

Proposition 4.1
The function σ defined on $\mathcal{O}_{\mathsf{U}}$ to $\mathbb{R}$ is a $\mathscr{C}^\infty$ strictly convex function, in the sense that its Hessian matrix $\partial^2_{\mathsf{U}}\sigma$ is symmetric positive definite. Moreover, the map $\mathsf{U} \mapsto \mathsf{V}$ is a $\mathscr{C}^\infty$ diffeomorphism from the open set $\mathcal{O}_{\mathsf{U}}$ onto the open set $\mathcal{O}_{\mathsf{V}} = \mathbb{R}^{n^s+9} \times (-\infty, 0)$.

Proof
It is easily established that the Hessian matrix $\partial^2_{\mathsf{U}}\sigma$ is positive definite and the inverse function theorem can then be applied [9,10]. □

We now obtain a partially symmetric form for the system of partial differential equations governing multicomponent magnetized reactive flows, making use of entropic variables.

Theorem 4.2
The function σ is an entropy function for system (22). Furthermore, the change of variables $\mathsf{U} \mapsto \mathsf{V}$ transforms system (22) into

$$\begin{aligned}\widetilde{\mathsf{A}}_0(\mathsf{V})\,\partial_t\mathsf{V} + \sum_{i\in\mathsf{C}} \widetilde{\mathsf{A}}_i(\mathsf{V})\partial_i\mathsf{V} &= \sum_{i,j\in\mathsf{C}} \partial_i(\widetilde{\mathsf{B}}_{ij}(\mathsf{V})(\partial_j\mathsf{V} + \widetilde{\mathsf{G}}_j(\mathsf{V}))) \\ &\quad - \sum_{i,j\in\mathsf{C}} \widetilde{\mathsf{M}}_i(\mathsf{V})^{\mathrm{T}}\,\widetilde{\mathsf{B}}_{ij}(\mathsf{V})(\partial_j\mathsf{V} + \widetilde{\mathsf{G}}_j(\mathsf{V})) + \widetilde{\Omega}_0(\mathsf{V})\end{aligned} \tag{47}$$

with $\widetilde{\mathsf{A}}_0 = \partial_{\mathsf{V}}\mathsf{U}$, $\widetilde{\mathsf{A}}_i = \mathsf{A}_i\partial_{\mathsf{V}}\mathsf{U}$, $\widetilde{\mathsf{B}}_{ij} = \mathsf{B}_{ij}\partial_{\mathsf{V}}\mathsf{U}$, $\widetilde{\mathsf{B}}^{\mathsf{s}}_{ij} = \mathsf{B}^{\mathsf{S}}_{ij}\partial_{\mathsf{V}}\mathsf{U}$, $\widetilde{\mathsf{B}}^{\mathsf{a}}_{ij} = \mathsf{B}^{\mathsf{a}}_{ij}\partial_{\mathsf{V}}\mathsf{U}$, $\widetilde{\mathsf{G}}_i = (\partial_{\mathsf{V}}\mathsf{U})^{-1}\mathsf{G}_i$, $\widetilde{\mathsf{M}}_i = \mathsf{M}_i$, and $\widetilde{\Omega}_0 = \Omega_0$, where the matrices $\widetilde{\mathsf{A}}_0(\mathsf{V})$, $\widetilde{\mathsf{A}}_i(\mathsf{V})$, $\widetilde{\mathsf{M}}_i(\mathsf{V})$, $i\in\mathsf{C}$, $\widetilde{\mathsf{B}}_{ij}(\mathsf{V})$, $\widetilde{\mathsf{B}}^{\mathsf{s}}_{ij}(\mathsf{V})$, $\widetilde{\mathsf{B}}^{\mathsf{a}}_{ij}(\mathsf{V})$, $i,j\in\mathsf{C}$, and the vectors $\widetilde{\mathsf{G}}_i(\mathsf{V})$, $i\in\mathsf{C}$, $\widetilde{\Omega}_0(\mathsf{V})$ are $\mathscr{C}^\infty$ functions. Note that the matrices $\mathsf{B}^{\mathsf{S}}_{ij}$ and $\mathsf{B}^{\mathsf{a}}_{ij}$ are the even and odd parts of B_{ij} with respect to the magnetic field $\mathbf{B}$. Finally, system (47) is of the partially symmetric form, that is, the vector and matrix coefficients verify properties (S_1)–(S_5).

Proof
The proof is lengthy and tedious and we refer the reader to References [9,10]. □

We investigate the nullspace of the matrix $\widetilde{\mathsf{B}}(\mathsf{V},\boldsymbol{\xi})$ in order to obtain a partially symmetric form for system (47).

Lemma 4.3
The nullspace of the matrix $\widetilde{\mathsf{B}}(\mathsf{V},\boldsymbol{\xi}) = \sum_{i,j\in\mathsf{C}}\widetilde{\mathsf{B}}^{\mathsf{s}}_{ij}(\mathsf{V})\xi_i\xi_j$, denoted by N, is of dimension 7 and does not depend on $\mathsf{V}\in\mathcal{O}_{\mathsf{V}}$ and $\boldsymbol{\xi}\in\Sigma^2$. This nullspace is spanned by the column vectors $(\mathsf{u}^{\mathrm{T}}, 0_{1,10})^{\mathrm{T}}$ and e^{n^s+k}, $k=1,\dots,6$, where $(\mathsf{e}^k)_{k=1,\dots,n^s+10}$ is the canonical basis of $\mathbb{R}^{n^s+10}$.

