Equivalent equations of lattice Boltzmann schemes

François Dubois*†

Summer School on the Lattice Boltzmann Method
Beijing Computational Science Research Center
Tsinghua University, Beijing, China
11 July 2012

* Conservatoire National des Arts et Métiers, LMSSC, Paris
† Department of Mathematics, University Paris-Sud, Orsay, France

□ LBE 2° ordre ; cas général fluide. (10/7/12 Pekin ①

$f_j(x, t+\Delta t) = f_j^*(x - v_j \Delta t, t)$; $m_k = \sum_j M_{kj} f_j$

$m_k(x, t+\Delta t) = \sum_j M_{kj} f_j^*(x - v_j \Delta t, t)$; $k > d$: $m_k^* = m_k + \underbrace{s_k}_{>0}(m_k^{eq} - m_k)$.

• ordre zéro. $m_k + O(\Delta) = m_k^* + O(\Delta) \quad \forall k$.

OK pour les conservés : $k=0 \leftrightarrow \rho$; $k = \alpha \leftrightarrow J_\alpha$.

$k > d$. $m_k^* - m_k = O(\Delta) \rightarrow s_k(m_k^{eq} - m_k) = O(\Delta)$. $m_k = m_k^{eq} + O(\Delta)$
$m_k^* = m_k^{eq} + O(\Delta)$.

• ordre un. $\sum_\alpha \cdots$

$m_k + \Delta t\, \partial_t m_k + O(\Delta^2) = \sum_{j,\ell} M_{kj} M^{-1}_{j\ell}\left[m_\ell^* - v_j^\alpha \partial_\alpha m_\ell^* \Delta t + O(\Delta^2)\right]$

$= m_k^* - \sum_{\ell, \alpha}\left(\sum_j M_{kj} v_j^\alpha M^{-1}_{j\ell}\right) \Delta t\, \partial_\alpha m_\ell^* + O(\Delta^2)$.

hyp : $M_{0j} \equiv 1$; $M_{\alpha j} \equiv v_j^\alpha$, $1 \leq \alpha \leq d$.

(def) Tensor of momentum velocity.

$\Lambda_{pq}^r = \sum_j M_{pj} \cdot M_{qj} \cdot M^{-1}_{jr}$. Alors $\Lambda_{0q}^r = \delta_{qr}$.

$m_k + \Delta t\, \partial_t m_k + O(\Delta^2) = m_k^* - \Delta t\, \Lambda_{k\alpha}^\ell \partial_\alpha m_\ell^* + O(\Delta^2)$ (très gal).

• $k = 0$ (masse) $m_0 \equiv m_0^* = \rho$. $\Lambda_{0\alpha}^\ell = \delta_{\alpha\ell}$

après division par Δt : $\partial_t \rho + O(\Delta) = -\partial_\alpha J_\alpha^* + O(\Delta)$.

mais $J_\alpha^* \equiv J_\alpha$ qui est conservé $\rightarrow \partial_t \rho + \partial_\alpha J_\alpha = O(\Delta)$.

Cours Tsinghua Univ ; F. DUBOIS.

(2)

- $k = \alpha$. Attention au changement de variable muette !

$$\partial_t J_\alpha + \Lambda^{\ell}_{\alpha\beta}\, \partial_\beta m^*_\ell = O(\Delta). \qquad m^*_\ell = m^{eq}_\ell + O(\Delta)\ldots$$

$$\partial_t J_\alpha + \Lambda^{\ell}_{\alpha\beta}\, \partial_\beta m^{eq}_\ell = O(\Delta). \qquad \partial_\beta m^*_\ell = \partial_\beta m^{eq}_\ell + O(\Delta) \text{ aussi !}$$

(on) $\Lambda^{\ell}_{\alpha\beta}\, m^{eq}_\ell = \sum_j M_{\alpha j} M_{\beta j} M^{-1}_{j\ell} \sum_p M_{\ell p} f^{eq}_p = \sum_j M_{\alpha j} M_{\beta j} f^{eq}_j \ldots$

{le truc technique à ne pas oublier !}

- $k > d$.

$$m_k + \Delta t\, \partial_t m^{eq}_k + O(\Delta^2) = m^*_k - \Delta t\, \Lambda^{\ell}_{k\alpha} (\partial_\alpha m^{eq}_\ell) + O(\Delta^2).$$

(def) Defect of equilibrium : $\theta_k \equiv \partial_t m^{eq}_k + \Lambda^{\ell}_{k\alpha}\, \partial_\alpha m^{eq}_\ell$.

$\theta_k = O(1)$ si $k > d$; $\theta_k = O(\Delta)$ si $k \leq d$. (éqs du 1^o ordre).

alors $m^*_k - m_k = s_k (m^{eq}_k - m_k) = \Delta t\, \theta_k + O(\Delta^2)$.

$$m_k = m^{eq}_k - \frac{\Delta t}{s_k}\, \theta_k + O(\Delta^2).$$

$$m^*_k = m^{eq}_k + (1 - s_k)(m_k - m^{eq}_k) = m^{eq}_k - \Delta t\, \frac{1 - s_k}{s_k}\, \theta_k + O(\Delta^2).$$

(def) Hénon's parameter ; $\sigma_k \equiv \frac{1}{s_k} - \frac{1}{2}$.

alors $1 - \frac{1}{s_k} = 1 - (\sigma_k + \frac{1}{2}) = \frac{1}{2} - \sigma_k$; $\boxed{m^*_k = m^{eq}_k + \Delta t \left(\frac{1}{2} - \sigma_k\right) \theta_k + O(\Delta^2)}$

- ordre deux.

