The Natural Growth Scale.

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Abstract : The present paper starts with the group of all germs of analytic self-mappings of $\mathbb{R}_{,+\infty}$ and concerns itself with its successive closures under (i) fractional iteration (ii) conjugation (iii) the solving of general composition equations.

Rather than attempting a systematic treatment, we focus on the typical difficulties attendant upon these extensions. On the formal side, power series make way first for transseries, then for ultraseries, involving finite resp. transfinite iterates of the exponential. On the analysis side, the first casualties are convergence and analyticity: from the start, we have to face generic *resurgence* (multicritical but of a weakly polarising type) and, further down the road, generic *cohesiveness* (a natural and very inclusive extension of Denjoy quasi-analyticity).

Nevertheless, none of these complications destroys the bi-constructive correspondence between the formal objects (series, transseries, ultraseries) and the geometric germs. We describe, and illustrate on numerous examples, the apparatus required for upholding this correspondence: mainly *accelero-summation*, which uses convolution-respecting integral transforms to ascend from one critical Borel plane to the next, and the so-called *display*, a semi-algebraic construct that supplements the genuine variable with a host of *pseudo-variables* and encapsulates in highly convenient form all the information about the resurgence pattern and Stokes constants of a given germ.

We also devote three sections to the (non-linear) *iso-differential* operators which, on top of their surprising algebraic properties, are uniquely adapted to germ composition, the analysis of deep convexity, and the description of the *universal asymptotics* of very slow- or fast-growing germs.

Lastly, we reflect on the seemingly unsurmountable indeterminacy inherent in the choice of transfinite exponential iterates, and on the implications of that indeterminacy for the *natural growth scale* (- by which we mean, roughly

speaking, the *ultimate extension*[1] of our groups of non-oscillating germs -): far from being the quintessential continuum that one would expect, the natural growth scale – on the formal as on the analysis side, in the large as well as locally – displays a granular, almost fractal-like structure.[2]

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[1]better envisioned as a *horizon* than as a frozen object with sharp contours.

[2]A first draft of this paper was posted on our WEB page in January 2016. The present version carries minor revisions made in April 2018.

1 Program: exploring/completing the natural growth scale.

1.1 Groups of real germs. Successive extensions.

The present paper purports to investigate the various groups $\mathbb{G}^{\text{ext}}$ of one-dimensional real germ mappings (for technical convenience, near $+\infty$ rather than $+0$) that can be obtained by starting from some elementary germ group $\mathbb{G}$ and then imposing closure under the resolution of various types of composition equations or systems – mainly the following four types $\mathcal{T}_i$ of increasingly general equations (where f denotes the unknown):

$$f^{\circ q} = f_0^{\circ p} \qquad (p/q \in \mathbb{Q}) \quad (\textit{fractional iteration}) \qquad (\mathcal{T}_1)$$

$$f \circ f_1 = f_2 \circ f \qquad (\textit{conjugation}) \qquad (\mathcal{T}_2)$$

$$id = f^{\circ n_r} \circ f_r \circ \cdots \circ f^{\circ n_1} \circ f_1 \qquad (n_i \in \mathbb{N}) \quad (\textit{positive composition}) \qquad (\mathcal{T}_3)$$

$$id = f^{\circ n_r} \circ f_r \circ \cdots \circ f^{\circ n_1} \circ f_1 \qquad (n_i \in \mathbb{Z}) \quad (\textit{general composition}) \qquad (\mathcal{T}_4)$$

As it happens, neither the general shape of the ultimate group extensions $\mathbb{G}^{\text{ext}}$ nor the sort of difficulties to arise along the way, significantly depend on the initial group $\mathbb{G}$, whether that be the group $\mathbb{G}_{\text{ana}}$ of all invertible real analytic germs $f : x \mapsto c_0\, x + \sum c_n\, x^{1-n} (c_0 > 0)$ at $+\infty$, or the group $\mathbb{G} :=< T, E >$ generated by the unit shift $T := x \mapsto x + 1$ and the exponential $E := exp$, or even the group $\mathbb{G} :=< T >$ generated by the sole unit shift and its *twins* (as defined in §13)! All these constructions result in kindred groups $\mathbb{G}^{\text{ext}}$, each of which can serve as a fairly satisfactory model for what we may call the *natural growth scale*. On the other hand, complicating all these constructions but also providing for excitement and surprises, two main difficulties will keep arising: the omnipresence of *divergence* and the unavoidability of *very fast-growing germs*.[3]

1.2 Non-oscillation and comparability.

We shall be working in a setting completely 'free of oscillations', in the sense that our initial groups $\mathbb{G}$ as well as their extensions $\mathbb{G}^{\text{ext}}$ shall contain only pair-wise comparable germs. The corresponding (strict) order will systematically be noted $\lessdot$:

$$\{f \lessdot g\} \Longleftrightarrow \{f(x) < g(x)\ , \quad \textit{for } x \textit{ large enough}\} \qquad (1.1)$$

[3] and of course of the very slow-growing reciprocal germs.

Moreover, the groups themselves and their extensions are going to be pairwise compatible, in the sense that they will generate over-groups:

$$\mathbb{G}_1, \mathbb{G}_2 \;\mapsto\; <\mathbb{G}_1, \mathbb{G}_2> \;\mapsto\; <\mathbb{G}_1^{\text{ext}}, \mathbb{G}_2^{\text{ext}}> \;\subset\; <\mathbb{G}_1, \mathbb{G}_2>^{\text{ext}} \tag{1.2}$$

in which the order $\lessdot$ still holds.[4]

1.3 Divergence: resurgent or/and cohesive.

Divergence, in this context, can only be of two sorts, *resurgent* or *cohesive*, and it has the saving grace of being always *resummable*: it complicates but does not destroy the connexion between our germs f as geometric objects, and their formal counterparts $\widetilde{f}$ as power series - or series of a far more general nature[5].

Resurgence, whether mono- or polycritical (i.e. forcing us to go through one or several intermediary models to perform resummation), always results in germs f that are real-analytic on some tapering complex neighbourhood of $]...,+\infty[$. In resurgence's wake come the so-called *alien derivations* and, dual to them, the *pseudo-variables*, which together generate a rich, flexible, and very useful algebraic-analytic apparatus.

Cohesiveness[6], on the other hand, is closely related with the frequent occurrence of finitely (resp. transfinitely) iterated exponentials in the formal objects $\widetilde{f}$, which then assume the form of transseries (resp. ultraseries). These generalised series $\widetilde{f}$ converge absolutely[7] on some strictly real neighbourhood of $+\infty$, and their sums f belong to a remarkable class of quasi-analytic functions – the so-called 'cohesive' class *COHES*.

1.4 The 'display' and its many uses.

To each resurgent germ f there corresponds an object, noted *Dpl* f ('display' of f), that involves the variable proper, but also a huge number of so-called *pseudo-variables*. The 'display' has many uses. Firstly, it carries all the local information about f, including its Stokes contants or holomorphic invariants. Secondly, it is the key to a complete understanding of the relation (upset by resurgence) between the *formal* and the *geometric* side, i.e. between the

[4] As the sequel will make clear, this is only a small part of what we mean when saying that the *extensions do not depend too much on the initial groups.*

[5] namely, transseries or ultraseries – see below.

[6] cohesiveness *stricto sensu*, i.e. non-analytic cohesiveness.

[7] either *directly*, if they carry only convergent power series, or *indirectly*, after the divergent-resurgent power series they carry (tucked away within the exponential towers) have been separately resummed.

(trans-)series $\tilde{f}$ and the germs f. Thirdly, any relation $R(f_1, ..., f_n) = id$ between germs immediately extends to an identity $R(Dpl\, f_1, ..., Dpl\, f_n) = id$ between their displays, unchanged in outward form but implying a much stronger set of constraints. This is hugely useful for establishing all sorts of transcendence and independence theorems.

1.5 Extensions: exponential or ultra-exponential.

We cannot have stability under $\mathcal{T}_1, \mathcal{T}_2$ (see §1.1), let alone under $\mathcal{T}_3, \mathcal{T}_4$, without introducing very fast or slow growing germs. There are actually two steps here.
(i) In the first step, we are content with introducing finite iterates of the exponential and logarithm:

$$E_n := E^{\circ n} \ , \ L_n = L^{\circ n} \quad (E := \exp \ , \ L := \log) \tag{1.3}$$

On the formal side, this leads to so-called *transseries*, and on the geometric side to *analysable* germs.
(ii) The second step has us introduce even more exotic newcomers, namely the transfinite iterates E_α and L_α, with an iteration order α running through the semi-open transfinite interval $[\omega, \omega^\omega[$, where ω stands for the first inaccessible ordinal. It is in fact enough to define the *ultraexponentials* $\mathcal{E}_n := E_{\omega^n}$ and their reciprocals, the *ultralogarithms* $\mathcal{L}_n := L_{\omega^n}$. They are required to verify

$$\mathcal{E}_1 := E = \exp \quad ; \quad \mathcal{E}_n(x+1) \equiv \exp(\mathcal{E}_{n-1}(x)) \tag{1.4}$$

$$\mathcal{L}_1 := L = \log \quad ; \quad -1 + \mathcal{L}_n(x) \equiv \mathcal{L}_{n-1}(\log(x)) \tag{1.5}$$

These relations, though not fully determining $\mathcal{E}_n$ and $\mathcal{L}_n$, yet suffice to rigidly constrain their growth regimen.

Even on the formal side, this leads to serious complications. It forces us to consider so-called *ultraseries*, which, unlike the more manageable transseries, admit not one but several competing canonical forms (yet remain pairwise comparable).

While *transseries* suffice for most purposes of *non-oscillating asymptotics*, in particular in differential calculus, *ultraseries* cannot be avoided if we demand closure under all composition equations $\mathcal{T}_3, \mathcal{T}_4$.

1.6 Functional incarnation of transfinite arithmetics.

Having got hold of a system – any system – of ultraexponentials and ultralogarithms, we easily define the corresponding general transfinite iterates E_α and L_α ($\alpha < \omega^\omega$). We can then replace the slow-growing L_α by suitable equivalence classes $[L_\alpha]$ so defined as to remove the indeterminacy inherent in the construction of the ultralogarithms. Next, we find that composition naturally carries over to the classes $[L_\alpha]$, giving rise to a semi-group $[\mathbb{L}]$, with transfinite iteration itself smoothly extending to $[\mathbb{L}]$. As it turns out, this double structure on $[\mathbb{L}]$ exactly reflects the semi-ring structure of the transfinite interval $[1, \omega^\omega[$, with its non-commutative addition, non-commutative multiplication, and semi-distributivity.[8]

1.7 Iso-differential operators and convexity.

We shall require a special class of operators, the so-called *iso-differential* operators $\mathrm{Dn}^{\{\boldsymbol{n}\}}$:

$$\mathrm{Dn}^{\{n_1,\dots,n_r\}} f := \prod_i (\mathrm{Dn}^{\{n_i\}} f) \quad \textit{with} \quad \mathrm{Dn}^{\{n_i\}} f := (-1)^{n_i}\, \partial^{n_i} \log(1/f') \tag{1.6}$$

They are indexed by *non-ordered sequences* of positive integers $\{\boldsymbol{n}\}$ and span a bialgebra *ISO* which is far better suited to germ composition and to the description of fast/slow germs than the larger bialgebra *DIFF* spanned by the ordinary differential operators $\mathrm{D}^{\{\boldsymbol{n}\}}$:

$$\mathrm{D}^{\{n_1,\dots,n_r\}} f := \prod_i f^{(n_i)} \qquad (n_i \in \mathbb{N}*) \tag{1.7}$$

DIFF and *ISO* both possess non-cocommutative co-products, respectively σ and χ, that reflect their action on germ composition $\circ$. They also possess (quite distinct) commutative products, respectively $\bullet$ and $\times$. The bialgebra *ISO* owes its name to the fact that its operators[9] have a double homogeneousness, measured by an *'isodegree'* $|\boldsymbol{n}| := \sum n_i$ simultaneously stable under σ and $\times$.

It is also useful to embed *ISO* into a vaster bialgebra $^\sharp$*ISO* spanned by operators $\mathbb{D}\mathrm{e}^{\langle \boldsymbol{n}\rangle}$ which are no longer strictly differential and whose indices $<\boldsymbol{n}>$ are now *ordered* integer sequences. On $^\sharp$*ISO*, both product and co-product assume much simpler expressions. Moreover, *ISO* and $^\sharp$*ISO* possess, as co-algebras, positive cones ISO^+ and $^\sharp ISO^+$ with bases $\mathrm{Da}^{\{\boldsymbol{n}\}}$ and $\mathbb{D}\mathrm{a}^{\langle \boldsymbol{n}\rangle}$ rich in unexpected algebraic-combinatorial properties and leading to a new notion of *iso-convexity* better adapted to germ compostion than ordinary convexity. To sum up, we have these four structures:

$$ISO \;\subset\; {}^\sharp ISO \quad ; \quad ISO^+ \;\subset\; {}^\sharp ISO^+ \tag{1.8}$$

[8]Thus, a logical-mathematical structure, which when first introduced met with fierce resistance on account of its supposedly ethereal character, reveals itself to be isomorphic to a very natural structure, firmly anchored in concrete, down-to-earth analysis.