Proof
This results from an explicit evaluation of the partially symmetrized system and we refer to References [9,10] for more details. □

The following proposition is a direct consequence of (31).

Proposition 4.4
System (47) can be rewritten in the form

$$\widetilde{\mathsf{A}}_0(\mathsf{V})\,\partial_t\mathsf{V} + \sum_{i\in\mathsf{C}}(\widetilde{\mathsf{A}}_i(\mathsf{V}) + \widetilde{\mathsf{A}}^{\mathsf{a}}_i(\mathsf{V}))\partial_i\mathsf{V} = \sum_{i,j\in\mathsf{C}}\partial_i(\widetilde{\mathsf{B}}_{ij}(\mathsf{V})\,\partial_j\mathsf{V}) + \widetilde{\Omega}(\mathsf{V}) \tag{48}$$

where

$$\widetilde{\mathsf{A}}^{\mathsf{a}}_i(\mathsf{V}) = \sum_{j\in\mathsf{C}}(\widetilde{\mathsf{M}}_j^{\mathrm{T}}\widetilde{\mathsf{B}}_{ji} - \widetilde{\mathsf{B}}_{ij}\widetilde{\mathsf{M}}_j - \partial_{\mathsf{V}}(\widetilde{\mathsf{B}}_{ij}\widetilde{\mathsf{M}}_j)\mathsf{V}) \tag{49}$$

and

$$\widetilde{\Omega}(\mathsf{V}) = -\widetilde{\mathsf{L}}(\mathsf{V})\mathsf{V} + \widetilde{\Omega}_0(\mathsf{V}), \quad \widetilde{\mathsf{L}}(\mathsf{V}) = \sum_{i,j\in\mathsf{C}}\widetilde{\mathsf{M}}_i{}^{\mathrm{T}}\widetilde{\mathsf{B}}_{ij}\widetilde{\mathsf{M}}_j$$

Moreover, the matrices $\widetilde{\mathsf{A}}^{\mathsf{a}}_i$, $i\in\mathsf{C}$, are such that for any x and y in $N(\widetilde{\mathsf{B}})$, we have $\langle x, \widetilde{\mathsf{A}}^{\mathsf{a}}_i y\rangle = 0$, $i\in\mathsf{C}$.

4.2. Partially normal form

In this section, we investigate a partially normal form for the system (48). Making use of the explicit basis of N, we define the matrix P by

$$\mathsf{P}=\begin{bmatrix} 1 & 0_{1,3} & 0_{1,3} & 0_{1,n^s-1} & 0_{1,3} & 0 \\ \check{\mathsf{u}} & 0_{n^s-1,3} & 0_{n^s-1,3} & \mathbb{I}_{n^s-1,n^s-1} & 0_{n^s-1,3} & 0_{n^s-1,1} \\ 0_{3,1} & 0_{3,3} & 0_{3,3} & 0_{3,n^s-1} & \mathbb{I} & 0_{3,1} \\ 0_{3,1} & \mathbb{I} & 0_{3,3} & 0_{3,n^s-1} & 0_{3,3} & 0_{3,1} \\ 0_{3,1} & 0_{3,3} & \mathbb{I} & 0_{3,n^s-1} & 0_{3,3} & 0_{3,1} \\ 0_{1,1} & 0_{1,3} & 0_{1,3} & 0_{1,n^s-1} & 0_{1,3} & 1 \end{bmatrix} \tag{50}$$

with $\check{\mathsf{u}}$ the vector of size n^s-1 defined by $\check{\mathsf{u}}=(1,\dots,1)^{\mathrm{T}}$. We may then introduce the auxiliary variable $\mathsf{U}'=\mathsf{P}^{\mathrm{T}}\mathsf{U}$ and the corresponding entropic variable $\mathsf{V}'=\mathsf{P}^{-1}\mathsf{V}$ given by

$$\mathsf{U}'=(\rho,\mathbf{E}^{\mathrm{T}},\mathbf{B}^{\mathrm{T}},\check{\varrho}^{\mathrm{T}},\rho\mathbf{v}^{\mathrm{T}},\rho e^{\mathrm{t}})^{\mathrm{T}}$$

with $\check{\varrho}=(\rho_2,\dots,\rho_{n^s})^{\mathrm{T}}$, and

$$\mathsf{V}'=\frac{1}{T}(g_1-\tfrac{1}{2}\mathbf{v}\cdot\mathbf{v},\varepsilon_0\mathbf{E}^{\mathrm{T}},\tfrac{1}{\mu_0}\mathbf{B}^{\mathrm{T}},g_2-g_1,\dots,g_{n^s}-g_1,\mathbf{v}^{\mathrm{T}},-1)^{\mathrm{T}}$$

