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A variation on the "infsup" condition

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Scope of the lecture

1) One field study

2) Two fields analysis

3) Three fields formulation

4) Answer to an old question put by J. F. Maître

A real Hilbert space H

$\mathcal{S}_H$ = unity sphere in H : $\{h \in H, \parallel h \parallel = 1\}$

$\mathcal{B}_H$ = unity ball in H : $\{h \in H, \parallel h \parallel \leq 1\}$

H' : topological dual of Hilbert space H

If $\zeta \in H'$, then $< \zeta, h > \in \mathbb{R}$ and $\parallel \zeta \parallel =$ sup $\{< \zeta, h >, h \in \mathcal{B}_H\}$.

Following I. Babuška (1971) :

consider two Hilbert spaces Y and Z

a continuous bilinear form $k : Y \times Z \longrightarrow \mathbb{R}$

introduce two associated linear operators

$$K : Y \longrightarrow Z' , \quad < Ky, z > = k(y, z), \quad y \in Y, z \in Z$$

$$K' : Z \longrightarrow Y' , \quad < y, K'z > = k(y, z), \quad y \in Y, z \in Z .$$

What are necessary and sufficient conditions to get $K \in \text{Isom}\,(Y, Z')$?

On one hand, K^{-1} must be continuous :

$$\exists\, \gamma > 0, \forall\, y \in Y, \ \| \, K y \, \| \geq \gamma \ \| \, y \, \|$$

equivalently $\quad \exists\, \gamma > 0, \, \forall\, y \in \mathcal{S}_Y, \, \exists\, z \in \mathcal{B}_Z, \ k(y,\, z) \ \geq \gamma$

equivalently $\quad \exists\, \gamma > 0, \ \inf_{y \in Y} \ \sup_{z \in Z} \dfrac{k(y,\, z)}{\| \, y \, \| \, \| \, z \, \|} \geq \gamma$

the famous "infsup" condition !

On the other hand,

if z is given in $\mathcal{S}_Z$, $\exists\, \zeta \in Z'$ such that $< \zeta,\, z > \neq 0$

the range of K is equal to Z' then $\exists\, y_0 \in Y, \, K y_0 \, = \, \zeta$

then $k(y_0,\, z) \, = < K y_0,\, z > = < \zeta,\, z > \neq 0$

and $\quad \forall\, z \in \mathcal{S}_Z, \ \sup_{y \in Y} k(y,\, z) \ = \ +\infty$

the *not so famous* "infinity" condition.

Babuška's theorem (1971) :

the infsup condition $\exists\, \gamma > 0, \forall\, y \in \mathcal{S}_Y, \exists\, z \in \mathcal{B}_Z,\ k(y,\, z) \geq \gamma$

and the infinity condition $\forall\, z \in \mathcal{S}_Z,\ \sup_{y \in Y} k(y,\, z) = +\infty$

are necessary and sufficient conditions to get $K \in \mathrm{Isom}\,(Y,\, Z')$.

Second fundamental result

we have the equivalence $K \in \mathrm{Isom}\,(Y,\, Z') \Longleftrightarrow K' \in \mathrm{Isom}\,(Z,\, Y')$

We deduce from these two theorems that

if K is an isomorphism from Y onto Z', we have the

second infsup condition $\exists\, \gamma' > 0, \forall\, z \in \mathcal{S}_Z, \exists\, y \in \mathcal{B}_Y,\ k(y,\, z) \geq \gamma'$

second infinity condition $\forall\, y \in \mathcal{S}_Y,\ \sup_{z \in Z} k(y,\, z) = +\infty$.

Classical references : O. Ladyzhenskaya (1963)
F. Brezzi (1974)
V. Girault and P.A. Raviart (1979, 1986)

Consider two Hilbert spaces X and M and
two continuous bilinear forms $a : X \times X \longrightarrow \mathbb{R}$

$$b : X \times M \longrightarrow \mathbb{R}$$

the associated linear operators

$$\begin{aligned}
A : X &\longrightarrow X' , & < Au, v > &= a(u, v), & u \in X, v \in X \\
B : X &\longrightarrow M' , & < Bu, q > &= b(u, q), & u \in X, q \in M \\
B' : M &\longrightarrow X' , & < u, B'q > &= b(u, q), & u \in X, q \in M \ .
\end{aligned}$$

In the framework of the first section : $Y = Z = X \times M$
and $k((u, p), (v, q)) = a(u, v) + b(u, q) + b(v, p)$.

