Persistent Approximation Property for controlled K-theory and large scale geometry

(jointwork with G. Yu)

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Operator propagation

- Let X be a proper metric space (i.e closed balls are compact) and let $\pi : C_0(X) \to \mathcal{L}(\mathcal{H})$ be a representation of $C_0(X)$ on a Hilbert space $\mathcal{H}$.
- Example : $\mathcal{H} = L^2(\mu, X)$ for μ Borelian measure on X and π the pointwise multiplication.

Definition

- *If T is an operator of $\mathcal{L}(\mathcal{H})$, then* Supp T *is the complementary of the open subset of $X \times X$*

$$\{(x,y) \in X \times X \text{ such that } \exists f \text{ and } g \in C_c(X) \text{ such that } f(x) \neq 0,\ g(y) \neq 0 \text{ and } \pi(f) \cdot T \cdot \pi(g) = 0\}$$

- *T has propagation less than r if $d(x,y) \leq r$ for all (x,y) in* Supp T*.*
- *if such r exists we say that T has finite propagation (less than r).*

Propagation and indices

- Let D be an elliptic differential operator on a compact manifold M.
- Let Q be a parametrix for D.
- Then $S_0 := Id - QD$ and $S_1 := Id - DQ$ are smooth kernel operators on $M \times M$:
- $$P_D = \begin{pmatrix} {S_0}^2 & S_0(Id + S_0)Q \\ S_1 D & Id - {S_1}^2 \end{pmatrix}$$

  is an idempotent with coefficients in smooth kernel operators on $M \times M$ and we can choose Q such that P_D has arbitrary small propagation.
- D is a Fredholm operator and

  $$\operatorname{Ind} D = [P] - \left[\begin{pmatrix} 0 & 0 \\ 0 & Id \end{pmatrix}\right] \in K_0(\mathcal{K}(L^2(M)) \cong \mathbb{Z}.$$

- How can we keep track of the propagation and have homotopy invariance?

Quasi-projections

Definition (Quasi-projection)

If X is a proper metric space and $\pi : C_0(X) \to \mathcal{L}(\mathcal{H})$ is a representation of $C_0(X)$ on a Hilbert space $\mathcal{H}$, $0 < \varepsilon < 1/4$ (control) and $r > 0$ (propagation). Then q in $\mathcal{L}(\mathcal{H})$ is an ε-r-projection if

- $q = q^*$;
- $\|q^2 - q\| < \varepsilon$;
- q has propagation less than r.

- If q is an ε-r-projection, then its spectrum has a gap around $1/2$.
- Hence there exists $\kappa : \operatorname{Sp} q \to \{0, 1\}$ continuous and such that $\kappa(t) = 0$ if $t < 1/2$ and $\kappa(t) = 1$ if $t > 1/2$.
- By continuous functional calculus, $\kappa(q)$ is a projection such that $\|\kappa(q) - q\| < 2\varepsilon$;

Quasi-projections and indices

- Let D be a differential elliptic operator on a manifold, let Q be a parametrix. Set $S_0 := Id - QD$ and $S_1 := Id - DQ$ and $P_D = \begin{pmatrix} {S_0}^2 & S_0(Id+S_0)Q \\ S_1 D & Id - {S_1}^2 \end{pmatrix}$ the idempotent with coefficients in smooth kernel operators that gives the index.Then

$$((2P_D^* - 1)(2P_D - 1) + 1)^{1/2} P_D ((2P_D^* - 1)(2P_D - 1) + 1)^{-1/2}$$

  is a projection conjugated to the idempotent P_D ;
- Choosing $Q = Q_{\varepsilon,r}$ with propagation small enought and approximating $((2P_D^* - 1)(2P_D - 1) + 1)^{1/2} P_D ((2P_D^* - 1)(2P_D - 1) + 1)^{-1/2}$ using a power serie, we can for all $0 < \varepsilon < 1/4$ and $r > 0$, construct a ε-r-projection $q_D^{\varepsilon,r}$ such that $\kappa(q_D^{\varepsilon,r})$ is canonically conjugated to P_D and hence

