Invariants of identity-tangent diffeomorphisms: explicit formulae and effective computation.

Abstract: *In this short Survey we revisit the subject of local, identity-tangent diffeomorphisms of $\mathbb{C}$ and their analytic invariants, under two viewpoints: that of explicit expansions, which necessarily involve multitangents and multizetas; and that of effective computation. Along the way, we stress the difference between the* collectors *(pre-invariant but of one piece) and the* connectors *(invariant but mutually unrelated). We also attempt to streamline the nomenclature and notations.*

Contents

1   Setting and notations.

1.1   Classical results.

We shall be concerned here with *local*[1] *identity-tangent diffeomorphisms* of $\mathbb{C}$, or *diffeos* for short, with the fixed-point located at ∞ for technical convenience:

$$f \;\; : \;\; z \mapsto z + \sum_{1 \leq s} f_s \, z^{1-s} \qquad a_s \in \mathbb{C} \qquad (1)$$

Unless f be the identity map, we can always subject it to an analytic (resp. formal) conjugation $f \mapsto f_1 = h \circ f \circ h^{-1}$, followed if necessary by an elementary ramification $\big(f_1(z^{1/p})\big)^p$, so as to give it the following *prepared* resp. *normal* form:

$$f_{\mathrm{prep}} : \;\; z \;\; \mapsto \;\; z + 1 - \rho z^{-1} + \sum_{2 < s_0 \leq s} f_{[s]} \, z^{1-s} \qquad \Big(s \in \frac{1}{p}\mathbb{N}^*\Big) \qquad (2)$$

$$f_{\mathrm{norm}} : \;\; z \;\; \mapsto \;\; z + 1 - \rho z^{-1} \qquad\qquad\qquad\qquad (3)$$

where s_0 may be chosen as large as one wishes.

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[1]i.e. *analytic germs of* –

The *tangency order* p and *iterative residue* ρ are the only *formal invariants* of identity-tangent diffeos. But our diffeos also possess countably many (independent) scalar *analytic invariants*, which are best defined as the Fourier coefficients of the so-called *connectors*.[2] These are pairs of germs of 1-periodic analytic mappings $\boldsymbol{\pi} = (\boldsymbol{\pi}_{\mathrm{no}}, \boldsymbol{\pi}_{\mathrm{so}})$ on upper/lower half-planes $\pm\Im(z) \gg 1$. There are p such pairs, corresponding to the p-fold ramification of z in (2). Here, *no* and *so* stand for *north* and *south*, i.e. the upper and lower half-planes.

We shall throughout prioritise the *standard case* $p = 1$, $\rho = 0$, i.e. focus on diffeos of the form:

$$f := l \circ g \quad with \quad l := z \mapsto z + 1 \quad and \quad g : z \mapsto z + \sum_{3 \leq s} g_s\, z^{1-s} \qquad (4)$$

and merely sketch the (minor) changes required to cover the general case.

Any standard f possesses two well-defined, mutually inverse so-called *iterators*, to wit $f^*_{\pm}$ (direct iterator) and $^*f_{\pm}$ (reciprocal iterator), defined on U-shaped domains[3] by the limits:

$$f^*_{\pm}(z) = \lim_{k \to \pm\infty} l^{-k} \circ f^k \quad ; \quad {}^*f_{\pm}(z) = \lim_{k \to \pm\infty} f^{-k} \circ l^k \qquad (5)$$

The connectors $\boldsymbol{\pi}^{\pm 1}$, with their northern and southern components, are then defined on $\pm\Im(z) \gg 1$ by:

$$\boldsymbol{\pi} := f^*_+ \circ {}^*f_- \quad ; \quad \boldsymbol{\pi}^{-1} := f^*_- \circ {}^*f_+ \qquad (6)$$

For reasons that will soon become apparent, we must also consider the infinitesimal generators f_* and $\boldsymbol{\pi}_*$ of f and $\boldsymbol{\pi}$. These are formal, generically divergent power resp. Fourier series. Of course, $\boldsymbol{\pi}_*$ is not constructed directly from $\boldsymbol{\pi}$, but via its northern and southern components. We thus have the three pairs:

$$\boldsymbol{\pi} := (\boldsymbol{\pi}_{\mathrm{no}}, \boldsymbol{\pi}_{\mathrm{so}}) \quad ; \quad \boldsymbol{\pi}^{-1} := (\boldsymbol{\pi}^{-1}_{\mathrm{no}}, \boldsymbol{\pi}^{-1}_{\mathrm{so}}) \quad ; \quad \boldsymbol{\pi}_* := (\boldsymbol{\pi}_{*\mathrm{no}}, \boldsymbol{\pi}_{*\mathrm{so}}) \qquad (7)$$

along with the relations

$$f(z) = \exp\big(f_*(z)\,\partial_z\big).\,z \qquad \big(f_*\,\partial_z f^* \equiv 1\big) \qquad (8)$$

$$\boldsymbol{\pi}^{\pm 1}_{\mathrm{no}}(z) = \exp\big(\pm\,\boldsymbol{\pi}_{*\mathrm{no}}(z)\,\partial_z\big).\,z \qquad (9)$$

$$\boldsymbol{\pi}^{\pm 1}_{\mathrm{so}}(z) = \exp\big(\pm\,\boldsymbol{\pi}_{*\mathrm{so}}(z)\,\partial_z\big).\,z \qquad (10)$$

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[2] In the context of identity-tangent diffeos, the connectors are sometimes referred to as *horn maps*, but the notion is more general: in resurgent analysis (see §1.2 *infra*) the connectors are the operators that take us from on sectorial model to the next.

[3] f^*_+ and $^*f_+$ are defined on a west-north-south domain, while f^*_- and $^*f_-$ are defined on an east-north-south domain.

In (8) f^* and $^*f$ denote of course the *formal iterators*, i.e. the power series solutions of the equations

$$f^* \circ f = l \circ f^* \qquad with \qquad f^*(z) = z + o(1) \tag{11}$$

$$f \circ {}^*f = {}^*f \circ l \qquad with \qquad {}^*f(z) = z + o(1) \tag{12}$$

normalised by the condition of carrying no constant term. Anticipating on the sequel, here is how the scalar invariants can be read off the Fourier expansions of the connectors:

$$\boldsymbol{\pi}_{\mathrm{no}}(z) = z + \sum_{\omega \in \Omega^-} A_\omega^+ \, e^{-\omega z} \quad ; \quad \boldsymbol{\pi}_{\mathrm{so}}(z) = z + \sum_{\omega \in \Omega^+} A_\omega^- \, e^{-\omega z} \tag{13}$$

$$\boldsymbol{\pi}_{\mathrm{no}}^{-1}(z) = z + \sum_{\omega \in \Omega^-} A_\omega^- \, e^{-\omega z} \quad ; \quad \boldsymbol{\pi}_{\mathrm{so}}^{-1}(z) = z + \sum_{\omega \in \Omega^+} A_\omega^+ \, e^{-\omega z} \tag{14}$$

$$\boldsymbol{\pi}_{*\mathrm{no}}(z) = +2\pi i \sum_{\omega \in \Omega^-} A_\omega \, e^{-\omega z} \quad ; \quad \boldsymbol{\pi}_{*\mathrm{so}}(z) = -2\pi i \sum_{\omega \in \Omega^+} A_\omega \, e^{-\omega z} \tag{15}$$

Pay attention to the altered position of $\pm$ in 13 and 14; the reasons for this apparent incoherence shall become clear in due course. The indices ω run through $\Omega := 2\pi i \mathbb{Z}^*$ or $\Omega^\pm := \pm 2\pi i \, \mathbb{N}^*$, and each of the three systems

$$\{A_\omega^+ \, , \, \omega \in \Omega\} \qquad , \qquad \{A_\omega^- \, , \, \omega \in \Omega\} \qquad , \qquad \{A_\omega \, , \, \omega \in \Omega\} \tag{16}$$

constitutes a *free* and *complete* system of analytic invariants.[4]

1.2   Reminder about resurgent functions.

We will have to be content here with a very sketchy presentation. The algebra of *resurgent fonctions* admits three different realisations or models:
(i) the *formal model*, consisting of formal power series $\tilde{\varphi}(z)$ of z^{-1} or of more general *transseries*;
(ii) the *convolutive model*, consisting of microfunctions[5] at $\zeta = 0$, whose *majors* $\check{\varphi}(\zeta)$ are constraint-free at the origin but whose *minors* $\hat{\varphi}(\zeta)$ have the

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[4]With the minor qualifier that, under a conjugation by a shift h of the form $l^\alpha(z) := z + \alpha$, the periodic germs $\boldsymbol{\pi}^\pm$ also undergo conjugation by the same shift, with obvious repercussions for their Fourier coefficients.

[5]i.e. minor-major pairs $(\hat{\varphi}(\zeta), \check{\varphi}(\zeta))$. The *majors* are defined up to regular germs at the origin, and the *minors* are related to them under $2\pi i \, \hat{\varphi}(\zeta) \equiv \check{\varphi}(\zeta \, e^{-\pi i}) - \check{\varphi}(\zeta \, e^{+\pi i})$ for $\zeta \sim 0$. In the present paper, we shall almost entirely dispense with majors, since we shall mostly be dealing with so-called *integrable* microfunctions, whose minors carry the whole information.

property of endless continuation[6] and exponential growth;[7]

(iii) the *geometric model(s)*, consisting of analytic germs $\varphi_\theta(z)$ defined on sectorial neighbourhoods of ∞ of bisectrix $arg(z^{-1}) = \theta$ and aperture at least π.

The natural algebra product in the z-models (i) and (iii) is of course multiplication. In the ζ-model (ii) it is convolution, defined first *locally*[8] by

$$(\hat{\varphi}_1 * \hat{\varphi}_2)(\zeta) := \int_0^\zeta \hat{\varphi}_1(\zeta_1)\,\hat{\varphi}_2(\zeta - \zeta_1)\,d\zeta_1 \qquad (\zeta \sim 0) \qquad (17)$$

and then *in the large* by analytic continuation.

In practice, one starts with elements $\tilde{\varphi}$ of model (i) obtained as formal solutions of differential or functional equations, and the aim is to resum them, i.e. to go to model (iii). Generally speaking, this is possible only over the detour through model (ii), with the *formal Borel tranform* $\mathcal{B}$

$$z^{-\sigma} \mapsto \frac{\zeta^{\sigma-1}}{\Gamma(\sigma)} \quad ; \quad (\partial_\sigma)^n z^{-\sigma} \mapsto (\partial_\sigma)^n \frac{\zeta^{\sigma-1}}{\Gamma(\sigma)} \quad ; \quad etc \qquad (18)$$

taking us from (i) to (ii), and the *polarised Laplace transform* $\mathcal{L}_\theta$

$$\varphi_\theta(z) = \int_{\arg(\zeta)=\theta} \hat{\varphi}(\zeta)\,e^{-\zeta z}\,d\zeta \qquad (19)$$

taking us from (ii) to (iii).

The most outstanding feature of the resurgence algebras is the existence on them of a rich array of so-called *alien operators* Δ_ω and $\Delta_\omega^\pm$, with indices ω running through $\mathbb{C}_\bullet := \widetilde{\mathbb{C} - \{0\}}$. These operators act on all three models[9], but are first defined in the convolutive model, where they have the effect of measuring the singularities of the (often highly ramified) minors $\hat{\varphi}$ at or rather *over* ω. Here is how they act:

$$(\Delta_\omega \hat{\varphi})(\zeta) \;\; := \;\; \sum_{\epsilon_1,\ldots,\epsilon_r} \frac{\epsilon_r}{2\pi i}\,\lambda_{\epsilon_1,\ldots,\epsilon_{r-1}}\,\hat{\varphi}^{\binom{\epsilon_1\,,\ldots,\,\epsilon_1}{\omega_1\,,\ldots,\,\epsilon_r}}(\omega + \zeta) \qquad (20)$$

$$(\Delta_\omega^\pm \hat{\varphi})(\zeta) \;\; := \;\; \sum_{\epsilon_1,\ldots,\epsilon_r} \pm\,\epsilon_r\,\lambda_{\epsilon_1,\ldots,\epsilon_{r-1}}^\pm\,\hat{\varphi}^{\binom{\epsilon_1\,,\ldots,\,\epsilon_1}{\omega_1\,,\ldots,\,\epsilon_r}}(\omega + \zeta) \qquad (21)$$

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[6]laterally along any *finite and finitely punctured* broken lines.

[7]i.e. *at most exponential*, along *infinite but finitely punctured broken lines*, with a suitable uniformity condition.

[8]When the *minors* $\hat{\varphi}$ are not integrable at the origin, one must modify the definition and draw in the *majors* $\check{\varphi}$.

[9]with the same symbols doing service in all three, since no confusion is possible.

with signs $\epsilon_j \in \{+, -\}$, with weights $\lambda_\bullet$, $\lambda_\bullet^+$, $\lambda_\bullet^-$ defined by

$$\lambda_{\epsilon_1, \ldots, \epsilon_{r-1}} \quad := \quad \frac{p! \, q!}{r!} \quad with \quad p := \sum_{\epsilon_i = +} 1 \, , \, q := \sum_{\epsilon_i = -} 1 \tag{22}$$

$$\lambda_{\epsilon_1, \ldots, \epsilon_{r-1}}^\epsilon \quad := \quad 1 \quad if \quad \epsilon_1 = \cdots = \epsilon_{r-1} = \epsilon \tag{23}$$
$$:= \quad 0 \quad otherwise$$

and with $\hat{\varphi}^{\binom{\epsilon_1, \ldots, \epsilon_1}{\omega_1, \ldots, \epsilon_r}}(\omega + \zeta)$ denoting the analytic continuation of $\hat{\varphi}$ from ζ to $\omega + \zeta$ under right (resp. left) circumvention of each intervening singularity ω_j if $\epsilon_j = +$ (resp. $\epsilon_j = -$). We start of course with a point ζ close enough to 0 on the axis $arg(\zeta) = arg(\omega)$, and extend the definition in the large by analytic continuation. The operators Δ_ω are *derivations*. Thus, in the convolutive and formal models, the identities hold:

$$\Delta_\omega(\hat{\varphi}_1 * \hat{\varphi}_2) \quad = \quad \Delta_\omega(\hat{\varphi}_1) * \hat{\varphi}_2 + \hat{\varphi}_1 * \Delta_\omega(\hat{\varphi}_2) \tag{24}$$
$$\Delta_\omega(\tilde{\varphi}_1 . \tilde{\varphi}_2) \quad = \quad \Delta_\omega(\tilde{\varphi}_1) . \tilde{\varphi}_2 + \tilde{\varphi}_1 . \Delta_\omega(\tilde{\varphi}_2) \tag{25}$$

When working in any one of the multiplicative models (formal or geometric), it is often convenient to phase-shift the alien operators, and to set:

$$\mathbb{\Delta}_\omega \quad := \quad e^{-\omega z} \Delta_\omega \qquad (\, [\partial_z, \mathbb{\Delta}_\omega] \equiv 0 \,) \tag{26}$$

$$\mathbb{\Delta}_\omega^\pm \quad := \quad e^{-\omega z} \Delta_\omega^\pm \qquad (\, [\partial_z, \mathbb{\Delta}_\omega^\pm] \equiv 0 \,) \tag{27}$$

The gain here is that the new operators commute with ∂_z. These phase-shifted operators are also the natural ingredients of the *axial operators* $\mathcal{R}_\theta$ and $\mathcal{R}_\theta^\pm$:

$$\mathcal{R}_\theta \quad = \quad \sum_{\arg(\omega) = \theta} \mathbb{\Delta}_\omega \tag{28}$$

$$\mathcal{R}_\theta^\pm \quad = \quad 1 + \sum_{\arg(\omega) = \theta} \mathbb{\Delta}_\omega^\pm \quad = \quad \exp\left(\pm 2\pi i \, \mathcal{R}_\theta \right) \tag{29}$$

which are the key to the *axis-crossing* identities :

$$\varphi_{\theta - \epsilon} = (\mathcal{R}_\theta^+ \, \varphi)_{\theta + \epsilon} \qquad ; \qquad (\Phi . \mathcal{R}_\theta^+)_{\theta - \epsilon} = (\mathcal{R}_\theta^+ . \Phi)_{\theta + \epsilon} \tag{30}$$

$$\varphi_{\theta + \epsilon} = (\mathcal{R}_\theta^- \, \varphi)_{\theta - \epsilon} \qquad ; \qquad (\Phi . \mathcal{R}_\theta^-)_{\theta - \epsilon} = (\mathcal{R}_\theta^- . \Phi)_{\theta + \epsilon} \tag{31}$$

that connect two sectorial germs $\varphi_{\theta - \epsilon}$ and $\varphi_{\theta + \epsilon}$ relative to Laplace integration right and left of any given singularity-carrying axis θ in the ζ-plane.[10]

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[10]In (24), (25), φ denotes any *resurgent function* and Φ any *resurgent operator* (such as multiplication or postcomposition by a resurgent function etc).

1.3   The Bridge equation.

The iterator f^* and $^*\!f$ characterised by the relations (11) and (12) verify the following resurgence equations

$$\Delta_\omega \, {}^*\!f(z) \;\; = \;\; +A_\omega \, \partial_z \, {}^*\!f(z) \qquad\qquad (\forall \omega \in \Omega) \qquad\qquad (32)$$

$$\Delta_\omega \, f^*(z) \;\; = \;\; -A_\omega \, e^{-\omega\,(f^*(z)-z)} \qquad\qquad (\forall \omega \in \Omega) \qquad\qquad (33)$$

with the very same scalar coefficients A_ω as in (15). For values of ω not in Ω, the alien derivatives would be $\equiv 0$. If we now introduce the differential operators:

$$\mathbb{A}_\omega := A_\omega \, e^{-\omega z} \, \partial_z \qquad\qquad (\forall \omega \in \Omega) \qquad\qquad (34)$$

the resurgence equations assume the form of the Bridge equation:[11]

$$\boldsymbol{\Delta}_\omega \, {}^*\!f(z) \;\; = \;\; +\mathbb{A}_\omega \, \partial_z \, {}^*\!f(z) \qquad\qquad\qquad (35)$$

$$\boldsymbol{\Delta}_\omega \, f^*(z) \;\; = \;\; -(\mathbb{A}_\omega \, z) \circ f^*(z) \qquad\qquad\qquad (36)$$

When expressed in terms of the subsitution operators F^* and $^*\!F$ associated with $^*\!f, f^*$, the Bridge equation takes an even more pleasant form

$$\big[\boldsymbol{\Delta}_\omega, F^*\big] \;\; = \;\; -\,F^* \, \mathbb{A}_\omega \qquad\qquad (F^* \, \varphi := \varphi \circ f^*) \qquad\qquad (37)$$

$$\big[\boldsymbol{\Delta}_\omega, {}^*\!F\,\big] \;\; = \;\; +\mathbb{A}_\omega \, {}^*\!F \qquad\qquad ({}^*\!F \, \varphi := \varphi \circ {}^*\!f) \qquad\qquad (38)$$

But whichever variant we may care to consider, the commutation identities $\big[\boldsymbol{\Delta}_{\omega_1}, \mathbb{A}_{\omega_2}\big] = 0$ make it easy to iterate the above resurgence equations. Thus from (35) we straightaway derive

$$\boldsymbol{\Delta}_{\omega_r} \dots \boldsymbol{\Delta}_{\omega_1} \, {}^*\!f(z) = \mathbb{A}_{\omega_1} \dots \mathbb{A}_{\omega_r} \, \partial_z \, {}^*\!f(z) \qquad (order\ reversion!) \qquad (39)$$

As a consequence, the effect on $^*\!f$ and f^* of the alien operators $\boldsymbol{\Delta}_\omega^\pm$ and of the axial operators $\mathcal{R}_\theta$ is easy to calculate. It is best written in terms of the substitution operators $^*\!F$ and F^* associated with $^*\!f, f^*$, and results in the so-called *axial* Bridge equation:

$$\mathcal{A}_\theta \;\; = \;\; \mathcal{R}_\theta \; - \; {}^*\!F \;\; \mathcal{R}_\theta \;\; F^* \qquad\qquad\qquad (40)$$

$$\mathcal{A}_\theta^+ \;\; = \;\; \mathcal{R}_\theta^+ \;\; {}^*\!F \;\; \mathcal{R}_\theta^- \;\; F^* = {}^*\!F \;\; \mathcal{R}_\theta^- \;\; F^* \;\; \mathcal{R}_\theta^+ \qquad\qquad (41)$$

$$\mathcal{A}_\theta^- \;\; = \;\; \mathcal{R}_\theta^- \;\; {}^*\!F \;\; \mathcal{R}_\theta^+ \;\; F^* = {}^*\!F \;\; \mathcal{R}_\theta^+ \;\; F^* \;\; \mathcal{R}_\theta^- \qquad\qquad (42)$$

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[11]so-called because it relates *ordinary* and *alien* derivatives of one and the same resurgent function. The Bridge equation has in fact much wider applications, and extends, in one form or another, to practically all *resonant* local objects, of which *identity-tangent diffeos* are but a special case. An entire book [E3] has been devoted to the subject.

