THE LOCAL REGULARITY OF SOAP FILMS
AFTER JEAN TAYLOR

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The following text is a minor modification of the transparencies that were used in the conference; please excuse the often telegraphic style.

The main goal of the series of lectures is a presentation (with some proofs) of Jean Taylor's celebrated theorem on the regularity of almost minimal sets of dimension 2 in $\mathbb{R}^3$, and a few more recent extensions or perspectives. Some of the results presented below are work of, or with T. De Pauw, V. Feuvrier A. Lemenant, and T. Toro.

The main references for these lectures are [D4] and [D5] (for the proofs), [D3] (for some of the questions), and the theses [Feu] and [Le].

CONTENTS

1. AN INTRODUCTION WITH THE PLATEAU PROBLEM

Plateau was indeed interested in soap films. The simplest version of Plateau's problem is: describe the soap films $E \subset \mathbb{R}^3$ bounded by a smooth curve Γ (existence and regularity).

Apparently, a (real life) soap film is composed of two layers of some molecules with a water-attracting head and a water-repelling tail, which align themselves head to head; the width of the film is roughly equal to the length of two molecules.

The general idea for a model is to minimize the area of a surface E spanned by Γ, but different descriptions of "soap films" and the boundary condition exist. We shall mention four.

A soap bubble would only be slightly different: due to different pessures on both sides, it does not exactly minimize area, and (in the smooth case), it has constant (instead of vanishing) mean curvature. And it is a typical example of what will be called an almost minimal set here.

1.a. Currents and the solution of Plateau's problem

We start with the most celebrated and successful model, provided by Currents. Work by Federer, Fleming, De Giorgi, and others.

A d-dimensional **current** is a continuous linear form on the space of smooth d-forms.

Main example: if S is smooth, oriented surface of dimension d, the current S' of integration on S is defined by $\langle S', \omega \rangle = \int_S \omega$. But we want a much larger class, which contains limits of objects like S', so that we get good compactness properties. As usual, the price to pay is that the general current is not really smooth, and we often need to prove the regularity of the current solutions of our problems.

Another useful example is the rectifiable current T defined on a d-dimensional rectifiable set E such that $H^d(E) < +\infty$, with a measurable orientation τ, and an integer-valued multiplicity m:

$$(1) \qquad \langle T, \omega \rangle = \int_E m(x)\, \omega(x) \cdot \tau(x)\, dH^d(x)$$

Recall that a rectifiable set of dimension d is a set E such that $E \subset N \cup \bigcup_{j \in \mathbb{N}} G_j$, where $H^d(N) = 0$ and each G_j is a C^1 emmbedded submanifold of dimension d. [Or the Lipschitz image of a subset of $\mathbb{R}^d$; this would yield an equivalent definition.]

See the definition of Hausdorff measure later.

The boundary of a d-dimensional current T is defined by

$$(2) \qquad \langle \partial T, \omega \rangle = \langle T, d\omega \rangle \ \text{ for every } (d-1)\text{-form } \omega,$$

were d denotes the exterior derivative. When S is a smooth oriented surface with boundary Γ, Green says that $\partial S' = \Gamma'$.

The classical way to state the Plateau problem in the current setting is to take a $(d-1)$-dimensional current Γ, with $\partial \Gamma = 0$, and minimize the mass of T, among d-dimensional currents T such that $\partial T = \Gamma$.

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The mass $\text{Mass}(T)$ is the operator norm of T, where we put a L^∞-norm on forms. When T is a rectifiable current given by (1),

$$(3) \qquad \text{Mass}(T) = \int_E |m(x)| \, dH^d(x)$$

A <u>normal current</u> is a rectifiable current T such that ∂T is rectifiable too. [This additional constraint won't disturb here because we know ∂T.]

The main interest of the setting is the following compactness theorem: if $\{T_k\}$ is a sequence of normal currents of dimension d, with supports in a fixed compact set in $\mathbb{R}^n$, and such that $\text{Mass}(T_k) + \text{Mass}(\partial T_k) \leq M$ for some fixed $M < +\infty$, then there is a subsequence that converges (in some weak norm) to some normal current T. Moreover, $\text{Mass}(T) \leq \liminf_{k \to +\infty} \text{Mass}(T_k)$ and $\text{Mass}(\partial T) \leq \liminf_{k \to +\infty} \text{Mass}(\partial T_k)$.

Important consequence: the existence of normal currents T that solve $\partial T = \Gamma$ and minimize $\text{Mass}(T)$.

Here we get a set E, the "support" of T, which is merey rectifiable a priori but then there are very strong regularity results for minimizers, and in particular E is a smooth submanifold when $d < 7$.

A great success for weak solutions and Geometric Measure Theory.

1.b. Size minimizers

Unfortunately, this is not a great model for soap films. See pictures from Ken Brakke's web page: http://www.susqu.edu/brakke/

So we want to describe minimal sets differently. If we want to keep the currents setting, we should <u>minimize the size</u> of solutions of $\partial T = S$, where

$$(4) \qquad \text{Size}(T) = H^d\big(\{x \in E \, ; \, m(x) \neq 0\}\big)$$

when T is given by (1).

Here H^d denotes the d-dimensional Hausdorff measure (think about surface measure, but $H^d(E)$ is also defined when E is not smooth). It is defined by

$$(5) \qquad H^d(E) = \lim_{\delta \to 0} H^d_\delta(E)$$

where

$$(6) \qquad H^d_\delta(E) = c_d \inf \left\{ \sum_{j \in \mathbb{N}} \text{diam}(D_j)^d \right\},$$

and the infimum is taken over all coverings of E by a countable collection $\{D_j\}$ of sets, with $\text{diam}(D_j) \leq \delta$ for all j. We choose the normalizing constant c_d so that H^d coincides with the Lebesgue measure on the subsets of $\mathbb{R}^d$.

When we try to minimize size, it is interesting to glue together two pieces of E when we can, because we don't pay for larger multiplicities. For instance, when $d = 1$, the union of two parallel segments that lie very close to each other minimizes the mass, but a

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size minimizer with the same boundary (two points marked $-$ on one side and two points marked $+$ on the other side) is composed of a long central segment with multiplicity 2, connected to the four ends by two pairs of short line segments with multiplicity 1 that make 120° angles with the central segment.

Similarly, when $d = 2$ and the boundary Γ is composed of two parallel circles that lie close to each other and have the same orientation, the mass-minimizing current corresponds to the union of the two parallel disks bounded by the circles, while the size-minimizer is easly guessed to be composed of a central disk, connected to the two circles by two small sections of catenoids. Again the three pieces make 120° angles along the circle where they meet.

Bad news: in this setting, no general existence theorem is known, even when $d = 2$, $n = 3$, and Γ comes from a smooth curve!

The difficulty is that we have no bounds on the masses, so the compactness theorem above does not apply.

Also, some soap films are not orientable (Möbius bands), and the problem $\partial T = \Gamma$ does not always fit. Various ad hoc solutions to this exist, but no general scheme.

Note that we could also want to minimize various intermediate "norms" between mass and size, for instance where we integrate some power $\alpha \in [0, 1]$ of the multiplicity $m(x)$ ($\alpha = 1$ corresponds to mass, and $\alpha = 0$ to size). This arises for instance with optimal irrigation networks. There are existence theorems in this context, proved in particular by T. De Pauw and R. Hardt.

1.c. Directly with sets

Return to $d = 2$, $n = 3$, and E is a surface spanned by the curve Γ. We want to minimize $H^2(E)$, but the difficulty is the definition of "spanned".

For Reifenberg (1960), E is a compact set that contains Γ, and the boundary condition is stated in terms of Čech homology on some commutative group G. We require the inclusion $i : \Gamma \to E$ to induce a trivial homomorphism from $H_1(\Gamma, G)$ to $H_1(E, G)$.

Then we minimize the area $H^2(E)$ under these constraints.

Reifenberg proves the existence of minimizers when $G = \mathbb{Z}_2$ or $G = \mathbb{R}/\mathbb{Z}$. Beautiful proof by hands, minimizing sequence, and haircuts. He also obtains higher-dimensional results, I think.

Very recently, De Pauw obtained the 2-dimensional case when $G = \mathbb{Z}$ (with currents).

The equivalence with the size minimizing problem is not clear, but the infimum is the same [De Pauw].

Let me propose a third definition, where we minimize $H^2(E)$ among all compact sets E obtained by deformation of an initial candidate E_0 with a <u>sliding boundary condition</u>. [Think about a rubber shower curtain.]

A deformation of E_0 with sliding boundary condition is a set $E = \varphi_1(E)$, where $\varphi_t : E_0 \to \mathbb{R}^3$, $0 \leq t \leq 1$ is a one-parameter family of functions such that:

(7) $\qquad\qquad (x, t) \to \varphi_t(x)$ is continuous: $E_0 \times [0, 1] \to \mathbb{R}^3$,

(8) $\qquad\qquad\qquad \varphi_0(x) = x$ for $x \in E_0$,

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$$(9) \qquad\qquad \varphi_t(x) \in \Gamma \text{ when } x \in \Gamma,$$

$$(10) \qquad\qquad \varphi_1 \text{ is Lipschitz.}$$

[We require (10) for safety and to follow the tradition of Almgren, but no bound on the Lipschitz constant is attached, and we may even be allowed to drop (10) altogether.]