[9]unlike those of *DIFF*.

1.8 Universal asymptotics of fast/slow germs.

With their natural adequation to germ composition, the iso-differential operators enlarge the circle of operations and equations at our disposal for carrying out group extensions $\mathbb{G} \mapsto \mathbb{G}^{\text{ext}}$. But their main utility lies in this: any iso-differential operator $\mathcal{D}$ acting on any ultra-slow germ $\mathcal{L}$ (say, on any transfinite iterate of L) produces a germ $\mathcal{D}.\mathcal{L}$ whose natural asymptotic expansion depends on $\mathcal{D}$ alone, not on $\mathcal{L}$.[10]

1.9 Stubborn indeterminacy in the realisation of ultra-exponentials.

The system (1.4)-(1.5) determines each pair $(\mathcal{L}_n, \mathcal{E}_n)$ in terms of $(\mathcal{L}_{n-1}, \mathcal{E}_{n-1})$, but only up to pre- resp. post-composition by a 1-periodic germ P.[11] That, plus the fact, just mentioned, of all ultra slow/fast germs sharing a universal asymptotics, dashes all hope of selecting a privileged solution $(\mathcal{L}_n, \mathcal{E}_n)$ based purely on real-asymptotic criteria. On the other hand, the possibility, however remote, cannot be dismissed off hand that *one* of these systems might possess extensions to the complex domain so regular or so distinctive as to single it (that system) out as clearly 'optimal'. To further complicate the picture, we shall find that all the 'reasonable' candidates for the first non-elementary pair $(\mathcal{L}_1, \mathcal{E}_1)$[12] are extremely close to one another. So the question is still open, and likely to remain so for quite a while.

1.10 Spirit of this paper: exploratory rather than systematic.

The present investigation is unapologetically exploratory in spirit and method. We isolate each of the main difficulties, describe in detail the methods for overcoming them (they involve a lot of fancy machinery), outline the unexpected features (there are quite a few of them), and illustrate everything on a series of select examples. But we do not attempt an exhaustive description of all possible extensions $\mathbb{G}^{\text{ext}}$ of all possible germ groups $\mathbb{G}$, especially where so doing would force us to grapple with the most general transseries or ultraseries. One excuse for this caution or restraint is that we are handling here an inflatable subject-matter and venturing into almost limitless territory, where *exhaustive* all too easily rhymes with *exhausting*, and *thorough*

[10] It is only the trans-asymptotic part of $\mathcal{D}.\mathcal{L}$ that depends on $\mathcal{L}$.

[11] More precisely, a germ P that commutes with the unit shift T.

[12] derived from the pair $(\mathcal{L}_0, \mathcal{E}_0) = (L, E)$.

implies *unreadable*[13].

But there is another reason, which is the danger of *diminishing returns*. Indeed, the extensions $\mathbb{G}^{\text{ext}}$ that we get by imposing full closure under $\mathcal{T}_1$-$\mathcal{T}_4$ (and under iso-differential equations for good measure), though huge, are also in a sense *sparsely populated*. They do not seem, for the moment at least, to contain all that many *native* germs of intrinsic interest, by which we mean remarkable germs arising *naturally and directly* within the new framework, as opposed to germs obtained by solving composition equations with external, pre-extension data.

To put it bluntly: these extensions, though huge, have a wasteland quality about them. They exhibit *low biodiversity*, compared with, say, classical complex analysis with its wealth of 'special functions'. This applies in particular to the rarefied ultra-exponential range, which would be hardest and most unrewarding to map out down to the last details and which for that reason shall receive here only a sketchy treatment.

2 Tools: resurgence, acceleration, cohesiveness, analysability.

This section presents - mainly for perspective and to settle notations - a very cursory survey of resurgence theory and its basic tools.

2.1 Resurgent functions. The three models.

Resurgent 'functions' live simultaneously in three models:
(i) in the *formal model*, as formal power series $\widetilde{\varphi}(z)$ or series of a more general type (here the tilda always stands for 'formal'),
(ii) in the *convolutive model*, as analytic germs $\widehat{\varphi}(\zeta)$ defined near the origin $0_\bullet$ of $\mathbb{C}_\bullet := \widetilde{\mathbb{C} - \{0\}}$; admitting an endless analytic continuation (usually highly ramified) laterally along any finite, finitely punctured broken line; possessing at most a discrete configuration of singular points ω; and growing at most exponentially when ζ goes to ∞ radially or ultimately radially,[14]
(iii) in the *geometric model(s)*, as analytic germs $\varphi_\theta(z)$ defined in certain sectorial neighbourhoods $|\arg(z^{-1}) - \theta| < \epsilon + \pi/2$ and admitting there $\widetilde{\varphi}(z)$

[13] Cf Voltaire: *"The secret of being a bore is to tell everything"*.

[14] i.e. following a broken line whose last segment is infinite.

as asymptotic series.

$$\begin{array}{llllll} & \widetilde{\varphi}(z) & \quad\cdots\quad & \varphi_\theta(z) & & z\text{-}plane\ (multiplication) \\ \text{Fig. 2.1} & \mathcal{B}\ \searrow & & \nearrow\ \mathcal{L}_\theta & & \\ & & \widehat{\varphi}(\zeta) & & & \zeta\text{-}plane\ \ (convolution) \end{array}$$

Despite its auxiliary character, the *convolutive model* or *'Borel plane'*[15] is where most obstacles to resummation assume tangible form in the shape of singular points ω ultimately responsible for the divergence of $\widetilde{\varphi}(z)$, and where these obstacles can be overcome. The product there is the finite-path *convolution* (2.1), unambiguously defined for small values of ζ, and then extended in the large by analytic continuation:

$$(\widehat{\varphi}_1 * \widehat{\varphi}_2)(\zeta) := \int_0^\zeta \widehat{\varphi}_1(\zeta_1) * \widehat{\varphi}_2(\zeta-\zeta_1)\, d\zeta_1 \tag{2.1}$$

Together, the *formal model* (our starting point) and the *geometric models* (our end goal) constitute the *multiplicative models*, where the product is ordinary multiplication. We go from one model to the next via algebra homomorphisms.

The first of these is the Borel transform $\mathcal{B}$. It acts term-wise and turns any power series $\widetilde{\varphi}(z)$ with coefficient growth of type Gevrey 1 into a power series $\widehat{\varphi}(\zeta)$ with non-zero radius of convergence

$$\mathcal{B}\ :\quad z^{-\sigma} \mapsto \zeta^{\sigma-1}/\Gamma(\sigma) \qquad\qquad (\sigma \notin -\mathbb{N}) \tag{2.2}$$

$$\mathcal{B}\ :\quad z^{n} \mapsto \delta^{(n)} \qquad\qquad (n \in \mathbb{N}\,,\ \delta = \mathit{Dirac}) \tag{2.3}$$

$$\mathcal{B}\ :\ \widetilde{\varphi}(z) = \sum a_n\, z^{-n} \quad\mapsto\quad \widehat{\varphi}(\zeta) = \sum a_n\, \zeta^{n-1}/(n-1)! \tag{2.4}$$

The second transform is the Laplace transform $\mathcal{L}$ or, for distinctiveness, $\mathcal{L}_\theta$:

$$\mathcal{L}_\theta :\ \ \widehat{\varphi}(\zeta) \quad\mapsto\quad \varphi_\theta(z) = \int_0^{e^{i\theta}\infty} \widehat{\varphi}(\zeta)\, e^{-z\,\zeta}\, d\zeta \qquad (\arg\zeta \equiv \theta) \tag{2.5}$$

Here are some elementary identities for future use:

$$\mathcal{B} : \widetilde{\varphi}_1.\widetilde{\varphi}_2 \mapsto \widehat{\varphi}_1 * \widehat{\varphi}_2 \tag{2.6}$$

$$\mathcal{B} : \partial\, \widetilde{\varphi}(z) \mapsto \widehat{\partial}\, \widehat{\varphi}(\zeta) := -\zeta\, \widehat{\varphi}(\zeta) \qquad (\partial := \mathrm{d/dz}) \tag{2.7}$$

$$\mathcal{B} : \psi(z) = \varphi(z+\gamma) \Rightarrow \widehat{\varphi}(\zeta) = \exp(-\gamma\,\zeta)\, \widehat{\psi}(\zeta) \tag{2.8}$$

$$\mathcal{B} : \psi(z) = (\varphi\circ h)(z) \Rightarrow \widehat{\psi}(\zeta) = (\widehat{\psi}\,\widehat{\circ}\,\widehat{h})(\zeta) := \widehat{\psi}(\zeta) + \sum \widehat{\hbar}^{*n}(\zeta) * \frac{(-\zeta)^n}{n!}\widehat{\psi}(\zeta) \tag{2.9}$$

This last identity (2.9) can be resorted to each time we must *post-compose* something by $h = id + \hbar$ with $\hbar(z) = o(1)$.

[15]'plane' is here something of a misnomer, since the functions $\widehat{\varphi}(\zeta)$ usually live on highly ramified Riemann surfaces *over* the 'Borel plane', or *over* a finite sector $|\arg(\zeta - \theta_0)| < \delta\theta$, or even *over* the positive real axis $\mathbb{R}^+$.

Minors and majors.

The convolution integral (2.1) makes sense only if each factor $\widehat{\varphi}_i(\zeta)$ is radially integrable at $0_\bullet$. When this is not the case, the germs $\widehat{\varphi}(\zeta)$ – the so-called *minors* – have to be supplemented by companion germs, the so-called *majors*, which are defined only modulo the space REG of regular germs at $0_\bullet$. They relate to the minors according to the formula:

$$\widehat{\varphi}(\zeta) = -\frac{1}{2\pi i}\left(\check{\varphi}(e^{\pi i}\zeta) - \check{\varphi}(e^{-\pi i}\zeta)\right) \qquad (\zeta \ near \ 0_\bullet) \tag{2.10}$$

Major convolution (compatible with *minor convolution* but of wider scope) is given by the rule:

$$(\check{\varphi}_1 *_u \check{\varphi}_2)(\zeta) = \frac{1}{2\pi i}\int_{\mathcal{I}(\zeta,u)} \check{\varphi}_1(\zeta_1)\,\check{\varphi}_2(\zeta-\zeta_1)\,d\zeta_1 \tag{2.11}$$

$$with \ \ \mathcal{I}(\zeta,u) = \left[\frac{1}{2}\zeta + e^{-\frac{\pi i}{2}}\,u\ ,\ \frac{1}{2}\zeta + e^{+\frac{\pi i}{2}}\,u\right] \qquad (0<\zeta<u\prec 1)$$

The definition makes good sense, since the small path $\mathcal{I}(\zeta,u)$ keeps clear of $0_\bullet$ and since, modulo REG, the integral on the left-hand side of (2.11) does not depend on the choice of u.

2.2 Convolution preserving averages.

Whenever the axis $\arg\zeta = \theta$ of Laplace integration carries singularities, the multivalued integrand $\widehat{\varphi}(\zeta)$ must be replaced by a univalued average $\mu\,\widehat{\varphi}(\zeta)$, so that the resummation scheme of Fig. 2.1 becomes:

$$\widetilde{\varphi}(z) \ \stackrel{\mathcal{B}}{\longrightarrow}\ \widehat{\varphi}(\zeta) \ \stackrel{\mu}{\longrightarrow}\ \mu\widehat{\varphi}(\zeta) \ \stackrel{\mathcal{L}}{\longrightarrow}\ \varphi(z) \tag{2.12}$$

Such an average $\mu : \widehat{\varphi} \mapsto \mu\,\widehat{\varphi}$ is defined via its weights $\mu^{\binom{\epsilon}{\omega}}$:

$$\mu\,\widehat{\varphi}(\zeta) := \sum_{\epsilon_i\in\{+,-\}} \mu^{\binom{\epsilon_1\ ,\dots,\ \epsilon_r}{\omega_1\ ,\dots,\ \omega_r}}\,\widehat{\varphi}^{\binom{\epsilon_1\ ,\dots,\ \epsilon_r}{\omega_1\ ,\dots,\ \omega_r}}(\zeta) \quad if \quad \omega_r<\zeta<\omega_{r+1} \tag{2.13}$$

where $\omega_1, \omega_2 \dots$ are the successive singular points on $\arg\zeta = \theta$ and where $\widehat{\varphi}^{\binom{\epsilon_1\ ,\dots,\ \epsilon_r}{\omega_1\ ,\dots,\ \omega_r}}(\zeta)$ denotes *the* determination of $\widehat{\varphi}(\zeta)$ on the interval $]\omega_r, \omega_{r+1}[$ that corresponds to the right (resp. left) circumvention of ω_i if $\epsilon_i = +$ (resp. $\epsilon_i = -$) starting from the origin. Crucially, the average must respect convolution

$$\mu(\widehat{\varphi}_1 * \widehat{\varphi}_2) \equiv (\mu\widehat{\varphi}_1) * (\mu\widehat{\varphi}_2) \qquad (\textit{first} * \textit{local}\,,\ \textit{second} * \textit{global}) \tag{2.14}$$

Although the above requirement imposes stringent algebraic constraints on the weights $\mu^{\binom{\epsilon}{\omega}}$, there is still a whole zoo of such averages. Let us mention only the most useful.

