K-homological finiteness for hyperbolic groups

Bogdan Nica

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Overview

Γ : hyperbolic group $\rightsquigarrow$ $C^*_r\Gamma$: C^*-algebra of the group Γ
$\partial\Gamma$: boundary of Γ $\rightsquigarrow$ $C(\partial\Gamma) \rtimes \Gamma$: C^*-algebra of the action $\Gamma \curvearrowright \partial\Gamma$

Strong finiteness property in K-homology for $C^*_r\Gamma$ and $C(\partial\Gamma) \rtimes \Gamma$

Hyperbolic groups ("rank-1") : most tractable with respect to

Geometric Group Theory $\rightsquigarrow$ Noncommutative Geometry
groups / group actions $\rightsquigarrow$ corresponding C^*-algebras

Lattices in higher rank : least tractable

C*-algebra of a group / group-action

- G : discrete countable group

$C^*_r G$: norm-closure of $\mathbb{C}G$ under λ

$\mathbb{C}G$: group algebra $\{\sum a_g g : \; a_g \in \mathbb{C}\}$, $\qquad (a g)(a' g') = aa' gg'$
λ : left regular representation of $\mathbb{C}G$ on $\ell^2 G$

- G : discrete countable group $\curvearrowright$ Ω : compact Hausdorff space

$C(\Omega) \rtimes_r G$: norm-closure of $C(\Omega) \rtimes_{alg} G$ under **any** λ_μ

$C(\Omega) \rtimes_{alg} G$: algebra $\{\sum \phi_g g : \; \phi_g \in C(\Omega)\}$, $\qquad (\phi g)(\phi' g') = \phi(g.\phi') gg'$
λ_μ : left regular representation of $C(\Omega) \rtimes_{alg} G$ on $\ell^2(G, L^2(\Omega, \mu))$
where μ Borel probability measure on Ω with full support

Hyperbolic groups

(Gromov) hyperbolicity is a coarse notion of negative curvature
space X : geodesic triangles are uniformly thin
group Γ : Γ acts geometrically on a hyperbolic space X

several **geometric models** X for a given Γ, possibly a preferred one

examples of hyperbolic groups

- × virtually cyclic groups (elementary) $\qquad$ (X : a point or a line)
- free groups $\qquad$ (X : a tree)
- uniform lattices in $SO(n,1)$, $SU(n,1)$, $Sp(n,1)$ $\qquad$ (X : $\mathbb{H}^n_{\mathbb{R}}$, $\mathbb{H}^n_{\mathbb{C}}$, $\mathbb{H}^n_{\mathbb{H}}$)
- $C'(1/6)$ small cancellation groups $\qquad$ (X : a Cayley graph)

A short history of $C^*_r\Gamma$

$\mathbb{F}$: free group

- *Conjecture*: $C^*_r\mathbb{F}$ has no nontrivial idempotents (Kadison ~1965)
- $C^*_r\mathbb{F}$ simple, unique trace (Powers 1975)
- $\mathbb{F}$ Rapid Decay (& a-T-menable) (Haagerup 1978)
- K-theory of $C^*_r\mathbb{F}$ ($\Rightarrow$ Kadison Conjecture) (Pimsner - Voiculescu 1980)

Foundations, early 1980's

- Connes: Noncommutative Geometry, much attention to $C^*_r G$
  e.g. *Baum - Connes Conjecture* on the K-theory of $C^*_r G$
- Gromov: hyperbolic groups, early Geometric Group Theory

Γ : hyperbolic group

- $C^*_r\Gamma$ simple, unique trace for Γ torsion-free (de la Harpe 1983)
- Γ Rapid Decay (Jolissaint, de la Harpe 1989)
- Baum - Connes Conjecture (Lafforgue, Mineyev - Yu 2002)
- Kadison Conjecture (Puschnigg 2002)

(T), a-T, and beyond

G is **a-T-menable** : some isometric action of G on a Hilbert space is proper (Gromov / Haagerup)

G has **property (T)** : every isometric action of G on a Hilbert space has bounded orbits (Serre / Kazhdan)

- among hyperbolic groups, both a-T and (T)
  - free groups (Haagerup 1978)
  - uniform lattices in $SO(n,1)$, $SU(n,1)$, $Sp(n,1)$ (Vershik +, Kostant ~1970)
  - $C'(1/6)$ small cancellation groups (Wise 2004)
- higher rank lattices have (T) (Kazhdan 1967, ...)