The axial Bridge equation[12] involves differential (resp. substitution) operators $\mathcal{A}_\theta$ (resp. $\mathcal{A}_\theta^\pm$):

$$\mathcal{A}_\theta \;=\; \sum_{\arg(\omega)=\theta} \mathbb{A}_\omega \tag{43}$$

$$\mathcal{A}_\theta^\pm \;=\; 1 + \sum_{\arg(\omega)=\theta} \mathbb{A}_\omega^\pm \;=\; \exp\left(\,\pm 2\pi i \,\mathcal{A}_\theta\right) \tag{44}$$

which are simply related to the differential (resp. substitution) operators $\boldsymbol{\Pi}_*$ (resp. $\boldsymbol{\Pi}^\pm$ associated with the connectors of §1.1:

$$\boldsymbol{\Pi_{no}} \;:=\; \mathcal{A}^+_{-\frac{\pi}{2}} \quad;\quad \boldsymbol{\Pi_{so}} \;:=\; \mathcal{A}^-_{+\frac{\pi}{2}} \tag{45}$$

$$\boldsymbol{\Pi_{no}^{-1}} \;:=\; \mathcal{A}^-_{-\frac{\pi}{2}} \quad;\quad \boldsymbol{\Pi_{so}^{-1}} \;:=\; \mathcal{A}^+_{+\frac{\pi}{2}} \tag{46}$$

$$\boldsymbol{\Pi_{*no}} \;:=\; +2\pi i \,\mathcal{A}_{-\frac{\pi}{2}} \quad;\quad \boldsymbol{\Pi_{*so}} \;:=\; -2\pi i \,\mathcal{A}_{+\frac{\pi}{2}} \tag{47}$$

The first identity (45) results from applying the direct axis-crossing formula (30) with $\theta = -\frac{\pi}{2}$ and $\varphi = {}^*\!f$ or $\Phi = {}^*\!F$, since ${}^*\!f_{\theta\pm\epsilon} = {}^*\!f_\pm$. The second identity (45) results from applying the inverse axis-crossing formula (31) with $\theta = +\frac{\pi}{2}$ and $\varphi = {}^*\!f$ or $\Phi = {}^*\!F$, since in that case ${}^*\!f_{\theta\pm\epsilon} = {}^*\!f_\mp$ (inversion!). The identities (46) and (46) immediately follow.

1.4   Invariants, connectors, collectors.

Let us survey in one table the main objects introduced so far or yet to come.

diffeo		*collectors*			*connectors*		*invariants*
$f = l \circ g$ $\xrightarrow{1'}$		$\mathfrak{p}$ $\xrightarrow{1''}$ $\boldsymbol{p}$	$\xrightarrow{1'''}$		$\boldsymbol{\pi} = (\boldsymbol{\pi}_{\mathrm{no}}, \boldsymbol{\pi}_{\mathrm{so}})$	$\xrightarrow{1''''}$	$\{A_\omega^\pm\}$
$\downarrow 2$		$\uparrow 4$			$\uparrow 5_{\mathrm{no}}$ $\uparrow 5_{\mathrm{so}}$		$\uparrow 6$
g_* $\xrightarrow{3'}$		$\mathfrak{p}_*$ $\xrightarrow{3''}$ $\boldsymbol{p}_*$	$\xrightarrow{3'''}$		$\boldsymbol{\pi}_* = (\boldsymbol{\pi}_{*\mathrm{no}}, \boldsymbol{\pi}_{*\mathrm{so}})$	$\xrightarrow{3''''}$	$\{A_\omega\}$

The upper row carries the objects of direct interest to us, while the lower row carries their infinitesimal counterparts, which are more in the nature of auxiliary tools.

The first, third and fourth columns carry objects already familiar to us. The second column, however, carries novel, highly interesting objects, the

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[12]The *singular* is appropriate here since (41) and (42) are merely exponential variants of (40). The commutation of the three automorphisms $\mathcal{A}_\theta^\pm$, $\mathcal{R}_\theta^\pm$, ${}^*\!F\;\;\mathcal{R}_\theta^\mp\;\;F^*$ is itself a consequence of the commutation of the three derivations $\mathcal{A}_\theta$, $\mathcal{R}_\theta$, ${}^*\!F\;\;\mathcal{R}_\theta\;\;F^*$.

collectors, which are very close in a sense to the *connectors*, yet should be, for the sake of conceptual cleanness, clearly held apart. The *collectors* may assume three distinct forms:

(i) formal series of multitangents, noted $\mathfrak{p}$;

(ii) formal series of monotangents, also noted $\mathfrak{p}$;

(iii) formal power series of z^{-1}, noted $\boldsymbol{p}$, and their Borel transforms $\hat{\boldsymbol{p}}$.

One goes from (i) to (ii) by multitangent reduction as in §2.3; and from (ii) to (iii) by the change $Te^{s_1} \mapsto z^{-s_1}$.

In any of their three incarnations, the collectors are but a step removed from the invariants. Yet they are not invariant themselves: they depend on the chart in which the diffeo f is taken. In fact, the diffeo can be reconstructed in totality from the collector – easily so if the collector is taken in the form (i), less easily if it is taken in the more *condensed* forms (ii) or (iii).

One last remark is in order here: although we are basically interested in the objects of the upper column, and more specifically in getting from f to its invariants $\{A_\omega^\pm\}$, we shall see that the most advantageous route is not the straight path through the arrows $1, 1', 1'', 1''''$, but any of the roundabout paths that start with 2 and 3': these indirect routes are much more economical in terms of calculations and also more respectful of the underlying symmetries and parities.

1.5   The reverse problem: canonical synthesis.

It can be shown that any pair $\boldsymbol{\pi} = (\boldsymbol{\pi}_{\mathrm{no}}, \boldsymbol{\pi}_{\mathrm{so}})$ is the connector of some standard diffeo f. This raises the problem of *synthesis*: how to reconstitute some f from a given set of invariants? And how to produce a canonical f among all possible choices? A semi-canonical synthesis was sketched in [E2] and a fully canonical one was constructed in [E4]. The latter depends on a single parameter c whose real part must be chosen large enough.[13] The construction produces a canonical $f_c := {}^*f_c \circ l \circ f_c^*$ from its iterator f_c^*, which in turn is explictely given, in operator form, by the formula

$$F_c^* := 1 + \sum_r \sum_{\omega_i \in \Omega} (-1)^r \, \mathcal{U}e_c^{\omega_1, \omega_2, \ldots, \omega_r}(z) \, \mathbb{A}_{\omega_r} \ldots \mathbb{A}_{\omega_2} \, \mathbb{A}_{\omega_1} \qquad (48)$$

with a careful re-arrangement of the terms[14] necessary to ensure convergence. The two ingredients in (48) are the invariants $\mathbb{A}_\omega$ taken in operator form (34),

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[13]Synthesis cannot be *absolute*, i.e. parameter-free.

[14]known as arborification-coarborification.

and some special resurgence monomials $\mathcal{U}e_c^{\boldsymbol{\omega}}(z)$ defined by

$$\mathcal{U}e_c^{\boldsymbol{\omega}}(z) := e^{||\boldsymbol{\omega}||z + c^2||\bar{\boldsymbol{\omega}}||z^{-1}} \mathrm{SPA} \int_0^\infty \frac{e^{-\sum(\omega_i t_i + c^2 \bar{\omega}_i t_i^{-1})}}{(t_r - t_{r-1})...(t_2 - t_1)(t_1 - z)} dt_1...dt_r \quad (49)$$

where *SPA* denotes a suitable average of all the 2^{r-1} possible integration multipaths that correspond to the 2^{r-1} manners in which the variables t_j may circumvent each other on their way from 0 to ∞.

2 Multitangents and multizetas.

The *multitangents* and *multizetas*, being the transcendental ingredient in the analytical expression of the invariants of identity-tangent diffeos[15], deserve a short excursus. But we must begin with a brief reminder about *moulds*, which are the proper tool for handling multi-indexed objects of whatever description.

2.1 Mould operations and mould symmetries.

Moulds are functions of finite sequences $\boldsymbol{\omega} = (\omega_1, ..., \omega_r)$ of any length $r \geq 0$, noted as right-upper indices and rendered, as mute variables, by a plain bold dot $\bullet$. Most moulds tend to fall into one or the other of four symmetry classes or types:

$$M^\bullet \mathit{symmetral}\,(\mathit{resp.\ alternal}) \;\Leftrightarrow\; \sum_{\boldsymbol{\omega} \in \mathrm{sha}(\boldsymbol{\omega}', \boldsymbol{\omega}'')} M^{\boldsymbol{\omega}} = M^{\boldsymbol{\omega}'} M^{\boldsymbol{\omega}''}\,(\mathit{resp.\ 0})$$

$$M^\bullet \mathit{symmetrel}\,(\mathit{resp.\ alternel}) \;\Leftrightarrow\; \sum_{\boldsymbol{\omega} \in \mathrm{she}(\boldsymbol{\omega}', \boldsymbol{\omega}'')} M^{\boldsymbol{\omega}} = M^{\boldsymbol{\omega}'} M^{\boldsymbol{\omega}''}\,(\mathit{resp.\ 0})$$

Here, $sha(\boldsymbol{\omega}', \boldsymbol{\omega}'')$ (resp. $she(\boldsymbol{\omega}', \boldsymbol{\omega}'')$) denotes the set of all sequences $\boldsymbol{\omega}$ deducible from $\boldsymbol{\omega}'$ and $\boldsymbol{\omega}''$ under plain (resp. contracting[16]) shufflings. Moulds can be *multiplied* and *composed*:

$$C^\bullet = A^\bullet \times B^\bullet \;\Longleftrightarrow\; C^{\boldsymbol{\omega}} = \sum_{\boldsymbol{\omega}'\boldsymbol{\omega}''=\boldsymbol{\omega}} A^{\boldsymbol{\omega}'} B^{\boldsymbol{\omega}''}$$

$$C^\bullet = A^\bullet \circ B^\bullet \;\Longleftrightarrow\; C^{\boldsymbol{\omega}} = \sum_{\boldsymbol{\omega}^1...\boldsymbol{\omega}^s=\boldsymbol{\omega}} A^{|\boldsymbol{\omega}^1|,...,|\boldsymbol{\omega}^s|} B^{\boldsymbol{\omega}^s} \ldots B^{\boldsymbol{\omega}^s} \qquad (|\boldsymbol{\omega}^i| \neq \boldsymbol{\emptyset})$$

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[15]and of much else – they are almost coextensive with the whole field of difference equations.

[16]ie allowing for order-compatible, pairwise contactions $(\omega_i', \omega_j'') \mapsto \omega_i' + \omega_j''$ of elements from the parent sequences.

with all the predictable relations, including

$$(A^\bullet \times B^\bullet) \circ C^\bullet = (A^\bullet \circ C^\bullet) \times (B^\bullet \circ C^\bullet)$$

. The multiplication resp. composition unit are the moulds $1^\bullet, I^\bullet$ defined by:

$$1^\emptyset := 1 \quad ; \quad 1^{\omega_1,...,\omega_r} := 0 \quad if \quad r \neq 0 \tag{50}$$

$$I^{\omega_1} := 1 \quad ; \quad I^{\omega_1,...,\omega_r} := 0 \quad if \quad r \neq 1 \tag{51}$$

2.2 Multizetas and multitangents.

In this subsection, all indices s_i are in $\mathbb{N}^*$ and, to preempt divergence, we (provisionally) assume $s_1 \neq 1$ for multizetas and $s_1, s_r \neq 1$ for multitangents.

We first consider two multizeta-valued moulds, the symmetrel $Ze^\bullet$ and symmetral $Za^\bullet$:

$$\mathrm{Ze}^{s_1,...,s_r} := \sum_{n_1 > ... > n_r > 0} n_1^{-s_1} ... n_r^{-s_r} \tag{52}$$

$$\mathrm{Za}^{s_1,...,s_r} := \sum_{n_1 \geq ... \geq n_r > 0} n_1^{-s_1} ... n_r^{-s_r} \frac{1}{r_1!} ... \frac{1}{r_t!} \tag{53}$$

In (53), the non-increasing sequences $(n_1, ..., n_r)$ involve t clusters of $r_1, ..., r_t$ identical integers $(1 \leq t \leq r)$. Clearly, $Za^\bullet = Ze^\bullet \circ (E^\bullet - 1^\bullet)$ with

$$(E-1)^\emptyset = 0 \quad and \quad (E-1)^{s_1,...,s_r} = \frac{1}{r!} \quad if \quad 1 \leq r \quad (\forall s_i) \tag{54}$$

In a similar vein, we introduce four multitangent-valued moulds, ranging over the four symmetry types:

$$\begin{array}{ccc}
\mathrm{Te}^\bullet & \xrightarrow{\ 1\ } & \mathrm{Ta}^\bullet \\
\downarrow 3 & & \downarrow 2 \\
\mathrm{Ten}^\bullet & \xrightarrow{\ 4\ } & \mathrm{Tan}^\bullet
\end{array}
\qquad
\begin{array}{ccc}
symmetrel & \xrightarrow{\ 1\ } & symmetral \\
\downarrow 3 & & \downarrow 2 \\
alternel & \xrightarrow{\ 4\ } & alternal
\end{array}$$

The two upper moulds are defined directly by[17]

$$\mathrm{Te}^{s_1,...,s_r}(z) := \sum_{n_1 > ... > n_r} (n_1 + z)^{-s_1} ... (n_r + z)^{-s_r} \tag{55}$$

$$\mathrm{Ta}^{s_1,...,s_r}(z) := \sum_{n_1 \geq ... \geq n_r} (n_1 + z)^{-s_1} ... (n_r + z)^{-s_r} \frac{1}{r_1!} ... \frac{1}{r_t!} \tag{56}$$

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[17]With the same r_j in (56) as in (53).

and the two lower moulds are their mould logarithms.[18] Thus:

$$\text{Ta}^\bullet = \text{Te}^\bullet \circ (\text{E}^\bullet - 1^\bullet) \tag{57}$$

$$\text{Tan}^\bullet = \text{Ten}^\bullet \circ (\text{E}^\bullet - 1^\bullet) \tag{58}$$

$$\text{Ten}^\bullet = \text{logmu}(\text{Te}^\bullet) \tag{59}$$

$$\text{Tan}^\bullet = \text{logmu}(\text{Ta}^\bullet) \tag{60}$$

In the sequel, we shall also consider the inverse $invTe^\bullet$ of $Te^\bullet$ (relative to mould multiplication). Clearly

$$\text{invTe}^{s_1,\ldots,s_r}(z) \ := \ \sum_{n_1 \leq \ldots \leq n_r} (-1)^r \, (n_1 + z)^{-s_1} \ldots (n_r + z)^{-s_r} \tag{61}$$

All four types of multitangents obviously verify

$$\text{T}^{s_1,\ldots,s_r}(-z) \equiv (-1)^{s_1+\ldots+s_r} \, \text{T}^{s_r,\ldots,s_1}(z) \qquad (\forall \text{T} \in \{\text{Te}, \text{Ta}, \text{Ten}, \text{Tan}\}) \tag{62}$$

In the case of $Tan^\bullet$, however, due to alternality we have an additional relation

$$\text{Tan}^{s_r,\ldots,s_1}(z) \equiv (-1)^{r-1} \, \text{Tan}^{s_1,\ldots,s_r}(z) \tag{63}$$

Combining (62), (63) we get the crucial *parity separation* property:

$$\text{Tan}^{s_1,\ldots,s_r}(-z) \equiv (-1)^{1+\sum d_i} \, \text{Tan}^{s_1,\ldots,s_r}(z) \qquad \textit{with } d_i := s_i - 1 \tag{64}$$

2.3 Multitangents in terms of multizetas.

Applying partial fraction decomposition to the series (55), one can easily expand any multitangent $Te^\bullet$ (hence also any $Ta^\bullet$, $Ten^\bullet$ or $Tan^\bullet$) into a finite sum of utterly elementary monotangents Te^{s_1}, also known as Eisenstein series. Here is the formula:

$$\text{Te}^{s_1,\ldots,s_r} = \sum_{\sigma=2}^{\sup(s_i)} \text{Teze}_\sigma^{s_1,\ldots,s_r} \, \text{Te}^\sigma = \sum_{i=1}^{r} \sum_{\sigma_i=2}^{s_i} \text{Teze}_{i,\sigma_i}^{s_1,\ldots,s_r} \, \text{Te}^{\sigma_i} \tag{65}$$

with

$$\text{Teze}_{i,\sigma_i}^{s_1,\ldots,s_r} = \sum_{\substack{\{ \substack{\sigma_i \leq s_i \\ s_j \leq \sigma_j (j \neq i)} \} }}^{\sum \sigma_k = \sum s_k} \text{Ze}^{\sigma_1,\ldots,\sigma_{i-1}} \, \text{Ze}^{\sigma_r,\ldots,\sigma_{i+1}} \prod_{j=1}^{i-1} (-1)^{\sigma_j} \prod_{1 \leq j \leq r}^{j \neq i} \frac{(-1)^{s_j}(\sigma_j - 1)!}{(\sigma_j - s_j)!(s_j - 1)!}$$

