Méthodes d'éléments finis mixtes pour pour les problèmes du second ordre

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Outline

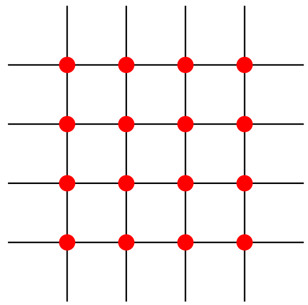
- A menagerie of approximation methods
- The model problem
- Continuous mixed formulation
- Approximation spaces
- Mixed finite element formulation
- The resulting linear system
- Reducing the mixed method to the finite volume method for rectangles
- Reducing the mixed method to the finite volume method for triangles
- A problem with difformed hexahedres
- The mixed-hybrid finite elements formulation
- Nonconforming finite elements

INTRODUCTION

Vertex-centered approximation methods

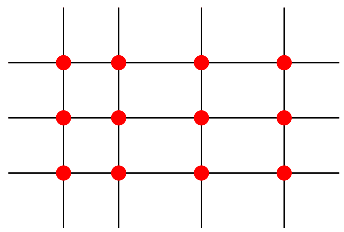
The degrees of freedom are located at the vertices of the mesh

Finite differences

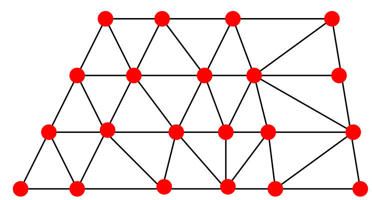


Finite elements

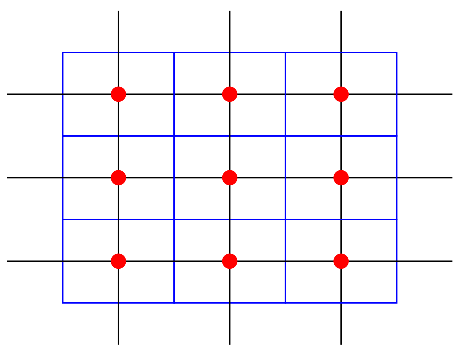
Q_1

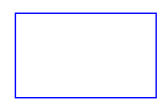


P_1



Vertex-centered finite volumes

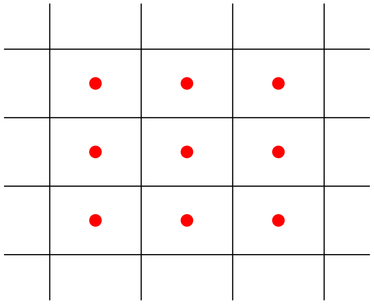




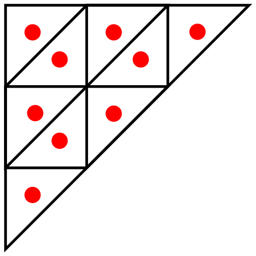
control volume

Cell-centered approximation methods

Mixed finite elements

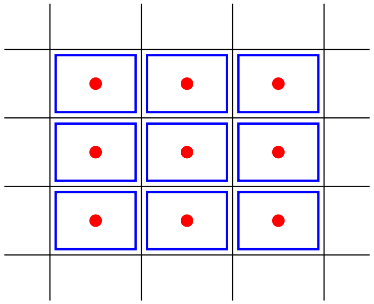


on rectangles



on triangles

Finite volumes

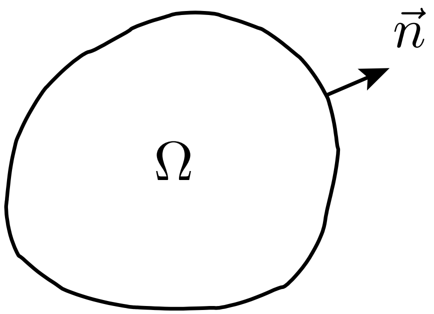


Unknowns : average value in each cell   Control volume = cell

Nodal methods in neutronics

The model problem

$$
\begin{aligned}
\text{div}(-K\,\vec{\text{grad}}\,p) &= f && \text{in } \Omega \\
p &= \overline{p} && \text{on } \partial\Omega \quad \text{if Dirichlet} \\
-K\frac{\partial p}{\partial n} &= g && \text{on } \partial\Omega \quad \text{if Neumann}
\end{aligned}
$$



For flow in porous media :

p, pressure K, permeability $\vec{u} = -K\,\vec{\text{grad}}\,p$, Darcy velocity

$$K(x) = \begin{bmatrix} k^1(x) & k^{12}(x) \\ k^{12}(x) & k^2(x) \end{bmatrix}, \quad 0 < \underline{\kappa}|\vec{v}|^2 \le (K(x)\vec{v},\vec{v}) \le \overline{\kappa}|\vec{v}|^2, \ \forall \vec{v} \in \mathbb{R}^2.$$

The Sobolev space $H^1(\Omega)$.

$$H^0(\Omega) = L^2(\Omega) \qquad \|q\|^2_{0,\Omega} = \int_\Omega q^2(x)dx$$

$$H^1(\Omega) = \{q \in L^2(\Omega);\ \vec{\text{grad}}\, q \in (L^2(\Omega))^2\}$$

$$\|q\|^2_{1,\Omega} = \|q\|^2_{0,\Omega} + |q|^2_{1,\Omega} \qquad |q|^2_{1,\Omega} = \int_\Omega |\vec{\text{grad}}\, q|^2(x)dx$$

The trace $q|_\Gamma$ of $q \in H^1(\Omega)$ is in $H^{1/2}(\Gamma)$.

The trace $\frac{\partial q}{\partial n}|_\Gamma$ of $q \in H^1(\Omega)$ is in $H^{-1/2}(\Gamma)$, the dual space of $H^{1/2}(\Gamma)$.

Weak primal formulation

Assume $K \in L^\infty(\Omega), f \in L^2(\Omega)$.

- Neumann boundary conditions: $g \in H^{-1/2}(\partial\Omega)$.

Find $p \in H^1(\Omega)$ such that

$$\int_\Omega K \, \vec{\text{grad}}\, p \cdot \vec{\text{grad}}\, q = \int_\Omega f q - < g, q >, \quad q \in H^1(\Omega).$$

- Dirichlet boundary conditions: $\overline{p} \in H^{1/2}(\partial\Omega)$.

Find $p \in V_{\overline{p}} = \{q \in H^1(\Omega), q = \overline{p} \text{ on } \partial\Omega\}$ such that

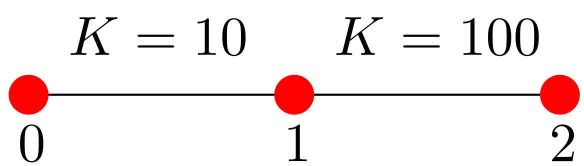
$$\int_\Omega K \, \vec{\text{grad}}\, p \cdot \vec{\text{grad}}\, q = \int_\Omega f q, \quad q \in V_0.$$

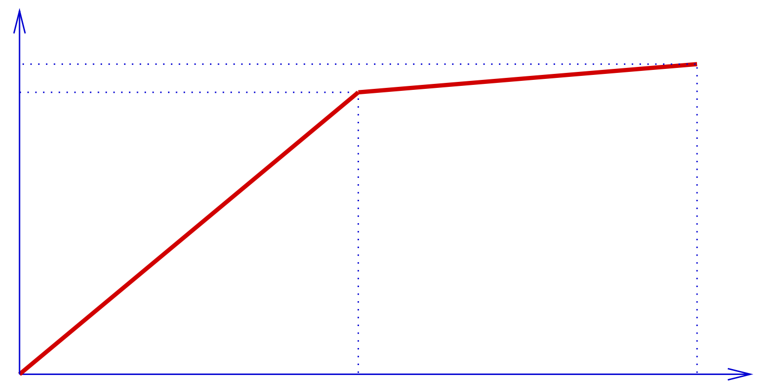
Problem : how to calculate $\vec{u}$ from p ?

APPROXIMATION WITH MIXED FINITE ELEMENTS

An example with a discontinuous K

In one dimension, $\Omega =]0,2[,\ f = 0,\ p(0) = 0,\ p(2) = 1$.





$$\frac{\partial u}{\partial x} = 0 \quad \Longrightarrow u \text{ constant thus very smooth.}$$

p continuous, piecewise linear, $\dfrac{\partial p}{\partial x}$ discontinuous at $x = 1 \Longrightarrow p$ is not smooth.

u has a physical meaning and is a good mathematical and numerical unknown.

