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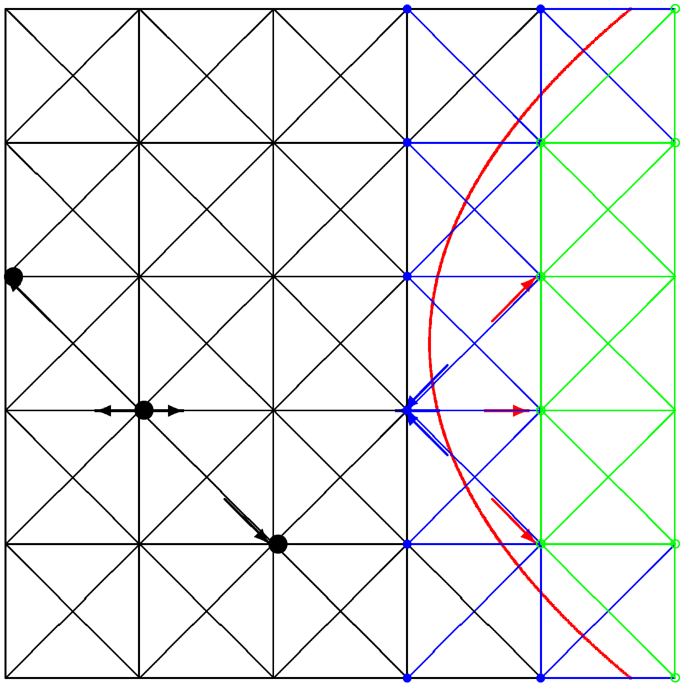
Some elements of Lattice Boltzmann method for hydrodynamic and anisotropic advection-diffusion problems

www.cemagref.fr

Paris, 20 December, 2006



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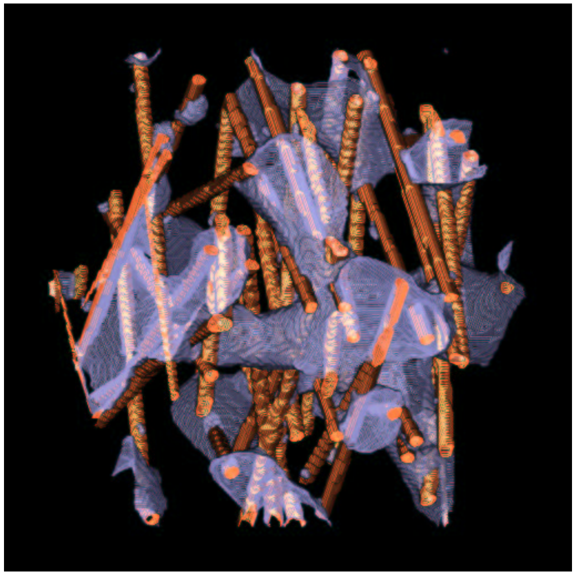
- I. Ginzburg, Equilibrium-type and Link-type Lattice Boltzmann models for generic advection and anisotropic-dispersion equation, Adv Water Resour, 28, 2005

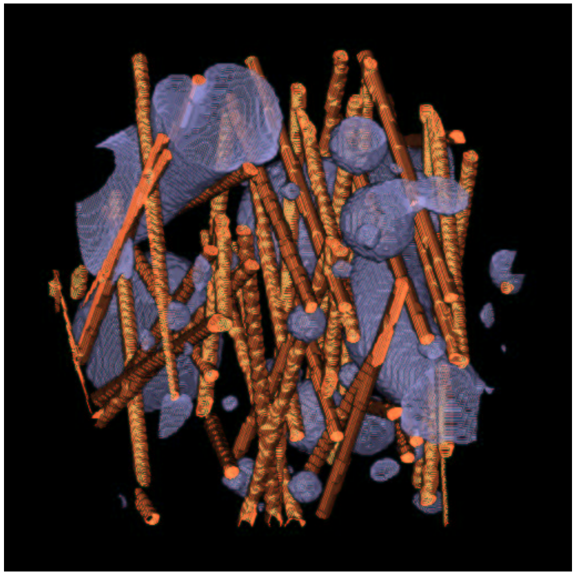
Oil distribution in an anisotropic fibrous material

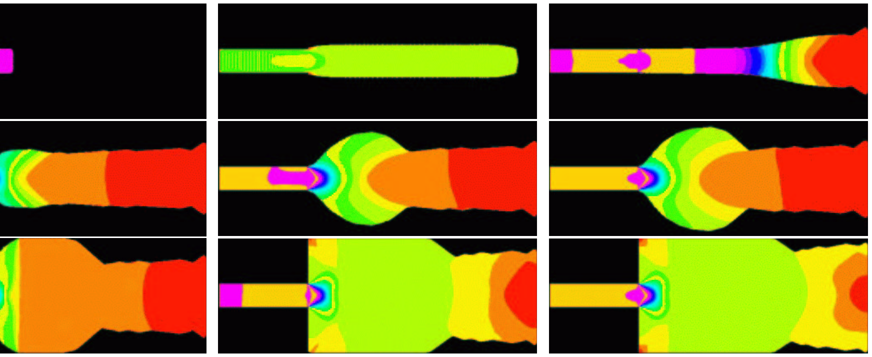
Fleece

oil is wetting

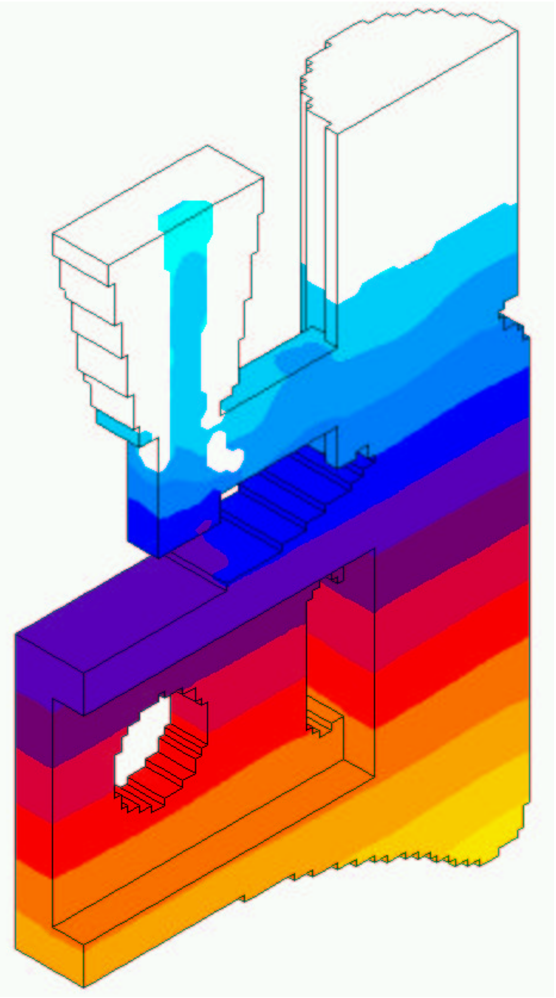


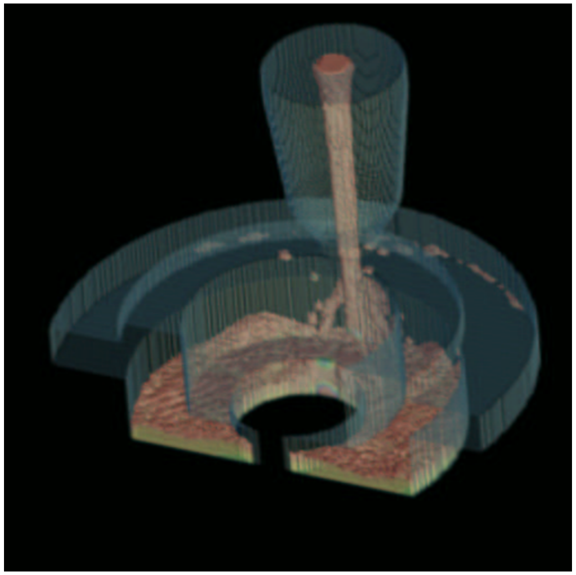
oil is non-wetting

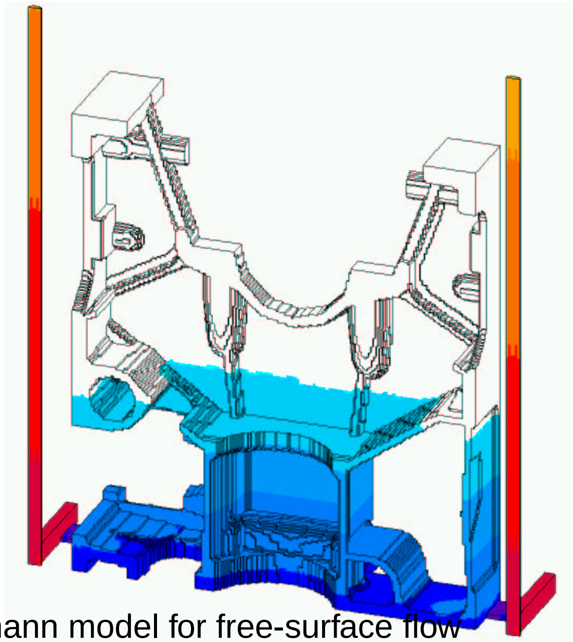




Free surface
Lattice Boltzmann method
for Newtonian and Bingham fluid







I. Ginzburg and K. Steiner, Lattice Bolzmann model for free-surface flow and its application to filling process in casting, **J.Comp.Phys.**, **185**, 2003

Key points

Basic LB method:

(1) Linear collision operators

(2) Chapman-Enskog expansion

(3) Boundary schemes

(4) Finite-difference type recurrence equations

(5) Knudsen layers

(6) Stability conditions

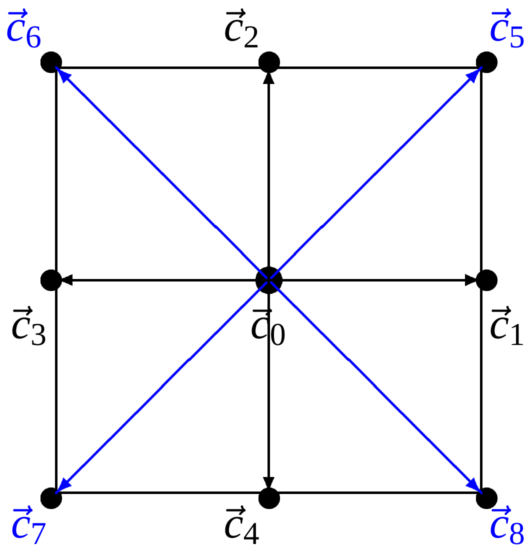
(7) Interface analysis

Applications:

- **Permeability computations in porous media**
- **Richard's equations for variably saturated flow in heterogeneous anisotropic aquifers**

Cubic velocity sets $\{\vec{c}_q, \quad q = 0,\ldots,Q-1\}$
$\vec{c}_q = \{c_{q\alpha}\ ,\ \alpha = 1,\ldots,d\}$

d2Q9



one rest (immobile):
$\vec{c}_0 = \vec{0} = (0,0)$
Q − 1 moving:
$\vec{c}_q = (\pm1,0),\ (0,\pm1),\ (\pm1,\pm1)$

- **d2Q5**: $\vec{0}$ and
  $(\pm1,0),\ (0,\pm1)$
- **d3Q7**: $\vec{0}$ and
  $(\pm1,0,0),\ (0,\pm1,0),\ (0,0,\pm1)$
- **d3Q13** : $\vec{0}$ and
  $(\pm1,\pm1,0),\ (0,\pm1,\pm1),\ (\pm1,0,\pm1)$
- **d3Q15** : $\vec{0}$ and
  $(\pm1,0,0),\ (0,\pm1,0),\ (0,0,\pm1),\ (\pm1,\pm1,\pm1)$
- **d3Q19** : $\vec{0}$ and
  $(\pm1,0,0),\ (0,\pm1,0),\ (0,0,\pm1),\ (\pm1,\pm1,0),\ (0,\pm1,\pm1),\ (\pm1,0,\pm1)$
- **d3Q27=d3Q19** $\cup$ **d3Q15**

Multiple-relaxation-time MRT-model

Velocity space		Moment space	
f_0	$\vec{c}_0$	$\widehat{f_0}$	$\mathbf{b}_1$
	$\dots$	$\dots$	
f_k	$\vec{c}_k$	$\widehat{f_k}$	$\mathbf{b}_k$
	$\dots$	$\dots$	
f_{Q-1}	$\vec{c}_{Q-1}$	$\widehat{f_{Q-1}}$	$\mathbf{b}_{Q-1}$

- **Moment (physical) space: basis vectors $\mathbf{b}_k$ and eigenvalues λ_k,** $k = 0, \dots, Q-1$.
- **Projection into moment space:** $\mathbf{f} = \sum_{k=0}^{Q-1} \widehat{f_k} \mathbf{b}_k$, $\widehat{f_k} = \langle f | \mathbf{b}_k \rangle$.
- **Collision in moment space:** $[\mathcal{A} \cdot (\mathbf{f} - \mathbf{e})]_q = \sum_{k=0}^{Q-1} \lambda_k (\widehat{f_k} - \widehat{e_k}) b_{kq}$.
- **Linear stability:** $-2 < \lambda_k < 0$.