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[18]with the natural definition $logmu(M^\bullet) = M^\bullet - \frac{1}{2}M^\bullet \times M^\bullet + \frac{1}{3}M^\bullet \times M^\bullet \times M^\bullet \ldots$.

or more symmetrically

$$\text{Teze}_{i,\sigma_i}^{s_1,\ldots,s_r} = \sum_{\substack{\{\,\sigma_i \leq s_i \\ s_j \leq \sigma_j (j \neq i)\,\}}}^{\sum \sigma_k = \sum s_k} \text{Ze}^{\sigma_1,\ldots,\sigma_{i-1}} (-1)^{s_i - \sigma_i} \text{viZe}^{\sigma_{i+1},\ldots,\sigma_r} \prod_{\substack{1 \leq j \leq r}}^{j \neq i} \frac{(\sigma_j - 1)!}{(\sigma_j - s_j)!(s_j - 1)!}$$

$$\text{viZe}^{s_1,\ldots,s_r} = (-1)^{s_1 + \cdots + s_r} \, \text{Ze}^{s_r,\ldots,s_1} \tag{66}$$

The leading monotangent $Te^1(z) = \frac{\pi}{\tan(\pi z)}$ generates all others under differentiation, and admits the following northern and southern expansions:

$$\text{Te}_{\text{no}}^1(z) = -\pi i - 2\pi i \sum_{0 < n} e^{+2\pi i n z} \qquad if \ \ \Im(z) > 0 \tag{67}$$

$$\text{Te}_{\text{so}}^1(z) = +\pi i + 2\pi i \sum_{0 < n} e^{-2\pi i n z} \qquad if \ \ \Im(z) < 0 \tag{68}$$

Since $\text{Te}^{s_1}(z) = \frac{(-1)^{s_1 - 1}}{(s_1 - 1)!} \partial_z^{s_1 - 1} \text{Te}^1(z)$, this yields

$$\text{Te}^{s_1}(z) = \sum_{\omega \in \Omega^{\mp}} \text{Te}_\omega^{s_1} \, e^{-\omega z} \qquad on \ each \ half\text{-}plane \ \ \pm \Im(z) > 0 \tag{69}$$

with

$$\text{Te}_\omega^{s_1} = \text{sign}(\Im(\omega)) \, 2\pi i \, \frac{\omega^{s_1 - 1}}{(s_1 - 1)!} \quad and \quad \Omega^{\mp} = 2\pi i \mathbb{Z}^{\mp} \tag{70}$$

All the above amounts to a simple procedure for calculating the Fourier expansions, north and south, of the four classes of multitangents. The classes $Te^{\bullet}$ and $Tan^{\bullet}$ shall be of direct concern to us:

$$\text{Te}_{\text{no}}^{\bullet}(z) = \sum_{\omega \in \Omega^-} \text{Te}_\omega^{\bullet} \, e^{-\omega z} \qquad ; \qquad \text{Te}_{\text{so}}^{\bullet}(z) = \sum_{\omega \in \Omega^+} \text{Te}_\omega^{\bullet} \, e^{-\omega z} \tag{71}$$

$$\text{Tan}_{\text{no}}^{\bullet}(z) = \sum_{\omega \in \Omega^-} \text{Tan}_\omega^{\bullet} \, e^{-\omega z} \qquad ; \qquad \text{Tan}_{\text{so}}^{\bullet}(z) = \sum_{\omega \in \Omega^+} \text{Tan}_\omega^{\bullet} \, e^{-\omega z} \tag{72}$$

Remark: advantages of $Tan^{\bullet}$ over $Te^{\bullet}$.
Unlike the symmetrel multitangents $Te^{\bullet}$, their alternal counterparts $Tan^{\bullet}$ admit of no simple, direct expansions of type (55), and their expression as superpositions of $Te^{\bullet}$ is very involved, as shown by the formulae of §5.1. The picture changes, however, after reduction into monotangents and symmetrel linearisation of the resulting multizetas: it is now $Tan^{\bullet}$ that gives rise, by and large, to the simpler expansions, as shown by the Table at the beginning of §5.2. Then $Tan^{\bullet}$ possesses a second advantage: that of having a definite

parity, which depends only on the total degree $\sum d_i$: cf (64) above. Thirdly, we shall see in the sections §3 and §4 that, when it comes to expressing the collectors or the invariants, $Tan^\bullet$ leads to decidedly simpler formulae than $Te^\bullet$, as immediately apparent from a comparison of (133) with (127)-(128) or of (142) with (141).

2.4   Multitangents in terms of resurgent monomials.

There exists an alternative, *resurgent* approach to multitangent reduction. In the convergent (i.e. $s_1, s_r \neq 1$) and non-ramified (i.e. $s_j \in \mathbb{N}^*$ rather than $\mathbb{Q}^*$) case, it hardly improves on the above procedure (see §2.3) but in the general case, especially when we go over to fractional indices s_j, the resurgent approach becomes the more flexible of the two methods and even, in a sense, the only practical one. For clarity, though, we first sketch the alternative method under retention of the two simplifying assumptions: no divergence, no ramification.

We begin by constructing two resurgent-valued, symmetrel, mutually inverse moulds, first in the *formal model*, via the induction:

$$\tilde{\mathrm{Se}}^\bullet(z) \;=\; \frac{e^{\partial_z}}{(1 - e^{\partial_z})}\left(\tilde{\mathrm{Se}}^\bullet(z) \times \mathrm{Je}^\bullet(z)\right) \tag{73}$$

$$\mathrm{inv}\tilde{\mathrm{Se}}^\bullet(z) \;=\; \frac{e^{\partial_z}}{(e^{\partial_z} - 1)}\left(\mathrm{Je}^\bullet(z) \times \mathrm{inv}\tilde{\mathrm{Se}}^\bullet(z)\right) \tag{74}$$

with the elementary mould $\mathrm{Je}^\bullet(z)$ defined by

$$\mathrm{Je}^\emptyset(z) := 0 \;\; ; \;\; \mathrm{Je}^{s_1}(z) := z^{-s_1} \;\; ; \;\; \mathrm{Je}^{s_1,\ldots,s_r}(z) := 0 \;\; (\forall\, r \geq 2) \tag{75}$$

Together with the conditions

$$\tilde{\mathrm{Se}}^\emptyset(z) = \mathrm{inv}\tilde{\mathrm{Se}}^\emptyset(z) = 1\,, \; \tilde{\mathrm{Se}}^{s_1,\ldots,s_r}(\infty) = \mathrm{inv}\tilde{\mathrm{Se}}^{s_1,\ldots,s_r}(\infty) = 0 \;\; (\forall\, r \geq 1) \tag{76}$$

the induction (73)-(74) uniquely defines each $\tilde{Se}^{\boldsymbol{s}}(z)$ and each $inv\tilde{Se}^{\boldsymbol{s}}(z)$ as a constant-free power series in z^{-1}.

In the *convolutive model* the induction becomes

$$\hat{\mathrm{Se}}^{s_1,\ldots,s_r}(\zeta) \;=\; \frac{e^{-\zeta}}{(1 - e^{-\zeta})}\int_0^\zeta \hat{\mathrm{Se}}^{s_1,\ldots,s_{r-1}}(\zeta - \zeta_r)\,\frac{\zeta_r^{s_r-1}}{\Gamma(s_r)}\,d\zeta_r \tag{77}$$

$$\mathrm{inv}\hat{\mathrm{Se}}^{s_1,\ldots,s_r}(\zeta) \;=\; \frac{e^{-\zeta}}{(e^{-\zeta} - 1)}\int_0^\zeta \frac{\zeta_1^{s_1-1}}{\Gamma(s_1)}\,\mathrm{inv}\hat{\mathrm{Se}}^{s_2,\ldots,s_r}(\zeta - \zeta_1)\,d\zeta_1 \tag{78}$$

Lastly, in the *geometric models* $+$ and $-$ (*east* and *west*), corresponding to Laplace integration along the axes $arg(\zeta) = 0$ and $arg(\zeta) = \pi$, we get

$$\text{Se}_+^{s_1,\dots,s_r}(z) := \sum_{0 < n_r < \dots < n_1} (n_1 + z)^{-s_1} \dots (n_r + z)^{-s_r} \qquad (79)$$

$$\text{Se}_-^{s_1,\dots,s_r}(z) := \sum_{n_1 \leq \dots \leq n_r \leq 0} (-1)^r (n_1 + z)^{-s_1} \dots (n_r + z)^{-s_r} \qquad (80)$$

$$\text{invSe}_+^{s_1,\dots,s_r}(z) := \sum_{0 < n_1 \leq \dots \leq n_r} (-1)^r (n_1 + z)^{-s_1} \dots (n_r + z)^{-s_r} \qquad (81)$$

$$\text{invSe}_-^{s_1,\dots,s_r}(z) := \sum_{n_r < \dots < n_1 \leq 0} (n_1 + z)^{-s_1} \dots (n_r + z)^{-s_r} \qquad (82)$$

From the structure of the induction (73), one directly (without calculation) infers that the $\tilde{Se}^\bullet(z)$ must verify resurgence equations of the form[19]

$$+2\pi i\, \Delta_\omega\, \text{Se}^\bullet(z) = \text{Ten}_\omega^\bullet \times \text{Se}^\bullet(z) \qquad (\forall \omega \in \Omega^+ = 2\pi i \mathbb{Z}^+) \quad (83)$$

$$-2\pi i\, \Delta_\omega\, \text{Se}^\bullet(z) = \text{Ten}_\omega^\bullet \times \text{Se}^\bullet(z) \qquad (\forall \omega \in \Omega^- = 2\pi i \mathbb{Z}^-) \quad (84)$$

$$\Delta_\omega^+\, \text{Se}^\bullet(z) = \text{Te}_\omega^\bullet \times \text{Se}^\bullet(z) \qquad (\forall \omega \in \Omega^+ = 2\pi i \mathbb{Z}^+) \quad (85)$$

$$\Delta_\omega^-\, \text{Se}^\bullet(z) = \text{Te}_\omega^\bullet \times \text{Se}^\bullet(z) \qquad (\forall \omega \in \Omega^- = 2\pi i \mathbb{Z}^-) \quad (86)$$

with scalar-valued moulds $Ten_\omega^\bullet$ (alternel) and $Te_\omega^\bullet$ which, for the moment, need not bear any relation to their namesakes in (71). However, applying the axis-crossing identity (30) to (85) with $\theta = +\frac{\pi}{2}$ and the reverse identity (31) to (86) with $\theta = -\frac{\pi}{2}$, and minding the fact that

$$\text{Se}_{\frac{\pi}{2} \pm \epsilon}^\bullet = \text{Se}_\mp^\bullet \quad (\textit{inversion!}) \quad ; \quad \text{Se}_{-\frac{\pi}{2} \pm \epsilon}^\bullet = \text{Se}_\pm^\bullet \quad (\textit{no inversion!})$$

we find respectively

$$\text{Te}_{\text{so}}^\bullet(z) \times \text{Se}_{-,\text{so}}^\bullet(z) = \text{Se}_{+,\text{so}}^\bullet(z) \quad \textit{with} \quad \text{Te}_{\text{so}}^\bullet(z) = \sum_{\omega \in \Omega^+} \text{Te}_\omega^\bullet\, e^{-\omega z} \quad (87)$$

$$Te_{\text{no}}^\bullet(z) \times \text{Se}_{-,\text{no}}^\bullet(z) = \text{Se}_{+,\text{no}}^\bullet(z) \quad \textit{with} \quad \text{Te}_{\text{no}}^\bullet(z) = \sum_{\omega \in \Omega^-} \text{Te}_\omega^\bullet\, e^{-\omega z} \quad (88)$$

Thus, whether "north" or "south", we arrive at the elementary identity

$$\text{Te}^\bullet(z) = \text{Se}_+^\bullet(z) \times \text{invSe}_-^\bullet(z) \qquad (89)$$

which of course can also be directly derived from the definitions (55), (79), (82). But we get an interesting extra – namely, that the moulds $Te_\omega^\bullet$ of

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[19] we drop the tilde for simplicity.

(85) and (86) coincide with those of (71) in the preceding subsection. If we now interpret the resurgence equations (83)-(86) in the convolutive model, we get an alternative expression of $Te_\omega^\bullet$ and $Ten_\omega^\bullet$ (and hence $Ta_\omega^\bullet$, $Tan_\omega^\bullet$) as finite integrals in the *zeta*-plane, which translates, after some work, into fast-convergent power series. This will stand us in good stead in the *divergent* and above all in the *ramified* cases. But we must first devote a short aside to the question of parity.

2.5 Respecting and harnessing parity.

When it comes to calculating what will turn out to be most basic and useful of all four multitangents, namely $Tan^\bullet$, all the above procedures must be adjudged *wasteful* in the sense that they derive the parity-separating $Tan^\bullet$ from the parity-mixing $Te^\bullet$. To remedy this, we shall replace the nearly odd function $\frac{e^\partial}{1-e^\partial}$ of (73)-(74) by the exactly odd $H(\partial)$:

$$H(\partial) := \frac{e^\partial}{1-e^\partial} + \frac{1}{2} = \frac{1}{2}\frac{1+e^\partial}{1-e^\partial} = -\frac{1}{2}\cotan\left(\frac{\partial}{2}\right) \tag{90}$$

and define two mutually inverse (but no longer exactly symmetrel) moulds $\tilde{See}^\bullet$ and $inv\tilde{See}^\bullet$ by the tweaked induction

$$\tilde{See}^\bullet(z) = +H(\partial_z)\left(\tilde{See}^\bullet(z) \times Je^\bullet(z)\right) \tag{91}$$

$$inv\tilde{See}^\bullet(z) = -H(\partial_z)\left(Je^\bullet(z) \times inv\tilde{See}^\bullet(z)\right) \tag{92}$$

Clearly, $\tilde{See}^{s_1,\ldots,s_r}(z)$ and $inv\tilde{See}^{s_1,\ldots,s_r}(z)$ are *even* (resp. *odd*) power series in z^{-1} iff the total "degree" $(\sum s_i) - r$ is *even* (resp. *odd*).

As in the preceding subsection, the induction immediately leads to resurgence equations of the form:

$$+2\pi i\,\Delta_\omega\,See^\bullet(z) = Teen_\omega^\bullet \times See^\bullet(z) \qquad (\forall\omega \in \Omega^+ = 2\pi i\mathbb{Z}^+) \tag{93}$$

$$-2\pi i\,\Delta_\omega\,See^\bullet(z) = Teen_\omega^\bullet \times See^\bullet(z) \qquad (\forall\omega \in \Omega^- = 2\pi i\mathbb{Z}^-) \tag{94}$$

$$\Delta_\omega^+\,See^\bullet(z) = Tee_\omega^\bullet \times See^\bullet(z) \qquad (\forall\omega \in \Omega^+ = 2\pi i\mathbb{Z}^+) \tag{95}$$

$$\Delta_\omega^-\,See^\bullet(z) = Tee_\omega^\bullet \times See^\bullet(z) \qquad (\forall\omega \in \Omega^- = 2\pi i\mathbb{Z}^-) \tag{96}$$

Moreover, (90) implies that the moulds $\tilde{Se}^\bullet(z)$ and $\tilde{See}^\bullet(z)$ are related as follows

$$\tilde{Se}^\bullet = \left(\tilde{See}^\bullet \circ D_{-\frac{1}{2}}^\bullet\right) \times \left(1^\bullet - \frac{1}{2}Je^\bullet(z) \circ D_{-\frac{1}{2}}^\bullet\right) \tag{97}$$

$$\tilde{See}^\bullet = \left(\tilde{Se}^\bullet \circ D_{+\frac{1}{2}}^\bullet\right) \times \left(1^\bullet + \frac{1}{2}Je^\bullet(z) \circ D_{+\frac{1}{2}}^\bullet\right) \tag{98}$$

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with

$$\mathrm{D}_a^{\emptyset} := 0 \quad ; \quad \mathrm{D}_a^{s_1,\dots,s_r} := a^{r-1} \quad (\forall r \geq 1) \tag{99}$$

However, the elementary second factors on the right-hand sides of (97) and (97) are convergent rather than resurgent-valued. As a consequence, the corresponding right and left germs coincide, and if we set

$$\mathrm{Tee}^\bullet(z) \;\; = \;\; \mathrm{See}_+^\bullet(z) \times \mathrm{invSee}_-^\bullet(z) \tag{100}$$

we shall have

$$\mathrm{Te}^\bullet(z) \;\; = \;\; \mathrm{Se}_+^\bullet(z) \times \mathrm{invSe}_-^\bullet(z) \;\; = \;\; \Big(\mathrm{See}_+^\bullet(z) \times \mathrm{invSee}_-^\bullet(z)\Big) \circ \mathrm{D}_{-\frac{1}{2}}^\bullet$$

and therefore

$$\mathrm{Te}^\bullet(z) \;\; = \;\; \mathrm{Tee}^\bullet(z) \circ \mathrm{D}_{-\frac{1}{2}}^\bullet \tag{101}$$

Postcomposing this by $\boldsymbol{E}^\bullet - \boldsymbol{1}^\bullet$ and defining an elementary, purely length-dependent mould $K^\bullet$ by:

$$\sum_{0 \leq r} K^{s_1,\dots,s_r} \, t^r := 2 \, \tan(\frac{t}{2}) \qquad (\forall s_i) \tag{102}$$

we get:

$$\mathrm{Ta}^\bullet(z) \;\; = \;\; \mathrm{Tee}^\bullet(z) \circ K^\bullet \tag{103}$$

Lastly, taking in the above identity the mould logarithm of both sides, we arrive at:

$$\mathrm{Tan}^\bullet(z) \;\; = \;\; \mathrm{Teen}^\bullet(z) \circ K^\bullet \tag{104}$$

$$\mathrm{Tan}_\omega^\bullet(z) \;\; = \;\; \mathrm{Teen}_\omega^\bullet(z) \circ K^\bullet \qquad \forall \omega \in \Omega = 2\,\pi i\, \mathbb{Z}^*) \tag{105}$$

We have thus expressed the basic alternal multitangents $Tan^\bullet$ directly in terms of auxiliary multitangents $Teen^\bullet$ which, though belonging to none of the basic symmetry types, share with $Tan^\bullet$ the same crucial property of parity separation: as functions of z, both $Teen^{s_1,\dots,s_r}(z)$ and $Tan^{s_1,\dots,s_r}(z)$ have the same parity as that of the number $1 - r + \sum s_i$.

