Pseudo pressure formulation: steady state and saturated zone

- **Conserved variable: extended water content**
  $\theta = \theta_s + \gamma(h - h_s)$, $\rho = \theta$, $P(\theta) = h$, $h \geq h_s$.
- **The linear extrapolation is regular if** $\gamma = \partial_h \theta(h_s) \neq 0$
- **The Richards' equation with the pseudo-compressible error** $\gamma \partial_t h$:
  $\gamma \partial_t h + \nabla \cdot \vec{j} = \nabla \cdot k(h) \mathsf{K}^a \cdot \nabla h$
  $\gamma \partial_t h \to 0$ when $t \to \infty$
- **Sub-steps are run in saturated points to reach the local steady state**: $\gamma \partial_t h \approx 0$

Semi-analytical 2D “multi-layer” method of M. Bakker & K. Hemker, based on exact Laplace solution along x

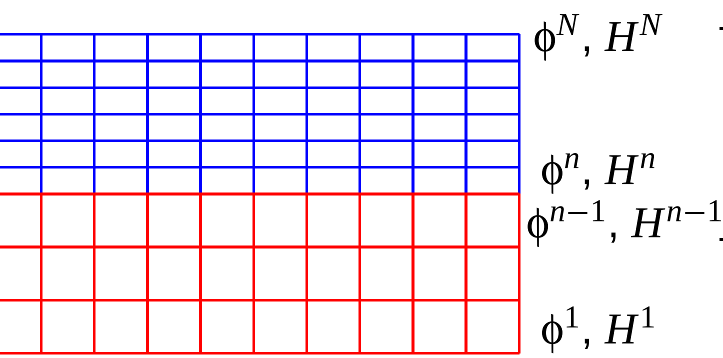
- **Solution:** $\vec{\phi} = \mathcal{E}\mathcal{H}\mathcal{E}^{-1}\vec{g}, \vec{g} = \left\{ \sigma \frac{K_{xy}^n}{K_{xx}^n} \right\}$

  $\mathcal{H} = \text{diag}\{x, \frac{\sinh(x/\sqrt{w_2})}{\sqrt{w_2}\cosh(X/\sqrt{w_2})}, \dots, \frac{\sinh(x/\sqrt{w_N})}{\sqrt{w_N}\cosh(X/\sqrt{w_N})}\}$, $\mathcal{E}$ **and** $\vec{w}$ **are eigenvectors/eigenvalues of tridiagonal matrix** $\mathcal{B}$

- **1D discretization along z:** $\frac{\partial^2 \phi}{\partial x^2} = \mathcal{B}\phi$,

  $$H^n K_{xx}^n \frac{\partial^2 \phi^n}{\partial x^2} = \frac{\phi^n - \phi^{n-1}}{c_n} + \frac{\phi^n - \phi^{n+1}}{c_{n+1}}$$

- **Assumptions: vertical approximation for the hydraulic resistance** $c_n = \frac{1}{2}\left(\frac{H^{n-1}}{K_{zz}^{n-1}} + \frac{H^n}{K_{zz}^n}\right)$



Compressible Navier-Stokes type equations.

- **MRT/TRT with forcing S_q^-:**

  $$f_q(\vec{r}+\vec{c}_q, t+1) = f_q(\vec{r}, t) + p_q + m_q + S_q^-.$$

- **N-S-E**: $\partial_t \rho + \nabla \cdot \vec{j} = O(\varepsilon^3)$

  $$\partial_t \vec{j} + \nabla \cdot (\frac{\vec{j} \otimes \vec{j}}{\rho}) =$$
  $$-\nabla P + \nabla \cdot (\nu \nabla \vec{j}) + \nabla(\nu_\xi \nabla \cdot \vec{j}) + \vec{F} + O(\varepsilon^3).$$

- **Force**: $\vec{F} = \sum_{q=1}^{Q-1} S_q^- \vec{c}_q$, $\vec{j} = \sum_{q=1}^{Q-1} f_q \vec{c}_q + \frac{1}{2}\vec{F}$.
- **Stress tensor**:

  $$\nu(\partial_\alpha j_\beta + \partial_\beta j_\alpha) = \Lambda^+ \sum_{q=1}^{Q-1} p_q c_{q\alpha} c_{q\beta}$$

- **TRT**:$\nu = \nu_\xi = \frac{1}{3}\Lambda^+$, **MRT**:$\nu_\xi \neq \nu$.