Here are possible advantages of this notion. First, we do not need to orient E, or choose a group. The definition could also be more flexible. That is, different choices of E_0 could lead to different solutions, so the choice of E_0 could be more precise than the selection of an algebraic class for instance.

But no existence result is know yet. Also, we still do not account for unrealistic deformations that would extend the film too far: some real films could be deformed into a point, but with a long homotopy.

It would also be nice to know which condition fits best the limit, when the width of the tubes tends to 0, of a Plateau problem with a tubular boundary.

Incidentally, all these definitions make sense in a much wider context (but then even less is known).

We now leave the Plateau problem with the conclusion that it is *not* solved.

2. ALMOST MINIMAL SETS

We shall focus more on the **regularity properties** of potential solutions away from the boundary. This includes J. Taylor's theorem.

Regularity near the boundary will possibly be a much more complicated matter, which (to my knowledge) was essentially not studied. I'll try to do this in the future, but with no guarantee of success.

Note that studying the regularity of solutions is both interesting in itself, and because this may be an important ingredient in proofs of existence. We'll try to give a hint of this in Section 8.

We shall start with a definition of local minimal and almost-minimal sets, and then give general regularity properties that hold in all dimensions. Taylor's theorem will be disccussed later.

2.a. Definitions

I think that the following definition gives a good local description of soap films and bubbles, and is also satisfied by the closed support of size-minimizing currents.

We work inside an open set $U \subset \mathbb{R}^n$, to make the definition local and avoid boundary problems; regularity near the boundary would be more complicated.

We use a small gauge function $h : \mathbb{R}_+ \to \mathbb{R}_+$ to account for perturbations (small additional forces, for instance, or gently unhomogeneous space). Feel free to take $h = 0$ (for minimal sets).

We consider closed sets $E \subset U$ with locally finite H^d-measure and (to simplify slightly) define competitors F for E only in compact balls $B \subset U$.

A <u>competitor</u> for E in B is a set $F = \varphi(E)$, where $\varphi : U \to U$ is Lipschitz (but no bounds required), with

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(1) $$\varphi(y) = y \text{ for } y \in U \setminus B, \text{ and } \varphi(B) \subset B.$$

This is a minor variation of Almgren's notion of competitor. Observe that F is a continuous deformation of E in U, because $F = \varphi_1(E)$, where $\varphi_t(y) = t\varphi(y) + (1 - t)y$, and thus $\varphi_t(B) \subset B$. Note that φ_1 is **not** required to be injective.

Definition. *Fix a gauge function $h : (0, +\infty) \to [0, +\infty)$, with $\lim\limits_{r \to 0} h(r) = 0$. An almost minimal set of dimension d in U, with gauge function h, is a closed set E, with $H^d(E \cap B) < +\infty$ for every closed ball $B \subset U$, such that*

(2) $$H^d(E \cap B) \leq H^d(F \cap B) + r^d h(r)$$

whenever B is a closed ball, $B \subset U$, r is its radius, and F is a competitor for E in B.

Remarks.

• The definition of competitors is as important as the accounting. In the standard definitin we also allow competitors in compact sets $K \subset U$ (not necessarily balls).

• Here we are allowed to merge pieces of E.

• This is very similar to Almgren and Taylor's definitions, even though the accounting in (2) is a little different.

• Some other definitions are equivalent. Some are probably not. See [D4].

• See below for a slightly larger set of competitors and hence a more restrictive notion of almost-minimal sets of codimension 1, based on the definition of global minimizers for the Mumford-Shah functional.

Simple examples.

• A line, a Y (three half lines in a plane, with 120° angles).

• But not two lines (even perpendicular), because Y junctions are more performant. Also it is not too hard to see that a cross is not a mass minimizer either ($\partial T = S$ is not the same as a connectedness condition, and incidentally mass minimizers fo $\partial T = S$, where S is a sum of two Dirac masses minus two other Dirac masses, would manage not to be supported by crossing segments).

• Exercise: what is the shortest connected set that contains three given points on the circle?

• A plane, a $\mathbb{Y}$ (i.e., a product of $Y \subset \mathbb{R}^2$ by a perpendicular line, or in other words a union of three half planes that make 120° angles along their common boundary) are 2-dimensional minimal sets.

• Less trivial: let $\mathbb{T} \subset \mathbb{R}^3$ be the (closed positive) cone of dimension 2 over the union of the edges of a regular tetrahedron centered at the origin; thus $\mathbb{T}$ has six faces that meet by sets of three along four half lines (the spine). Then $\mathbb{T}$ is a minimal set. The minimality of $\mathbb{T}$ was proved by J. Taylor (with an argument like the proof in Section 5, or by Morgan-Lawlor (with a calibration).

• Catenoids (or other minimal surfaces), with $d = 2$, $n = 3$, $h(r) = 0$ for r small, but $h(r)$ we need to take $h(r)$ large for r large.

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• Expected: soap films and bubbles, minimizers of functional with a main term like $H^d(E)$, and under topological constraints that allows deformations; see below.

Reduction (= cleaning).

For $E \subset \mathbb{R}^n$ closed, with locally finite H^d measure, denote by E^* the closed support of E. That is,

$$(3) \qquad E^* = \big\{ x \in E \,;\, H^d(E \cap B(x,r)) > 0 \text{ for all } r > 0 \big\}.$$

We say that E is <u>reduced</u> when $E = E^*$.

If E is almost minimal, then E^* is almost minimal, with the same gauge h, because $H^d(E \setminus E^*) = 0$. So it is safe to focus on reduced sets.

This simplifies things; otherwise we would get uglier statements because if E is almost minimal, then $E \cup Z$ is also almost minimal for any closed Z such that $H^d(Z) = 0$.

But we should keep in mind that $E^* \setminus E$ can play a role in some topological problems.

At any rate, from now on all our sets will be reduced.

2.b. Cones and Minimal sets in $U = \mathbb{R}^3$

By minimal set of dimension d in $\mathbb{R}^n$, we just mean an almost minimal set, with $U = \mathbb{R}^n$ and $h(r) = 0$. We don't expect so many of them to exist, because only few asymptotic behaviours at ∞ are allowed.

By minimal cone, we just mean a minimal set which is a (positive) cone. Minimal cone are very important; they arise as blow-up limits of almost minimal sets, and hopefully they are much simpler (see later). For instance, you see them in soap films and bubbles. Maybe in some other examples in nature (honeycombs clearly, radiolaria less obviously).

When $d = 1$ and $n = 2$, the nonempty (reduced) minimal sets are the lines and unions Y of three half lines with the same endpoint, that make $120°$ angles at that point. Proof by hand.

Same thing when $d = 1$ and $n > 2$. The spine of a $\mathbb{T}$ (four half lines leaving from a point with maximal equal angles) is not minimal.

When $d = 2$, $n = 3$, the nonempty (reduced) minimal cones are: the planes, and the cones $\mathbb{Y}$ and $\mathbb{T}$ as above.

Ernest Lamarle, Jean Taylor and A. Heppes proved that there is no other (reduced) minimal cone of dimension 2 in $\mathbb{R}^3$. Idea: the intersection with the unit sphere sort of minimizes length, it is a union of arcs of great circles that meet only with $120°$ angles; make a list and eliminate the ones that are not minimal. For instance, a great circle is the intersection of $\partial B(0,1)$ with a 2-plane through 0.

See Ken Brakke's home page for pictures of minimal cones and some unlucky candidates (like the cone over the edges of a cube):
<div align="center">http://www.susqu.edu/brakke/</div>

Question: is there a list of minimal cones of dimension 2 in $\mathbb{R}^4$?

So far, I have a first description (as a union of faces that meet with $120°$ angles, or the cone over a net of great circles), but not very precise, and very few examples. [Such as the union of two orthogonal planes or the product two Y-sets.]

Question: are there other minimal sets of dimension 2 in $\mathbb{R}^3$?

This is unlikely, but I don't know a proof. At least, here is a partial result (to which we shall return later later).

A <u>MS-competitor</u> for $E \subset \mathbb{R}^n$ in the closed ball B is another closed set F such that F coincides with E on $\mathbb{R}^n \setminus B$ and

(4)
$$F \text{ separates } x \text{ from } y \text{ whenever } x, y \in \mathbb{R}^n \setminus (E \cap B)$$
$$\text{lie in different connected components of } \mathbb{R}^n \setminus (E \cap B).$$

A <u>MS-minimal set</u> is a closed set $E \subset \mathbb{R}^n$ such that

(5)
$$H^{n-1}(E \cap B) \le H^{n-1}(F \cap B)$$

whenever B is a closed ball and F is a MS-competitor for E in B.

MS-minimal sets arise naturally in the study of the Mumford-Shah functional: they are the sets E such that the pair $(0, E)$ is a global Mumford-Shah minimizer in $\mathbb{R}^n$.

Almgren competitors for E are MS-competitors, hence MS-minimal sets are Almgren minimal sets.

It turns out that every nonempty MS-minimal set in $\mathbb{R}^3$ is a minimal cone, i.e., a plane, a **Y**, or a **T**. This should have been known, but I did not find it. A proof will be sketched below, which looks like the argument of J. Taylor for the minimality of $\mathbb{T}$.