$$\operatorname{Ind} D = [\kappa(q_D^{\varepsilon,r})] - \left[\begin{pmatrix} 0 & 0 \\ 0 & Id \end{pmatrix}\right]$$

  in $K_0(\mathcal{K}(L^2(M))) \cong \mathbb{Z}$ (recall that $\kappa(q_D^{\varepsilon,r})$ is the spectral proj. of $q_D^{\varepsilon,r}$).

Higher Indices

The receptacles of higher indices of elliptic differential operators are K-theory of C^*-algebras which encode the (large scale) geometry of the underlying spaces.

Example

- Group C^*-algebra of a discrete group Γ : higher indices for equivariant elliptic differential operators on cocompact covering space with group Γ;
- Crossed product C^*-algebras : higher indices for longitudinally elliptic differential operators;
- Roe algebras : higher indices for elliptic differential operators on complete noncompact Riemannian manifolds.

These algebras are endowed with a propagation structure arising from the geometric structure.Differential operators are local and these higher indices can be defined using ε-r-projections.

Aim: find obstructions for K-theory elements to be realized as higher indices.

The framework : Filtered algebras

Definition

A filtered C^*-algebra A is a C^*-algebra equipped with a family $(A_r)_{r>0}$ of closed linear subspaces:

- $A_r \subset A_{r'}$ if $r \leqslant r'$;
- A_r is closed under involution;
- $A_r \cdot A_{r'} \subset A_{r+r'}$;
- the subalgebra $\bigcup_{r>0} A_r$ is dense in A.

- If A is unital, we also require that the identity 1 is an element of A_r for every positive number r.
- The elements of A_r are said to have propagation less than r.

Examples

- $\mathcal{K}(L^2(X,\mu))$ for X a metric space and μ probability measure on X. More generally $A \otimes \mathcal{K}(L^2(X,\mu))$ for A is a C^*-algebra.
- Roe algebras:
  - Σ proper discrete metric space, $\mathcal{H}$ separable Hilbert space
  - $C[\Sigma]_r$: space of locally compact operators on $\ell^2(\Sigma) \otimes \mathcal{H}$ (i.e T satisfies fT and Tf compact for all $f \in C_c(\Sigma)$) and with propagation less than r.
  - The Roe algebra of Σ is $C^*(\Sigma) = \overline{\cup_{r>0} C[\Sigma]_r} \subset \mathcal{L}(\ell^2(\Sigma) \otimes \mathcal{H})$ (filtered by $(C[\Sigma]_r)_{r>0}$).
- C^*-algebras of groups and cross-products:
  - If Γ is a discrete finitely generated group equipped with a word metric. Set

$$\mathbb{C}[\Gamma]_r = \{x \in \mathbb{C}[\Gamma] \text{ with support in } B(e,r)\}.$$

    Then $C^*_{red}(\Gamma)$ and $C^*_{max}(\Gamma)$ are filtered by $(\mathbb{C}[\Gamma]_r)_{r>0}$.
  - More generally, if Γ acts on a A by automorphisms, then $A \rtimes_{red} \Gamma$ and $A \rtimes_{max} \Gamma$ are filtered C^*-algebras.

Almost projections and almost unitaries

Let $A = (A_r)_{r>0}$ be a unital filtered C^*-algebra, $r > 0$ (propagation) and $0 < \varepsilon < 1/4$ (control):

- $p \in A_r$ is a ε-r-projection if $p \in A_r$, $p = p^*$ and $\|p^2 - p\| < \varepsilon$.
- a ε-r projection p gives rise by functional calculus to a projection $\kappa(p)$ such that $\|p - \kappa(p)\| < 2\varepsilon$.
- $u \in A_r$ is a ε-r-unitary if $u \in A_r$, $\|u^* \cdot u - 1\| < \varepsilon$ and $\|u \cdot u^* - 1\| < \varepsilon$.
- any ε-r-unitary is invertible.

