High order equivalent equations of lattice Boltzmann scheme

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We have used the dispersion equation of a Lattice Boltzmann scheme [LL00] to derive equivalent equations of such methods for several classical schemes (D1Q3 for thermal problem, D1Q3 for acoustic model, d2Q5 for thermal problem). We have compared the results with the direct Taylor apprach suggested by one of us [Du07]. We observe a complete agreement between the two algorithms. Note that intensive use of formal calculus has been necessary for this study.
We study in a first part the so-called D1Q3 model. It is defined by the following matrix M :

$$M \,=\, \begin{pmatrix} 1 & 1 & 1 \\ -\lambda & 0 & \lambda \\ \lambda^2/2 & 0 & \lambda^2/2 \end{pmatrix}$$

For example, the equivalent equations of the acoustic D1Q3 model are the following ones at the order five (with $\lambda \equiv \Delta x / \Delta t$, $c_s^2 = \alpha\, \lambda^2$ and $\sigma \equiv 1/s - 1/2$) :

$$\frac{\partial \rho}{\partial t} \,+\, \frac{\partial q}{\partial x} \,-\, \frac{\lambda^2\, \Delta t^2}{12}\,(1-\alpha)\,\frac{\partial^3 q}{\partial x^3} \,-\, \frac{\lambda^4\, \Delta t^3}{12}\,\sigma\,\alpha\,(1-\alpha)\,\frac{\partial^4 \rho}{\partial x^4}$$

$$-\,\frac{\lambda^4\, \Delta t^4}{120}\,(1-\alpha)\,(20\alpha\sigma^2 - 10\sigma^2 - \alpha - 1)\,\frac{\partial^5 q}{\partial x^5} \,=\, 0$$

$$\frac{\partial q}{\partial t} \,+\, \alpha\,\lambda^2\,\frac{\partial \rho}{\partial x} \,-\, \sigma\,\lambda^2\,\Delta t\,(1-\alpha)\,\frac{\partial^2 q}{\partial x^2} \,-\, \frac{\lambda^4\, \Delta t^2}{6}\,\alpha\,(1-\alpha)\,(6\sigma^2 - 1)\,\frac{\partial^3 \rho}{\partial x^3}$$

$$-\,\frac{\lambda^4\, \Delta t^3}{12}\,\sigma\,(1-\alpha)\,(24\alpha\,\sigma^2 - 12\sigma^2 - 4\alpha + 1)\,\frac{\partial^4 q}{\partial x^4}$$

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$$- \frac{\lambda^6 \, \Delta t^4}{120} \, \alpha \, (1-\alpha) \, (360\alpha\sigma^4 - 90\alpha\sigma^2 - 240\sigma^4 + 50\sigma^2 + 4\alpha - 1) \, \frac{\partial^5 \rho}{\partial x^5} \, = \, 0 \, .$$

For the advection-diffusion equation with the d1q3 model with

$$m_2 \, = \, V \, \rho \, ,$$

and

$$m_3 \, = \, \alpha \, \frac{\lambda^2}{2} \, \rho \, ,$$

a relaxation of the second and third momentum components associated with the parameters τ and σ :

$$\tau \, = \, \frac{1}{s_2} - \frac{1}{2} \, ,$$

$$\sigma \, = \, \frac{1}{s_3} - \frac{1}{2} \, ,$$

we obtain the following equivalent equation at the third order :

$$\frac{\partial \rho}{\partial t} \, + \, V \, \frac{\partial \rho}{\partial x} \, - \, \tau \, \Delta t \, (\alpha \, \lambda^2 - V^2) \, \frac{\partial^2 \rho}{\partial x^2}$$

$$- \frac{V \, \Delta t^2}{12} \Big(2(1 - 12 \, \tau^2 \, V^2) + (24\alpha \, \tau^2 + 12\sigma\alpha\tau - 12\sigma\tau + 1 - 3\alpha)\lambda^2 \Big) \, \frac{\partial^3 \rho}{\partial x^3} \, = \, 0 \, .$$

For the acoustic D2Q9 system, we obtain at the , order of accuracy :

$$\frac{\partial \rho}{\partial t} \, + \, \frac{\partial q_x}{\partial x} \, + \, \frac{\partial q_y}{\partial y} \, - \, \frac{\lambda^2 \, \Delta t^2}{18} \, \Delta(\mathrm{div} q) \, - \, \frac{\lambda^4 \, \Delta t^3}{108} \, (\sigma_3 + \sigma_8) \, \Delta^2 \rho \, = \, 0 \, .$$

$$\frac{\partial q_x}{\partial t} + \frac{\lambda^2}{3} \, \frac{\partial \rho}{\partial x} - \frac{\lambda^2 \, \Delta t}{3} \Big(\sigma_3 \, \frac{\partial}{\partial x}(\mathrm{div} q) + \sigma_8 \, \Delta q_x \Big) - \frac{\lambda^4 \, \Delta t^2}{27} \, (3\sigma_3^2 + 3\sigma_8^2 - 1) \, \frac{\partial}{\partial x}(\Delta \rho) \, = \, 0 \, .$$

$$\frac{\partial q_y}{\partial t} + \frac{\lambda^2}{3} \, \frac{\partial \rho}{\partial y} - \frac{\lambda^2 \, \Delta t}{3} \Big(\sigma_3 \, \frac{\partial}{\partial y}(\mathrm{div} q) + \sigma_8 \, \Delta q_y \Big) - \frac{\lambda^4 \, \Delta t^2}{27} \, (3\sigma_3^2 + 3\sigma_8^2 - 1) \, \frac{\partial}{\partial y}(\Delta \rho) \, = \, 0 \, .$$

[*Du*07] F. Dubois. "Equivalent partial differential equations of a lattice Boltzmann scheme", *Computers and Mathematics with Applications*, ICMMES-2005, to appear, 2007.
[*LL*00] P. Lallemand, L.S. Luo. "Theory of the lattice Boltzmann method: Dispersion, dissipation, isotropy, Galilean invariance, and stability", *Physical Review E*, vol. **61**, n° 6, p. 6546-6562, june 2000.