Finite volume method for linear and non linear elliptic problems with discontinuities

Franck BOYER and Florence HUBERT

L.A.T.P. - Marseille, FRANCE

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OUTLINES

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THE SCHEME FV4

Approximate the solution (for Dirichlet BC for instance) of

$$-\Delta u = f \qquad (*)$$

in an open bounded set Ω discretized by control volumes $\mathcal{K}$ (ex : triangles).

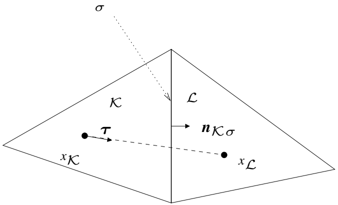
The finite volume scheme principle

- Integrate (*) overall control volumes :

$$\int_{\mathcal{K}} f = \int_{\mathcal{K}} -\Delta u = -\sum_{\sigma \subset \partial \mathcal{K}} \int_{\sigma} \nabla u \cdot \boldsymbol{n}_{\mathcal{K}\sigma}.$$

- Approximate normal fluxes

$$\int_{\sigma} \nabla u \cdot \boldsymbol{n}_{\mathcal{K}\sigma}$$



- Taylor expansion for $\sigma = \mathcal{K}|\mathcal{L}$

$$|\sigma| \frac{u(x_{\mathcal{L}}) - u(x_{\mathcal{L}})}{d_{\mathcal{K}\mathcal{L}}} \sim \int_{\sigma} \nabla u \cdot \boldsymbol{\tau}_{\mathcal{K}\mathcal{L}} \text{ where } \boldsymbol{\tau}_{\mathcal{K}\mathcal{L}} = \frac{\vec{x_{\mathcal{L}} x_{\mathcal{K}}}}{\|\vec{x_{\mathcal{L}} x_{\mathcal{K}}}\|}$$

THE FV4 SCHEME

Approximate the solution (for Dirichlet BC for instance) of

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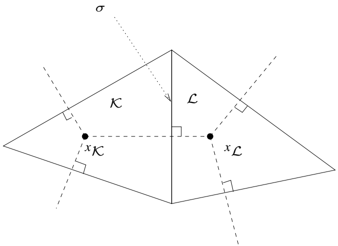
The classical FV4 scheme

$$\int_{\mathcal{K}} f = \int_{\mathcal{K}} -\Delta u = -\sum_{\sigma \subset \partial \mathcal{K}} \int_{\sigma} \nabla u \cdot \boldsymbol{n}_{\mathcal{K}\sigma} \approx \sum_{\sigma \subset \partial \mathcal{K}} |\sigma| \frac{u_{\mathcal{L}} - u_{\mathcal{K}}}{d_{\mathcal{K}\mathcal{L}}}.$$

CONSISTENCY : YES if $[x_{\mathcal{K}}, x_{\mathcal{L}}] \perp \sigma$.



⇒ Such meshes are called admissible.

The FV4 scheme

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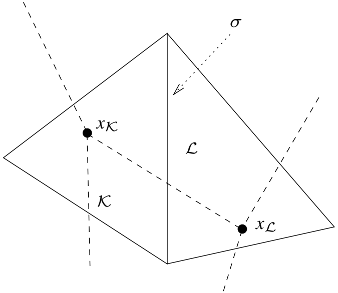


CONSISTENCY : NO if $[x_{\mathcal{K}} x_{\mathcal{L}}] \not\perp \sigma$.

$\Rightarrow$ Such control volumes are said to be non admissible.

Error estimates for the FV4 scheme

Admissible Meshes

Theorem

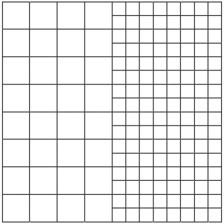
The error of the FV4 scheme in case of admissible meshes, is bounded by $C\text{size}(\mathcal{T})$.

ERROR ESTIMATES FOR THE FV4 SCHEME

NON ADMISSIBLE MESHES

THEOREM

If non admissible control volumes are located along a curve Γ*, the error of the FV4 scheme is bounded by* $C\text{size}(\mathcal{T})^{\frac{1}{2}}$.



Example of non admissible meshes.

ANISOTROPIC OPERATOR

Can we only use admissible meshes ?

NO

1. To few meshes satisfy the admissibilty condition (triangles, Voronoï, ...).
2. For the anisotropic operator

$$-\mathrm{div}(A\nabla u) = f$$

   the admissibility condition becomes : $A[x_{\mathcal{K}} x_{\mathcal{L}}] \mathbin{/\!/} \boldsymbol{n}$ ...
3. How write these geometrical condition in case of variable diffusion tensor ?

$$-\mathrm{div}(A(z)\nabla u) = f.$$

A solution is to approximate the two componants of the gradient.

NONLINEAR DIFFUSION OPERATOR

Example : p-laplacian

Approximate in "$W_0^{1,p}(\Omega)$" the unique solution of

$$-\text{div}(|\nabla u|^{p-2}\nabla u) = f, \quad 1 < p < +\infty.$$

Finite volume approach requires a consistant approximation of

$$\int_\sigma |\nabla u|^{p-2}\nabla u \cdot \boldsymbol{n}.$$

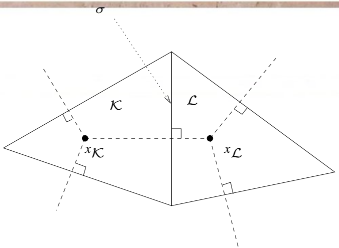
Impossible to obtain only with the two values $u_\mathcal{K}$ and $u_\mathcal{L}$.

$\rightsquigarrow$ **We still need an approximation of the whole gradient** .

THE PROBLEM OF DISCONTINUOUS COEFFICIENTS

$$-\mathrm{div}(k(z)\nabla u) = f,$$

$$k(z) \in \mathbb{R}.$$



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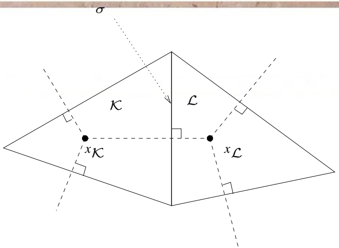

If k is smooth, the finite volume FV4 writes :

$$\int_{\mathcal{K}} f = \int_{\mathcal{K}} -\mathrm{div}(k(z)\nabla u)\, dz = -\sum_{\sigma\subset\partial\mathcal{K}} \int_\sigma \underbrace{(k(s)\nabla u)}_{=\text{flux}} \cdot \boldsymbol{n} ds \approx \sum_{\sigma\subset\partial\mathcal{K}} |\sigma| k_\sigma \frac{u_{\mathcal{L}} - u_{\mathcal{K}}}{d_{\mathcal{K}\mathcal{L}}},$$

where k_σ is an approximation of k on the edge σ

$$k_\sigma = k(x_\sigma) \;\; \text{where} \;\; k_\sigma = \frac{1}{|\sigma|}\int_\sigma k(s)\, ds.$$

THE PROBLEM OF DISCONTINUOUS COEFFICIENTS

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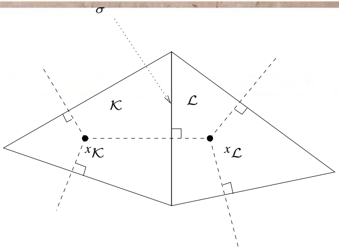


If k is discontinuous across σ : $k_{\mathcal{K}}$ ans $k_{\mathcal{L}}$ on $\mathcal{K}$ et $\mathcal{L}$:
How to write the scheme ? We look for k_σ such that

$$|\sigma| k_\sigma \frac{u_{\mathcal{L}} - u_{\mathcal{K}}}{d_{\mathcal{K}\mathcal{L}}} \approx \int_\sigma (k(s)\nabla u(s)) \cdot \boldsymbol{n} ds.$$

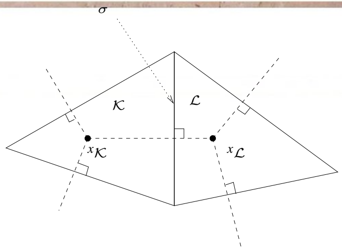
The *simple* choices of $k_\sigma = k_{\mathcal{K}}$, $k_\sigma = k_{\mathcal{L}}$ where $k_\sigma = \frac{1}{2}(k_{\mathcal{K}} + k_{\mathcal{L}})$ lead to non consistent fluxes.