Mixed formulation

Write the elliptic problem as a system of first order equations:

$$\text{div}\,\vec{u} = f, \quad \vec{u} = -K\,\vec{\text{grad}}\,p, \text{ in } \Omega$$

$$p = \overline{p} \text{ on } \Gamma_D, \quad \vec{u}\cdot\vec{n} = g \text{ on } \Gamma_N, \quad \Gamma_N \cup \Gamma_D = \Gamma = \partial\Omega.$$

Assume that $f \in L^2(\Omega)$ so div$\vec{u} \in L^2(\Omega)$.
Therefore we take $\vec{u} \in H(\text{div}, \Omega) = \{\vec{v} \in (L^2(\Omega))^2; \text{div}\,\vec{v} \in L^2(\Omega)\}$.

Multiply the second equation by K^{-1}, then by $\vec{v}$, integrate over Ω and by parts. We obtain $\displaystyle\int_\Omega (K^{-1}\vec{u})\cdot\vec{v} - \int_\Omega p\text{div}\vec{v} = - < p, \vec{v}\cdot\vec{n} >$

Recall Green's formula: $\displaystyle\int_\Omega \vec{\text{grad}}\,q\cdot\vec{v} + \int_\Omega q\,\text{div}\vec{v} = \int_\Gamma q\vec{v}\cdot\vec{n}$.

It is sufficient to take $p \in \mathcal{M} = L^2(\Omega), \vec{u} \in \mathcal{W} = H(\text{div}, \Omega)$.

Properties of $\mathcal{W} = H(\textbf{div}, \Omega)$

$H(\text{div}, \Omega) = \{\vec{v} \in (L^2(\Omega))^2; \text{div } \vec{v} \in L^2(\Omega)\}$ is an Hilbert space with norm

$$\|\vec{v}\|_{H(\text{div},\Omega)} = \|\vec{v}\|_{L^2(\Omega)} + \|\text{div}\vec{v}\|_{L^2(\Omega)}$$

Traces $(\vec{v} \cdot \vec{n})|_\Gamma$ of functions $\vec{v}$ of $H(\text{div}, \Omega)$ are in $H^{-1/2}(\Gamma)$,

so boundary data must be such that $\overline{p} \in H^{1/2}(\Gamma_D), \quad g \in H^{-1/2}(\Gamma_N)$.

Also the space $\mathcal{V} = \{\vec{v} \in \mathcal{W}; \text{div } \vec{v} = 0\}$ will play an important role.

Notations

The data are $f \in L^2(\Omega), \overline{p} \in H^{1/2}(\Gamma_D), g \in H^{-1/2}(\Gamma_N)$.

The spaces are $\mathcal{M} = L^2(\Omega)$, $\mathcal{W} = H(\text{div}, \Omega)$, $\mathcal{W}_g = \{\vec{v} \in \mathcal{W}; \vec{v} \cdot \vec{n} = g \text{ on } \Gamma_N\}$.

Introduce the forms

$$
\begin{array}{llll}
a: & (L^2(\Omega))^2 \times (L^2(\Omega))^2 & \longrightarrow \mathbb{R}, & a(\vec{u}, \vec{v}) = \int_\Omega (K^{-1}\vec{u}) \cdot \vec{v}, \\
b: & \mathcal{W} \times \mathcal{M} & \longrightarrow \mathbb{R}, & b(\vec{v}, q) = \int_\Omega q \, \text{div} \vec{v}, \\
l_{\mathcal{W}}: & \mathcal{W} & \longrightarrow \mathbb{R}, & l_{\mathcal{W}}(\vec{v}) = \int_{\Gamma_D} -\overline{p} \, \vec{v} \cdot \vec{n}, \\
l_{\mathcal{M}}: & \mathcal{M} & \longrightarrow \mathbb{R}, & l_{\mathcal{M}}(\vec{v}) = \int_\Omega f q.
\end{array}
$$

Mixed formulation

The problem is

$$(\mathcal{P}_m)\left\{\begin{array}{llll} \multicolumn{4}{l}{\text{Find } \vec{u}\in\mathcal{W}_g \text{ and } p\in\mathcal{M} \text{ such that}} \\ a(\vec{u},\vec{v}) \quad -b(\vec{v},p) & = & l_{\mathcal{W}}(\vec{v}), & \vec{v}\in\mathcal{W}_0, \\ b(\vec{u},q) & = & l_{\mathcal{M}}(q), & q\in\mathcal{M}. \end{array}\right.$$

$a,\ b,\ l_{\mathcal{W}},\ l_{\mathcal{M}}$ continuous

a $\mathcal{V}-$elliptic i.e. $a(v,v)\geq\overline{\kappa}\|v\|_0^2$ for all $v\in\mathcal{V}$

inf-sup condition $\inf_{\{q\in\mathcal{M}:\|q\|_{\mathcal{M}}=1\}}\ \sup_{\{v\in\mathcal{W}:\|v\|_{\mathcal{W}}=1\}} b(v,q)>0$

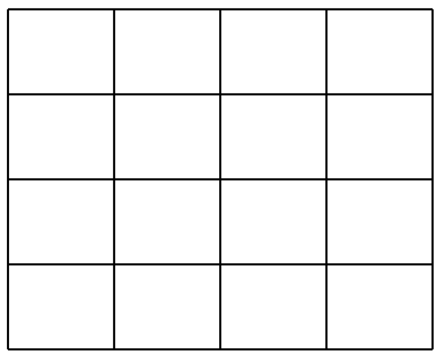
Brezzi's theorem $\Longrightarrow$ $\exists$! solution to the problem $(\mathcal{P}_m)$.

Discretization of the domain

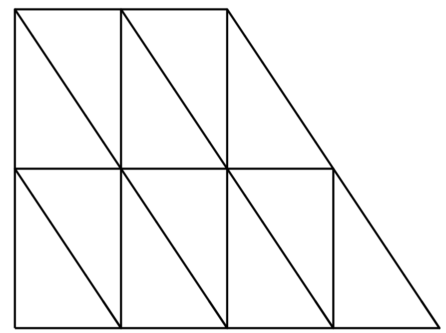
1. Let $\mathcal{T}_h$ be a discretization of Ω.
   $\mathcal{A}_h$ be the set of edges.
   h the largest diameter of the cells.

   Card$(\mathcal{T}_h)$ $= ne =$ number of cells.
   Card$(\mathcal{A}_h)$ $= na =$ number of edges

a rectangular mesh



a triangular mesh



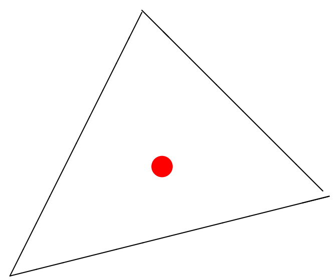
2. Define

- an approximation p_h of p in $\mathcal{M}_h$, a finite dimensional subset of $\mathcal{M}$
- an approximation $\vec{u}_h$ of $\vec{u}$ in $\mathcal{W}_h$, a finite dimensional subset of $\mathcal{W}$.

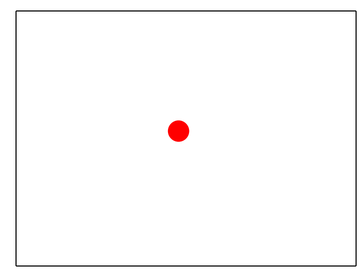
Approximation space for the scalar unknown

$\mathcal{M}_h$ = the space of functions q_h in $\mathcal{M}$ which are

constant over each triangle



constant over each rectangle



dim $\mathcal{M}_h$ = ne = number of elements

The degrees of freedom are
p_T an approximation of the average value of p over the cell T, $\quad T \in \mathcal{T}_h$.

A basis is $\{\chi_T\}_{T \in \mathcal{T}_h}$ such that $\chi_T(x) = \begin{cases} 1 & \text{if } x \in T \\ 0 & \text{otherwise} \end{cases}$

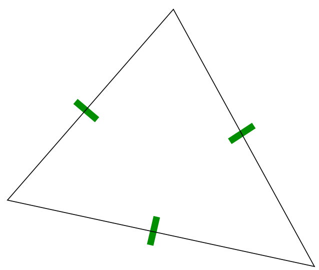
Then $p_h = \sum_{T \in \mathcal{T}_h} p_T \chi_T$

Approximation of the vector unknown

$\mathcal{W}_h = \{\vec{v}_h \in \mathcal{W}; \vec{v}_h|T \in \mathcal{W}_T, T \in \mathcal{T}_h\}$.

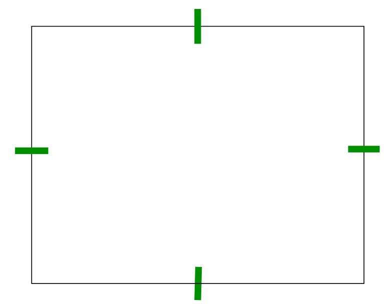
on each triangle

$$\mathcal{W}_T = \{\vec{v}_h; \vec{v}_{h|T} = a \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} + \begin{pmatrix} b \\ c \end{pmatrix}\}$$



on each rectangle

$$\mathcal{W}_T = \{\vec{v}_h; \vec{v}_{h|T} = \begin{pmatrix} ax_1 + b \\ cx_2 + d \end{pmatrix}\}$$



Functions $\vec{v} \in \mathcal{W}_T$ are uniquely defined by $\int_E \vec{v} \cdot \vec{n}_T, E \subset \partial T$.