- **Grid space**: $\Delta r_\alpha = 1, \alpha = 1, \dots, d$
- **Time**: $\Delta t = 1$ (1 update)
- **Population vector:** $\mathbf{f}(\vec{r}, t) = (f_q)$, $q = 0, .., Q-1$
- **Equilibrium function:** $\mathbf{e}(\vec{r}, t) = (e_q)$, $q = 0, .., Q-1$
- $Collision_q(\vec{r}, t) = [\mathcal{A} \cdot (\mathbf{f} - \mathbf{e})]_q$, $\mathcal{A}[Q \times Q]$-matrix
  $\tilde{f}_q(\vec{r}, t) = f_q(\vec{r}, t) + Collision_q$
- **Propagation:** $f_q(\vec{r} + \vec{c}_q, t+1) = \tilde{f}_q(\vec{r}, t)$

MRT basis of $d2Q9$ model

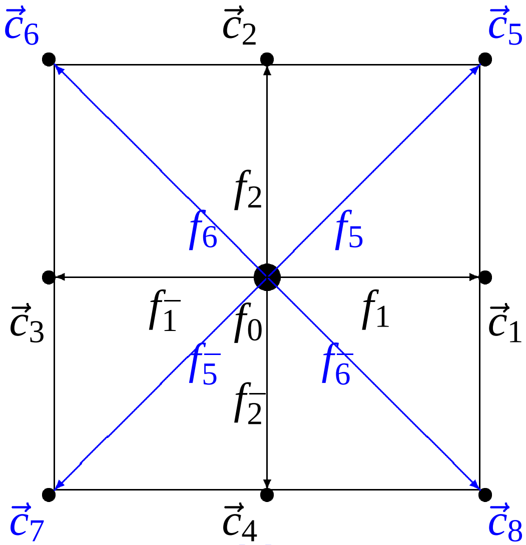
$$d2Q9 : \mathbf{b}_k\ ,\ k = 1, \ldots, 9$$

$$
\begin{aligned}
(\mathbf{b}_1)_q &= 1 \\
(\mathbf{b}_2)_q &= c_{qx} \\
(\mathbf{b}_3)_q &= c_{qy} \\
(\mathbf{b}_4)_q &= 3c_q^2 - 4, c_q^2 = c_{qx}^2 + c_{qy}^2 \\
(\mathbf{b}_5)_q &= 2c_{qx}^2 - c_q^2 \\
(\mathbf{b}_6)_q &= c_{qx}c_{qy} \\
(\mathbf{b}_7)_q &= c_{qx}(3c_q^2 - 5) \\
(\mathbf{b}_8)_q &= c_{qy}(3c_q^2 - 5) \\
(\mathbf{b}_9)_q &= \frac{1}{2}(9c_q^4 - 21c_q^2 + 8).
\end{aligned}
$$

$\mathbf{b}_1$	$\mathbf{b}_2$	$\mathbf{b}_3$	$\mathbf{b}_4$	$\mathbf{b}_5$	$\mathbf{b}_6$	$\mathbf{b}_7$	$\mathbf{b}_8$	$\mathbf{b}_9$
1	0	0	-4	0	0	0	0	4
1	1	0	-1	1	0	-2	0	-2
1	1	1	2	0	1	1	1	1
1	0	1	-1	-1	0	0	-2	-2
1	-1	1	2	0	-1	1	1	1
1	-1	0	-1	1	0	-2	0	-2
1	-1	-1	2	0	1	1	1	1
1	0	-1	-1	-1	0	0	-2	-2
1	1	-1	2	0	-1	1	1	1
Eigenvalues								
λ_0^+	λ_1^-	λ_2^-	λ_1^+	λ_2^+	λ_3^+	λ_3^-	λ_4^-	λ_4^+

$$\lambda_0^+ \to 0,\ \lambda_1^- \to 0,\ \lambda_2^- \to 0,$$
$$\lambda_1^+ \to \nu_\xi,\ \lambda_2^+ \to \nu,\ \lambda_3^+ \to \nu.$$

Link-model LM, (2005)



- **Decomposition:**
  $$f_q = f_q^+ + f_q^-$$
- **Symmetric part**: $f_q^+ = \frac{1}{2}(f_q + f_{\bar{q}})$
- **Antisymmetric part**: $f_q^- = \frac{1}{2}(f_q - f_{\bar{q}})$

- **Link:** $\{\vec{c}_q, \vec{c}_{\bar{q}}\}$, $\vec{c}_q = -\vec{c}_{\bar{q}}$
- $Collision_q = p_q + m_q$
- **Symmetric collision part**: $p_q = \lambda_q^+ (f_q^+ - e_q^+)$
- **Antisymmetric collision part**: $m_q = \lambda_q^- (f_q^- - e_q^-)$
- **Local equilibrium**: $e_q = e_q^+ + e_q^-$, $e_q = e_q(\mathbf{f})$

Microscopic conservation laws with Link Model

- **Mass+momentum with LM: two-relaxation-time operator, TRT only !!!**

- **BGK:** $\lambda^+ = \lambda^- = \lambda$

  (Qian, d'Humières & Lallemand, 1992)

  $[\mathcal{A} \cdot (\mathbf{f} - \mathbf{e})]_q = \lambda(f_q - e_q)$

$$\begin{cases} \text{BGK} \subset \text{TRT} \subset \text{MRT} \\ \text{BGK} \subset \text{TRT} \subset \text{LM} \end{cases}$$

- **Conserved mass quantity** $\rho(\vec{r}, t)$:

  Let $\rho(\vec{r}, t) = \sum_{q=0}^{Q-1} f_q = \sum_{q=0}^{Q-1} f_q^+ = \sum_{q=0}^{Q-1} e_q = \sum_{q=0}^{Q-1} e_q^+$, **then**
  $\sum_{q=0}^{Q-1} \lambda_q^+ (f_q^+ - e_q^+) = 0$ if $\lambda_q^+ = \lambda^+$.

- **Conserved** $d-$**dimensional momentum quantity** $\vec{j}(\vec{r}, t)$:

  Let $\vec{j}(\vec{r}, t) = \sum_{q=1}^{Q-1} f_q \vec{c}_q = \sum_{q=1}^{Q-1} f_q^- \vec{c}_q = \sum_{q=1}^{Q-1} e_q \vec{c}_q = \sum_{q=1}^{Q-1} e_q^- \vec{c}_q$, **then**

  $\sum_{q=1}^{Q-1} \lambda_q^- (f_q^- - e_q^-) \vec{c}_q = 0$ if $\lambda_q^- = \lambda^-$.

Following idea of Chapman-Enskog (1916-1917)

- Let $\varepsilon = \frac{1}{L}$ with L as a characteristic length

- Let $x' = \varepsilon x$

- Let $t_1 = \varepsilon t,\ t_2 = \varepsilon^2 t,$ …

$$\partial_t = \varepsilon \partial_{t_1} + \varepsilon^2 \partial_{t_2} + \dots$$

- **Population expansion around the equilibrium:**

$$f_q = e_q + \varepsilon f_q^{[1]} + \varepsilon^2 f_q^{[2]} + \dots$$

- **Collision components** :

$$p_q^{[n]} = [\mathcal{A}\, \mathbf{f}^{[n]}]_q^+,$$
$$m_q^{[n]} = [\mathcal{A}\, \mathbf{f}^{[n]}]_q^-.$$

- **Macroscopic laws:**

$$\sum_{q=0}^{Q-1} [p_q^{[1]} + p_q^{[2]}] = 0,$$
$$\sum_{q=0}^{Q-1} [m_q^{[1]} + m_q^{[2]}] \vec{c}_q = 0.$$

Directional Taylor expansion, $\partial_q = \nabla \cdot \vec{c}_q = \varepsilon \partial_{q'}$

- **Evolution equation**: $f_q(\vec{r}+\vec{c}_q, t+1) - f_q(\vec{r}, t) = \sum_n \varepsilon^n (p_q^{[n]}(\vec{r}, t) + m_q^{[n]}(\vec{r}, t))$.

- **Inversion is trivial for LM** :

$$f_q^{+[n]} = \frac{p_q^{[n]}}{\lambda^+},$$
$$f_q^{-[n]} = \frac{m_q^{[n]}}{\lambda_q^-}.$$

- **First order expansion**:

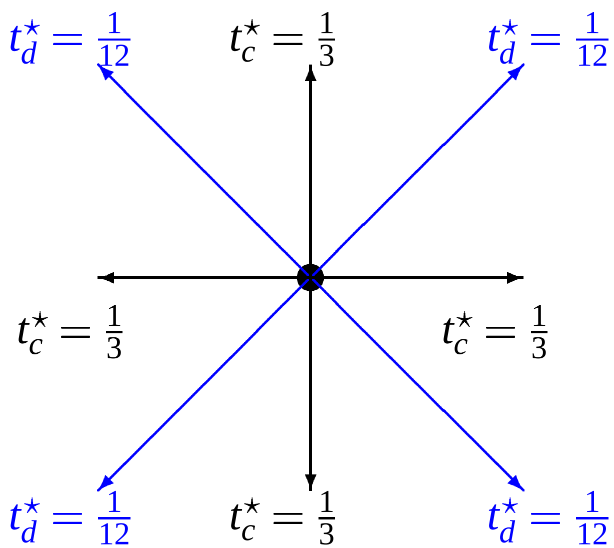
$$\varepsilon p_q^{[1]} = \varepsilon(\partial_{t_1} e_q^+ + \partial_{q'} e_q^-),$$
$$\varepsilon m_q^{[1]} = \varepsilon(\partial_{t_1} e_q^- + \partial_{q'} e_q^+).$$

- **Second order expansion**:

$$\varepsilon^2 p_q^{[2]} = \varepsilon^2 \partial_{t_2} e_q^+ - \varepsilon(\partial_{t_1} \Lambda^+ p_q^{[1]} + \partial_{q'} \Lambda_q^- m_q^{[1]}),$$
$$\varepsilon^2 m_q^{[2]} = \varepsilon^2 \partial_{t_2} e_q^- - \varepsilon(\partial_{t_1} \Lambda_q^- m_q^{[1]} + \partial_{q'} \Lambda^+ p_q^{[1]}).$$

- **Eigenvalue functions:**

$$\Lambda^+ = -\left(\frac{1}{2} + \frac{1}{\lambda^+}\right) > 0,$$
$$\Lambda_q^- = -\left(\frac{1}{2} + \frac{1}{\lambda_q^-}\right) > 0.$$



Isotropic weights:

$$\sum_{q=1}^{Q-1} t_q^\star c_{q\alpha} c_{q\beta} = \delta_{\alpha\beta},$$
$$\sum_{q=1}^{Q-1} t_q^\star c_{q\alpha}^2 c_{q\beta}^2 = \frac{1}{3},$$
$\alpha \neq \beta$.

TRT $+$ **isotropic weights**

- $e_q = t_q^\star(P(\rho) + j_q)\ ,\ j_q = \vec{j} \cdot \vec{c}_q\ ,\ \vec{j} = \sum_{q=1}^{Q-1} t_q^\star j_q \vec{c}_q$

  (1) Stokes equation for pressure P and momentum $\vec{j}$:
  Kinematic viscosity: $\nu = \frac{1}{3}\Lambda^+$
  (2) Isotropic linear convection-diffusion equation
  when $P = c_e\rho\ (0 < c_e < 1)$ **and** $\vec{j} = \rho\vec{U}$
  Diffusion coefficients: $D_{\alpha\alpha} = c_e\Lambda^-, \forall\alpha = 1, \cdots, d$

- $e_q = t_q^\star(P(\rho) + \frac{3j_q^2 - |j|^2}{2\rho} + j_q)$

  (1) Navier-Stokes equation
  (2) Isotropic linear convection-diffusion equation
  without second order numerical diffusion $O(\Lambda^- U_\alpha U_\beta)$

"Magic parameter" $\Lambda^\pm = \Lambda^+\Lambda^-$ **is free for both equations.**

Let $P = c_s^2 \rho$

Mach number:

$Ma = \frac{U}{c_s}$,

U **is characteristic velocity**.

Sound velocity:

$0 < c_s^2 < 1$

best : $c_s^2 = \frac{1}{3}$

(Lallemand & Luo, 2000)

Incompressible Navier-Stokes equation

– **MRT/TRT/BGK with forcing** S_q^-**:**

$$f_q(\vec{r} + \vec{c}_q, t+1) = f_q(\vec{r}, t) + p_q + m_q + S_q^- .$$

Incompressible limit, $Ma \to 0$**:**

$$\rho = \rho_0(1 + Ma^2 P') \;,\; P' = \frac{P(\rho) - P(\rho_0)}{\rho_0 U^2},$$

$$\nabla \cdot \vec{u} = O(-Ma^2 \partial_t P') = O(\varepsilon^3) \text{ if } U = O(\varepsilon)$$

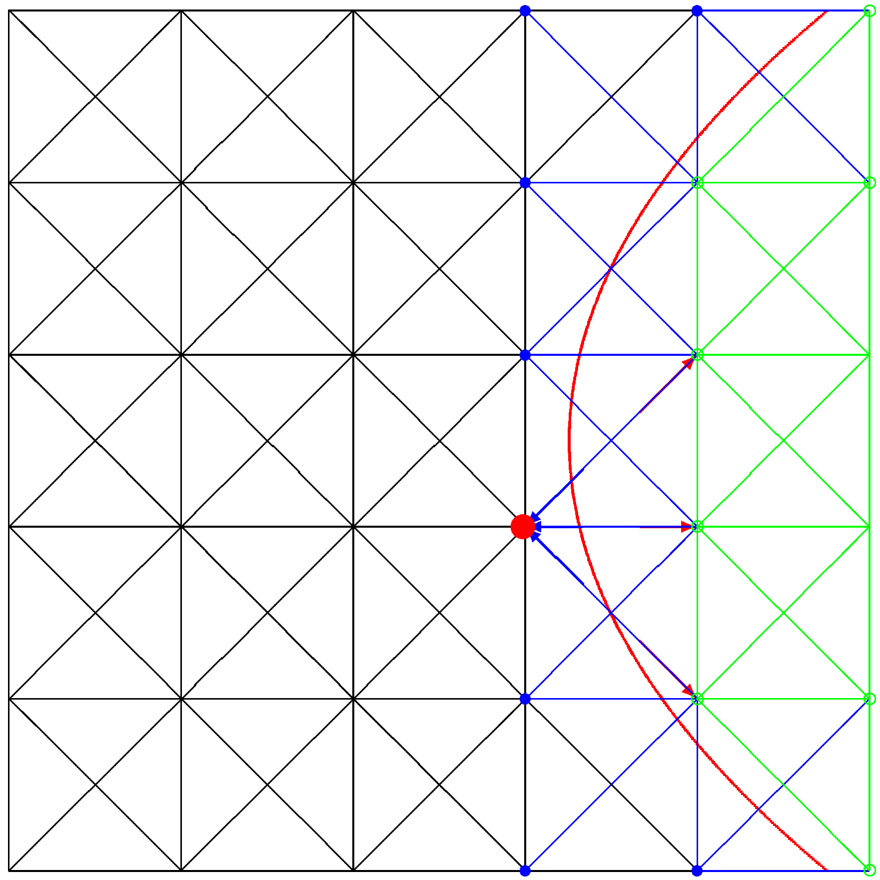
$$\rho_0 \partial_t \vec{u} + \rho_0 \nabla \cdot (\vec{u} \otimes \vec{u}) =$$
$$-\nabla P + \nabla \cdot (\rho_0 \nu \nabla \vec{u}) + \vec{F} + O(\varepsilon^3) + O(Ma^2)$$