Indeed $\nabla u \cdot \boldsymbol{n}$ is discontinuous across σ !

THE PROBLEM OF DISCONTINUOUS COEFFICIENTS

$$-\text{div}(k(z)\nabla u) = f,$$

$$k(z) \in \mathbb{R}.$$



TAKE A NEW UNKNOWN u_σ ON THE EDGE σ :

Write the continuity of the approximate fluxes across σ.

$$F_{\mathcal{K}\mathcal{L}} \stackrel{\text{def}}{=} |\sigma| k_{\mathcal{L}} \frac{u_{\mathcal{L}} - u_\sigma}{d_{\mathcal{L}\sigma}} = |\sigma| k_{\mathcal{K}} \frac{u_\sigma - u_{\mathcal{K}}}{d_{\mathcal{K}\sigma}}.$$

Eliminate the fictive unknown u_σ :

$$u_\sigma = \frac{k_{\mathcal{L}} d_{\mathcal{K}\sigma} u_{\mathcal{L}} + k_{\mathcal{K}} d_{\mathcal{L}\sigma} u_{\mathcal{K}}}{k_{\mathcal{L}} d_{\mathcal{K}\sigma} + k_{\mathcal{K}} d_{\mathcal{L}\sigma}}$$

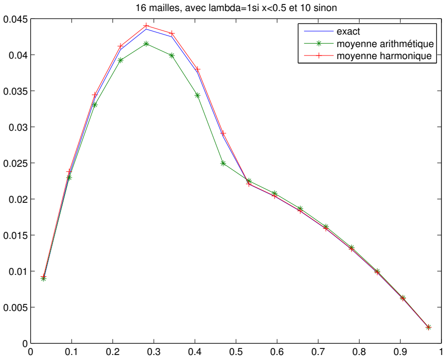
$$\Longrightarrow F_{\mathcal{K}\mathcal{L}} = |\sigma| k_\sigma \frac{u_{\mathcal{L}} - u_{\mathcal{K}}}{d_{\mathcal{K}\mathcal{L}}}, \text{ with } k_\sigma = \frac{k_{\mathcal{K}} k_{\mathcal{L}} (d_{\mathcal{K}\sigma} + d_{\mathcal{L}\sigma})}{k_{\mathcal{L}} d_{\mathcal{K}\sigma} + k_{\mathcal{K}} d_{\mathcal{L}\sigma}},$$ harmonic mean value.

THE PROBLEM OF DISCONTINUOUS COEFFICIENTS

In presence of discontinuities, the scheme converges but the order of convergence depends on the choice of k_σ :

CAS 1D : $-\frac{d}{dx}\left(k(x)\frac{d}{dx}u_e\right) = f$, with $k(x) = \begin{cases} k^+ & \text{if } x > 0.5 \\ k^- & \text{if } x < 0.5 \end{cases}$

- k_σ arithmetic mean value : order $\frac{1}{2}$
- k_σ harmonic mean value : order 1



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  - **Eymard, Gallouët, Herbin (00)**
- **Gradient reconstructions**
  - MPFA schemes. **Aavatsmark (98/04), Lepotier (05),...**
  - Gradient FV schemes. **Eymard, Gallouët, Herbin (06), ...**
  - Mixte FV scheme **Droniou, Eymard (06)**
  - Diamond schemes, DDFV schemes. **Coudière (99), Hermeline (00), Domelevo & Omnès (05), Pierre (06), Delcourte & al (06), ABH (07), ...**
- **Anisotropic problems with discontinuities**
  - **Hermeline (03)**
  - **BH (07)**
  - **Benchmark - FVCA5 Aussois june 2008**

OUTLINES

Aim and notations

- The DDFV scheme (Discrete Duality Finite Volume) for

$$\begin{cases} -\text{div}\,(\varphi(z, \nabla u_e(z))) = f(z), & \text{in } \Omega, \\ u_e = 0, \text{ on } \partial\Omega, \end{cases}$$

  - Ω in an open bounded polygonal set $\mathbb{R}^2$.
  - $u \mapsto -\text{div}(\varphi(\cdot, \nabla u))$ is an monotonic and coercitive (of Leray-Lions type) operator.

ASSUMPTIONS ON φ

- Let $p \in]1, \infty[$, $p' = \frac{p}{p-1}$ and $f \in L^{p'}(\Omega)$. ► $p \geq 2$ to simplify.
- $\varphi : \Omega \times \mathbb{R}^2 \to \mathbb{R}^2$ is a Caratheorory function such that :

$$(\varphi(z,\xi),\xi) \geq C_\varphi \left(|\xi|^p - 1\right), \tag{$\mathcal{H}_1$}$$

$$|\varphi(z,\xi)| \leq C_\varphi \left(|\xi|^{p-1} + 1\right). \tag{$\mathcal{H}_2$}$$

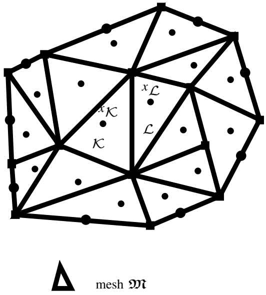
$$(\varphi(z,\xi) - \varphi(z,\eta), \xi - \eta) \geq \frac{1}{C_\varphi}|\xi - \eta|^p. \tag{$\mathcal{H}_3$}$$

$$|\varphi(z,\xi) - \varphi(z,\eta)| \leq C_\varphi \left(1 + |\xi|^{p-2} + |\eta|^{p-2}\right)|\xi - \eta|. \tag{$\mathcal{H}_4$}$$

- φ is lipschitz continuous with respect to z.

THE DDFV MESHES

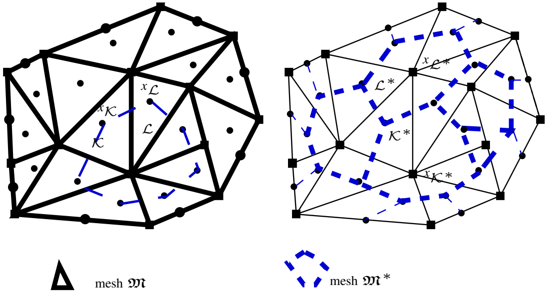
primal, dual and "diamond".



Primal control volumes

$\rightsquigarrow (u_{\mathcal{K}})_{\mathcal{K}\in\mathfrak{M}}$

primal, dual and "diamond".



Primal control volumes
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Dual control volumes
$\rightsquigarrow (u_{\mathcal{K}^*})_{\mathcal{K}^*\in\mathfrak{M}^*}$

THE DDFV MESHES

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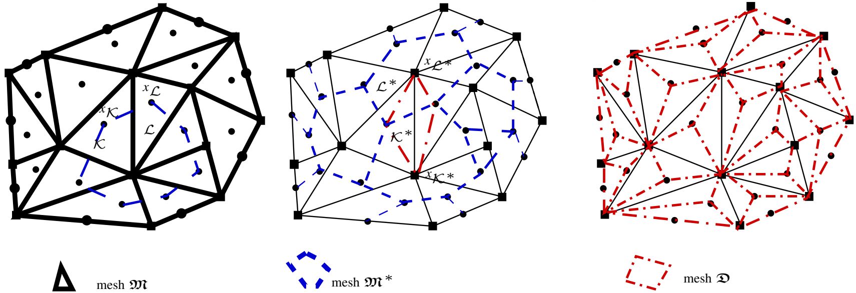


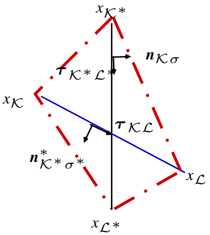
Primal control volumes
$\rightsquigarrow$ $(u_\mathcal{K})_{\mathcal{K}\in\mathfrak{M}}$