On remarque que div$\mathcal{W}_h = \mathcal{M}_h$.

dim$\mathcal{W}_h$ = na = number of edges.

The degrees of freedom for $\mathcal{W}_h$ are

u_E an approximation of the flow rate of $\vec{u}$ across E, $\displaystyle\int_E \vec{u} \cdot \vec{n}_E,\ E \in \mathcal{A}_h$,

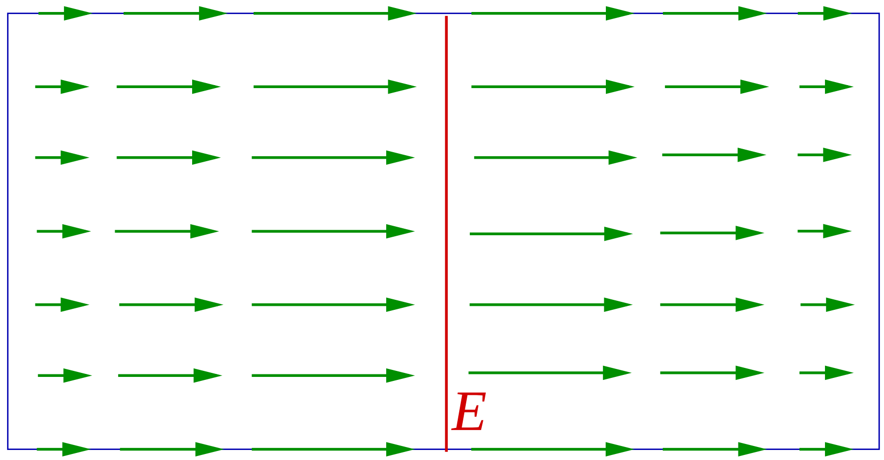
$\vec{n}_E$ a chosen unit normal to E.

A basis of $\mathcal{W}_h$ is $\{\vec{v}_E\}_{E \in \mathcal{A}_h}$ such that $\displaystyle\int_F \vec{v}_E \cdot \vec{n}_F = \delta_{E,F}, F \in \mathcal{A}_h$.

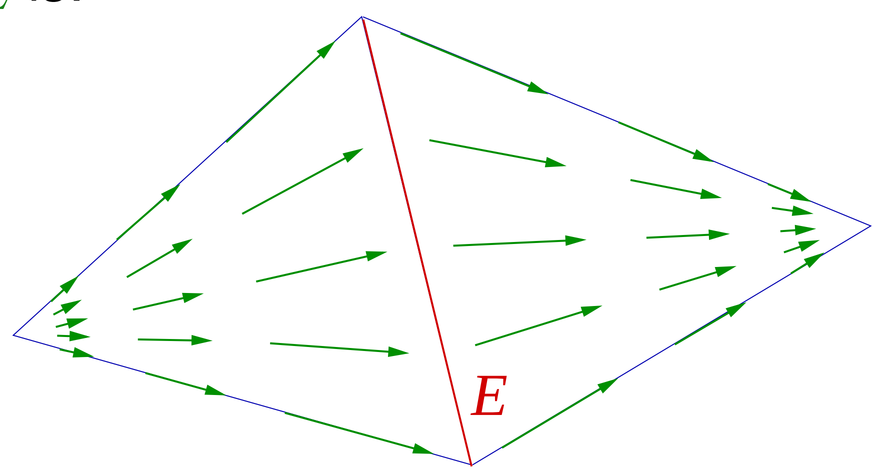
Then, $\displaystyle\vec{u}_h = \sum_{E \in \mathcal{A}_h} u_E \vec{v}_E$.

Basis functions of $\mathcal{W}_h$

- For rectangles $\vec{v}_E$ is:



- For triangles $\vec{v}_E$ is:



The approximation problem

Assume the data $\overline{p}$, g are piecewise constant on the edges $E \subset \Gamma$.

Introduce $\mathcal{W}_{hg} = \{\vec{v} \in \mathcal{W}_h ; \vec{v} \cdot \vec{n} = g \text{ on } \Gamma_N\}$

The approximation problem is

$$(\mathcal{P}_{mh}) \begin{cases} \text{Find } \vec{u}_h \in \mathcal{W}_{hg} \text{ and } p_h \in \mathcal{M}_h \text{ such that} \\ a(\vec{u}_h, \vec{v}_h) \quad -b(\vec{v}_h, p_h) \quad = \quad l_{\mathcal{W}}(\vec{v}_h), \qquad \vec{v}_h \in \mathcal{W}_{h0}, \\ b(\vec{u}_h, q) \quad = \quad l_{\mathcal{M}}(q_h), \qquad q_h \in \mathcal{M}_h. \end{cases}$$

Then $\exists$ a constant C independent of h such that

$$\|p - p_h\|_{\mathcal{M}} + \|\vec{u} - \vec{u}_h\|_{\mathcal{W}} \leq C \left\{ \inf_{q_h \in \mathcal{M}_h} \|p - q_h\|_{\mathcal{M}} + \inf_{\vec{v}_h \in \mathcal{W}_h} \|\vec{u} - \vec{v}_h\|_{\mathcal{W}} \right\}.$$

Keypoint for the error estimates: the discrete inf-sup condition

- Interpolation operators

1) $\pi_h : L^2(\Omega) \longrightarrow \mathcal{M}_h; \;\; \pi_h(q) = \sum_{T \in \mathcal{T}_h} q_T \chi_T, \quad q_T = \frac{1}{|T|}\int_T q$

2) $\Pi_h : (H^1(\Omega))^n \longrightarrow \mathcal{W}_h; \;\; \Pi_h(\vec{v}) = \sum_{E \in \mathcal{A}_h} v_E \vec{v}_E, \quad v_E = \int_E \vec{v} \cdot \vec{n}_T$.

- The following diagram commutes:

$$\begin{array}{ccc} (H^1(\Omega))^n \subset H(\text{div}, \Omega) & \xrightarrow{\text{div}} & L^2(\Omega) \\ \downarrow \Pi_h & & \downarrow \pi_h \\ \mathcal{W}_h & \xrightarrow{\text{div}} & \mathcal{M}_h. \end{array}$$

- Norm of Π_h independent of h

$\Longrightarrow$

inf-sup condition on approximation spaces with a constant independent of h.

Error bounds

Interpolation Theorem *If* $\{\mathcal{T}_h : h \in \mathcal{H}\}$ *is a regular family of triangulations of* $\overline{\Omega}$*, then* $\exists\, C > 0$, *independent of* h*, such that*

$$\|q - \pi_h(q)\|_{0,\Omega} \leq C\, h|q|_{1,\Omega}, \quad \forall q \in H^1(\Omega),$$

$$\|\vec{v} - \Pi_h\vec{v}\|_{0,\Omega} \leq C\, h|\vec{v}|_{1,\Omega}, \quad \forall \vec{v} \in (H^1(\Omega))^n,$$

$$\|\mathrm{div}\vec{v} - \mathrm{div}\Pi_h\vec{v}\|_{0,\Omega} \leq C\, h|\mathrm{div}\vec{v}|_{1,\Omega}, \quad \forall \vec{v} \in (H^1(\Omega))^n \textit{with}\ \mathrm{div}\vec{v} \in H^1(\Omega).$$

$\Longrightarrow \exists$ a constant C independent of h such that

$$\|p - p_h\|_{\mathcal{M}} + \|\vec{u} - \vec{u}_h\|_{\mathcal{W}} \leq Ch[\ |q|_{1,\Omega} + |\vec{v}|_{1,\Omega} + |\mathrm{div}\vec{v}|_{1,\Omega}\ ].$$

Discrete equations

The unknowns are the degrees of freedom:

Find $\{p_T\}_{T\in\mathcal{T}_h}, \{u_E\}_{E\in\mathcal{A}_h}$ such that

$$\begin{aligned}
\int_\Omega K^{-1}\sum_{F\in\mathcal{A}_h} u_F\vec{v}_F\cdot\vec{v}_E - \int_\Omega \sum_{T\in\mathcal{T}_h} p_T\chi_T \,\text{div}\,\vec{v}_E &= \int_{\Gamma_D} -\bar{p}\,\vec{v}_E\cdot\vec{n}, \quad E\in\mathcal{A}_h, E\not\subset\Gamma_N \\
\int_\Omega \text{div}\sum_{E\in\mathcal{A}_h} u_E\vec{v}_E\,\chi_T &= \int_\Omega f\,\chi_T, \qquad T\in\mathcal{T}_h
\end{aligned}$$