The trivial lateral averages.

The right average μ_+ and left average μ_- involve only one determination:

$$\mu_\pm^{\left({\epsilon_1 \atop \omega_1}{,\ldots, \atop ,\ldots,}{\epsilon_r \atop \omega_r}\right)} = 1 \quad (\textit{resp.}\, 0) \quad \textit{if} \quad \epsilon_1 = \cdots = \epsilon_r = \pm \qquad (\textit{resp. otherwise}) \tag{2.15}$$

These elementary 'averages' have simplicity going for them, but they fail to respect *realness*: when $\theta = 0$ and $\widetilde{\varphi}(z)$ is real, $\mu_+\,\widehat{\varphi}(\zeta)$ and $\mu_-\,\widehat{\varphi}(\zeta)$ are not, except in the trivial case when $\widehat{\varphi}(\zeta)$ is regular and uniform on $\mathbb{R}^+$.

The 'standard' average.

Its weights, solely dependent on the ϵ_i's, are given by the direct formula:

$$\mu^{\left({\epsilon_1 \atop \omega_1}{,\ldots, \atop ,\ldots,}{\epsilon_r \atop \omega_r}\right)} := \frac{\Gamma(p+\frac{1}{2})\,\Gamma(q+\frac{1}{2})}{\Gamma(r+1)\,\Gamma(\frac{1}{2})\,\Gamma(\frac{1}{2})} = \frac{(2\,p)!\,(2\,q)!}{4^{p+q}\,p!\,q!\,(p+q)!} \tag{2.16}$$

$$\textit{with} \qquad p := \sum_{\epsilon_i=+} 1 \;\;,\;\; q := \sum_{\epsilon_i=-} 1 \qquad (p+q=r) \tag{2.17}$$

The 'organic' average.

Its weights are given by the inductive formula:

$$\mu^{\left({\epsilon_1 \atop \omega_1}{,\ldots, \atop ,\ldots,}{\epsilon_r \atop \omega_r}\right)} := \mu^{\left({\epsilon_1 \atop \omega_1}{,\ldots, \atop ,\ldots,}{\epsilon_{r-1} \atop \omega_{r-1}}\right)}\,\frac{1}{2}\,\Big(1+\epsilon_{r-1}\,\epsilon_r\,\frac{\omega_{r-1}}{\omega_r}\Big) \tag{2.18}$$

$$\textit{with} \qquad \mu^{\left({+ \atop \omega_1}\right)} := \mu^{\left({- \atop \omega_1}\right)} := \frac{1}{2} \tag{2.19}$$

The 'standard' and 'organic' averages both respect convolution and realness. The simpler standard average is sufficient for most intents and purposes, but in some (fairly rare) cases one must resort to the organic average (or to any one of a host of so-called *well-behaved* averages) in order to get a function $\mu.\widehat{\varphi}(\zeta)$ that *does not grow faster* than the lateral determinations $\mu_\pm.\widehat{\varphi}(\zeta)$.

2.3 Alien derivations.

To capture the always important, and often remarkable, behaviour of $\widehat{\varphi}(\zeta)$ near[16] its singular points ω, we require a system of linear operators $\widehat{\Delta}_\omega$ carrying indices $\omega \in \mathbb{C}_\bullet$, behaving à la Leibniz with respect to convolution

$$\widehat{\Delta}_\omega\,(\widehat{\varphi}_1 * \widehat{\varphi}_2) \equiv (\widehat{\Delta}_\omega\,\widehat{\varphi}_1) * \widehat{\varphi}_2 + \widehat{\varphi}_1 * (\widehat{\Delta}_\omega\,\widehat{\varphi}_2) \tag{2.20}$$

[16] or, due to multivaluedness, *above* ω.

and yielding 0 whenever the test function $\widehat{\varphi}(\zeta)$ has no singularities above ω. The action of these Δ-operators, known as *alien derivations*, is given[17] by a formula reminiscent of (2.13):

$$\widehat{\Delta}_\omega\, \widehat{\varphi}\,(\zeta) := \sum_{\epsilon_i=\pm} \frac{\epsilon_r}{2\pi i}\, \delta^{\binom{\epsilon_1\ ,\dots,\ \epsilon_r}{\omega_1\ ,\dots,\ \omega_r}}\, \widehat{\varphi}^{\binom{\epsilon_1\ ,\dots,\ \epsilon_r}{\omega_1\ ,\dots,\ \omega_r}}(\zeta+\omega) \qquad (\omega_r := \omega) \tag{2.21}$$

with the weights $\delta^{\binom{\epsilon}{\omega}}$ subject to strong algebraic constraints in order to ensure (2.20). Here are the main systems of alien derivations :

The 'standard' alien derivations.

Their weights depend only on the sign sequence $(\epsilon_1, \dots, \epsilon_{r-1})$:

$$\delta^{\binom{\epsilon_1\ ,\dots,\ \epsilon_r}{\omega_1\ ,\dots,\ \omega_r}} := \frac{p!\, q!}{(p{+}q{+}1)!} \quad \textit{with} \quad p := \sum_{\epsilon_i=+}^{1\leqslant i\leqslant r-1} 1 \ ;\ p := \sum_{\epsilon_i=-}^{1\leqslant i\leqslant r-1} 1 \tag{2.22}$$

The 'organic' alien derivations.

Their weights are given by:

$$\delta^{\binom{\epsilon_1\ ,\dots,\ \epsilon_r}{\omega_1\ ,\dots,\ \omega_r}} := \begin{cases} (\omega_{p+1}-\omega_p)/(2\,\omega_r) & \textit{if}\ \ (\epsilon_1,\dots,\epsilon_r) = ((+)^p, (-)^q, \epsilon_r) \\ (\omega_{q+1}-\omega_q)/(2\,\omega_r) & \textit{if}\ \ (\epsilon_1,\dots,\epsilon_r) = ((-)^q, (+)^p, \epsilon_r) \\ 0 & \textit{otherwise} \end{cases}$$

Alien derivations in the multiplicative models.

For use in the multiplicative models, we set :

$$\Delta_\omega := \mathcal{B}^{-1}\, \widehat{\Delta}_\omega\, \mathcal{B} \qquad (\textit{formal model}) \tag{2.23}$$

$$\Delta_\omega := \mathcal{L}_\theta\, \widehat{\Delta}_\omega\, \mathcal{L}_\theta^{-1} \qquad (\textit{geometric models}) \tag{2.24}$$

$$\boldsymbol{\Delta}_\omega := e^{-\omega\, z}\, \Delta_\omega \qquad (\textit{formal and geometric models}) \tag{2.25}$$

The Leibniz rule now looks even more Leibnizian : [18]

$$\Delta_\omega\,(\varphi_1\,.\,\varphi_2) \equiv (\Delta_\omega\,\varphi_1).\varphi_2 + \varphi_1.(\Delta_\omega\,\varphi_2) \qquad (z\textit{-plane}) \tag{2.26}$$

$$\boldsymbol{\Delta}_\omega\,(\varphi_1\,.\,\varphi_2) \equiv (\boldsymbol{\Delta}_\omega\,\varphi_1).\varphi_2 + \varphi_1.(\boldsymbol{\Delta}_\omega\,\varphi_2) \qquad (z\textit{-plane}) \tag{2.27}$$

Thanks to their exponential factor $e^{-\omega z}$, the 'bold-face' or 'invariant' operators $\boldsymbol{\Delta}_\omega$ have the great advantage of commuting with the ordinary z-differentiation $\partial := \partial_z$

$$[\widehat{\Delta}_\omega, \widehat{\partial}] = -\omega\, \widehat{\Delta}_\omega \implies [\Delta_\omega, \partial] = -\omega\, \Delta_\omega \implies [\boldsymbol{\Delta}_\omega, \partial] = 0 \tag{2.28}$$

[17]first for small ζ on the axis $\arg\zeta = \arg\omega$, then in the large by analytic continuation.
[18]We drop the tilda or the polarisation angle θ.

and of behaving optimally simply ('invariantly') under post-composition by an identity tangent g:

$$\mathbf{\Delta}_\omega (f \circ g) \equiv (\mathbf{\Delta}_\omega f) \circ g + (\partial f) \circ g \,.\, (\mathbf{\Delta} g) \qquad \textit{if} \qquad g(z) \sim z \tag{2.29}$$

2.4 Pseudovariables and displays.

Pseudovariables $\mathbf{Z}^{\boldsymbol{\omega}} = \mathbf{Z}^{\omega_1,\dots,\omega_r}$.

The notion of pseudovariable is dual to that of alien derivation (of the bold-face or invariant sort). Pseudovarialbles carry as upper indices sequences $\boldsymbol{\omega} := (\omega_1, ..., \omega_r)$ of arbitrary length r. Multiplication for them reduces to sequence shuffling, while differentiation (ordinary or alien) and post-composition obey the predictable rules:

$$\mathbf{Z}^{\boldsymbol{\omega}'} . \mathbf{Z}^{\boldsymbol{\omega}''} = \sum \mathbf{Z}^{\boldsymbol{\omega}} \qquad \big(\boldsymbol{\omega} \in \text{shuffle}(\boldsymbol{\omega}', \boldsymbol{\omega}'')\big) \tag{2.30}$$

$$\partial_z \mathbf{Z}^{\omega_1,\dots,\omega_r} = 0 \tag{2.31}$$

$$\mathbf{\Delta}_{\omega_0} \mathbf{Z}^{\omega_1,\dots,\omega_r} = \begin{cases} \mathbf{Z}^{\omega_2,\dots,\omega_r} & \textit{if} \;\; \omega_0 = \omega_1 \\ 0 & \textit{if} \;\; \omega_0 \neq \omega_1 \end{cases} \tag{2.32}$$

$$\mathbf{Z}^{\boldsymbol{\omega}} \circ g = \mathbf{Z}^{\boldsymbol{\omega}} \qquad \textit{if} \quad g(z) = z + o(z) \tag{2.33}$$

The display.

The display is best thought of as some sort of 'alien Taylor expansion':

$$\text{Dpl}\, \varphi := \varphi + \sum_r \sum_{\omega_j} \mathbf{Z}^{\omega_1,\dots,\omega_r} \, \mathbf{\Delta}_{\omega_r} \dots \mathbf{\Delta}_{\omega_1} \varphi \tag{2.34}$$

It has a double character − both local (via its z-dependence) and global (via its $\mathbf{Z}$-dependence). It encodes, in ultra-compact and user-friendly form, a huge amount of information about the function $\widehat{\varphi}(\zeta)$, describing as it does the behaviour of $\widehat{\varphi}(\zeta)$ at each ω and on each of its various Riemann sheets. What is more, any relation R between resurgent functions automatically extends to their displays :

$$\{\, R(\varphi_1, \dots, \varphi_s) \equiv 0 \,\} \implies \{\, R(\text{Dpl}\, \varphi_1, \dots, \text{Dpl}\, \varphi_s) \equiv 0 \,\} \tag{2.35}$$

which is fantastically helpful for establishing transcendence or independence results.