2.6 The divergent case. Normalisation.

If we now drop the condition that ensured convergence, namely $s_1, s_r \neq 1$, and yet insist on retaining all properties and symmetries of our moulds, we

must do two things to our infinite series: *truncate* them and *correct* them. Concretely, we must set

$$\text{Te}^\bullet(z) \ := \ \lim_{k\to\infty} \text{Te}^\bullet_k(z) \quad := \lim_{k\to\infty} \ \text{invcoSe}^\bullet_k \times \text{doTe}^\bullet_k(z) \times \text{coSe}^\bullet_k$$

$$\text{Se}^\bullet_\pm(z) \ := \ \lim_{k\to\infty} \text{Se}^\bullet_{k,\pm}(z) \quad := \lim_{k\to\infty} \ \text{invcoSe}^\bullet_k \times \text{doSe}^\bullet_{k,\pm}(z)$$

$$\text{invSe}^\bullet_\pm(z) := \lim_{k\to\infty} \text{invSe}^\bullet_{k,\pm}(z) := \lim_{k\to\infty} \ \text{invdoSe}^\bullet_{k,\pm}(z) \times \text{coSe}^\bullet_k$$

Here, the symmetrel *dominant* factors $Te^\bullet$, $doSe^\bullet_{k,\pm}$, $invdoSe^\bullet_{k,\pm}$ are defined as in (55) and (79)-(82) but with sums truncated at $\pm k$ instead of $\pm\infty$. Thus

$$\text{doTe}_k^{s_1,\dots,s_r}(z) \ := \ \sum_{-k\le n_r<\dots<n_1\le k} (n_r+z)^{-s_r}\dots(n_1+z)^{-s_1} \quad (\forall s_i) \ (106)$$

As for the symmetrel, z-constant *corrective* factors $coSe^\bullet_{k\pm}$ and $invcoSe^\bullet_{k\pm}$, their definition reduces to

$$\text{coSe}_k^{s_1,\dots,s_r} \ := \ \frac{(\,c+\log k\,)^r}{r!} \qquad if \ \ (s_1,\dots,s_r)=(1,\dots,1) \quad (107)$$

$$\text{invcoSe}_k^{s_1,\dots,s_r} \ := \ \frac{(-c-\log k)^r}{r!} \qquad if \ \ (s_1,\dots,s_r)=(1,\dots,1) \quad (108)$$

$$\text{coSe}_k^{s_1,\dots,s_r} \ = \ \text{invcoSe}_k^{s_1,\dots,s_r} := \ 0 \quad if \ \ (s_1,\dots,s_r)\ne(1,\dots,1) \ (109)$$

In the formal model, the resurgent-valued moulds $\tilde{Se}^\bullet$ and $inv\tilde{Se}^\bullet$ are still uniquely defined by the induction (73)-(74) together with the condition

$$\tilde{\text{Se}}^{\boldsymbol{s}}(z) \ , \ \ \text{inv}\tilde{\text{Se}}^{\boldsymbol{s}}(z) \ \ \in \mathbb{Q}[[z^{-1}]] \otimes \mathbb{Q}[(c+\log z)] \ \dot{-} \ \mathbb{Q} \qquad (\forall \boldsymbol{s}\ne\boldsymbol{\emptyset}) \quad (110)$$

The normalising condition, in other words, is that $\tilde{\text{Se}}^{\boldsymbol{s}}(z)$ and $\text{inv}\tilde{\text{Se}}^{\boldsymbol{s}}(z)$, as formal series in z^{-1} and polynomials in the bloc $(c+log\,z)$, should have no constant term.

In the sectorial models, the c-normalisation implies:

$$Se_\pm^{\overbrace{1,\dots,1}^{r\,\text{times}}}(0) = \frac{(\gamma-c)^r}{r!} \quad ; \quad invSe_\pm^{\overbrace{1,\dots,1}^{r\,\text{times}}}(0) = \frac{(c-\gamma)^r}{r!} \qquad (111)$$

with

$$\gamma = \lim_{k\to\infty}(1+\frac{1}{2}+\dots+\frac{1}{k}-\log k) = 0.577215\dots = Euler \ constant \quad (112)$$

For multitangents, we may still formally apply the procedure (65)-(66) of §2.3 to reduce them into combinations of monotangents and mutizetas,

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but this time we are liable to get formally divergent multizetas. The c-normalisation then amounts to setting $\zeta(1) = Ze^1 := \gamma - c$ and to adopting for all divergent multizetas[20] the unique *symmetrel extension* compatible with that initial choice.[21]

There are two natural choices for the normalisation constant c:

(i) Either we set $c = 0$, in which case we eschew γ in the formal model but at the cost of introducing it in the convolutive and sectorial models. It also complicates the definition of the multitangents and multizetas, since it forces us to set $Ze^1 = \gamma$, which however is not entirely unnatural, in view of the formula

$$\sigma\,\Gamma(\sigma) = \exp\left(-\gamma\,\sigma + \sum_{2 \leq n}(-1)^n\frac{\zeta(n)}{n}\sigma^n\right) \qquad (113)$$

(ii) Or we set $c = \gamma$, which forcibly introduces γ into the formal model but rids us of it everywhere else, including in the definition of multitangents and multizetas, since it amounts to setting $Ze^1 = 0$. This shall be our preferred choice.

2.7 The ramified case.

This is the case of fractional *weights* s_i in $p^{-1}\mathbb{N}^*$ and no longer in $\mathbb{N}^*$. Everything carries over to that case, except the *finite* reduction of multitangents into monotangents and multizetas.

The formulae (65)-(66) still make formal sense but lead to expansions which are not only infinite but also divergent. When properly re-summed, they yield the correct expressions, but from the point of view of calculational efficiency, this approach is worthless.

Of course, straightforward Fourier analysis in the upper and lower halves of the z-plane would yield the coefficients $Te_\omega^\bullet$ along with all the others, but not in the form of nice convergent series, and again at great cost.

The resurgence approach of §2.4 and §2-5, on the other hand, survives ramification without any modification. When pursued to the end, this approach even leads to some sort of functional equation for multizetas, that is to say, to something resembling the classical relation between $\zeta(s)$ and $\zeta(1 - s)$.

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[20]i.e. for all multizetas with initial index $s_1 = 1$.

[21]Thus $Ze^{1,1} := -\frac{1}{2}Ze^2 + \frac{1}{2}(\gamma - c)^2$, $Ze^{1,2} := -Ze^{2,1} - Ze^3 + (\gamma - c)\,Ze^2$ etc. There exist simple formulae for calculating the symmetrel extension of all multizetas relative to any given choice of Ze^1.

3 Collectors and connectors in terms of f.

3.1 Operator relations.

To the composition identity between germs $f^* = l^{-1} \circ f^* \circ f \equiv l^{-1} \circ f^* \circ l \circ g$ there answers the operatorial relation[22]

$$F^* \;=\; G\; F^*_{:1} \qquad with \quad F^*_{:1} := L\, F^* L^{-1} \tag{114}$$

To solve (114) while respecting the basic symmetry between f, g and f^{-1}, g^{-1}, we set as our basic 'infinitesimals' the following operators

$$\underline{G}^+_{:n} \;:=\; L^n.(G\,-\,1).L^{-n} \qquad (n_i \in \mathbb{Z}) \tag{115}$$

$$\underline{G}^-_{:n} \;:=\; L^n.(G^{-1}-1).L^{-n} \qquad (n_i \in \mathbb{Z}) \tag{116}$$

This leads straightaway to the formal expansions

$$F^*_+ \;=\; 1 + \sum_{1 \leq r} \;\; \sum_{0 \leq n_r < ... < n_1} \underline{G}^+_{:n_r} \cdots \underline{G}^+_{:n_1} \qquad (n_i \in \mathbb{Z}) \tag{117}$$

$$F^*_- \;=\; 1 + \sum_{1 \leq r} \;\; \sum_{n_1 < ... < n_r < 0} \underline{G}^-_{:n_r} \cdots \underline{G}^-_{:n_1} \qquad (n_i \in \mathbb{Z}) \tag{118}$$

$$^*F_+ \;=\; 1 + \sum_{1 \leq r} \;\; \sum_{0 \leq n_1 < ... < n_r} \underline{G}^-_{:n_r} \cdots \underline{G}^-_{:n_1} \qquad (n_i \in \mathbb{Z}) \tag{119}$$

$$^*F_- \;=\; 1 + \sum_{1 \leq r} \;\; \sum_{n_r < ... < n_1 < 0} \underline{G}^+_{:n_r} \cdots \underline{G}^+_{:n_1} \qquad (n_i \in \mathbb{Z}) \tag{120}$$

$$\mathbf{\Pi}^+ := {}^*F_- . F^*_+ \;=\; 1 + \sum_{1 \leq r} \;\; \sum_{n_r < ... < n_1} \underline{G}^+_{:n_r} \cdots \underline{G}^+_{:n_1} \qquad (n_i \in \mathbb{Z}) \tag{121}$$

$$\mathbf{\Pi}^- := {}^*F_+ . F^*_- \;=\; 1 + \sum_{1 \leq r} \;\; \sum_{n_1 < ... < n_r} \underline{G}^-_{:n_r} \cdots \underline{G}^-_{:n_1} \qquad (n_i \in \mathbb{Z}) \tag{122}$$

For standard diffeos f, the above expansions for $F^*, {}^*F$ (resp. $\mathbf{\Pi}^{\pm 1}$) are easily shown to *converge* when they are made to act on test functions that are defined on suitably extended U-shaped domains (resp. on suitably distant half-planes $|\Im(z)| \gg 1$).

The challenge is now to extract from these expansions (- first in the standard, then in the general case -) *theoretically appealing, analytically transparent*, and *computationally manageable* expressions for (in that order) the *collectors, connectors,* and *invariants.*

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[22]To diffeos $f, g...$ we associate the operators $F, G...$ of postcomposition by $f, g...$

3.2 The direct, non-symmetrical scheme.

To express our 'infinitesimals' $\underline{G}_{:n}^{\pm}$ in terms of the diffeo's Taylor coefficients, we first set

$$\underline{G}_{:n}^{\pm} = \sum_{1 \le k} \frac{1}{k!} \left(g^{\pm 1}(z+n) - (z+n) \right)^k \partial_z^k \tag{123}$$

and then

$$\left(g(z) - z \right)^n = \sum_{1 \le d} g_{n,1+d}^+ z^{-d} = \sum_{2 \le s} g_{n,s}^+ z^{-s+1} \qquad (d = \text{``degree''}) \tag{124}$$

$$\left(g^{-1}(z) - z \right)^n = \sum_{1 \le d} g_{n,1+d}^- z^{-d} = \sum_{2 \le s} g_{n,s}^- z^{-s+1} \qquad (s = \text{``weight''}) \tag{125}$$

Next, to account for the action of the *derivation* operators ∂_z implicit in the definition of the *substitution* operators $\underline{G}_{:n}^{\pm}$, we require integers $\delta_{\bullet}^{\bullet}$ defined by

$$\sum_{\sum(l_i - n_i) = 1} \delta_{n_1,\ldots,n_r}^{l_1,\ldots,l_r} x_1^{l_1} \ldots x_r^{l_r} \equiv x_1^{n_2}.(x_1+x_2)^{n_3} \ldots (x_1+\ldots x_{r-1})^{n_r} \tag{126}$$

Letting the operators on both sides of (121) resp. (122) act on the test function z, and harvesting all r-linear summands, we find the sought-after expansions for the collectors $\mathfrak{p}^{\pm}$:

$$\mathfrak{p}(z) = z + \sum_{1 \le r} \sum_{\substack{0 \le l_i \\ 1 \le n_i}}^{n_i + l_i \le s_i} (-1)^{n-1} \delta_{n_1,\ldots,n_r}^{l_1,\ldots,l_r} \, \mathbf{Te}^{s_1,\ldots,s_r}(z) \prod_{1 \le i \le r} \frac{(s_i-1)! \, g_{n_i,s_i-l_i+1}^+}{(s_i-l_i-1)!} \tag{127}$$

$$\mathfrak{p}^{-1}(z) = z + \sum_{1 \le r} \sum_{\substack{0 \le l_i \\ 1 \le n_i}}^{n_i + l_i \le s_i} (-1)^{n-1} \delta_{n_1,\ldots,n_r}^{l_1,\ldots,l_r} \, \mathbf{Te}^{s_1,\ldots,s_r}(z) \prod_{1 \le i \le r} \frac{(s_i-1)! \, g_{n_i,s_i-l_i+1}^-}{(s_i-l_i-1)!} \tag{128}$$

with $n := n_1 + \ldots n_r$.

3.3 The indirect, symmetrical scheme.

As already pointed out in §1.4, the only way to fully respect the underlying symmetries between $g, \mathfrak{p}$ and $g^{-1}, \mathfrak{p}^{-1}$ is to switch to the formal generators $g_*, \mathfrak{p}_*$. That means expressing the substitution operators $\underline{G}_{:n}$ in terms of the (generically divergent) series

$$g_*(z) = \sum_{1 \le d} g_{*1+d} z^{-d} = \sum_{2 \le s} g_{*s} z^{-s+1} \qquad (d = degree, s = weight) \tag{129}$$

and setting

$$\underline{G}_{:n} = \sum_{1 \leq k} \frac{1}{k!} \left(g_*(z+n)\, \partial_z \right)^k \tag{130}$$

This time around, we require much simpler coefficients $\delta^\bullet$ and $\delta_1^\bullet$ with only one sequence of (upper) indices:

$$\sum_{l_i \geq 0,\, \sum l_i = r-1} \delta^{l_1,\ldots,l_r}\, x_1^{l_1} \ldots x_r^{l_r} \;\equiv\; x_1.(x_1+x_2)\ldots(x_1+\cdots+x_{r-1}) \tag{131}$$

$$\sum_{l_i \geq 0,\, \sum l_i = r} \delta_1^{l_1,\ldots,l_r}\, x_1^{l_1} \ldots x_r^{l_r} \;\equiv\; x_1.(x_1+x_2)\ldots(x_1+\cdots+x_r) \tag{132}$$

We then let the formal derivation operator $log\mathbf{\Pi}$ (rather than the algebra automorphism $\mathbf{\Pi}$ as in §3.2) act on the test function z. *Without* resp. *with* ulterior derivation by z, we obtain these two expansions:

$$\mathfrak{p}_*(z) = \sum_{1 \leq r} (-1)^{r-1} \sum_{0 \leq l_i < s_i} \delta^{l_1,\ldots,l_r}\, \mathbf{Tan}^{s_1,\ldots,s_r}(z) \prod_{1 \leq i \leq r} \frac{(s_i-1)!\, g_{*s_i-l_i+1}}{(s_i-l_i-1)!} \tag{133}$$

$$\mathfrak{p}_*'(z) = \sum_{1 \leq r} (-1)^{r} \sum_{0 \leq l_i < s_i} \delta_1^{l_1,\ldots,l_r}\, \mathbf{Tan}^{s_1,\ldots,s_r}(z) \prod_{1 \leq i \leq r} \frac{(s_i-1)!\, g_{*s_i-l_i+1}}{(s_i-l_i-1)!} \tag{134}$$

Of these expansions, the first is computationally more advantageous (it carries less summands) while the second is formally more appealing (its multitangents $Tan^\bullet$ have exactly the same total weight $\sum s_j$ as the accompanying coefficient clusters). We may note that while it would be possible (but rather pointless) to produce similar expansions for all derivatives $\mathfrak{p}_*^{(n)}$, nothing analogous exists for the indefinite integrals $\mathfrak{p}_*, \mathfrak{p}_* \ldots$. In any case, and despite involving $g_*, \mathfrak{p}_*$ rather than $g, \mathfrak{p}$, the 'symmetrical' expansions (133) and (134) are theoretically simpler and more basic and computationally more efficient than their 'asymmetrical' counterparts (127) and (128).

3.4 Parity separation.

Formula (133) may be rewritten as

$$\mathfrak{p}_*(z) = \sum_{1 \leq r} (-1)^{r-1} \sum_{0 \leq l_i,\, 1 \leq d_i}^{\sum l_i = r-1} \delta^{l_1,\ldots,l_r}\, \mathbf{Tan}^{d_1+l_1,\ldots,d_r+l_r}(z) \prod_{1 \leq i \leq r} \frac{(d_i+l_i-1)!\, g_{*1+d_i}}{(d_i-1)!}$$

Here, multitangents $Tan^\bullet$ of total "degree" $-r+\sum(d_i+l_i) = -1+\sum d_i$ sit in front of coefficient clusters $\prod g_{*1+d_i}$ of total "degree" $\sum d_i$. As *functions*, therefore, these multitangents always have the *same* parity as the accompanying coefficient clusters.

3.5   From collectors to connectors.

From $\mathfrak{p}$ to $\boldsymbol{\pi} = (\boldsymbol{\pi}_{\mathrm{no}}, \boldsymbol{\pi}_{\mathrm{so}})$:
If we retain on the right-hand side of (127) the sole summands of weight[23] less than s and call $\mathfrak{p}_{[s]}(z)$ the value of this finite sum, then, as s goes to ∞ and for $K_{\pm}$ large enough, the functions $\mathfrak{p}_{[s]}(z)$ converge on half-planes $\Im(z) > K_+$ resp. $\Im(z) < -K_-$, to the northern resp. southern component of $\boldsymbol{\pi}$.

From $\mathfrak{p}_*$ to $\boldsymbol{\pi}_* = (\boldsymbol{\pi}_{*\mathrm{no}}, \boldsymbol{\pi}_{*\mathrm{so}})$:
The 'infinitesimal' $\mathfrak{p}_*$ and $\boldsymbol{\pi}_*$ are formal, and generically divergent, series. But we may still consider the finite truncated sums $\mathfrak{p}_{*[s]}(z)$ with their Fourier coefficients $\mathfrak{p}_{*\omega[s]}$, and observe that, when $s \to$, these $\mathfrak{p}_{*\omega[s]}$ do tend to finite limits $\mathfrak{p}_{*\omega}$, which are the Fourier coefficients of the sought-after infinitesimal connector $\boldsymbol{\pi}_*$, or rather those of its northern and southern components.

Remark: Despite being very close to the *connectors*, the *collectors* differ from them in two fundamental respects: they are *not invariant* and they are *of one piece*.

The *non-invariance* is fairly obvious for $\mathfrak{p}$ taken in its *natural* multitangent expansion; but it remains true of $\mathfrak{p}$ after its *natural* reduction to a monotangent expansion. Indeed, if we go from the 'monotangential' $\mathfrak{p}(z)$ to the power series $\boldsymbol{p}(z)$ by changing each $Te^s(z)$ to z^{-s} and then formally Borel-transform $\boldsymbol{p}(z)$ to $\hat{\boldsymbol{p}}(\zeta)$ by changing z^{-s} to $\zeta^{s-1}/(s-1)!$, we end up with an entire function $\hat{\boldsymbol{p}}(\zeta)$ whose *only* invariant values are the ones it assumes on the set $\Omega := 2\pi i\, \mathbb{Z}^*$. See §4.1 *infra*.

As for being *of one piece*, this is a property not so much of the collectors as of their constituent multitangents or monotangents[24], which are meromorphic over the whole of $\mathbb{C}$, in complete contrast to the connectors, whose northern and southern components are usually completely unrelated: each one may a priori be *anything*.

3.6   Reflexive and unitary diffeomorphisms.

In §3.4 we observed that in the expansion (133) of $\mathfrak{p}_*$, coefficient clusters $\prod g_{*1+d_i}$ of even (resp. odd) total degree $\sum d_i$ accompany multitangents $Tan^\bullet$ that are even functions with real Fourier coefficients (resp. odd functions with

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[23]the 'weight' in question is that of the coefficient clusters. But the weight of the accompanying multitangents (or, after reduction, of the multizeta-monotangent combinations) differs from the first only by one unit.