Operator $\Phi : \; X \times M \longrightarrow X' \times M'$

associated with the bilinear form $k(\bullet, \bullet)$ is defined by blocs :

$$\Phi = \begin{pmatrix} A & B' \\ B & 0 \end{pmatrix} .$$

In order to study if Φ is an isomorphism, consider $f \in X'$, $g \in Z'$ and try to solve the system :

$$\begin{array}{lll} (1) & Au + B'p & = f \\ (2) & Bu & = g \, . \end{array}$$

Of course the kernel V of operator B has a crucial role ;

define $V = \ker B = \{ v \in X, \forall \, q \in M, \, b(v, \, q) = 0 \}$,

use the orthogonality decomposition in Hilbert spaces :

if $u \in X$, consider $u^0 \in V$ and $u^1 \in V^\perp$ such that $u = u^0 + u^1$.

Observe that the polar set $V^0 \equiv \{ \zeta \in X', \forall \, v \in V, < \zeta, \, v >= 0 \}$ can be identified with the dual space $(V^\perp)'$ of its orthogonal.

$$\begin{array}{ll}(1) & Au + B'p \;=\; f \\ (2) & Bu \qquad\quad\; = \; g\,.\end{array}$$

the equation (2) takes the form : $\quad (3) \quad u^1 \in V^{\perp}, \quad Bu^1 \;=\; g\,.$

natural **hypothesis (i)** to solve (3) : $\quad B \in \mathrm{Isom}\,(V^{\perp},\, M')$

then report u^1 inside equation (1) and test this equation against $v \in V$ to eliminate the *so-called* pressure p :

$$(4) \quad u^0 \in V, \quad \forall\, v \in V, < Au^0,\, v > = < f - Au^1,\, v >$$

natural **hypothesis (ii)** to solve (4) : $\quad A \in \mathrm{Isom}\,(V,\, V')$

observe that equation (4) can also be written as $\quad f - Au \in V^0$

then equation (1) takes the form : $\quad (5) \quad p \in M, \quad B'p \;=\; f - Au$

and has a unique solution

due to the hypothesis (i) : $\quad B' \in \mathrm{Isom}\,(M,\, V^0)$

and the fact that the right hand side in (5) belongs to polar space V^0.

"my version" of the Girault and Raviart's theorem (1986) :

$\Phi = \begin{pmatrix} A & B' \\ B & 0 \end{pmatrix}$ is an isomorphism from $X \times M$ onto its dual

if and only if the hypotheses (i) $B \in \mathrm{Isom}\,(V^{\perp},\, M')$
and (ii) $A \in \mathrm{Isom}\,(V,\, V')$ are satisfied.

Other expression of hypothesis (i) : $B' \in \mathrm{Isom}\,(M,\, (V^{\perp})' \equiv V^{0}))$

Infsup condition associated with this formulation of hypothesis (i) :

$$\exists\, \beta' > 0,\ \forall\, p \in \mathcal{S}_M,\ \exists\, v \in \mathcal{B}_{V^{\perp}},\ b(v,\, p) \geq \beta'$$

equivalently : $\exists\, \beta' > 0,\ \forall\, p \in \mathcal{S}_M,\ \exists\, v \in \mathcal{B}_X,\ b(v,\, p) \geq \beta'$

equivalently : $\exists\, \beta' > 0,\ \inf_{p \in M} \sup_{v \in X} \dfrac{b(v,\, p)}{\| v \| \, \| p \|} \geq \beta'$ classical !

Observe that the infinity condition

$\forall\, v \in \mathcal{B}_{V^{\perp}},\ \sup_{p \in M} b(v,\, p) = +\infty$ is trivial !

Girault and Raviart's theorem (1986), formulated by the authors :

$\Phi = \begin{pmatrix} A & B' \\ B & 0 \end{pmatrix}$ is an isomorphism from $X \times M$ onto its dual

if and only if the hypotheses

(i) $\exists\, \beta' > 0, \ \inf_{p \in M} \sup_{v \in X} \dfrac{b(v,\, p)}{\| \, v \, \| \, \| \, p \, \|} \geq \beta'$

and (ii) $A \in \mathrm{Isom}\,(V,\, V')$ are satisfied.