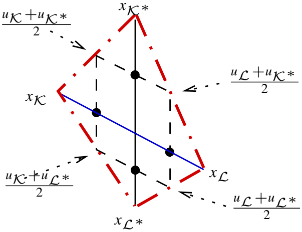
Dual control volumes
$\rightsquigarrow$ $(u_{\mathcal{K}^*})_{\mathcal{K}^*\in\mathfrak{M}^*}$

Diamond cells
$\rightsquigarrow$ Discrete gradient

THE DISCRETE GRADIENT

$$\nabla_{\mathcal{D}}^{\mathcal{T}} u^{\mathcal{T}} = \frac{1}{\sin \alpha_{\mathcal{D}}} \left(\frac{u_{\mathcal{L}} - u_{\mathcal{K}}}{|\sigma^*|} \boldsymbol{n}_{\mathcal{K}\sigma} + \frac{u_{\mathcal{L}^*} - u_{\mathcal{K}^*}}{|\sigma|} \boldsymbol{n}_{\mathcal{K}^*\sigma} \right), \quad \forall \, \text{diamond cell } \mathcal{D}.$$





Equivalent definition
$$\begin{cases} \nabla_{\mathcal{D}}^{\mathcal{T}} u^{\mathcal{T}} \cdot (x_{\mathcal{L}} - x_{\mathcal{K}}) = u_{\mathcal{L}} - u_{\mathcal{K}}, \\ \nabla_{\mathcal{D}}^{\mathcal{T}} u^{\mathcal{T}} \cdot (x_{\mathcal{L}^*} - x_{\mathcal{K}^*}) = u_{\mathcal{L}^*} - u_{\mathcal{K}^*}. \end{cases}$$

THE DDFV SCHEME

THE DISCRETE GRADIENT

$$\nabla_{\mathcal{D}}^{\mathcal{T}} u^{\mathcal{T}} = \frac{1}{\sin \alpha_{\mathcal{D}}} \left(\frac{u_{\mathcal{L}} - u_{\mathcal{K}}}{|\sigma^*|} \boldsymbol{n}_{\mathcal{K}\sigma} + \frac{u_{\mathcal{L}^*} - u_{\mathcal{K}^*}}{|\sigma|} \boldsymbol{n}_{\mathcal{K}^*\sigma} \right), \quad \forall \text{ diamond cell } \mathcal{D}.$$

THE STANDARD DDFV SCHEME

Classical finite volume formulation :

$$- \sum_{\sigma \in \mathcal{E}_{\mathcal{K}}} |\sigma| \left(\varphi_{\mathcal{D}}(\nabla_{\mathcal{D}}^{\mathcal{T}} u^{\mathcal{T}}), \boldsymbol{n}_{\mathcal{K}\sigma} \right) = \int_{\mathcal{K}} f(z)\, dz, \quad \forall \mathcal{K} \in \mathfrak{M},$$

$$- \sum_{\sigma^* \in \mathcal{E}_{\mathcal{K}^*}} |\sigma^*| \left(\varphi_{\mathcal{D}}(\nabla_{\mathcal{D}}^{\mathcal{T}} u^{\mathcal{T}}), \boldsymbol{n}_{\mathcal{K}^*\sigma} \right) = \int_{\mathcal{K}^*} f(z)\, dz, \forall \mathcal{K}^* \in \mathfrak{M}^*,$$

with

$$\varphi_{\mathcal{D}}(\xi) = \frac{1}{|\mathcal{D}|} \int_{\mathcal{D}} \varphi(z, \xi)\, dz, \quad \text{approximate flux on the diamond cell}$$

The DDFV scheme

The discrete gradient

$$\nabla_{\mathcal{D}}^{\mathcal{T}} u^{\mathcal{T}} = \frac{1}{\sin \alpha_{\mathcal{D}}} \left(\frac{u_{\mathcal{L}} - u_{\mathcal{K}}}{|\sigma^*|} \boldsymbol{n}_{\mathcal{K}\sigma} + \frac{u_{\mathcal{L}^*} - u_{\mathcal{K}^*}}{|\sigma|} \boldsymbol{n}_{\mathcal{K}^*\sigma} \right), \quad \forall \text{ diamond cell } \mathcal{D}.$$

The standard DDFV scheme

Classical finite volume formulation :

$$-\sum_{\sigma \in \mathcal{E}_{\mathcal{K}}} |\sigma| \left(\varphi_{\mathcal{D}}(\nabla_{\mathcal{D}}^{\mathcal{T}} u^{\mathcal{T}}), \boldsymbol{n}_{\mathcal{K}\sigma} \right) = \int_{\mathcal{K}} f(z)\, dz, \quad \forall \mathcal{K} \in \mathfrak{M},$$

$$-\sum_{\sigma^* \in \mathcal{E}_{\mathcal{K}^*}} |\sigma^*| \left(\varphi_{\mathcal{D}}(\nabla_{\mathcal{D}}^{\mathcal{T}} u^{\mathcal{T}}), \boldsymbol{n}_{\mathcal{K}^*\sigma} \right) = \int_{\mathcal{K}^*} f(z)\, dz, \ \forall \mathcal{K}^* \in \mathfrak{M}^*,$$

or

$$-\mathrm{div}_{\mathcal{K}}^{\mathcal{T}} \left(\varphi_{\mathfrak{D}}(\nabla_{\mathcal{D}}^{\mathcal{T}} u^{\mathcal{T}}) \right) = \int_{\mathcal{K}} f(z)\, dz, \quad \forall \mathcal{K} \in \mathfrak{M},$$

$$-\mathrm{div}_{\mathcal{K}^*}^{\mathcal{T}} \left(\varphi_{\mathfrak{D}}(\nabla_{\mathcal{D}}^{\mathcal{T}} u^{\mathcal{T}}) \right) = \int_{\mathcal{K}^*} f(z)\, dz, \ \forall \mathcal{K}^* \in \mathfrak{M}^*,$$

THE DDFV SCHEME

THE DISCRETE GRADIENT

$$\nabla_{\mathcal{D}}^{\mathcal{T}} u^{\mathcal{T}} = \frac{1}{\sin \alpha_{\mathcal{D}}} \left(\frac{u_{\mathcal{L}} - u_{\mathcal{K}}}{|\sigma^*|} \boldsymbol{n}_{\mathcal{K}\sigma} + \frac{u_{\mathcal{L}^*} - u_{\mathcal{K}^*}}{|\sigma|} \boldsymbol{n}_{\mathcal{K}^*\sigma} \right), \quad \forall \text{ diamond cell } \mathcal{D}.$$

THE STANDARD DDFV SCHEME

Thanks to a Green formulae we have :

$\rightsquigarrow$ Variational formulation (discrete duality) :

$$2 \sum_{\mathcal{D} \in \mathfrak{D}} |\mathcal{D}| \left(\varphi_{\mathcal{D}}(\nabla_{\mathcal{D}}^{\mathcal{T}} u^{\mathcal{T}}), \nabla_{\mathcal{D}}^{\mathcal{T}} v^{\mathcal{T}} \right) = \int_{\Omega} f v^{\mathfrak{M}} dz + \int_{\Omega} f v^{\mathfrak{M}^*} dz, \quad \forall v^{\mathcal{T}} \in \mathbb{R}^{\mathcal{T}}.$$

CONSEQUENCES

- Existence and uniqueness of a solution to the scheme (monotonicity).
- Variational structure preserved if $\varphi = \nabla_{\xi} \Phi$.