Remark

- if q is a ε-r-projection of A, there exists h an ε-r-projection of $C([0,1], M_2(A))$ such that $h(0) = I_2$ and $h(1) = \mathrm{diag}(q, 1-q)$;
- if u and v are ε-r-unitaries in A, there exists W a 3ε-$2r$-unitary of $C([0,1], M_2(A))$ such that $W(0) = \mathrm{diag}(u, v)$ and $W(1) = \mathrm{diag}(uv, 1)$.

Notations

- $P^{\varepsilon,r}(A)$ is the set of ε-r-projections of A.
- $U^{\varepsilon,r}(A)$ is the set of ε-r-unitaries of A.
- $P^{\varepsilon,r}_\infty(A) = \bigcup_{n\in\mathbb{N}} P^{\varepsilon,r}(M_n(A))$ for $P^{\varepsilon,r}(M_n(A)) \hookrightarrow P^{\varepsilon,r}(M_{n+1}(A));\ x \mapsto \mathrm{diag}(x,0)$.
- $U^{\varepsilon,r}_\infty(A) = \bigcup_{n\in\mathbb{N}} U^{\varepsilon,r}(M_n(A))$ for $U^{\varepsilon,r}(M_n(A)) \hookrightarrow U^{\varepsilon,r}(M_{n+1}(A));\ x \mapsto \mathrm{diag}(x,1)$.

Quantitative K-groups

Define for a unital C^*-algebra A, $r > 0$ and $0 < \varepsilon < 1/4$ the (stably)-homotopy equivalence relations on $P_\infty^{\varepsilon,r}(A) \times \mathbb{N}$ and $U_\infty^{\varepsilon,r}(A)$ (recall that $P_\infty^{\varepsilon,r}(A) = \bigcup_{n\in\mathbb{N}} P^{\varepsilon,r}(M_n(A))$ and $U_\infty^{\varepsilon,r}(A) = \bigcup_{n\in\mathbb{N}} U^{\varepsilon,r}(M_n(A))$):

- $(p,l) \sim (q,l')$ if there exists $k \in \mathbb{N}$ and $h \in P_\infty^{\varepsilon,r}(C([0,1],A))$ s.t $h(0) = \mathrm{diag}(p, I_{k+l'})$ and $h(1) = \mathrm{diag}(q, I_{k+l})$.
- $u \sim v$ if there exists $h \in U_\infty^{3\varepsilon,2r}(C([0,1],A))$ s.t $h(0) = u$ and $h(1) = v$.

Definition

1. $K_0^{\varepsilon,r}(A) = P^{\varepsilon,r}(A)/\sim$ *and* $[p,l]_{\varepsilon,r}$ *is the class of* (p,l) *mod.* $\sim$;
2. $K_1^{\varepsilon,r}(A) = U^{\varepsilon,r}(A)/\sim$ *and* $[u]_{\varepsilon,r}$ *is the class of* u *mod.* $\sim$.

- $K_0^{\varepsilon,r}(A)$ is an abelian group for $[p,l]_{\varepsilon,r} + [p',l']_{\varepsilon,r} = [\mathrm{diag}(p,p'), l+l']_{\varepsilon,r}$;
- $K_1^{\varepsilon,r}(A)$ is an abelian group for $[u]_{\varepsilon,r} + [v]_{\varepsilon,r} = [\mathrm{diag}(u,v)]_{\varepsilon,r}$.

The non-unital case

Lemma

$K_0^{\varepsilon,r}(\mathbb{C}) \stackrel{\cong}{\to} \mathbb{Z}$; $[p,l]_{\varepsilon,r} \mapsto \text{rank}\, \kappa(p) - l$; $\quad K_1^{\varepsilon,r}(\mathbb{C}) \cong \{0\}$.