Force: $\vec{F} = \sum_{q=1}^{Q-1} S_q^- \vec{c}_q$

$$\vec{j} = \sum_{q=1}^{Q-1} f_q \vec{c}_q + \frac{1}{2}\vec{F}, \; \vec{u} = \frac{\vec{j}}{\rho_0}$$

Kinetic boundary problem

Boundary nodes:
fluid nodes with at least one outside neighbor



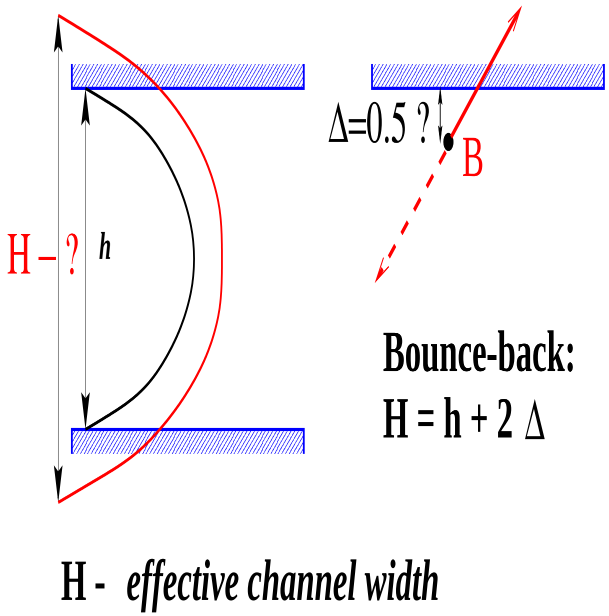
Boundary conditions

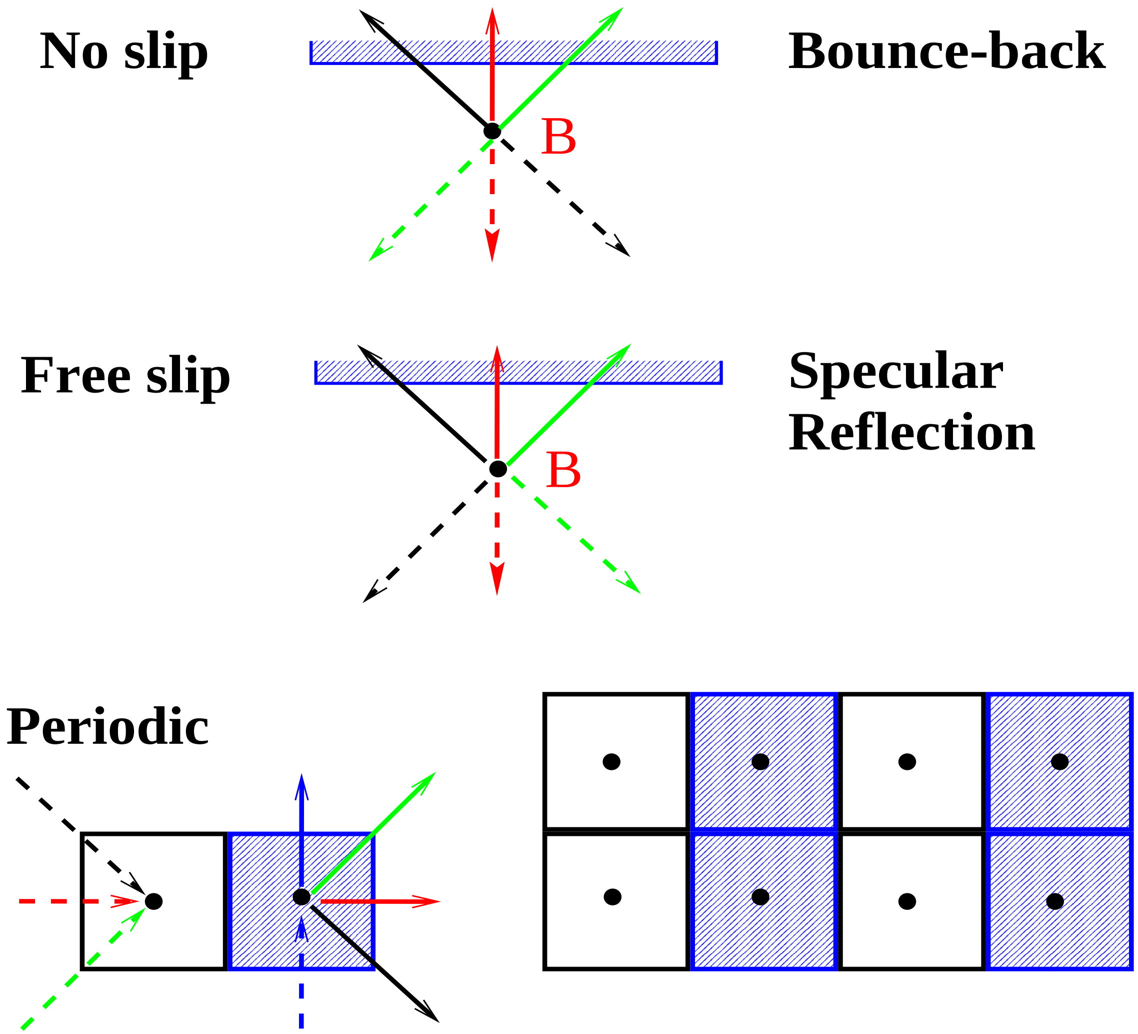
- **Dirichlet velocity condition** via the bounce-back

$$f_{\bar{q}}(\vec{r}_{\mathrm{b}}) = \tilde{f}_q(\vec{r}_{\mathrm{b}}) - 2e_q^{-}(\vec{r}_{\mathrm{b}} + \delta_q \vec{c}_q)$$

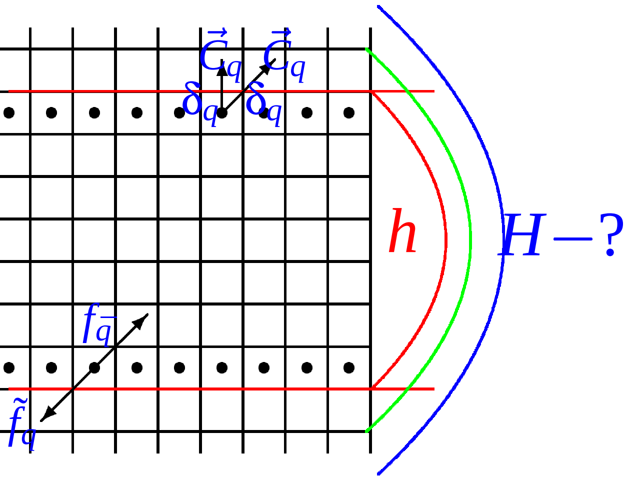
- **Dirichlet pressure condition** via the anti-bounce-back

$$f_{\bar{q}}(\vec{r}_{\mathrm{b}}) = -\tilde{f}_q(\vec{r}_{\mathrm{b}}) + 2e_q^{+}(\vec{r}_{\mathrm{b}} + \delta_q \vec{c}_q)$$





No-slip condition via bounce-back reflection
I. Ginzburg & P. M. Adler, J.Phys.II France, **1994**



$H < h$ **if** $\Lambda^{\pm} < \frac{3}{16}$

$H = h$ **if** $\Lambda^{\pm} = \frac{3}{16}$

$H > h$ **if** $\Lambda^{\pm} > \frac{3}{16}$

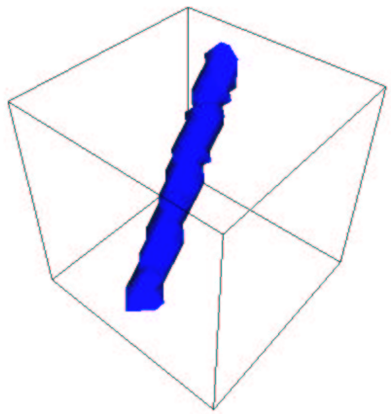
$H \to \infty$ **if** $\Lambda^{+} = \Lambda^{-}$
and $\nu \to \infty$ **(BGK)**

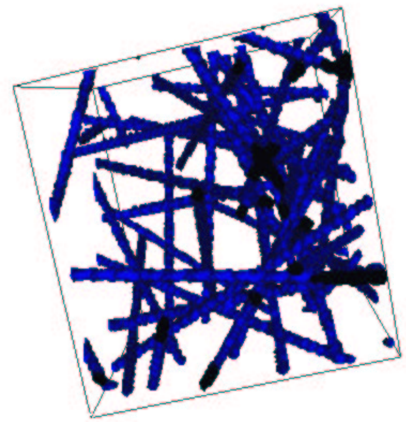
- **First order closure relation:**
  $$j_q(\vec{r}_{\mathrm{b}}) + \frac{1}{2}\partial_q j_q(\vec{r}_{\mathrm{b}}) = O(\varepsilon^2),\ \delta_q = \frac{1}{2}$$
- **Second order closure relation:**
  $$j_q(\vec{r}_{\mathrm{b}}) + \frac{1}{2}\partial_q j_q(\vec{r}_{\mathrm{b}}) + \frac{1}{2}\frac{4}{3}\Lambda^{\pm}\partial_q^2 j_q(\vec{r}_{\mathrm{b}}) = O(\varepsilon^3)$$
- **For Poiseuille flow, effective width** H **of the channel is**
  $$H^2 = h^2 + \frac{16}{3}\Lambda^{\pm} - 1$$

Permeability measurements with the bounce-back reflection

$$\frac{k(\Lambda^+) - k(\Lambda^+ = \frac{1}{2})}{k(\Lambda^+ = \frac{1}{2})}, \ \nu = \frac{1}{3}\Lambda^+$$

- **Darcy law:** $\nu \overline{\vec{j}} = \mathbf{K}\overline{(\vec{F} - \nabla P)}$
- **Permeability is viscosity dependent for BGK,** $\Lambda^{\pm} = 9\nu^2$
- **Permeability is absolutely viscosity independent for TRT if** $\Lambda^- = \Lambda^{\pm}/\Lambda^+$ **and** $\Lambda^{\pm}$ **is fixed**
- **WHY ??**





Λ^+	$20^3, \quad \phi \approx 0.965$		$90^3, \quad \phi \approx 0.941$	
	TRT	**BGK**	**TRT**	**BGK**
$1/8$	10^{-13}	-0.077	10^{-13}	-0.083
$15/2$	-2.8×10^{-12}	4.699	-10^{-13}	2.236

Steady recurrence equations (2006)

– **Equivalent link-wise finite-difference form:**

$$p_q = \lambda^+ n_q^+ = \bar{\Delta}_q e_q^- - \Lambda^- \Delta_q^2 e_q^+ + (\Lambda^\pm - \frac{1}{4})\Delta_q^2 p_q$$

$$m_q = \lambda^- n_q^- = \bar{\Delta}_q e_q^+ - \Lambda^+ \Delta_q^2 e_q^- + (\Lambda^\pm - \frac{1}{4})\Delta_q^2 m_q$$

– **where link-wise f.d. operators are:**

$$\bar{\Delta}_q \phi(\vec{r}) = \frac{1}{2}(\phi(\vec{r}+\vec{c}_q) - \phi(\vec{r}-\vec{c}_q))$$

$$\Delta_q^2 \phi(\vec{r}) = \phi(\vec{r}+\vec{c}_q) - 2\phi(\vec{r}) + \phi(\vec{r}-\vec{c}_q)\ ,\ \forall\ \phi\ .$$

Look for solution as expansion around the equilibrium:

$$p_q = p_q(\mathbf{e}^-) - 2\Lambda^- p_q(\mathbf{e}^+) \,,$$
$$m_q = m_q(\mathbf{e}^+) - 2\Lambda^+ m_q(\mathbf{e}^-) \,,$$

where

$$p_q(\mathbf{e}^+) = \sum_{k=1,2,\dots} T_q^{(2k)}(\mathbf{e}^+) \,,$$
$$p_q(\mathbf{e}^-) = \sum_{k=1,2,\dots} T_q^{(2k-1)}(\mathbf{e}^-) \,,$$
$$m_q(\mathbf{e}^-) = \sum_{k=1,2,\dots} T_q^{(2k)}(\mathbf{e}^-) \,,$$
$$m_q(\mathbf{e}^+) = \sum_{k=1,2,\dots} T_q^{(2k-1)}(\mathbf{e}^+) \,,$$

and

$$T_q^{(2k)}(\mathbf{e}) = \frac{a_{2k}\partial_q^{2k} e_q}{(2k)!} \,,$$
$$T_q^{(2k-1)}(\mathbf{e}) = \frac{a_{2k-1}\partial_q^{2k-1} e_q}{(2k-1)!}$$

Solution for the coefficients of the series, $k \geq 1$:

$$a_{2k-1} = 1 + \\ +2(\Lambda^{\pm} - \tfrac{1}{4}) \Sigma_{1 \leq n < k} a_{2n-1} \frac{(2k-1)!}{(2n-1)!(2(k-n))!}$$

$$a_{2k} = 1 + 2(\Lambda^{\pm} - \tfrac{1}{4}) \Sigma_{1 \leq n < k} a_{2n} \frac{(2k)!}{(2n)!(2(k-n))!}$$

