General third order Chapman-Enskog expansion of lattice Boltzmann schemes

François Dubois[ab]

[a] *Laboratoire de Mathématiques d'Orsay, Université Paris-Sud, CNRS, Université Paris-Saclay, Orsay, France.*

[b] *Conservatoire National des Arts et Métiers, LMSSC laboratory, F-75003 Paris, France.*

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The lattice Boltzmann scheme in his actual form has been developed with the contributions of Lallemand, Succi, d'Humières, Luo [1, 2, 3, 4] and many others. In order to derive the equivalent partial differential equations, a classical of the Chapman Enskog expansion is popular in the lattive Boltzmann community (see *e.g.* [4]). A main drawback of this approach is the fact that multiscale expansions are used without a clear mathematical signification of the various variables and functions. Independently of this framework, we have proposed in [5, 6] the Taylor expansion method to obtain formally equivalent partial differential equations. The infinitesimal variable is simply the time step (proportional to the space step with the acoustic scaling). This approach has been experimentaly validated in various contributions [7, 8]. A third order extension for fluid flow has been proposed in [9] and an efficient implementation up to fourth order accuracy is presented in [10].

In this contribution, we consider a regular lattice $\mathcal{L}$ composed by vertices x separated by distances that are simple expressions of the space step Δx. A discrete time t is supposed to be an integer multiple of a time step $\Delta t > 0$. A very general lattice Boltzmann scheme with q discrete velocities of the form

$$f_j(x,\, t+\Delta t) = f_j^*(x - v_j\, \Delta t,\, t)\,, \qquad 0 \leq j < q\,.$$

The distribution f^* after relaxation is defined with moments m such that

$$m_k = \sum_j M_{k\ell}\, f_j\,.$$

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The d'Humières matrix [3] M is invertible and we decompose the moments in the following way:

$$m \equiv \begin{pmatrix} W \\ Y \end{pmatrix} .$$

The conserved variables W are not modified after relaxation: $W^* = W$. The microscopic variables Y are changed in a nonlinear way by the relaxation process:

$$Y^* = Y + S\,(\Phi(W) - Y)\,.$$

The matrix S is invertible, and ofter chosen as diagonal. It is supposed to be fixed in the asymptotic process presented hereafter. The equilibrium values $Y^{\mathrm{eq}} = \Phi(W)$ are given smooth functions of the conserved variables. When Y^* is evaluated, we have simply

$$f^* = M^{-1}\, m^*\,.$$

We introduce the momentum-velocity operator matrix Λ defined by the relation

$$\Lambda_{k\ell} = \sum_{j,\,\alpha} M_{kj}\; v_j^\alpha\; (M^{-1})_{j\ell}\, \partial_\alpha\,, \quad 0 \le k,\, \ell < q\,.$$

It is nothing else than the advection operator seen in the space of moments. Then we have an exponential form of the discrete iteration of the lattice Boltzmann scheme:

$$m(x, t + \Delta t) = \exp(-\Delta t\, \Lambda)\; m^*(x,\, t)\,.$$

With this general framework, we follow in this contribution the Chapman-Enskog formalism proposed by Chen–Doolen [11] and Qian–Zhou [12]. We suppose that $\Delta t \equiv \varepsilon$ is an infinitesimal parameter and we expand the nonconserved moments as differential nonlinear function of the conserved variables:

$$Y = \Phi(W) + \varepsilon\, \Psi_1(W) + \varepsilon^2\, \Psi_2(W) + \mathrm{O}(\varepsilon^3)\,.$$

Then we suppose that a multi-scale approach is present for the time dynamics:

$$\partial_t = \partial_{t_1} + \varepsilon\, \partial_{t_2} + \varepsilon^2\, \partial_{t_3} + \mathrm{O}(\varepsilon^3)\,.$$

Then we prove that the conserved quantities W follow the following multi-time dynamics :

$$\partial_{t_1} W + \Gamma_1(W) = 0\,, \quad \partial_{t_2} W + \Gamma_2(W) = 0\,, \quad \partial_{t_3} W + \Gamma_3(W) = 0\,.$$

The differential operators $\Gamma_1(W)$, $\Psi_1(W)$, $\Gamma_2(W)$, $\Psi_2(W)$ and $\Gamma_3(W)$ of this expansion are recursively determined as a function of the data v_j, M, $\Phi(W)$ and S. We compare our result with the particular third order expansion proposed in [9] and the linear approach presented in [10]. The previous operators $\Gamma_j(W)$ and $\Psi_i(W)$ are of order j and are exactly the ones derived in our fourth order expansion of lattice Boltzmann schemes with the Taylor expansion method [13].

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