Definition

If A is a non unital filtered C^*-algebra and $\tilde{A}$ the unitarization of A,

- $K_0^{\varepsilon,r}(A) = \ker : \; K_0^{\varepsilon,r}(\tilde{A}) \to K_0^{\varepsilon,r}(\mathbb{C}) \cong \mathbb{Z}$;
- $K_1^{\varepsilon,r}(A) = K_1^{\varepsilon,r}(\tilde{A})$;

Definition

If A and B are filtered C^*-algebras with respect to $(A_r)_{r>0}$ and $(B_r)_{r>0}$, a homomorphism $f : A \to B$ is filtered if $f(A_r) \subset B_r$.

- A filtered $f : A \to B$ induces $f_*^{\varepsilon,r} : K_*^{\varepsilon,r}(A) \to K_*^{\varepsilon,r}(B)$;
- $A \hookrightarrow A \otimes \mathcal{K}((\ell^2(\mathbb{N})))$; $a \mapsto a \otimes e_{1,1}$ induces the Morita equivalence

$$K_*^{\varepsilon,r}(A) \stackrel{\cong}{\to} K_*^{\varepsilon,r}(A \otimes \mathcal{K}(\ell^2(\mathbb{N}))).$$

Structure homomorphisms

For any filtered C^*-algebra A, $0 < \varepsilon \leqslant \varepsilon' < 1/4$ and $0 < r \leqslant r'$, we have natural (compatible) structure homomorphisms

- $\iota_0^{\varepsilon,r} : K_0^{\varepsilon,r}(A) \longrightarrow K_0(A);\ [p,l]_{\varepsilon,r} \mapsto [\kappa(p)] - [I_l];$
- $\iota_1^{\varepsilon,r} : K_1^{\varepsilon,r}(A) \longrightarrow K_1(A);\ [u]_{\varepsilon,r} \mapsto [u];$ (ε-r-unitaries are invertible);
- $\iota_*^{\varepsilon,r} = \iota_0^{\varepsilon,r} \oplus \iota_1^{\varepsilon,r};$
- $\iota_0^{\varepsilon,\varepsilon',r,r'} : K_0^{\varepsilon,r}(A) \longrightarrow K_0^{\varepsilon',r'}(A);\ [p,l]_{\varepsilon,r} \mapsto [p,l]_{\varepsilon',r'};$
- $\iota_1^{\varepsilon,\varepsilon',r,r'} : K_1^{\varepsilon,r}(A) \longrightarrow K_1^{\varepsilon',r'}(A);\ [u]_{\varepsilon,r} \mapsto [u]_{\varepsilon',r'}.$
- $\iota_*^{\varepsilon,\varepsilon',r,r'} = \iota_0^{\varepsilon,\varepsilon',r,r'} \oplus \iota_1^{\varepsilon,\varepsilon',r,r'}.$

For any $\varepsilon \in (0,1/4)$ and any projection p (resp. unitary u) in A, there exists $r > 0$ and q (resp. v) an ε-r-projection (resp. an ε-r-unitary) of A such that $\kappa(q)$ and p are closed and hence homotopic projections (resp. u et v are homotopic invertibles)

Consequence

For every $\varepsilon \in (0,1/4)$ and $y \in K_*(A)$, there exists r and x in $K_*^{\varepsilon,r}(A)$ such that $\iota_*^{\varepsilon,r}(x) = y$.