- **Non-dimensional steady solutions on the fixed grid**

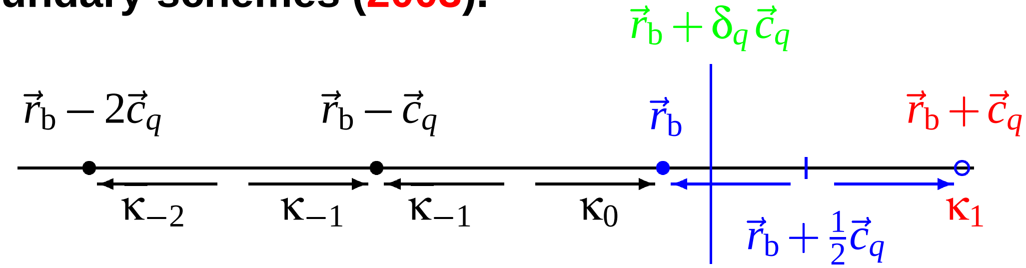
  $\vec{j}' = \frac{\vec{j}}{\rho_0 U}$, $P' = \frac{P - P_0}{\rho_0 U^2}$ **are the same if**

  $Ma = \frac{U}{c_s}$, $Fr = \frac{U^2}{gL}$, $Re = \frac{UL}{\nu}$ **and** $\Lambda^{\pm}$ **are fixed**

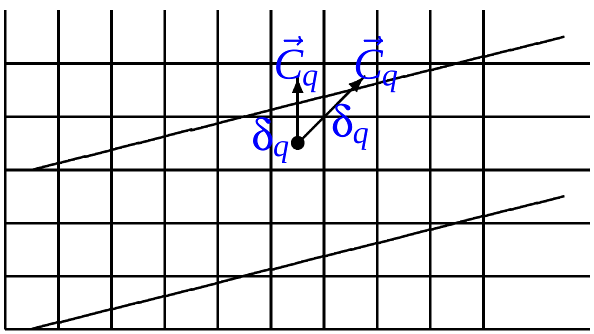
- **Provided that this property is shared by the microscopic boundary schemes,**

  the permeability is the same if $\Lambda^{\pm}$ **is fixed !!!**

Multi-reflection Dirichlet boundary schemes (2003).



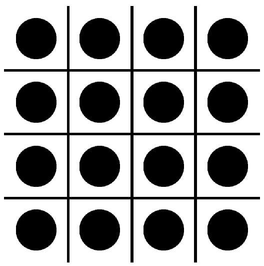
Boundary surface cuts at $\vec{r}_\mathrm{b} + \delta_q \vec{c}_q$ the link between boundary node $\vec{r}_\mathrm{b}$ and an outside one at $\vec{r}_\mathrm{b} + \vec{c}_q$.

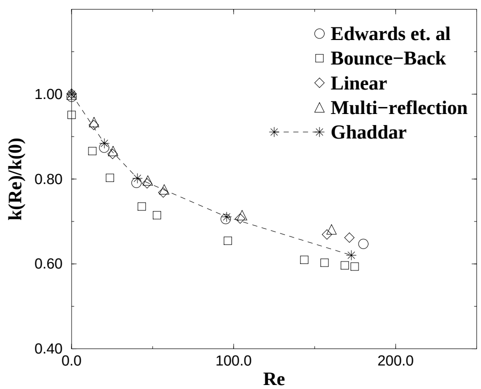


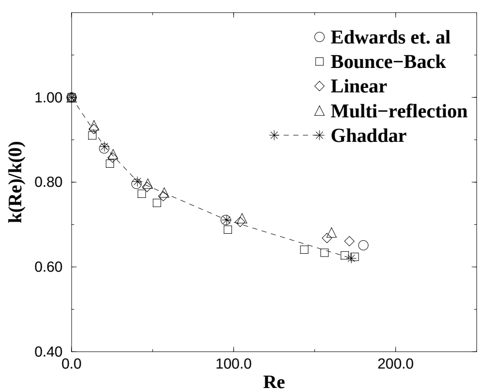
$$
\begin{aligned}
f_{\bar{q}}(\vec{r}_\mathrm{b}, t+1) \;=\;& \kappa_1 f_q(\vec{r}_\mathrm{b} + \vec{c}_q, t+1) \\
&+ \kappa_0 f_q(\vec{r}_\mathrm{b}, t+1) \\
&+ \kappa_{-1} f_q(\vec{r}_\mathrm{b} - \vec{c}_q, t+1) \\
&+ \bar{\kappa}_{-1} f_{\bar{q}}(\vec{r}_\mathrm{b} - \vec{c}_q, t+1) \\
&+ \bar{\kappa}_{-2} f_{\bar{q}}(\vec{r}_\mathrm{b} - 2\vec{c}_q, t+1) \\
&+ k_b e_q^{\pm}(\vec{r}_\mathrm{b} + \delta_q \vec{c}_q, t+1) + f_q^{\mathrm{p.c.}}
\end{aligned}
$$

Coefficients are adjusted to fit a prescribed Dirichlet value via the Taylor expansion along a link:

- **Linear schemes:** **exact for linear velocity/constant pressure.**
- **MR schemes:** **exact for parabolic velocity/linear pressure.**







Stokes and Navier-Stokes flow in a square array of cylinders.

Error (in percents) of different methods in Stokes regime in 66^2 box					
ϕ	Edwards, FE	Bounce-back	Linear	Multi-reflection	Ghaddar, FE
0.2	2.54	-1.63	5.5×10^{-2}	-6.5×10^{-2}	-2.4×10^{-2}
0.3	0.53	0.78	0.51	2.8×10^{-2}	9.8×10^{-3}
0.4	-0.64	-4.86	0.13	-9.2×10^{-2}	-2.2×10^{-2}
0.5	-2.54	-1.1	-0.95	-8.9×10^{-3}	3.4×10^{-2}
0.6	-8.36	-6.9	0.55	-2.1×10^{-1}	1.3×10^{-2}

– **Stokes quasi-analytical solution (Hasimoto, 1959)** : $\frac{F^d}{l} = \frac{4\pi\mu\overline{U}}{k^\star(\phi)}$, $k = \frac{V}{4\pi l}k^\star$, ϕ is the relative solid square fraction ($\phi_{max} = \pi/4$).

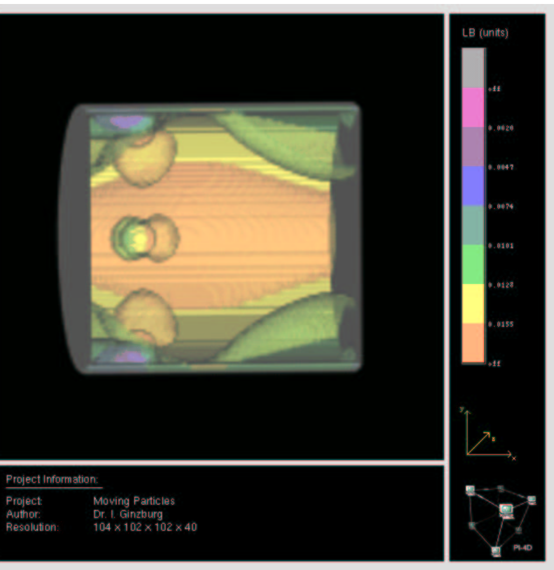
– **Apparent (NSE) permeability is computed from Darcy Law.**

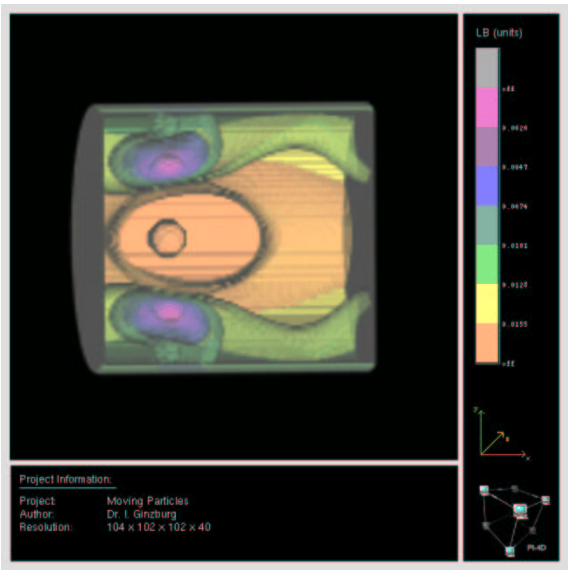
– **Dimensionless permeability versus Re number is plotted:**

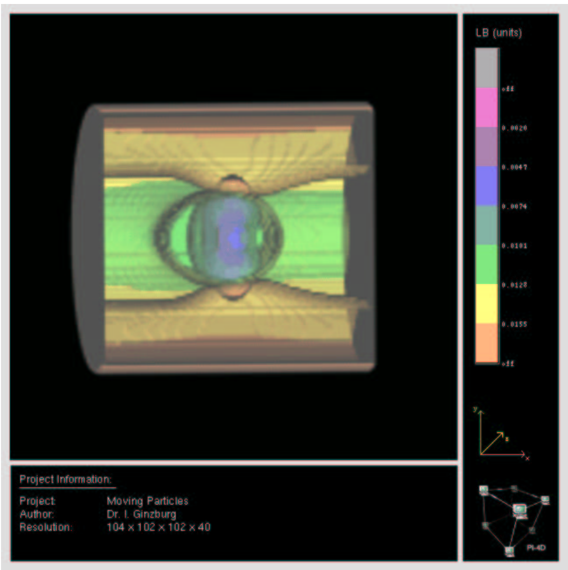
(1) **Top picture: NSE permeability/Stokes quasi-analytical solution.**

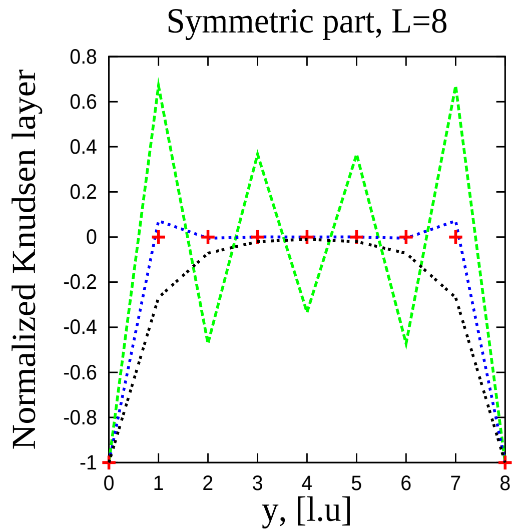
(2) **Bottom picture:NSE permeability/Stokes numerical value.**

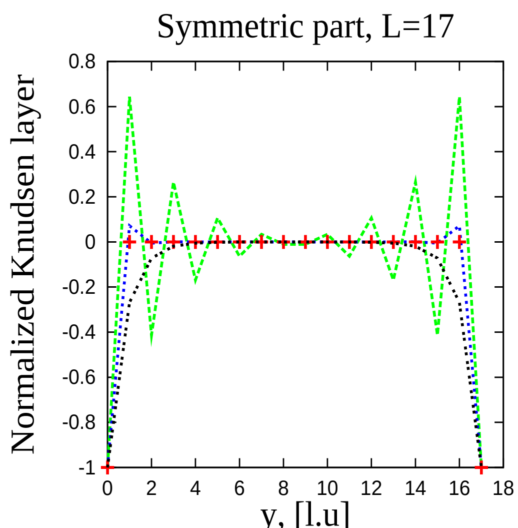
Pressure distribution around three spheres moving in a circular tube











$\Lambda^{\pm} < \frac{1}{4}$: **accommodation oscillates**, ($\Lambda^{\pm} = \frac{3}{256}$, $\Lambda^{\pm} = \frac{3}{16}$)

$\Lambda^{\pm} > \frac{1}{4}$: **it decreases exponentially** ($\Lambda^{\pm} = \frac{3}{2}$)

Solutions beyond the Chapman-Enskog expansion

$$p_q = p_q^{ch} + g_q^{+},\; m_q = m_q^{ch} + g_q^{-}$$

$$g_q^{+} = (\Lambda^{\pm} - \tfrac{1}{4})\Delta_q^2 g_q^{+} \;,\; \sum_{q=0}^{Q-1} g_q^{+} = 0$$

$$g_q^{-} = (\Lambda^{\pm} - \tfrac{1}{4})\Delta_q^2 g_q^{-} \;,\; \sum_{q=1}^{Q-1} g_q^{-}\vec{c}_q = 0$$

Example of Knudsen layer in horizontal channel:

e.g.,**exact Poiseuille flow using non-linear equilibrium**

$$g_q^{+} = (3c_{qx}^2 - 1)t_q^{\star}K^{+}(y)c_{qy}^2$$

$$g_q^{-} = (3c_{qx}^2 - 1)t_q^{\star}K^{-}(y)c_{qy}$$

$$K^{\pm}(y) = k_1^{\pm} r_0^{y} + k_2^{\pm} r_0^{-y},\; r_0 = \frac{2\sqrt{\Lambda^{\pm}}+1}{2\sqrt{\Lambda^{\pm}}-1}$$

r_0 and $1/r_0$ obey: $(r+1)^2 = 4\Lambda^{\pm}(r-1)^2$

$\Lambda^{\pm} = \frac{1}{4}$: **accommodation in boundary node**

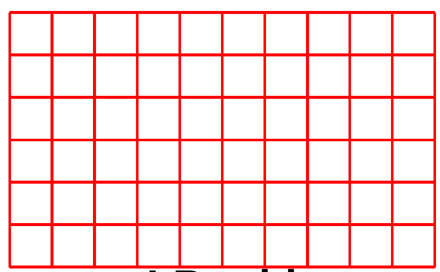
- $h[L] = -\psi(\theta)/\rho g$
  $\psi(\theta)$ **capillary pressure**
- $K[L\,T^{-1}]$ **hydraulic conductivity**, $K = K_r K_s$
- $K_s[L\,T^{-1}]$ **saturated hydraulic conductivity**, $K_s = k\rho g/\mu$
- $K_r(h) = k_{rw}$ **dimensionless relative hydraulic conductivity**
- $k\mathsf{K}^a$ **permeability tensor**
  K^a **is dimensionless tensor**
  $\mathsf{K}^a = \mathsf{I}$ **in isotropic case**

Richards' equation: $\partial_t\theta + \nabla\cdot\vec{u} = 0$,
$\vec{u} = -K(h)\mathbf{K}^a\cdot(\nabla h + \vec{1}_g)$.