2.5 Multicritical resurgence and accelero-summation.

When the full formal solution of a local analytic equation or system (say, a singular ODE) involves, alongside the familiar power series of z^{-1}, a mixture of several non comparable exponential blocks, for instance blocks of the form $u_i\, e^{\sigma_{ij}\, z_j}$ with

$$z_1 := h_1(z) \prec z_2 := h_2(z) \prec \cdots \prec z_r := h_r(z) \quad (e.g.\ \ z_j \equiv z^{\alpha_j}\ \textit{with}\ 0 < \alpha_j \uparrow)$$

one is usually confronted with *multi-critical resurgence.* Concretely, this means that, instead of applying the simple, mono-critical resummation scheme (2.12), one must go successively through a number of distinct Borel planes ζ_j – as many as there are distinct *'critical times'* z_j. The intermediary functions $\widehat{\varphi}_j(\zeta_j)$ generically possess faster than exponential growth at ∞ and each transition $\widehat{\varphi}_j(\zeta_j) \to \widehat{\varphi}_{j+1}(\zeta_{j+1})$ is via a so-called *acceleration transform* $\mathcal{C}_{j,j+1}$. These two complications aside, the situation in each ζ_j-plane remains much the same as in the mono-critical case : on each intermediary function $\widehat{\varphi}_j(\zeta_j)$ there act specific alien derivations, generating their own resurgence equations and contributing their own Stokes constants. The overall scheme reads:

$$\begin{array}{ccccccccc}
\widetilde{\varphi}_1(z_1) & \leftarrow & \widetilde{\varphi}(z) & & \varphi(z) & \leftarrow & \varphi_r(z_r) \\
\downarrow \mathcal{B} & & & & & & \mathcal{L}\uparrow \\
\widehat{\varphi}_1(\zeta_1) & \underset{\mathcal{C}_{1,2}}{\rightarrow} & \widehat{\varphi}_2(\zeta_2) & \underset{\mathcal{C}_{2,3}}{\rightarrow \cdots \rightarrow} & \widehat{\varphi}_{r-1}(\zeta_{r-1}) & \underset{\mathcal{C}_{r-1,r}}{\rightarrow} & \widehat{\varphi}_r(\zeta_r)
\end{array} \qquad \text{Fig. 2.2}$$

2.6 Acceleration and deceleration transforms.

A single pair C_F , C^F of integral kernels does service for the four combinations of minor/major, ac/decelerations, but with a characteristic diagonal 'flip':

$$\text{Fig.2.3} \quad \begin{bmatrix} & \textit{acceleration} & \textit{deceleration} \\ \textit{minor} & C_F & C^F \\ \textit{major} & C^F & C_F \end{bmatrix} \Big(z_1 \prec z_2\ ,\ z_1 = F(z_2) \Big)$$

These kernels depend on the germ F that expresses the *slower* 'time' z_1 in terms of the *faster* one z_2.

$$C_F(\zeta_2, \zeta_1) \ := \ \frac{1}{2\pi i} \int_{c-i\infty}^{c+i\infty} e^{z_2\zeta_2 - z_1\zeta_1}\, dz_2 \quad \textit{with}\ \ z_1 \equiv F(z_2) \qquad (2.36)$$

$$C^F(\zeta_2, \zeta_1) \ := \ \int_{+u}^{+\infty} e^{-z_2\zeta_2 + z_1\zeta_1}\, dz_2 \quad \textit{with}\ \ z_1 \equiv F(z_2)\ \textit{and}\ 1 \prec u \qquad (2.37)$$

Acceleration from ζ_1 to ζ_2 with $z_1 = F(z_2)$ and $1 \prec F(x) \prec x$:

$$\widehat{\varphi}_2(\zeta_2) = \int_{+0}^{+\infty} C_F(\zeta_2,\zeta_1)\, \widehat{\varphi}_1(\zeta_1)\, d\zeta_1 \tag{2.38}$$

$$\check{\varphi}_2(\zeta_2) = \frac{1}{2\pi i}\int_{c-i\infty}^{c+i\infty} C^F(\zeta_2,\zeta_1)\, \check{\varphi}_1(\zeta_1)\, d\zeta_1 \tag{2.39}$$

Deceleration from ζ_2 to ζ_1 with $z_1 = F(z_2)$ and $1 \prec F(x) \prec x$:

$$\zeta_1\, \widehat{\varphi}_1(\zeta_1) = \frac{1}{2\pi i}\int_{0_1}^{0_2} \zeta_2\, \widehat{\varphi}_2(\zeta_2)\, C^F(\zeta_2,\zeta_1)\, d\zeta_2 \quad (u > 0) \tag{2.40}$$

$$\zeta_1\, \check{\varphi}_1(\zeta_1) = \int_{+0}^{+v} \zeta_2\, \check{\varphi}_2(\zeta_2)\, C_F(\zeta_2,\zeta_1)\, d\zeta_2 \tag{2.41}$$

Here again, we notice a flip of finite/infinite, path/loop integrals. Integration in (2.38) is along an infinite path, in (2.41) along a finite one. Integration in (2.39) is along an infinite loop that encircles 0 anticlockwise, in (2.40) along a finite loop from 0 to 0 that encircles $\zeta_1 > 0$ anticlockwise.

But the basic, really useful transform is of course *minor acceleration* (2.38), and the crucial point to note here is that the lower kernel $C_F(\zeta_2,\zeta_1)$ has exactly the right faster-than-exponential rate of decrease (as $\zeta_1 \to +\infty$) to make the acceleration integral (2.38) convergent for small enough values of $\zeta_2 > 0$. This defines a germ $\widehat{\varphi}_2(\zeta_2)$ which then must, and can, be continued in the large, *over* the whole of $\mathbb{R}^+$.

2.7 Pseudo-acceleration and -deceleration transforms.

Here, the change is between two *equivalent* 'times', denoted for distinction by $z_{1\_}$ and z_1 with $z_1 = z_{1\_} + F(z_{1\_})$ and $1 \prec F(x) \prec x$ as above.[19] The new transforms serve a totally different purpose that will be made clear in §2.8, but their integral kernels C_{id+F} , C^{id+F} are closely related to the old ones:

$$C_{id+F}(\zeta_{1\_},\zeta_1) = C_F(\zeta_{1\_} - \zeta_1,\zeta_1) \tag{2.42}$$

$$C^{id+F}(\zeta_{1\_},\zeta_1) = C^F(\zeta_{1\_} - \zeta_1,\zeta_1) \tag{2.43}$$

In keeping with the more elementary character of the new transforms, all integration paths/loops now become finite.

Pseudodeceleration from ζ_1 to $\zeta_{1\_}$ with $z_1 = (id+F)(z_{1\_})$:

$$\widehat{\varphi}_{1\_}(\zeta_{1\_}) = \int_{+0}^{\zeta_1} C_{id+F}(\zeta_{1\_},\zeta_1)\, \widehat{\varphi}_1(\zeta_1)\, d\zeta_1 \tag{2.44}$$

$$\check{\varphi}_{1\_}(\zeta_{1\_}) = \frac{1}{2\pi i}\int_{v_1}^{v_2} C^{id+F}(\zeta_{1\_},\zeta_1)\, \check{\varphi}_1(\zeta_1)\, d\zeta_1 \tag{2.45}$$

[19]The case when $z_{1\_}$ and z_1 are too close, i.e. when $F(x) = o(1)$, is uninteresting.

Pseudoacceleration from ζ_{1_-} to ζ_1 with $z_1 = (id+F)(z_{1_-})$:

$$\zeta_1\,\widehat{\varphi}_1(\zeta_1) \;=\; \frac{1}{2\pi i}\int_{0_1}^{0_2} \zeta_{1_-}\,\widehat{\varphi}_{1_-}(\zeta_{1_-})\,C^{id+F}(\zeta_{1_-},\zeta_1)\,d\zeta_{1_-} \tag{2.46}$$

$$\zeta_1\,\check{\varphi}_1(\zeta_1) \;=\; \int_{\zeta_0}^{v} \zeta_{1_-}\,\check{\varphi}_{1_-}(\zeta_{1_-})\,C_{id+F}(\zeta_{1_-},\zeta_1)\,d\zeta_{1_-} \tag{2.47}$$

2.8 Cohesiveness and the Regularity Scale.

Each intermediary step $\widehat{\varphi}_i(\zeta_i) \mapsto \widehat{\varphi}_{i+1}(\zeta_{i+1})$ of the accelero-summation scheme (see Fig. 2.2) is actually three steps in one:

Substep 1. If the accelerand $\widehat{\varphi}_i(\zeta_i)$ is ramified over $\mathbb{R}^+$, it must be averaged to $\mu\,\widehat{\varphi}_i(\zeta_i)$ relative to some convolution-respecting average μ.

Substep 2. We calculate the acceleration integral (2.38), but with $\mu.\widehat{\varphi}_i(\zeta_i)$ in place of $\widehat{\varphi}_i(\zeta_i)$. The integral converges for ζ_{i+1} small enough and > 0.

Substep 3. To turn the new germ $\widehat{\varphi}_{i+1}(\zeta_{i+1})$ into a global function *over* $\mathbb{R}^+$, we must *continue* it forward along $\mathbb{R}^+$ and *circumvent* every intervening singularity ω to the right and to the left.

Obviously, the two operations in substep 3, namely *continuation* and *circumvention of singularities*, require some form of quasi-analyticity. Most of the time there is no problem, because most of the time we have analyticity – but not always. This is where *cohesiveness* comes in and saves the day.

Cohesive functions.

We define the class *COHES* of *cohesive functions* by first extending the classical Denjoy classes ${}^{\alpha}DEN$ to all transfinite orders $\alpha < \omega^\omega$ and then going to the limit: [20]

$$^{\alpha}\mathrm{DEN} \;:=\; \{f \;;\; |f^{(n)}(t)| < c_{0,f}\,(c_{1,f})^n \left(\log'_{\alpha+1}(n)\right)^{-n}\} \tag{2.48}$$

$$\mathrm{COHES} \;:=\; \cup_{\alpha<\omega^\omega}\, {}^{\alpha}\mathrm{DEN} \tag{2.49}$$

Like each ${}^{\alpha}$DEN, the limit *COHES* is stable under $+, \times, \circ, \partial$ and most other operations. Crucially, it is also *quasi-analytic* : two cohesive functions defined on a real interval J coincide as soon they coincide on a subinterval $I \subset J$.

[20] Despite the latitude in the analytic incarnation of the transfinite iterates $\log_{\alpha+1}$ (see §8), each class ${}^{\alpha}$DEN is unambiguously defined: the indeterminacy is absorbed by the constant $c_{1,f}$ in (2.48).

Cohesive continuation.

Any cohesive function given on an interval $]0,\zeta[$ (viewed for the circumstance as part of $\mathbb{R}^+$ in some Borel plane) can be constructively continued to its maximal interval of cohesiveness $]0,\zeta_*[$ by a suitably weak deceleration followed by the reverse weak acceleration. See [E6], §3.11 or [E8], §9a, pp 93-94.

Cohesive singularities and their circumvention.

In some contexts like the Dulac problem, accelero-summation may produce strictly cohesive germs on $\mathbb{R}^+$ in some Borel planes[21] with any number of cohesive singularities there. To proceed with accelero-summation, the germs in question have to be cohesively continued (multivaluedly so) up to $+\infty$, which means bypassing all intervening singularities to the right and to the left, while being prohibited from leaving the real axis! This sounds an impossibility, but is not. See [E6], §3.12 or [E8], §9b, pp 94-95.

2.8 Time changes and the Great Divide.

As pointed out, the really useful transforms are, paradoxically, the *accelerations* and *pseudo-decelerations*. Indeed, despite going 'in opposite directions', both share a common regularising effect, albeit of crucially different force. To adequately describe that common effect together with that difference in regularising potency, we must distinguish three sub-classes for each :

strong accelerations	$\log z_1/\log z_2 \to 0$	e.g. $z_1 = \log z_2$
medium accelerations	$\log z_1/\log z_2 \to \alpha \in]0,1[$	e.g. $z_1 = (z_2)^\alpha$
weak accelerations	$\log z_1/\log z_2 \to 1$	e.g $z_1 = \frac{z_2}{\log z_2}$
strong pseudodeceler.	$\log z_1/\log(z_{1\_} - z_1) \to 1$	e.g. $z_1 = z_{1\_} + \frac{z_{1\_}}{\log z_{1\_}}$
medium pseudodeceler.	$\log z_1/\log(z_{1\_} - z_1) \to \alpha$	e.g. $z_1 = z_{1\_} + (z_{1\_})^\alpha$
weak pseudodeceler.	$\log z_1/\log(z_{1\_} - z_1) \to 0$	e.g. $z_1 = z_{1\_} + \log z_{1\_}$

Whatever the nature of the accelerand $\widehat{\varphi}_1$ (provided it has the proper accelerable growth at infinity), the corresponding accelerate $\widehat{\varphi}_2$ is automatically guaranteed a minimum of quasi-analytic smoothness – the weaker the accelaration, the less the smoothness.
(i) Strong accelerates are always analytic in a spiralling neighbourhood of 0.

[21] This is never the case, though, with composition equations, because these, as we shall see, are either non-polarising or weakly polarising.

with infinite aperture.
(ii) Medium accelerates are always analytic in a neighbourhood of $0_\bullet$ with at least finite aperture.
(iii) Weak accelerates are always cohesive in a real right-neighbourhood $]0,...[$ of $0_\bullet$, but may lack an extension to the complex domain.

With pseudo-decelerations, the picture is the same, but on the other side of the great cohesive/non-cohesive divide: whatever the nature of the pseudo-decelerand $\widehat{\varphi}_1$, one can always, by suitably strenghtening the pseudo-deceleration, ensure in the pseudo-decelerate $\widehat{\varphi}_{1_-}$ *any given* degree of smoothness, *short of cohesive*.

Another difference is this: *accelerations* completely upset the singularity landscape (they remove the old singular points and may create new ones) whereas *pseudo-decelerations* keep all singular points ω in place and merely smoothen the singularities there.

Smooth accelero-summation.

On the practical side, we can take advantage of the regularising effect of pseudo-decelerations to replace the accelero-summation scheme Fig. 2.2 by an improved scheme

$$\begin{array}{ccccccccc}
\widetilde{\varphi}_{1_-}(z_{1_-}) & \leftarrow & \widetilde{\varphi}(z) & & & & \varphi(z) & \leftarrow & \varphi_r(z_{r_-}) \\
\downarrow \mathcal{B} & & & & & & & & \mathcal{L}\uparrow \\
\widehat{\varphi}_{1_-}(\zeta_{1_-}) & \underset{\mathcal{C}_{1_-,2_-}}{\rightarrow} & \widehat{\varphi}_{2_-}(\zeta_{2_-}) & \rightarrow\cdots\rightarrow & & & \widehat{\varphi}_{(r-1)_-}(\zeta_{(r-1)_-}) & \underset{\mathcal{C}_{(r-1)_-,r_-}}{\rightarrow} & \widehat{\varphi}_r(\zeta_{r_-})
\end{array}$$

where, thanks to the selection of suitably *slow times* $z_{i_-} \sim z_i$ in each critical time class $[z_i]$, we ensure the smoothness of the minors $\widehat{\varphi}_{i_-}$ and all their alien derivatives $\widehat{\Delta}_{\omega_r}.\ldots.\widehat{\Delta}_{\omega_1}.\widehat{\varphi}_{i_-}$ $(\omega_k \in \mathbb{R}^+)$ and, by the same token, render the corresponding majors redundant.