From Theorem 3.7, normal variables are in the form $\mathsf{W}=(\psi_{\mathrm{I}}(\mathsf{U}'_{\mathrm{I}}),\phi_{\mathrm{II}}(\mathsf{V}'_{\mathrm{II}}))^{\mathrm{T}}$, where U'_{I} is the first seven components of U' and $\mathsf{V}'_{\mathrm{II}}$ the last n^s+3 components of V'. For convenience, we choose the variable W given by

$$\mathsf{W}=\left(\rho,\mathbf{E}^{\mathrm{T}},\mathbf{B}^{\mathrm{T}},\log\frac{\rho_2^{r_2}}{\rho_1^{r_1}},\dots,\log\frac{\rho_{n^s}^{r_{n^s}}}{\rho_1^{r_1}},\mathbf{v}^{\mathrm{T}},T\right)^{\mathrm{T}} \tag{51}$$

Proposition 4.5
The map $\mathsf{V}\mapsto\mathsf{W}$ is a $\mathscr{C}^\infty$ diffeomorphism from the open set $\mathscr{O}_{\mathsf{V}}$ onto the open set $\mathscr{O}_{\mathsf{W}}=(0,\infty)\times\mathbb{R}^6\times\mathbb{R}^{n^s-1}\times\mathbb{R}^3\times(0,\infty)$.

Proof
We only have to prove that the map $\mathsf{Z}\mapsto\mathsf{W}$ is a $\mathscr{C}^\infty$ diffeomorphism, thanks to Propositions 2.1 and 4.1. It is readily $\mathscr{C}^\infty$ and, in order to describe its range, let $\mathsf{w}\in\mathscr{O}_{\mathsf{W}}$ and define $\mathsf{z}_{n^s+10}=\mathsf{w}_{n^s+10}$, $\mathsf{z}_{n^s+6+\mu}=\mathsf{w}_{4+\mu}$, $\mathsf{z}_{n^s+3+\mu}=\mathsf{w}_{1+\mu}$, $\mathsf{z}_{n^s+\mu}=\mathsf{w}_{n^s+6+\mu}$, $\mu=1,2,3$, z_1 by the following equation:

$$\mathsf{z}_1+\sum_{2\leqslant k\leqslant n^s}\mathsf{z}_1^{r_1/r_k}\exp(\mathsf{w}_{6+k}/r_k)=\mathsf{w}_1$$

which admits a unique positive solution and $\mathsf{z}_k=(\mathsf{z}_1^{r_1}\exp\mathsf{w}_{6+k})^{1/r_k}$, $2\leqslant k\leqslant n^s$. Evaluating of $\partial_{\mathsf{Z}}\mathsf{W}$ and applying the inverse function theorem, we then deduce that the map $\mathsf{Z}\mapsto\mathsf{W}$ is a $\mathscr{C}^\infty$ diffeomorphism onto $\mathscr{O}_{\mathsf{W}}$. □

In order to separate hyperbolic and parabolic variables, we introduce the partitioning of $\{1,\dots,n^s+10\}$ into $\mathrm{I}=\{1,\dots,7\}$ and $\mathrm{II}=\{8,\dots,n^s+10\}$, since $n_0=7$, and we use the vector and matrix block structure induced by this partitioning. We have $\mathsf{W} = (\mathsf{W}_{\mathrm{I}}{}^{\mathrm{T}}, \mathsf{W}_{\mathrm{II}}{}^{\mathrm{T}})^{\mathrm{T}}$, where W_{I} corresponds to the hyperbolic variables and W_{II} to the parabolic variables

$$\mathsf{W}_{\mathrm{I}} = \left(\rho, \mathbf{E}^{\mathrm{T}}, \mathbf{B}^{\mathrm{T}}\right)^{\mathrm{T}}, \quad \mathsf{W}_{\mathrm{II}} = \left(\log \frac{\rho_2^{r_2}}{\rho_1^{r_1}}, \ldots, \log \frac{\rho_{n^s}^{r_{n^s}}}{\rho_1^{r_1}}, \mathbf{v}^{\mathrm{T}}, T\right)^{\mathrm{T}} \tag{52}$$

Theorem 4.6
The change of variables $\mathsf{V} \mapsto \mathsf{W}$ transforms the system (47) into

$$\overline{\mathsf{A}}_0 \partial_t \mathsf{W} + \sum_{i\in \mathsf{C}} (\overline{\mathsf{A}}_i + \overline{\mathsf{A}}_i^{\mathrm{a}}) \partial_i \mathsf{W} = \sum_{i,j\in \mathsf{C}} \partial_i (\overline{\mathsf{B}}_{ij} \partial_j \mathsf{W}) + \overline{\mathscr{T}} + \overline{\Omega} \tag{53}$$