(T), a-T, and beyond

Theorem (Yu 2005, N. 2013)

Let Γ be hyperbolic. Then Γ admits a proper isometric action on an L^p-space for p large enough.

– Yu : $\ell^p(\Gamma \times \Gamma)$

– N. : $L^p(\partial\Gamma \times \partial\Gamma, \nu_{\mathrm{BM}})$
key analytic fact: $\partial\Gamma$ has "polynomial growth" (*Ahlfors regularity*)
p : metric dimension of $\partial\Gamma$

Theorem (Bader - Gelander - Furman - Monod 2007)

Let G be a lattice in higher rank. Then every isometric action of G on an L^p-space has bounded orbits.

Next: a rank-1/higher rank finiteness contrast for group C*-algebras

K-homological finiteness

Theorem (Emerson - N. 2013)

Let Γ be one of the following:

- a free group,
- a torsion-free uniform lattice in $\mathrm{SO}(n,1)$ or $\mathrm{SU}(n,1)$,
- a torsion-free $C'(1/6)$ group with #gen $= 1 +$ #rel.

Then every K-homology class for $\mathrm{C}^*_\mathrm{r}\Gamma$ is p-summable over $\mathbb{C}\Gamma$ for $p \gg 1$.

(vaguely akin to L^p-finiteness for actions)

Theorem (Puschnigg 2011)

Let G be a lattice in higher rank. Then no (non-zero) K-homology class for $\mathrm{C}^*_\mathrm{r} G$ is finitely summable over $\mathbb{C}G$.

(formally related to lack of L^p-finiteness for actions)

K-homology

- a cohomology theory for C*-algebras, formally dual to K-theory
- more analytically flavoured, notion of finiteness / summability

A : unital C*-algebra

Definition (Atiyah, Kasparov, Connes)

An odd / even **Fredholm module** (π, T) for A consists of
· $\pi : A \to \mathscr{B}(H)$ representation of A on a Hilbert space H
· $T \in \mathscr{B}(H)$ projection / unitary mod $\mathscr{K}(H)$
such that $[T, \pi(a)] \in \mathscr{K}(H)$ for all a in A

(π, T) is a p-**summable Fredholm module** if
· $T \in \mathscr{B}(H)$ projection / unitary mod $\mathscr{L}^p(H)$
such that $[T, \pi(a)] \in \mathscr{L}^p(H)$ for all a in a dense subalgebra of A

$K^*(A) = \{ \text{Fredholm modules} \} / \sim$
$\sim$: unitary equivalence, operator homotopy, degenerates

Uniform summability in K-homology

Classical Fact. Let M be a compact smooth manifold. Then the K-homology of $C(M)$ is **uniformly summable**: every class has a representative which is p-summable over $C^\infty(M)$ for $p > \dim M$

Theorem (Emerson - N. 2013)

Let Γ be torsion-free hyperbolic. Then
- $\gamma K^1(C^*_r\Gamma)$ is uniformly summable over $\mathbb{C}\Gamma$;
- $\gamma K^0(C^*_r\Gamma)$ is uniformly summable over $\mathbb{C}\Gamma$, for $\chi(\Gamma) = 0$ or 'good' γ.

· in general $\gamma \neq 1$ (Skandalis), but $\gamma = 1$ for Γ a-T-menable (Higson - Kasparov), e.g. free groups, uniform lattices in $SO(n,1)$ or $SU(n,1)$, $C'(1/6)$ group

· concrete γ models for free groups, lattices in $SO(n,1)$ or $SU(n,1)$

· *structural* theorem vs. previous sporadic examples

The role of the boundary

· summability for K-homology of $C^*_r\Gamma$ comes from the boundary!