- **Conserved variable**: **moisture content** $\theta(\vec{r},t)$.
- **Characteristic scaling**: $h^{lb} = \mathcal{L}h^{phys}$, $K^{lb} = \mathcal{U}K^{phys}$.
- **Coordinate scaling:** $\Delta\vec{r}^{lb} = \mathcal{L}\mathsf{H}\cdot\Delta\vec{r}^{phys} = 1$,
  $\mathsf{H} = \mathrm{diag}(l_x, l_y, l_z)$.
- **LB grid:**

$$\partial_t\theta - \nabla\cdot K^{lb}(\theta)\mathbf{K}^{\mathbf{lb}}\cdot\vec{1}_g = \nabla\cdot K^{lb}(\theta)\mathbf{K}^{\mathbf{a\ lb}}\cdot\nabla h^{lb},$$

$$K^{lb}(\theta) = \mathcal{U}K^{phys}(\theta), \quad K^{lb}_{\alpha\beta} = [K^a_{\alpha\beta}]^{phys} l_\alpha, \quad , K^{a\ lb}_{\alpha\beta} = [K^a_{\alpha\beta}]^{phys} l_\alpha l_\beta$$

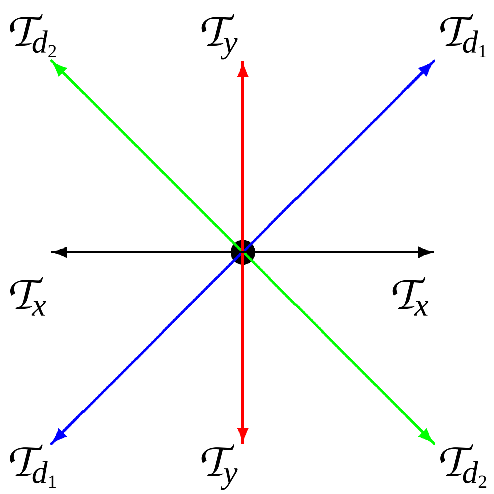


LB grid

$l_z = 4l_x$

Solution grid

Generic advection and anisotropic dispersion equation (AADE).



- **LM**: $f_q(\vec{r}+\vec{c}_q, t+1) = f_q(\vec{r}, t) + m_q + p_q + S_q^+$
- **Equilibrium**: $e_q = t_q P(\rho) + t_q^\star j_q, \; q = 1, \ldots, Q-1$
- **Immobile population:** $e_0 = \rho - \sum_{q=1}^{Q-1} e_q^+$
- **AADE**: $\partial_t \rho + \nabla \cdot \vec{j} = \nabla \cdot \vec{D} + M, \; M = \sum_{q=0}^{Q-1} S_q^+$

* **Diffusive flux:** $-\vec{D} = -\sum_{q=1}^{Q-1} \Lambda_q^- m_q \vec{c}_q,$
  $D_\alpha = \mathsf{D}_{\alpha\beta} \partial_\beta P(\rho), \; \alpha = 1, \ldots, d \;, \; \beta = 1, \ldots, d$
* **Diffusion tensor:** $\mathsf{D}_{\alpha\beta} = \sum_{q=1}^{Q-1} \mathcal{T}_q c_{q\alpha} c_{q\beta}, \; \mathcal{T}_q = \Lambda_q^- t_q$

- **LM-operator** has $(Q-1)/2$ Λ_q^- **-freedoms for** $\mathsf{D}_{\alpha\beta}$
- **MRT-operator** has **only** d **eigenvalue freedoms for** $\mathsf{D}_{\alpha\alpha}$

Richards' equation via the AADE.

$$\rho = \theta,\ e_q = t_q P(\rho) + t_q^{\star} j_q,\ \vec{j} = -K(\theta)\mathbf{K}\cdot\vec{1}_g.$$

$$\partial_t \rho + \nabla\cdot\vec{j} = \nabla\cdot k(P)\mathbf{K^a}\nabla P(\rho).$$

- **Mixed form, θ/h-based**

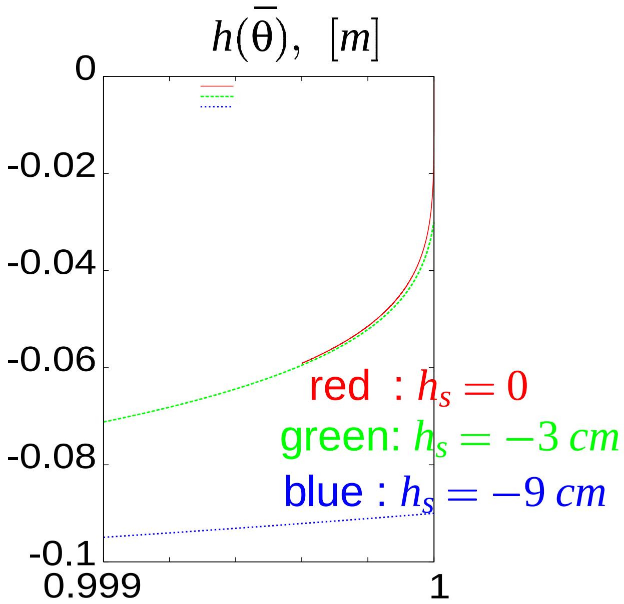
$$P = h(\theta),\ k(P) = K(\theta).$$

- **Moisture content form, θ-based**

$$P = \theta,\ k(P) = K(\theta)\partial_\theta h(\theta).$$

- **Kirchoff transform, $\theta/P-$based**

$$P = \int_{-\infty}^{h(\theta)} K(h')dh',\ k(P) = 1.$$



VGM Sandy soil:
$\alpha = 3.7m^{-1}$, $n = 5$

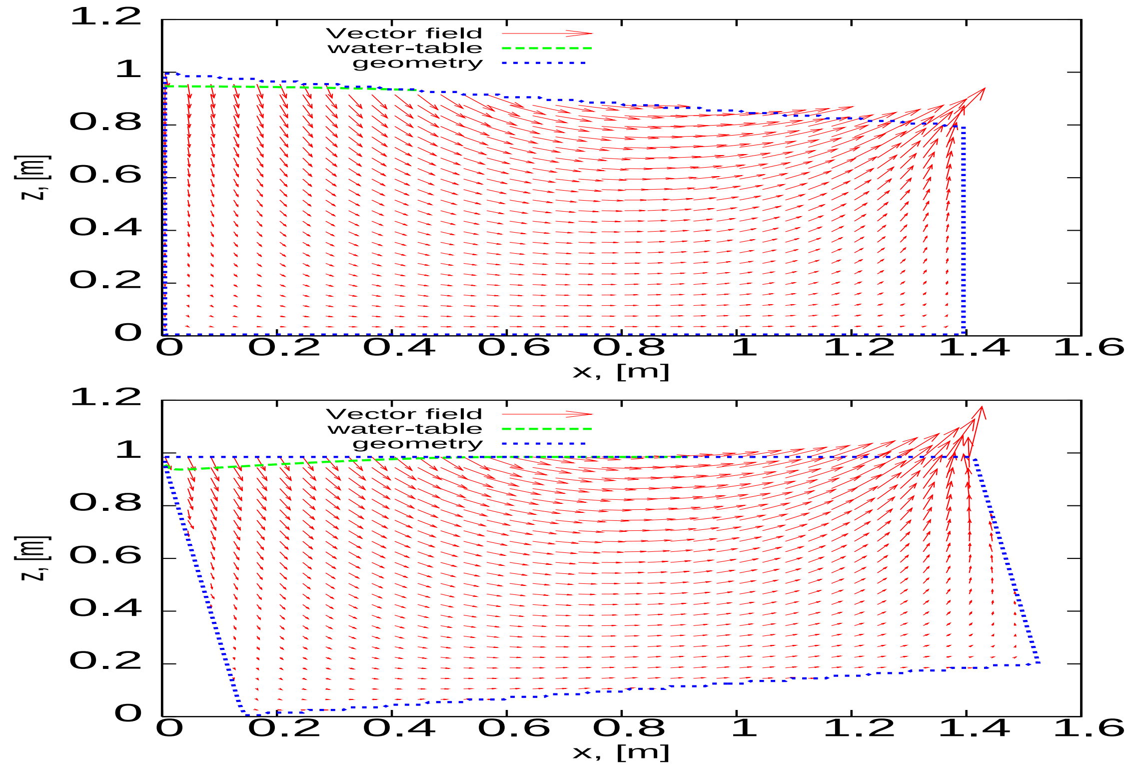
Original VGM (1980):
$\partial_\theta h(\bar{\theta} = 1)$ **is unbounded**
$1/\gamma = \partial_\theta h(1 - 10^{-6}) = 3566.24\ m$

Modified VGM (T. Vogel et all, 2001): $\partial_\theta h(h_s) < \infty$, $h_s < 0$

Heavy rainfall episodes,
Project “Dynamics of shallow water tables”,
http://www-rocq.inria.fr/estime/DYNAS.

- **Compared to finite element solutions,**
  E. Beaugendre et.al, 2006.
- **No-flow condition except for open surface.**
- **Seepage face conditions on open surface.**
- **Explicit in time,**
  Multi-reflection boundary schemes.

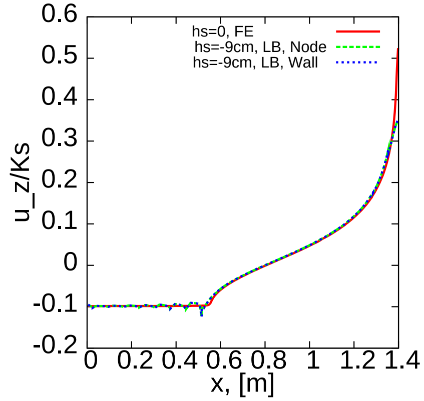
physical axis parallel to LB axis.



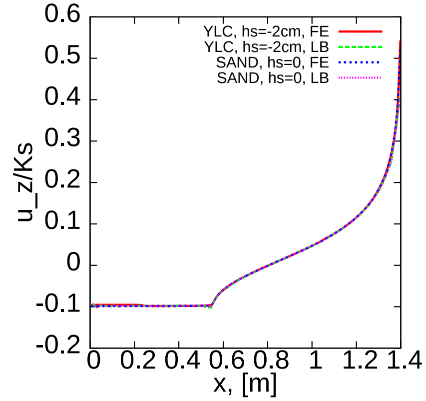
open surface parallel to LB axis.

Reduced vertical velocity on open surface, u_z/K_s.

SAND on non-aligned grid



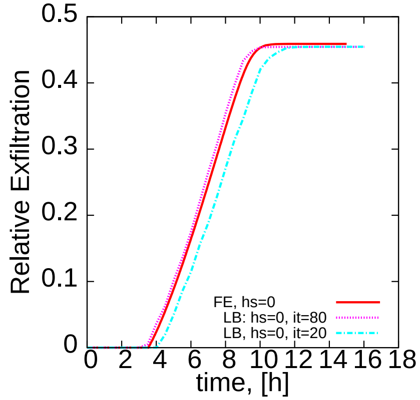
YLC and SAND, on aligned grid



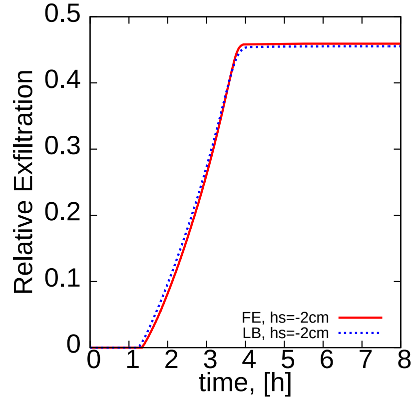
- **Compared to finite element solutions,** **E. Beaugendre et.al, 2006.**
- **Rainfall intensity is** $q_{in} = 0.1K_s$ **for all soils.**
- **FE grid with** 280 **nodes on the open surface.**
- **LB grid with** 70 **nodes on the open surface.**

Relative ex-filtration fluxes

SCL on non-aligned grid

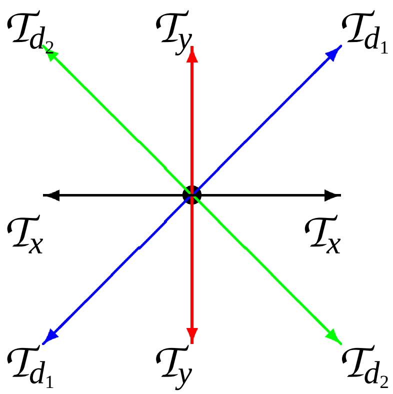


YLC on aligned grid



Solution for $\mathcal{T}_q = \Lambda_q^- t_q$, $D_{\alpha\beta} = \sum_{q=1}^{Q-1} \mathcal{T}_q c_{q\alpha} c_{q\beta}$.

- **d2Q4** : **2** links for D_{xx}, D_{yy}
- **d3Q7** : **3** links for D_{xx}, D_{yy}, D_{zz}



- **d2Q9** : **4** links for D_{xx}, D_{yy}, D_{xy}
- **d3Q13** : **6** links for 6 diff. coeff.
- **d3Q15** : **7** links for 6 diff. coeff.
- **d3Q19** : **9** links for 6 diff. coeff.

