A new method for the boundary vorticity

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Abstract. We consider the bidimensional Stokes problem for incompressible fluids in stream function-vorticity formulation. For this problem, the classical finite elements method of degree one converges with an order of only 1/2 for the quadratic norm of the vorticity. We propose to change the method of approximation and add harmonic functions to approach vorticity along the boundary. Numerical results are very satisfying and we prove that this new numerical scheme leads to an error of order 1 for the natural norm of the vorticity and under more regularity assumptions from 3/2 to 2 for the quadratic norm of the vorticity.

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1. Introduction

Let Ω be a bidimensional open bounded domain assumed sufficiently regular. The Navier-Stokes equations modelize the equilibrium of an incompressible and viscous fluid. When the viscosity is sufficiently important or the velocity of the fluid sufficiently small, we can neglect convection terms and we obtain the stationary Stokes problem which is (in primitive variables *i.e* velocity u and pressure p) :

$$\left\{\begin{array}{rcll} -\nu\Delta\mathbf{u}+\nabla p & = & \mathbf{f} & \text{in} \quad \Omega \\ \operatorname{div}\mathbf{u} & = & 0 & \text{in} \quad \Omega \\ \mathbf{u} & = & 0 & \text{on} \quad \partial\Omega \end{array}\right.$$

where ν is the kinematic viscosity and $\mathbf{f}$ a field of given external forces. This problem is important as it is often necessary to solve it for computing a solution to the Navier-Stokes non-linear problem.

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If Ω is supposed simply connected and as velocity is divergence free, this two-dimensional problem is often rewritten with stream function and vorticity variables. Velocity is the curl of some stream function and vorticity is the curl of the velocity. The classical way of discretizing this problem in stream function-vorticity formulation is to choose a finite element method with polynomials of degree one for both variables. But it is well-known that this scheme is not stable and that the vorticity is unsatisfyingly computed on the boundary of the domain when the meshes are unstructured. Moreover, the convergence rate of the quadratic norm of the error on the vorticity is only of order 1/2 (see Figure 2), which is not optimal (see [7]). Amara and Bernardi [3] show that it is possible to stabilize this formulation and improve convergence, but we prefer to directly work on the other stream function-vorticity variational formulation of the Stokes problem which is well-posed (Bernardi-Girault-Maday [4]). We propose in the sequel to study this well-posed formulation, in order to obtain a simple numerical scheme, using harmonic functions (obtained by integral representation), and convergent with an order at least one for the natural norm of the vorticity.

2. Variational formulation

We shall consider the space $L^2(\Omega)$ of all classes of square integrable functions and the space $H^1(\Omega)$ of functions in $L^2(\Omega)$ whose gradient is in $L^2(\Omega)$. The space $H^1_0(\Omega)$ is composed of functions of $H^1(\Omega)$ whose trace on the boundary of Ω is null. In the following, $(\bullet,\bullet)$ denotes the standard inner product in $L^2(\Omega)$ and $<\bullet,\bullet>_{-1,1}$ the duality product between $H^1_0(\Omega)$ and its topological dual space $H^{-1}(\Omega)$. We finally introduce the space $M(\Omega)=\{\varphi\in L^2(\Omega),\Delta\varphi\in H^{-1}(\Omega)\}$ and the well-posed weak formulation of the stationary and homogeneous Stokes problem [4]. It consists in finding the vorticity field ω in $M(\Omega)$ and the stream function ψ in $H^1_0(\Omega)$ such that :

$$\left\{\begin{array}{lcll}(\omega,\varphi)+<\Delta\varphi,\psi>_{-1,1} & = & 0 & \forall\varphi\in M(\Omega)\\ <-\Delta\omega,\xi>_{-1,1} & = & (\mathbf{f},\mathbf{rot}\ \xi) & \forall\xi\in H^1_0(\Omega)\end{array}\right.$$

The vectorial field $\mathbf{f}$ of external forces is given in $(L^2(\Omega))^2$.