2.c. General regularity properties

Here is a rapid survey on the general regularity properties of almost-minimal sets. These will be less precise than J. Taylor's theorem, but they hold in all dimensions and codimensions (and they will be needed for the proof).

In what follows, E is a reduced almost minimal set of dimension d in $U \subset \mathbb{R}^n$, with gauge function h. The constants C will depend on h, n, and d, but not on E or U.

Local Ahlfors-regularity

Almgren and D.-Semmes: there exists $C \ge 1$ such that

(6)
$$C^{-1} r^d \le H^d(E \cap B(x, r)) \le C r^d$$

whenever

(7)
$$x \in E \text{ and } r \in (0, 1) \text{ are such that } B(x, 2r) \subset U.$$

Thus this is just a size condition. The hard part is the lower bound (if E is too thin, we can deform it to an even smaller set). This property is very useful in estimates; its proof uses comparisons with competitors obtained by applying Federer-Fleming projections onto d-dimensional skeletta of dyadic cubes.

Local uniform rectifiability

Some uniform way of saying that E is almost as smooth as a Lipschitz graph. In fact, E is locally uniformly rectifiable, with big pieces of Lipschitz graphs [D.-Semmes], which

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means that there exists $C \geq 1$ such that, when (7) holds, we can find a C-Lipschitz graph G of dimension d such that

$$(8) \qquad H^d(E \cap G \cap B(x,r)) \geq C^{-1}r^d.$$

A C-Lipschitz graph is just the graph of some C-Lipschitz function A defined on a d-plane V and with values in $V^\perp$.

The statement forgets many things that we know, like the fact that E does not have holes. But it implies various other (weak) regularity properties, and it is used in the following.

2.d. Uniform concentration and stability under limits

The following "uniform concentration property" holds for E. It was introduced by Dal Maso, Morel, and Solimini [DMS] in the slightly different context of the Mumford-Shah functional, to get the lowersemicontinuity of H^d along some minimizing sequences and prove existence results.

For each $\varepsilon > 0$, there is a constant $C \geq 1$ such that for each $(x,r) \in E \times (0,1)$ such that (7) holds, we can find $y \in E$ and $t > 0$ such that

$$(9) \qquad t \geq C^{-1}r\,, \quad B(y,t) \subset B(x,r)\,, \text{ and}$$

$$(10) \qquad H^d(E \cap B(y,t)) \geq (1 - \varepsilon)H^d(P \cap B(y,t))$$

for any d-plane P through y. [Thus E is $(1 - \varepsilon)$-concentrated in $B(y,t)$.] It is important that C depends only on h, n, and d.

The property is obtained [D1] as a simple consequence of uniform rectifiability, and gives the following result.

Theorem [D1]. *Let $\{E_k\}$ be a sequence of reduced almost minimal sets in U, with the same gauge function h. Suppose that $\{E_k\}$ converges to the closed set E locally in U. Then*

$$(11) \qquad H^d(E \cap V) \leq \liminf_{k \to +\infty} H^d(E_k \cap V) \text{ for every open set } V \subset U,$$

and

$$(12) \qquad E \text{ is a reduced almost minimal set, with gauge function } h.$$

We say that the sequence $\{E_k\}$ converges to E locally in the open set U when

$$(13) \qquad \lim_{k \to +\infty} d_{x,r}(E, E_k) = 0 \text{ for every ball } B(x,r) \subset\subset U,$$

where we set

$$(14) \qquad \begin{aligned} d_{x,r}(E, E_k) = {}& r^{-1} \sup \big\{ \mathrm{dist}(y, E)\,;\, y \in E_k \cap B(x,r) \big\} \\ & + r^{-1} \sup \big\{ \mathrm{dist}(y, E_k)\,;\, y \in E \cap B(x,r) \big\}. \end{aligned}$$

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Comments.

● The main part is (11), which follows directly from (10) by a covering argument [DMS]. Then (12) follows by constructing appropriate competitors.

● (11) would fail for general sets : dotted lines can converge to lines.

● Converging subsequences are easy to find, so this looks like a nice way to start proofs of existence for minimizers of functionals with a main surface term, and this was the point of [DMS] in the context of the Mumford-Shah functional. Unfortunately problems arise when we need to check that the limits are still acceptable competitors.

● This theorem will help with blow-up and blow-in limits, and for various proofs by contradiction and compactness.

We can also hope that it will be used as a replacement for the compactness theorem for normal currents that was mentioned above.

2.e. Quasiminimal sets

It may be useful to consider the larger class of quasiminimal sets (introduced by Almgren and called "restricted sets").

As before, E is a closed set in U, with $H^d(E \cap B) < +\infty$ for every compact ball $B \subset U$. We still compare E with sets $F = \varphi(E)$, where φ is Lipschitz and such that if

$$(14) \qquad W_\varphi = \big\{ x \in \mathbb{R}^n \, ; \, \varphi(x) \neq x \big\},$$

then $W_\varphi \cup \varphi(W_\varphi) \subset\subset U$, but the accounting will be different.

We say that E is a quasiminimal set with constants M and scale δ_0 if, whenever $\varphi : U \to U$ is as above and $\mathrm{diam}(W_\varphi \cup \varphi(W_\varphi) \leq \delta_0$,

$$(15) \qquad H^d(E \cap W_\varphi) \leq M H^d(\varphi(E \cap W_\varphi)).$$

We may even add an error term like $h(\delta)\delta^d$, where $\delta = \mathrm{diam}(W_\varphi \cup \varphi(W_\varphi))$.

Examples:

● the image of an almost-minimal set under a bilipschitz mapping;

● a minimizer of a fonctional like $\int_E g(x) dH^d(x)$, where we only know that $C^{-1} \leq g(x) \leq C$ everywhere.

The advantage of this notion is a greater flexibility, and the fact that the theorems above extend to quasiminimal sets (but take $r < \delta_0$ and let C depend on M as well).

3. JEAN TAYLOR'S THEOREM (1976)

3.a. A statement

Theorem [Ta]. *Let E be a reduced local almost minimal set of dimension 2 in some open set $U \subset \mathbb{R}^3$, with gauge function $h(r) = Cr^\alpha$ ($\alpha > 0$). Then for each $x \in E$, there is a ball $B(x, r)$ where E is the C^1-diffeomorphic image of a plane, a $\mathbb{Y}$, or a $\mathbb{T}$.*

Comments.

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• The conclusion means that there is a minimal cone T through x (but not necessarily centered at x) and a C^1-diffeomorphism Φ defined on $B(x, 2r)$, say, and with values in $\mathbb{R}^3$, such that $|\Phi(z) - z| \leq 10^{-3}r$, and so that $E \cap B(x, r) = \Phi(T) \cap B(x, r)$. We also control $||D\Phi - Id||$.

But the simplest description is that near $B(x, r)$, E is composed of C^1 faces, which meet along C^1 arcs with $120°$ angles and the same combinatorics as T.

• Recall that the singularies above occur in real soap films.

• Alas, the result does not give a concrete way to estimate r.

• Some (Dini) constraint on h is needed, but $h(r) = Cr^\alpha$ is not meant to be optimal.

• You can get more regularity than this, especially if h is very small or vanishing and in the regions where E is C^1-diffeomorphic to a plane.

• The bi-Hölder (as opposed to C^1) local equivalence to a minimal cone is easier to get, and extends to 2-dimensional almost minimal set in $\mathbb{R}^n$ [D4]. For the full C^1 equivalence, something like epiperimetry seems to be required and, when $n > 3$, I can only prove it under some nondegeneracy condition. That is, I assume that some blow-up limit X of E at x has the following full length property. See [D5].

Write $K = X \cap \partial B(0, 1)$ as a union of great circles or arcs of great circles. Cut them in 2 or 3 when needed, to make them less than $9\pi/10$ long. Call V the set of vertices where the different arcs meet. Consider any $\varphi : V \to \partial B(0, 1)$, with $\sup_{x \in V} |\varphi(x) - x|$ small. When γ is an arc of K with endpoints x and $y \in V$, call $\varphi_*(\gamma)$ the geodesic from $\varphi(x)$ to $\varphi(y)$. Set $\varphi_*(K) = \cup_\gamma \varphi_*(\gamma)$ and call $\varphi_*(X)$ the cone over $\varphi_*(K)$. We require the existence of $c > 0$ such that, whenever φ is such that $H^1(\varphi_*(K)) > H^1(K)$, there is a deformation $\widetilde{X}$ of $\varphi_*(X)$ in $B(0, 1)$ such that

$$H^2(\widetilde{X} \cap B(0, 1)) \leq H^2(\varphi_*(X) \cap B(0, 1)) - c[H^1(\varphi_*(K)) - H^1(K)].$$

Note that the minimal cones in $\mathbb{R}^3$ and the known minimal cones on $\mathbb{R}^n$ have the full length property. But the list of minimal cone is not known when $n > 3$.

Personal motivations

The theorem is a very nice regularity result in itself, but here are other reasons why I was interested in understanding a (new) proof.