Convergence of DDFV scheme

Theorem

Let $f \in L^{p'}(\Omega)$ *and* $\mathcal{T}_n$ *a family of meshes such that* $\mathrm{size}(\mathcal{T}_n)$ *tends to* 0 *with*

$$\mathrm{reg}(\mathcal{T}_n) = \max\left(\max_{\mathcal{D}\in\mathfrak{D}} \frac{\mathrm{d}_{\mathcal{D}}}{\sqrt{|\mathcal{D}|}}, \max_{\mathcal{K}\in\mathfrak{M}} \frac{\mathrm{d}_{\mathcal{K}}}{\sqrt{|\mathcal{K}|}}, \max_{\mathcal{K}^*\in\mathfrak{M}^*} \frac{\mathrm{d}_{\mathcal{K}^*}}{\sqrt{|\mathcal{K}^*|}}, \ldots \right)$$

bounded. Then

- $u^{\mathcal{T}_n} \underset{n\to\infty}{\longrightarrow} u_e$ *strongly in* $L^p(\Omega)$.
- $\nabla^{\mathcal{T}_n} u^{\mathcal{T}_n} \underset{n\to\infty}{\longrightarrow} \nabla u_e$ *strongly in* $L^p(\Omega)$.
- $\varphi(\cdot, u^{\mathcal{T}_n}) \underset{n\to\infty}{\longrightarrow} \varphi(\cdot, u_e)$ *strongly in* $L^{p'}(\Omega)$.

REGULAR COEFFICIENTS

- The laplacian (i.e. φ is linear with $p = 2$) :

  Domelevo & Omnès (M^{2}AN, 05)

  $\Rightarrow$ Estimate in $O(h)$ under few restrictions on the meshes.

- General case :

  Andreianov, Boyer & H. (Num. Meth. for PDEs, 07)

THEOREM

If $u_e \in W^{2,p}(\Omega)$ *and if*

$$\varphi \text{ is Lip. for all } \Omega, \text{ with } \left|\frac{\partial \varphi}{\partial z}(z,\xi)\right| \leq C_\varphi \left(1 + |\xi|^{p-1}\right), \ \ \forall \xi \in \mathbb{R}^2, \qquad (\mathcal{H}_5)$$

then

$$\|u_e - u^{\mathfrak{M}}\|_{L^p} + \|u_e - u^{\mathfrak{M}^*}\|_{L^p} + \|\nabla u_e - \nabla^{\mathcal{T}} u^{\mathcal{T}}\|_{L^p} \leq C \,\text{size}(\mathcal{T})^{\frac{1}{p-1}}.$$

OUTLINES

AIMS

(F. Boyer & F. H., preprint, 06)

If φ is discontinuous with respect to z

- u_e **does not** belong to $W^{2,p}(\Omega)$.
- The consistency is lost along the discontinuity.
- The DDFV scheme converges but slowly.

WE ASSUME THAT φ IS PIECEWISE LIPSCHITZ CONTINUOUS.

We modify the DDFV scheme in order to recover the consistency.

THE PROBLEM IN 1D

$$\Omega =]-1,1[,\ \varphi(x,\cdot) = \begin{cases} \varphi_-(\cdot), \text{if } x < 0, \\ \varphi_+(\cdot), \text{if } x > 0. \end{cases}$$

$$-\partial_x(\varphi(x,\partial_x u_e)) = f \ \text{ in } \Omega \iff \begin{cases} -\partial_x(\varphi_-(\partial_x u_e)) = f, & \text{on }]-1,0[, \\ -\partial_x(\varphi_+(\partial_x u_e)) = f, & \text{on }]0,1[, \\ \varphi_+(\partial_x {u_e}^+(0)) = \varphi_-(\partial_x {u_e}^-(0)). \end{cases}$$

THE PROBLEM IN 1D

$$\Omega =]-1,1[,\ \varphi(x,\cdot) = \begin{cases} \varphi_-(\cdot), \text{if } x < 0, \\ \varphi_+(\cdot), \text{if } x > 0. \end{cases}$$

Let $x_0 = -1 < \ldots < x_N = 0 < \ldots < x_{N+M} = 1$ be a discretization of $[-1,1]$. The finite volume scheme in 1D writes for $i \in \{0, N+M-1\}$:

$$-F_{i+1} + F_i = \int_{x_i}^{x_{i+1}} f(x)\,dx. \tag{1}$$

with

$$F_i = \varphi(x_i, \nabla_i u^{\mathcal{T}}),\ \ \nabla_i u^{\mathcal{T}} = \frac{u_{i+\frac{1}{2}} - u_{i-\frac{1}{2}}}{x_{i+\frac{1}{2}} - x_{i-\frac{1}{2}}},\ \ \forall i \neq N, \tag{2}$$

QUESTION : How to define F_N ?

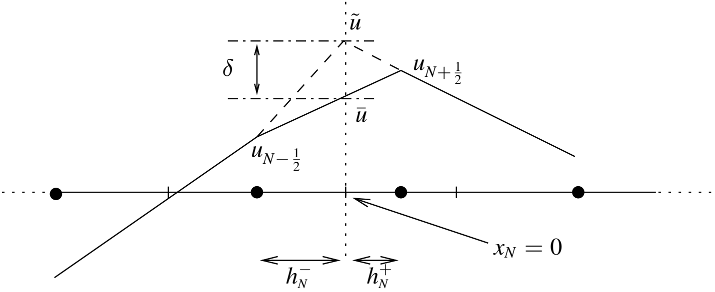
THE NEW GRADIENT 1D

We look for $\tilde{u}$ such that

$$\nabla_N^+ u^{\mathcal{T}} = \frac{u_{N+\frac{1}{2}} - \tilde{u}}{h_N^+}, \quad \nabla_N^- u^{\mathcal{T}} = \frac{\tilde{u} - u_{N-\frac{1}{2}}}{h_N^-},$$

we have

$$\varphi_+(\nabla_N^+ u^{\mathcal{T}}) = \varphi_-(\nabla_N^- u^{\mathcal{T}}).$$



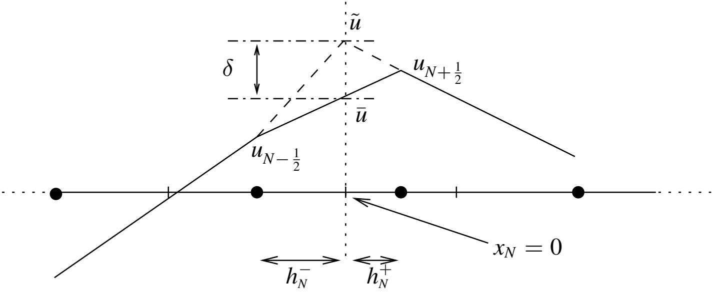
THE NEW GRADIENT 1D

In fact we can rewrite $\tilde{u}$ as

$$\tilde{u} = \bar{u} + \delta, \quad \text{with } \bar{u} = \frac{h_N^- u_{N+\frac{1}{2}} + h_N^+ u_{N-\frac{1}{2}}}{h_N^- + h_N^+}.$$

so that

$$\nabla_N^+ u^{\mathcal{T}} = \nabla_N u^{\mathcal{T}} - \frac{\delta}{h_N^+}, \quad \text{and } \nabla_N^- u^{\mathcal{T}} = \nabla_N u^{\mathcal{T}} + \frac{\delta}{h_N^-}.$$



THE NEW GRADIENT IN 1D

THEOREM (CASE $p \geq 2$)

- *For all $u^{\mathcal{T}} \in \mathbb{R}^N$, there exists a unique δ such that*

$$F_N \stackrel{\text{def}}{=} \varphi_-\left(\nabla_N u^{\mathcal{T}} + \frac{\delta}{h_N^-}\right) = \varphi_+\left(\nabla_N u^{\mathcal{T}} - \frac{\delta}{h_N^+}\right),$$

  we note it $\delta_N(\nabla_N u^{\mathcal{T}})$.
- *The new scheme admits a unique solution.*
- *The flux F_N is consistent with an error in $h^{\frac{1}{p-1}}$.*

THE PROOF RELIES ON : Monotonicity, coercivity, ...