Controlled index map

- Recall that if D is an elliptic differential operator on a compact manifold M, then for every $0 < \varepsilon < 1/4$ and $r > 0$, there exists $q_D^{\varepsilon,r}$ a ε-r-projection in $\mathcal{K}(L^2(M))$ s.t. $\operatorname{Ind} D = [\kappa(q_D^{\varepsilon,r})] - \left[\begin{pmatrix} 0 & 0 \\ 0 & Id \end{pmatrix}\right]$;
- We can define in this way a controlled index $\operatorname{Ind}^{\varepsilon,r} D = [q_D^{\varepsilon,r}, 1]$ in $K_0^{\varepsilon,r}(\mathcal{K}(L^2(M)))$ such that $\operatorname{Ind} D = \iota_0^{\varepsilon,r}(\operatorname{Ind}^{\varepsilon,r} D)$;

More generally, we have:

Lemma

Let X be a cpct metric space, then for any $0 < \varepsilon < 1/4$ and any $r > 0$, there exists a controlled index map $\operatorname{Ind}_{X,*}^{\varepsilon,r} : K_*(X) \to K_*^{\varepsilon,r}(\mathcal{K}(L^2(X)))$ s.t

1. $\iota_*^{\varepsilon,\varepsilon',r,r'} \circ \operatorname{Ind}_{X,*}^{\varepsilon,r} = \operatorname{Ind}_{X,*}^{\varepsilon',r'}$;
2. *the composition*

$$K_0(X) \longrightarrow K_0^{\varepsilon,r}(\mathcal{K}(L^2(X))) \xrightarrow{\iota_0^{\varepsilon,r}} K_0(\mathcal{K}(L^2(X))) \cong \mathbb{Z}$$

is the index map.

Behaviour for small propagation

Theorem

Let X be a finite simplicial complex equipped with a metric. Then there exists $0 < \varepsilon_0 < 1/4$ such that the following holds :

For every $0 < \varepsilon < \varepsilon_0$, there exists $r_0 > 0$ such that for any $0 < r < r_0$ then

$$\mathrm{Ind}_{X,*}^{\varepsilon,r} : K_*(X) \to K_*^{\varepsilon,r}(\mathcal{K}(L^2(X)))$$

is an isomorphism.

Under this identification the usual index map $\mathrm{Ind}_X : K_0(X) \to \mathbb{Z}$ is given by

$$\iota_0^{\varepsilon,r} : K_0^{\varepsilon,r}(\mathcal{K}(L^2(X))) \longrightarrow \mathbb{Z};\ [p,l]_{\varepsilon,r} \mapsto \mathrm{rang}\,\kappa(p) - l.$$

Persistent Approximation Property

Recall that for every $\varepsilon \in (0, 1/4)$ and $y \in K_*(A)$, there exist r and x in $K_*^{\varepsilon,r}(A)$ s.t $\iota_*^{\varepsilon,r}(x) = y$. How faithfull this approximation is?

Lemma

For any ε small enough, any $r > 0$ and any x in $K_*^{\varepsilon,r}(A)$ s.t $\iota_*^{\varepsilon,r}(x) = 0$ then there exists $r' \geqslant r$ such that $\iota_*^{\varepsilon,\lambda\varepsilon,r,r'}(x) = 0$ in $K_*^{\lambda\varepsilon,r'}(A)$ for some universal $\lambda \geqslant 1$.

Does r' depend on x?

Definition (Persistent Approximation Property)

For A a filtered C^*-algebra and positive numbers ε, ε', r and r' such that $0 < \varepsilon \leqslant \varepsilon' < 1/4$ and $0 < r \leqslant r'$, define :

$\mathcal{PA}_*(A, \varepsilon, \varepsilon', r, r')$: for any $x \in K_*^{\varepsilon,r}(A)$, then $\iota_*^{\varepsilon,r}(x) = 0$ in $K_*(A)$ implies that $\iota_*^{\varepsilon,\varepsilon',r,r'}(x) = 0$ in $K_*^{\varepsilon',r'}(A)$.