Implicit interface continuity conditions (1993,2005):

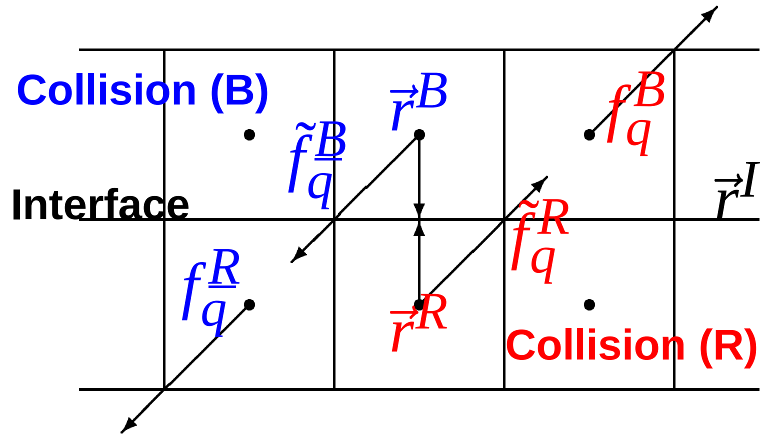
(1) **Symmetric equilibrium part:** $S_q^R(\vec{r}^R) = S_{\bar{q}}^B(\vec{r}^B)$

with $S_q = e_q^+ + \frac{1}{2}m_q - \Lambda^+ p_q + \frac{1}{2}\mathcal{S}_q{}^+$

(2) **Antisymmetric equilibrium part:** $G_q^R(\vec{r}^R) = -G_{\bar{q}}^B(\vec{r}^B)$

with $G_q = e_q^- + \frac{1}{2}p_q - \Lambda_q^- m_q + \frac{1}{2}\mathcal{S}_q{}^-$

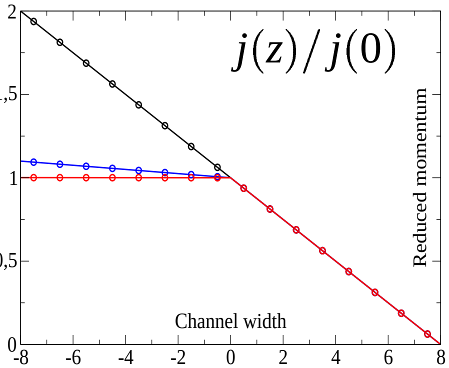
$$\begin{cases} f_q^B(\vec{r}^B, t+1) & = & \tilde{f}_q^R(\vec{r}^R, t) \\ f_{\bar{q}}^R(\vec{r}^R, t+1) & = & \tilde{f}_{\bar{q}}^B(\vec{r}^B, t) \end{cases}$$



N-S-E: interface continuity conditions

- **Two phase Couette flow**:
  $\partial_{zz} j = 0,\ j(-H) = 0,\ j(+H) = 1$
  $j^B(0^+) = j^R(0^-)$
  $\nu^B \partial_z j^B(0^+) = \nu^R \partial_z j^R(0^-)$

- **Exact for any viscosity ratio when the interface is midway two lattice rows**



$\nu^R = \nu^B$, $\nu^R/\nu^B = 10$, $\nu^R/\nu^B = 10^3$

(1) Symmetric equilibrium part: $e_q^{+R}(\vec{r}^I) = e_q^{+B}(\vec{r}^I) + O(\varepsilon^3)$
Continuity of the pressure P (without surface tension):
$P^R(\vec{r}^I) = P^B(\vec{r}^I) + O(\varepsilon^2)$ **only if** $t_q^R = t_q^B$

(2) **Continuity of the tangential stress components:**

$$\nu^R \mathcal{D}^R_{\alpha z}(\vec{r}^I) = \nu^B \mathcal{D}^B_{\alpha z}(\vec{r}^I)\ ,\ \mathcal{D}_{\alpha z} = (\partial_\alpha j_z + \partial_z j_\alpha)$$

(3) **Continuity of the tangential momentum components:**

$$j^R_\alpha(\vec{r}^I) = j^B_\alpha(\vec{r}^I) + O(\varepsilon^2)\ ,\ \alpha = \{x, y\}$$