· first, we deal with the C*-algebra of the boundary action:

Theorem (Emerson - N. 2013)

Let Γ be torsion-free hyperbolic. Then the K-homology of $C(\partial\Gamma) \rtimes \Gamma$ is uniformly summable.

· then we apply the following 'Gysin sequence' in K-homology

Theorem (Emerson - N. 2013)

Let Γ be torsion-free hyperbolic. Then there is an exact sequence:

$$0 \longrightarrow \mathrm{K}_1(B\Gamma) \longrightarrow \mathrm{K}^0(C(\partial\Gamma) \rtimes \Gamma) \xrightarrow{i^*} \gamma\mathrm{K}^0(C^*_r\Gamma) \longrightarrow \mathrm{K}_0(B\Gamma)$$

$$\longrightarrow \mathrm{K}^1(C(\partial\Gamma) \rtimes \Gamma) \xrightarrow{i^*} \gamma\mathrm{K}^1(C^*_r\Gamma) \longrightarrow 0$$

$C(\partial\Gamma) \rtimes \Gamma$

Let Γ be torsion-free hyperbolic. Then the K-homology of $C(\partial\Gamma) \rtimes \Gamma$ is uniformly summable.

- $C(\partial\Gamma) \rtimes \Gamma$ nuclear, purely infinite simple – since $\Gamma \curvearrowright \partial\Gamma$ amenable, minimal & convergence (Adams 1994, Anantharaman-Delaroche 1997, Laca - Spielberg 1996)
- for Γ torsion-free, $C(\partial\Gamma) \rtimes \Gamma$ satisfies Poincaré duality

$$K_*\big(C(\partial\Gamma) \rtimes \Gamma\big) \simeq K^{*+1}\big(C(\partial\Gamma) \rtimes \Gamma\big)$$

via cup-cap products with 'fundamental classes' (Emerson 2003)

similarity with Cuntz - Krieger C*-algebras O_A

Visual structure

- $\partial\Gamma$ is a topological object, but Fredholm modules require *analysis*
- geometric models for Γ have homeomorphic boundaries at infinity $\rightsquigarrow$ $\partial\Gamma$ canonical compact Hausdorff space, e.g.
  - Γ free group — $\partial\Gamma$: a Cantor set
  - Γ uniform lattice in $SO(n,1)$ — $\partial\Gamma : S^{n-1}$
- **any** concrete realization of $\partial\Gamma$ as ∂X, X a geometric model for Γ, carries metric - measure structure coming from within X:
  – a scale of Hölder equivalent *visual metrics* on ∂X
  – their associated Hausdorff measures, equivalent *visual measures*
- key fact: visual metric/measure structure has 'polynomial growth'

$$\mu(r\text{-ball}) \asymp r^{\mathrm{hdim}\,(\partial X, d)}$$

for μ visual measure, d visual metric on ∂X (Coornaert 1993)

Visual dimension

our summability results use a notion of **metric dimension** for ∂X:

$$\mathrm{visdim}\,\partial X = \inf\{\mathrm{hdim}\,(\partial X, d) : d \text{ visual metric on } \partial X\}$$

when attained, finer summability for Fredholm modules

'Cantor' Example
X tree, ∂X topologically a Cantor set
$\mathrm{visdim}\,\partial X = 0$, not attained

'Carnot' Example
$X = \mathbb{H}^n_K$, ∂X topologically S^{kn-1} $(k = 1, 2, 4)$
$\mathrm{visdim}\,\partial X = kn + k - 2$, attained by Carnot metric

Is $\inf\{\mathrm{visdim}\,\partial X : X \text{ geometric model for } \Gamma\}$ an invariant for Γ ?