- Checking that every MS-minimal set in $\mathbb{R}^3$ is a cone;

- Getting an extension of Ambrosio, Fusco, and Pallara's theorem on Mumford-Shah minimal segmentations in $\mathbb{R}^3$, where knowing the proof seemed necessary. This is now a theorem of A. Lemenant (see below);

- Getting an extension to higher ambient dimensions;

- Getting existence results for Plateau-Like problems. [First examples by V. Feuvrier, but no boundary yet; see below.]

Next we want to describe the main ingredients for the proof. We already mentionned the general regularity properties that lead to the limiting theorem above. The next ingredient is a standard of Geometric Measure Theory.

3.b. Monotonicity of density

Theorem. *Let E be a minimal set of dimension d in $U \subset \mathbb{R}^n$. Set*

$$(1) \qquad \theta(x,r) = r^{-d} H^d(E \cap B(x,r))$$

for $x \in E$ and $r > 0$ such that $B(x,r) \subset U$. Then for each x,

$$(2) \qquad \theta(x,\cdot) \ \text{ is nondecreasing.}$$

Idea of proof. Observe that $r \to H^d(E \cap B(x,r))$ is nondecreasing, so it is the integral of its derivative (seen as a Stiljes measure), which is no less than its almost-everywhere derivative. Thus it is enough to check that

$$(3) \qquad r^{-d}\frac{\partial}{\partial r}\big(H^d(E \cap B(0,r))\big) \geq d\, r^{-d-1} H^d(E \cap B(0,r))$$

almost everywhere. But

$$(4) \qquad \frac{\partial}{\partial r}\big(H^d(E \cap B(0,r))\big) \geq H^{d-1}(E \cap \partial B(0,r))$$

(think about C^1 surfaces, for which we could check (4) first)), so it is enough to show that

$$(5) \qquad H^d(E \cap B(0,r)) \leq \frac{r}{d}\, H^{d-1}(E \cap \partial B(0,r)).$$

But

$$(6) \qquad \frac{r}{d} H^{d-1}(E \cap \partial B(0,r)) = H^d(\Gamma \cap B(0,r)),$$

where Γ denotes the cone over $E \cap \partial B(0,r))$.

Now $[\Gamma \cap B(0,r)] \cup E \setminus B(0,r)]$ is not directly a competitor for E, but we can approximate it by Lipschitz deformations of E in $B(0,r)$ [Choose a radial deformation that expands a lot near $\partial B(0,r)$ and contracts most of $B(0,r)$ to the origin.] The comparison yields (5) and the monotonicity of θ.

Comments and extensions

● This is a classical and easy argument, but always very useful. Many results in Geometric Measure Theory depend on the existence of a suitable monotonicity formula.

● If E is merely almost minimal but h is small enough, θ is almost nondecreasing, i.e., $\theta(x,r)\exp\left\{C\int_0^r h(2t)\frac{dt}{t}\right\}$ is nondecreasing.

● If θ is constant, E coincides with a minimal cone in $B(0,1)$. [This requires an unpleasant bit of additional work however.]

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● Here is a simple consequence obtained by a typical compactness argument. For each small $\delta > 0$, we can find $\varepsilon > 0$ such that, if $x \in E$, $B(x, 2r) \subset U$, $h(2r) \leq \varepsilon$, and

$$\theta(x, 2r) \leq \lim_{t \to 0} \theta(x, t) + \varepsilon \tag{7}$$

then there is a minimal cone Z centered at x such that

$$d_{x,r}(E, Z) \leq \delta \tag{8}$$

and

$$\left| H^d(E \cap B(y, t)) - H^d(Z \cap B(y, t)) \right| \leq C\delta r^d. \tag{9}$$

Idea of proof. Fix $\delta > 0$. Suppose not. Assume that for $\varepsilon = 2^{-k}$ there is an almost minimal set E_k and a pair (x_k, r_k) that satisfies the hypotheses but not the conclusion.

By dilation invariance, we can assume that $x_k = 0$, $r_k = 1$, and $U_k \supset B(0, 2)$.

If needed, replace $\{E_k\}$ with a subsequence that converges in $B = B(0, 19/10)$ to some limit E.

By the limiting theorem and because $h_k(2t_k) \leq 2^{-k}$, E is minimal in B.

By the lowersemicontinuity of H^d,

$$H^d(E \cap B(y, t)) \leq \liminf_{k \to +\infty} H^d(E_k \cap B(y, t)) \tag{10}$$

for $B(y, t) \subset B$. By the proof of the limiting theorem,

$$H^d(E \cap \overline{B}(y, t)) \geq \limsup_{k \to +\infty} H^d(E_k \cap \overline{B}(y, t)) \tag{11}$$

when $B(y, t) \subset B$. If not, we could use the limit E to construct a better competitor.

By the almost-constant density condition (7), the density $\theta_E(0, t)$ is constant on $(0, 18/10)$.

By the equality case in monotonicity, E coincides with a minimal cone Z in $B(0, 18/10)$.

Thus (8) holds for k large, and (9) fails.

Finally (10), (11), and a uniformity argument imply (9) for k large (a contradiction). QED.

3.c. Blow-up limits

By almost monotonicity, there exists

$$d(x) = \lim_{r \to 0} \theta(x, r) \tag{12}$$

and when E is a minimal set in $\mathbb{R}^n$, $d_\infty = \lim_{r \to +\infty} \theta(r)$. Note that $C^{-1} \leq d(x) \leq d_\infty \leq C$ (for $x \in E$), by local Ahlfors-regularity.

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A <u>blow-up limit</u> of E at x is any set $F = \lim_{k \to +\infty} \dfrac{1}{r_k} [E - x]$, for a sequence $\{r_k\}$ that tends to 0. There may be more than one blow-up limits of E at a given x (for instance, if E slowly spiral around x), but this does not happen when J. Taylor's theorem holds.

Every blow-up limit of E is a minimal cone with density $d(x)$. Indeed, by the limiting theorem, F is a minimal set and its density $\rho^{-d} H^d(F \cap B(0, \rho))$ is constant and equal to $d(x)$. Hence F is a minimal cone.

Similarly, if E is minimal in $\mathbb{R}^n$ and F is a blow-in limit of E (make $\{r_k\}$ tend to $+\infty$ above), then F is a minimal cone with constant density d_∞.

3.d. An extension of Reifenberg's topological disk theorem

The following extension of Reifenberg's topological disk theorem will be useful to get the final parameterization. [Joint work [DDT] with T. de Pauw and T. Toro.]

For every small $\tau > 0$, we can find $\varepsilon_0 > 0$ such that the following happens.

Let $E \subset \mathbb{R}^3$ be a closed set, with $0 \in E$. Suppose that for each $x \in E \cap B(0, 2)$ and $r \in (0, 2]$, we can find a minimal cone $Z = Z(x, r)$ (a plane, a $\mathbb{Y}$ or a $\mathbb{T}$, not necessarily centered at x) such that

$$(13) \qquad\qquad\qquad\qquad d_{x,r}(E, Z) \leq \varepsilon_0.$$

Then there is a minimal cone Z_0 and a bi-Hölder homeomorphism $f : B(0,1) \to f(B(0,1))$ such that
$$|f(x) - x| \leq \tau \;\; \text{for } x \in B(0,1),$$
$$(1 - \tau)|x - y|^{1+\tau} \leq |f(x) - f(y)| \leq (1 + \tau)|x - y|^{1-\tau} \;\; \text{for } x, y \in B(0,1),$$
$$E \cap B(0, 1 - \tau) \subset f(Z_0 \cap B(0,1)) \subset E \cap B(0, 1 + \tau).$$

Comments.

When all the cones Z are planes, we get the well known theorem of Reifenberg.

The main case is when $Z(0,2)$ is centered at 0. Then we can take $Z_0 = Z(0,2)$.

Same sort of proof here as in the Reifenberg case, but with three layers: identify the "spine" of E, show that it is Reifenberg-flat, construct f first on the spine of Z_0, and then extend to Z_0 and to $\mathbb{R}^3$. Do all this scale by scale (from large ones to small ones); each time push $f_k(Z_0)$ in the direction of E, using the minimal cones Z and a partition of $\mathbf{1}$.

Some coherence between the various $Z(x,r)$ is forced by (13).

When $d_{x,r}(E, Z)$ tends to 0 with some definite faster speed, we get a better (for instance C^1) parameterization.

Not surprising but useful: global metric and topological information is derived from approximate information at all scales and locations. [Compare with John and Nirenberg's thorem and results of Cheeger and Colding.]

So, in order to find the parameterization promised by (the bi-Hölder version of) J. Taylor's theorem, we just need to find the good cones $Z(y,t)$ for y close to x and t small.

First lines of the proof in the Reifenberg case.

We shall get a map $f : Z_0 \to E$, obtained by composing successive mappings g_k that push points in the direction of E.

For each scale 2^{-k}, $k \geq 0$, we select a collection of points z_j, $j \in J_k$, in E, with $|z_i - z_j| \geq 2^{-k-3}$ for $i \neq j$, and $E \cap B(0,2) \subset \cup_j B(z_j, 2^{-k-2})$.

Then we construct a partition of $\mathbf{1}$ near $E \cap B(0,2)$, composed of smooth functions θ_j, $j \in J_k$, such that $\mathrm{supp}(\theta_j) \subset B(z_j, 2^{-k-1})$ and $\sum_j \theta_j = 1$ near $E \cap B(0,2)$.

