Haagerup property for arbitrary von Neumann algebras

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related to work by R. Okayasu, R. Tomatsu

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Equivalent notions of the Haagerup property

A group G has the Haagerup property if:

- There exists a net of positive definite normalized functions in $C_0(G)$ converging to 1 uniformly on compacta
- G admits a proper affine action on a Hilbert space
- There exists a proper, conditionally negative function on G

Examples

- Amenable groups
- F_n (Haagerup, '78/'79)
- $SL(2, \mathbb{Z})$
- Haagerup property + Property (T) implies compactness

HAP for von Neumann algebras

Definition Haagerup property (Choda '83, Jolissaint '02)

A finite von Neumann algebra (M, τ) has HAP if there exists a net $(\Phi_i)_i$ of normal cp maps $\Phi_i : M \to M$ such that:

- $\tau \circ \Phi_i \leq \tau$
- The map $T_i : x\Omega_\tau \mapsto \Phi_i(x)\Omega_\tau$ is compact
- $T_i \to 1$ strongly

Remark:

- In the definition (M, τ) has HAP than Φ_i's can be chosen unital and such that $\tau \circ \Phi_i = \tau$.

HAP for groups versus HAP for vNA's

Theorem (Choda '83)

A discrete group G has HAP $\Leftrightarrow$ The group von Neumann algebra $\mathcal{L}(G)$ has HAP

Idea of the proof: (Haagerup)

$\Rightarrow$ φ_i the positive definite functions $\Rightarrow \Phi_i : \mathcal{L}(G) \to \mathcal{L}(G) : \lambda(f) \mapsto \lambda(\varphi_i f)$.

$\Leftarrow$ Φ_i cp maps $\Rightarrow$ use the 'averaging technique':

$$\varphi_i(s) = \tau(\lambda(s)^* \Phi_i(\lambda(s))).$$

HAP for von Neumann algebras

Definition Haagerup property

A σ-finite von Neumann algebra (M, φ) has HAP if there exists a net $(\Phi_i)_i$ of normal cp maps $\Phi_i : M \to M$ such that:

- $\varphi \circ \Phi_i \leq \varphi$
- The map $T_i : x\Omega_\varphi \mapsto \Phi_i(x)\Omega_\varphi$ is compact
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HAP for von Neumann algebras

Definition Haagerup property (MC, Skalski)

An arbitrary von Neumann algebra (M, φ) with nsf weight φ has HAP if there exists a net $(\Phi_i)_i$ of normal cp maps $\Phi_i : M \to M$ such that:

- $\varphi \circ \Phi_i \leq \varphi$
- The map $T_i : \Lambda_\varphi(x) \mapsto \Lambda_\varphi(\Phi_i(x))$ is compact
- $T_i \to 1$ strongly

Remark:

- In our approach it is essential to treat weights instead of states.

Motivating examples

- Brannan '12: Free orthogonal and free unitary quantum groups have HAP.

  Kac case $\Rightarrow$ Semi-finite.

- De Commer, Freslon, Yamashita '13:

  Non-Kac case of this result $\Rightarrow$ Non-semi-finite.

- Houdayer, Ricard '11: Free Araki-Woods factors.

Problems arising?

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Questions:

- Does the definition depend on the choice of the weight?
- Can the maps Φ_i be taken ucp and φ-preserving?
- Can we always assume that $\Phi_i \circ \sigma_t^\varphi = \sigma_t^\varphi \circ \Phi_i$?

Weight independence

Theorem (MC, A. Skalski)

The HAP is independent of the choice of the n.s.f. weight: (M, φ) has HAP iff (M, ψ) has HAP.

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Idea of the proof:

- Treat the semi-finite case using Radon-Nikodym derivatives.

$$\varphi(h \cdot h) = \psi(\cdot)$$

Let φ have cp maps Φ_i. Then formally,

$$\Phi_i'(\cdot) := h^{-1}\Phi_i(h \cdot h)h^{-1},$$

will yield the cp maps for ψ.

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- Let α be any φ-preserving action of $\mathbb{R}$ on (M, φ). If $(M \rtimes \mathbb{R}, \hat{\varphi})$ has HAP then (M, φ) has HAP.

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- Use crossed product duality to conclude the converse.
- Conclude from the semi-finite case (Step 1).

Crossed products

Consequence

Let α be any action of a group G on M.