$$m_k(x, t + \Delta t) = \sum_{j,\ell} M_{kj} M^{-1}_{j\ell}\, m^*_\ell (x - v_j \Delta t, t).$$

$$m_k + \Delta t\, \partial_t m_k + \frac{\Delta t^2}{2} \partial_t^2 m_k + O(\Delta^3) = \sum_{j\ell} M_{kj} M^{-1}_{j\ell} \Big[m^*_\ell - v^\alpha_j \Delta t\, \partial_\alpha m^*_\ell + v^\alpha_j v^\beta_j \frac{\Delta t^2}{2}\, \partial^2_{\alpha\beta} m^*_\ell \Big] + O(\Delta^3)$$

③

o. $m_k + \Delta t\, \partial_t m_k + \frac{\Delta t^2}{2} \partial_t^2 m_k = m_k^* - \Delta t\, \Lambda_{k\alpha}^{\ell}\, \partial_\alpha m_\ell^*$
$+ \frac{\Delta t^2}{2} \left[\sum_j M_{kj} \cdot \Pi_{\alpha j} \Pi_{\beta j} \Pi^{-1}_{j\ell} \right] \partial^2_{\alpha\beta} m_\ell^* + O(\Delta^3)$

o $k=0$ $\quad m_0 = m_0^* = \rho$; $\Lambda_{0\alpha}^{\ell} = \delta_{\alpha\ell}$; $\sum_j \cdots = \Lambda_{\alpha\beta}^{\ell} \cdot M_{0j} \equiv 1$

$\partial_t \rho + \frac{\Delta t}{2} \partial_t^2 \rho + O(\Delta^2) = -\partial_\alpha J_\alpha + \frac{\Delta t}{2} \Lambda_{\alpha\beta}^{\ell}\, \partial^2_{\alpha\beta} m_\ell^{eq} + O(\Delta^2)$.

$\partial_t^2 \rho = \partial_t(-\partial_\alpha J_\alpha) + O(\Delta) = -\partial_\alpha(\partial_t j_\alpha) + O(\Delta) = -\partial_\alpha(-\Lambda_{\alpha\beta}^{\ell}\, \partial_\beta m_\ell^{eq}) + O(\Delta)$.

compensation des deux termes $\to \partial_t \rho + \partial_\alpha J_\alpha = O(\Delta)$.

o $k = \alpha \leqslant d$. $m_\alpha = m_\alpha^* = J_\alpha$. $\partial_\beta m_\ell^* = \partial_\beta m_\ell^{eq} + \Delta t \left(\frac{1}{2} - \sigma_\ell\right)(\partial_\beta \theta_\ell) + O(\Delta^2)$.

le point technique que j'avais négligé pour le D1Q3 !

$\partial_t^2 J_\alpha = \partial_t(\partial_t J_\alpha) = \partial_t(-\Lambda_{\alpha\beta}^{\ell}\, \partial_\beta m_\ell^{eq}) + O(\Delta) = -\Lambda_{\alpha\beta}^{\ell}\, \partial_\beta(\partial_t m_\ell^{eq}) + O(\Delta)$

$= -\Lambda_{\alpha\beta}^{\ell}\, \partial_\beta [\theta_\ell - \Lambda_{\ell\gamma}^{p}\, \partial_\gamma m_p^{eq}] = -\Lambda_{\alpha\beta}^{\ell}(\partial_\beta \theta_\ell) + \Lambda_{\alpha\beta}^{\ell} \Lambda_{\ell\gamma}^{p}\, \partial^2_{\beta\gamma} m_p^{eq} + O(\Delta)$.

$\sum_\ell \Lambda_{\alpha\beta}^{\ell} \Lambda_{\ell\gamma}^{p} = \sum_j \Pi_{\alpha j} \Pi_{\beta j} \Pi^{-1}_{j\ell} \sum_q \Pi_{\ell q} \Pi_{\gamma q} \Pi^{-1}_{qp} = \sum_j \Pi_{\alpha j} \Pi_{\beta j} \Pi_{\gamma j} \Pi^{-1}_{jp}$.

$\partial_t^2 J_\alpha = -\Lambda_{\alpha\beta}^{\ell}(\partial_\beta \theta_\ell) + \left(\sum_j \Pi\Pi\Pi\Pi^{-1}\right) \partial^2_{\beta\gamma} m_\ell^{eq} + O(\Delta)$.

$\partial_t J_\alpha + \frac{\Delta t}{2} \left\{ -\Lambda_{\alpha\beta}^{\ell}(\partial_\beta \theta_\ell) + (\Pi\Pi\Pi\Pi^{-1})\, \partial^2_{\beta\gamma} m_\ell^{eq} \right\} + O(\Delta^2) =$

$-\Lambda_{\alpha\beta}^{\ell} \left[\partial_\beta m_\ell^{eq} + \Delta t \left(\frac{1}{2} - \sigma_\ell\right)(\partial_\beta \theta_\ell)\right] + \frac{\Delta t}{2} (\Pi\Pi\Pi\Pi^{-1})\, \partial^2_{\beta\gamma} m_\ell^{eq} + O(\Delta^2)$.

$\partial_t J_\alpha + \Lambda_{\alpha\beta}^{\ell}\, \partial_\beta m_\ell^{eq} = \Delta t\, \Lambda_{\alpha\beta}^{\ell} (\partial_\beta \theta_\ell) \left[\frac{1}{2} - \left(\frac{1}{2} - \sigma_\ell\right)\right] = \sigma_\ell \Delta t\, \Lambda_{\alpha\beta}^{\ell} (\partial_\beta \theta_\ell) + O(\Delta^2)$.