$u_E = g|E|, \quad E\subset\Gamma_N$ (assuming $\vec{n}_E = \vec{n}$)

Find $\{p_T\}_{T\in\mathcal{T}_h}, \{u_E\}_{E\in\mathcal{A}_h}$ such that

$$\begin{aligned}
\sum_{F\in\mathcal{A}_h} u_F\int_\Omega K^{-1}\vec{v}_F\cdot\vec{v}_E - \sum_{T\in\mathcal{T}_h} p_T\int_\Omega \chi_T\,\text{div}\,\vec{v}_E &= \int_{\Gamma_D} -\bar{p}\,\vec{v}_E\cdot\vec{n}, \quad E\in\mathcal{A}_h, E\not\subset\Gamma_N \\
\sum_{E\in\mathcal{A}_h} u_E\int_\Omega \text{div}\vec{v}_E\,\chi_T &= \int_\Omega f\,\chi_T, \qquad T\in\mathcal{T}_h
\end{aligned}$$

$u_E = g|E|, \quad E\subset\Gamma_N.$

Algebraic system

This leads to the linear system

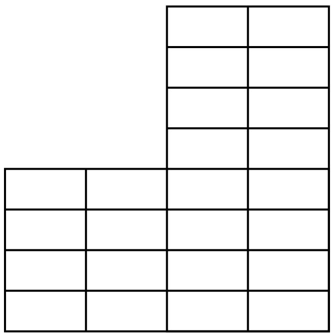
$$\begin{bmatrix} A & -{}^tD \\ D & 0 \end{bmatrix} \begin{bmatrix} U \\ P \end{bmatrix} = \begin{bmatrix} F_v \\ F_q \end{bmatrix}$$

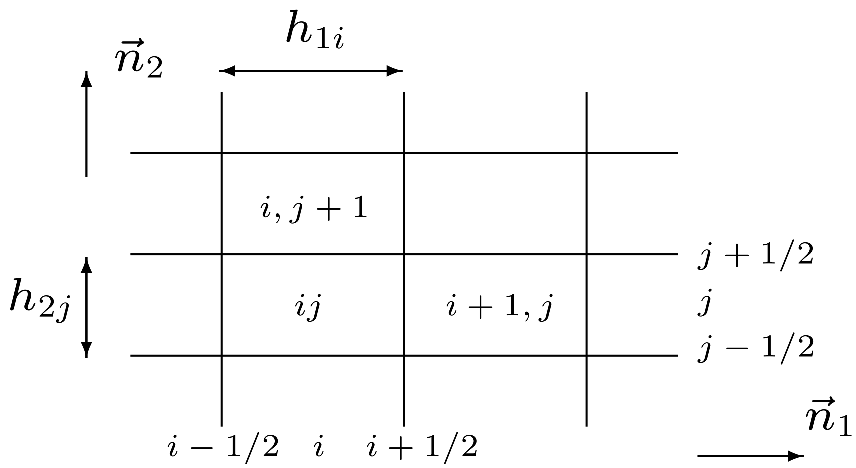
with $P = \{p_T\}_{T \in \mathcal{T}_h}, U = \{u_E\}_{E \in \mathcal{A}_h, E \not\subset \Gamma_N}$.

This linear system is not positive-definite.

- For triangles A has 5 nonzero entries per row
- For quadrilaterals A has 7 nonzero entries per row

On a rectangular mesh





We take $\vec{n}_E = \vec{n}_1$ if E is vertical, $\vec{n}_E = \vec{n}_2$ if E is horizontal.

Note that

$$\sum_{E \in \mathcal{A}_h} u_E \int_\Omega \text{div} \vec{v}_E \, \chi_T = \sum_{E \subset \partial T} u_E \int_T \text{div} \vec{v}_E.$$

Thus the second discrete equation gives

$$u_{i+1/2,j} - u_{i,j-1/2} + u_{i,j+1/2} - u_{i-1/2,j} = \int_{T_{ij}} f$$

Consider now the first discrete equation.

Denote $\mathcal{N}(E)$ the set of the $\begin{array}{ll} 2 & \text{cells adjacent to } E \text{ if } E \not\subset \Gamma \\ 1 & \text{cell adjacent to } E \text{ if } E \subset \Gamma \end{array}$

- If $E = E_{i+1/2,j}, E \not\subset \Gamma_D$:

$$\sum_{T \in \mathcal{T}_h} p_T \int_\Omega \chi_T \operatorname{div} \vec{v}_E = \sum_{T \in \mathcal{N}(E)} p_T \int_T \operatorname{div} \vec{v}_E = p_{ij} - p_{i+1,j}.$$

- If $E = E_{i+1/2,j}, E \subset \Gamma_D$:

$\sum_{T \in \mathcal{T}_h} p_T \int_\Omega \chi_T \operatorname{div} \vec{v}_E = p_{ij}$ assuming E lies on the right of the domain,

$\sum_{T \in \mathcal{T}_h} p_T \int_\Omega \chi_T \operatorname{div} \vec{v}_E = -p_{i+1,j}$ assuming E lies on the left of the domain.

- If $E = E_{i+1/2,j}, E \not\subset \Gamma$:

$$\sum_{F \in \mathcal{A}_h} u_F \int_\Omega K^{-1} \vec{v}_F \cdot \vec{v}_E = \sum_{T \in \mathcal{N}(E)} \sum_{F \in \partial T} u_F \int_\Omega K^{-1} \vec{v}_F \cdot \vec{v}_E =$$

$$u_{i+1/2,j} \int_{T_{ij}} K^{-1} \vec{v}_{i+1/2,j} \cdot \vec{v}_{i+1/2,j} + u_{i-1/2,j} \int_{T_{ij}} K^{-1} \vec{v}_{i-1/2,j} \cdot \vec{v}_{i+1/2,j} +$$

$$u_{i,j+1/2} \int_{T_{ij}} K^{-1} \vec{v}_{i,j+1/2} \cdot \vec{v}_{i+1/2,j} + u_{i,j-1/2} \int_{T_{ij}} K^{-1} \vec{v}_{i,j-1/2} \cdot \vec{v}_{i+1/2,j} +$$

$$u_{i+1/2,j} \int_{T_{i+1,j}} K^{-1} \vec{v}_{i+1/2,j} \cdot \vec{v}_{i+1/2,j} + u_{i+3/2,j} \int_{T_{i+1,j}} K^{-1} \vec{v}_{i+3/2,j} \cdot \vec{v}_{i+1/2,j} +$$

$$u_{i+1,j+1/2} \int_{T_{i+1,j}} K^{-1} \vec{v}_{i+1,j+1/2} \cdot \vec{v}_{i+1/2,j} + u_{i+1,j-1/2} \int_{T_{i+1,j}} K^{-1} \vec{v}_{i+1,j-1/2} \cdot \vec{v}_{i+1/2,j}$$

- If $E \subset \Gamma$, say for instance $i = 0$ (left boundary):

$$\sum_{F \in \mathcal{A}_h} u_F \int_\Omega K^{-1} \vec{v}_F \cdot \vec{v}_E = \sum_{T \in \mathcal{N}(E)} \sum_{F \in \partial T} u_F \int_\Omega K^{-1} \vec{v}_F \cdot \vec{v}_E =$$

$$u_{1/2,j} \int_{T_{1,j}} K^{-1} \vec{v}_{1/2,j} \cdot \vec{v}_{1/2,j} + u_{3/2,j} \int_{T_{1,j}} K^{-1} \vec{v}_{3/2,j} \cdot \vec{v}_{1/2,j} +$$

$$u_{1,j+1/2} \int_{T_{1,j}} K^{-1} \vec{v}_{1,j+1/2} \cdot \vec{v}_{1/2,j} + u_{1,j-1/2} \int_{T_{1,j}} K^{-1} \vec{v}_{1,j-1/2} \cdot \vec{v}_{1/2,j}$$

Denote $K^{-1} = \begin{bmatrix} \alpha^1 & \alpha^{12} \\ \alpha^{12} & \alpha^2 \end{bmatrix}$,

with $\alpha^1 = k^2/\kappa, \alpha^2 = k^1/\kappa, \alpha^{12} = -k^{12}/\kappa, \kappa = k^1 k^1 - (k^{12})^2$.

$$\int_{T_{ij}} K^{-1}\vec{v}_{i+1/2,j} \cdot \vec{v}_{i+1/2,j} = \frac{\alpha^1_{ij}}{3}\frac{h_{1i}}{h_{2j}}$$

$$\int_{T_{ij}} K^{-1}\vec{v}_{i-1/2,j} \cdot \vec{v}_{i+1/2,j} = \frac{\alpha^1_{ij}}{6}\frac{h_{1i}}{h_{2j}}$$

$$\int_{T_{ij}} K^{-1}\vec{v}_{i,j+1/2} \cdot \vec{v}_{i+1/2,j} = \frac{\alpha^{12}_{ij}}{4}$$

When K is diagonal

Products of basis functions for vertical edges by basis functions for horizontal edges vanish.