Tangential velocity $u_\alpha = j_\alpha/\rho$ **is discontinuous**,
$u^R_\alpha(\vec{r}^I) \neq u^B_\alpha(\vec{r}^I)$ **if density are different** $\rho^R \neq \rho^B$,
$P = c_s^2 \rho$, $c_s^{2R}/c_s^{2B} = \rho^B/\rho^R$

Two phase Poiseuille flow, example

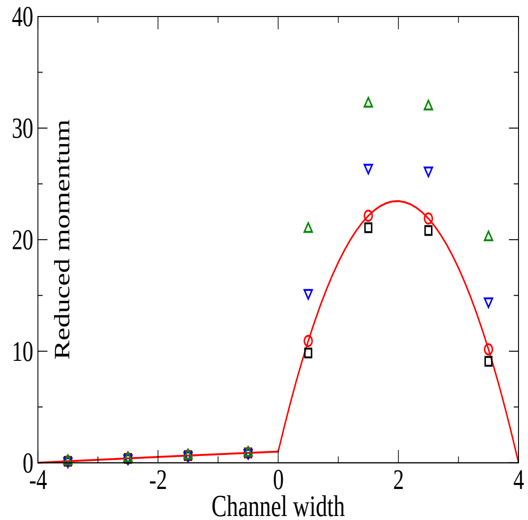
$j(z)/j(0)$ **when** $\nu^R/\nu^B = 10^2$**,** $\partial_{zz} j^B/\partial_{zz} j^R = 10^3$

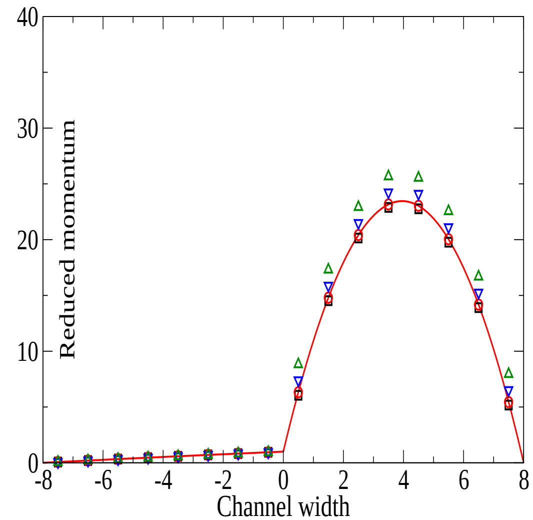
- $\nu^{(i)}\partial_{zz} j^{(i)} = -F^{(i)},\ j^{(i)}(\pm H) = 0$
  $j^B(0^+) = j^R(0^-)$
  $\nu^B\partial_z j^B(0^+) = \nu^R\partial_z j^R(0^-)$

- **No-slip condition via bounce-back**

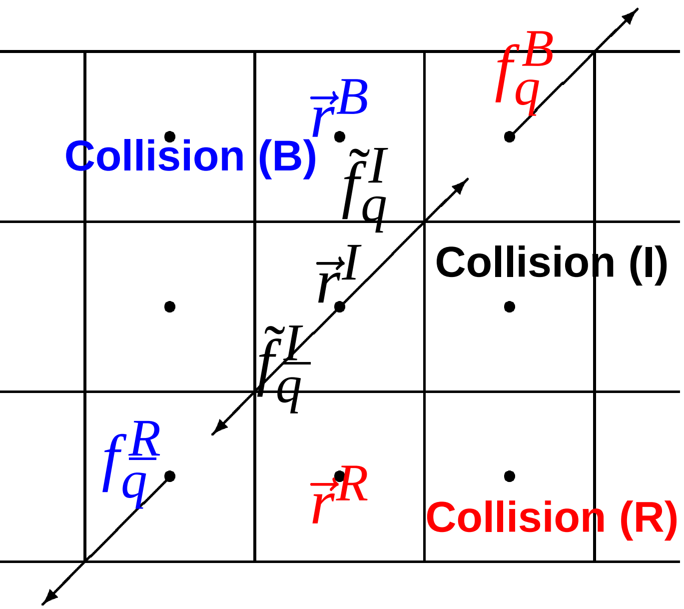
- **Continuity condition** $j^B(0^+) = j^R(0^-)$ **is exact midway the lattice nodes when two curvatures differ** only when $\Lambda^{\pm} = \frac{3}{16}$

$$\Lambda^{\pm} = \frac{3}{64},\ \Lambda^{\pm} = \frac{3}{16},\ \Lambda^{\pm} = \frac{3}{4},\ \Lambda^{\pm} = \frac{3}{2}$$