Persistent Approximation Property

$\mathcal{PA}_*(A, \varepsilon, \varepsilon', r, r')$: for any $x \in K_*^{\varepsilon,r}(A)$, then $\iota_*^{\varepsilon,r}(x) = 0$ in $K_*(A)$ implies that $\iota_*^{\varepsilon,\varepsilon',r,r'}(x) = 0$ in $K_*^{\varepsilon',r'}(A)$

is equivalent to:

the restriction of $\iota_*^{\varepsilon',r'} : K_*^{\varepsilon',r'}(A) \longrightarrow K_*(A)$ to $\iota_*^{\varepsilon,\varepsilon',r,r'}(K_*^{\varepsilon,r}(A))$ is one-to-one.

Example

If $A = \mathcal{K}(\ell^2(\Sigma))$ for Σ discrete metric set.

- $\mathcal{PA}_0(A, \varepsilon, \varepsilon', r, r')$ holds if for any ε-r-projections q and q' in $\mathcal{K}(\ell^2(\Sigma) \otimes \mathcal{H})$ such that rang $\kappa(q) =$ rang $\kappa(q')$, then q and q' are homotopic ε'-r'-projections up to stabilization.
- $\mathcal{PA}_1(A, \varepsilon, \varepsilon', r, r')$ holds if any ε-r-unitary in $\mathcal{K}(\ell^2(\Sigma) \otimes \mathcal{H}) + \mathbb{C}Id)$ is homotopic to Id as a ε'-r'-unitary.

Examples

Definition (Universal example for proper actions)

A locally compact space Z is a universal example for proper actions of Γ if for any locally compact space X provided with a proper action of Γ, there exists $f : X \to Z$ continuous and equivariant, and any two such maps are equivariantly homotopic.

Every group admits a universal example for proper actions.

Theorem

Let Γ be a finitely generated discrete group. Assume that

- *Γ satisfies the Baum-Connes conjecture with coefficients;*
- *Γ has a cocompact universal example for proper action;*

Then for a universal $\lambda > 1$, any $\varepsilon \in (0, \frac{1}{4\lambda})$ and any $r > 0$, there exists $r' > r$ such that $\mathcal{PA}_*(A \rtimes_{red} \Gamma, \varepsilon, \lambda\varepsilon, r, r')$ holds for any Γ-C^*-algebra A.

Examples: Γ hyperbolic, Γ Haagerup with cocompact universal example.

The geometric case

Observation : we can identify $C_0(\Gamma) \rtimes \Gamma$ as a filtered C^*-algebra to $\mathcal{K}(\ell^2(\Gamma))$ and (recall that $\kappa(q)$ is the spectral projection affiliated to q)
$\iota_0^{\varepsilon,r} : K_0^{\varepsilon,r}(\mathcal{K}(\ell^2(\Gamma)) \to K_0(\mathcal{K}(\ell^2(\Gamma)) \cong \mathbb{Z} : [q,l]_{\varepsilon,r} \mapsto \text{rang}\, \kappa(q) - l$.

Corollary

Let Γ be a finitely generated discrete group. Assume that

- *Γ satisfies the Baum-Connes conjecture with coefficients;*
- *Γ has a cocompact universal example for proper action.*

Then for a universal $\lambda > 1$, any $\varepsilon \in (0, \frac{1}{4\lambda})$ and any $r > 0$, there exists $r' > r$ such that $\mathcal{PA}_*(A \otimes \mathcal{K}(\ell^2(\Gamma)), \varepsilon, \lambda\varepsilon, r, r')$ holds for any C^*-algebra A.

- The Gromov group does not satisfy the conclusion of the corollary.
- This statement is purely geometric.

Coarse geometry

Let (Σ, d) be a proper discrete metric space;

- Σ has bounded geometry if for all $r > 0$, there exists an integer N such that any ball of radius r has cardinal less than N (example : $|\Gamma|$, the underlying metric space of a finitely generated group Γ equipped with any word metric) ;
- Let (Σ', d') be another proper discrete metric space. A map $f : \Sigma \to \Sigma'$ is coarse if
  - f is proprer ;
  - $\forall r > 0$, $\exists s > 0$ such that $d(x, y) < r \Rightarrow d'(f(x), f(y)) < s$;
- A coarse map $f : \Sigma \to \Sigma'$ is a coarse equivalence if there is a coarse map $g : \Sigma' \to \Sigma$ and $M > 0$ such that $d(f \circ g(y), y) < M$ and $d(g \circ f(x), x) < M \quad \forall x \in X$ and $\forall y \in Y$.