The space $M(\Omega)$ can be decomposed as follows : $M(\Omega)=H^1_0(\Omega)\oplus\mathcal{H}(\Omega)$ where $\mathcal{H}(\Omega)$ is composed of harmonic functions of $L^2(\Omega)$. Using the above-mentioned decomposition of $M(\Omega)$, the previous problem is equivalent to find $\omega=\omega^0+\omega^\Delta$ in $H^1_0(\Omega)\oplus\mathcal{H}(\Omega)$ such that :

$$\left\{\begin{array}{lclcll}\text{find } \omega^0\in H^1_0(\Omega): & & & & & \\ \langle-\Delta\omega^0,\xi\rangle_{-1,1} & = & (\nabla\omega^0,\nabla\xi) & = & (\mathbf{f},\mathbf{rot}\ \xi) & \forall\xi\in H^1_0(\Omega).\\ \text{find } \omega^\Delta\in\mathcal{H}(\Omega): & & & & & \\ & & (\omega^\Delta,\varphi) & = & -(\omega^0,\varphi) & \forall\varphi\in\mathcal{H}(\Omega).\\ \text{find } \psi\in H^1_0(\Omega): & & & & & \\ \langle-\Delta\chi,\psi\rangle_{-1,1} & = & (\nabla\psi,\nabla\chi) & = & (\omega^0+\omega^\Delta,\chi) & \forall\chi\in H^1_0(\Omega).\end{array}\right.$$

So we propose to discretize directly the two spaces $H^1_0(\Omega)$ and $\mathcal{H}(\Omega)$ obtained in the decomposition of $M(\Omega)$.

3. Numerical method

Numerically, the space $H_0^1(\Omega)$ is approached by the space of continuous functions on $\overline{\Omega}$, null on the boundary, polynomials of degree one in each triangle. Space $\mathcal{H}(\Omega)$ is approached by the finite-dimensional space $\mathcal{H}_\mathcal{T}$, composed of harmonic functions whose trace on the boundary is the characteristic function of an edge of a triangle. Its dimension is then equal to the number N of edges of the mesh that are on the boundary. It means that $\mathcal{H}_\mathcal{T}$ is equal to Span $\{\varphi_i,\ i = 1,\ N\}$ with φ_i such that :

$$\left\{ \begin{array}{rcl} \Delta\varphi_i & = & 0 \quad \text{in } \Omega \\ \varphi_i & = & \chi_{ij} \text{ on } \partial\Omega, \end{array} \right.$$

and χ_{ij} the function which is null on all edges of the boundary except the i-th denoted by Γ_i. To obtain the values of the harmonic function φ_i inside Ω, we use the simple layer potential representation (see Nédélec [9]) : $\varphi_i(x) = \dfrac{1}{2\pi}\displaystyle\int_{\Gamma_i} \ln|x-y|\,\mathrm{d}\gamma_y,\ \ \forall x \in \overline{\Omega}$. So, we have just replaced the first degree polynomial functions associated with the boundary degrees of freedom by the same number of harmonic functions.

We prove that using harmonic functions in the numerical scheme give stability, a convergence rate of order one in $M-$norm for the vorticity and in H^1-norm for the stream function (even on unstructured meshes) (see [5]). Moreover, if the domain is convex, the convergence rate in quadratic norm is of order 3/2 and, in more regular cases, of order 2 for both variables [1].

4. Numerical results

We present here one numerical experiment, which has been performed on a unit square with an analytical solution (test of Bercovier-Engelman [8]) :

$$f_1(x,y) = 256\left(x^2(x-1)^2(12y-6) + y(y-1)(2y-1)(12x^2-12x+2)\right)$$
$$f_2(x,y) = -\,f_1(y,x)$$

for which we have :

$$\psi(x,y) = -128(y^2(y-1)^2x^2(x-1)^2)$$
$$\omega(x,y) = 256(y^2(y-1)^2(6x^2-6x+1) + x^2(x-1)^2(6y^2-6y+1)).$$

Other numerical experiments are given in [2].