[24]Things get more tangled in the *ramified case*, i.e. for diffeos with a contact order $p > 1$. In that case, the multitangents still exist and still possess uniform determinations on each upper/lower half-plane of the p-ramified z-plane, but one can no longer go from one determination to the next by crossing the real axis between two consecutive integers.

purely imaginary Fourier coefficients). As a consequence, there is no simple condition on the coefficients g_{*1+d_i} of g_* capable of ensuring that $\mathfrak{p}_*$ be *odd*, whereas three elementary conditions may ensure that it be *even*, namely:

(i) all coefficients g_{*1+d_i} of odd degree d_i vanish and those of even degree are real

(ii) all coefficients g_{*1+d_i} of even degree d_i vanish and those of odd degree are purely imaginary

(iii) all coefficients g_{*1+d_i} of even degree d_i are real and those of odd degree are purely imaginary

No special significance attaches to case (ii), but the cases (i) and (iii) present interesting stability properties, with collectors and connectors inheriting the nature of f. This is an incentive for singling out the following three types of diffeos f whose inverses f^{-1} either coincide with, or are analytically conjugate to, the image of f under an elementary involution:

$$
\begin{array}{llcllll}
reflexive & : & \check{f} = f^{-1} & || & weakly\ reflexive & : & \check{f} \overset{\text{an. cj.}}{\sim} f^{-1} \\
unitary & : & \bar{f} = f^{-1} & || & weakly\ unitary & : & \bar{f} \overset{\text{an. cj.}}{\sim} f^{-1} \\
counitary & : & \bar{\check{f}} = f^{-1} & || & weakly\ counitary & : & \bar{\check{f}} \overset{\text{an. cj.}}{\sim} f^{-1}
\end{array}
$$

Here, $\bar{f}$ denotes the complex conjugate of f, and $\check{f} := \sigma \circ f \circ \sigma$ with $\sigma(z) \equiv -z$. Conjugation by τ, with $\tau(z) \equiv i\,z$, clearly exchanges *unitary* and *counitary*, so that *weakly unitary* is equivalent to *weakly counitary*. Though *unitariness* seems a more natural notion, we shall work here with *counitariness*, which is better adapted to the correspondance $f \mapsto \boldsymbol{\pi}$ and enables us to take f in standard form $f = l \circ g$.

P$_1$: f is reflexive iff the power series f_* resp. f^* are *even* resp. *odd*, in which case $f_{*\pm}(-z) \equiv f_{*\mp}(z)$ and $f_\pm^*(-z) \equiv -f_\mp^*(z)$. Likewise, f is counitary iff the power series f_* resp. f^* are of the form $f_{*\mathrm{re}} \circ \tau$ resp. $\tau^{-1} \circ f_{\mathrm{re}}^* \circ \tau$ with real $f_{*\mathrm{re}}, f_{\mathrm{re}}^*$, in which case $\bar{f}_{*\pm}(-z) \equiv f_{*\mp}(z)$ and $\bar{f}_\pm^*(-z) \equiv -f_\mp^*(z)$.

P$_2$: If a standard f is reflexive resp. counitary, then its conjugate $l^{+\frac{1}{2}} \circ f \circ l^{-\frac{1}{2}}$ is of the standard form $f = l \circ g$ with reflexive resp. counintary factors l and $g := l^{-\frac{1}{2}} \circ f \circ l^{-\frac{1}{2}}$.

P$_3$: If f is (weakly or strictly) reflexive resp. counitary, then its connector $\boldsymbol{\pi}$ is (strictly) reflexive resp. counitary. This is geometrically obvious, from the relations **P$_1$** injected into the definition (6), but the remarkable fact is that the analytical procedure (133) also respects this conservation of reflexivity or counitariness at every single step. Thus, if we apply it to the decomposition $f = l \circ g$ (as in **P$_2$**) of a reflexive f, we have to do with an *even* infinitesimal generator g_* that carries only coefficients g_{*1+d} of *even* degree d, and (133) automatically produces an *even* $\boldsymbol{p}_*$. The diffeo g itself is of mixed parity, but its coefficients of g_{*1+d} of *odd* degree are fully determined by the earlier

coefficients of *even* degree, and can thus be used in place of the g_{*1+d}. Either way, for reflexive diffeos the calculation of the invariants is a much more pleasant affair than for general diffeos, due to the drastic reduction in the mass of coefficients and (provided f be real) to the realness of $\mathfrak{p}_*$ and $\boldsymbol{\pi}_*$.

P₄: Conversely, any reflexive resp. counitary $\boldsymbol{\pi}$ is the invariant of some reflexive resp. counitary f. This follows from the *canonical synthesis* (see §1.4) which, for c real and large enough, automatically produces diffeos f_c of the required type.[25]

P₅: (Reinhard Schäffke). The product or quotient of two reflexive (res. unitary) diffeomorphisms is obviously conjugate to a reflexive (res. unitary) diffeomorphisms, but the converse is also true: any weakly reflexive (resp. unitary) f can, for any consecutive integers n_j, be represented as a quotient of two strictly reflexive (resp. unitary) diffeos f_j:

$$f := f_1 \circ f_2^{-1} \quad with \quad f(z) := z + 1 + o(1),\ f_j(z) := z + n_j + o(1),\ n_1 - n_2 = 1$$

and that too with explicit factors f_j:

$$f \ weakly \ reflexive \qquad || \qquad f \ weakly \ counitary$$

$$f_j := ({}^*f) \circ l^{n_j} \circ \sigma \circ (f^*) \circ \sigma \ || \ f_j := ({}^*f) \circ l^{n_j} \circ \sigma \circ (\bar{f}^*) \circ \sigma \quad (135)$$
$$= f^{n_j} \circ ({}^*f) \circ \sigma \circ (f^*) \circ \sigma \ || \quad = f^{n_j} \circ ({}^*f) \circ \sigma \circ (\bar{f}^*) \circ \sigma \quad (136)$$
$$= f^{n_j} \circ h^{-1} \circ \sigma \circ h \circ \sigma \quad || \quad = f^{n_j} \circ h^{-1} \circ \sigma \circ \bar{h} \circ \sigma \quad (137)$$

Indeed, the equivalent definitions (135), (136), (137) make it clear, respectively:

– that f_1, f_2 are reflexive (resp. counitary);
– that $f = f_1 \circ f_2^{-1}$;
– that f_1, f_2 are analytic.[26]

P₆: Piecing together all the above, we see that the commutative, non-associative[27] operation mix_c:

$$\mathrm{mix_c} \ : \ (\boldsymbol{\pi_1}, \boldsymbol{\pi_2}) \mapsto \boldsymbol{\pi} := \boldsymbol{\pi}_{f_{1,c} \circ f_{2,c}} = \boldsymbol{\pi}_{f_{2,c} \circ f_{1,c}} \quad (138)$$

---

[25]As pointed out to us by Reinhard Schäffke, this can also be deduced from the bifactorisation of f in **P₅** below, provided we admit the existence of a pre-image f for any given $\boldsymbol{\pi}$, which fact again follows from the canonical synthesis, but may also be established more directly.

[26]The analytic h in (137) conjugates the weakly reflexive/counitary f with a strictly reflexive/counitary f_0, i.e. $h \circ f = f_0 \circ h$. By definition, such a pair h, f_0 exists. We may note in passing that the factorisation $f = f_1 \circ f_2^{-1}$ would still hold for complex (in the reflexive case) or real (in the unitary case) values of n_j, but in that case the above formulae break down (f_1, f_2 are no longer analytic) and we must take recourse to another, more involved construction.

[27]$mix_c(\boldsymbol{\pi_1}, \boldsymbol{\pi_2})$ is doubly germinal: for a given $(\boldsymbol{\pi_1}, \boldsymbol{\pi_2})$, it is defined for c large enough, and for a given c , it is defined for $(\boldsymbol{\pi_1}, \boldsymbol{\pi_2})$ close enough to (id, id).

(where $f_{j,c}$ stands for the c-canonical pre-image of $\boldsymbol{\pi}_j$) respects reflexivity and counitariness.

4   Scalar invariants in terms of f.

4.1   The invariants A_ω as entire functions of f.

Let $\boldsymbol{\pi}_\omega^\pm$ and $\boldsymbol{\pi}_{*\omega}$ be the Fourier coefficient of the *connectors*, as defined in §3.5 by weight-wise truncation of the *collectors* and passage to the limit:

$$If +\Im(z) \gg 1\colon\ \boldsymbol{\pi}^{\pm 1}(z) = z + \sum_{\omega \in \Omega^-} \boldsymbol{\pi}_\omega^\pm\, e^{-\omega z}\ ;\ \boldsymbol{\pi}_*(z) = \sum_{\omega \in \Omega^-} \boldsymbol{\pi}_{*\omega}\, e^{-\omega z} \quad (139)$$

$$If -\Im(z) \gg 1\colon\ \boldsymbol{\pi}^{\pm 1}(z) = z + \sum_{\omega \in \Omega^+} \boldsymbol{\pi}_\omega^\pm\, e^{-\omega z}\ ;\ \boldsymbol{\pi}_*(z) = \sum_{\omega \in \Omega^+} \boldsymbol{\pi}_{*\omega}\, e^{-\omega z} \quad (140)$$

In view of (127)-(128) and (133), these Fourier coefficients are given by the convergent series[28]

$$\boldsymbol{\pi}_\omega^\pm = \sum_{1 \le r} \sum_{\substack{0 \le l_i \\ 1 \le n_i}}^{n_i + l_i \le s_i} (-1)^{n-1}\, \delta_{n_1,\dots,n_r}^{l_1,\dots,l_r}\ \mathbf{Te}_\omega^{s_1,\dots,s_r} \prod_{1 \le i \le r} \frac{(s_i-1)!\, g_{n_i, s_i - l_i + 1}^\pm}{(s_i - l_i - 1)!} \quad (141)$$

$$\boldsymbol{\pi}_{*\omega} = \sum_{1 \le r} (-1)^{r-1} \sum_{0 \le l_i < s_i} \delta^{l_1,\dots,l_r}\ \mathbf{Tan}_\omega^{s_1,\dots,s_r} \prod_{1 \le i \le r} \frac{(s_i-1)!\, g_{*s_i - l_i + 1}}{(s_i - l_i - 1)!} \quad (142)$$

However, the need to define the alien operators $\Delta_\omega^\pm$ and Δ_ω in uniform manner for all ω clashes with the need to associate within one and the same pair $(\boldsymbol{\pi}_{\mathrm{no}}, \boldsymbol{\pi}_{\mathrm{so}})$ resp. $(\boldsymbol{\pi}_{\mathrm{no}}^{-1}, \boldsymbol{\pi}_{\mathrm{so}}^{-1})$ northern and southern components originating from the same collector $\mathfrak{p}$ or $\mathfrak{p}^{-1}$. This clash leads to a regrettable but unavoidable disharmony in the correspondance between the invariants $A_\omega^\pm$ and A_ω, as defined from the resurgence equations, and the Fourier coefficients of the connectors, as derived from the collectors. This correspondance takes the form:

$$\forall \omega \in \Omega^-\quad:\quad A_\omega^+ = \boldsymbol{\pi}_\omega^+\quad;\quad A_\omega^- = \boldsymbol{\pi}_\omega^-\quad;\quad +2\pi i\, A_\omega = \boldsymbol{\pi}_{*\omega}$$

$$\forall \omega \in \Omega^+\quad:\quad A_\omega^- = \boldsymbol{\pi}_\omega^+\quad;\quad A_\omega^+ = \boldsymbol{\pi}_\omega^-\quad;\quad -2\pi i\, A_\omega = \boldsymbol{\pi}_{*\omega}$$

Alternatively, if we take the Borel transform $\hat{\boldsymbol{p}}(\zeta)$ of the collector $\boldsymbol{p}(z)$ in its power series variant, and observe that:

$$\boldsymbol{\pi}_\omega^\pm \equiv sign(\Im\omega)\, 2\pi i\, \widehat{\boldsymbol{p}}^\pm(\omega) \quad and \quad \boldsymbol{\pi}_{*\omega} \equiv sign(\Im\omega)\, 2\pi i\, \widehat{\boldsymbol{p}}_*(\omega)$$

---

[28]Regarding the nature of their convergence, see the remark at the end of the present subsection.

then the correspondance assumes the following form, with yet another twist:

$$\forall \omega \in \Omega^- \quad : \quad A_\omega^+ = -2\pi i\, \widehat{\boldsymbol{p}}^+(\omega) \quad ; \quad A_\omega^- = -2\pi i\, \widehat{\boldsymbol{p}}^-(\omega) \quad ; \quad A_\omega = -\widehat{\boldsymbol{p}}_*(\omega)$$

$$\forall \omega \in \Omega^+ \quad : \quad A_\omega^- = +2\pi i\, \widehat{\boldsymbol{p}}^+(\omega) \quad ; \quad A_\omega^+ = +2\pi i\, \widehat{\boldsymbol{p}}^-(\omega) \quad ; \quad A_\omega = -\widehat{\boldsymbol{p}}_*(\omega)$$

Remark: nature of the convergence.
(i) Weight truncation: if we truncate (141) to $\boldsymbol{\pi}^\pm_{\omega,[s]}$ or (142) to $\boldsymbol{\pi}_{*\omega,[s]}$ by retaining the sole terms of total weight $\leq s$ and then let s go to ∞, we have convergence of the truncated series to the correct limits.
(ii) If in (141) we regroup all terms corresponding to identical (up to order) monomial clusters $\prod g_{s_i}^{k_i}$ into single blocks, then the series consisting of these blocks is always convergent, and may be regarded the Taylor expansion of $\boldsymbol{\pi}^\pm_\omega$ viewed as an entire function of g.
(iii) The same applies to (142) but *only* if we switch from the coefficients g_{*s} (which have Gevrey-1 growth) to the g_s (which have Gevrey-0 i.e. analytic growth) before forming the weight-homogeneous blocks. This does not contradict the point (i) above: when truncating (142), there is no need of switching to the g_s, since truncation at weight s is exactly the same whether performed relative to the coefficients of g_* or those of g.

4.2 The case $\rho(f) \neq 0$. Normalisation.

For diffeos of the form $f(z) = z + 1 - \rho z^{-1} + \mathcal{O}(z^{-2})$ with a non-vanishing 'iterative residue' ρ, the defining relation (5) for the right and left iterators must be changed to

$$f_\pm^*(z) = \lim_{k \to \pm\infty} f^k(z) - k \pm \rho\,(c + \log|k|) \tag{143}$$

with the normalisation constant c as in §2.6. In the formal model, this leads to

$$\tilde{f}^*(z) = z + \rho\,(c + \log z) + \mathcal{O}(z^{-1}) \tag{144}$$

That apart, nothing changes and all the previous results and formulae still hold, including the explicit expansions (127)-(133) and (141)-(142), provided we set $Ze^1 := \gamma - c$ and normalise all multizetas and multitangents accordingly. As mentioned in §2.6, the recommended choice is $c = \gamma$, since it amounts to setting $Ze^1 := 0$.

4.3 The case $p(f) \neq 1$. Ramification.

Here again, the transition is straightforward. The 'prepared' form (2) for the diffeo now carries fractional exponents $s \in p^{-1}\mathbb{N}^*$. As a consequence, the

multiplicative z-plane and the convolutive ζ-plane are now p-ramified, and so is the index set Ω, which is embedded in the ζ-plane. We still have one single collector $\mathfrak{p}$ (resp. $\mathfrak{p}_*$), *ramified* yet *of one piece*, but p distinct pairs of connectors, $\boldsymbol{\pi} = (\boldsymbol{\pi}_{\mathrm{no}}, \boldsymbol{\pi}_{\mathrm{so}})$ (resp. $\boldsymbol{\pi}_* = (\boldsymbol{\pi}_{*\mathrm{no}}, \boldsymbol{\pi}_{*\mathrm{so}})$), separately *unramified* and mutually *unrelated*. The invariants $\boldsymbol{\pi}_\omega^\pm$ or $\boldsymbol{\pi}_{*\omega}$ are still given by the familiar formulae (141), (142), but with coefficients Te_ω and Tan_ω that are best calculated by resurgent analysis, as in §2.7, and are no longer finite sums of multizetas, even of ramified ones.

The transition to the most general case, with (ρ, p) any element of $(\mathbb{C}, \mathbb{N}^*)$, follows exactly the same lines, and merely combines the partial adjustments of the present and preceding subsections.

4.4 Growth properties of the invariants.

Growth in ω for a given analytic f :
For a diffeo f in prepared form (2), any majorisation of its coefficients easily translates into a majorisation of its invariants:

$$\{ \, |f_{[s]}| \le c_0 \, c_1^s \, \} \Longrightarrow \{ \, |A_\omega^\pm| \le C_0 \, C_1^{|\omega|} \, \} \tag{145}$$

Rough estimates of (C_0, C_1) in terms of (c_0, c_1) were given in [E2] and sharper ones in [B]. These results can be derived from a geometric analysis in the z-plane or from a resurgent analysis in the ζ-plane. Things change, though, when we go over to the Gevrey case.
Growth in ω for a given f of Gevrey class:
Formal diffeos f (in prepared form) of Gevrey class τ are easily shown to be stable under formal conjugations (also in prepared form) of the same Gevrey class. For $0 < \tau$, the Gevrey class is non-analytic, and Gevrey conjugacy turns out to be strictly stronger than formal conjugacy if and only if $\tau < 1$. This implies, for $0 < \tau < 1$, the existence of Gevrey conjugation invariants. These, however, can no longer be defined in the z-plane, since f is purely formal and has no geometric realisation there. In the ζ-plane, though, the Borel tranforms of the iterators ${}^*f$ and f^* still exist (again, assuming $\tau < 1$); still extend to uniform analytic functions on $\mathbb{C} - 2\pi i \, \mathbb{Z}$; still verify the familiar resurgence equations (32)-(33); and still unambigously defined define invariants $A_\omega^\pm$ and A_ω, which are still given by the explicit expansions (141)-(142). The only difference lies in the faster than exponential growth of $\hat{f}^*(\zeta)$ and ${}^*\hat{f}(\zeta)$ as $|\zeta| \to \infty$, and in the faster than exponential growth of $A_\omega^\pm$ as $|\omega| \to \infty$. More precisely, for $0 < \tau < 1$, the earlier implication (145)

becomes[29]:

$$\{\, |f_{[s]}| \le c_0 \, c_1^s \, s^{\tau s} \,\} \implies \{\, |A_\omega^\pm| \le C_0 \, C_1^{|\omega|} \, \exp(C_2 \, |\omega|^{\frac{1}{1-\tau}}) \}\tag{146}$$

Growth in f for a given ω:
We may now fix ω and ask how $A_\omega^+(f), A_\omega^-(f), A_\omega(f)$ behave as functions of f or, to simplify, as entire functions of any given coefficient $f_{[s]}$ $(s \ge 2)$ relative to a prepared form (2). Unlike with the ω-growth, there is little difference here between $A_\omega^\pm$ and A_ω.
(i) If $s > 2$, all three entire functions $A_\omega^+(f_{[s]}), A_\omega^-(f_{[s]}), A_\omega(f_{[s]})$ have at most exponential growth in $|f_{[s]}|^{\frac{1}{s-1}}$.
(ii) If $s = 2$, the corresponding coefficient coincides up to sign with the iterative residue (i.e. $f_{[2]} = -\rho$), and the entire functions $A_\omega^+(\rho), A_\omega^-(\rho), A_\omega(\rho)$ have at most exponential growth in $|\rho \log \rho|$. The result appears to be sharp.[30]

These results are almost "special cases" of the following statement: at any given point ζ_0 on $\widetilde{\mathbb{C} - \Omega}$, the Borel transform of the direct iterator assumes a value $\hat{f}^*(\zeta_0)$ which, as an entire function of $f_{[s]}$, is exactly of exponential type in $|f_{[s]}|^{\frac{1}{s-1}}$. This applies even for $s = 2$. The difference between the cases $s \ne 2$ and $s = 2$ makes itself felt only when we move ζ_0 to some point ω_0 located over Ω, to investigate the leading singularity there and infer from it the value of the invariants. When $\rho = 0$, the leading singularity in question is a simple pole $a_{\omega_0}(\zeta - \omega_0)^{-1}$, but when $\rho \ne 0$ it is of the form $a_{\omega_0}(\zeta - \omega_0)^{\rho\,\omega_0 - 1}/\Gamma(\rho\,\omega_0)$ and can be quite violent if ρ has an imaginary part.