2.10 Transseries and transmonomials.

Three co-dependent notions are relevant here: *transmonomials*, prime or composite, and *transseries*. Being formal objects, they all bear tildas, and since, in case of convergence or re-summability, they represent real germs on $]...,+\omega[$, their variable will be noted x.
(a) The *prime transmonomials* $\widetilde{P}$ cannot be factored into simpler elements, and must be viewed as being *large*.

(b) The *composite transmonomials* $\widetilde{M}$ can be uniquely factored into a well-ordered product of prime transmonomials:

$$\widetilde{M}(x) = (\widetilde{P}_0(x))^{a_{\tilde{P}_0}} \prod_{\widetilde{P}_0 \succ \widetilde{P}} (\widetilde{P}(x))^{a_{\tilde{P}}} \qquad (a_{\widetilde{P}_0} \in \mathbb{R}^*, a_{\widetilde{P}} \in \mathbb{R}) \tag{2.50}$$

$\widetilde{M}$ is said to be *large* (resp. small) if its leading factor $\widetilde{P}_0$ is raised to a positive (resp. negative) power $a_{\widetilde{P}_0}$.
(c) The *transseries* $\widetilde{S}$ can be uniquely expanded as well-ordered sums of transmonomials:

$$\widetilde{S}(x) = a_{\widetilde{M}_0}\, \widetilde{M}_0(x) + \sum_{\widetilde{M}_0 \succ \widetilde{M}} a_{\widetilde{M}}\, \widetilde{M}(x) \qquad (a_{\widetilde{M}_0} \in \mathbb{R}^*, a_{\widetilde{M}} \in \mathbb{R}) \tag{2.51}$$

$\widetilde{S}$ is said to be *large* (resp. *small*) if its leading term $\widetilde{M}_0$ is itself *large* (resp. *small*), and it is *positive* (resp. *negative*) if $a_{\widetilde{M}_0}$ is > 0 (resp. $0 < 0$). Each transseries splits into three part $\widetilde{S}(x) = \widetilde{S}^+(x) + s_0 + \widetilde{S}^-(x)$ with $s_0 \in \mathbb{R}$ and with $\widetilde{S}^+(x)$ resp. $\widetilde{S}^-(x)$ carrying only *large* resp. *small* transmonomials.

We must first define, inductively on n, the *logarithm-free* objects of *exponential depth* n.
(a_0) The only *log*-free prime transmonomial of *exp*-depth 0 is x.
(b_0) The only *log*-free transmonomials of *exp*-depth 0 are the $x^a (a \in \mathbb{R}^*)$.
(c_0) All *log*-free transseries of *exp*-depth 0 are well-ordered series of the form:

$$\widetilde{S}(x) = a_{\sigma_0}\, x^{-\sigma_0} + \sum_{\sigma_0 < \sigma} a_\sigma\, x^{-\sigma} \qquad (\sigma_0\,, \sigma \in \mathbb{R}\,,\ \sigma \uparrow) \tag{2.52}$$

(a_n) Each *log*-free prime transmonomial $\widetilde{P}$ of *exp*-depth n can be written uniquely as

$$\widetilde{P}(x) = e^{\widetilde{M}(x)} \quad \text{with } \widetilde{M} \text{ a } \textit{large} \text{ transmonomial of } \textit{exp}\text{-depth } n\text{–}1. \tag{2.53}$$

(b_n) Each *log*-free transmonomial $\widetilde{M}$ of *exp*-depth n can be written uniquely either as a well-ordered product or, what amounts to the same, as the exponential of a purely large transseries multiplied by an ordinary power :

$$\widetilde{M}(x) = (\widetilde{P}_0(x))^{a_{\tilde{P}_0}} \prod_{\widetilde{P}_0 \succ \widetilde{P}} (\widetilde{P}(x))^{a_{\tilde{P}}} = e^{\widetilde{S}^+(x)}\, x^{a_*} \qquad (\textstyle\prod \textit{well-ordered}) \tag{2.54}$$

with a leading prime transmonomial $\widetilde{P}_0$ of *exp*-depth n and other factors $\widetilde{P}$ of *exp*-height $\leqslant n$, or with a purely large transseries $\widetilde{S}^+$ of *exp*-height $n-1$.

(c_n) Each *log*-free transseries $\widetilde{S}$ of *exp*-depth n can be uniquely expanded as a well-ordered sum of transmonomials $\widetilde{M}$ of *exp*-depth $\leqslant n$, one of which at least[22] must be exactly of *exp*-depth n

$$\widetilde{S}(x) = a_{\widetilde{M}_0}\,\widetilde{M}_0(x) + \sum_{\widetilde{M}_0 > \widetilde{M}} a_{\widetilde{M}}\,\widetilde{M}(x) \qquad (\textstyle\sum \textit{well-ordered}) \tag{2.55}$$

Let $\widetilde{TRANS}^+$ be the space of all *log*-free transseries and let L_n be the n^{th} iterate of log. Taking advantage of the natural embedding:

$$\widetilde{TRANS}^+ \circ L_n \subset \widetilde{TRANS}^+ \circ L_{n+1} \tag{2.56}$$

we can define the trialgebra $\widetilde{TRANS}$ of real transseries:

$$\widetilde{TRANS} := \bigcup_{0 \leqslant n} \widetilde{TRANS}^+ \circ L_n \tag{2.57}$$

to which the operations $+, \times, \partial, \circ$ extend without difficulty.[23] Since these four operations actually reduce to three,[24] we can think of $\widetilde{TRANS}$ as a *trialgebra.*

Canonical form of a transseries.

The composition $\widetilde{F}, \widetilde{G} \mapsto \widetilde{F} \circ \widetilde{G}$ does not yield $\widetilde{F} \circ \widetilde{G}$ directly in canonical form, but we get there by expelling the *small* transmonomials from all exponentials and all logarithms (simple or iterated) by repeated use of the identities:

$$\exp \widetilde{G}(x) = e^{\widetilde{G}^+(x) + a + \widetilde{G}^-(x)} = e^a\, e^{\widetilde{G}^+(x)} \big(1 + \sum_{1 \leqslant n} \frac{(\widetilde{G}^-(x))^n}{n!}\big) \tag{2.58}$$

$$\log(\widetilde{G}(x)) = \log\big(b\, e^{\widetilde{\Gamma}^+(x)}\,(1 + \widetilde{\Gamma}^-(x)\big) = \log b + \widetilde{\Gamma}^+(x) + \sum_{1 \leqslant n} (-1)^{n-1} \frac{(\widetilde{\Gamma}^-(x))^n}{n}$$

Transseries in real-ordered form.

Any transseries can be uniquely written as a well-ordered *mock power series*

$$\widetilde{T}(x) = \widetilde{T}_{\sigma_0}(x)\,(\widetilde{P}(x))^{-\sigma_0} + \sum_{\sigma_0 < \sigma \uparrow} \widetilde{T}_\sigma(x)\,(\widetilde{P}(x))^{-\sigma} \qquad (\sigma_0, \sigma \in \mathbb{R}) \tag{2.59}$$

[22] either the first one (if it is large) or all the last ones from a certain rank on (if they are small).

[23] as long as they make formal sense: thus $\widetilde{S} \circ \widetilde{T}$ is defined only is the second factor is $\widetilde{T}$ is large and positive, i.e. starts with a large transmonomial $\widetilde{M}_0$ and a positive coefficient $a_{\widetilde{M}_0}$ in front of it.

[24] The composition $\circ$ is essentially expressible in terms of $\times$ and ∂: see (2.9).

Here $\widetilde{P} = e^{\widetilde{M}}$ denotes the *largest* prime transmonomial[25] present in $\widetilde{T}$, and the *mock coefficients* $\widetilde{T}_\sigma$ in front of the powers $\widetilde{P}^{-\sigma}$ are 'relatively negligible' transseries, i.e. transseries containing only prime transmonomials $\prec \widetilde{P}$.
(i) $\widetilde{P}(x)$ is said to be the *ruling* prime transmonomial of the transseries $\widetilde{T}(x)$, or its *ruler* for short.
(ii) $\widetilde{M}(x) := \log \widetilde{P}(x)$, as a well-defined, usually composite (i.e. non-prime) transmonomial, is said to be the *ruling time* of $\widetilde{T}(x)$.
(iii) The real-indexed transseries $\widetilde{T}_\sigma(x)$ in (2.59) are said to be the *pseudo-coefficients* of $\widetilde{T}(x)$.

Derivations on transseries.

Besides ordinary *total* differentiation ∂ (with respect to the variable x), partial differentiation ∂_P with respect to *prime transmonomials* P defines as many independent formal derivations on the algebras of transseries of a given logarithmic depth (with predictable rules governing the behaviour of ∂_P under post-composition). Then, on the *analysable* transseries, we also have the huge host of *alien derivations* Δ_ϖ for ϖ of the form $c\,M$ with any scalar c and any (not necessarily prime) transmonomial M. Despite the similarity in their indexation, the two systems of derivations have of course nothing in common.

2.11 Analysable germs. The complexity hierarchy.

Most transseries $\widetilde{S}$ are fated to remain formal, but there is an important subclass that can be re-summed to analytic germs S on $]..., +\infty[$. The corresponding sums deserve to be known as *analysable* germs, since they can be completely *formalised*, i.e. reduced bi-constructively and without loss of information to a formal object $\widetilde{S}$ and so, ultimately, to a set of real coefficients arranged in a tree-like structure. The correspondance $\widetilde{S} \leftrightarrow S$ is indeed constructive in both directions:
(i) as *accelero-synthesis* in the direction $\widetilde{S} \to S$
(ii) as *decelero-analysis* in the direction $S \to \widetilde{S}$

However, even for re-summable transseries, one must distinguish at least seven degrees in the severity of the divergence liable to occur, each degree bringing its own complications and calling for specific (but, thank Goodness, mutually compatible) remedies. Here is the list:
$\mathcal{C}_1$ *Direct convergence.*

[25] i.e. present in a least one of the transmonomials $\widetilde{M}$ of $\widetilde{T}$'s canonical expansion of type (2.55). There always exist one such largest $\widetilde{P}$.

$\mathcal{C}_2$ *Graded convergence.*
$\mathcal{C}_3$ *Seriable (or compensable) divergence.*
$\mathcal{C}_4$ *Resurgent divergence of non-polarising type.*
$\mathcal{C}_5$ *Resurgent divergence of polarising type.*
$\mathcal{C}_6$ *Infinite criticity.*
$\mathcal{C}_7$ *Infinite exponential depth.*

Pending the case by case discussion to be given in §3, here are a few general remarks.

Unlike *decelero-analysis*, which, 'being of the nature of destruction', is a relatively straightforward affair, *accelero-synthesis* must of necessity be a gradual, arduous process, in the course of which more and more 'parts' of $\widetilde{S}$ shed their formal character (*'they drop their tilda'*) and turn into functions living in various Borel planes[26](*'they acquire a hat'*), to finally contribute to the total germ S (*'they drop their hat'*). The process, roughly, goes like this:
(i) we find and order, from smaller to larger, all the prime transmonomials $\widetilde{P}_i = e^{\widetilde{M}_i}$ present or 'nested' in $\widetilde{S}$, in whatever position, at whatever exponential height,
(ii) for each such $\widetilde{P}_i = e^{\widetilde{M}_i}$ we isolate all transseries $\widetilde{T}$, present or nested in $\widetilde{S}$ (canonically expanded), and admitting $\widetilde{P}_i$ as *ruling* prime transmonomial,
(iii) we write each such $\widetilde{T}$ in the real-ordered form (2.59) and we realise it in a Borel plane or Borel axis ξ_i conjugate to some 'time' x_i that is equivalent ($\sim$) to the *ruling time* $M_i(x)$ and slow enough to ensure smoothness.[27] This makes perfect sense since at this stage not only $\widetilde{M}_i(x)$ but all the pseudo-coefficients $\widetilde{T}_\sigma(x)$ of $\widetilde{T}$ have already be *de-formalised* (turned into bona fide functions) in the previous steps of accelero-synthesis,
(iv) we then proceed to the next Borel plane or axis ξ_{i+1} conjugate to $x_{i+1} \sim M_{i+1}(x)$ by accelerating each $\widehat{T}_{\sigma,i}(\xi_i)$ to $\widehat{T}_{\sigma,i+1}(\xi_{i+1})$,
(v) by the time we reach the last Borel plane or Borel axis ξ_{last}, the whole of $\widetilde{S}(x)$ has been *de-formalised* and we can apply Laplace to get from $\widehat{S}_{\text{last}}(\xi_{\text{last}})$ to $S_{\text{last}}(x_{\text{last}})$ and from there to $S(x)$.