- If $E = E_{i+1/2,j}, E \not\subset \Gamma$:

$$\sum_{F \in \mathcal{A}_h} u_F \int_\Omega K^{-1}\vec{v}_F \cdot \vec{v}_E = \sum_{T \in \mathcal{N}(E)} \sum_{F \in \partial T} u_F \int_\Omega K^{-1}\vec{v}_F \cdot \vec{v}_E =$$

$$u_{i+1/2,j}\left[\int_{T_{ij}} \frac{1}{k^1_{ij}}\vec{v}_{i+1/2,j} \cdot \vec{v}_{i+1/2,j} + \int_{T_{i+1,j}} \frac{1}{k^1_{i+1,j}}\vec{v}_{i+1/2,j} \cdot \vec{v}_{i+1/2,j}\right] +$$

$$u_{i-1/2,j}\int_{T_{ij}} \frac{1}{k^1_{ij}}\vec{v}_{i-1/2,j} \cdot \vec{v}_{i+1/2,j} + u_{i+3/2,j}\int_{T_{i+1,j}} \frac{1}{k^1_{i+1,j}}\vec{v}_{i+3/2,j} \cdot \vec{v}_{i+1/2,j}$$

- If $E \subset \Gamma$, say for instance $i = 0$ (left boundary):

$$\sum_{F \in \mathcal{A}_h} u_F \int_\Omega K^{-1}\vec{v}_F \cdot \vec{v}_E = \sum_{T \in \mathcal{N}(E)} \sum_{F \in \partial T} u_F \int_\Omega K^{-1}\vec{v}_F \cdot \vec{v}_E =$$

$$u_{1/2,j}\int_{T_{1,j}} \frac{1}{k^1_{1j}}\vec{v}_{1/2,j} \cdot \vec{v}_{1/2,j} + u_{3/2,j}\int_{T_{1,j}} \frac{1}{k^1_{1,j}}\vec{v}_{3/2,j} \cdot \vec{v}_{1/2,j}$$

Using V, H as indices for the vertical and the horizontal edges we can write the linear system as

$$\begin{bmatrix} A_V & 0 & -{}^tD_V \\ 0 & A_H & -{}^tD_H \\ D_V & D_H & 0 \end{bmatrix} \begin{bmatrix} U_V \\ U_H \\ P \end{bmatrix} = \begin{bmatrix} F_{vV} \\ F_{vH} \\ F_q \end{bmatrix}$$

Matrices A_V et A_H are tridiagonal.

Mixed finite elements vs finite volumes

Assume still that K is diagonal.

Use trapezoidal rule to calculate the coefficients of A. Then

$$\left[\int_{T_{ij}} \frac{1}{k^1_{ij}} \vec{v}_{i+1/2,j} \cdot \vec{v}_{i+1/2,j} + \int_{T_{i+1,j}} \frac{1}{k^1_{i,j+1}} \vec{v}_{i+1/2,j} \cdot \vec{v}_{i+1/2,j}\right] \simeq \frac{1}{2h_{2j}} \left(\frac{h_{1i}}{k^1_{ij}} + \frac{h_{1,i+1}}{k^1_{i+1,j}}\right)$$

$$\int_{T_{ij}} \frac{1}{k^1_{ij}} \vec{v}_{i-1/2,j} \cdot \vec{v}_{i+1/2,j} \simeq 0$$

$$\int_{T_{i+1,j}} \frac{1}{k^1_{i,j+1}} \vec{v}_{i+3/2,j} \cdot \vec{v}_{i+1/2,j} \simeq 0$$

Therefore the matrix A_V becomes diagonal. Its rows correspond to the equations:

$$u_{i+\frac{1}{2},j} = -\left(\frac{k^1}{h_1}\right)_{i+\frac{1}{2},j} (p_{i+1,j} - p_{i,j})h_{2j}$$

with $$\left(\frac{k^1}{h_1}\right)_{i+\frac{1}{2},j} = \frac{1}{\frac{1}{2}\left(\frac{h_{1i}}{k^1_{ij}} + \frac{h_{1,i+1}}{k^1_{i+1,j}}\right)}$$ = the harmonic average of $\frac{k^1}{h_1}$.

This formula for $u_{i+\frac{1}{2},j}$ is slightly different from that given before for a standard finite volume method using harmonic average of K.

It is natural since one can realize that the coefficient in front of $(p_{i+1,j} - p_{i,j})h_{2j}$ is actually like k^1/h_1 (and not just k_1).

It gives slightly better results in cases where there is also a sharp change in h_1.

What did we actually do to obtain finite volumes from mixed finite elements ?

We approximated a by a_h such that

$$\begin{aligned} a(\vec{u}_h, \vec{v}_h) &= \int_\Omega K^{-1}\vec{u}_h \cdot \vec{v}_h = \sum_{T\in\mathcal{T}_h} \int_T K^{-1}\vec{u}_h \cdot \vec{v}_h \\ a_h(\vec{u}_h, \vec{v}_h) &= \sum_{T\in\mathcal{T}_h} \oint_T K^{-1}\vec{u}_h \cdot \vec{v}_h \end{aligned}$$

where $\oint_T$ is an approximate integral over T calculated with the trapezoidal rule in x_1 for the vertical edges and in x_2 for horizontal edges.

The bilinear form a_h can be rewritten as

$$a_h(\vec{u}_h, \vec{v}_h) = \sum_{T \in \mathcal{T}_h} \sum_{F \subset \partial T} \alpha_{T,F} \int_F \vec{u}_h \cdot \vec{n}_F \int_F \vec{v}_h \cdot \vec{n}_F,$$

with $\alpha_{T(E),E} = \frac{1}{2|E|} \frac{h^1_{T(E)}}{k^1_{T(E)}}$ for a vertical edge E.

This gives a matrix A_h corresponding to a_h which is diagonal.

$$\begin{array}{rl} a_h(\vec{v}_E, \vec{v}_E) = & \alpha_{T_1(E),E} + \alpha_{T_2(E),E}, \quad T_1(E), T_2(E) \in \mathcal{N}(E), \quad \text{if } E \not\subset \Gamma, \\ a_h(\vec{v}_E, \vec{v}_E) = & \alpha_{T(E),E} \quad \text{if } E \subset \Gamma, \\ a_h(\vec{v}_E, \vec{v}_F) = & 0 \quad \text{if } E \neq F. \end{array}$$

The new approximate formulation is

$$(\mathcal{P}^*_{mh}) \left\{ \begin{array}{lcll} \multicolumn{4}{l}{\text{Find } \vec{u}^*_h \in \mathcal{W}_{hg} \text{ and } p^*_h \in \mathcal{M}_h \text{ such that}} \\ a_h(\vec{u}^*_h, \vec{v}_h) - b(\vec{v}_h, p^*_h) & = & l_{\mathcal{W}}(\vec{v}_h), & \vec{v}_h \in \mathcal{W}_{h0}, \\ b(\vec{u}^*_h, q_h) & = & l_{\mathcal{M}}(q_h), & q_h \in \mathcal{M}_h. \end{array} \right.$$

which is equivalent to the cell-centered finite volume formulation on rectangles.

The algebraic system is now

$$\begin{bmatrix} A_h & -{}^tD \\ D & 0 \end{bmatrix} \begin{bmatrix} U^* \\ P^* \end{bmatrix} = \begin{bmatrix} F_v \\ F_q \end{bmatrix}.$$

The row equation associated with $\vec{v}_h = \vec{v}_E$ reads now

$$\begin{aligned} &(\alpha_{T_1(E),E} + \alpha_{T_2(E),E}) u_E^* - p_{T_1(E)}^* + p_{T_2(E)}^* = 0, \quad E \not\subset \Gamma, \\ &\alpha_{T,E}\, u_E^* - p_{T_1(E)}^* + \overline{p}_E = 0, \quad E \subset \Gamma. \end{aligned}$$

One can know eliminate U^* to obtain the linear system for P^*

$$(DA_h^{-1}\, {}^tD)\; P* = F_q - DA_h^{-1} F_v$$

where $DA_h^{-1\,t}D$ is still a sparse matrix (5 diagonals).