For each $j \in J_k$, set $P_j = Z(z_j, 2^{-k})$ (the good plane), and denote by π_j the orthogonal projection onto P_j. Then set

$$(14) \qquad g_k(x) = x + \sum_{j \in J_k} \theta_j(x) \left[\pi_j(x) - x \right].$$

The goal of this map is to push points in the direction of E (because E lies close to the P_j). We should observe that

- P_i is close to P_j when $\mathrm{supp}(\theta_i)$ meets $\mathrm{supp}(\theta_j)$ (because both planes are close to E)

- $g_k(x) - x$ is not too large, because θ_j is supported near z_j. It is even smaller if we know that x lies close to E.

Next we define f_k on Z_0 by

$$(15) \qquad f_0(x) = x \quad \text{and} \quad f_{k+1} = g_k \circ f_k \text{ for } k \geq 0.$$

We should check by induction that $f_k(x)$ lies within $C \varepsilon_0 2^{-k}$ of E for $x \in Z_0 \cap B(0, 3/2)$, then that $\{f_k\}$ converges uniformly to a mapping $f : Z_0 \cap B(0, 3/2) \to E$.

The slightly unpleasant part is to check that f is bi-Hölder. Also we should extend f to a map defined in $B(0,1)$, and take care of the cones $Z(x,r)$ that are not planes.

End of the description of the tool kit.

4. A PROOF OF JEAN TAYLOR'S THEOREM

Recall that we are given an almost minimal set $E \subset U$ and a point $x_0 \in E$, and we want to find $r_0 > 0$ such that E is bi-Hölder equivalent to a minimal cone (centered at x) in $B(x_0, r_0)$.

We may assume that $x_0 = 0$. We want to apply the extension of Reifenberg's theorem, so it is enough to find minimal cones $Z(x,r)$, when $x \in B(0, 2r_0)$ and $0 < r \leq 2r_0$, such that $d_{x,r}(E, Z) \leq \varepsilon_0$.

We whall assume that E is minimal (to simplify).

4.a. Regularity of E near a P-point

We start with the simplest special case when 0 is a P-point, i.e., when $d(0) = \pi$.

Recall that $\theta(y,t) = t^{-2} H^2(E \cap B(y,t))$ for $y \in E$ and $t > 0$.

Pick $\varepsilon > 0$ very small, and choose r_0 so that $\theta(0, 16 r_0) \leq d(0) + \varepsilon$.

By almost-constant density, E is very close to a plane in $B(0, 8 r_0)$ (as in (3.8) and (3.9)). In particular, $\theta(x, 4 r_0) \leq d(0) + \tau$ for $x \in B(0, 2 r_0)$, where $\tau > 0$ is as small as we want.

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By monotonicity, $\theta(x, r) \leq d(0) + \tau$ for $x \in B(0, 2r_0)$ and $0 < r \leq 4r_0$.

Thus every $x \in E \cap B(0, 2r_0)$ is of type P.

And, for $x \in E \cap B(0, 2r_0)$, $\theta(x, \cdot)$ is almost-constant on $(0, 4r_0]$.

If τ is chosen small enough, almost-constant density implies that for $x \in E \cap B(0, 2r_0)$ and $0 < r \leq 2r_0$, E is ε_0-close to a minimal cone (in fact, a plane) in $B(x, r)$.

The standard Reifenberg theorem applies, and $E \cap B(0, r_0)$ is bi-Hölder equivalent to a disk, as needed.

Here we got lucky because the density was already close to its absolute minimum. If the origin is a Y-point, we get that E is close to a $\mathbb{Y}$, and $\theta(x, r) \leq \pi + \tau$ for points that are far from the spine of the **Y**. For the other points x, we cannot apply the almost constant-density principle as easily, so we need the next section.

4.b. Existence of Y-points

Still assume that E is minimal (to simplify).

Existence Lemma. *Suppose that there is a cone Y of type $\mathbb{Y}$, centered at 0, such that $d_{0,4}(E, Y) \leq \varepsilon$. Then (if ε small enough) $E \cap B(0, 1)$ contains at least a Y-point.*

We assume not, and will get a contradiction by topology. First we check that

(16) $$\theta(x, 1) \leq \frac{3\pi}{2} + \tau \ \text{ for } x \in B(0, 2) \text{ and } 0 < r \leq 1,$$

with $\tau > 0$ as small as we want.

The proof (that we vaguely sketch here) is by compactness, as for the almost constant-density principle near (3.7): suppose not, find a sequence $\{E_k\}$ that converges to a $\mathbb{Y}$ in $B(0, 4)$ but for which (16) fails. Then use the lower semicontinuity of Hausdorff measure and the proof of the limiting theorem to get an estimate like (3.11) and a contradiction.

Choose τ so that $\frac{3\pi}{2} + \tau < d_+$, where d_+ denotes the density of a cone of type $\mathbb{T}$.

By monotonicity, $d(x) < d_+$ for $x \in E \cap B(0, 2)$ so x is never a T-point.

Since there is no Y-point, x is a P-point, and E is bi-Hölder equivalent to a plane near x.

We may assume that the spine of Y is the vertical axis.

Call S the unit circle in the horizontal plane through 0, and denote by a_1, a_2, a_3 the points of $S \cap Y$.

Since E is close to a plane near a_i, we can apply the proof of regularity for P-points, and get the bi-Hölder equivalence of E to a plane in $B(a_i, 10^{-1})$. Then we can modify S slightly, to get a simple arc γ that crosses E exactly three times (one near each a_i), transversally in bi-Hölder coordinates.

Since E is bi-Hölder equivalent to a plane near every $x \in E \cap B(0, 2)$, we can deform γ into a point (inside $B(0, 2)$), and the number of intersections of γ with E only jumps by multiples of ± 2. [In the detailed argument, it is easier to discretize and get transversality.]

But we started with 3 intersections, (odd) and end with 0 (even); this contradiction proves the lemma. $\qquad\square$

A modification of this degree argument even works for 2-dimensional sets in $\mathbb{R}^n$.

4.c. Regularity near Y- and T-points

Once we have the existence lemma, it is possible to prove the regularity of E near a Y-point or a T-point roughly as we did for the P-points.

The existence lemma is useful, because we find many balls where the density is close to $3\pi/2$ and E looks like a $\mathbb{Y}$, and we can apply the constant density argument as soon as we know that they are centered at a Y-point. [Think about balls centered near the spine of $\mathbf{Y}$.]

We do not have an existence lemma for T-points (as above), but for J. Taylor's theorem, we do not need one, essentially because the only T-point nearby is already given to us. See the section about minimal sets though.

For higher-dimensional analogues, we could be in trouble here.

When 0 is a T-point, we would get the bi-Hölder equivalence of E to a cone of type $\mathbb{T}$ near 0 roughly as follows.

First assume that $\theta(0, Cr_0)$ is close to d_+, and use the constant density argument to control E in all the balls centered at 0.

Use this control and the existence lemma to obtain lots of Y-points y (along the spines of the approximationg cones of type $\mathbb{T}$), and balls $B(y, C^{-1}|y|)$ centered on them where the density of E is close to $3\pi/2$.

By monotonicity, this stays true for smaller balls centered at y.

By the constant density argument, we control E in these smaller balls centered at y.

Next we use some previous ball (centered at 0 or a y) to control $B(x, C^{-1}d)$ when $x \in E$ and d is its distance to the set of Y-points.

Finally use the constant density argument to control the smaller balls centered at x.

And check that this is enough to control E in every small ball, and apply the extension of Reifenberg's Topological Disk Theorem to get a parameterization. $\qquad\square$

5. MS-MINIMAL SETS IN $\mathbb{R}^3$

We turn to applications, and start with the fact that

(1) every reduced MS-minimal set in $\mathbb{R}^3$ is a minimal cone.

Recall that a MS-minimal set in $\mathbb{R}^3$ is a closed set $E \subset \mathbb{R}^3$, with locally finite H^2-measure, such that

$$H^2(E \cap B) \leq H^2(F \cap B)$$

whenever B is a closed ball and F is a MS-competitor for E in B. This means that $E = F$ out of B, and (as in (2.4))

(2)
$$F \text{ separates } x \text{ from } y \text{ whenever } x, y \in \mathbb{R}^n \setminus (E \cap B)$$
$$\text{lie in different connected components of } \mathbb{R}^n \setminus (E \cap B).$$

Deformations of E inside B satisfy (2), so every Almgren-competitor is a MS-competitor, and E is Almgren minimal. The converse may be false, and we don't know how to prove that every two-dimensional Almgren-minimal set in $\mathbb{R}^3$ is a cone.

Although I did not find a proof in the literature, there are similar arguments in [Morgan82] and even [Ta].

For the proof, let E be MS-minimal.

Recall that all blow-in limits of E are cones, with constant density equal to $d_\infty = \lim\limits_{r \to +\infty} \theta(x,r) \in \{\pi, \dfrac{3\pi}{2}, d_+\}$ (the limit does not depend on x).

If we can find $x \in E$ with $d(x) = d_\infty$, E is a cone by monotonicity and constant density. This settles the case when $d_\infty = \pi$.