The extra complications arising from *infinite criticity* or *infinite exponential depth*, to be discussed in §3, do not radically alter the overall picture.

2.12 Multicritical displays.

With each (mono- or poly-critical) resurgent transseries $\widetilde{S}$ we associate an extremely useful object: the *display*, noted $Dpl.\widetilde{S}$ on the formal side and

[26]sometimes reduced to 'Borel axes' $\mathbb{R}^+$, often with ramification there.
[27]See §2.8 *supra*.

$(Dpl.S)_{\boldsymbol{\tau}}$ on the geometric side. The display combines two things: the so-called *pseudo-variables* $\mathbf{Z}^{\boldsymbol{\varpi}}$ (a notion dual to that of alien derivation) and the alien derivatives, of all orders and relative to all critical times z_j, of our $\widetilde{S}$ resp. S. The definition reads:

$$\begin{aligned} \mathrm{Dpl}.\widetilde{S} \;&=\; \widetilde{S} + \sum_{1\leqslant r} \sum_{\varpi_1,\dots,\varpi_r} \mathbf{Z}^{\varpi_1,\dots,\varpi_r}\, \boldsymbol{\Delta}_{\varpi_r} \dots \boldsymbol{\Delta}_{\varpi_1} \widetilde{S} && (2.60)\\ \downarrow \mathcal{A}_{\boldsymbol{\tau}} & && \\ (\mathrm{Dpl}.S)_{\boldsymbol{\tau}} \;&=\; S_{\boldsymbol{\tau}} + \sum_{1\leqslant r} \sum_{\varpi_1,\dots,\varpi_r} \mathbf{Z}^{\varpi_1,\dots,\varpi_r}\, (\boldsymbol{\Delta}_{\varpi_r} \dots \boldsymbol{\Delta}_{\varpi_1} S)_{\boldsymbol{\tau}} && (2.61) \end{aligned}$$

Here $\mathcal{A}_{\boldsymbol{\tau}}$ denotes accelero-summation $\widetilde{S} \mapsto S_{\boldsymbol{\tau}}$ relative to some multipolarisation $\boldsymbol{\tau}$, which in each critical Borel plane prescribes an integration axis $\arg \zeta_j = \theta_j$ and a convolution average μ_j.

The main novelty, however, is this: unlike the indices $\omega_i \in \mathbb{C}_\bullet$ of the monocritical display (2.34), the new indices are of the form $\varpi_i := \omega_i\, M_i$, with ω_i still in $\mathbb{C}_\bullet$ but followed by a transmonomial factor M_i that characterises the critical time class.

In our all-real context, we often restrict the sums (2.60)-(2.61) to the sole indices ϖ_j of positive ω_i-part, in which case we get the *lesser* or *all-real display dpl*, with fixed polarisation angles $\theta_j = 0$ and only the choice of the averages μ_j left to our discretion.

As the definitions make clear, the display carries – *displays*, as it were – the complete collection of our object's *Stokes constants*.[28] In that sense, it contains, in ultra compact and algebraically operative form, "everything there is to know" about the object. In fact, it is only at the level of displays that the correspondance *formal* ↔ *geometric* reaches perfection, as becomes obvious on the following *trans-polarisation formulae*:

$$\mathcal{A}_{\boldsymbol{\tau}'} \equiv \mathcal{P}_{\boldsymbol{\tau}',\boldsymbol{\tau}} \,.\, \mathcal{A}_{\boldsymbol{\tau}} \qquad \textit{with} \qquad \boldsymbol{\tau} = \begin{pmatrix} \theta_1 \,, \dots, \theta_s \\ \mu_1 \,, \dots, \mu_s \end{pmatrix} \;,\; \boldsymbol{\tau}' = \begin{pmatrix} \theta'_1 \,, \dots, \theta'_s \\ \mu'_1 \,, \dots, \mu'_s \end{pmatrix} \quad (2.62)$$

$$\mathcal{P}_{\boldsymbol{\tau}',\boldsymbol{\tau}} \,.\, \mathbf{Z}^{\varpi_1,\dots,\varpi_r} = \sum_j \mathbf{Z}^{\varpi_1,\dots,\varpi_j}\, \mathbf{P}^{\varpi_{j+1},\dots,\varpi_r}_{\boldsymbol{\tau}',\boldsymbol{\tau}} \qquad \textit{with} \qquad \mathbf{P}^{\boldsymbol{\varpi}}_{\boldsymbol{\tau}',\boldsymbol{\tau}} \in \mathbb{R} \quad (2.63)$$

or, in the short-hand of mould notations:

$$\mathcal{P}_{\boldsymbol{\tau}',\boldsymbol{\tau}} \,.\, \mathbf{Z}^\bullet = \mathbf{Z}^\bullet \times \mathbf{P}^\bullet_{\boldsymbol{\tau}'\,\boldsymbol{\tau}} \qquad \textit{with} \qquad \mathbf{P}^\bullet_{\boldsymbol{\tau}',\boldsymbol{\tau}} \textit{ symmetral}$$

These formulae say, in essence, that it is enough to know *one* polarised sum to know all the others. Indeed, they show how to derive any $(Dpl.S)_{\boldsymbol{\tau}'}$ from

[28] also known, depending on the viewpoint, as *resurgence coefficients* or *holomorphic invariants*.

any given $(Dpl.S)_{\boldsymbol{\tau}}$ by subjecting the latter to the purely formal operation $\mathcal{P}_{\boldsymbol{\tau}',\boldsymbol{\tau}}$ which acts on the sole pseudo-variables via *universal constants* $\mathbf{P}^{\bullet}_{\boldsymbol{\tau}',\boldsymbol{\tau}}$ that depend only on the trans-polarisation pair $(\boldsymbol{\tau}',\boldsymbol{\tau})$ and nothing else.[29]

Another crucial property is that any relation between resurgent objects automatically carries over to identical relations between their displays:

$$\{R(\widetilde{S}_1,\ldots,\widetilde{S}_s)=0\} \Longrightarrow \{R(dpl.\widetilde{S}_1,\ldots,dpl.\widetilde{S}_s)=R(Dpl.\widetilde{S}_1,\ldots,Dpl.\widetilde{S}_s)=0\}$$

This in turn implies a huge number of new constraints on the coefficients of the $\widetilde{S}_i$, a fact that can be very helpful for proving transcendence and independence results

2.13 Ultraseries and ultramonomials.

There exist composition equations, beginning with the simpler types $\mathcal{T}_1,\mathcal{T}_2$ (iteration and conjugation), that cannot be solved within the framework of transseries - even if we allow infinite exponential depth. To deal with such situations and achieve the dream goal of *full compositional closure*[30], it is necessary, but also sufficient, to introduce a coherent system of *transfinite* iterates of E and L, with iteration orders α less than ω^ω. Unfortunately, the formal construction, though unique, admits many distinct analytic realisations, with no clear privileged choice (but once that choice – any choice – is made, the correspondance $\widetilde{S} \leftrightarrow S$ holds without restriction). This complication, strictly speaking, affects only the analysis side. But even on the formal side, the ultra-exponentials lead to the replacement of our transseries by less tractable *ultrasseries*, made up of *ultramonomials* with no straightforward decomposition into prime factors. Nonetheless, the new objects – ultraseries and ultramonomials – remain pairwise comparable, on the formal as on the analysis side; non-oscillation is preserved; and the order $\lessdot$ survives.

2.14 Main germ groups and main extensions.

All our germ groups $\mathbb{G}$ and extensions $\mathbb{G}^{\mathrm{ext}}$ shall be defined by some combination of three things: $<$ *generators*$\|$*extensors*$\|$*restrictors* $>$. It usually matters, of course, whether one applies a given extensor *before* or *after* a given restrictor (though one tends to privilege those *restrictions* that are stable under most *extensions*) and so one should always specify that order within the brackets $< ..\|..\|.. >$.

[29]This also applies to the *lesser* display *dpl*.

[30]not just *closure under composition*, which is no big deal, but closure under the solving of all equations or systems that involve the composition $\circ$.

Our main *generators* are going to be:
(gen_1) The unit shift T
(gen_2) The group $GenT$ of all real shifts $T_\sigma : x \mapsto x + \sigma$.
(gen_3) The group $GenP$ of all real power functions $P_\sigma : x \mapsto x^\sigma$.
(gen_4) The exponential $E := exp$.
(gen_5) The ultra-exponentials $\mathcal{E}_n = E_{\omega^n} := exp^{\circ\omega^n}$ $(n \in \mathbb{N})$
(gen_6) The composition group $CvgPow$ of convergent real power series of the form $x \mapsto a_* \, x \, (1 + \sum a_n x^{-n})$ with $a_* > 0$.
(gen_7) The composition group $CvgTrans$ of gradedly-convergent transseries with a large positive leading term.[31]
(gen_8) The composition group $CvgUltra$ of gradedly-convergent ultraseries with a large positive leading term.

Our *extensors* or *enlargers* will consist in demanding closure under the solving of a given type of composition equations or systems, mainly:
(ext_1) Iteration equations (see $\mathcal{T}_1$ in §1.1)..
(ext_2) Conjugation equations (see $\mathcal{T}_2$ in §1.1)..
(ext_3) Positive composition equations (see $\mathcal{T}_3$ in §1.1).
(ext_4) General composition equations (see $\mathcal{T}_4$ in §1.1).
(ext_5) General composition systems.
We shall also pay special attention to an important subclass, the so-called *twins* equations or *siblings* systems:
(ext_6) $W(f, g) = id$ $\quad (\{f, g\}$: unknown 'twins').
(ext_7) $W_1(f_1, .., f_r) = ... = W_{r-1}(f_1, .., f_r) = id$ $\quad (\{f_i\}$: unknown 'siblings').
Twins equations or *siblings* systems contain only unknowns and seem underdetermined[32] but in fact, in the most interesting cases,[33] they exhibit *sporadicity*,[34] with all the fascination that attaches to sporadic objects.

Lasty, our main *restrictors* or *qualifiers* will be:
(rst_1) identity-tangency (i.e $f(x) \sim x + o(x)$).
(rst_2) shift-tangency (i.e $f(x) \sim x + \sigma + o(1)$).
(rst_3) 0-exponentiality (i.e. $lim.stat.L_n \circ f \circ E_n = id$ as $n \to +\infty$).
(rst_4) finite formal criticity (finitely many distinct prime transsmonomials).
(rst_5) finite analytic criticity (finitely many *critical times*).
(rst_6) non-polarisation (no singularities on any of the real Borel axes).
(rst_7) finite exponential depth.
(rst_8) analyticity on $]..., +\infty[$ (as opposed to mere cohesiveness).

[31] i.e. a leading term of the form $a_{M_0} \, M_0(x), a_{M_0} > 0$.
[32] their *non-trivial* solutions, when they exist at all, are determined only up to a common conjugation.
[33] e.g. when one looks for identity-tangent solutions.
[34] in the sense that very few such equations or systems possess *non-trivial* solutions.

3 Groups of analysable germs. Complexity hierarchy.

3.1 Degrees of divergence. Rising complexity.

We have a neat hierarchy with seven degrees $\mathcal{C}_j$ of rising complexity. Each degree is defined inductively, relative to the prime transmonomials $P(x)$ present in a given transseries and taken in increasing order. Each degree reflects the properties of the mock power series $\widetilde{T}(x)$ of (2.59) ruled by these $P(x)$ and of their mock coefficients $T_\sigma(x)$. We can legitimately drop the tildas,[35] since *at that stage of the inductive re-summation* the $T_\sigma(x)$ (and of course $P(x)$ itself) have already be re-summed.

$\mathcal{C}_1$ Direct convergence.

There is a common abscissa of convergence $x_0 < +\infty$ such that all mock power series $\widetilde{T}(x)$ converge uniformly on $[x_0 + \epsilon, +\infty[$.

$\mathcal{C}_2$ Graded convergence.

Each mock power series $\widetilde{T}(x)$ has its own abscissa $x_{\widetilde{T}} < +\infty$ of absolute convergence. These $x_{\widetilde{T}}$ may not be bounded, but there is a common $x_0 < +\infty$ such that all $\widetilde{T}(x)$ can be continued (analytically or cohesively) to the whole interval $]x_0, +\infty[$.

$\mathcal{C}_3$ Seriable divergence.

Some of the mock power series $\widetilde{T}(x)$ ruled by $P(x) = e^{M(x)}$ may have no finite convergence abscissae, but are simultaneously Borel summable relative
(i) to any time x_* equivalent to the *ruling time* $M(x)$ but slow enough in the class $[M(x)]$,
(ii) to any time x_{**} faster than $M(x)$, e.g. all $x_\alpha = (P(x))^\alpha (\alpha > 0)$,
(iii) with sums independent of the choice of x_* or x_{**}.
Given this huge latitude, one cannot speak of 'critical time classes', nor are there any resurgence phenomena or Stokes constants attached to this very peculiar, 'soft' type of divergence.

[35] on the $T_\sigma(x)$ and on $P(x)$, though not yet on $\widetilde{T}(x)$.