- **Coordinate links:** $\mathcal{T}_\alpha = \frac{1}{2}(D_{\alpha\alpha} - s_\alpha)$, $\alpha = 1, \ldots, d$
- **Free parameters:** $s_\alpha = 2\sum_{q(diag)} \mathcal{T}_q c_{q\alpha}^2$
- **Diagonal links:**

  d2Q9 : $\mathcal{T}_q = \frac{1}{4}(s_\alpha + D_{xy} c_{qx} c_{qy})$, $s_\alpha = s_x = s_y$

  d3Q19 :$\mathcal{T}_q = \frac{1}{4}(s_{\alpha\beta} + D_{\alpha\beta} c_{q\alpha} c_{q\beta})$, $s_{\alpha\beta} = \frac{s_\alpha + s_\beta - s_\gamma}{2}$

- **Positivity of the equilibrium weights** $t_q \geq 0$ ($\mathcal{T}_q = \Lambda_q^- t_q \geq 0$): $|D_{\alpha\beta}| \leq s_{\alpha\beta}$, $s_\alpha = (s_{\alpha\beta} + s_{\alpha\gamma}) \leq D_{\alpha\alpha}$, $D_{\alpha\alpha} \geq 0$ **may restrict the range of the off-diagonal coefficients:**

- **d2Q9** : $|D_{xy}| \leq \min\{D_{xx}, D_{yy}\} \Longrightarrow$ **positive definite**
- **d3Q19**: $|D_{\alpha\beta}| + |D_{\alpha\gamma}| \leq D_{\alpha\alpha} \Longrightarrow$ **positive definite**

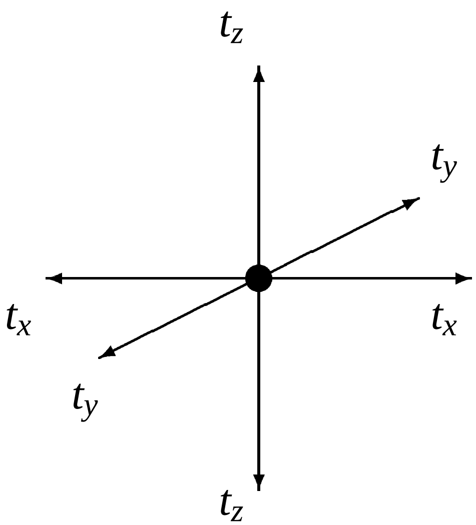
Linear (von Neumann) stability analysis (2004-)

- **Periodic, linear in space solution:** $\mathbf{f}(\vec{r},t) = \Omega^t K_x^x K_y^y K_z^z \mathbf{f}^*$

- **Evolution equation:** $(I + \mathcal{A} \cdot (I - \mathcal{E})) \cdot \mathbf{f}^* = \Omega\, \mathcal{K} \cdot \mathbf{f}^*$,
$\mathcal{K} = \text{diag}(K_x^{c_{qx}}, K_y^{c_{qy}}, K_z^{c_{qz}})$

- **If** $|\Omega| > 1$ **for any wave-vectors** (K_x, K_y, K_z) **the model is unstable, otherwise the model is stable:**
$\Omega \mathbf{f}^* = \mathcal{K}^{-1} \cdot (I + \mathcal{A} \cdot (I - \mathcal{E})) \cdot \mathbf{f}^*$

- **Principal analytical result** (**with help of Miller's Theorems, 1971**):
For advection-diffusion TRT model,
if $\Lambda^{\pm} = \Lambda^+ \Lambda^- = \frac{1}{4}$, i.e. $\lambda^+ + \lambda^- = -2$,
then condition $\Omega^2 = 1$ **is equivalent for any** Λ^+ **and** Λ^-



minimal stencils

– If $e_q^+ \to$

$t_q\rho(1+\frac{3U_q^2-|U|^2}{2})$, then

$D_{\alpha\alpha} \to D_{\alpha\alpha} + \frac{U_\alpha^2 \Delta_t}{2}$

LB with $\Lambda^+ = \Lambda^- = \frac{1}{2} \Leftrightarrow$

MFTCS or Lax-Wendroff

Diffusion dominant criteria for LB:

– **Positivity of immobile weight:** $0 \le \frac{e_0}{\rho} \le 1$

– **Minimal stencils:**

$$\frac{e_0}{\rho} = 1 - \sum_{q=1}^{Q-1} t_q, \quad \frac{\Delta_t}{\Delta_x^2} \sum_{\alpha=1}^{d} D_{\alpha\alpha} = \Lambda^- \sum_{q=1}^{Q-1} t_q$$

– $\forall\ \Delta_t$ **and** $\forall\ \Delta_x$ **the model is stable if**

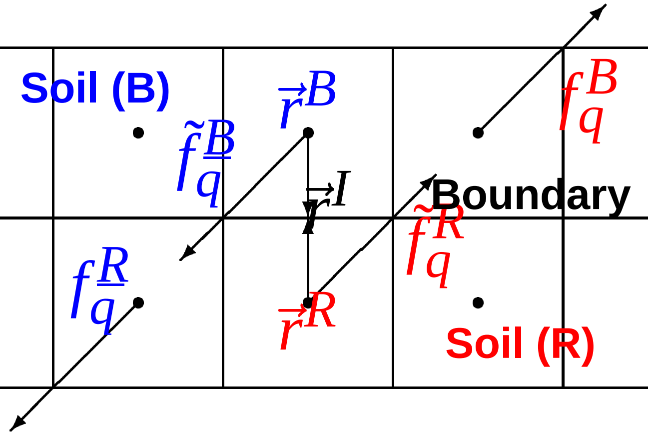
$$\Lambda^- > \frac{\Delta_t \sum_{\alpha=1}^{d} D_{\alpha\alpha}}{\Delta_x^2},$$

– **or**, $\Delta_t < \Lambda^- \frac{\Delta_x^2}{\sum_{\alpha=1}^{d} D_{\alpha\alpha}}$, Λ^- **is arbitrary.**

– **Stability/accuracy is adjusted with** Λ^+ $(\Lambda^\pm = \frac{1}{4})$.

– **LB with** $\Lambda^+ = \Lambda^- = \frac{1}{2} \Leftrightarrow$
Forward-time central scheme (FTCS)

Richards' equation in heterogeneous media



– **First order:** $[P^R \sum_{q\in I} t_q^R](\vec{r}^I) = [P^B \sum_{q\in I} t_q^B](\vec{r}^I)$

Continuity of the diffusion variable in stratified soil:

$P^R(\vec{r}^I) = P^B(\vec{r}^I) + O(\varepsilon^2)$ **if only**

$\sum_{q\in I} t_q^R = \sum_{q\in I} t_q^B$

- **Mixed form:**

  $P^R = h^R$, $P^B = h^B$ **then** $h^R(\vec{r}^I) = h^B(\vec{r}^I)$

- **Moisture content form:** $P^R = \theta^R$, $P^B = \theta^B$ **then**

  $\theta^R(\vec{r}^I) = \theta^B(\vec{r}^I)$, $h^R(\vec{r}^I) \neq h^B(\vec{r}^I)$

- **Kirchoff transform:** $P(\theta) = \int_{-\infty}^{h(\theta)} K(h')dh'$

  $h^R(\vec{r}^I) \neq h^B(\vec{r}^I)$ **if** $K^R(h) \neq K^B(h)$ or $h^R(\theta) \neq h^B(\theta)$

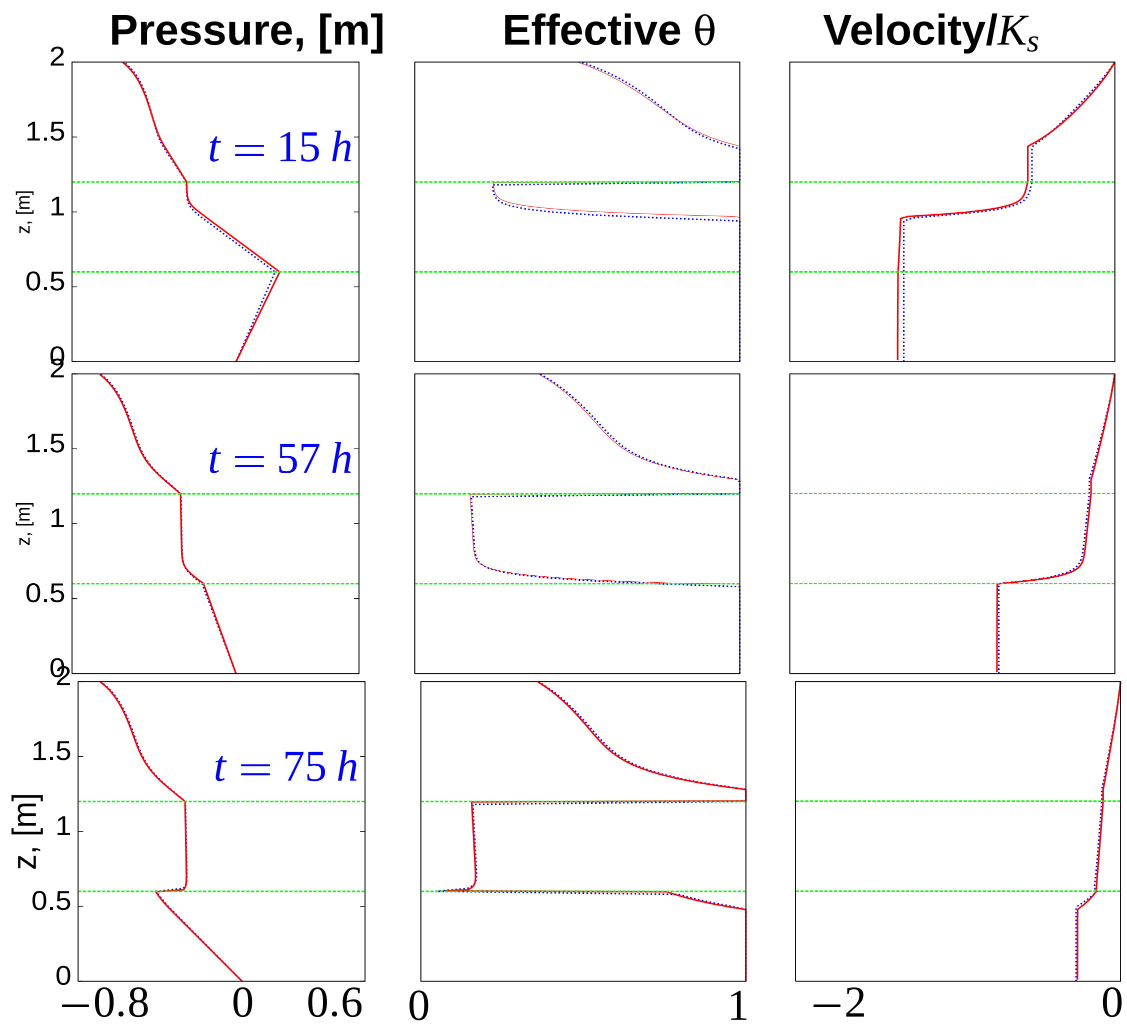
Drainage tube from Marinelli & Durnford (1998)

- Pressure head at the base ($z = 0$) is reduced from the hydrostatic to the atmospheric value

- medium-grained sand is in the middle between fine-grained sands: $K_s^{middle}/K_s^{top} = 100$

- Mixed LB formulation with $\Delta x = 1/150\ m$, $\Delta t = 1/150h$

- Minimum Δx of the adaptive implicit RK method is $\Delta x \approx 10^{-8} - 10^{-7} m$, $\Delta t = 3h$

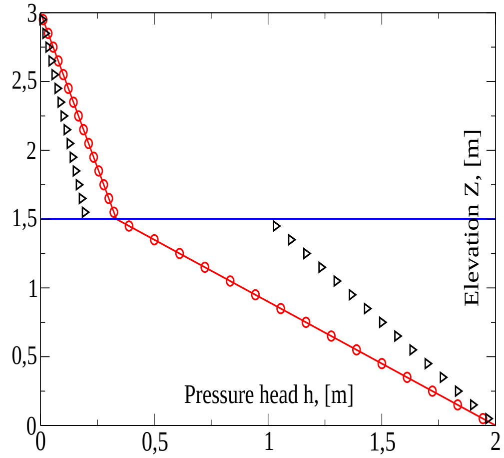


Anisotropic weights (TRT-A) or Anisotropic eigenvalues (LM-I)

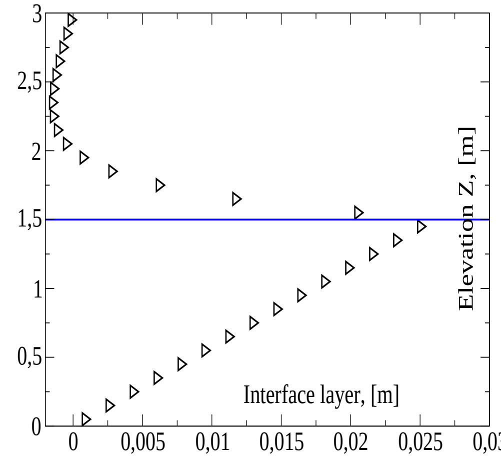
– **TRT-A** :
isotropic $\{\Lambda_q^-\} = \Lambda^-, \forall q$
anisotropic weights $\{t_q\}$
$P^R(\vec{r}^I) = P^B(\vec{r}^I)$ **if only**
$$\frac{\Lambda^{-B}}{\Lambda^{-R}} = \frac{\mathsf{D}^B_{zz}}{\mathsf{D}^R_{zz}}$$

– **LM-I** :
isotropic weights:
$$t_q^R = t_q^B = c_e t_q^\star, \forall q$$
anisotropic $\{\Lambda_q^-\}$
Exact only if
$$\frac{\Lambda_q^{-B}}{\Lambda_q^{-R}} = \frac{\mathsf{D}^B_{zz}}{\mathsf{D}^R_{zz}}$$

– **No interface layers if only** $\mathcal{T}_q^R \partial_q P^R = \mathcal{T}_q^B \partial_q P^B$, $\mathcal{T}_q = \Lambda_q^- t_q$
Vertical flow, necessary: $\mathcal{T}_q^B / \mathcal{T}_q^R = [t_q \Lambda_q^-]^B / [t_q \Lambda_q^-]^R = \mathsf{D}^B_{zz} / \mathsf{D}^R_{zz}$



TRT-A: discontinuous
e.g, $\Lambda^{-B} = \Lambda^{-R}$
$$h^R(\vec{r}^I)/h^B(\vec{r}^I) = \mathsf{D}^B_{zz}/\mathsf{D}^R_{zz} = 5$$