Next suppose that $d_\infty = \frac{3\pi}{2}$. Since densities go to the limit, every blow-in limit of E is a cone of type $\mathbb{Y}$.

Let R be very large, so that E is very close to a $\mathbf{Y}$ in $B(0,R)$.

By the existence lemma in Section 4.b, there is a Y-point in $B(0,R/4)$. Hence E is a cone (as above).

So we can assume that $d_\infty = d_+$ and that there is no T-point in E.

Let R be large, and pick a cone T of type $\mathbb{T}$ so that $d_{0,2R}(E,T)$ is very small.

By the regularity theorem, each point of E has a neighborhood where E is bi-Hölder-equivalent to a plane or a $\mathbb{Y}$.

Set $E_Y = \left\{ y \in E \,;\, d_0(y) = \frac{3\pi}{2} \right\}$. Each $y \in E_Y$ has a neighborhood where E is bi-Hölder equivalent to a $\mathbb{Y}$, with at most three components of $\mathbb{R}^3 \setminus E$ around. Call $H(y)$ this set of components.

Notice that near $x \in E \setminus E_Y$, E is bi-Hölder equivalent to a plane; so we could distinguish points of E_Y topologically.

Near $y \in E_Y$, E_Y is a simple curve (bi-Hölder equivalent to a line). This comes from the equivalence of E to a $\mathbb{Y}$ near y.

And $H(y)$ is constant along that curve.

Return to the specific situation of $E \cap B(0,2R)$ and T.

Call $a_1, \cdots, a_4$ the four points of $\partial B(0,R) \cap \mathrm{spine}(T)$. Apply the proof of the regularity theorem near each a_i. We get that E is bi-Hölder equivalent to a $\mathbf{Y}$ in $B(a_i, R/10)$, say.

And there is exactly one branch γ_i of E_Y near a_i.

We can also check that E_Y does not approach the rest of $\partial B(0,R)$ (there are only P-points there, as we can check with density).

Follow γ_i as it enters $B(0,R)$. It stays a simple curve (by our local description of E_Y), and it does not stop. Eventually it leaves $B(0,R)$ as some other γ_j.

Call H_i the constant value of $H(y)$ on γ_i. So we can find $i \neq j$ so that $H_i = H_j$.

There are four big apparent components in $B(0,2R) \setminus E$, but are they all different? By the bi-Hölder description of E near a_i, H_i corresponds to the three big apparent components in $B(0,2R) \setminus E$ that get close to a_i, and since $H_i = H_j$, at least two big components coincide.

Then the definition of MS-competitors allows us to remove the big wall of E between these two components and save some area. This is impossible, by MS-minimality of E. Note that this step is easy, but it fails with standard (Almgren) minimal sets!

Remark. With topology alone, we cannot prove that a set that is close to $\mathbf{T}$ in $B(0,2R)$ cannot be locally equivalent to a plane or a $\mathbf{Y}$: it is not hard to find counterexamples [R. Hard]. These sets are probably never minimal, but I don't know a proof.

This sort of topological trouble will probably prevent us from extending the local regularity results to higher dimensions. For instance, existence lemmas for a T-point could be hard to get!

And again we don't know whether every minimal set in $\mathbb{R}^3$ is a cone.

6. C^1 REGULARITY AND EPIPERIMETRY

Just a few words about the proof (see [D5] for details). Given a closed set $E \subset U$ and $B(x, r) \subset U$, set

$$(1) \qquad \beta_E(x, r) = \inf_Z d_{x,r}(E, Z),$$

where the infimum is taken over all minimal cones Z (not necessarily centered at x or 0) that contain x. [This is a bilateral variant of numbers introduced by P. Jones, but we allow other minimal cones.]

The biHölder estimate holds as soon as we show that $\beta_E(y, t) \leq \varepsilon_0$ for $y \in E \cap B(x, 2r)$ and $0 < t \leq 2r$. For a C^1 estimate, we need to show that the $\beta_E(y, t)$ decay at some definite rate (like a power of t), uniformly in $y \in B(x, 2r)$.

Since the $\beta_E(y, t)$ are not so easy to control, we use a different quantity, namely the density defect

$$
\begin{aligned}
(2) \qquad f_x(r) &= \theta(x, r) - d(x) \\
&= \frac{1}{r^2} H^2(E \cap B(x, r)) - \lim_{t \to 0} \frac{1}{t^2} H^2(E \cap B(x, t)).
\end{aligned}
$$

Recall that if E is minimal, θ is nondecreasing and $f_x(r) \geq 0$.

Now we want to shows that $f_x(r)$ decay at some definite rate, and for instance that $r f_x'(r) \geq c f_x(r)$.

Before, we compared with the cone over $E \cap \partial B(x, r)$ to get that $f_x'(r) = \theta'(x, r) \geq 0$. Now we want to assume that $f(r) > 0$ and find a competitor that is even better than the cone, and makes us save $c r^2 f_x(r)$.

Most of the proof is the construction of a competitor. [And roughly the same one is used to show that $f_x(r)$ controls $\beta_E(x, r/2)$].

We assume that $x = 0$ and $r = 1$, and (to simplify) that E is minimal.

We may assume that E is close to a minimal cone X in $B(0, 2)$ (by compactness, or we include this in a scheme). Set $K = X \cap \partial B(0, 1)$; this is a union of arcs of great circles g_j (contained in 2-planes through 0).

We first use the biHölder equivalence (or some topology) to find a net of curves $\gamma_j \subset E \cap \partial B(0, 1)$, that lie close to the g_j and have the same separation properties as the g_j. We may even make them simple and disjoint.

We then replace the γ_j with small Lipschitz graphs Γ_j with the same endpoints, and

$$(3) \qquad H^1(\Gamma_j \setminus \gamma_j) \leq H^1(\gamma_j \setminus \Gamma_j) \leq C \Delta L_j$$

where $\Delta L_j = H^1(\gamma_j) - H^1(\rho_j)$ is the amount of extra length compared to the geodesic ρ_j with the same endpoint as γ_j and Γ_j.

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Denote by $\widehat{\Gamma}_j$ the cone over Γ_j, and set $\widehat{\Gamma} = B(0,1) \cap \left[\cup_j \widehat{\Gamma}_j \right]$.

The main point of the argument is that if $\Delta L_j > 0$, we can replace each Γ_j with the graph G_j of a harmonic function with the same boundary values, and save about $\Delta L = \sum_j \Delta L_j$ in length. [See later.]

The competitor.

We leave E as it is out of $B(0,1)$.

We replace E with $G = \cup_j G_j$ in $B(0,1)$, except that we leave out a very small annulus $A = B(0,1) \setminus B(0,1-\varepsilon)$ where we deform continuously E onto a subset of G.

First accounting. We compute what we win in the comparizon against the cone over $E \cap \partial B(0,1)$.

We save $cH^1(E \setminus \gamma)$, with $\gamma = \cup_j \gamma_j$ because it is sent to G. [We will account for A in the losses.]

We do not loose length when we replace γ_j with Γ_j.

We win $c\Delta L_j$ when we replace Γ_j with G_j.

We need to add surface to account for A. The area there is roughly the integral of the length of the trajectories of the points that were moved, i.e.

$$(4) \qquad \begin{aligned} \text{Loss} &\leq C \int_{E \cap \partial B(0,1) \setminus \Gamma_j} \text{dist}(z, \Gamma_j) \, dH^1(z) \\ &\leq \eta H^1(E \cap \partial B(0,1) \setminus \Gamma_j) \leq C\eta\Delta L \end{aligned}$$

with η as small as we want.

Altogether, the loss is compensated by the gain, and we win about $c\Delta_L + cH^1(E \setminus \gamma)$ (and observe that $H^1(\gamma \setminus \Gamma) \leq C\Delta L$).

More accounting. If $E \cap \partial B(0,1) = \gamma = \Gamma$ and each Γ_j is equal to the geodesic ρ_j, we appear to win nothing (but if E coincides with a minimal cone on $\partial B(0,1)$, then $f(r) = 0$ by direct comparizon.

In addition, the actual computation is slightly different: to get the differential inequality with f_x, we need to compare $d(0)$ (the density of a cone like x) with $\frac{1}{2}H^1(\rho)$ (the density of the cone over the ρ_j).

If $H^1(\rho) \leq 2d(0)$, the estimate above works.

If $H^1(\rho) > 2d(0)$, the estimate is not good enough, but it turns out that some angle in ρ is wrong (different from $2\pi/3$), and we can save extra length with a modification of the cone.

This final argument boils down to a simple geometric property of the minimal cones (a simple version of epiperimetry).

For 2-dimensional minimal cones in $\mathbb{R}^n$, $n \geq 4$, I don't know whether they all have it.

Graphs of harmonic functions. Recall we had a small Lipschitz graph Γ_j, and the cone $\widehat{\Gamma}_j$ over Γ_j, which we want to improve.

We may assume that both ends of Γ_j lie on the horizontal plane.

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Then $\widehat{\Gamma}_j$ is a small Lipschitz surface, vaguely triangular, bounded by two segments and Γ_j.

By construction, the angle of the sector is $\leq 9\pi/10$.