$\mathcal{C}_4$ **Non-polarising resurgent divergence.**

Some at least of the mock power series $\widetilde{T}(x)$ exhibit effective resurgence, mono- or even polycritical, usually of critical time(s) $x_{\widetilde{T}} := P(x)$ or $x_{\widetilde{T},\alpha} := (P(x))^\alpha$, but without singular points on the main axis $\mathbb{R}^+$ of the corresponding Borel plane(s) $\xi_{\widetilde{T}}$ or $\xi_{\widetilde{T},\alpha}$.

$\mathcal{C}_5$ **Polarising resurgent divergence.**

Same as above, but with singular points on the axes $\mathbb{R}^+$ of at least *some* Borel planes. When finitely (resp. infinitely) many alien derivations $\mathbf{\Delta}_\omega$ with $\omega > 0$ act effectively on (at least) *one and the same* mock power series, we speak of weakly (resp. strongly) polarising resurgence.[36]

$\mathcal{C}_6$ **Infinite criticity.**

The number of critical time classes is not bounded.

$\mathcal{C}_7$ **Infinite exponential depth.**

There is no bound on the height of the exponential towers present in the transseries. More precisely, the transseries carries prime transmonomials $\widetilde{P}_n$ of unbounded exponentially: $\limsup \mathit{expo}(\widetilde{P}) = +\infty$.

Enlargement by conjugation, continuous iteration, extraction.

As we shall see:
(i) Composition and reciprocation (i.e. taking the composition inverse) respect each of the above seven degrees.
(ii) Conjugation or continuous iteration of germs of zero-exponentiality often generates resurgence, but always of non-polarising type.
(iii) 'Extraction' (i.e. the solving of composition equations or systems), when all the factors g_i have zero-exponentiality, often generates resurgence, sometimes even of the weakly (but never strongly) polarising type.
(iv) Conjugation of germs of identical but non-zero exponentiality, or more generally 'extraction', whenever possible in the transserial framework,[37] generically introduces infinite exponential depth in the formal solutions and replaces analyticity by cohesiveness in the germ solutions.

[36]Saying that infinitely many $\mathbf{\Delta}_\omega$ with $\omega > 0$ act effectively on $\widetilde{\varphi}(z)$ is much stronger than saying that $\widehat{\varphi}(\zeta)$ has infinitely many singular points over $\mathbb{R}^+$ (relative to *forward* analytic continuation).

[37]That is the case *iff* after the substitution of E_n for f and E_{n_i} for g_i ($n_i := \mathrm{expo}(g_i)$) the composition equation has a solution $n \in \mathbb{Z}$ or is trivially verified for all n.

(v) We do not know at the moment of any composition equation or system that would generate *seriable* divergence in their solutions.[38]

3.2 Groups of convergent analysable germs.

Convergent transseries: direct convergence.

Their stability under composition and reciprocation is elementary. This applies is particular to the groups $< GenT, E >$ and $< PowSer, E >$ generated by the exponential and all real shifts resp. all real analytic map germs at $+\infty$. Both groups already contain transseries which, once written in canonical form, are of a very general form. In particular, the first of these two groups is dense in the second, whether in the natural topology of formal or in that of convergent transseries.

Convergent transseries: graded convergence.

Elementary transseries of type

$$\sum_{0\leqslant n} e^{-n.x}.(x+\sigma_n)^{-1} = \sum_{0\leqslant n} e^{-n.x} \sum_{0\leqslant k} (-\sigma_n)^k x^{-k-1} \qquad (0<\sigma\uparrow) \tag{3.1}$$

present us with the simplest instance of *graded convergence*: no uniform convergence abscissa x_0, yet no ambiguity at all as to the proper sum.

To illustrate how much *graded convergence* differs from true divergence, let us consider two similar looking difference equations:

$$A_1(x) - A_1(x+1) = a_1(x) \quad := e^{-x^{\alpha}} \qquad (0<\alpha<1) \tag{3.2}$$

$$A_2(x) - A_2(x+1) = a_2(x) \quad := e^{-x^{1+\alpha}} \qquad (0<\alpha<1) \tag{3.3}$$

with their transserial solutions expanded in canonical form:

$$A_1(x) \;=\; e^{-x^{\alpha}}\, x^{\beta}\, S_1(x) \qquad \textit{with} \quad \beta = 1-\alpha > 0 \tag{3.4}$$

$$A_2(x) \;=\; e^{-x^{1+\alpha}} \sum_{0\leqslant n} e^{-n\,(1+\alpha)\,x^{\alpha}}\, c_n(x) \tag{3.5}$$

Here, $S_1(x)$ is a formal, *divergent* and *resurgent* power series (in x^{-1} and $x^{-\beta}$) implicitely defined as the constant-free solution of the difference equation

$$S_1(x) - R_1(x)\, S_1(x+1) = x^{-\beta} \quad \textit{with} \quad R_1(x) := (1+x^{-1})^{\beta}\, e^{x^{\alpha}-(x+1)^{\alpha}} \tag{3.6}$$

[38]but the possibility cannot be completely ruled out in the case of highly *alternate* composition equations or systems, since their classification in a way runs parallel to that of differential equations or systems, and these sometimes (though extremely rarely) do generate seriable divergence.

with $R_1(x)$ viewed as convergent series in $\mathbb{C}[[x^{-1}, x^{-\beta}]]$.

On the other hand, the coefficients $c_n(x)$ in the expansion of A_2 are *convergent* power series explicitely given by

$$c_n(x) = \exp(-\gamma_n(x)) \quad , \quad \gamma_n(x) = (x+n)^{1+\alpha} - x^{1+\alpha} - n\,(1+\alpha)\,x^{\alpha} \tag{3.7}$$

with $\gamma_n(x)$ and $c_n(x)$ viewed as series of decreasing powers of x. Their domains of convergence $|x| > n$, however, decrease as n increases, so that we have *graded* rather than *direct* convergence in the transseries $A_2(x)$

$$A_1(x) \;\in\; x^{\beta} e^{-x^{\alpha}}\, \mathbb{C}[[x^{-1}, x^{-\beta}]] \qquad (\beta := 1-\alpha > 0) \tag{3.8}$$

$$A_2(x) \;\in\; e^{-x^{1+\alpha}}\, \mathbb{C}[[e^{-(1+\alpha)\,x^{\alpha}}, x^{-1}, x^{-\beta}]] \qquad (\beta := 1-\alpha > 0) \tag{3.9}$$

However, it would be confusing to lump A_1 and A_2 into the same 'divergent' category: A_1 exhibits true resurgence, possesses genuine invariants, and can boast a non-trivial display, whereas A_2 falls short on all three counts.

3.3 Groups of seriable analysable germs.

Let $\overset{m}{*}$ denote the multiplicative convolution

$$(h_1 \overset{m}{*} h_2)(x) := \int_1^x h_1(x_1)\, h_2(\frac{x}{x_1}) \frac{dx_1}{x_1} \tag{3.10}$$

We define the 'compensators' $x^{-\boldsymbol{\sigma}}$ as follows:

$$x^{-\sigma_0, -\sigma_1, \ldots, -\sigma_r} := x^{-\sigma_0} \overset{m}{*} x^{-\sigma_1} \overset{m}{*} \cdots \overset{m}{*} x^{-\sigma_r} \qquad (\sigma_i \in \mathbb{R}) \tag{3.11}$$

For distinct exponents σ_i we have

$$x^{-\sigma_0, -\sigma_1, \ldots, -\sigma_r} = \sum_{0 \leqslant i \leqslant r} x^{\sigma_i} \prod_{j \neq i} (\sigma_j - \sigma_i)^{-1} \tag{3.12}$$

When σ_i occurs $1 + n_i$ times, the formula becomes

$$x^{-\sigma_0^{[1+n_0]}, \ldots, -\sigma_r^{[1+n_r]}} = (n_0! \ldots n_r!)^{-1}\, (-\partial_{\sigma_0})^{n_0} \ldots (-\partial_{\sigma_r})^{n_r}\, x^{-\sigma_0, \ldots, -\sigma_r} \tag{3.13}$$

The easy inequalities

$$|x^{-\sigma_0, \ldots, -\sigma_r}| \;\leqslant\; \frac{1}{r!} |\log x|^r |x|^{-\sigma_*} \qquad (\sigma_i > 0\,,\; \sigma_* = \inf \sigma_i) \tag{3.14}$$

$$\Big| \frac{z^{-\sigma_0, \ldots, -\sigma_r}}{x_0^{-\sigma_0, \ldots, -\sigma_r}} \Big| \;\leqslant\; \Big| \frac{\log z}{\log x_0} \Big|^r \Big| \frac{z}{x_0} \Big|^{-\sigma_*} \qquad (x_0 < |z| < 1\,,\; z \in \mathbb{C}^*) \tag{3.15}$$

show that a series

$$\varphi(z) = \sum_{\boldsymbol{\sigma}} a_{\boldsymbol{\sigma}}\, z^{-\boldsymbol{\sigma}} \qquad (\textit{each } \boldsymbol{\sigma} \textit{ a finite sequence}) \tag{3.16}$$

can be convergent as a series of compensators, yet divergent as a power series, due to the proximity of some indices σ_i inside the same sequences $\boldsymbol{\sigma}$. To deal with such series, we take our cue from the inequalities (3.14)-(3.15) and consider slowly spiralling neighbourhoods $\mathcal{D}$ of infinity on $\mathbb{C}_\bullet$:

$$\mathcal{D}_{x_0,\kappa_0} = \{z \mid |z|.|\log z|^{-\kappa_0} > |x_0|.|\log x_0|^{-\kappa_0}\} \tag{3.17}$$

Next, using the *sup* norm $\|.\|^{\mathcal{D}}$ on these domains, we define the smaller compensation norm $\|.\|^{\mathcal{D}}_{\text{comp}}$:

$$\|\varphi\|^{\mathcal{D}}_{\text{comp}} := \inf\left\{\sum |a_{\boldsymbol{\sigma}}|.\|z^{-\boldsymbol{\sigma}}\|^{\mathcal{D}}\right\} \leqslant \|\varphi\|^{\mathcal{D}} \tag{3.18}$$

with an *inf* taken over all possible expansions of φ into infinite sums (3.16) of compensatrors. The compensation norm is multiplicative:

$$\|\varphi_1\,\varphi_2\|^{\mathcal{D}}_{\text{comp}} \leqslant \|\varphi_1\|^{\mathcal{D}}_{\text{comp}}\,\|\varphi_2\|^{\mathcal{D}}_{\text{comp}} \tag{3.19}$$

That follows from the formula:

$$z^{-\sigma'_0,\dots,-\sigma'_r}\, z^{-\sigma''_0,\dots,-\sigma''_s} \;=\; \sum z^{-\sigma_0,\dots,-\sigma_{r+s}} \tag{3.20}$$

$$\sigma_n = \sigma'_{i_n} + \sigma''_{i_n} \qquad , \qquad i_n + j_n \equiv n\,, \tag{3.21}$$

$$0 = \sigma'_{i_0} \leqslant \sigma'_{i_1} \leqslant \dots \leqslant \sigma'_{i_r} = r \quad , \quad 0 = \sigma''_{i_0} \leqslant \sigma''_{i_1} \leqslant \dots \leqslant \sigma''_{i_r} = s \tag{3.22}$$

which linearises the product $z^{-\boldsymbol{\sigma}'}\, z^{-\boldsymbol{\sigma}''}$ of two compensators of lengths r', r'' into a sum of $\frac{(r'+r'')!}{r'!\,r''!}$ compensators $z^{-\boldsymbol{\sigma}}$ of length $r'+r''$. As a consequence, we can speak of the *algebra of compensable power series*, i.e. power series with a finite compensation norm $\|.\|^{\mathcal{D}}_{\text{comp}}$ for some $\mathcal{D} = \mathcal{D}_{x_0,\kappa_0}$.

All well and good, except that taking the *inf* of all representations (3.16) is clearly not a practical re-summation strategy. Fortunately each compensable series $\widetilde{\varphi}(z)$ is Borel resummable, and very flexibly so: Borel summation works not only relative to all variables $z_* := \mathit{log} z - \kappa_0\, \mathit{log}\, \mathit{log} z$ for κ_0 large enough, but also relative to all variables $z_{**} \succ z_*$ (e.g. $z_\alpha := z^\alpha$), and in all cases yields the same sum as does the decomposition into sums of compensators.

The definition of compensability, along with the re-summation procedure, extends to the mock power series, leading to the larger notion of *seriability*. *Seriability* occupies a position midway between *convergence* and *strict resurgence*. With the latter it shares divergence, but lacks precisely defined critical times, exhibits no polarisation, and generates no Stokes constants. It is of common occurence in differential geometry: to most objects ridden with Louivillian small denominators yet having an unambiguous geometric existence,[39] there tend to correspond, on the formal side, *compensable* or *seriable*

[39] like the *transit maps* associated with limit cycles of ODEs in planar geometry.

expansions.

3.4 Groups of non-polarised analysable germs.

Non-polarised analysable germs.