The geometrical Persistent Approximation Property

Definition

Let (Σ, d) *a proper discrete metric space. We say that* Σ *satisfies the geometrical Persistent Approximation Property if there exists* $\lambda > 1$ *such that for any* $0 < \varepsilon \leqslant \frac{1}{4\lambda}$ *and any* $r > 0$*, there exists* $r' > r$ *and* $\varepsilon' \in [\varepsilon, 1/4)$ *such that* $\mathcal{PA}_*(A \otimes \mathcal{K}(\ell^2(\Sigma)), \varepsilon, \varepsilon', r, r')$ *holds for any* C^**-algebra* A*.*

Remark

The geometrical Persistent Approximation Property is invariant under coarse equivalence.

Example

If Γ (finitely generated) satisfies the Baum-Connes conjecture with coefficients and admits a cocompact universal example for proper action, then $|\Gamma|$ satisfies the geometrical Persistent Approximation Property.

Uniform coarse contractibility property

Let (Σ, d) be a discrete metric space with bounded geometry. Recall that the Rips complex of degree r is the set $P_r(\Sigma)$ of probability measures on Σ with support of diameter less than r (notice that $P_r(\Sigma) \subset P_{r'}(\Sigma)$ if $r \leqslant r'$).

Definition

Σ has the uniform coarse contractibility property if for any $r > 0$, there exists $r' > r$ such that every compact subset in $P_r(\Sigma)$ lies in a contractible compact subset of $P_{r'}(\Sigma)$.

Remark

This is the topological counterpart of the Persistent Approximation Property!

Example : Σ Gromov hyperbolic.

Coarse embedding in a Hilbert space

Definition

Σ coarsely embeds in a Hilbert space $\mathcal{H}$ if there exists $f : \Sigma \to \mathcal{H}$ s.t : for all $R > 0$, there exists $S > 0$ s.t $d(x, y) < R \Rightarrow \|f(x) - f(y)\| < S$ and $\|f(x) - f(y)\| < R \Rightarrow d(x, y) < S$.

Examples : Σ Gromov hyperbolic, Γ amenable group, exact, linear...

Theorem

Let Σ be a discrete metric space with bounded geometry. Assume that

- *Σ coarsely embeds in a Hilbert space;*
- *Σ satisfies the uniform coarse contractibility property.*

Then Σ satisfies the geometrical Persistent Approximation Property.

Persistence approximation property and homotopy groups

- For A a unital C^*-algebras, we set $U_\infty(A) = \cup_{n\in\mathbb{N}} U_n(A)$ for $U_n(A) \hookrightarrow U_{n+1}(A)$; $x \mapsto \mathrm{diag}(x,1)$ and $GL_\infty(A) = \cup_{n\in\mathbb{N}} GL_n(A)$ for $GL_n(A) \hookrightarrow GL_{n+1}(A)$; $x \mapsto \mathrm{diag}(x,1)$.
- Recall that $U_n(\mathbb{C})$ and $GL_n(\mathbb{C})$ (and therefore $U_\infty(\mathbb{C})$ and $GL_\infty(\mathbb{C})$) are homotopy equivalent. Hence $\pi_k(U_\infty(\mathbb{C})) = \pi_k(GL_\infty(\mathbb{C}))$ for all integer k;
- Bott Periodicity : $\pi_{2k}(U_\infty(\mathbb{C})) = \{0\}$ and $\pi_{2k+1}(U_\infty(\mathbb{C})) \cong \mathbb{Z}$.