$\widehat{\Gamma}_j$ is the graph of a homogeneous function F defined on a base D, and in first approximation $H^2(\widehat{\Gamma}_j) \sim H^2(D) + \int_D |\nabla F|^2$.

We replace F with the harmonic extension H of $F_{|\partial D}$ to D, and get a new graph G_j. Then $H^2(G_j) \sim H^2(D) + \int_D |\nabla H|^2$, and $\int_D |\nabla H|^2 \leq (1-\eta) \int_D |\nabla F|^2$ by computations on the Fourier expansion of $F_{|\partial D}$. [The only case of equality would be when F is linear, which is forbidden by the boundary values.]

So we get significant improvement. In fact, $\int_D |\nabla F|^2 \geq c\Delta L$, by Parseval.

7. THE MUMFORD-SHAH FUNCTIONAL

Here we want to explain the connection between J. Taylor's theorem and regularity results for the minimal segmentations of the Mumford-Shah functional in $\mathbb{R}^3$. We shall focus on a recent result of Antoine Lemenant.

First define the Mumford-Shah functional in $\mathbb{R}^n$. We are given a simple domain $\Omega \subset \mathbb{R}^n$, a bounded function $g \in L^\infty(\Omega)$, and we set

$$(1) \qquad J_g(u,K) = H^{n-1}(K) + \int_{\Omega \setminus K} |\nabla u|^2 + \int_{\Omega \setminus K} |u-g|^2$$

for $(u,K) \in \mathcal{A}$, the set of acceptable pairs (u,K) such that $K \subset \Omega$ is closed in Ω, and $u \in W^{1,2}(\Omega \setminus K)$ has one derivative in L^2 on $\Omega \setminus K$.

The functional was introduced by Mumford and Shah (≤ 1989), at least in dimension $n=2$, for image segmentation. It was also considered as a tool for modelling cracks when $n=3$.

In the context of image segmentation, Ω is a screen, g is a given image, and u defines a segmentation for g. If (u,K) minimizes J, u should give a good compromise between the following three constraints:
- $u-g$ is small
- u is simple (varies slowly), but may have jumps along a singular set K (which we see as describing edges in the picture), but
- K is not too complicated.

Comments concerning image segmentation:
- Segmentation $\neq$ compression: it is fine if u and K only give some simplified idea of g.
- We could give different weights to the three terms, but the difference can be scaled out by multiplying u and g by a constant, and composing with a dilation.
- Lots of variants exist, but often with a term like $H^{n-1}(K)$.
- This works fine because, as conjectured by Mumford and Shah, K is automatically regular (instead of just being short) when (u,K) is a minimal pair.
- this gives an automatic and context free algorithm for segmentation. This is good but then we cannot expect too much from this.

7.a. Regularity statements

Concerning the mathematics, the interest lies in the interplay between the two main terms of the functional (H^{n-1} and energy). Often the proofs for the Mumford-Shah functional contain a proof of the (easier) analogous property for almost minimal sets.

The **existence** of minimal segmentations (minimizers for J_g) is known from Ambrosio [A] and De Giorgi-Carriero-Leaci [DCL], and uses compactness properties of SBV.

Some **regularity** properties are known for reduced minimal segmentations. These typically concern K, because once we know K, studying u is easy.

For instance, K is locally Ahlfors-regular (Dal Maso-Morel-Solimini [DMS] and Carriero-Leaci [CL]) and uniformly rectifiable [DS2].

When $n = 2$, Mumford and Shah conjectured that K is a finite union of C^1 arcs, that can only meet with $120°$ angles.

In higher dimensions, the best general regularity result is the following.

Theorem (Ambrosio-Fusco-Pallara [AFP1]). *There exists $\varepsilon_0 > 0$, depending only on n, such that if (u, K) is a reduced minimizer for J_g in $\Omega \subset \mathbb{R}^n$, $x \in K$, $r||\nabla g||_\infty^2 \leq \varepsilon_0$, $B(x, r) \subset \Omega$,*

$$(2) \qquad \int_{B(x,r)\setminus K} |\nabla u|^2 \leq \varepsilon_0 r^{n-1},$$

and there is a hyperplane P through x such that

$$(3) \qquad d_{x,r}(K, P) \leq \varepsilon_0,$$

then K coincides with a C^1 submanifold in $B(x, r/2)$.

It is not so hard to see that for H^{n-1}-almost every $x \in K$, (2) and (3) hold for arbitrarily small radii r, so almost every $x \in K$ has a neighborhood where K is a C^1 submanifold. In fact, the set of points x where this does not happen is of Hausdorff dimension $< n - 1$ [Ri].

We could dispense with (2) by a compactness argument, but let us not bother.

In the situation of the theorem, it turns out that u is fairly regular in $B(x, r/2)$, so the main term of J_g is $H^{n-1}(K)$, and K tends to be almost-minimal. In effect, the theorem contains a regularity result for almost minimal sets such that (3) holds.

We want to discuss the following generalization (when $n = 3$).

Theorem [Lemenant]. *There exists $\varepsilon_0 > 0$ such that if (u, K) is a reduced minimizer for J_g in $\Omega \subset \mathbb{R}^3$, $x \in K$, $r||\nabla g||_\infty^2 \leq \varepsilon_0$, $B(x, r) \subset \Omega$, (2) holds, and there is a minimal cone Z centered at x such that*

$$(4) \qquad d_{x,r}(K, Z) \leq \varepsilon_0,$$

then K is C^1-equivalent to Z in $B(x, r/2)$.

Before we discuss part of the proof, let us say a few words about blow-up limits and global minimizers.

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7.b. Global Mumford-Shah minimizers

We also have a theorem about limits in this context, and A. Bonnet showed that every blow-up limit of (u, K) (a reduced minimizer for J_g) is a global Mumford-Shah minimizer in $\mathbb{R}^n$, with the definition below.

A normalization procedure for u is used (to allow u to go to a limit), but let us skip the details.

Let $\mathcal{A}$ denote the set of admissible pairs (u, K), where $K \subset \mathbb{R}^n$ is closed, $H^{n-1}(K \cap B(0, R)) < +\infty$ for $R > 0$, $u \in W_{loc}^{1,2}(\mathbb{R}^n \setminus K)$, and $\int_{B(0,R) \setminus F} |\nabla u(x)|^2 dx$ for $R > 0$.

A <u>competitor</u> for $(u, K) \in \mathcal{A}$ in the closed ball B is a pair $(v, G) \in \mathcal{A}$ such that $(v, G) = (u, K)$ out of B, and (as in (5.2))

(5)
$$G \text{ separates } x \text{ from } y \text{ whenever } x, y \in \mathbb{R}^n \setminus (K \cap B)$$
$$\text{lie in different connected components of } \mathbb{R}^n \setminus (K \cap B).$$

A <u>global Mumford-Shah minimizer</u> in $\mathbb{R}^n$ is a pair $(u, K) \in \mathcal{A}$ such that

(6)
$$H^{n-1}(K \cap B) + \int_{B \setminus K} |\nabla u(x)|^2 dx$$
$$\leq H^{n-1}(G \cap B) + \int_{B \setminus G} |\nabla v(x)|^2 dx$$

for every closed ball B and every competitor for (u, K) in B.

There are notions of almost- and quasi-minimizers in this context too, but let us not bother. See [D2].

Notice that when $u = 0$, the pair $(0, K)$ is a global Mumford-Shah minimizer in $\mathbb{R}^n$ precisely when K is a MS-minimal set in $\mathbb{R}^n$ (because we can always replace v with 0 above). So, when $n = 3$, K is a minimal cone.

The interest of global Mumford-Shah minimizers is that they are simpler (no function g), and knowing them will help us understand the Mumford-Shah minimizers in a domain.

The theorems of Ambrosio-Fusco-Pallara and Lemenant are still valid in this context of global minimizers (and we don't need r to be small).

7.c. Comments about the proof

We start from a nice situation (as in (2) and (4)), and want to obtain the decay of various quantities. We are interested in

(7)
$$\beta_K(x, r) = \inf_Z d_{x,r}(K, Z)$$

where the infumum is taken over all minimal cones through x. But it is easier to rely on

(8)
$$\omega(x, r) = r^{-2} \int_{B(x,r)} |\nabla u|^2.$$

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There are other quantities, for instance to measure the size of holes (or how well K disconnects space), or a gauge function (for almost-minimality) but let us skip them.

The main idea is to fix x and show that $\omega(x, r)$ decays like a power when r tends to 0. [This is essentially the only way to make sure it stays small.] Then we use $\omega(x, r)$ to control the other quantities, and make sure that they stay small.

We shall just see various symptomatic cases; the general case is a more complicated mixture.

Decay for $\omega(x, r)$.

Suppose K is a minimal cone, and u is an energy minimizer in $\mathbb{R}^n \setminus K$. Thus $\Delta u = 0$ and $\frac{\partial u}{\partial n} = 0$ on K.

Then $\omega(x, r)$ decays like a power. When $K = \emptyset$, this is easy, because ∇u is locally constant and by the way $\omega(x, r)$ is normalized. When K is a plane, just use the fact that u has a harmonic extension on the other side, which is smooth. Thus $\int_{B(x,r)} |\nabla u|^2 \leq C r^3$. When K is another cone, we use spherical harmonics and estimate the first eigenvalue of the Laplacian on $\partial B(0, 1) \setminus K$.