EXAMPLE

For a p-laplacian like equation :

$$\varphi_-(\xi) = k_-|\xi + G_-|^{p-2}(\xi + G_-),$$
$$\varphi_+(\xi) = k_+|\xi + G_+|^{p-2}(\xi + G_+),$$

where $k_-, k_+ \in \mathbb{R}^+$ and $G_-, G_+ \in \mathbb{R}^2$. We obtain

$$F_{_N} = \left(\frac{k_-^{\frac{1}{p-1}} k_+^{\frac{1}{p-1}} (h_{_N}^- + h_{_N}^+)}{h_{_N}^+ k_-^{\frac{1}{p-1}} + h_{_N}^- k_+^{\frac{1}{p-1}}} \right)^{p-1} \left|\nabla_{_N} u^{\mathcal{T}} + \overline{G}\right|^{p-2} \left(\nabla_{_N} u^{\mathcal{T}} + \overline{G}\right),$$

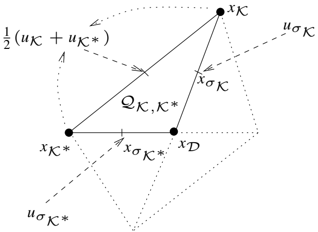
where $\overline{G}$ is a weighted arythmetic mean value of G_- and G_+ defined by

$$\overline{G} = \frac{h_{_N}^- G_- + h_{_N}^+ G_+}{h_{_N}^- + h_{_N}^+}.$$

Warning : the expression of δ_N can not in general be explicited.

- $\nabla_{\mathcal{D}}^{\mathcal{N}} u^{\mathcal{T}}$ is constant on each quarter of diamond

$$\nabla_{\mathcal{D}}^{\mathcal{N}} u^{\mathcal{T}} = \sum_{\mathcal{Q} \in \mathfrak{Q}_{\mathcal{D}}} 1_{\mathcal{Q}} \nabla_{\mathcal{Q}}^{\mathcal{N}} u^{\mathcal{T}},$$



$$\nabla_{\mathcal{Q}_{\mathcal{K},\mathcal{K}^*}}^{\mathcal{N}} u^{\mathcal{T}} = \frac{2}{\sin \alpha_{\mathcal{D}}} \left(\frac{u_{\sigma_{\mathcal{K}^*}} - \frac{1}{2}(u_{\mathcal{K}} + u_{\mathcal{K}^*})}{|\sigma_{\mathcal{K}}|} \boldsymbol{n}_{\mathcal{K}\mathcal{L}} + \frac{u_{\sigma_{\mathcal{K}}} - \frac{1}{2}(u_{\mathcal{K}} + u_{\mathcal{K}^*})}{|\sigma_{\mathcal{K}^*}|} \boldsymbol{n}_{\mathcal{K}^*\mathcal{L}^*} \right)$$

$$\rightsquigarrow \nabla_{\mathcal{Q}}^{\mathcal{N}} u^{\mathcal{T}} = \nabla_{\mathcal{D}}^{\mathcal{T}} u^{\mathcal{T}} + B_{\mathcal{Q}} \delta^{\mathcal{D}}, \forall \mathcal{Q} \subset \mathcal{D}.$$

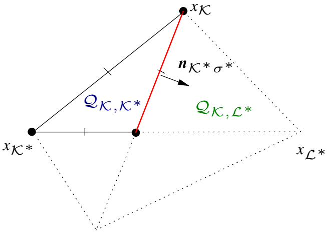
- $B_{\mathcal{Q}}$ is a 2×4 matrix that only depends on the geometry.
- $\delta^{\mathcal{D}}$ are a family of new intermediate unknowns to be determined.

We impose the conservativity of the fluxes

Note

$$\varphi_{\mathcal{Q}}(\xi) = \frac{1}{|\mathcal{Q}|} \int_{\mathcal{Q}} \varphi(z,\xi)\, dz.$$

We look for $\delta^{\mathcal{D}} \in \mathbb{R}^4$ such that



$$\left(\varphi_{\mathcal{Q}_{\mathcal{K},\mathcal{K}^*}}(\nabla^{\mathcal{T}}_{\mathcal{D}} u^{\mathcal{T}} + B_{\mathcal{Q}_{\mathcal{K},\mathcal{K}^*}} \delta^{\mathcal{D}}), \boldsymbol{n}_{\mathcal{K}^*\sigma^*}\right) = \left(\varphi_{\mathcal{Q}_{\mathcal{K},\mathcal{L}^*}}(\nabla^{\mathcal{T}}_{\mathcal{D}} u^{\mathcal{T}} + B_{\mathcal{Q}_{\mathcal{K},\mathcal{L}^*}} \delta^{\mathcal{D}}), \boldsymbol{n}_{\mathcal{K}^*\sigma^*}\right)$$

WE IMPOSE THE CONSERVATIVITY OF THE FLUXES

Note

$$\varphi_{\mathcal{Q}}(\xi) = \frac{1}{|\mathcal{Q}|}\int_{\mathcal{Q}} \varphi(z,\xi)\,dz.$$

We look for $\delta^{\mathcal{D}} \in \mathbb{R}^4$ such that

$$\begin{aligned}
\left(\varphi_{\mathcal{Q}_{\mathcal{K},\mathcal{K}^*}}(\nabla_{\mathcal{D}}^{\mathcal{T}} u^{\mathcal{T}} + B_{\mathcal{Q}_{\mathcal{K},\mathcal{K}^*}}\delta^{\mathcal{D}}), \boldsymbol{n}_{\mathcal{K}^*\sigma^*}\right) &= \left(\varphi_{\mathcal{Q}_{\mathcal{K},\mathcal{L}^*}}(\nabla_{\mathcal{D}}^{\mathcal{T}} u^{\mathcal{T}} + B_{\mathcal{Q}_{\mathcal{K},\mathcal{L}^*}}\delta^{\mathcal{D}}), \boldsymbol{n}_{\mathcal{K}^*\sigma^*}\right)\\
\left(\varphi_{\mathcal{Q}_{\mathcal{L},\mathcal{K}^*}}(\nabla_{\mathcal{D}}^{\mathcal{T}} u^{\mathcal{T}} + B_{\mathcal{Q}_{\mathcal{L},\mathcal{K}^*}}\delta^{\mathcal{D}}), \boldsymbol{n}_{\mathcal{K}^*\sigma^*}\right) &= \left(\varphi_{\mathcal{Q}_{\mathcal{L},\mathcal{L}^*}}(\nabla_{\mathcal{D}}^{\mathcal{T}} u^{\mathcal{T}} + B_{\mathcal{Q}_{\mathcal{L},\mathcal{L}^*}}\delta^{\mathcal{D}}), \boldsymbol{n}_{\mathcal{K}^*\sigma^*}\right)\\
\left(\varphi_{\mathcal{Q}_{\mathcal{K},\mathcal{K}^*}}(\nabla_{\mathcal{D}}^{\mathcal{T}} u^{\mathcal{T}} + B_{\mathcal{Q}_{\mathcal{K},\mathcal{K}^*}}\delta^{\mathcal{D}}), \boldsymbol{n}_{\mathcal{K}\sigma}\right) &= \left(\varphi_{\mathcal{Q}_{\mathcal{L},\mathcal{K}^*}}(\nabla_{\mathcal{D}}^{\mathcal{T}} u^{\mathcal{T}} + B_{\mathcal{Q}_{\mathcal{L},\mathcal{K}^*}}\delta^{\mathcal{D}}), \boldsymbol{n}_{\mathcal{K}\sigma}\right)\\
\left(\varphi_{\mathcal{Q}_{\mathcal{K},\mathcal{L}^*}}(\nabla_{\mathcal{D}}^{\mathcal{T}} u^{\mathcal{T}} + B_{\mathcal{Q}_{\mathcal{K},\mathcal{L}^*}}\delta^{\mathcal{D}}), \boldsymbol{n}_{\mathcal{K}\sigma}\right) &= \left(\varphi_{\mathcal{Q}_{\mathcal{L},\mathcal{L}^*}}(\nabla_{\mathcal{D}}^{\mathcal{T}} u^{\mathcal{T}} + B_{\mathcal{Q}_{\mathcal{L},\mathcal{L}^*}}\delta^{\mathcal{D}}), \boldsymbol{n}_{\mathcal{K}\sigma}\right)
\end{aligned}$$