- **When** $\Lambda^{\pm} \neq \frac{3}{16}$**, "double" error from the bounce-back and interface conditions**



Interface collision operator

(1) **Interface symmetric equilibrium part:**

$$m_q^I + \frac{1}{2}(\mathcal{S}_q^I - \mathcal{S}_{\bar{q}}^I) = (S_q^B - S_{\bar{q}}^R)(\vec{r}^I)$$

with $S_q = e_q^+ + \frac{1}{2}m_q - \Lambda^+ p_q + \frac{1}{2}\mathcal{S}_q$

(2) **Interface anti-symmetric equilibrium part:**

$$p_q^I + \frac{1}{2}(\mathcal{S}_q^I + \mathcal{S}_{\bar{q}}^I) = (G_q^B + G_{\bar{q}}^R)(\vec{r}^I)$$

with $G_q = e_q^- + \frac{1}{2}p_q - \Lambda_q^- m_q + \frac{1}{2}\mathcal{S}_q$

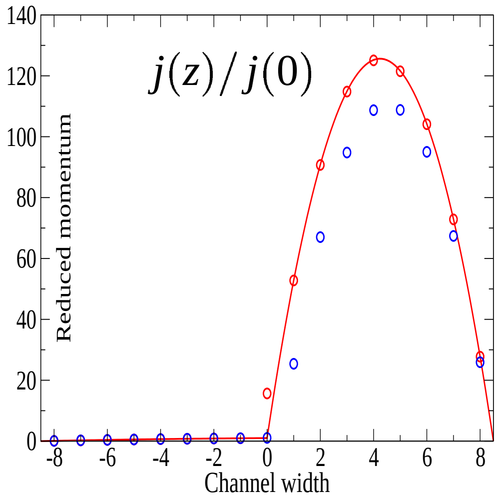
Find interface collision components,

p_q^I (or λ^+, e_q^{+I}), m_q^I (or λ^-, e_q^{-I}) and source $\mathcal{S}_q^I$,

from the prescribed interface conditions

Two phase Poiseuille flow

- **Harmonic mean:** $j^R(0^-) = j^B(0^+)$ exactly if $\Lambda^{\pm I} = \Lambda^{\pm R} = \Lambda^{\pm B} = \frac{3}{8}$
- **Arithmetic mean:** continuity of the stress is not exact at the interface



$\nu^R/\nu^B = 10^2$, $F^R = F^B$

N-S-E: interface collision operator (1994,2005)

– **Prescribed continuity conditions**:

Pressure: $e_q^{+R}(\vec{r}^I) = e_q^{+B}(\vec{r}^I) + O(\varepsilon^3)$

Tangential stress components: $\nu^R \mathcal{D}^R_{\alpha z}(\vec{r}^I) = \nu^B \mathcal{D}^B_{\alpha z}(\vec{r}^I)$

Tangential momentum components: $j^R_\alpha(\vec{r}^I) = j^B_\alpha(\vec{r}^I) + O(\varepsilon^2)$

– **Interface collision components:**

(1) $e_q^{+I} = \frac{1}{2}(e_q^{+R} + e_q^{+B})$, $\Lambda^{+I} = \frac{2\Lambda^{+R}\Lambda^{+B}}{\Lambda^{+R}+\Lambda^{+B}}$

Harmonic mean interface viscosity: $\nu^I = \frac{2\nu^R\nu^B}{\nu^R+\nu^B}$

(2) $m_q^I = \frac{1}{2}(m_q^R + m_q^B)$

(3) Forcing: $\mathcal{S}_q^I = \frac{1}{2}(\mathcal{S}_q^R + \mathcal{S}_q^B)$

(4) **Deficiency**: $j_\alpha(\vec{r}^I) = \frac{1}{2}(j^R_\alpha + j^B_\alpha)(\vec{r}^I) + \Delta j_\alpha$, $\Delta j_\alpha = \frac{1}{4}(\partial_z j^B_\alpha - \partial_z j^R_\alpha)$