where

$$\begin{aligned}
&\overline{\mathsf{A}}_0 = (\partial_{\mathsf{W}}\mathsf{V})^{\mathrm{T}}\, \widetilde{\mathsf{A}}_0\, (\partial_{\mathsf{W}}\mathsf{V}), \quad \overline{\mathsf{B}}_{ij} = (\partial_{\mathsf{W}}\mathsf{V})^{\mathrm{T}}\, \widetilde{\mathsf{B}}_{ij}\, (\partial_{\mathsf{W}}\mathsf{V})\\
&\overline{\mathsf{B}}_{ij}^{\mathsf{s}} = (\partial_{\mathsf{W}}\mathsf{V})^{\mathrm{T}}\, \widetilde{\mathsf{B}}_{ij}^{\mathsf{s}}\, (\partial_{\mathsf{W}}\mathsf{V}), \quad \overline{\mathsf{B}}_{ij}^{\mathsf{a}} = (\partial_{\mathsf{W}}\mathsf{V})^{\mathrm{T}}\, \widetilde{\mathsf{B}}_{ij}^{\mathsf{a}}\, (\partial_{\mathsf{W}}\mathsf{V})\\
&\overline{\mathsf{A}}_i = (\partial_{\mathsf{W}}\mathsf{V})^{\mathrm{T}}\, \widetilde{\mathsf{A}}_i\, (\partial_{\mathsf{W}}\mathsf{V}), \quad \overline{\Omega} = (\partial_{\mathsf{W}}\mathsf{V})^{\mathrm{T}}\, \widetilde{\Omega}\\
&\overline{\mathsf{M}}_i = (\partial_{\mathsf{W}}\mathsf{V})^{-1}\, \widetilde{\mathsf{M}}_i\, (\partial_{\mathsf{W}}\mathsf{V}), \quad \overline{\mathsf{G}}_i = (\partial_{\mathsf{W}}\mathsf{V})^{-1}\, \widetilde{\mathsf{G}}_i\\
&\overline{\mathsf{A}}_i^{\mathrm{a}} = (\partial_{\mathsf{W}}\mathsf{V})^{\mathrm{T}}\, \widetilde{\mathsf{A}}_i^{\mathrm{a}}\, (\partial_{\mathsf{W}}\mathsf{V}), \quad \overline{\mathscr{T}} = -\sum_{i,j\in\mathsf{C}} \partial_i (\partial_{\mathsf{W}}\mathsf{V})^{\mathrm{T}}\, \widetilde{\mathsf{B}}_{ij}\, (\partial_{\mathsf{W}}\mathsf{V})\, \partial_j \mathsf{W}
\end{aligned}$$

and

$$\overline{\mathsf{A}}_i^{*} = \sum_{j\in\mathsf{C}} (\overline{\mathsf{M}}_j{}^{\mathrm{T}} \overline{\mathsf{B}}_{ji} - \overline{\mathsf{B}}_{ij} \overline{\mathsf{M}}_j - (\partial_{\mathsf{W}}\mathsf{V})^{\mathrm{T}} \partial_{\mathsf{W}} (\widetilde{\mathsf{B}}_{ij} \widetilde{\mathsf{M}}_j) \mathsf{V}) \tag{54}$$

where the matrices $\overline{\mathsf{A}}_0(\mathsf{W})$, $\overline{\mathsf{A}}_i(\mathsf{W})$, $\overline{\mathsf{A}}_i^{\mathrm{a}}(\mathsf{W})$, $i\in\mathsf{C}$, $\overline{\mathsf{B}}_{ij}(\mathsf{W})$, $\overline{\mathsf{B}}_{ij}^{\mathsf{s}}(\mathsf{W})$, $\overline{\mathsf{B}}_{ij}^{\mathsf{a}}(\mathsf{W})$, $i,j\in\mathsf{C}$, and vectors $\overline{\mathsf{G}}_i(\mathsf{W})$, $i\in\mathsf{C}$, $\overline{\Omega}(\mathsf{W})$, $\overline{\mathscr{T}}(\mathsf{W}, \boldsymbol{\partial}_{\mathbf{x}}\mathsf{W})$ are $\mathscr{C}^\infty$ functions of $\mathsf{W}\in\mathscr{O}_{\mathsf{W}}$ and $\boldsymbol{\partial}_{\mathbf{x}}\mathsf{W}\in\mathbb{R}^{3(n^s+10)}$. Furthermore, system (53) is of the partially normal form, that is the matrices $\overline{\mathsf{A}}_0$, $\overline{\mathsf{A}}_i^{\mathrm{a}}$, $i\in\mathsf{C}$, $\overline{\mathsf{B}}_{ij}$, $i,j\in\mathsf{C}$, and vector $\overline{\mathscr{T}}$ satisfy property (Nor_1)–(Nor_3).