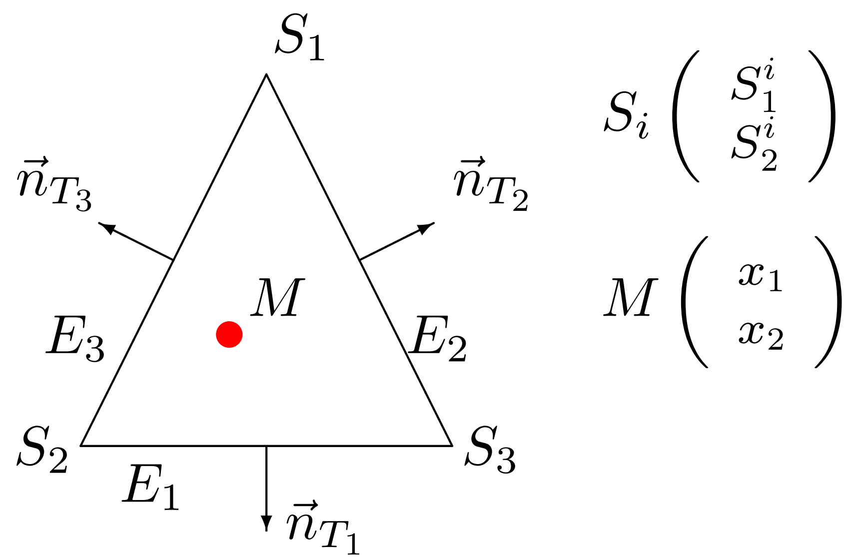
Did we lose accuracy by replacing a by a_h ?

Le cas des triangles

Base de $\vec{\mathcal{W}}_T$:

$$\vec{v}_{T,E_i} = \frac{1}{2|T|}\begin{pmatrix} x_1 - S_1^i \\ x_2 - S_2^i \end{pmatrix} = \frac{1}{2|T|}\vec{S_iM},$$

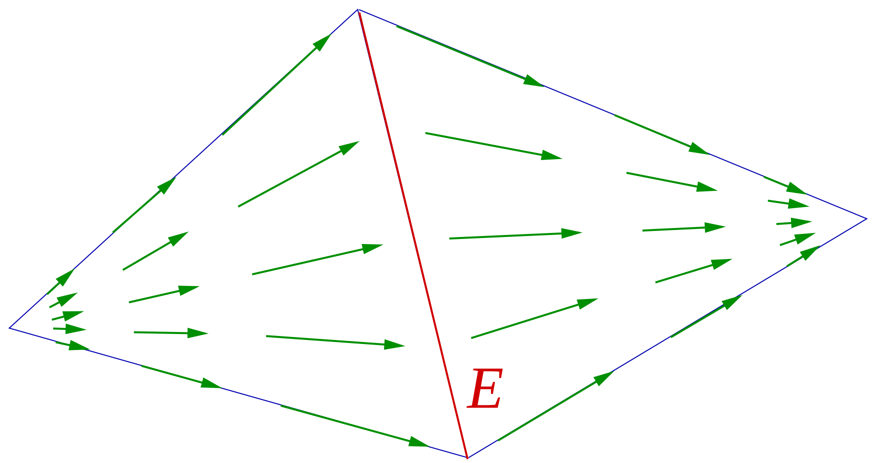
$i = 1, 2, 3$



Les $\vec{v}_{T,E_i}$ vérifient $\displaystyle\int_{E_j} \vec{v}_{T,E_i} \cdot \vec{n}_{T_j} = \delta_{ij}$.

Base de $\mathcal{W}_h$:

$$\vec{v}_E(x) = \begin{cases} \quad \vec{v}_{T_1(E),E}(x) & \text{si } x \in T_1(E) \\ - \quad \vec{v}_{T_2(E),E}(x) & \text{si } x \in T_2(E) \\ \quad 0 & \text{ailleurs} \end{cases}$$



Volumes finis triangulaires

Comme pour les rectangles on approche a par a_h de sorte que

$$\begin{aligned} a(\vec{u}_h, \vec{v}_h) &= \sum_{T \in \mathcal{T}_h} \underbrace{\int_T K^{-1} \vec{u}_h \cdot \vec{v}_h}_{a^T(\vec{u}_h, \vec{v}_h)} \\ a_h(\vec{u}_h, \vec{v}_h) &= \sum_{T \in \mathcal{T}_h} \underbrace{\sum_{i=1}^{3} \alpha_{T,E_i} \int_{E_i} \vec{u}_h \cdot \vec{n}_{E_i} \int_{E_i} \vec{v}_h \cdot \vec{n}_{E_i}.}_{a_h^T(\vec{u}_h, \vec{v}_h)} \end{aligned}$$

La matrice de a_h est diagonale.

Trouver $\vec{u}_h^* \in \mathcal{W}_h$ et $p_h^* \in M_h$ tels que

$$\begin{array}{llcll} a_h(\vec{u}_h^*, \vec{v}_h) & -b(p_h^*, \vec{v}_h) & = & g(\vec{v}_h), & \vec{v}_h \in \mathcal{W}_h, \\ b(\vec{u}_h^*, q_h) & & = & f(q_h), & q_h \in \vec{M}_h. \end{array}$$

Le système algébrique s'écrit maintenant

$$\begin{bmatrix} A_h & -{}^tB \\ B & 0 \end{bmatrix} \begin{bmatrix} U^* \\ P^* \end{bmatrix} = \begin{bmatrix} \overline{F} \\ F \end{bmatrix}.$$

La ligne du système correspondant à $\vec{v}_E$ s'écrit maintenant

$$(\alpha_{T_1(E),E} + \alpha_{T_2(E),E}) u_E^* - p_{T_1(E)}^* + p_{T_2(E)}^* = 0, \quad E \not\subset \partial\Omega,$$

$$\alpha_{T,E}\, u_E^* - p_{T_1(E)}^* + \overline{p}_E = 0, \quad E \subset \partial\Omega.$$

On peut donc maintenant éliminer U^* en maintenant la structure creuse de la matrice en P^*.

Reste à choisir les coefficients α_{T,E_i} de sorte que la précision ne soit pas affectée.

Les coefficients

$$\alpha_{T,E_1} = -\frac{1}{4|T|}(K_T^{-1}\vec{S_1S_3}) \cdot \vec{S_2S_1},$$

$$\alpha_{T,E_2} = -\frac{1}{4|T|}(K_T^{-1}\vec{S_2S_1}) \cdot \vec{S_3S_2}, \qquad \alpha_{T,E_3} = -\frac{1}{4|T|}(K_T^{-1}\vec{S_3S_2}) \cdot \vec{S_1S_3}.$$

Ce choix permet de préserver l'ordre de l'erreur.

Estimations d'erreur

a et b sont comme avant, vérifiant les hypothèses de continuité, de $\mathcal{V}$-ellipticité, et la condition inf-sup .

Théorème : Hypothèses sur a_h : il existe A^*, α^* indépendantes de h telles que

(H1) $a_h(\vec{u}_h, \vec{v}_h) \leq A^* \|\vec{u}_h\|_{\mathcal{W}} \|\vec{v}_h\|_{\mathcal{W}}, \quad \vec{u}_h, \vec{v}_h \in \mathcal{W}_h$

(H2) $a_h(\vec{v}_h, \vec{v}_h) \geq \alpha^* \|\vec{v}_h\|^2_{\mathcal{W}},\ \vec{v}_h \in \mathcal{V}_h = \{\vec{v}_h \in \mathcal{W}_h \mid b(\vec{v}_h, q_h) = 0,\ q_h \in \mathcal{M}_h\}.$

Alors il existe C telle que

$$\|\vec{u} - \vec{u}_h^*\|_{\mathcal{W}} + \|p - p_h^*\|_{\mathcal{M}}$$

$$\leq C \left\{ \inf_{\vec{v}_h \in \mathcal{W}_h} \left(\|\vec{u} - \vec{v}_h\|_{\mathcal{W}} + \sup_{\vec{\eta}_h \in \mathcal{W}_h} \frac{|a(\vec{v}_h, \vec{\eta}_h) - a_h(\vec{v}_h, \vec{\eta}_h)|}{\|\vec{\eta}_h\|_{\mathcal{W}}} \right) + \inf_{q_h \in \mathcal{M}_h} (\|p - q_h\|_{\mathcal{M}}) \right\}.$$

On connait déjà les erreurs d'interpolation :

$$\inf_{q_h \in \mathcal{M}_h} \|p - q_h\|_{\mathcal{M}} \leq Ch\|p\|_{H^1(\Omega)}, \quad \inf_{\vec{v}_h \in \mathcal{W}_h} (\|\vec{u} - \vec{v}_h\|_{\mathcal{W}}) \leq Ch(\|\vec{u}\|_{H_1(\Omega)} + \|\mathrm{div}\vec{u}\|_{H_1(\Omega)}).$$

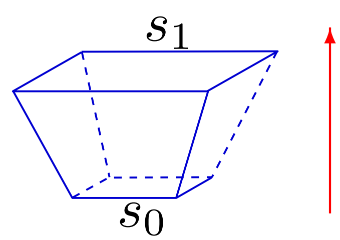
Il reste à vérifier les hypothèses (H1) et (H2) pour appliquer le théorème, et à évaluer l'erreur $a - a_h$.