The infinity hypothesis for B' operator is lost in this formulation due to the particularity of the situation !

Motivation :

vorticity-velocity-pressure formulation of the Stokes problem

FD (1992, 2002), FD, S. Salaün and S. Salmon (2000, 2003) :

$$\begin{aligned} \omega - \operatorname{curl} u &= 0 \\ \nu \operatorname{curl} \omega + \nabla p &= g \\ \operatorname{div} u &= 0 \,. \end{aligned}$$

Integrate by parts and multiply by *ad hoc* coefficients : abstract form

W : space for vorticity, U for velocity, P for pressure

three continuous linear forms

$$\begin{aligned} j : W \times W \ni (\omega, \varphi) &\longmapsto j(\omega, \varphi) \in \mathbb{R} \\ r : W \times U \ni (\omega, v) &\longmapsto r(\omega, v) \in \mathbb{R} \\ d : U \times P \ni (u, q) &\longmapsto d(u, q) \in \mathbb{R} \end{aligned}$$

bilinear form $k : (W \times U \times P) \times (W \times U \times P) \longrightarrow \mathbb{R}$

$k((\omega, u, p), (\varphi, v, q)) = j(\omega, \varphi) + r(\omega, v) + r(\varphi, u) + d(u, q) + d(v, p)$

associated linear operators :

$$
\begin{array}{lllll}
J : W \longrightarrow W', & < J\omega, \varphi >= j(\omega, \varphi), & \omega \in W, \varphi \in W \\
R : W \longrightarrow U', & < R\omega, v >= r(\omega, v), & \omega \in W, v \in U \\
R' : U \longrightarrow W', & < \omega, R'v >= r(\omega, v), & \omega \in W, v \in U \\
D : U \longrightarrow P', & < Du, q >= d(u, q), & u \in U, q \in P \\
D' : P \longrightarrow U', & < u, D'q >= d(u, q), & u \in U, q \in P
\end{array}
$$

linear system to solve :

$$
\begin{array}{lll}
(6) & J\,\omega + R'u & = f \\
(7) & R\omega + D'p & = g \\
(8) & Du & = h
\end{array}
$$

orthogonal decomposition of the velocity :

$$u = u^0 + u^1, \; u^0 \in \ker D, \; u^1 \in (\ker D)^\perp$$

orthogonal decomposition of the vorticity :

$$\omega = \omega^0 + \omega^1, \; \omega^0 \in \ker R, \; \omega^1 \in (\ker R)^\perp \; .$$

$$
\begin{array}{lll}
(6) & J\,\omega + R'u & = \; f \\
(7) & R\omega + D'p & = \; g \\
(8) & Du & = \; h
\end{array}
$$

Equation (8) takes the form : $(9) \quad u^1 \in (\mathrm{ker} D)^\perp, \quad Du^1 \; = \; h$

natural **hypothesis (iii)** to solve (9) : $D \in \mathrm{Isom}\,((\mathrm{ker} D)^\perp,\, P')$

test second equation against $v \in \mathrm{ker} D$ to eliminate the pressure p :

$$< D'p,\, v > = < p,\, Dv > = \; 0$$

$$(10) \quad \omega^1 \in (\mathrm{ker} D)^\perp, \quad \forall\, v \in \mathrm{ker} D, < R\omega^1,\, v > = < g,\, v >$$

natural **hypothesis (iv)** to solve (10) : $R \in \mathrm{Isom}\,((\mathrm{ker} R)^\perp,\, (\mathrm{ker} D)')$

then equation (10) implies that $g - R\omega \in (\mathrm{ker} D)^0$

and equation (7) takes now the form $(11) \quad D'p \; = \; g - R\omega$

then due to hypothesis (iii) : $D' \in \mathrm{Isom}\,(P,\, (\mathrm{ker} D)^0)$

equation (11) has a unique solution

$$\begin{array}{lll} (6) & J\,\omega + R'u & = f \\ (7) & R\omega + D'p & = g \\ (8) & Du & = h \end{array}$$

The fields u^1, ω^1 and p are known.