On the two following figures, we compare the numerical solutions obtained by the classical method (only first degree polynomial functions for the vorticity) with the method we propose (using harmonic functions on the boundary for the vorticity). We can observe on Figure 1 that the boundary vorticity is well computed and that the numerical oscillations, observed with the classical method, have disappeared. Convergence

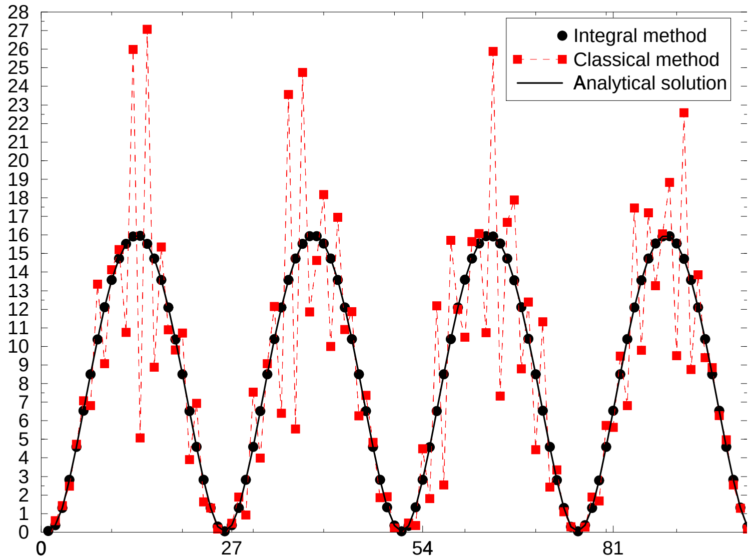

Figure 1. Vorticity on the boundary (Bercovier-Engelman test) - Horizontal axis : curvilinear abscissa along the boundary, vertical axis : value of the vorticity, theoretical extremum=+16 .

results are in agreement with the theory and exhibit the improvement of the computation of the vorticity (see Figure 2).

We are working now on the application of this numerical method to a more general formulation, including the three fields of vorticity, velocity and pressure [6]. This formulation, which is an extension of the stream function-vorticity one presents the same instability on unstructured meshes. The first numerical results, using harmonic vorticity on the boundary, are in complete agreement with the ones presented here.

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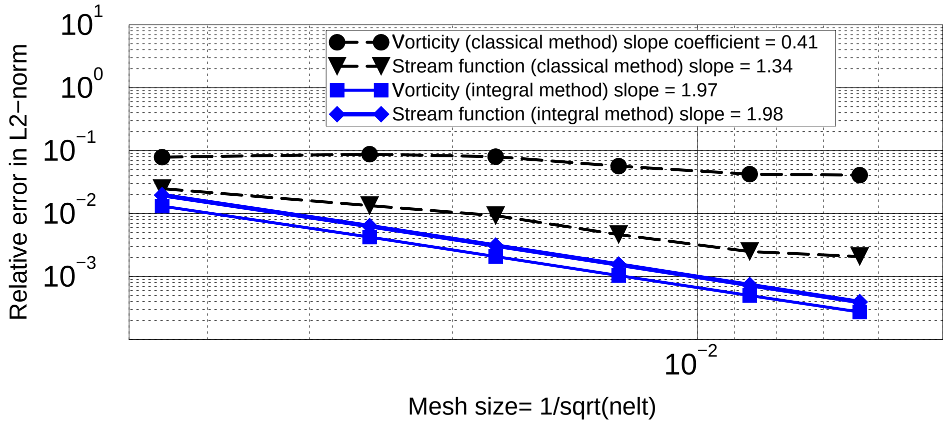

Figure 2. Convergence orders in quadratic norms (Bercovier-Engelman test).

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