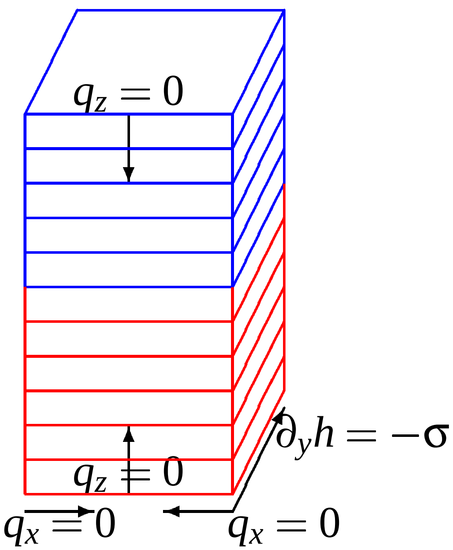


LM-I: correction to solution
e.g., **diagonal links:** $\mathcal{T}_q^B = \mathcal{T}_q^R = \mathcal{T}_\perp^R$,
vertical links: $\mathcal{T}_\perp^B / \mathcal{T}_\perp^R = 7$

Anisotropic heterogeneous stratified 3D box
following M. Bakker & K. Hemker, Adv. Water. Res. 2004

Anisotropic principal $xy-$ axis:

$$\begin{pmatrix} K_{xx}^{(i)} & K_{xy}^{(i)} & 0 \\ K_{xy}^{(i)} & K_{yy}^{(i)} & 0 \\ 0 & 0 & K_{zz}^{(i)} \end{pmatrix}$$



- **Problem:** $\nabla \cdot \mathbf{K}^R \nabla h^R = 0$, $z < 0$, $\nabla \cdot \mathbf{K}^B \nabla h^B = 0$, $z > 0$
- **Interface conditions:** $h^R(0^-) = h^B(0^+)$, $K_{zz}^R \partial_z h^R(0^-) = K_{zz}^B \partial_z h^B(0^+)$
- **Boundary conditions:**
  $q_x = -[K_{xx}\partial_x h + K_{xy}\partial_y h] \mid_{\pm X} = 0$, $\partial_y h \mid_{\pm Y} = -\sigma$, $q_z = -K_{zz}\partial_z h \mid_{\pm Z} = 0$
- **From 3D to 2D:** $h(x,y,z) = \phi(x,z) - \sigma y + h_r$, $h_r = h(0,0,0)$,
  $\partial_x \phi^{(i)}(\pm X) = g^{(i)}$, $g^{(i)} = K_{xy}^{(i)} / K_{xx}^{(i)} \sigma$
- **Three solutions can be distinguished for 2 layered system:**
  Invariant along x and z: $\phi(x,z) = 0$, if $g^B = g^R = 0$
  Linear along x, invariant along z: $\phi(x,z) = \frac{g^B + g^R}{2} x$, if $g^B = g^R$
  Non-linear: $\phi(x,z) = (\frac{g^B + g^R}{2} x + (g^B - g^R)\phi^\star(x,z))$, **if** $g^B \neq g^R$,
  $\partial_x \phi^\star(\pm X, z) = \frac{1}{2}\text{sign}(z)$

Ground water whirls: $\vec{u}(x,z) = (-K_{xx}^{(i)}\partial_x\phi^\star, -K_{zz}^{(i)}\partial_z\phi^\star)$

- **Analytical solution (2005) via Fourier series for** $\phi^\star(x,z)$

  $K_{xx}^R\partial_{xx}\phi^\star + K_{zz}^R\partial_{zz}\phi^\star = 0,\ z < 0$

  $K_{xx}^B\partial_{xx}\phi^\star + K_{zz}^B\partial_{zz}\phi^\star = 0,\ z > 0$

  $\forall K_{xx}^R, K_{xx}^B$ and K_{zz}^R, K_{zz}^B

- **Boundary:**

  $\partial_x\phi^\star(\pm X, z) = \frac{1}{2}\textbf{sign}(z)$

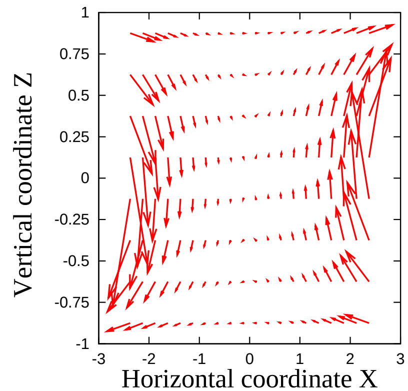
  $\partial_z\phi^\star(x, \pm Z) = 0$

- **Interface:**

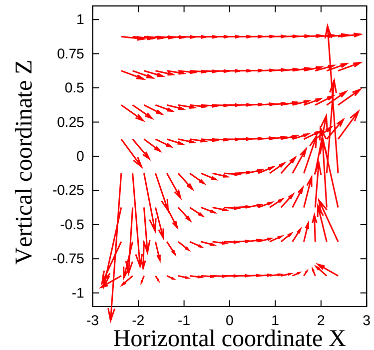
  $\phi^\star(x,0^+) = \phi^\star(x,0^-)$

  $K_{zz}^B\partial_z\phi^\star(x,0^+) = K_{zz}^R\partial_z\phi^\star(x,0^-)$

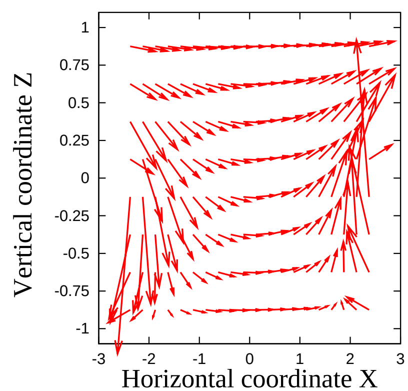
Isotropic tensors



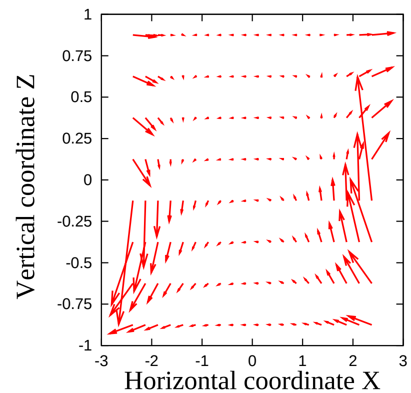
Proportional, $K_{\alpha\alpha}^B/K_{\alpha\alpha}^R = 5$



Horizontal, $K_{xx}^B/K_{xx}^R = 5$



Vertical, $K_{zz}^B/K_{zz}^R = 5$



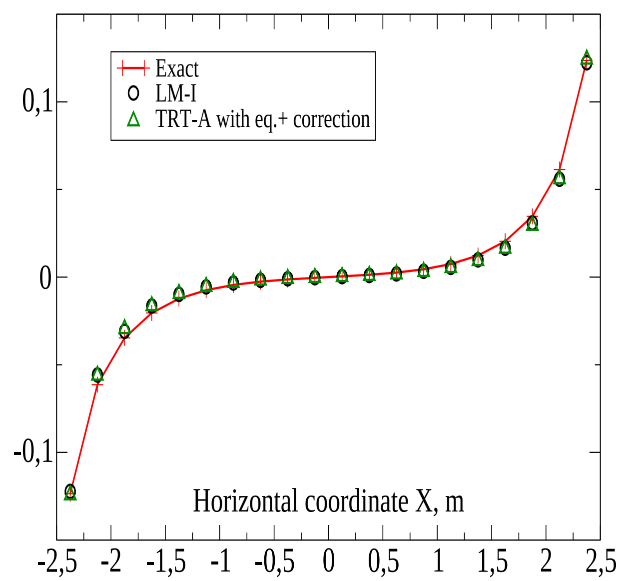
3D computations without interface flux corrections

$K_{zz}^B/K_{zz}^R = 5$, $K_{\tau\tau}^B = K_{\tau\tau}^R$, $g^{(i)} = K_{xy}^{(i)}/K_{xx}^{(i)} = \frac{1}{8}\mathbf{sign}(z)$

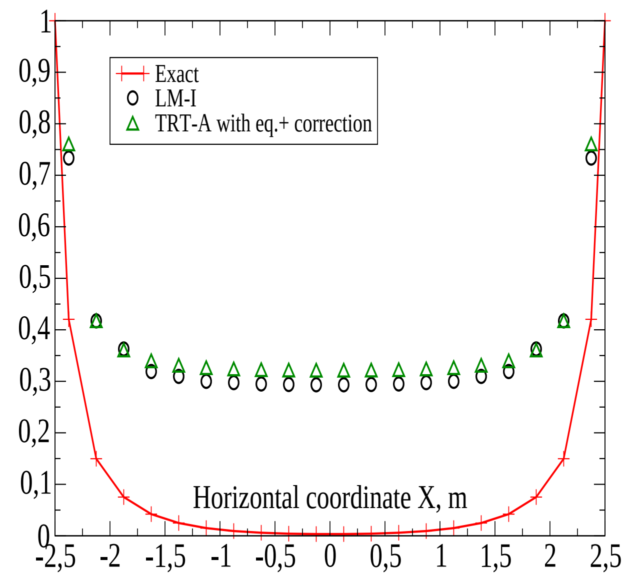
solution: $\phi(x,z) = (g^B - g^R)\phi^\star(x,z)$, $\partial_x\phi^\star(\pm X,z) = \frac{1}{2}\mathbf{sign}(z)$

interface links: $t_q^R\Lambda_q^{-R} \neq t_q^B\Lambda_q^{-B}$, $\Phi_{qx}^R(0^-) \neq \Phi_{qx}^B(0^+)$

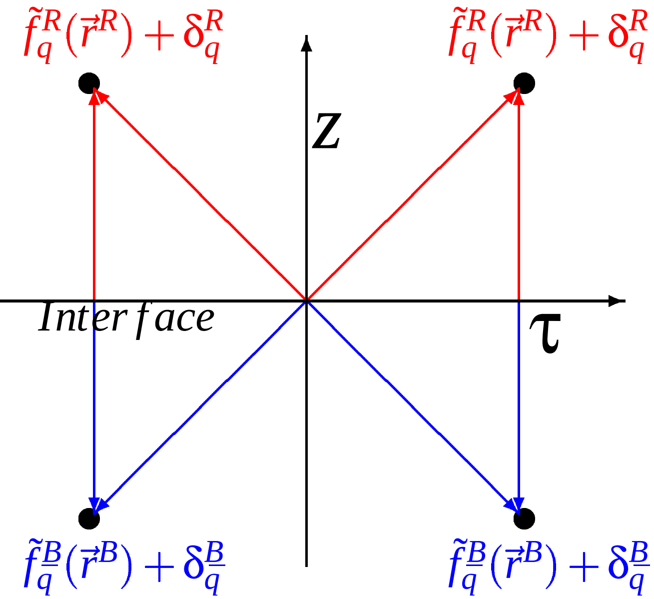
$\phi^\star(x,0^+) - \phi^\star(x,0^-)$



$\partial_x\phi^\star(x,0^+) - \partial_x\phi^\star(x,0^-)$



Leading order interface corrections



- **Piece-wise linear solution for $i^{th}-$ layer**:
  $f_q^{(i)} = [e_q + t_q \partial_q P(x,y,z)/\lambda_q^-]^{(i)}$, $P^R(\vec{r}^I) = P^B(\vec{r}^I)$
  Then $\partial_\tau P^R(\vec{r}^I) = \partial_\tau P^B(\vec{r}^I)$, and $D_{zz}^R \partial_z P^R(\vec{r}^I) = D_{zz}^B \partial_z P^B(\vec{r}^I)$
- **No interface layers if each flux component is continuous**:
  $\Phi_{q\alpha}^R = \Phi_{q\alpha}^B$, i.e. $t_q^R \Lambda_q^{-R} \partial_\alpha P^R c_{q\alpha} = t_q^B \Lambda_q^{-B} \partial_\alpha P^B c_{q\alpha}$, $\forall q \in I$
- **Piece-wise linear solutions for any anisotropy and heterogeneity via the interface corrections**

$$f_q(\vec{r}^B, t+1) = \tilde{f}_q^R(\vec{r}^R, t) + (\delta_q^{+\,R} + \delta_q^{-\,R}),$$
$$f_{\bar{q}}(\vec{r}^R, t+1) = \tilde{f}_{\bar{q}}^B(\vec{r}^B, t) + (\delta_{\bar{q}}^{+\,B} + \delta_{\bar{q}}^{-\,B})$$

Diffusion variable:$\delta_q^{+\,R} = S_q^R(r_E^R - 1)$, $S_q^R = e_q^{+R} + \frac{1}{2} m_q^R \approx e_q^{+R}(\vec{r}^I)$

Fluxes: $\delta_q^{-\,R} = \sum_{\alpha=\{x,y,z\}} \delta_{q\,\alpha}^{-\,R}$, $\delta_{q\,\alpha}^{-\,R} = \Phi_{q\alpha}^R (r_E^R r_\Lambda^R r_\alpha^R - 1)$
with ratios: $r_E^R = t_q^B / t_q^R$, $r_\Lambda^R = \Lambda_q^{-B} / \Lambda_q^{-R}$, $r_\alpha^R = [\partial_\alpha P^B / \partial_\alpha P^R](\vec{r}^I)$

- Neumann conditions via the modified bounce-back:
  $f_{\bar{q}}(\vec{r}_{\rm b}, t+1) =$
  $\tilde{f}_q(\vec{r}_{\rm b}, t) + \delta_n(\vec{r}_{\rm b}, t) + \delta_\tau(\vec{r}_{\rm b}, t)$