PROPOSITION

For all $u^{\mathcal{T}} \in \mathbb{R}^{\mathcal{T}}$ and all diamond cell $\mathcal{D}$, there exists a **unique** $\delta^{\mathcal{D}} \in \mathbb{R}^4$ *that ensures the conservativity condition.*

THE M-DDFV SCHEME

We changethe approximate flux of the DDFV scheme :

$$\varphi_{\mathcal{D}}(\nabla_{\mathcal{D}}^{\mathcal{T}} u^{\mathcal{T}}) = \frac{1}{|\mathcal{D}|} \int_{\mathcal{D}} \varphi(z, \nabla_{\mathcal{D}}^{\mathcal{T}} u^{\mathcal{T}})\, dz,$$

by

$$\varphi_{\mathcal{D}}^{\mathcal{N}}(\nabla_{\mathcal{D}}^{\mathcal{T}} u^{\mathcal{T}}) = \frac{1}{|\mathcal{D}|} \sum_{\mathcal{Q} \in \mathfrak{Q}_{\mathcal{D}}} |\mathcal{Q}| \varphi_{\mathcal{Q}}(\underbrace{\nabla_{\mathcal{D}}^{\mathcal{T}} u^{\mathcal{T}} + B_{\mathcal{Q}} \delta^{\mathcal{D}}(\nabla_{\mathcal{D}}^{\mathcal{T}} u^{\mathcal{T}})}_{= \nabla_{\mathcal{Q}}^{\mathcal{N}} u^{\mathcal{T}}}),$$

DISCRETE DUALITY FORMULATION ON THE DIAMOND CELLS

$$2 \sum_{\mathcal{D} \in \mathfrak{D}} |\mathcal{D}| \left(\varphi_{\mathcal{D}}^{\mathcal{N}}(\nabla_{\mathcal{D}}^{\mathcal{T}} u^{\mathcal{T}}), \nabla_{\mathcal{D}}^{\mathcal{T}} v^{\mathcal{T}}\right) = \int_{\Omega} f v^{\mathfrak{M}} dz + \int_{\Omega} f v^{\mathfrak{M}^*} dz, \quad \forall v^{\mathcal{T}} \in \mathbb{R}^{\mathcal{T}}. \tag{3}$$

EXAMPLE

If φ is :

- **linear** i.e. $\varphi(z,\xi) = A(z)\xi$.
- **constant on primal cells**, $A(z) = A_{\mathcal{K}}$ sur $\mathcal{K}$.

We find the schemes proposed in **Hermeline (03)** for which all the computations can be done **explicitly**.

• $A(z) = \lambda(z)\mathrm{Id}$, λ constant on the primal cells, $\alpha_{\mathcal{D}} = \frac{\pi}{2}$

$$
\begin{aligned}
(\varphi_{\mathcal{D}}^{\mathcal{N}}, \boldsymbol{\nu}) &= \frac{\lambda_{\mathcal{K}}\lambda_{\mathcal{L}}}{\frac{|\sigma_{\mathcal{K}}|}{|\sigma_{\mathcal{K}}|+|\sigma_{\mathcal{L}}|}\lambda_{\mathcal{K}} + \frac{|\sigma_{\mathcal{L}}|}{|\sigma_{\mathcal{K}}|+|\sigma_{\mathcal{L}}|}\lambda_{\mathcal{L}}} \frac{u_{\mathcal{L}} - u_{\mathcal{K}}}{|\sigma_{\mathcal{K}}| + |\sigma_{\mathcal{L}}|}, \\
(\varphi_{\mathcal{D}}^{\mathcal{N}}, \boldsymbol{\nu}^*) &= \left(\frac{|\sigma_{\mathcal{K}^*}|}{|\sigma_{\mathcal{K}^*}| + |\sigma_{\mathcal{L}^*}|}\lambda_{\mathcal{K}} + \frac{|\sigma_{\mathcal{L}^*}|}{|\sigma_{\mathcal{K}^*}| + |\sigma_{\mathcal{L}^*}|}\lambda_{\mathcal{L}} \right) \frac{u_{\mathcal{L}^*} - u_{\mathcal{K}^*}}{|\sigma_{\mathcal{K}^*}| + |\sigma_{\mathcal{L}^*}|}.
\end{aligned}
$$

Properties of the m-DDFV scheme

The scheme finally can be written as

$$\mathcal{F}\left(\left(\nabla_{\mathcal{D}}^{\mathcal{T}} u^{\mathcal{T}} + B_{\mathcal{Q}} \delta^{\mathcal{D}}(\nabla_{\mathcal{D}}^{\mathcal{T}} u^{\mathcal{T}})\right)_{\mathcal{Q}\in\mathfrak{Q}}\right) = \text{sources terms},$$

with on each diamond cells

$$\delta^{\mathcal{D}}(\xi) = \mathcal{G}_{\xi}^{-1}(0).$$

Theorem (Case $p > 2$)

- *The scheme m-DDFV admits a* **unique** *solution $u^{\mathcal{T}}$.*
- *If φ piecewise smooth and if u_e is smooth on each quarter of diamond $\mathcal{Q}$, we have*

$$\|u_e - u^{\mathcal{T}}\|_{L^p} + \|\nabla u_e - \nabla^{\mathcal{N}} u^{\mathcal{T}}\|_{L^p} \leq C\, h^{\frac{1}{p-1}}.$$

OUTLINES

REMARKS ON THE POTENTIEL CASE

If φ dérives from a potentiel Φ

$$\begin{cases} \varphi(z,\xi) &= \nabla_\xi \Phi(z,\xi), \quad \text{for all } \xi \in \mathbb{R}^2 \ \text{ a.e. } z \in \Omega, \\ \Phi(z,0) &= 0, \quad \text{a. e. } z \in \Omega. \end{cases}$$

PROPOSITION

The solution $u^{\mathcal{T}}$ of the (3) *is the unique minimum of*

$$J^{\mathcal{T}}(v^{\mathcal{T}}) = 2 \sum_{\mathcal{D}\in\mathfrak{D}} \sum_{\mathcal{Q}\in\mathfrak{Q}_{\mathcal{D}}} |\mathcal{Q}| \Phi_{\mathcal{Q}}(\nabla^{\mathcal{N}}_{\mathcal{Q}} v^{\mathcal{T}}) - \sum_{\mathcal{K}} |\kappa| f_{\mathcal{K}} v_{\mathcal{K}} - \sum_{\mathcal{K}^*} |\kappa^*| f_{\mathcal{K}^*} v_{\mathcal{K}^*}, \ \forall v^{\mathcal{T}} \in \mathbb{R}^{\mathcal{T}}$$

with $\Phi_{\mathcal{Q}}(\cdot) = \int_{\mathcal{Q}} \Phi(z,\cdot) dz.$

REMARKS ON THE POTENTIEL CASE

PROPOSITION

$(u^{\mathcal{T}}, (\delta^{\mathcal{D}}(\nabla^{\mathcal{T}}_{\mathcal{D}} u^{\mathcal{T}}))_{\mathcal{D}})$ is the unique minimun of the functional

$$J^{\mathcal{T},\Delta}(v^{\mathcal{T}}, \tilde{\delta}) = 2 \sum_{\mathcal{D}\in\mathfrak{D}} \sum_{\mathcal{Q}\in\mathfrak{Q}_{\mathcal{D}}} |\mathcal{Q}|\Phi_{\mathcal{Q}}(\nabla^{\mathcal{T}}_{\mathcal{D}} v^{\mathcal{T}} + B_{\mathcal{Q}}\tilde{\delta}^{\mathcal{D}})$$
$$- \sum_{\mathcal{K}} |\kappa| f_{\mathcal{K}} v_{\mathcal{K}} - \sum_{\mathcal{K}^*} |\kappa^*| f_{\mathcal{K}^*} v_{\mathcal{K}^*}, \quad \forall v^{\mathcal{T}} \in \mathbb{R}^{\mathcal{T}}, \ \forall \tilde{\delta} \in \Delta.$$

A DÉCOMPOSITION-COORDINATION ALGORITHM

NON QUADRATIC FUNCTIONAL (see Glowinsky & al.)