4.5   Alternative computational strategies.

(i) Direct Fourier analysis in the multiplicative plane.
The methods amounts to calculating the limit:[31]

$$A_\omega^{\mp\epsilon(\omega)} = \boldsymbol{\pi}_\omega^\pm = \lim_{k\to\pm\infty} \int_{z_0}^{1+z_0} \left(l^{-k} \circ f^{2k} \circ l^{-k}(z) - z \right) e^{\omega\,z}\, dz \tag{147}$$

with $\epsilon(\omega) := sign(\Im(\omega))$. Although the parenthesised part of the integrand converges to $\boldsymbol{\pi}^\pm(z) - z$ for $|\Im(z)|$ large enough, the above scheme, even after optimisation in the choice of z_0, is computationally costly (integral instead of series) and inefficient (arithmetical convergence) as well as theoretically

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[29]For details, see [E2], p 424.
[30]See the argument in §8 of [BEE].
[31]If $\rho(f) \ne 0$, the shift l^{-k} should of course be replaced by $l^{-k+(c+\log k)\rho}$, with $c = \gamma$ as recommended choice for the normalisation constant c. See §2.6.

opaque (it sheds no light on the internal structure of the invariants as functions of f).

(ii) Asymptotic coefficient analysis in the formal model.

The method starts with the inductive calculation of the first N coefficients of the direct iterator $f^*(z)$ from its functional equation (11). One then switches to the Borel transform $\hat{f}^*(\zeta)$ and uses the method of *coefficient asymptotics*[32] to derive the form of the two singularities[33] closest to the origin (they are located over $\pm 2\pi i$). When applied to a parameter-free diffeo f with proper optimising precautions, the method is superbly efficient for computing $A_{\pm 2\pi i}$, even for diffeos f that are 'large', i.e. distant from the identity. Thus, with N taken in the region of 200 or 300, one typically gets $A_{\pm 2\pi i}$ with 100 exact digits or more, in less than half an hour of Maple time.

(iii) Resurgent analysis in the Poincaré plane.

The method is based on the resurgence equation (33) verified by the direct iterator f^*. But instead of interpreting that resurgence equation, as usual, in the highly ramified ζ-plane, one performs a conformal transform $\zeta \to \xi$ derived from the classical modular function λ:

$$\zeta = q(\xi) := -\log(1 - \lambda(\xi)) = -\log \lambda(-\frac{1}{\xi}) = 16 \sum_{n \text{ odd}} q_n\, e^{2\pi i \xi} \qquad (148)$$

$$q_n := \sum_{d|n} \frac{1}{d} = \frac{1}{n} \sum_{d|n} d \qquad (149)$$

That comformal transform does three things:
(*) it maps the Riemann surface $\widetilde{\mathbb{C} - 2\pi i\, \mathbb{Z}}$ of the ζ variable uniformly onto the Poincaré half-plane $\Im(\xi) > 0$;
(**) it changes the power series $\hat{f}^*(\zeta)$ with finite radius of convergence into a Fourier series $\widehat{f}^*(\xi) := \hat{f}^* \circ q(\xi)$ that converges on the entire Poincaré half-plane.
(***) it turns the alien operators into finite superpositions of post-composition operators – more precisely, post-composition by simple homographies $h_{\omega,j}^\pm$ or $h_{\omega,j}^\pm$ with entire coefficients:

$$\Delta_\omega^\pm \widehat{\varphi}(\xi) := \widehat{\varphi} \circ h_{\omega,1}^\pm(\xi) - \widehat{\varphi} \circ h_{\omega,2}^\pm(\xi) \qquad (150)$$

$$\Delta_\omega \widehat{\varphi}(\xi) := \sum_{1 \le j \le 2^r} m_{\omega,j}\, \widehat{\varphi} \circ h_{\omega,j}(\xi) \qquad (r := |\frac{\omega}{2\pi i}|, m_{\omega,j} \in \mathbb{Q}) \quad (151)$$

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[32]For a brief exposition of the method, see for ex. the section §2.3 of *Power Series with sum-product Taylor coefficients and their resurgence algebra*, J. Ecalle and S. Sharma, Ed. Scuola Normale Superiore, Pisa, 2011.
[33]or of the $2\,p$ closest singularities when $p(f) \ne 0$.

The method is efficient enough for small values of ω, but as $r := \left|\frac{\omega}{2\pi i}\right|$ increases, the minima

$$H^{\pm}(\omega) \;\; := \;\; \inf_{\Im(\xi)>0} \left\{ \Im(\xi)\,,\, \Im(h_{\omega,1}^{\mp}(\xi))\,,\, \Im(h_{\omega,2}^{\mp}(\xi)) \right\} \tag{152}$$

$$H(\omega) \;\; := \;\; \inf_{\Im(\xi)>0} \left\{ \Im(\xi)\,,\, \Im(h_{\omega,1}(\xi))\,,\,\ldots,\, \Im(h_{\omega,2^r}(\xi)) \right\} \tag{153}$$

rapidly decrease to zero, making it necessary to evaluate our Fourier series for $\widehat{f}^*(\xi)$ close to the boundary of their domain of convergence, i.e. the real axis, which of course is computationally costly.

(iv) Explicit multizetaic expansions.
This method, to which the present paper was devoted, has the advantage of explicitness and theoretical transparency, expressing as it does the invariants in terms of universal transcendental constants (the multizetas) and of the diffeo's Taylor coefficients. It has the further advantage of handling large values of ω almost as efficiently as small ones. But the method's chief drawback would seem to be this: it involves expansions which converge very fast (faster than geometrically) once they reach 'cruising speed', but which often take a damn long time to reach that speed. This is the case, not so much for ω large, but for f large, i.e. for diffeos too distant from id.

4.6 Concluding remarks.

(i) The invariants as autark functions.
Local, analytic, resonant vector fields X ranging through a *fixed* formal conjugacy class, possess holomorphic invariants A_ω which are *autark* functions of X, that is to say, of any given *free*[34] Taylor coefficient of X. Autark functions, very informally, are entire functions whose asymptotic behaviour in every sector of exponential increase or decrease admits a complete description, with dominant exponential terms accompanied by divergent-resurgent power series, which in turn verify a *closed* system of resurgence equations. Whether the invariants A_ω of diffeos are autark, too, seems likely but is yet unproved. Be that as it may, one would like to fully understand the asymptotic behaviour of A_ω as f grows, or as any given coefficient or parameter in f grows, since for very 'large' diffeos f the *direct* computation of the invariants would in any case be very costly.

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[34]i.e. of each coefficient that may freely vary without causing X to leave its formal conjugacy class.

(ii) Formal multizetas: dynamical vs arithmetical variants.
There exist several distinct but most probably equivalent notions of *arithmetical formal multizetas*, like the multizeta symbols subject to the two systems of so-called *quadratic multizeta relations*, or again to the *pentagonal, hexagonal* and *digonal relations*. But there also exists a demonstrably distinct and *weaker* notion of *dynamical formal multizetas* (and *multitangents*), by which we mean any system $\mathbb{S}$ of scalar-valued multizeta symbols (resp. function-valued multitangent symbols) that, when inserted into the expansions (142) (resp. (127)) guarantees, first, the convergence of these expansions, and, second, the invariance of the A_ω (resp. $\boldsymbol{\pi}$) so produced. This immediately suggests a programme: to repeat for the dynamical multizetas what has been successfully done for their arithmetical counterparts, in particular to construct *explicit, complete and canonical systems of irreducibles*.

(iii) Abstract invariants.
Let $\{^{\mathbb{S}}A_\omega,\, \omega \in \Omega\}$ be the system of 'abstract' invariants induced by a system $\mathbb{S}$ of dynamical multizetas as above. Since the system of natural invariants $\{A_\omega,\, \omega \in \Omega\}$ is complete, there necessarily exist conversion formulae of the form:

$$^{\mathbb{S}}A_{\omega_0} = \sum_{1 \leq r} \sum_{\omega_1 + \ldots \omega_r = \omega_0} H_{\mathbb{S}}^{\omega_1, \ldots, \omega_r} A_{\omega_1} \ldots A_{\omega_r} \qquad (154)$$

that respect the basic ω-gradation and carry interesting 'universal' structure constants $H_{\mathbb{S}}^\bullet$. These constants ought to be of particular significance in the case of the system $\mathbb{S}_0$ of 'rational' dynamical multizetas which is analogous, on the dynamical side, to the canonical system of 'rational'[35] multizetas on the arithmetical side.

4.7 Some historical background.

(i) Identity-tangent diffeos in holomorphic dynamics.
The iteration of one-dimensional analytic mappings – whether local or global; identity-tangent or not – has a long history going back a century or more. Fatou, for one, knew about the analytic classes of identity-tangent diffeos and had formed a clear, geometry-based idea of their invariants. The subject then when into something of a hibernation, until the advent of high-power computation, which brought about an explosive revival of holomorphic dynamics, one- and many-dimensional. For the specific subject of analytic invariants, however, the main impetus for renewal came from an unexpected quarter:

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[35]they become rational, of course, only after an homogeneous rescaling that amounts to setting $\pi := 1$.

resurgent analysis.

(ii) Identity-tangent diffeos and resurgent analysis.
The fact is that identity-tangent diffeos possess generically divergent but always resurgent iterators and fractional iterates, with an interesting, non-linear pattern of resurgence or self-reproduction at the singular points in the Borel plane, and it was in the process of sorting out these phenomena that resurgence theory was born, and later applied to general local objects and much else. In a sense, this involved a retreat from dynamics proper, since it meant focusing on the Borel plane, where the key dynamic notions of trajectory, fixed point etc admit no simple interpretation. For the invariants A_ω, however, the shift in focus brought a definite advantage, since in the Borel plane these invariants are automatically *localised* and *isolated* (they appear as coefficients of the leading singularities over the point ω) whereas in the multiplicative plane they are *diffuse* and *intertwined* (they make themselves felt only collectively and indirectly, via Stokes phenomena and the like, and the only way to isolate them is by Fourier analysis of type (147), which is but a half-hearted way of doing what Borel analysis does neatly and efficiently). This applies not just to identity-tangent diffeos, but to a huge range of local objects and equations. It also works in both directions: in that of "analysis", i.e. calculating and investigating the invariants of a given object; and in that of "synthesis", i.e. starting from an admissible system of prospective 'invariants' and constructing an object of which they are the actual invariants. And it has to be said that in both directions resurgence theory performs rather better than geometry. It leads in particular to a privileged or "canonical" synthesis, a notion which eludes geometry.

(iii) Identity-tangent diffeos and the resuscitation of multizetas.
Multizetas, especially of length 2, were first considered by Euler as an isolated curiosity, and later fell into a protracted oblivion for want of applications. They resurfaced only in the late 1970s and early 1980s, precisely in the context of holomorphic dynamics and identity-tangent diffeos, as *the* transcendental ingredient in the make-up of their invariants. Ten years later, the multizetas started cropping up in half a dozen, largely unconnected contexts: braid groups and knot theory; Feynman diagrams; Galois theory; mixed Tate motives; arithmetical dimorphy; ARI/GARI and the flexion structure, etc. At the moment, all these strands are in the process of merging or at least cross-fertilising, and constitute a vibrant, active field of research.

(iv) Identity-tangent diffeos and the actual computation of their invariants.

The sections of [E2] devoted to the invariants of identity-tangent diffeos were written with no computational applications in mind, and no attempt was made to optimise the bounds in formulae such as (145) or (146). On the contrary, the PhD thesis [B], which revisits the subject 30 years on, lays its main emphasis on these neglected aspects and provides effective Maple programmes for the computations of the invariants; it also offers copious asides on the algebraic aspects of multitangents, which largely, but not exactly, mirror those of multizetas.

5   Tables.

5.1   Multitangents: from symmetrel to alternal.

We express $Tan^\bullet$ in terms of $Te^\bullet$ according to the defining relations

$$\{\mathrm{Tan}^\bullet = (\mathrm{logmu}.\mathrm{Te}^\bullet) \circ (E^\bullet - \mathbf{1}^\bullet)\} \Longleftrightarrow \{(E^\bullet - \mathbf{1}^\bullet) \circ \mathrm{Tan}^\bullet = \mathrm{Te}^\bullet \circ (E^\bullet - \mathbf{1}^\bullet)\}$$

$$\mathrm{Tan}^{n_1} = \mathrm{Te}^{n_1}$$

$$\mathrm{Tan}^{n_1,n_2} = \tfrac{1}{2}\mathrm{Te}^{n_1,n_2} - \tfrac{1}{2}\mathrm{Te}^{n_2,n_1}$$

$$\mathrm{Tan}^{n_1,n_2,n_3} = \tfrac{1}{3}\mathrm{Te}^{n_1,n_2,n_3} - \tfrac{1}{6}\mathrm{Te}^{n_1,n_3,n_2} - \tfrac{1}{6}\mathrm{Te}^{n_2,n_1,n_3} - \tfrac{1}{6}\mathrm{Te}^{n_2,n_3,n_1} - \tfrac{1}{6}\mathrm{Te}^{n_3,n_1,n_2}$$
$$+ \tfrac{1}{3}\mathrm{Te}^{n_3,n_2,n_1} - \tfrac{1}{6}\mathrm{Te}^{n_1+n_3,n_2} + \tfrac{1}{12}\mathrm{Te}^{n_1,n_2+n_3} + \tfrac{1}{12}\mathrm{Te}^{n_1+n_2,n_3}$$
$$+ \tfrac{1}{12}\mathrm{Te}^{n_3,n_1+n_2} + \tfrac{1}{12}\mathrm{Te}^{n_2+n_3,n_1} - \tfrac{1}{6}\mathrm{Te}^{n_2,n_1+n_3}$$

$$\mathrm{Tan}^{n_1,n_2,n_3,n_4} = \tfrac{1}{4}\mathrm{Te}^{n_1,n_2,n_3,n_4} - \tfrac{1}{12}\mathrm{Te}^{n_1,n_2,n_4,n_3} - \tfrac{1}{12}\mathrm{Te}^{n_1,n_3,n_2,n_4} - \tfrac{1}{12}\mathrm{Te}^{n_1,n_3,n_4,n_2}$$
$$- \tfrac{1}{12}\mathrm{Te}^{n_1,n_4,n_2,n_3} + \tfrac{1}{12}\mathrm{Te}^{n_1,n_4,n_3,n_2} - \tfrac{1}{12}\mathrm{Te}^{n_2,n_1,n_3,n_4} + \tfrac{1}{12}\mathrm{Te}^{n_2,n_1,n_4,n_3}$$
$$- \tfrac{1}{12}\mathrm{Te}^{n_2,n_3,n_1,n_4} - \tfrac{1}{12}\mathrm{Te}^{n_2,n_3,n_4,n_1} + \tfrac{1}{12}\mathrm{Te}^{n_2,n_4,n_1,n_3} + \tfrac{1}{12}\mathrm{Te}^{n_2,n_4,n_3,n_1}$$
$$- \tfrac{1}{12}\mathrm{Te}^{n_3,n_1,n_2,n_4} - \tfrac{1}{12}\mathrm{Te}^{n_3,n_1,n_4,n_2} + \tfrac{1}{12}\mathrm{Te}^{n_3,n_2,n_1,n_4} + \tfrac{1}{12}\mathrm{Te}^{n_3,n_2,n_4,n_1}$$
$$- \tfrac{1}{12}\mathrm{Te}^{n_3,n_4,n_1,n_2} + \tfrac{1}{12}\mathrm{Te}^{n_3,n_4,n_2,n_1} - \tfrac{1}{12}\mathrm{Te}^{n_4,n_1,n_2,n_3} + \tfrac{1}{12}\mathrm{Te}^{n_4,n_1,n_3,n_2}$$
$$+ \tfrac{1}{12}\mathrm{Te}^{n_4,n_2,n_1,n_3} + \tfrac{1}{12}\mathrm{Te}^{n_4,n_2,n_3,n_1} + \tfrac{1}{12}\mathrm{Te}^{n_4,n_3,n_1,n_2} - \tfrac{1}{4}\mathrm{Te}^{n_4,n_3,n_2,n_1}$$
$$+ \tfrac{1}{12}\mathrm{Te}^{n_1,n_2,n_3+n_4} - \tfrac{1}{12}\mathrm{Te}^{n_1,n_3,n_2+n_4} - \tfrac{1}{12}\mathrm{Te}^{n_2,n_3,n_1+n_4} + \tfrac{1}{12}\mathrm{Te}^{n_2,n_4,n_1+n_3}$$
$$- \tfrac{1}{12}\mathrm{Te}^{n_3,n_1,n_2+n_4} + \tfrac{1}{12}\mathrm{Te}^{n_3,n_2,n_1+n_4} + \tfrac{1}{12}\mathrm{Te}^{n_4,n_2,n_1+n_3} - \tfrac{1}{12}\mathrm{Te}^{n_4,n_3,n_1+n_2}$$
$$+ \tfrac{1}{12}\mathrm{Te}^{n_1,n_2+n_3,n_4} - \tfrac{1}{12}\mathrm{Te}^{n_1,n_2+n_4,n_3} - \tfrac{1}{12}\mathrm{Te}^{n_2,n_1+n_3,n_4} + \tfrac{1}{12}\mathrm{Te}^{n_2,n_1+n_4,n_3}$$
$$- \tfrac{1}{12}\mathrm{Te}^{n_3,n_1+n_4,n_2} + \tfrac{1}{12}\mathrm{Te}^{n_3,n_2+n_4,n_1} + \tfrac{1}{12}\mathrm{Te}^{n_4,n_1+n_3,n_2} - \tfrac{1}{12}\mathrm{Te}^{n_4,n_2+n_3,n_1}$$
$$+ \tfrac{1}{12}\mathrm{Te}^{n_1+n_2,n_3,n_4} - \tfrac{1}{12}\mathrm{Te}^{n_1+n_3,n_2,n_4} - \tfrac{1}{12}\mathrm{Te}^{n_1+n_3,n_4,n_2} + \tfrac{1}{12}\mathrm{Te}^{n_2+n_4,n_1,n_3}$$
$$+ \tfrac{1}{12}\mathrm{Te}^{n_2+n_4,n_3,n_1} - \tfrac{1}{12}\mathrm{Te}^{n_1+n_4,n_2,n_3} + \tfrac{1}{12}\mathrm{Te}^{n_1+n_4,n_3,n_2} - \tfrac{1}{12}\mathrm{Te}^{n_3+n_4,n_2,n_1}$$
$$+ \tfrac{1}{24}\mathrm{Te}^{n_1+n_2,n_3+n_4} - \tfrac{1}{12}\mathrm{Te}^{n_1+n_3,n_2+n_4} + \tfrac{1}{12}\mathrm{Te}^{n_2+n_4,n_1+n_3} - \tfrac{1}{24}\mathrm{Te}^{n_3+n_4,n_1+n_2}$$

$$\mathrm{Tan}^{n_1,\ldots,n_5} = \textit{540 summands.}$$

$$\mathrm{Tan}^{n_1,\ldots,n_6} = \textit{3688 summands.}$$

$$\mathrm{Tan}^{n_1,\ldots,n_7} = \textit{47292 summands.}$$

5.2 Parity/imparity of alternal multitangents.

We begin by comparing the number of summands in the monotangent reductions $red_1(Te^\bullet)$ and $red_1(Tan^\bullet)$ (resp. $red_2(Te^\bullet)$ and $red_2(Tan^\bullet)$) of $Te^\bullet$ and $Tan^\bullet$ *before* (resp. *after*) symmetrel linearisation of the resulting multizetas. N.B. A further reduction $red_3(Te^\bullet)$ and $red_3(Tan^\bullet)$, corresponding to a complete decomposition of the multizeta into *arithmetical irreducibles*,

would yield even fewer summands.