Test equation (6) against $\varphi \in \ker R$: $< R'u, \varphi > = < u, R\varphi > = 0$

and report the value of ω^1 :

$$(12) \qquad \omega^0 \in \ker R, \quad \forall\, \varphi \in \ker R, < J\omega^0, \varphi > = < f - J\omega^1, \varphi >$$

natural **hypothesis (v)** to solve (12) : $J \in \mathrm{Isom}\,(\ker R, (\ker R)')$

then equation (12) implies that $f - J\omega \in (\ker R)^0$

and equation (6) takes now the form

$$(13) \qquad u^0 \in \ker R,\ R'u^0 = f - J\omega - R'u^1$$

due to hypothesis (iv) : $R' \in \mathrm{Isom}\,(\ker D, (\ker R)^0)$

equation (13) has a unique solution.

Note the algorithm induced by this approach : u^1, ω^1 , p , ω^0 , u^0.

Isomorphism Theorem with three fields

Let K be defined from $W \times U \times P$ to $W' \times U' \times P'$

$$\text{by the matrix } K = \begin{pmatrix} J & R' & 0 \\ R & 0 & D' \\ 0 & D & 0 \end{pmatrix}$$

then K is an isomorphism if and only if the three hypotheses

(v) $J \in \mathrm{Isom}\,(\ker R,\ (\ker R)')$

(iv) $R \in \mathrm{Isom}\,((\ker R)^\perp,\ (\ker D)')$

(iii) $D \in \mathrm{Isom}\,((\ker D)^\perp,\ P')$ are satisfied

We can replace (iii) by $D' \in \mathrm{Isom}\,(P,\ (\ker D)^0)$

and (iv) by $R' \in \mathrm{Isom}\,(\ker D,\ (\ker R)^0)$.

Note that the infinity condition associated to (iv) :

$$\forall\, \varphi \in \mathcal{S}_{(\ker R)^\perp},\ \sup_{u \in \ker D} r(\varphi,\, u) = +\infty$$ is *a priori* not trivial !

J.F. Maître (Giens, Canum 1993) :
“What is the link between the three fields infsup conditions and the classical two fields infsup conditions ?”

In other terms, $\begin{pmatrix} J & R' & 0 \\ R & 0 & D' \\ 0 & D & 0 \end{pmatrix} = \begin{pmatrix} A & B' \\ B & 0 \end{pmatrix}$?

ok when $X = W \times U, \; M = P \,, \; A = \begin{pmatrix} J & R' \\ R & 0 \end{pmatrix} \,, \; B = \begin{pmatrix} 0 & D \end{pmatrix} .$

the infsup condition for B' operator $\exists\, \beta' > 0, \; \inf_{p \in M} \sup_{v \in X} \dfrac{b(v,\, p)}{\| \, v \, \| \, \| \, p \, \|} \geq \beta'$

takes the analogous form for D’ : $\exists\, \delta' > 0, \; \inf_{p \in P} \sup_{v \in U} \dfrac{d(v,\, p)}{\| \, v \, \| \, \| \, p \, \|} \geq \delta'$

At what precise conditions operator A is an isomorphism from $\ker B = W \times \ker D$ onto its dual $(\ker B)' = W' \times (\ker D)'$?

Make attention that R' is not exactly equal to R' restricted to $\ker D$!
The exact isomorphism condition $R' \in \mathrm{Isom}\,(\ker D,\,(\ker R)^0)$
leads to an infsup condition

$$\exists\,\rho' > 0,\ \inf_{u \in \ker D}\ \sup_{\varphi \in (\ker R)^\perp} \frac{r(\varphi,\,u)}{\|\,\varphi\,\|\,\|\,u\,\|} \geq \delta'$$

that can be written equivalently

$$\exists\,\rho' > 0,\ \inf_{u \in \ker D}\ \sup_{\varphi \in W} \frac{r(\varphi,\,u)}{\|\,\varphi\,\|\,\|\,u\,\|} \geq \delta'$$

But the associated **infinity condition**

$$\forall\,\varphi \in \mathcal{S}_{(\ker R)^\perp},\ \sup_{u \in \ker D} r(\varphi,\,u) = +\infty$$

remains *a priori* not trivial and has not to be dropped !

Many thanks to Christine Bernardi for a very fruitfull interaction !