For any finite set X, any $0 < \varepsilon \leqslant \varepsilon' < 1/4$ and $0 < r \leqslant r'$, the inclusions

$$U_\infty^{\varepsilon,r}(\mathcal{K}(\ell^2(X)) \subseteq U_\infty^{\varepsilon',r'}(\mathcal{K}(\ell^2(X)) \subseteq GL_\infty(\mathcal{K}(\ell^2(X)) \cong GL_\infty(\mathbb{C})$$

gives rise for any integer k to

- $\jmath_k^{\varepsilon,\varepsilon',r,r'} : \pi_k(U_\infty^{\varepsilon,r}(\mathcal{K}(\ell^2(X)))) \to \pi_k(U_\infty^{\varepsilon',r'}(\mathcal{K}(\ell^2(X))))$
- $\jmath_k^{\varepsilon,r} : \pi_k(U_\infty^{\varepsilon,r}(\mathcal{K}(\ell^2(X)))) \to \pi_k(GL_\infty(\mathbb{C}))$

such that $\jmath_k^{\varepsilon',r'} \circ \jmath_k^{\varepsilon,\varepsilon',r,r'} = \jmath_k^{\varepsilon,r}$.

Persistence approximation property and π_k

Definition

For $\mathcal{F}$ a family of finite metric spaces and k integer consider: $\mathcal{PA}_k(\mathcal{F},\varepsilon,\varepsilon',r,r')$: for any X in $\mathcal{F}$ and any $x \in \pi_k(U_\infty^{\varepsilon,r}(\mathcal{K}(\ell^2(X))))$, then $\jmath_k^{\varepsilon,r}(x) = 0$ in $\pi_k(GL_\infty(\mathbb{C}))$ implies that $\jmath_k^{\varepsilon,\varepsilon',r,r'}(x) = 0$ in $\pi_k(U_\infty^{\varepsilon,r}(\mathcal{K}(\ell^2(X))))$.

Theorem

Let $\mathcal{F}$ be a family of finite metric spaces. Then for some $\varepsilon_0 > 0$ (independent on $\mathcal{F}$) the following assertions are equivalent:

1. *for any integer k, any $\varepsilon > \varepsilon_0$ and any $r > 0$, there exists $\varepsilon' > \varepsilon$ and $r' > r$ such that $\mathcal{PA}_k(\mathcal{F},\varepsilon,\varepsilon',r,r')$ holds;*
2. *for $k = 0, 1$, any $\varepsilon > \varepsilon_0$ and any $r > 0$, there exists $\varepsilon' > \varepsilon$ and $r' > r$ such that $\mathcal{PA}_k(\mathcal{F},\varepsilon,\varepsilon',r,r')$ holds;*
3. *for any $\varepsilon > \varepsilon_0$ and any $r > 0$, there exists $\varepsilon' > \varepsilon$ and $r' > r$ such that $\mathcal{PA}_*(\mathcal{K}(\ell^2(X)),\varepsilon,\varepsilon',r,r')$ holds for any X in $\mathcal{F}$;*

Applications

1. If $\mathcal{F}$ is a family of finite δ-hyperbolic spaces for some fixed δ, then for any integer k, any $\varepsilon > \varepsilon_0$ and any $r > 0$, there exists $\varepsilon' > \varepsilon$ and $r' > r$ such that $\mathcal{PA}_k(\mathcal{F}, \varepsilon, \varepsilon', r, r')$ holds;
2. If $\mathcal{F}$ is the family of a finite subsets of a finitely generated group Γ and with some conditions on RIps complexes, then Baum-Connes conjecture with coefficients for Γ implies that for any integer k, any $\varepsilon > \varepsilon_0$ and any $r > 0$, there exists $\varepsilon' > \varepsilon$ and $r' > r$ such that $\mathcal{PA}_k(\mathcal{F}, \varepsilon, \varepsilon', r, r')$ holds.