Remarque : Pour que l'analyse ci-dessous fonctionne il faut que les coefficients $\alpha_{T,E_i}, i = 1, 2, 3$ soient strictement positifs

$\Longrightarrow$ les angles des triangles de $\mathcal{T}_h$ doivent être tous aigus.

A problem with difformed hexahedrons (and rectangles)

Raviart-Thomas-Nédélec mixed finite elements do not contain constant velocities.



$$\vec{u} = \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}$$

An example due to T. Russell

Exact flow rate through an horizontal section B_z, for $0 \leq z \leq 1$:

$$\int_{B_z} \vec{u} \cdot \vec{n}_z = ((1-z)s_0 + zs_1)^2.$$

Flow rate calculated with $\vec{u}_h$ the image of $\vec{u}$ by Piola's transformation :

$$\int_{B_z} \vec{u}_h \cdot \vec{n}_z = (1-z)s_0^2 + zs_1^2.$$

. Constant velocities are not invariant for Π_h

$\Longrightarrow$

Interpolation results do not hold (Bramble-Hilbert lemma can't be applied)

The method does not converge.

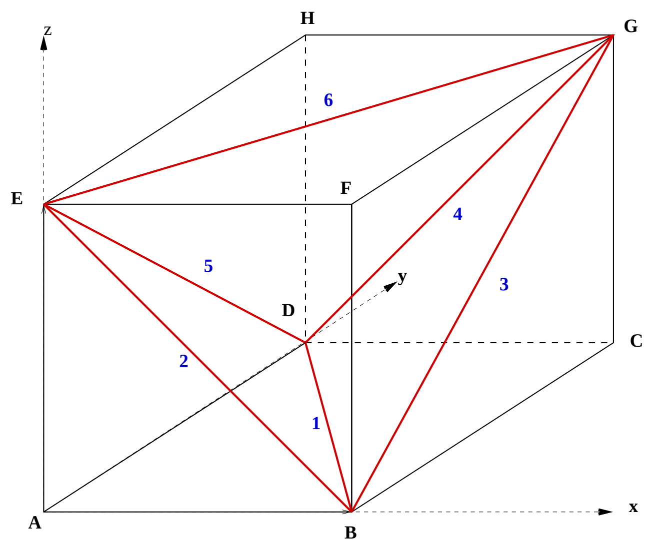
A mixed finite element due to Kuznetsov and Repin (2003)

Remark : With tetrahedrons, a constant velocity field lies in $\mathcal{W}_h$.

$\Longrightarrow$ Build a macroelement of an hexaedron H by dividing it into 5 tetrahedrons.

$$T = T_1 \cup T_2 \cup T_3 \cup T_4 \cup T_5$$

T_1 : ABDE, T_2 : BEFG, T_3 : BCGD, T_4 : DEHG, T_5 : BEDG



The new approximation space $\mathcal{W}_T$

$$\mathcal{W}_T = \{\vec{v} \in H(\text{div}, T);\ \vec{v}|_{T_i} \in RTN_0(T_i), i = 1, \cdots, 5,\ \text{div}\,\vec{v}\ \text{const.}\}.$$

Degrees of freedom for pressure and velocity are the same:

- average pressure in the hexahedron (1),
- flow rates through the faces (6).

Conditions on $\vec{v}$

$$\vec{v} = a \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} + \begin{pmatrix} b \\ c \\ d \end{pmatrix}$$

- $\vec{v}|_{T_i} \in RTN_0, i = 1, \cdots, 5 \rightarrow$ 20 d.o.f.
- $\vec{v} \in H(\text{div}, T) \rightarrow$ 4 conditions
- div $\vec{v}$ constant $\rightarrow$ 4 conditions
- constant flux on each face of $T \rightarrow$ 6 conditions

$\vec{v}$ is uniquely defined.

Why 5 tetrahedrons ?

- ✓ 6 tetrahedrons $\Rightarrow$ too many degrees of freedom.
- ✓ 5 tetrahedrons $\Rightarrow$ right number of degrees of freedom

A constant velocity field is indeed in $\mathcal{W}_T$ since:

- ✓ it lies in $RTN_0(T_i), i = 1, \cdots, 5$
- ✓ it lies in $H(\text{div}, T)$
- ✓ its divergence is constant in T

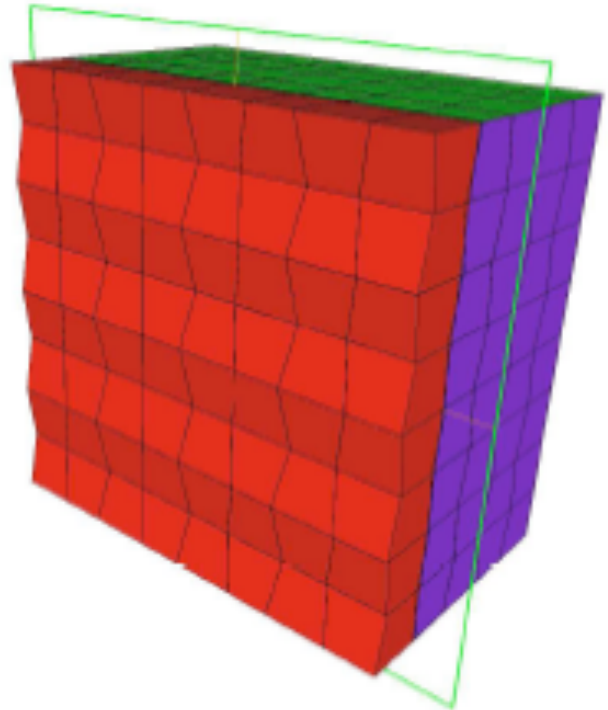
Résultats numériques

Solution exacte: $p = x(1-x)y^2(1-y)^2z(z-1)sin(\pi x)sin(\pi y)sin(\pi z)$

Maillage cubique

	Elément fini RTN				Elément fini KR			
maillage	$\vert\vert p_h - \pi_h p \vert\vert_{0,\Omega}$		$\vert\vert u_h - \Pi_h u \vert\vert_{0,\Omega}$		$\vert\vert p_h - \pi_h p \vert\vert_{0,\Omega}$		$\vert\vert u_h - \Pi_h u \vert\vert_{0,\Omega}$	
	erreur	ordre	erreur	ordre	erreur	ordre	erreur	ordre
4	0.01639		0.00107		0.01640		0.00119	
8	0.00445	1.88	0.00024	2.15	0.00445	1.88	0.00028	2.06
16	0.00113	1.97	6e-5	2.	0.00113	1.97	7.23e-5	1.97
32	0.00028	1.99	1.5e-5	2.	0.00028	1.99	1.8e-5	2.00
64	7.1e-5	2.	3.8e-6	1.98	7.1e-5	2.	4.5e-6	2.