- If $M \rtimes_\alpha G$ has HAP then so has M
- If M has HAP and G amenable then $M \rtimes_\alpha G$ has HAP

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Comments:

- $M \rtimes_\alpha G$ has HAP implies that G has HAP in case G discrete
- $\mathbb{Z}^2 \rtimes SL(2, \mathbb{Z})$ does not have HAP whereas $SL(2, \mathbb{Z})$ has HAP and is weakly amenable

Markov property

Let M be a von Neumann algebra with normal state φ. We say that a normal map $\Phi : M \to M$ is *Markov* if it is a ucp φ-preserving map.

Theorem (MC, A. Skalski)

The following are equivalent:

- (M, φ) has HAP
- (M, φ) has HAP and the cp maps Φ_i are Markov

Corollary: If (M_1, φ_1) and (M_2, φ_2) have HAP then so does the free product $(M_1 \star M_2, \varphi_1 \star \varphi_2)$. (following Boca '93).

Modular HAP

We say that (M, φ) has the modular HAP if the cp maps Φ_i commute with $\sigma_t, t \in \mathbb{R}$.

Theorem (MC, Skalski)

(M, φ) is the von Neumann algebra of a compact quantum group with Haar state φ. TFAE:

- (M, φ) has HAP
- (M, φ) has the modular HAP

Questions:

- Does the definition depend on the choice of the weight? NO
- Can the maps Φ_i be taken ucp and φ-preserving (Markov)? YES if φ is a state.
- Can we always assume that $\Phi_i \circ \sigma_t^\varphi = \sigma_t^\varphi \circ \Phi_i$? YES in every known example.

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Question: Can we find Markov maps in case $(B(H), \mathrm{Tr})$?

Equivalent Haagerup properties

- Haagerup property via standard forms (Okayasu-Tomatsu) see also [COST, C.R. Adad. Sci. Paris 2014]

Symmetric Haagerup property

An arbitrary von Neumann algebra (M, φ) with nsf weight φ has symmetric HAP if there exists a net $(\Phi_i)_i$ of normal cp maps $\Phi_i : M \to M$ such that:

- $\varphi \circ \Phi_i \leq \varphi$
- The map $T_i : D_\varphi^{\frac{1}{4}} x D_\varphi^{\frac{1}{4}} \mapsto D_\varphi^{\frac{1}{4}} \Phi_i(x) D_\varphi^{\frac{1}{4}}$ is compact
- $T_i \to 1$ strongly

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- $T_i \to 1$ strongly or $\Phi_i \to 1$ in the point σ-weak topology

Equivalent Haagerup properties

Definition

Let $(\Phi_t)_{t\geq 0}$ be a semigroup of cp maps on M. $(\Phi_t)_{t\geq 0}$ is called Markov if $\Phi_t, t \geq 0$ is Markov. It is called KMS-symmetric if $T_t : D_\varphi^{\frac{1}{4}} x D_\varphi^{\frac{1}{4}} \mapsto D_\varphi^{\frac{1}{4}} x D_\varphi^{\frac{1}{4}}$ is self-adjoint. It is called immediately L^2-compact if $T_t, t > 0$ is compact.

Theorem: HAP via Markov semigroups (MC, Skalski)

M von Neumann algebra with normal state φ. TFAE:

- (M, φ) has HAP.
- There exists an immediately L^2-compact KMS-symmetric Markov semigroup $(\Phi_t)_{t\geq 0}$ on M.

Comment: Proof via symmetric HAP + ideas of Jolissaint-Martin '04/Cipriani Sauvageot '03.

Equivalent Haagerup properties

The next result is the non-commutative analogue of the existence of a proper conditionally negative definite function on a discrete group. $\Rightarrow$ quadratic form: 'quantum Dirichlet form'.

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Theorem (MC, Skalski)

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- M has HAP
- $L^2(M, \varphi)$ admits an orthonormal basis $\{e_n\}_n$ and a non-decreasing sequence of non-negative numbers $\{\lambda_n\}_n$ such that $\lim_n \lambda_n \to \infty$ and

$$Q(\xi) = \sum_{n=1}^{\infty} \lambda_n |\langle e_n, \xi \rangle|^2, \qquad \xi \in \mathrm{Dom}(Q),$$

where $\mathrm{Dom}(Q) = \{\xi \in L^2(M, \varphi) \mid \sum_n \lambda_n |\langle e_n, \xi \rangle|^2 < \infty\}$ defines a conservative completely Dirichlet form.