Proof
This theorem is a direct application of Theorem 3.7. □

The partially normal form (53) will be used for local existence theorems. This form, however, is insufficient in order to establish global existence results around constant equilibrium states [3,4,7]. More specifically, consider an equilibrium state W^{e} such that $\mathbf{E}^{\mathsf{e}} = \mathbf{B}^{\mathsf{e}} = \mathbf{v}^{\mathsf{e}} = 0$. One can then establish that the antisymmetric contributions of dissipative matrices vanish $\overline{\mathsf{B}}_{ij}^{\mathsf{ae}} = 0$, $i,j\in\mathsf{C}$, and that we also have $(\partial_{\mathsf{V}}(\overline{\mathsf{B}}_{ij}\overline{\mathsf{M}}_j)\mathsf{V})^{\mathsf{e}} = 0$, $i\in\mathsf{C}$. Furthermore, the linearized source term $\overline{\mathsf{L}}^{\mathsf{e}} = -(\partial_{\mathsf{W}}\overline{\Omega})^{\mathsf{e}}$ is symmetric positive semidefinite thanks to the structural assumptions on thermochemistry. However, it can be shown that the antisymmetric contributions $\sum_{j\in\mathsf{C}}(\overline{\mathsf{M}}_j{}^{\mathrm{T}}\overline{\mathsf{B}}_{ji} - \overline{\mathsf{B}}_{ij}\overline{\mathsf{M}}_j)^{\mathsf{e}}$, $i\in\mathsf{C}$, *never vanish*, so that we cannot apply the existence theorems of [3,4,7].

Proposition 4.7
The system in normal form (53) can be split into a hyperbolic subsystem and a parabolic subsystem

$$\begin{cases} \overline{\mathsf{A}}_0^{\mathrm{I,I}}\ \partial_t \mathsf{W}_{\mathrm{I}} = -\sum\limits_{i\in\mathsf{C}} \overline{\mathsf{A}}_i^{\mathrm{I,I}}\ \partial_i \mathsf{W}_{\mathrm{I}} + \overline{\Gamma}_{\mathrm{I}} \\ \overline{\mathsf{A}}_0^{\mathrm{II,II}}\ \partial_t \mathsf{W}_{\mathrm{II}} = -\sum\limits_{i\in\mathsf{C}} \left(\overline{\mathsf{A}}_i^{\mathrm{II,I}} + \overline{\mathsf{A}}_i^{\mathrm{aII,I}}\right) \partial_i \mathsf{W}_{\mathrm{I}} + \sum\limits_{i,j\in\mathsf{C}} \partial_i(\overline{\mathsf{B}}_{ij}^{\mathrm{II,II}} \partial_j \mathsf{W}_{\mathrm{II}}) + \overline{\Gamma}_{\mathrm{II}} \end{cases} \tag{55}$$

where

$$\overline{\Gamma}_{\mathrm{I}} = \overline{\Omega}_{\mathrm{I}} - \sum_{i\in\mathsf{C}} (\overline{\mathsf{A}}_i^{\mathrm{aI,II}} + \overline{\mathsf{A}}_i^{\mathrm{I,II}}) \partial_i \mathsf{W}_{\mathrm{II}}, \quad \overline{\Gamma}_{\mathrm{II}} = \overline{\Omega}_{\mathrm{II}} + \overline{\mathscr{T}}_{\mathrm{II}} - \sum_{i\in\mathsf{C}} (\overline{\mathsf{A}}_i^{\mathrm{aII,II}} + \overline{\mathsf{A}}_i^{\mathrm{II,II}}) \partial_i \mathsf{W}_{\mathrm{II}}$$

Moreover, the matrices $\overline{\mathsf{A}}_0^{\mathrm{I,I}}(\mathsf{W})$, $\overline{\mathsf{A}}_0^{\mathrm{II,II}}(\mathsf{W})$, $\overline{\mathsf{A}}_i^{\mathrm{I,I}}(\mathsf{W})$, $\overline{\mathsf{A}}_i^{\mathrm{II,I}}(\mathsf{W})$, $\overline{\mathsf{A}}_i^{\mathrm{aII,I}}(\mathsf{W})$, $i\in\mathsf{C}$, $\overline{\mathsf{B}}_{ij}^{\mathrm{II,II}}(\mathsf{W})$, $i,j\in\mathsf{C}$, are $\mathscr{C}^\infty$ functions of $\mathsf{W}\in\mathscr{O}_{\mathsf{W}}$, and the vectors $\overline{\Gamma}_{\mathrm{I}}(\mathsf{W}, \boldsymbol{\partial}_{\mathbf{x}}\mathsf{W}_{\mathrm{II}})$, $\overline{\Gamma}_{\mathrm{II}}(\mathsf{W}, \boldsymbol{\partial}_{\mathbf{x}}\mathsf{W}_{\mathrm{II}})$ are $\mathscr{C}^\infty$ functions of $\mathsf{W}\in\mathscr{O}_{\mathsf{W}}$ and $\boldsymbol{\partial}_{\mathbf{x}}\mathsf{W}_{\mathrm{II}} \in \mathbb{R}^{3(n^s+3)}$.

Proof
It is a direct application of Theorem 4.6. □

4.3. Existence theorem

We now apply Theorem 3.8 to system modelling multicomponent reactive magnetized dissipative flows. Note that the local existence result of Kawashima [4] also applies to the system (55).

Theorem 4.8
Consider the Cauchy problem for system (55) in $\mathbb{R}^3$ with initial conditions

$$\mathsf{W}(0,\mathbf{x}) = \mathsf{W}^0(\mathbf{x}), \quad \mathbf{x}\in\mathbb{R}^3 \tag{56}$$

where $\mathsf{W}^0 \in V_l(\mathbb{R}^3)$, $l>9/2$, $\inf_{\mathbb{R}^3} \rho^0 > 0$ and $\inf_{\mathbb{R}^3} T^0 > 0$.