Let $\mathcal{A} = (A_\mathcal{Q})_{\mathcal{Q}\in\mathfrak{Q}}$ be a family of 2×2 positive definite matrices.

$$L_\mathcal{A}^{\mathcal{T},\Delta}(v^\mathcal{T}, \tilde{\delta}, g, \lambda) \stackrel{\text{def}}{=} 2\sum_{\mathcal{Q}\in\mathfrak{Q}} |\mathcal{Q}|\Phi_\mathcal{Q}(g_\mathcal{Q}) - \sum_{\mathcal{K}} |\mathcal{K}| f_\mathcal{K} v_\mathcal{K} - \sum_{\mathcal{K}^*} |\mathcal{K}^*| f_{\mathcal{K}^*} v_{\mathcal{K}^*}$$
$$+ 2\sum_{\mathcal{Q}\in\mathfrak{Q}} |\mathcal{Q}|(\lambda_\mathcal{Q}, g_\mathcal{Q} - \nabla_\mathcal{D}^\mathcal{T} v^\mathcal{T} - B_\mathcal{Q}\tilde{\delta}^\mathcal{D})$$
$$+ \sum_{\mathcal{Q}\in\mathfrak{Q}} |\mathcal{Q}| \Big(A_\mathcal{Q}(g_\mathcal{Q} - \nabla_\mathcal{D}^\mathcal{T} v^\mathcal{T} - B_\mathcal{Q}\tilde{\delta}^\mathcal{D}), (g_\mathcal{Q} - \nabla_\mathcal{D}^\mathcal{T} v^\mathcal{T} - B_\mathcal{Q}\tilde{\delta}^\mathcal{D})\Big),$$
$$\forall v^\mathcal{T} \in \mathbb{R}^\mathcal{T}, \forall \tilde{\delta} \in \Delta, \forall g, \lambda \in (\mathbb{R}^2)^\mathfrak{Q}.$$

THEOREM

The solution $u^\mathcal{T}$ of the m-DDFV scheme is obtained from the unique saddle-point of the lagragian $L_\mathcal{A}^{\mathcal{T},\Delta}$.

REMARK : Standart choice of the augmention parameter : $A_\mathcal{Q} = r\mathrm{Id}$.

THE ALGORITHM

- *Step 1 :* Find $(u^{\mathcal{T},n}, \delta^n_{\mathcal{D}})$ solution of

$$2\sum_{\mathcal{Q}\in\mathfrak{Q}} |\mathcal{Q}|\Big(A_{\mathcal{Q}}(\nabla^{\mathcal{T}}_{\mathcal{D}} u^{\mathcal{T},n} + B_{\mathcal{Q}}\delta^n_{\mathcal{D}} - g^{n-1}_{\mathcal{Q}}), \nabla^{\mathcal{T}}_{\mathcal{D}} v^{\mathcal{T}}\Big)$$
$$= \sum_{\mathcal{K}} |\mathcal{K}| f_{\mathcal{K}} v_{\mathcal{K}} + \sum_{\mathcal{K}^*} |\mathcal{K}^*| f_{\mathcal{K}^*} v_{\mathcal{K}^*} + 2\sum_{\mathcal{Q}\in\mathfrak{Q}} |\mathcal{Q}|(\lambda^{n-1}_{\mathcal{Q}}, \nabla^{\mathcal{T}}_{\mathcal{D}} v), \quad \forall v^{\mathcal{T}} \in \mathbb{R}^{\mathcal{T}}.$$

$$\sum_{\mathcal{Q}\in\mathfrak{Q}_{\mathcal{D}}} |\mathcal{Q}|{}^t B_{\mathcal{Q}} A_{\mathcal{Q}}(B_{\mathcal{Q}}\delta^n_{\mathcal{D}} + \nabla^{\mathcal{T}}_{\mathcal{D}} u^{\mathcal{T},n} - g^{n-1}_{\mathcal{Q}}) - \sum_{\mathcal{Q}\in\mathfrak{Q}_{\mathcal{D}}} |\mathcal{Q}|{}^t B_{\mathcal{Q}} \lambda^{n-1}_{\mathcal{Q}} = 0, \ \forall \mathcal{D} \in \mathfrak{D}.$$

- *Step 2 :* On each $\mathcal{Q}$, find $g^n_{\mathcal{Q}}$ solution of

$$\varphi_{\mathcal{Q}}(g^n_{\mathcal{Q}}) + \lambda^{n-1}_{\mathcal{Q}} + A_{\mathcal{Q}}(g^n_{\mathcal{Q}} - \nabla^{\mathcal{T}}_{\mathcal{D}} u^{\mathcal{T},n} - B_{\mathcal{Q}}\delta^n_{\mathcal{D}}) = 0.$$

- *Step 3 :* On each $\mathcal{Q}$ compute $\lambda^n_{\mathcal{Q}}$ as

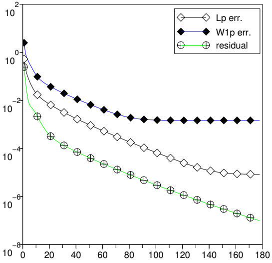
$$\lambda^n_{\mathcal{Q}} = \lambda^{n-1}_{\mathcal{Q}} + A_{\mathcal{Q}}(g^n_{\mathcal{Q}} - \nabla^{\mathcal{T}}_{\mathcal{D}} u^{\mathcal{T},n} - B_{\mathcal{Q}}\delta^n_{\mathcal{D}}).$$

Convergence of the iterative method

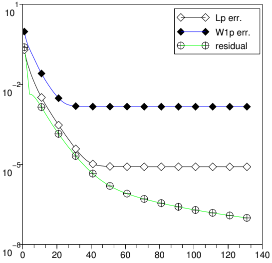
Theorem

For all family of augmentation parameter $\mathcal{A}$, the previous algorithm converges towards the unique solution of the m-DDFV scheme.

The algorithm is still valid in the **non-potential** case.



isotropic augmentation
$A_{\mathcal{Q}} = r\,\mathrm{Id}$.



naisotropic augmention
$A_{\mathcal{Q}}$ adapted to the problem

OUTLINES

Behaviour of DDFV scheme

Anisotropic operators ⇝ Benchmark Finite volume Schemes on general grids for anisotropic and heterogeneous diffusions problems. FVCA5 juin 2008

$$-\mathrm{div}(A(z)\nabla u_e) = f$$

$$A(z_1, z_2) = \frac{1}{|z|^2}\begin{pmatrix} z_1^2 + \delta z_2^2 & (1-\delta)z_1 z_2 \\ (1-\delta)z_1 z_2 & \delta z_1^2 + z_2^2 \end{pmatrix}$$

For $u_e(x, y) = \sin(\pi x)\sin(\pi y)$. For the DDFV scheme

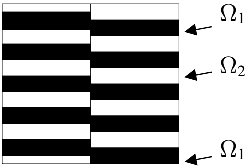
δ	Order L^2 rectangle	Order H^1 rectangle	Order L^2 triangle	Order H^1 triangle
2	2.01	1.95	2.0	1.0
10^{-1}	2.03	1.91	2.0	1.0
10^{-3}	1.88	1.64	2.0	1.0

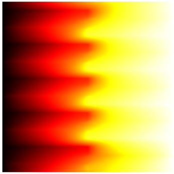
BEHAVIOUR OF DDFV SCHEME

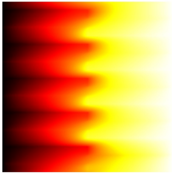
ANISOTROPIC OPERATORS ⇝ Benchmark Finite volume Schemes on general grids for anisotropic and heterogeneous diffusions problems. FVCA5 juin 2008

$$-\mathrm{div}(A(z)\nabla u_e) = f$$

$$A = \begin{pmatrix} \alpha & 0 \\ 0 & \beta \end{pmatrix}, \text{ with } \begin{pmatrix} \alpha \\ \beta \end{pmatrix} = \begin{pmatrix} 10^2 \\ 10 \end{pmatrix} \text{ on } \Omega_1, \begin{pmatrix} \alpha \\ \beta \end{pmatrix} = \begin{pmatrix} 10^{-2} \\ 10^{-3} \end{pmatrix} \text{ on } \Omega_2$$