$(n_1, ..., n_r)$	$\#\mathrm{red}_1(\mathrm{Te}^\bullet)$	$\#\mathrm{red}_1(\mathrm{Tan}^\bullet)$	$\#\mathrm{red}_2(\mathrm{Te}^\bullet)$	$\#\mathrm{red}_2(\mathrm{Tan}^\bullet)$
$(5, 7, 4)$	34	40	30	17
$(5, 7, 14)$	133	124	106	65
$(5, 7, 4, 5)$	148	141	177	80
$(8, 11, 4, 9)$	580	679	1127	454
$(8, 11, 7, 12)$	824	741	1154	452
$(4, 5, 4, 5, 4)$	42	54	389	98
$(3, 4, 5, 6, 7)$	455	874	2748	760

The following six examples of multitangent reduction (of type red_2) illustrate the phenomenon of *parity separation* in $Tan^\bullet$, as opposed to $Te^\bullet$.

Example 1 : $\mathrm{Te}^{2,6,4}$ is neither even nor odd.

$\mathrm{Te}^{2,6,4}(z) = \sum_{2 \leq m \leq 6} \mathrm{Teze}_m^{2,6,4} \, \mathrm{Te}^m(z) \ with$

$\mathrm{Teze}_1^{2,6,4} = +20 \, \mathrm{Ze}^{11} + 56 \, \mathrm{Ze}^{9,2} - 70 \, \mathrm{Ze}^{8,3} - 112 \, \mathrm{Ze}^{3,8} + 54 \, \mathrm{Ze}^{7,4} + 42 \, \mathrm{Ze}^{4,7}$
$\qquad\qquad - 20 \, \mathrm{Ze}^{6,5} - 20 \, \mathrm{Ze}^{5,6}$
$\qquad = 0$
$\mathrm{Teze}_2^{2,6,4} = +14 \, \mathrm{Ze}^{10} + 35 \, \mathrm{Ze}^{8,2} + 56 \, \mathrm{Ze}^{2,8} - 40 \, \mathrm{Ze}^{7,3} - 28 \, \mathrm{Ze}^{3,7} + 35 \, \mathrm{Ze}^{6,4}$
$\qquad\qquad + 39 \, \mathrm{Ze}^{4,6} - 32 \, \mathrm{Ze}^{5,5}$
$\mathrm{Teze}_3^{2,6,4} = +8 \, \mathrm{Ze}^9 + 20 \, \mathrm{Ze}^{7,2} + 14 \, \mathrm{Ze}^{2,7} - 20 \, \mathrm{Ze}^{6,3} - 22 \, \mathrm{Ze}^{3,6} + 8 \, \mathrm{Ze}^{5,4} + 8 \, \mathrm{Ze}^{4,5}$
$\mathrm{Teze}_4^{2,6,4} = +5 \, \mathrm{Ze}^8 + 10 \, \mathrm{Ze}^{6,2} + 11 \, \mathrm{Ze}^{2,6} - 8 \, \mathrm{Ze}^{5,3} - 8 \, \mathrm{Ze}^{3,5} + 6 \, \mathrm{Ze}^{4,4}$
$\mathrm{Teze}_5^{2,6,4} = +2 \, \mathrm{Ze}^7 + 4 \, \mathrm{Ze}^{5,2} + 4 \, \mathrm{Ze}^{2,5} - 2 \, \mathrm{Ze}^{4,3} - 2 \, \mathrm{Ze}^{3,4}$
$\mathrm{Teze}_6^{2,6,4} = +\mathrm{Ze}^6 + \mathrm{Ze}^{4,2} + \mathrm{Ze}^{2,4}$

Example 2 : $Tan^{2,6,4}$ is even since $deg(Tan^{2,6,4}) = 2+6+4-3 = 9 = odd$.

$\mathrm{Tan}^{2,6,4}(z) = \mathrm{Tanze}_2^{2,6,4} \, \mathrm{Te}^2(z) + \mathrm{Tanze}_4^{2,6,4} \, \mathrm{Te}^4(z) + \mathrm{Tanze}_6^{2,6,4} \, \mathrm{Te}^6(z) \ with$

$\mathrm{Tanze}_2^{2,6,4} = +5 \, \mathrm{Ze}^{10} - \frac{7}{3} \, \mathrm{Ze}^{8,2} + \frac{56}{3} \, \mathrm{Ze}^{2,8} - 40 \, \mathrm{Ze}^{7,3} - 28 \, \mathrm{Ze}^{3,7} + 9 \, \mathrm{Ze}^{6,4}$
$\qquad\qquad + 13 \, \mathrm{Ze}^{4,6} - 32 \, \mathrm{Ze}^{5,5}$
$\mathrm{Tanze}_4^{2,6,4} = +3 \, \mathrm{Ze}^8 + \frac{8}{3} \, \mathrm{Ze}^{6,2} + \frac{11}{3} \, \mathrm{Ze}^{2,6} - 8 \, \mathrm{Ze}^{5,3} - 8 \, \mathrm{Ze}^{3,5} + 2 \, \mathrm{Ze}^{4,4}$
$\mathrm{Tanze}_6^{2,6,4} = +\frac{2}{3} \, \mathrm{Ze}^6 + \frac{1}{3} \, \mathrm{Ze}^{4,2} + \frac{1}{3} \, \mathrm{Ze}^{2,4}$

Example 3: $\text{Te}^{2,7,4}(z)$ is neither even nor odd.

$$\text{Te}^{2,7,4}(z) = \sum_{2 \le m \le 7} \text{Teze}_m^{2,7,4} \, \text{Te}^m(z) \ \textit{with}$$

$$\begin{aligned}
\text{Teze}_1^{2,7,4} &= +30\,\text{Ze}^{12} + 84\,\text{Ze}^{10,2} - 112\,\text{Ze}^{9,3} - 168\,\text{Ze}^{3,9} + 112\,\text{Ze}^{8,4} \\
&\quad + 84\,\text{Ze}^{4,8} - 104\,\text{Ze}^{7,5} - 112\,\text{Ze}^{5,7} + 100\,\text{Ze}^{6,6} \\
&= 0 \\
\text{Teze}_2^{2,7,4} &= +20\,\text{Ze}^{11} + 56\,\text{Ze}^{9,2} + 84\,\text{Ze}^{2,9} - 70\,\text{Ze}^{8,3} - 56\,\text{Ze}^{3,8} + 54\,\text{Ze}^{7,4} \\
&\quad + 56\,\text{Ze}^{4,7} - 20\,\text{Ze}^{6,5} - 20\,\text{Ze}^{5,6} \\
\text{Teze}_3^{2,7,4} &= +14\,\text{Ze}^{10} + 35\,\text{Ze}^{8,2} + 28\,\text{Ze}^{2,8} - 40\,\text{Ze}^{7,3} - 42\,\text{Ze}^{3,7} + 35\,\text{Ze}^{6,4} \\
&\quad + 35\,\text{Ze}^{4,6} - 32\,\text{Ze}^{5,5} \\
\text{Teze}_4^{2,7,4} &= +8\,\text{Ze}^{9} + 20\,\text{Ze}^{7,2} + 21\,\text{Ze}^{2,7} - 20\,\text{Ze}^{6,3} - 20\,\text{Ze}^{3,6} + 8\,\text{Ze}^{5,4} + 8\,\text{Ze}^{4,5} \\
\text{Teze}_5^{2,7,4} &= +5\,\text{Ze}^{8} + 10\,\text{Ze}^{6,2} + 10\,\text{Ze}^{2,6} - 8\,\text{Ze}^{5,3} - 8\,\text{Ze}^{3,5} + 6\,\text{Ze}^{4,4} \\
\text{Teze}_6^{2,7,4} &= +2\,\text{Ze}^{7} + 4\,\text{Ze}^{5,2} + 4\,\text{Ze}^{2,5} - 2\,\text{Ze}^{4,3} - 2\,\text{Ze}^{3,4} \\
\text{Teze}_7^{2,7,4} &= +\text{Ze}^{6} + \text{Ze}^{4,2} + \text{Ze}^{2,4}
\end{aligned}$$

Example 4: $Tan^{2,7,4}$ is even since $deg(Tan^{2,7,4}) = 2+7+4-3 = 10 = odd$.

$$\text{Tan}^{2,7,4}(z) = \text{Tanze}_3^{2,7,4}\,\text{Te}^3(z) + \text{Tanze}_5^{2,7,4}\,\text{Te}^5(z) + \text{Tanze}_7^{2,7,4}\,\text{Te}^7(z) \ \textit{with}$$

$$\begin{aligned}
\text{Tanze}_1^{2,7,4} &= +36\,\text{Ze}^{12} + 84\,\text{Ze}^{10,2} - 112\,\text{Ze}^{9,3} - 168\,\text{Ze}^{3,9} + 56\,\text{Ze}^{8,4} \\
&\quad + 28\,\text{Ze}^{4,8} - 104\,\text{Ze}^{7,5} - 112\,\text{Ze}^{5,7} + \tfrac{100}{3}\,\text{Ze}^{6,6} \\
&= 0 \\
\text{Tanze}_3^{2,7,4} &= +11\,\text{Ze}^{10} + \tfrac{49}{3}\,\text{Ze}^{8,2} + \tfrac{28}{3}\,\text{Ze}^{2,8} - 40\,\text{Ze}^{7,3} - 42\,\text{Ze}^{3,7} + \tfrac{35}{3}\,\text{Ze}^{6,4} \\
&\quad + \tfrac{35}{3}\,\text{Ze}^{4,6} - 32\,\text{Ze}^{5,5} \\
\text{Tanze}_5^{2,7,4} &= +\tfrac{10}{3}\,\text{Ze}^{8} + \tfrac{10}{3}\,\text{Ze}^{6,2} + \tfrac{10}{3}\,\text{Ze}^{2,6} - 8\,\text{Ze}^{5,3} - 8\,\text{Ze}^{3,5} + 2\,\text{Ze}^{4,4} \\
\text{Tanze}_7^{2,7,4} &= +\tfrac{2}{3}\,\text{Ze}^{6} + \tfrac{1}{3}\,\text{Ze}^{4,2} + \tfrac{1}{3}\,\text{Ze}^{2,4}
\end{aligned}$$

Example 5: $Tan^{2,5,2,4}$ is even since $deg(Tan^{2,5,2,4}) = 2+5+2+4-4 = 9 = odd$.

$$\text{Tan}^{2,5,2,4}(z) = \text{Tanze}_2^{2,5,2,4}\,\text{Te}^2(z) + \text{Tanze}_4^{2,5,2,4}\,\text{Te}^4(z) \ \textit{with}$$

$$\begin{aligned}
\text{Tanze}_2^{2,5,2,4} &= +\tfrac{14}{3}\text{Ze}^{9,2} + \tfrac{2}{3}\text{Ze}^{2,9} + 8\,\text{Ze}^{8,3} + \tfrac{4}{3}\text{Ze}^{3,8} + \tfrac{50}{3}\text{Ze}^{7,4} - \text{Ze}^{4,7} + 5\,\text{Ze}^{6,5} \\
&\quad + \tfrac{20}{3}\text{Ze}^{5,6} + \tfrac{40}{3}\text{Ze}^{7,2,2} + \tfrac{65}{3}\text{Ze}^{2,7,2} - \tfrac{20}{3}\text{Ze}^{6,3,2} - \tfrac{20}{3}\text{Ze}^{6,2,3} - \tfrac{10}{3}\text{Ze}^{3,6,2} \\
&\quad + \tfrac{70}{3}\text{Ze}^{2,6,3} + \tfrac{20}{3}\text{Ze}^{3,2,6} + \tfrac{20}{3}\text{Ze}^{2,3,6} + \tfrac{20}{3}\text{Ze}^{5,4,2} + \tfrac{32}{3}\text{Ze}^{5,2,4} + 7\,\text{Ze}^{4,5,2} \\
&\quad + \tfrac{32}{3}\text{Ze}^{2,5,4} + 5\,\text{Ze}^{2,4,5} + 6\,\text{Ze}^{4,4,3} + 4\,\text{Ze}^{4,3,4} + 2\,\text{Ze}^{3,4,4} \\
&\quad - 32\,\text{Ze}^{5,3,3} - 12\,\text{Ze}^{3,5,3} \\
\text{Tanze}_4^{2,5,2,4} &= +\text{Ze}^{7,2} - \tfrac{1}{3}\text{Ze}^{2,7} + 2\,\text{Ze}^{6,3} + \tfrac{2}{3}\text{Ze}^{3,6} + \tfrac{8}{3}\text{Ze}^{5,4} + \tfrac{8}{3}\text{Ze}^{5,2,2} + \tfrac{7}{3}\text{Ze}^{2,5,2} \\
&\quad - \tfrac{2}{3}\text{Ze}^{3,4,2} + 2\,\text{Ze}^{2,4,3} + \tfrac{4}{3}\text{Ze}^{3,2,4} + \tfrac{4}{3}\text{Ze}^{2,3,4}
\end{aligned}$$

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Example 6 : $Tan^{2,3,2,5}$ is even since $deg(Tan^{2,3,2,5}){=}2{+}3{+}2{+}5{-}4{=}8{=}$ *even.*

$$Tan^{2,3,2,5}(z) = Tanze_3^{2,3,2,5} \, Te^3(z) + Tanze_5^{2,3,2,5} \, Te^5(z) \; with$$

$$\begin{aligned}
Tanze_1^{2,3,2,5} = &+\tfrac{2}{3}Ze^{9,2} + \tfrac{10}{3}Ze^{2,9} - 11\,Ze^{8,3} - \tfrac{8}{3}Ze^{3,8} - 7\,Ze^{7,4} + 2\,Ze^{4,7} - \tfrac{10}{3}Ze^{5,6} \\
&-10\,Ze^{7,2,2} + 10\,Ze^{2,7,2} - \tfrac{25}{3}Ze^{6,3,2} - \tfrac{40}{3}Ze^{6,2,3} + 10\,Ze^{3,6,2} - \tfrac{25}{3}Ze^{2,6,3} \\
&-20\,Ze^{3,2,6} - 20\,Ze^{2,3,6} - 2\,Ze^{5,4,2} - 8\,Ze^{5,2,4} + 5\,Ze^{4,5,2} + 10\,Ze^{2,5,4} \\
&-10\,Ze^{4,2,5} + 5\,Ze^{2,4,5} + 20\,Ze^{5,3,3} + 30\,Ze^{3,5,3} + 10\,Ze^{3,3,5} - 6\,Ze^{4,4,3} \\
&-15\,Ze^{4,3,4} - 6\,Ze^{3,4,4} \\
Tanze_3^{2,3,2,5} = &+\tfrac{2}{3}Ze^{7,2} + \tfrac{4}{3}Ze^{2,7} - \tfrac{2}{3}Ze^{6,3} - \tfrac{2}{3}Ze^{3,6} + \tfrac{1}{3}Ze^{5,4} + Ze^{4,5} - \tfrac{2}{3}Ze^{5,2,2} \\
&+\tfrac{17}{3}Ze^{2,5,2} - Ze^{4,3,2} - 4\,Ze^{4,2,3} + 2\,Ze^{3,4,2} + 2\,Ze^{2,4,3} - 4\,Ze^{3,2,4} \\
&-Ze^{2,3,4} + 4\,Ze^{3,3,3} \\
Tanze_5^{2,3,2,5} = &+\tfrac{1}{3}Ze^{5,2} + \tfrac{1}{3}Ze^{2,5} + Ze^{2,3,2}
\end{aligned}$$

5.3 The invariants as entire functions of f: the general case.

We write down, up to weight 10 inclusively, the expansion of the collector $\mathfrak{p}_*$ in terms of the g_*. We assume $p(f) = 1$ but impose no restriction on $\rho(f) \equiv -g_{*2}$. In this and all further examples, we order the terms according to their total weight and, within a given total weight, we start with the lowest monotangents.

Example 1: $\mathfrak{p}_*$ up to weight 10 for $f = l \circ g$ with $g_*(z) = \sum_{1 \le d} g_{*1+d} z^{-d}$.