Then there exits $t_0>0$, such that (55), (56) admit a unique solution $\mathsf{W} = (\mathsf{W}_{\mathrm{I}}^{\mathrm{T}}, \mathsf{W}_{\mathrm{II}}^{\mathrm{T}})^{\mathrm{T}}$ with $\mathsf{W}(t,\mathbf{x})\in\mathscr{O}_{\mathsf{W}}$ defined on the strip $\bar{Q}_{t_0} = [0,t_0]\times\mathbb{R}^3$, continuous in $\bar{Q}_{t_0}$ with its derivatives of first-order in t and second-order in $\mathbf{x}$, and for which the following inequalities hold:

$$\sup_{0\leqslant t\leqslant t_0} \left[\|\rho(t)\|_l + \sum_{k=2}^{n^s} \left\| \log\frac{\rho_k^{r_k}}{\rho_1^{r_1}}(t) \right\|_l + \sum_{i\in\mathsf{C}} (\|v_i(t)\|_l + \|E_i(t)\|_l + \|B_i(t)\|_l) + \|T(t)\|_l \right] < +\infty$$

$$\inf_{\bar{Q}_{t_0}} \rho(t,\mathbf{x}) > 0, \quad \inf_{\bar{Q}_{t_0}} T(t,\mathbf{x}) > 0$$

$$\sup_{0\leqslant t\leqslant t_0} \left[\|\partial_t\rho(t)\|_{l-1} + \sum_{i\in\mathsf{C}} (\|\partial_t E_i(t)\|_{l-1} + \|\partial_t B_i(t)\|_{l-1}) \right] < +\infty$$

$$\int_0^{t_0}\left[\sum_{k=2}^{n^s}\left\|\partial_t \log \frac{\rho_k^{r_k}}{\rho_1^{r_1}}(\tau)\right\|_{l-1}^2 + \sum_{i\in\mathsf{C}}\|\partial_t v_i(\tau)\|_{l-1}^2 + \|\partial_t T(\tau)\|_{l-1}^2\right.$$

$$\left. + \sum_{k=2}^{n^s}\left\|\log \frac{\rho_k^{r_k}}{\rho_1^{r_1}}(\tau)\right\|_{l+1}^2 + \sum_{i\in\mathsf{C}}\|v_i(\tau)\|_{l+1}^2 + \|T(\tau)\|_{l+1}^2\right]\mathrm{d}\tau < +\infty$$

Moreover, either t_0 is as large as one wants, or there exists t_1 such that the theorem is true for any $t_0 < t_1$ and such that for $t_0 \to t_1^-$, either the following quantity:

$$\|\rho(t_0)\|_{1,\infty} + \sum_{k=2}^{n^s}\left\|\log \frac{\rho_k^{r_k}}{\rho_1^{r_1}}(t_0)\right\|_{2,\infty} + \|T(t_0)\|_{2,\infty} + \sum_{i\in\mathsf{C}}(\|v_i(t_0)\|_{2,\infty} + \|E_i(t_0)\|_{1,\infty} + \|B_i(t_0)\|_{1,\infty}) \tag{57}$$

or $\sup_{\bar{Q}_{t_0}} 1/T$ is unbounded.

We only have to verify that the assumptions of Theorem 3.8 are satisfied.

Proof

We define the function $\mathscr{L}$ by $\mathscr{L}(\mathsf{W}_{\mathrm{I}},\mathsf{W}_{\mathrm{II}}) = 1/\rho + 1/T$. As we assume that $\mathsf{W}^0 \in V_l(\mathbb{R}^3)$, $\inf_{\mathbb{R}^3} \rho^0 > 0$, $l > 9/2$, and $\inf_{\mathbb{R}^3} T^0 > 0$, we obtain that property (Ex_1) holds. Proposition 4.7 implies readily properties (Ex_2)–(Ex_3) and (Ex_6) on the regularity of matrices and vectors. Properties (Ex_4)–(Ex_5) concerning symmetry of matrices $\overline{\mathsf{A}}_0^{\mathrm{I,I}}$, $\overline{\mathsf{A}}_0^{\mathrm{II,II}}$ and $\overline{\mathsf{A}}_i^{\mathrm{I,I}}$, $i \in \mathsf{C}$, are obtained by using properties (S_1)–(S_2) obtained in Theorem 4.6.