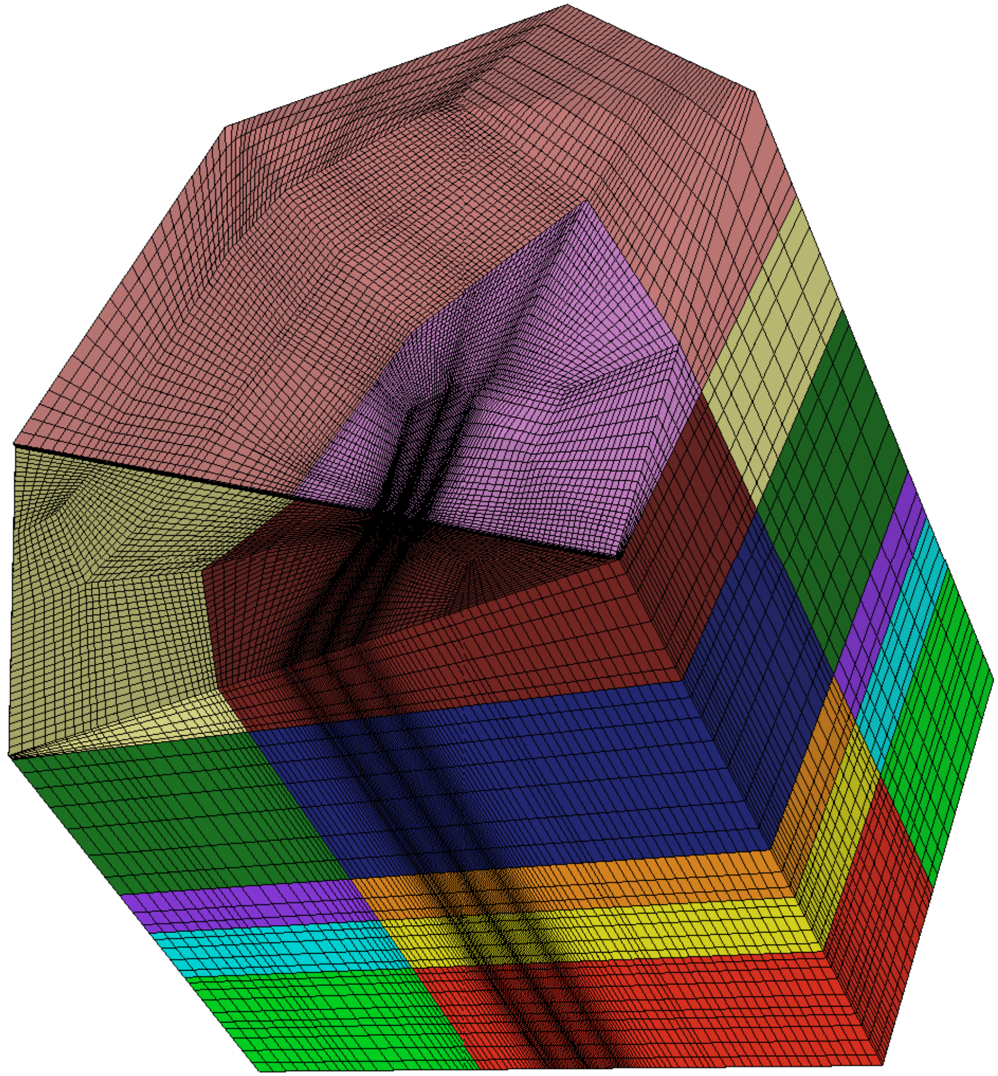
Maillage déformé



	Elément fini RTN				Elément fini KR			
maillage	$\|\| p_h - \pi_h p \|\|_{0,\Omega}$ erreur	ordre	$\|\| u_h - \Pi_h u \|\|_{0,\Omega}$ erreur	ordre	$\|\| p_h - \pi_h p \|\|_{0,\Omega}$ erreur	ordre	$\|\| u_h - \Pi_h u \|\|_{0,\Omega}$ erreur	ordre
4	0.01716		0.07885		0.016072		0.039495	
8	0.00565	1.6	0.04048	0.96	0.00423	1.92	0.01319	1.58
16	0.00294	0.62	0.02632	0.62	0.00106	1.99	0.00421	1.64
32	0.00238	0.3	0.02343	0.16	0.00026	2.	0.00136	1.62
64	0.00226	0.08	0.02289	0.03	6.5e-5	2.01	0.00046	1.32

A mesh with difformed hexahedrons

Over 500000
hexahedrons



MIXED-HYBRID FINITE ELEMENTS

Introduction

Instead of calculating $\vec{u}_h \in \mathcal{W}_h$, we now calculate

$$\vec{u}_h^\star \in \mathcal{W}_h^\star = \{\vec{v}_h \in (L^2(\Omega))^2; \vec{v}_h|T \in \mathcal{W}_T, T \in \mathcal{T}_h\}$$

Functions of $\mathcal{W}_h^\star$ are not required to have their flux continuous across the edges.

Continuity of the flux will now be written explicitely.

We need also

$$\mathcal{N}_h = \{\mu_h \in \Pi_{E \in \mathcal{A}_h} \mu_E,\ \mu_E \in \mathbb{R}\}.$$

The mixed-hybrid formulation is

Find $\vec{u}_h^\star \in \mathcal{W}_{hg}^\star, p_h^\star \in \mathcal{M}_h, \lambda_h \in \mathcal{N}_h$ such that

$$\int_T K^{-1}\vec{u}_h^\star \cdot \vec{v}_h - \int_T p_h^\star \,\mathrm{div}\, \vec{v}_h + \sum_{E\in\partial T} \int_E \lambda_h \vec{v}_h \cdot \vec{n}_T = \int_{\Gamma_D} -\overline{p}\, \vec{v}_h \cdot \vec{n}_T,$$

$$\vec{v}_h \in \mathcal{W}_h^\star, T \in \mathcal{T}_h$$

$$\int_T \mathrm{div}\, \vec{u}_h \; q_h = \int_T f \; q_h, \quad q_h \in \mathcal{M}_h, \; T \in \mathcal{T}_h$$

$$- \sum_{T\in\mathcal{T}_h, \partial T \supset E} \vec{u}_h^\star \cdot \vec{n}_T \; \mu_h = 0, \quad E \in \mathcal{A}_h, E \not\subset \Gamma, \mu_h \in \mathcal{N}_h$$

$$\vec{u}_h^\star \cdot \vec{n}|_E = g|E|, \quad E \subset \Gamma_N,$$

$$\lambda_h|_E = \overline{p}, \quad E \subset \Gamma_D.$$

λ_h represents a trace of the pressure on the edges $E \in \mathcal{A}_h$.

We check easily that $p_h^\star = p_h, \vec{u}_h^\star|_T = \vec{u}_h|_T, \; T \in \mathcal{T}_h$.

The linear system

$$\begin{bmatrix} A^\star & -{}^tD & -{}^tB \\ D & 0 & 0 \\ B & 0 & I_D \end{bmatrix} \begin{bmatrix} U^\star \\ P \\ \Lambda \end{bmatrix} = \begin{bmatrix} F_v \\ F_q \\ F_\mu \end{bmatrix}$$

$A^\star$ is block diagonal; we can eliminate $U^\star = A^{\star(-1)}(F_v + {}^tDP + {}^tB\Lambda)$ to get

$$\begin{bmatrix} DA^{\star(-1)}\,{}^tD & DA^{\star(-1)}\,{}^tB \\ BA^{\star(-1)}\,{}^tD & BA^{\star(-1)}\,{}^tB + I_D \end{bmatrix} \begin{bmatrix} P \\ \Lambda \end{bmatrix} = \begin{bmatrix} F_q - DA^{\star(-1)}F_v \\ F_\mu - BA^{\star(-1)}F_v \end{bmatrix}$$

The matrix $DA^{\star(-1)}\,{}^tD$ is diagonal, so we can eliminate P:
$P = (DA^{\star(-1)}\,{}^tD)^{-1}[F_q - DA^{\star(-1)}F_v - (DA^{\star(-1)}\,{}^tB)\Lambda]$ to obtain

$$H\Lambda = G \quad (H \text{ sparse})$$

where $$\begin{aligned} H &= BA^{\star(-1)}\,{}^tB + I_D - (BA^{\star(-1)}\,{}^tD)(DA^{\star(-1)}\,{}^tD)^{-1}(DA^{\star(-1)}\,{}^tB) \\ G &= F_\mu - BA^{\star(-1)} - (BA^{\star(-1)}\,{}^tD)(DA^{\star(-1)}\,{}^tD)^{-1}(F_q - DA^{\star(-1)}F_v) \end{aligned}$$

Properties of the matrix H

- H is sparse

The number of nonzeros in the line E is equal to the number of neighbouring edges + 1 (for E itself) (7 for a rectangular mesh).

- H is positive definite

To prove it, assuming $(H\Lambda, \Lambda) = 0$ we have to show that this implies $\Lambda = 0$. Then

$$((BA^{\star(-1)}\,{}^tB + I_D)\Lambda, \Lambda) - ((BA^{\star(-1)}\,{}^tD)(DA^{\star(-1)}\,{}^tD)^{-1}(DA^{\star(-1)}\,{}^tB)\Lambda, \Lambda) = 0$$

Introduce $P = (DA^{\star(-1)}\,{}^tD)^{-1}(-(DA^{\star(-1)}\,{}^tB)\Lambda)$. We obtain

$$(A^{\star(-1)}\,{}^tB\Lambda, {}^tB\Lambda) + (I_D\Lambda, \Lambda) - (A^{\star(-1)}\,{}^tDP, B\Lambda) = 0$$

But equation for P implies that

$$((DA^{\star(-1)}\,{}^tD)P, P) + ((DA^{\star(-1)}\,{}^tB)\Lambda, P) = 0.$$

Adding to the previous equation gives

$$(A^{\star(-1)}({}^tDP + {}^tB\Lambda), {}^tDP + {}^tB\Lambda) + (I_D\Lambda, \Lambda) = 0$$

Since $A^{\star(-1)}$ is positive definite and I_D is positive semi-definite, this implies that ${}^tDP + {}^tB\Lambda = 0$ and $\lambda_E = 0, E \subset \Gamma_D$.

Equation ${}^tDP + {}^tB\Lambda = 0$ says actually that

$$P_T - \lambda_E = 0, E \supset \partial T, T \in \mathcal{T}_h$$

which means that the pressure is constant over Ω.

But from $\lambda_E = 0, E \subset \Gamma_D$ it follows that $P = 0$ and $\Lambda = 0$.

Non conforming finite elements

Once that the U and P have been eliminated, we end up with a system in TP.

Therefore the mixed-hybrid method can be interpreted as a non-conforming finite element method whose degrees of freedom are the average pressure on the edges.

CONCLUSION

The mixed finite element method

- is locally conservative,
- works with difformed and unstructured grids,
- handles nondiagonal tensors,
- does not satisfy the maximum principle, even on rectangular grids.

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