Equivalent Haagerup properties

The next result is the non-commutative analogue of the existence of a proper conditionally negative definite function on a discrete group. $\Rightarrow$ quadratic form: 'quantum Dirichlet form'.

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- Explicit example for free orthogonal quantum group (following Cipriani-Kula-Franz '13).

Quantum groups

Locally compact quantum groups (Kustermans, Vaes)

A **von Neumann algebraic quantum group** $\mathbb{G}$ consists of:

- a von Neumann algebra $L^\infty(\mathbb{G})$;
- a comultiplication, i.e. a unital normal $*$-homomorphism $\Delta\colon L^\infty(\mathbb{G}) \to L^\infty(\mathbb{G}) \otimes L^\infty(\mathbb{G})$ such that $(\Delta \otimes \iota)\Delta = (\iota \otimes \Delta)\Delta$;
- two normal semi-finite faithful Haar weights $\varphi, \psi : L^\infty(\mathbb{G})^+ \to [0, \infty]$, i.e.

$$
\begin{aligned}
(\iota \otimes \varphi)\Delta(x) &= \varphi(x)1, \qquad \forall x \in L^\infty(\mathbb{G})^+, \\
(\psi \otimes \iota)\Delta(x) &= \psi(x)1, \qquad \forall x \in L^\infty(\mathbb{G})^+.
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- two normal semi-finite faithful Haar weights $\varphi, \psi : L^\infty(\mathbb{G})^+ \to [0, \infty]$, i.e.

$$(\iota \otimes \varphi)\Delta(x) = \varphi(x)1, \qquad \forall x \in L^\infty(\mathbb{G})^+,$$
$$(\psi \otimes \iota)\Delta(x) = \psi(x)1, \qquad \forall x \in L^\infty(\mathbb{G})^+.$$

Classical examples:

- $L^\infty(G)$ with $\Delta_G(f)(x, y) = f(xy)$ and $\varphi(f) = \int f(x)d_lx$ Haar measure.
- $VN(G)$, $\Delta(\lambda_x) = \lambda_x \otimes \lambda_x$, $\varphi(\lambda_f) = f(e)$ Plancherel weight.

Quantum groups

Haagerup property for quantum groups (Daws, Fima, Skalski, White)

A quantum group $\mathbb{G}$ has the Haagerup property if:

- $c_0(\mathbb{G})$ admits an approximate unit build from 'positive definite functions' [DS]
- $\mathbb{G}$ admits a mixing representation weakly containing the trivial representation
- $\mathbb{G}$ admits a proper real cocycle

[DS] Daws, Salmi: Completely positive definite functions and Bochner's theorem for locally compact quantum groups, '13.

Open question: $\mathbb{G}$ has HAP if and only if $L^\infty(\hat{\mathbb{G}})$ has HAP

Quantum groups

Theorem (MC)

The quantum group $SU_q(1,1)$ (=non-compact+non-discrete+non-amenable) has the following properties:

- HAP
- Weakly amenable
- Coamenable

Comment: Proof based on Plancherel decomposition of the left multiplicative unitary by Groenevelt-Koelink-Kustermans '10 + De Canniere-Haagerup '85.

Quantum groups

Definition: weak amenability

A quantum group $\mathbb{G}$ is called weakly amenable if there exists a net $a_i \in A(\mathbb{G})$ such that,

$$\|a_i x - x\|_{A(\mathbb{G})} \to 0, \qquad x \in A(\mathbb{G}),$$

and $\|a_i\|_{M_0(A(\mathbb{G}))} \leq \Lambda$.

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- One can find a sequence $a_i \in A(\mathbb{G})^+$ commuting with the scaling group τ such that,

$$\|a_i x - x\|_{C_0(\mathbb{G})} \to 0, \qquad x \in A(\mathbb{G}),$$

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- Then work to turn $C_0(\mathbb{G})$-norm to $A(\mathbb{G})$-norm.

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  and $\|a_i\|_{M_0(A(\mathbb{G}))} \leq \Lambda$.
- Then work to turn $C_0(\mathbb{G})$-norm to $A(\mathbb{G})$-norm. Remark:

$$\| \cdot \|_{C_0(\mathbb{G})} \leq \| \cdot \|_{A(\mathbb{G})}$$