In order to establish that (Ex_7) holds, we consider ϕ_{II} in $W_2^1(\mathbb{R}^3)$ written in the form $\phi_{\mathrm{II}} = (\phi^{\varrho\mathrm{T}}, \phi^{\mathrm{vT}}, \phi^{\mathrm{T}})^{\mathrm{T}}$, with $\phi^{\varrho} = (\phi_2^{\varrho}, \dots, \phi_{n^s}^{\varrho})^{\mathrm{T}}$, $\phi^{\mathrm{v}} \in \mathbb{R}^3$ and $\phi^{\mathrm{T}} \in \mathbb{R}$. For the diffusive part $\overline{\mathsf{B}}_{ij}^{\mathrm{II,II}}$, $i,j \in \mathfrak{C}$, we obtain

$$\sum_{i,j\in\mathsf{C}} (\partial_i \phi_{\mathrm{II}})^{\mathrm{T}} \overline{\mathsf{B}}_{ij}^{\mathrm{diff,II,II}} (\partial_j \phi_{\mathrm{II}}) = \frac{p}{T}\psi^{\|\mathrm{T}} A^{\|}\psi^{\|} + \frac{p}{T}\sum_{k\in\mathsf{C}} \psi_k^{\perp\mathrm{T}} A^{\perp}\psi_k^{\perp}$$

where $\psi^{\|} = \sum_{i\in\mathsf{C}} \mathscr{B}\cdot\mathbf{e}_i \partial_i \phi^{\mathrm{d}}$, $\psi_k^{\perp} = \sum_{i\in\mathsf{C}} (\mathbf{e}_i - \mathscr{B}\cdot\mathbf{e}_i\mathscr{B})\cdot\mathbf{e}_k \;\; \partial_i\phi^{\mathrm{d}}$, ϕ^{d} is defined by $\phi^{\mathrm{d}} = (\frac{1}{T}\phi^{\mathrm{T}}, 0, \frac{T}{p}\rho_2(\phi_2^{\varrho} + \frac{r_2}{T}\phi^{\mathrm{T}}), \dots, \frac{T}{p}\rho_{n^s}(\phi_{n^s}^{\varrho} + \frac{r_{n^s}}{T}\phi^{\mathrm{T}}))^{\mathrm{T}}$, and $(\mathbf{e}_1, \mathbf{e}_2, \mathbf{e}_3)$ is the canonical basis of $\mathbb{R}^3$. We now use (Tr_3) concerning $A^{\|}$ et $A^{\perp}$. Since the second component of ϕ^{d} vanish, ϕ^{d} is proportional to $(0, \varrho^{\mathrm{T}})^{\mathrm{T}}$ only if $\phi^{\mathrm{d}} = 0$, in such a way that

$$\sum_{i,j\in\mathsf{C}} (\partial_i \phi_{\mathrm{II}})^{\mathrm{T}} \overline{\mathsf{B}}_{ij}^{\mathrm{diff,II,II}} (\partial_j \phi_{\mathrm{II}}) \geqslant \alpha \sum_{k\in\mathsf{C}} [|\mathbf{e}_k\cdot(\boldsymbol{\partial}_{\mathbf{x}}\phi^{\mathrm{T}})^{\|}|^2 + |\mathbf{e}_k\cdot(\boldsymbol{\partial}_{\mathbf{x}}\phi^{\mathrm{T}})^{\perp}|^2]$$

$$+ \alpha\sum_{l=1}^{n^s}\sum_{k\in\mathsf{C}} [|\mathbf{e}_k\cdot(\boldsymbol{\partial}_{\mathbf{x}}\phi_l^{\varrho})^{\|}|^2 + |\mathbf{e}_k\cdot(\boldsymbol{\partial}_{\mathbf{x}}\phi_l^{\varrho})^{\perp}|^2]$$

and, for W in a compact set of $\mathcal{O}_{\mathsf{W}}$, we have uniformly

$$\sum_{i,j\in\mathsf{C}} (\partial_i \phi_{\mathrm{II}})^{\mathrm{T}} \overline{\mathsf{B}}_{ij}^{\mathrm{diff,II,II}} (\partial_j \phi_{\mathrm{II}}) \geqslant \alpha \left(|\boldsymbol{\partial}_{\mathbf{x}}\phi^{\mathrm{T}}|^2 + \sum_{2\leqslant l\leqslant n^s} |\boldsymbol{\partial}_{\mathbf{x}}\phi_l^{\varrho}|^2 \right) \tag{58}$$

Concerning the viscous part $\overline{\mathsf{B}}_{ij}^{\text{II},\text{II}}$, $i,j\in\mathsf{C}$, we have

$$\sum_{i,j\in\mathsf{C}}(\partial_i\phi_{\text{II}})^{\mathrm{T}}\overline{\mathsf{B}}_{ij}^{\text{visc},\text{II},\text{II}}(\partial_j\phi_{\text{II}})$$

$$=\frac{\kappa}{T}\left[\sum_{i\in\mathsf{C}}\mathbf{e}_i\cdot\partial_i\phi^{\mathbf{v}}\right]^2+\frac{\eta_1+\eta_3}{3T}\left[\sum_{i\in\mathsf{C}}(\mathbf{e}_i\cdot(\partial_i\phi^{\mathbf{v}})-3\mathbf{e}_i^{\|}\cdot(\partial_i\phi^{\mathbf{v}})^{\|})\right]^2$$