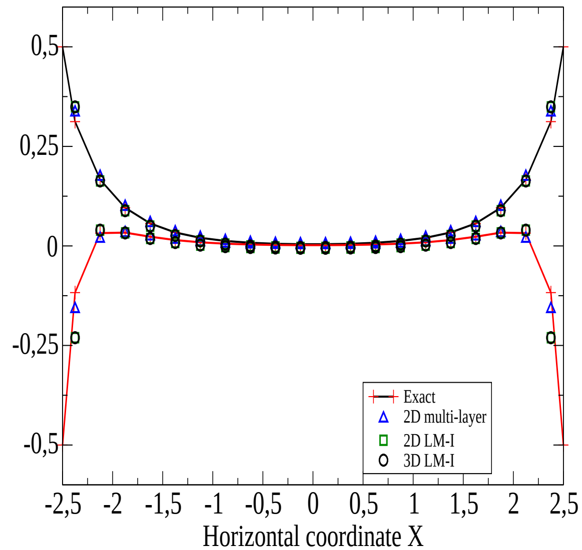
- Prescribed normal derivative:
  $\delta_n = -2\mathcal{T}_q \partial_n P(\vec{r}_{\rm w}, t) C_{qn}$

- Relaxed tangential derivatives:
  $\delta_\tau = -2\mathcal{T}_q \sum_\tau \partial_\tau P(\vec{r}_{\rm b}, t) C_{q\tau}$,
  $\partial_\tau P$ is derived from the current population solution

LM-I-model with interface corrections

Comparison with exact and "multi-layer" solutions for

$\partial_x \phi^{\star B}(x, 0^+)$ **and** $\partial_x \phi^{\star R}(x, 0^-)$, $K_{xy}^{(i)} = [K_{xx}^{(i)} \textbf{sign}(z)]/2$

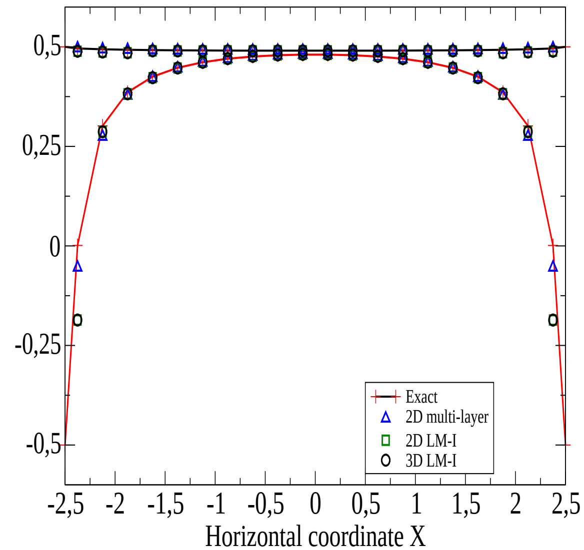


$K_{zz}^B / K_{zz}^R = 500$, $K_{\tau\tau}^B = K_{\tau\tau}^R$

diagonal links:

$\mathcal{T}_{d1}^B / \mathcal{T}_{d1}^R = \mathcal{T}_{d2}^R / \mathcal{T}_{d2}^B = 7$

vertical links: $\mathcal{T}_\perp^B / \mathcal{T}_\perp^R = 1498$



$K_{zz}^B / K_{zz}^R = 500$, $K_{\tau\tau}^B / K_{\tau\tau}^R = 100$

diagonal links:

$\mathcal{T}_{d1}^B / \mathcal{T}_{d1}^R = \mathcal{T}_{d2}^R / \mathcal{T}_{d2}^B = 100$

vertical links: $\mathcal{T}_\perp^B / \mathcal{T}_\perp^R = 1300$

Steady-state unconfined flow

computed with $h-$formulation on anisotropic grids

- **Reference solution:**
  Clement et al, J.Hydrol., 181, 1996

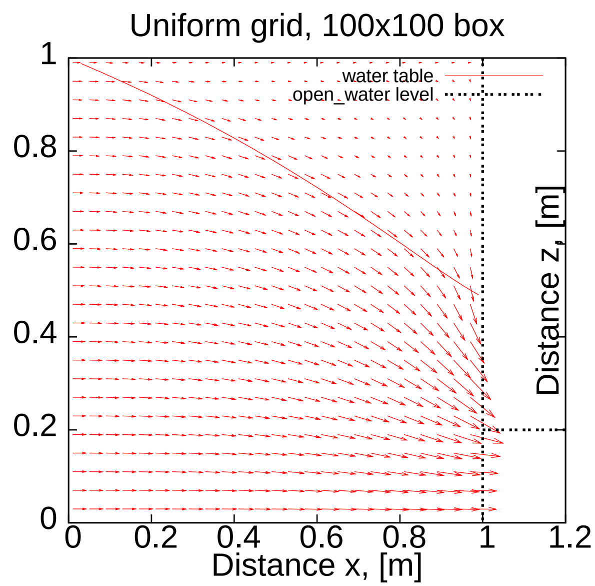
- **Boundary conditions:**

  Top/bottom: No-flow via bounce-back reflection
  $f_{\bar{q}}(\vec{r}_{\rm b}, t+1) = \tilde{f}_q(\vec{r}_{\rm b}, t)$

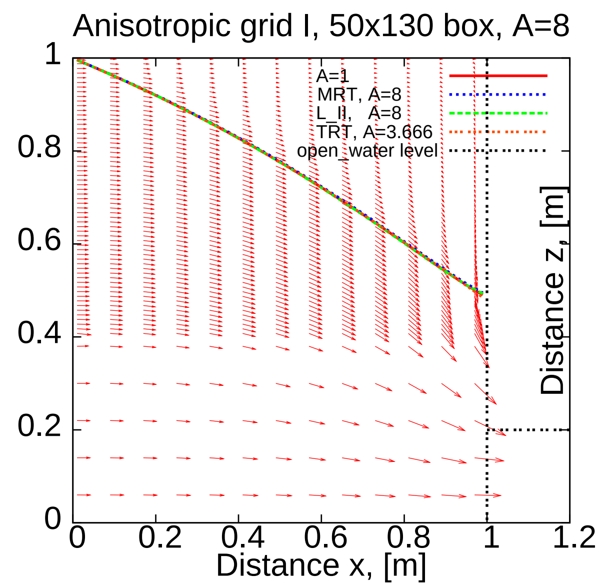
  West: Hydrostatic, $h^b(z) + z = 1\ m$ via **anti-bounce-back reflection**
  $f_{\bar{q}}(\vec{r}_{\rm b}, t+1) = -\tilde{f}_q(\vec{r}_{\rm b}, t) + 2t_q h^b(z)$

  East: Seepage above $z = 0.2\ m$

Uniform refining in $100^2 = 1^2\ m^2$ box



Anisotropic non-uniform refining $l_z^{bottom} = 0.5$, $l_z^{top} = 4$, $A = l_z^{top}/l_z^{bottom} = 8$



Solutions on the anisotropic grids

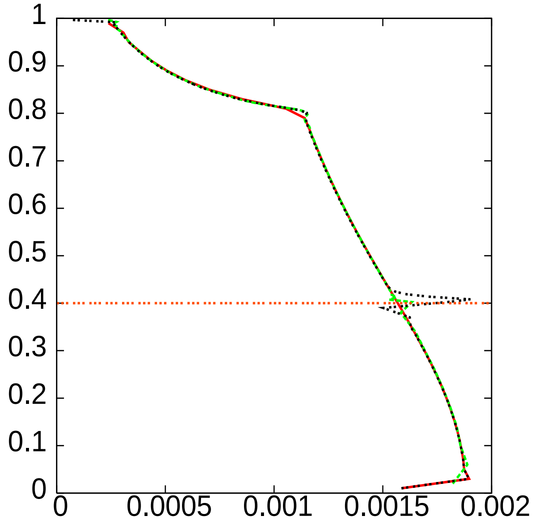
Uniform refining, $A = l_z^{top}/l_z^{bottom} = 1$

LM-I: $A = 8$ (isotropic weights, anisotropic ratios $\Lambda_q^{-top}/\Lambda_q^{-bottom}$)

TRT: $A = 3, (6)$ (anisotropic weights, isotropic ratio $\Lambda^{-top}/\Lambda^{-bottom}$)

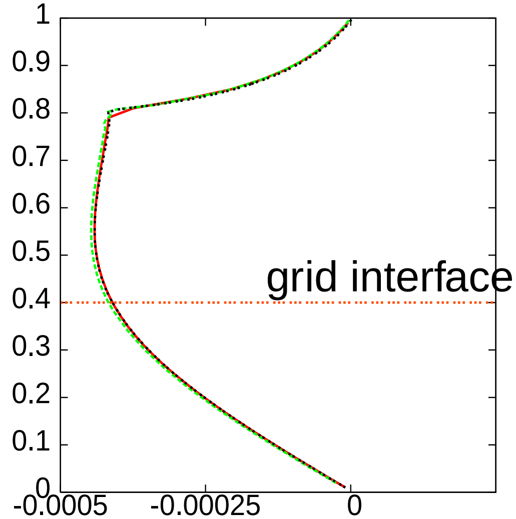
Horizontal velocity $u_x^{phys}(z)$

$[u_x^{lb^{bottom}}/u_x^{lb^{top}}](\vec{r}^I) = A$

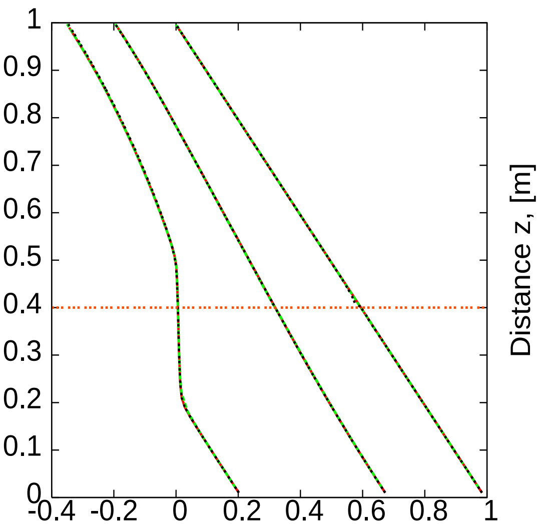


Vertical velocity $u_z^{phys}(z)$

$u_z^{lb^{bottom}}(\vec{r}^I) = u_z^{lb^{top}}(\vec{r}^I)$



Pressure head $h(z)$



AADE: interface collision operator (2005)

– **Prescribed continuity conditions:**

Diffusion variable: $e_q^{+R}(\vec{r}^I) = e_q^{+B}(\vec{r}^I) + O(\varepsilon^2)$

Advective-diffusive flux components:

$e_q^{-R} - \Lambda_q^{-R} m_q^R = e_q^{-B} - \Lambda_q^{-B} m_q^B$

Interface collision components:

(1) $e_q^{-I} = \frac{1}{2}(e_q^{-R} + e_q^{-B})$

Harmonic means:

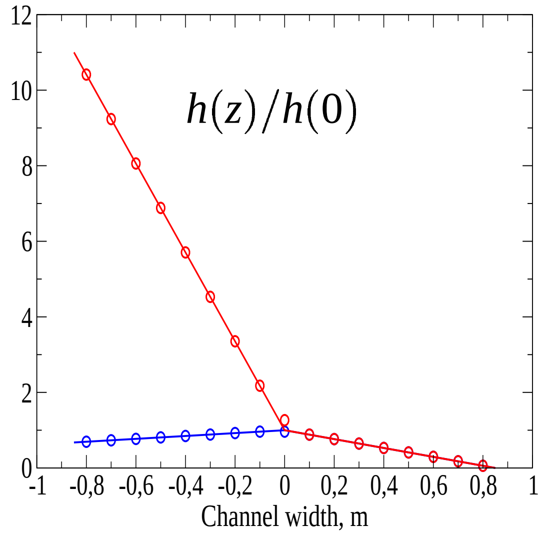
LM-I : $\Lambda_q^{-I} = \frac{2\Lambda_q^{-B}\Lambda_q^{-R}}{\Lambda_q^{-B}+\Lambda_q^{-R}}$ if $\Lambda_q^{-R}/\Lambda_q^{-B} = m_q^B/m_q^R$

TRT-A: $\Lambda^{-I} = \frac{2\Lambda^{-B}\Lambda^{-R}}{\Lambda^{-B}+\Lambda^{-R}}$ if $\Lambda^{-R}/\Lambda^{-B} = \sum_{q\in I} m_q^B / \sum_{q\in I} m_q^R$

(2) $p_q^I = \frac{1}{2}(p_q^R + p_q^B)$

(3) **Mass source:** $Q_q^{+I} = \frac{1}{2}(Q_q^{+R} + Q_q^{+B})$

(4) **Deficiency:** $P^I = \frac{1}{2}(P^{(R)} + P^{(B)}) + \Delta P$, $\Delta P = \frac{1}{4}(\partial_z P^{(B)} - \partial_z P^{(R)})$



Harmonic mean:

$$K_{zz}^I = \frac{2K_{zz}^R K_{zz}^B}{K_{zz}^R + K_{zz}^B}$$

if

$\Lambda_q^{-R}/\Lambda_q^{-B} = \Lambda^{-R}/\Lambda^{-B} = K_{zz}^R/K_{zz}^B$

With or without convection:

$\nabla \cdot \mathbf{K}\nabla(h + \vec{1}_g) = 0$ or

$\nabla \cdot \mathbf{K}\nabla h = 0$

Topics of International Conference for Mesoscopic Methods in Engineering and Science (ICMMES), www.icmmes.org

LB Method:

- Kinetic schemes
- Finite volume and finite-difference LB
- Adaptive grids
- Thermal (hybrid) schemes
- Comparative studies of LBE, FE and FV

• Difficult problems:

- Stabilization for high Reynolds numbers
- Stabilization for high density ratios
- Stability of boundary schemes.

Applications:

- Porous media: flow+dispersion, capillary functions, relative permeabilities, acoustic properties,...
- Direct Numerical Simulations including Large-Eddy Simulations (LES), e.g. for external aerodynamics of a car
- Rheology for complex fluids:
  - (1) particulate suspensions
  - (2) foaming process
  - (3) multi-phase and multi-component fluids
  - (4) non-Newtonian and bio-fluids
- Flow-structure interactions and Micro-fluidics (non-continuum effects)
- Parallel, physically based animations of fluids.

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