$+\mathbf{Te}^1 \left[g_{*2} \right] + \mathbf{Te}^2 \left[g_{*3} \right] + \mathbf{Te}^3 \left[g_{*4} \right] + \mathbf{Te}^4 \left[g_{*5} \right] + \mathbf{Te}^2 \left[6\,\zeta(3)\, g_{*2}\, g_{*4} - 6\,\zeta(3)\, g_{*3}^2 \right]$

$+\mathbf{Te}^5 \left[g_{*6} \right] + \mathbf{Te}^3 \left[6\,\zeta(3)\, g_{*2}\, g_{*5} - 6\,\zeta(3)\, g_{*3}\, g_{*4} \right[+ \mathbf{Te}^6 \left[g_{*7} \right]$

$+\mathbf{Te}^2 \left[30\,\zeta(5)\, g_{*4}^2 - \dfrac{5}{2}\,\zeta(5)\, g_{*2}^4 + 10\,\zeta(5)\, g_{*2}\, g_{*6} - 40\,\zeta(5)\, g_{*3}\, g_{*5} \right]$

$+\mathbf{Te}^3 \left[\dfrac{4}{3}\,\zeta(2)^2\, g_{*2}\, g_{*3}^2 - \dfrac{4}{3}\,\zeta(2)^2\, g_{*2}^2\, g_{*4} \right] + \mathbf{Te}^4 \left[3\,\zeta(3)\, g_{*4}^2 + \dfrac{1}{4}\,\zeta(3)\, g_{*2}^4 - 10\,\zeta(3)\, g_{*3}\, g_{*5} \right.$

$\left. +7\,\zeta(3)\, g_{*2}\, g_{*6} \right] + \mathbf{Te}^5 \left[-\dfrac{2}{3}\,\zeta(2)\, g_{*2}\, g_{*3}^2 + \dfrac{2}{3}\,\zeta(2)\, g_{*2}^2\, g_{*4} \right] + \mathbf{Te}^7 \left[g_{*8} \right]$

$+\mathbf{Te}^2 \left[36\,\zeta(3)^2\, g_{*3}^3 - \dfrac{32}{5}\,\zeta(2)^3\, g_{*3}^3 + 18\,\zeta(3)^2\, g_{*5}\, g_{*2}^2 + \dfrac{48}{5}\,\zeta(2)^3\, g_{*2}\, g_{*3}\, g_{*4} \right.$

$\left. -54\,\zeta(3)^2\, g_{*2}\, g_{*3}\, g_{*4} - \dfrac{16}{5}\,\zeta(2)^3\, g_{*5}\, g_{*2}^2 \right] + \mathbf{Te}^3 \left[20\,\zeta(5)\, g_{*4}\, g_{*5} + 10\,\zeta(5)\, g_{*2}\, g_{*7} \right.$

$\left. -30\,\zeta(5)\, g_{*3}\, g_{*6} - 5\,\zeta(5)\, g_{*2}^3\, g_{*3} \right] + \mathbf{Te}^4 \left[-\dfrac{1}{5}\,\zeta2^2\, g_{*3}^3 - \dfrac{21}{10}\,\zeta(2)^2\, g_{*2}^2\, g_{*5} \right.$

$\left. +\dfrac{23}{10}\,\zeta(2)^2\, g_{*2}\, g_{*3}\, g_{*4} \right] + \mathbf{Te}^5 \left[8\,\zeta(3)\, g_{*2}\, g_{*7} - 12\,\zeta(3)\, g_{*3}\, g_{*6} + 4\,\zeta(3)\, g_{*4}\, g_{*5} \right.$

$\left. +\zeta(3)\, g_{*2}^3\, g_{*3} \right] + \mathbf{Te}^6 \left[-\dfrac{1}{3}\,\zeta(2)\, g_{*3}^3 + \dfrac{3}{2}\,\zeta(2)\, g_{*2}^2\, g_{*5} - \dfrac{7}{6}\,\zeta(2)\, g_{*2}\, g_{*3}\, g_{*4} \right]$

$+\mathbf{Te}^8 \left[g_{*9} \right] + \mathbf{Te}^2 \left[210\,\zeta(7)\, g_{*4}\, g_{*6} - 140\,\zeta(7)\, g_{*5}^2 - 84\,\zeta(7)\, g_{*3}\, g_{*7} + 14\,\zeta(7)\, g_{*2}\, g_{*8} \right.$

$\left. -\dfrac{133}{3}\,\zeta(7)\, g_{*2}^3\, g_{*4} + \dfrac{133}{3}\,\zeta(7)\, g_{*2}^2\, g_{*3}^2 \right] + \mathbf{Te}^3 \left[36\,\zeta(3)^2\, g_{*3}^2\, g_{*4} - 9\,\zeta(3)^2\, g_{*2}\, g_{*4}^2 \right.$

$\left. +21\,\zeta(3)^2\, g_{*2}^2\, g_{*6} + \dfrac{3}{4}\,\zeta(3)^2\, g_{*2}^5 - \dfrac{32}{5}\,\zeta(2)^3\, g_{*3}^2\, g_{*4} - \dfrac{64}{15}\,\zeta(2)^3\, g_{*2}^2\, g_{*6} \right.$

$\left. +\dfrac{32}{3}\,\zeta(2)^3\, g_{*2}\, g_{*3}\, g_{*5} - 48\,\zeta(3)^2\, g_{*2}\, g_{*3}\, g_{*5} \right] + \mathbf{Te}^4 \left[45\,\zeta(5)\, g_{*4}\, g_{*6} - 20\,\zeta(5)\, g_{*5}^2 \right.$

$\left. -36\,\zeta(5)\, g_{*3}\, g_{*7} + 11\,\zeta(5)\, g_{*2}\, g_{*8} - \dfrac{10}{3}\,\zeta(5)\, g_{*2}^3\, g_{*4} - \dfrac{25}{6}\,\zeta(5)\, g_{*2}^2\, g_{*3}^2 \right]$

$+\mathbf{Te}^5 \left[\dfrac{10}{3}\,\zeta(2)^2\, g_{*2}\, g_{*3}\, g_{*5} - \dfrac{2}{5}\,\zeta(2)^2\, g_{*3}^2\, g_{*4} - \dfrac{44}{15}\,\zeta(2)^2\, g_{*2}^2\, g_{*6} \right] + \mathbf{Te}^6 \left[9\,\zeta(3)\, g_{*2}\, g_{*8} \right.$

$\left. -14\,\zeta(3)\, g_{*3}\, g_{*7} + 5\,\zeta(3)\, g_{*4}\, g_{*6} + \dfrac{1}{2}\,\zeta(3)\, g_{*2}^2\, g_{*3}^2 + 2\,\zeta(3)\, g_{*2}^3\, g_{*4} \right]$

$+\mathbf{Te}^7 \left[\dfrac{8}{3}\,\zeta(2)\, g_{*2}^2\, g_{*6} - \dfrac{5}{3}\,\zeta(2)\, g_{*2}\, g_{*3}\, g_{*5} - \zeta(2)\, g_{*3}^2\, g_{*4} \right] + \mathbf{Te}^9 \left[g_{*10} \right]$

5.4 The invariants as entire functions of f: the reflexive case.

As in Example 1, we write down the expansion of the collector $\mathfrak{p}_*$ in terms of the g_*, but this time for a reflexive g. We still assume $p(f) = 1$ and reflexivity automatically implies $\rho(f) \equiv -g_{*2} \equiv 0$. There being fewer coefficients

g_{*s}, we reach weight 13.

Example 2: $\mathfrak{p}_*$ up to weight 13 for $f = l \circ g$ with $g_*(z) = \sum_{1 \leq d} g_{*1+2d} z^{-2d}$.

$+\mathbf{Te}^2 \left[g_{*3} \right] + \mathbf{Te}^4 \left[g_{*5} \right] + \mathbf{Te}^2 \left[-6 \zeta(3) g_{*3}^2 \right] + \mathbf{Te}^6 \left[g_{*7} \right] + \mathbf{Te}^2 \left[-40 \zeta(5) g_{*3} g_{*5} \right]$

$+\mathbf{Te}^4 \left[-10 \zeta(3) g_{*3} g_{*5} \right] + \mathbf{Te}^2 \left[36 \zeta(3)^2 g_{*3}^3 - \frac{32}{5} \zeta(2)^3 g_{*3}^3 \right] + \mathbf{Te}^4 \left[-\frac{1}{5} \zeta(2)^2 g_{*3}^3 \right]$

$+\mathbf{Te}^6 \left[-\frac{1}{3} \zeta(2) g_{*3}^3 \right] + \mathbf{Te}^8 \left[g_{*9} \right] + \mathbf{Te}^2 \left[-140 \zeta(7) g_{*5}^2 - 84 \zeta(7) g_{*3} g_{*7} \right]$

$+\mathbf{Te}^4 \left[-20 \zeta(5) g_{*5}^2 - 36 \zeta(5) g_{*3} g_{*7} \right] + \mathbf{Te}^6 \left[-14 \zeta(3) g_{*3} g_{*7} \right]$

$+\mathbf{Te}^2 \left[-\frac{15648}{175} \zeta(2)^4 g_{*3}^2 g_{*5} - 80 \zeta(6,2) g_{*3}^2 g_{*5} + 800 \zeta(3) \zeta(5) g_{*3}^2 g_{*5} \right]$

$+\mathbf{Te}^4 \left[-\frac{272}{21} \zeta(2)^3 g_{*3}^2 g_{*5} + 80 \zeta(3)^2 g_{*3}^2 g_{*5} \right] + \mathbf{Te}^6 \left[-\frac{34}{15} \zeta(2)^2 g_{*3}^2 g_{*5} \right]$

$+\mathbf{Te}^8 \left[-\frac{5}{3} \zeta(2) g_{*3}^2 g_{*5} \right] + \mathbf{Te}^{10} \left[g_{*11} \right] + \mathbf{Te}^2 \left[-144 \zeta(9) g_{*3} g_{*9} - 1008 \zeta(9) g_{*5} g_{*7} \right.$

$\left. -210 \zeta(9) g_{*3}^4 - 216 \zeta(3)^3 g_{*3}^4 + \frac{576}{5} \zeta(3) \zeta(2)^3 g_{*3}^4 \right]$

$+\mathbf{Te}^4 \left[\frac{18}{5} \zeta(3) \zeta(2)^2 g_{*3}^4 + 14 \zeta(7) g_{*3}^4 - 78 \zeta(7) g_{*3} g_{*9} - 210 \zeta(7) g_{*5} g_{*7} \right]$

$+\mathbf{Te}^6 \left[6 \zeta(2) \zeta(3) g_{*3}^4 - \frac{10}{3} \zeta(5) g_{*3}^4 - 28 \zeta(5) g_{*5} g_{*7} - 44 \zeta(5) g_{*3} g_{*9} \right]$

$+\mathbf{Te}^8 \left[-18 \zeta(3) g_{*3} g_{*9} \right] + \mathbf{Te}^2 \left[-168 \zeta(8,2) g_{*3}^2 g_{*7} - 280 \zeta(8,2) g_{*3} g_{*5}^2 \right.$

$-\frac{125056}{385} \zeta(2)^5 g_{*3} g_{*5}^2 - \frac{375168}{1925} \zeta(2)^5 g_{*3}^2 g_{*7} + 1760 \zeta(5)^2 g_{*3} g_{*5}^2$

$\left. +1056 \zeta(5)^2 g_{*3}^2 g_{*7} + 3360 \zeta(3) \zeta(7) g_{*3} g_{*5}^2 + 2016 \zeta(3) \zeta(7) g_{*3}^2 g_{*7} \right]$

$+\mathbf{Te}^4 \left[1080 \zeta(3) \zeta(5) g_{*3}^2 g_{*7} + \frac{23824}{175} \zeta(2)^4 g_{*3}^2 g_{*7} + 180 \zeta(6,2) * g_{*3}^2 g_{*7} \right.$

$\left. +\frac{6544}{525} \zeta(2)^4 g_{*3} g_{*5}^2 + 100 \zeta(6,2) g_{*3} g_{*5}^2 + 200 \zeta(3) \zeta(5) g_{*3} g_{*5}^2 \right]$

$+\mathbf{Te}^6 \left[-\frac{3064}{105} \zeta(2)^3 g_{*3}^2 g_{*7} - 140 \zeta(3)^2 g_{*3}^2 g_{*7} + \frac{88}{21} \zeta(2)^3 g_{*3} g_{*5}^2 \right]$

$+\mathbf{Te}^8 \left[\frac{8}{15} \zeta(2)^2 g_{*3} g_{*5}^2 - \frac{39}{5} \zeta(2)^2 g_{*3}^2 g_{*7} \right]$

$+\mathbf{Te}^{10} \left[-\frac{2}{3} \zeta(2) g_{*3} g_{*5}^2 - 4 \zeta(2) g_{*3}^2 g_{*7} \right) + \mathbf{Te}^{12} \left[g_{*13} \right]$

5.5 The invariants as entire functions of f: one-parameter cases.

Example 3: $\mathfrak{p}_*$ up to weight 12 for $f = l \circ g$ with $g_{(z)} = z + g_2 \, z^{-1}$.

$+\mathbf{Te}^1 \; g_2 + \mathbf{Te}^3 \left[\frac{1}{2}\right] g_2^2 + \mathbf{Te}^2 \left[3\,\zeta(3)\right] g_2^3 + \mathbf{Te}^5 \left[\frac{1}{2}\right] g_2^3 + \mathbf{Te}^2 \left[10\,\zeta(5)\right] g_2^4$

$+\mathbf{Te}^3 \left[-\frac{2}{3}\,\zeta(2)^2\right] g_2^4 + \mathbf{Te}^4 \left[\frac{9}{2}\,\zeta(3)\right] g_2^4 + \mathbf{Te}^5 \left[\frac{1}{3}\,\zeta(2)\right] g_2^4 + \mathbf{Te}^7 \left[\frac{7}{12}\right] g_2^4 + \mathbf{Te}^2 \left[\frac{77}{2}\,\zeta(7)\right] g_2^5$

$+\mathbf{Te}^3 \left[9\,\zeta(3)^2 - \frac{32}{15}\,\zeta(2)^3\right] g_2^5 + \mathbf{Te}^4 \left[16\,\zeta 5\right] g_2^5 + \mathbf{Te}^5 \left[-\frac{22}{15}\,\zeta(2)^2\right] g_2^5 + \mathbf{Te}^6 \left[\frac{15}{2}\,\zeta(3)\right] g_2^5$

$+\mathbf{Te}^7 \left[\frac{4}{3}\,\zeta(2)\right] g_2^5 + \mathbf{Te}^9 \left[\frac{2}{3}\right] g_2^5 + \mathbf{Te}^2 \left[151\,\zeta(9)\right] g_2^6 + \mathbf{Te}^3 \left[54\,\zeta(3)\,\zeta(5)\right] - \frac{14758}{2625}\,\zeta(2)^4$

$\qquad + 3\,\zeta(6,2)\right] g_2^6 + \mathbf{Te}^4 \left[\frac{271}{4}\,\zeta(7) - 6\,\zeta(3)\,\zeta(2)^2\right] g_2^6 + \mathbf{Te}^5 \left[27\,\zeta(3)^2 - \frac{1052}{175}\,\zeta(2)^3\right] g_2^6$

$+\mathbf{Te}^6 \left[\frac{55}{2}\,\zeta(5) + 5\,\zeta(2)\,\zeta(3)\right] g_2^6 + \mathbf{Te}^7 \left[-\frac{134}{75}\,\zeta(2)^2\right] g_2^6 + \mathbf{Te}^8 \left[\frac{193}{16}\,\zeta(3)\right] g_2^6$

$+\mathbf{Te}^9 \left[\frac{53}{15}\,\zeta(2)\right] g_2^6 + \mathbf{Te}^{11} \left[\frac{13}{20}\right] g_2^6$

Example 4: $\mathfrak{p}_*$ up to weight 12 for $f = l \circ g$ with $g(z) = z \left[1 + 2\,g_{*2}z^{-2}\right]^{\frac{1}{2}}$.

$+\mathbf{Te}^1 \left[g_{*2}\right] + \mathbf{Te}^2 \left[-\frac{5}{2}\,\zeta(5)\right] g_{*2}^4 + \mathbf{Te}^4 \left[\frac{1}{4}\,\zeta(3)\right] g_{*2}^4 + \mathbf{Te}^3 \left[\frac{3}{4}\,\zeta(3)^2\right] g_{*2}^5$

$+\mathbf{Te}^2 \left[\frac{3}{2}\,\zeta(3)^3 - \frac{4}{5}\,\zeta(3)\,\zeta(2)^3 + \frac{47}{6}\,\zeta(9)\right] g_{*2}^6 + \mathbf{Te}^4 \left[-\frac{21}{40}\,\zeta(3)\,\zeta(2)^2 - \frac{63}{64}\,\zeta(7)\right] g_{*2}^6$

$+\mathbf{Te}^6 \left[\frac{3}{8}\,\zeta(2)\,\zeta(3) + \frac{1}{16}\,\zeta(5)\right] g_{*2}^6 + \mathbf{Te}^8 \left[-\frac{1}{16}\,\zeta(3)\right] g_{*2}^6$

Example 5: $\mathfrak{p}_*$ up to weight 15 for $f = l \circ g$ with $g(z) = z\left[1 + 3\,g_{*3}z^{-3}\right]^{\frac{1}{3}}$.

$$+\mathbf{Te}^2\left[g_{*3}\right] + \mathbf{Te}^2\left[-6\,\zeta(3)\right]g_{*3}^2 + \mathbf{Te}^2\left[-\frac{32}{5}\,\zeta(2)^3 + 36\,\zeta(3)^2\right]g_{*3}^3 + \mathbf{Te}^4\left[-\frac{1}{5}\,\zeta(2)^2\right]g_{*3}^3$$

$$+\mathbf{Te}^6\left[-\frac{1}{3}\,\zeta(2)\right]g_{*3}^3 + \mathbf{Te}^2\left[-210\,\zeta(9) + \frac{576}{5}\,\zeta(3)\,\zeta(2)^3 - 216\,\zeta(3)^3\right]g_{*3}^4$$

$$+\mathbf{Te}^4\left[14\,\zeta(7) + \frac{18}{5}\,\zeta(3)\,\zeta(2)^2\right]g_{*3}^4 + \mathbf{Te}^6\left[6\,\zeta(2)\,\zeta(3) - \frac{10}{3}\,\zeta(5)\right]g_{*3}^4$$

$$+\mathbf{Te}^2\left[1960\,\zeta(7)\,\zeta(5) + 5880\,\zeta(3)\,\zeta(9) - \frac{6912}{5}\,\zeta(3)^2\,\zeta(2)^3 - \frac{23054144}{125125}\,\zeta(2)^6\right.$$
$$\left.+1296\,\zeta(3)^4 - 420\,\zeta(10,2)\right]g_{*3}^5$$

$$+\mathbf{Te}^4\left[-\frac{216}{5}\,\zeta(3)^2\,\zeta(2)^2 - 434\,\zeta(3)\,\zeta 7\right] + \frac{1332224}{28875}\,\zeta(2)^5 - 38\,\zeta(5)^2 - 49\,\zeta(8,2)\right]g_{*3}^5$$

$$+\mathbf{Te}^6\left[-72\,\zeta(2)\,\zeta(3)^2 + \frac{340}{3}\,\zeta(3)\,\zeta(5) + \frac{1007}{1575}\,\zeta(2)^4 - \frac{50}{3}\,\zeta(6,2)\right]g_{*3}^5$$

$$+\mathbf{Te}^8\left[\frac{193}{75}\,\zeta(2)^3\right]g_{*3}^5 + \mathbf{Te}^{10}\left[\frac{16}{15}\,\zeta(2)^2\right]g_{*3}^5 + \mathbf{Te}^{12}\left[\frac{7}{45}\,\zeta(2)\right]g_{*3}^5$$

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