Basic Fredholm module

Γ : hyperbolic group, X : geometric model for Γ
μ : visual probability measure on ∂X
λ_μ : regular representation of $C(\partial\Gamma) \rtimes \Gamma$ on $\ell^2(\Gamma, L^2(\partial X, \mu))$
$P_{\ell^2\Gamma}$: projection of $\ell^2(\Gamma, L^2(\partial X, \mu))$ onto $\ell^2\Gamma$

$(\lambda_\mu, P_{\ell^2\Gamma})$

- an odd Fredholm module for $C(\partial\Gamma) \rtimes \Gamma$,
- p-summable for every $p > \max\{2, \operatorname{visdim} \partial X\}$,
- represents the Poincaré dual of $[1] \in \mathrm{K}_0(C(\partial\Gamma) \rtimes \Gamma)$.

· Fredholmness due to $\Gamma \curvearrowright \partial X$ convergence (topological dynamics)

· summability relies on Ahlfors regularity

· the K-homology class of $(\lambda_\mu, P_{\ell^2\Gamma})$ independent of μ and X

· K-homology class of $(\lambda_\mu, P_{\ell^2\Gamma})$ vanishes iff $\chi(\Gamma) = \pm 1$; it has infinite order iff $\chi(\Gamma) = 0$ (Γ torsion-free)

Representing K-homology

by Poincaré duality, twisting the basic Fredholm module by K-theory of $C(\partial\Gamma) \rtimes \Gamma$ exhausts K-homology of $C(\partial\Gamma) \rtimes \Gamma$:

Let Γ torsion-free hyperbolic. Then:
• every class in $\mathrm{K}^1(C(\partial\Gamma) \rtimes \Gamma)$ has a representative of the form

$$\left(\lambda_\mu, P_{\ell^2\Gamma}\rho_\mu(e)P_{\ell^2\Gamma}\right), \qquad e \text{ projection in } C(\partial\Gamma) \rtimes \Gamma,$$

and $e \in C(\partial\Gamma) \rtimes \Gamma$ can be chosen so that the Fredholm module is p-summable for every $p > \max\{2, \operatorname{visdim} \partial X\}$.

• every class in $\mathrm{K}^0(C(\partial\Gamma) \rtimes \Gamma)$ has a representative of the form

$$\left(\lambda_\mu, P_{\ell^2\Gamma}\rho_\mu(u)P_{\ell^2\Gamma} + (1 - P_{\ell^2\Gamma})\right), \qquad u \text{ unitary in } C(\partial\Gamma) \rtimes \Gamma,$$

and $u \in C(\partial\Gamma) \rtimes \Gamma$ can be chosen so that the Fredholm module is p-summable for every $p > \max\{2, \operatorname{visdim} \partial X\}$.

ρ_μ : *right* regular representation on $\ell^2(\Gamma, L^2(\partial X, \mu))$

Summary

Let Γ be torsion-free hyperbolic. Then every K-homology class for $C(\partial\Gamma) \rtimes \Gamma$ is p-summable for each $p > \max\{2, \operatorname{visdim} \partial X\}$, where X is any geometric model for Γ.

The same is true for $C^*_r\Gamma$ under further assumptions.

Examples

• Let Γ be a torsion-free uniform lattice in $\mathrm{Isom}(\mathbb{H}^n_K)$. Then every K-homology class for $C(S^{kn-1}) \rtimes \Gamma$ is $(kn+k-2)^+$-summable over $\mathrm{Lip}(S^{kn-1}) \rtimes_{\mathrm{alg}} \Gamma$, where S^{kn-1} carries the Carnot metric

• Let Γ be a torsion-free uniform lattice in $\mathrm{SO}(n,1)$ / $\mathrm{SU}(n,1)$. Then every K-homology class for $C^*_r\Gamma$ is n^+-summable / $(2n)^+$-summable over $\mathbb{C}\Gamma$.

caveat: slightly weaker for $\mathbb{H}^2_{\mathbb{R}}$

Questions

Let Γ be a hyperbolic group.

- Is the K-homology of $C^*_r\Gamma$ uniformly summable over $\mathbb{C}\Gamma$?
- Does γ act as the identity on the K-homology of $C^*_r\Gamma$?