Extension by compactness: $\omega(x, r)$ decays like a power when K is a generalized Reifenberg-flat set, which means that $\beta_0(x, r) \leq \varepsilon_0$ for all $x \in K$ and $r \leq 1$, and where the constant ε_0 is chosen small enough.

Observe that generalized Reifenberg-flat sets separate space well into 2, 3, or 4 components (and the decay occurs component by component).

But why should K be generalized Reifenberg-flat?

Control on $\beta_K(x, r)$.

Now suppose that $\omega(x, r)$ is very small, or even that u is locally constant.

Then K is a locally almost minimal set, and the proof of Jean Taylor's theorem says that $\beta_K(x, r)$ decays like a power. The estimates are not good enough to be used like this, but here is a statement that can be used.

Lemma. *There exist ε_1 (small) and $C \geq 1$ such that, if*

$$(9) \qquad \qquad \beta_K(x, \rho) \leq 10\varepsilon_1 \quad \text{for } 2r \leq \rho \leq Cr$$

and K is almost-minimal in $B(x, Cr)$, with a sufficiently small gauge function, then

$$(10) \qquad \qquad \beta_K(x, r) \leq \varepsilon_1.$$

This gives a nice way to check that (10) stays true at smaller scales (if $\omega(x, r)$ is small enough).

But it can also be used backwards: if for some reason (10) fails, then K is not almost-minimal with a small gauge function, and in fact (by a compactness argument) we can find a deformation $\widetilde{K}$ of K in $B(x, Cr)$, with a noticeably smaller area!

Extensions of u.

If we want to use a deformation $\widetilde{K}$ of K in a ball B to construct a competitor for (u, K), we need to be able to associate a function $\widetilde{u} \in W^{1,2}_{loc}(\Omega \setminus \widetilde{K})$, with $\widetilde{u} = u$ out of B.

This could be a pain, because we never know in which local component of $B \setminus \widetilde{K}$ a given point y lies.

But in the complement of a generalized Reifenberg-flat set, this can be done. We use Whitney cubes and partitions of $\mathbf{1}$, and brutally define separate extensions of the restrictions of u to each component.

The additional energy of the extension is dominated by $\int |\nabla u|^2$ (and is often even smaller).

A stopping time argument.

Then we need to put everything together.

We start from $B(0,1)$, where (2) and (4) hold, and proceed scale by scale (starting from the large one). We stop whenever (10) becomes false. That is, we cover the set of $x \in E$ such that (10) fails for some r by essentially maximal balls $B(x_j, r_j)$ where (10) fails.

We wish to show that $\sum_j r_j^2$ is small (depending on ω). So we use the lemma, extend u as before, and get a competitor for (u, K). We do not lose too much energy (if ω was small enough), and we save $\eta \sum_j r_j^2$ surface, as needed.

Away from the B_j we still have the decay of ω mentioned above, by a compactness argument. Except for an additional problem: we may have had some communication between the components of $\mathbb{R}^3 \setminus K$, through the B_j, and for technical reasons we need to separate them, by adding all the ∂B_j for which B_j meets some sphere $\partial B(0, \rho)$, with $1/2 \leq \rho \leq 2/3$.

We can choose ρ by Chebyshev, so the cost is less than $C \sum_j r_j^3$, which is much smaller than $\sum_j r_j^2$. So we get some decay, unless ω is much smaller than $\sum_j r_j^2$.

We distinguish cases, fix the small lies above, put together various inequalities, and get the result. The main point of the stopping time is again to be able to use all the expected estimates before we know for sure that they will hold everywhere.

See Carleson, Jones, Semmes, Léger for other uses of similar stopping time arguments.

8. SIMPLER VARIANTS OF THE PLATEAU PROBLEM

We end with a potential application (my main motivation now). So far the scheme described below only works in simple special cases.

Fix a simple domain $\Omega \subset \mathbb{R}^n$ (or possibly manifold), preferably closed and with some holes, and a continuous bounded function $g : \Omega \to [1, M]$, and set

$$(1) \qquad J_g(E) = \int_E g(x) \, dH^d(x) \ \text{ for } E \subset \Omega.$$

Also fix a closed set $E_0 \subset \Omega$, and call $\mathcal{F}(E_0)$ the class of continuous deformations of E_0 in Ω.

That is, $E \in \mathcal{F}(E_0)$ if $E = \varphi_1(E_0)$, where we set $\varphi_t(x) = \varphi(x,t)$ and $\varphi : E_0 \times [0,1] \to \Omega$ is continuous, with $\varphi(x,0) = x$ for $x \in E_0$.

We may (or not) require that φ_1 be Lipschitz (but with no bounds attached).

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We shall try to minimize J_g in the class $\mathcal{F}(E_0)$, i.e., we try to find $E \in \mathcal{F}(E_0)$ such that $J_g(E)$ is minimal.

Let us always choose Ω, d, and E_0 so that

$$0 < \inf_{E \in \mathcal{F}(E_0)} J_g(E) < +\infty.$$

In particular, we do not choose E_0 contractible in Ω.

Example 1: $n = 2$, $d = 1$, $\Omega = \mathbb{R}^2 \setminus [B_1 \cup B_2]$ (two disjoint open balls with the same diameter, say), $g = 1$, and $E_0 = \partial B(0, R)$ (with R large). Two cases occur, depending on whether the balls are far from each other.

Notice that taking Ω closed is better, because E likes to contain boundary pieces.

Exercise: try to encode $\mathcal{F}(E_0)$ with a topological condition.

Example 2. [A higher dimensional variant.] Take $n = 3$, $d = 2$, and $\Omega = \mathbb{R}^3 \setminus A$ for some open solid torus A, $g = 1$, and $E_0 = \partial A$. Again two cases occur, and sometimes $E^* \neq E$ for topological reasons.

Comments.
- We may add smaller terms to J_g.
- In these two examples, the existence of minimizers is perhaps easy to get, even if $g \neq 1$, because the codimension is 1 and we can use the compactness properties of BV. This is less clear in general.
- In all cases the minimizers, if they exist, are almost minimal (with h coming from the regularity of g). Thus, when $d = 2$ and $n = 3$, Jean Taylor's theorem will apply if g is Hölder-continuous. [Proof of a variant Almgren-quasiminimality, and of almost-minimality if g is continuous.]
- Many other classes $\mathcal{F}$ could be tried, even in this simple context (separation when $d = n - 1$, linking conditions of two types, algebraic topology, etc.). We chose one for which standard methods seem hard to apply.
- Plateau problems will probably be harder; intermediate problems exist (soap between walls, manifolds).

Existence results for J_g?

How can J. Taylor's result help prove existence results in this context? Partial results when $d = 2$, $n = 3$, and g is Hölder-continuous (so far) by Vincent Feuvrier [Feu].

Try the stupid way, with a minimizing sequence $\{E_k\}$ in $\mathcal{F}(E_0)$.

We [= V. Feuvrier] can choose $\{E_k\}$, or modify it, so that the E_k are quasi-minimal sets, with uniform bounds, as follows.

Construct "dyadic nets" $\mathcal{B}_k$ adapted to the E_k, with smaller and smaller mesh sizes, but uniform "regularity" (no small angles, etc.).

Deform E_k into a union $\widetilde{E}_k$ of 2-faces in $\mathcal{B}_k$ that minimizes J_g; the existence of a minimizer $\widetilde{E}_k$ comes from the finiteness of the net, the fact $J_g(\widetilde{E}_k)$ is almost as small as $J_g(E_k)$ comes from the adaptation of the net to the initial E_k, and the quasiminimality comes from the minimality of $\widetilde{E}_k$ and the regularity of the net.

Let us suppose first that $E_k^* = E_k$. Extract a subsequence, so that $\{E_k\}$ converges to some E.

Then $J_g(E) \leq \liminf_{k \to +\infty} J_g(E_k)$, by the lowersemicontinuity part of the limiting result. So it is enough to show that $E \in \mathcal{F}(E_0)$.

By the limiting results, because $\{E_k\}$ is minimizing, and because g is Hölder-continuous, E should be almost-minimal with a small gauge function.

By J. Taylor's theorem, E is locally C^1-equivalent to a plane, a $\mathbf{Y}$, or a $\mathbf{T}$. Then there is a retraction from a neighborhood of E onto E (this is a local property).

So E is a deformation of E_k for k large (or contains such a deformation). Hence E (or a subset) lies in $\mathcal{F}(E_0)$.

If $E_k^* \neq E_k$, take E to be a limit of the E_k^*, and then simply let $E_k^* \setminus E_k$ follow the Lipschitz retraction.

This should give existence in some cases that are not given by the more standard methods with currents [R. Hardt + T. De Pauw].

Rapid list of questions (some of which may take some time to answer)
- Give a list of 2-dimensional minimal cones in $\mathbb{R}^4$;
- prove that every minimal set of dimension 2 in $\mathbb{R}^3$ is a cone;
- study minimal cones and sets of codimension 1 in $\mathbb{R}^4$;
- deal with the boundary regularity (typically, with a sliding boundary condition), and then solve Plateau problems.

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