Non admissible coarse mesh   Mesh 320×320

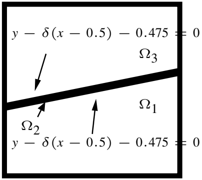
COMPARISON OF DDFV AND M-DDFV SCHEMES

ANISOTROPIC OPERATORS ⇝ Benchmark Finite volume Schemes on general grids for anisotropic and heterogeneous diffusions problems. FVCA5 juin 2008

$$-\text{div}(A(z)\nabla u_e) = f$$

$$A = R_\theta \begin{pmatrix} \alpha & 0 \\ 0 & \beta \end{pmatrix} R_\theta^{-1}, \delta = 0.2, u_e(x,y) = -x - \delta, \delta = \tan\theta$$

$$\begin{pmatrix} \alpha \\ \beta \end{pmatrix} = \begin{pmatrix} 10^2 \\ 10 \end{pmatrix} \text{ on } \Omega_2, \quad \begin{pmatrix} \alpha \\ \beta \end{pmatrix} = \begin{pmatrix} 1 \\ 10^{-1} \end{pmatrix} \text{ on } \Omega_1 \cup \Omega_3.$$



Mesh following the discontinuity : 210 quadrangles

- The DDFV scheme

$$erl2 \sim 1.18 \times 10^{-3}, \; ergradl2 \sim 1.33 \times 10^{-2}$$

- The m-DDFV scheme is exact :

$$erl2 \sim 5.45 \times 10^{-16}, \; ergradl2 \sim 3.88 \times 10^{-15}$$

OTHER APPLICATION

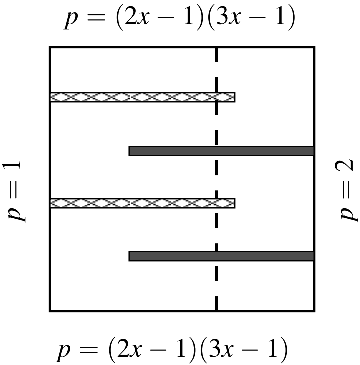
FLOWS IN FRACTURED POROUS MEDIA (GDR MOMAS)

$$\begin{cases} \operatorname{div} v = 0, & \text{Mass conservation,} \\ v = -\mathbf{K}\nabla p, & \text{Darcy law.} \end{cases}$$

Porous media : $\mathbf{K} = \mathrm{Id}$.
Fractures (Left) : $\mathbf{K} = 10^{-2}\mathrm{Id}$.
Fractures (Right) :

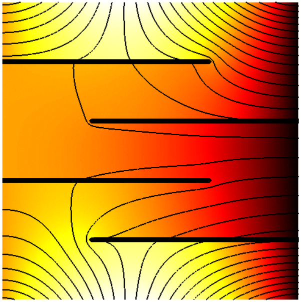
$$\mathbf{K} = \begin{pmatrix} 10^2 & 0 \\ 0 & 10^{-2} \end{pmatrix}.$$

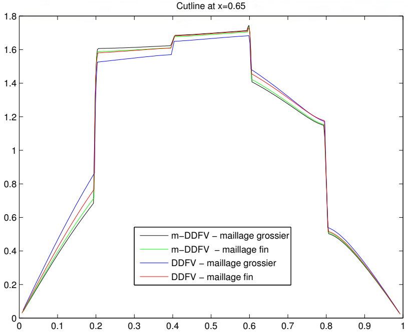
Domain $\Omega =]0,1[^2$
Aperture of the fractures : 10^{-2}



OTHER APPLICATION

CUTLINES OF THE PRESSION





Coarse mesh : 17760 cells

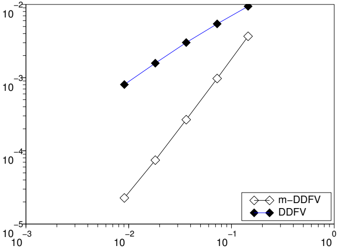
Fine mesh : 68160 cells

NONLINEAR PROBLEMS

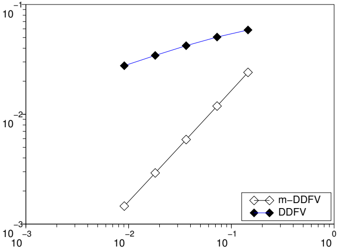
THE M-DDFV FOR A NONLINEAR TRANSMISSION PROBLEM

$$\begin{cases} \text{if } z_1 < 0.5, & \varphi(z,\xi) = |\xi|^{p-2}\xi, \\ \text{if } z_1 > 0.5, & \varphi(z,\xi) = (A\xi,\xi)^{\frac{p-2}{2}} A\xi, \text{ with } A = \begin{pmatrix} 2 & 0 \\ 0 & 5 \end{pmatrix}. \end{cases}$$

ILLUSTRATION FOR P=3.0



Error in L^∞ - orders 1.71 and 0.97



Error in $W^{1,p}$ - orders 1.0 and 0.3

MORE GENERAL TRANSMISSION PROBLEMS

$\Omega = \Omega_1 \cup \Omega_2$ with $\Omega_1 =]0, 0.5[\times]0, 1[$ and $\Omega_2 =]0.5[\times]0, 1[$

$$\varphi(z, \xi) = |\xi|^{p_i - 2}\xi \text{ on } \Omega_i$$

$$u_e(x, y) = \begin{cases} x\left(\left(\lambda^{\frac{p_2-1}{p_1-1}} - 1\right)(2x-1) + 1\right) & \text{for } x \leq 0.5 \\ (1-x)((1+\lambda)(2x-1)+1) & \text{for } x \geq 0.5 \end{cases}$$

$\rightsquigarrow$ Large discontinuities of the gradients along the interface

For $p_1 = 2, p_2 = 4$

h	DDFV $L^p(\Omega)$	m-DDFV $L^p(\Omega)$	DDFV $W^{1,p}(\Omega)$	m-DDFV $W^{1,p}(\Omega)$
7.25E-02	4.70E-01	3.61E-02	2.5E+01	1.41
3.63E-02	2.36E-01	9.14E-02	2.03E+01	6.62E-01
1.81E-02	1.19E-01	2.24E-03	1.65E+01	3.11E-01
9.07E-03	6.01E-02	4.46E-04	1.34E+01	1.47E-01
order	0.98	2.11	0.30	1.08

CONCLUSIONS

- The approach DDFV or m-DDFV allows the use of a large variety of meshes and of elliptic operators.
- The structure of the continuous problem is preserved so that it can be successfully used for nonlinear problems.
- In case of discontinuous coefficients, a good convergence order is recovered by the use of the m-DDFV scheme.
- We derive an efficient nonlinear algorithm to solve such schemes.
- We can couple a linear operator with a non linear ones.
- Drawback : the maximum principle is not fullfilled.

OTHER WORKS ON DDFV SCHEMES

- More general boundary conditions (Neumann, Fourier) S. Krell, master report 2007.
- Coupling with a domain decomposition method S. Krell, master report 2007.
- Extension to 3D Works on Hermeline or Andreianov and *all* or with Y. Coudière et *all*.
- Extension to the div $-$ rot problem and to the Stokes problem Works of S. Delcourte, P. Omnès and *all*.

PERSPECTIVES

- Boundary condition of Ventcel type and coupling with a domain decomposition algorithm Work in progress with L. Halpern.
- Extension to nonlinear stokes equations, PHD thesis of Stella Krell.
- Nonlinear tests functions to recover the maximum principle.
- Optimal strategy for the choice of the augmentation parameter.
- Error estimates in the case where the regularity of the solution in only of Besov type.

Thank you !