$$+\frac{\eta_1-\eta_3}{T}\left[\left[\sum_{i\in\mathsf{C}}\mathbf{e}_i^{\perp}\cdot(\partial_i\phi^{\mathbf{v}})^{\perp}\right]^2+\left[\sum_{i\in\mathsf{C}}\mathbf{e}_i^{\perp}\cdot(\partial_i\phi^{\mathbf{v}})^{\odot}\right]^2-2\sum_{i,j\in\mathsf{C}}(\mathbf{e}_i^{\perp}\cdot\mathbf{e}_j^{\odot}(\partial_i\phi^{\mathbf{v}})^{\perp}\cdot(\partial_j\phi^{\mathbf{v}})^{\odot})\right]$$

$$+\frac{\eta_1+\eta_4}{T}\left[\sum_{i\in\mathsf{C}}(\mathscr{B}\cdot\mathbf{e}_i(\partial_i\phi^{\mathbf{v}})^{\perp}+\mathscr{B}\cdot\partial_i\phi^{\mathbf{v}}\,\mathbf{e}_i^{\perp}\right]\cdot\left[\sum_{j\in\mathsf{C}}(\mathscr{B}\cdot\mathbf{e}_j(\partial_j\phi^{\mathbf{v}})^{\perp}+\mathscr{B}\cdot\partial_j\phi^{\mathbf{v}}\,\mathbf{e}_j^{\perp}\right]$$

In order to establish that this quantity is bounded from below, we use the invariance property with respect to orthogonal co-ordinate transforms and we chose a new co-ordinate system $(\mathbf{e}_1,\mathbf{e}_2,\mathbf{e}_3)$ such that $\mathscr{B}=\mathbf{e}_1$. We then have

$$\sum_{i,j\in\mathsf{C}}(\partial_i\phi_{\text{II}})^{\mathrm{T}}\ \overline{\mathsf{B}}_{ij}^{\text{visc},\text{II},\text{II}}(\partial_j\phi_{\text{II}})=\frac{\kappa}{T}(\boldsymbol{\partial}_{\mathbf{x}}\cdot\phi^{\mathbf{v}})^2+\frac{\eta_1+\eta_4}{T}[(\mathbf{S}_{12}^{\phi})^2-(\mathbf{S}_{13}^{\phi})^2]$$

$$+\frac{3}{4}\frac{\eta_1+\eta_3}{T}(\mathbf{S}_{11}^{\phi})^2+\frac{\eta_1-\eta_3}{T}[\tfrac{1}{4}(\mathbf{S}_{22}^{\phi}-\mathbf{S}_{33}^{\phi})^2+(\mathbf{S}_{23}^{\phi})^2]$$

where $\mathbf{S}^{\phi}$ is defined by $\mathbf{S}^{\phi}=\boldsymbol{\partial}_{\mathbf{x}}\phi^{\mathbf{v}}+(\boldsymbol{\partial}_{\mathbf{x}}\phi^{\mathbf{v}})^{\mathrm{T}}-\frac{2}{3}(\boldsymbol{\partial}_{\mathbf{x}}\cdot\phi^{\mathbf{v}})\mathbb{I}$. Thanks to (Tr_2) about strict dissipativity and concerning $\eta_1+\eta_4$, $\eta_1+\eta_3$, $\eta_1-\eta_3$, and assuming that κ, is strictly positive, we have

$$\sum_{i,j\in\mathsf{C}}(\partial_i\phi_{\text{II}})^{\mathrm{T}}\overline{\mathsf{B}}_{ij}^{\text{visc},\text{II},\text{II}}(\partial_j\phi_{\text{II}})\geqslant\alpha\sum_{i,j\in\mathsf{C}}(\partial_i\phi_j^{\mathbf{v}}+\partial_j\phi_i^{\mathbf{v}})^2 \tag{59}$$

uniformly in W in a compact set of $\mathscr{O}_{\mathsf{W}}$. Combining these estimates (58) and (59) with

$$\int\sum_{i,j\in\mathsf{C}}(\partial_i\phi_j^{\mathbf{v}}+\partial_j\phi_i^{\mathbf{v}})^2\,\mathrm{d}\mathbf{x}=\frac{1}{2}\int\sum_{i,j\in\mathsf{C}}(\partial_i\phi_j^{\mathbf{v}})^2\,\mathrm{d}\mathbf{x}+\frac{1}{2}\int\left(\sum_{i\in\mathsf{C}}\partial_i\phi_i^{\mathbf{v}}\right)^2\mathrm{d}\mathbf{x}$$

valid for $\phi_i^{\mathbf{v}}\in W_2^1(\mathbb{R}^3)$, $i\in\mathsf{C}$, we obtain (Ex_8). The case where κ vanishes can be reduced to the case where κ is strictly positive proceeding as in Reference [6].

Finally, we note that from the conservation of ρ, we have

$$\rho(t,\mathbf{x})\geqslant\inf_{\mathbb{R}^3}\ \rho^0(\mathbf{x})\exp\left(-\int_0^t\|\boldsymbol{\partial}_{\mathbf{x}}\cdot\mathbf{v}(s)\|_{0,\infty}\mathrm{d}s\right)$$

and thus $\inf_{\mathbb{R}^3}\rho(t,\mathbf{x})>0$ as long as (57) remains finite, so that only T may reach the boundary of $\mathscr{O}_{\mathsf{W}}$. □

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