This is the case when, on top of some or all of the previous complications, some of the mock power series may exhibit (mono- or polycritical) resurgence, but without any derivation $\mathbf{\Delta}_\omega$ ($\omega \in \mathbb{R}^+$) acting *directly* on them.[40]

Displays of non-polarised germs.

As a consequence, there are no singular points on $\mathbb{R}^+$ in any of the Borel planes; no need for convolution averages; no polarisation in the re-summed germs or their displays.

Frequent existence of a "geometric construction" (for the solution).

Non-polarisation often goes hand in hand with the existence of a geometric construction for the solution f a given composition equation, in the form of a limit $f = \lim f_n$, with f_n simply defined from the equation's factors g_i. This is definitely the case for iteration, conjugation, and some composition equations. These constructions, however, "do not mix" under composition, and if we want to perform regular group extensions, investigate the properties of these extensions, compare their elements pairwise, establish non-oscillation, etc, there is in the end no substitute for the transseries approach.

3.5 Groups of polarised analysable germs.

Weak and strong polarisation.

We say that there is weak (resp. strong) polarisation, when there is resurgence with finitely (resp. infinitely) many $\mathbf{\Delta}_\omega$ ($\omega \in \mathbb{R}^+$) acting simultaneously on at least one mock power series. Composition equations or systems can at most generate weakly polarising resurgence. There is thus no need to resort to well-behaved convolution averages, and there is a clearly privileged sum, corresponding to the standard convolution average.

[40] But they may act *indirectly*, as initial factors in operator strings $\mathbf{\Delta}_\omega \mathbf{\Delta}_{\omega_r} \dots \mathbf{\Delta}_{\omega_1}$, with $\omega \in \mathbb{R}^+, \omega_i \notin \mathbb{R}^+$.

The lesser display.

The *lesser display* (if need be, polycritical) *dpl*, which takes into account the sole derivations $\mathbf{\Delta}_\omega$ with $\omega \in \mathbb{R}^+$, suffices for a comparison of all possible sums, relative to all possible choices of convolution averages, for each effective critical time.

Integration of large transmonomials.

The integration $a(x) \mapsto A(x) = \partial^{-1}a(x)$ of large transmonomials $a(x)$, or of small transmonomials *larger* than some $L'_k(x)$, is a major source of weakly polarising resurgence. Even convergent transmonomials produce resurgence,[41] but of the simplest possible type, with only a single active alien derivation $\mathbf{\Delta}_1$, relative to a single critical time x_0 given by

$$x_0 = \text{stat.lim}_{r\to+\infty}\Big(\Big|\log\frac{a(x)}{L'_r(x)}\Big|\Big) \qquad \big(L_r := \log^{\circ r}\big) \tag{3.23}$$

The limit here is 'stationary', since for r large enough the germs on the right-hand side of (3.23) become equivalent at ∞.

$$x_0 \;=\; \big|\log a(x)\big| \qquad\qquad \textit{if} \qquad 1 < \lim\frac{\log a(x)}{\log x} \leqslant +\infty \tag{3.24}$$

$$x_0 \;=\; \log x \qquad\qquad \textit{if} \qquad 0 \leqslant \lim\frac{\log a(x)}{\log x} < 1 \tag{3.25}$$

Consider for instance this equation with $A(x)$ as unknown:

$$A'(x) = a(x) = L'_3(x)\, b(x) \quad \textit{with} \quad \text{expo}(b) \leqslant -4 \tag{3.26}$$

Any monomial b, large or small, of exponentiality $\leqslant -4$, will do. For instance:

$$\begin{aligned} b(x) \;&:=\; \big(L_4(x)\big)^{\alpha_4}\,\big(L_5(x)\big)^{\alpha_5}\,\big(L_6(x)\big)^{\alpha_6} \qquad \textit{or} \\ b(x) \;&:=\; \exp\Big(\big(L_5(x)\big)^{\beta_5}\,\big(L_6(x)\big)^{\beta_6}\,\big(L_7(x)\big)^{\beta_7}\Big) \qquad (\beta_5 > 0) \end{aligned}$$

The critical time here is $x_0 = L_4(x)$ and relative to that critical variable (3.26) becomes

$$A'_0(x_0) = e^{x_0}\, b_0(x_0) \qquad\qquad \big(A_0(x_0) \equiv A(x)\ ,\ b_0(x_0) \equiv b(x)\big)$$

The formal solution is given by

$$A_0(x_0) = e^{x_0}\, B_0(x_0) \;=\; e^{x_0}\,(1+\partial_{x_0})^{-1}\big(b_0(x_0)\big)$$

[41] About the sole exceptions are $e^{\alpha x}\, x^n$ or $x^\alpha\,(\log x)^n$ with $n \in \mathbb{N}$.

with $(1+\partial)^{-1}$ expanded straightforwardly in positive powers of ∂, while the resummation is given by the Laplace transform of

$$\widehat{B}_0(\xi_0) = (1-\xi_0)^{-1}\,\widehat{b}_0(\xi_0)$$

The definition of $\widehat{b}_0$ here is unproblematic, since the monomial $b_0(x_0) = b(E_4(x_0))$ is automatically subexponential in x_0, and Laplace summation too is unproblematic, since there is only one singularity on the positive real axis in the Borel plane.

3.6 Infinite exponential depth.

This last complication, which takes us beyond the framework of proper transseries as defined in §2.10, creates few complications on the formal side, but tends to substitute cohesiveness for analyticity in the sums.

3.7 Accelero-summation commutes with composition.

Proving this commutation establishes that large, re-summable transseries, taking in any of the seven categories $\mathcal{C}_i$ listed above, constitute a semi-group under composition. Then stability under reciprocation (taking the composition inverse) has to be proven. As the formal composition or reciprocation of transseries resolves itself into several steps,[42] it is enough to check that accelero-summation commutes with each one of them. The checks are tedious enough (see [E5] in a rather special case) but demand little more than dogged patience.

4 Conjugation/iteration of zero-exponentiality germs.

4.1 The three steps of conjugation.

The aim here is to show that any (large, positive) analysable germ f:

$$f(z) = a(z) + A(z) \qquad \textit{with} \quad a(z) \succ A(z) \quad ; \quad \mathrm{expo}(a) = 0. \qquad (4.1)$$

[42]Repeatedly resorting to the Taylor formula; rephrasing composition and reciprocation in terms of the operations ∂ and $\times$ in the multiplicative plane (resp. $\widehat{\partial}$ and $*$ in the Borel plane); expelling all infinitesimals from the exponentials and logarithms; and lastly re-arranging the terms so produced in accordance with the well order of transseries.

is *analysably* conjugate to the unit shift T. That will automatically take care of the mutual conjugation of any two such germs, and also settle for them the matter of continuous iteration.

Conjugating f to T is a process that is best broken down to three steps.

Step one:

We may assume $f(z) - z$ to be ultimately positive (if not, we replace f by $f^{\circ -1}$). Regardless of whether $f(z)$ is $\sim z$ or $\sim (1 + \mathit{const}).z$ or $\succ z$, there always exits large enough integers n such that the variable change

$$z = E_n(z_1) \quad , \quad z_1 = L_n(z) \tag{4.2}$$

turns f into a *strongly* identity-tangent germ f_1:

$$f_1(z_1) = L_n \circ f \circ E_n(z_1) = z_1 + b(z_1) + B(z_1) \quad ; \quad 1 \succ b(z_1) \succ B(z_1) \tag{4.3}$$

Here, b denotes the leading transmonomial of $f_1(z_1) - z_1$ (together with the real scalar in front of it) and B the remaining transseries. If the original transseries of f is convergent, the change of variable keeps it that way. If it is divergent (and resummable), it does not 'add' to its divergence (and keeps it resummable).

Step two:

A second change of variable

$$z_1 = h_{1,2}(z_2) \quad , \quad z_2 = h_{2,1}(z_1) \quad \mathit{with} \quad h_{2,1} := \partial^{-1}\frac{1}{b} = \int \frac{1}{b} \tag{4.4}$$

turns f_1 into a moderately identity-tangent germ f_2, whose second transmonomial is exactly 1:

$$\begin{aligned} f_2(z_2) &= h_{2,1} \circ f_1 \circ h_{1,2}(z_2) & (4.5)\\ &= z_2 + (h'_{2,1}\, b) \circ h_{1,2}(z_2) + o\big((h'_{2,1}\, b) \circ h_{1,2}(z_2)\big) & (4.6)\\ &= z_2 + \frac{b \circ h_{1,2}(z_2)}{h'_{1,2}(z_2)} + o\Big(\frac{b \circ h_{1,2}(z_2)}{h'_{1,2}(z_2)}\Big) & (4.7)\\ &= z_2 + 1 + \varphi_2(z_2) \quad \mathit{with} \quad \varphi_2(z_2) = o(1) & (4.8) \end{aligned}$$

In nearly all cases this step creates divergence[43], but always of resurgent-resummable type

$$\mathbf{\Delta}_{\varpi_0}\, h_{2,1} = c_0 = \mathrm{Const} \qquad \mathit{with} \qquad \varpi_0 = \sigma_0\, \beta(z_1) \quad , \quad \sigma_0 > 0) \tag{4.9}$$

$$\mathrm{Dpl}(h_{2,1}) = c_0\, \mathbf{Z}^{\varpi_0} + h_{2,1} \tag{4.10}$$

[43]the exceptional (but obviously important) transmonomials that admit convergent indefinite integrals are $z^{-\sigma}, z^n\,(\log z)^m, e^{-\sigma z} z^m$ $(\sigma \in \mathbb{C}, n, m\mathbb{N})$.

Step three:

We conjugate the germ f_2 to the unit shift T by solving the equation:

$$f_2^* \circ f_2(z_2) = 1 + f_2^*(z_2) \quad \textit{with} \quad f_2^*(z_2) = z_2 + \sum_{1\leqslant n} (\delta_n\, f_2^*)(z_2) \qquad (4.11)$$

where $\delta_n\, f_2^*$ denotes the term that is n-linear in the remainder transseries φ_2 of (4.8). These n-linear terms are given inductively by the following system of difference equations:

$$(\delta_1 f_2^*)(z_2) - (\delta_1 f_2^*)(z_2+1) \;=\; \varphi_2(z_2) \qquad (4.12)$$
$$(\delta_n f_2^*)(z_2) - (\delta_n f_2^*)(z_2+1) \;=\; \sum_{1\leqslant p<1} \frac{1}{p!}\big(\varphi_2(z_2)\big)^p \left(\delta_{n-p} f_2^*\right)^{(p)}(z_2+1) \quad (\forall n>1)$$

We have to split φ_2 into five parts φ_k:

$$f_2(z_2) := z_2 + 1 + \sum_{1\leqslant k\leqslant 5} \varphi_2^{[k]}(z_2) \quad \textit{with} \quad \varphi_2^{[k]} \in \mathcal{F}_k \qquad (4.13)$$

each of which behaves very differently under the solving of difference equations (we drop the lower index 2 for simplicity):

$$\Phi^{[k]}(z) - \Phi^{[k]}(z+1) = \varphi^{[k]}(z) \qquad (1 \leqslant k \leqslant 5) \qquad (4.14)$$

Each $\varphi^{[k]}$ consists of small transmonomials b belonging to one of five transmonomial intervals $\mathcal{F}_k$ characterised by a specific rate of decrease at $+\infty$, on the real axis (hence the choice of x as variable):

transmo^{al}.		*asymptotic behaviour*	*rate of decrease*
$b \in \mathcal{F}_1$	$\Leftrightarrow$	$\lim \frac{\log(1/b(x))}{x} = 0$	*subexponential*
$b \in \mathcal{F}_2$	$\Leftrightarrow$	$0 < \lim \frac{\log(1/b(x))}{x} < \infty$	*exponential*
$b \in \mathcal{F}_3$	$\Leftrightarrow$	$\lim \frac{\log(1/b(x))}{x} = \infty\,,\ \lim \frac{\log(1/b(x))}{x\log x} < \infty$	*weakly overexp^{al}*
$b \in \mathcal{F}_4$	$\Leftrightarrow$	$\lim \frac{\log(1/b(x))}{x\log c} = \infty\,,\ \lim \frac{\log\log(1/b(x))}{x} < \infty$	*moderately overexp^{al}*
$b \in \mathcal{F}_5$	$\Leftrightarrow$	$\lim \frac{\log\log(1/b(x))}{x} = \infty$	*strongly overexp^{al}*

More tellingly, in each case the transmonomial b falls into one of the following increasing intervals:

$$b \in \mathcal{F}_1 \;\Leftrightarrow\; b(x) \in \bigcup_{r=1}^{\infty} \left[\tfrac{1}{L_r(x)} \;,\; \exp(-\tfrac{x}{L_r(x)})\right]$$
$$b \in \mathcal{F}_2 \;\Leftrightarrow\; b(x) \in \bigcup_{r=1}^{\infty} \left[\exp(-\tfrac{1}{r}\,x) \;,\; \exp(-r\,x)\right]$$
$$b \in \mathcal{F}_3 \;\Leftrightarrow\; b(x) \in \bigcup_{r=1}^{\infty} \left[\exp(-x\,L_r(x) \;,\; \exp(-r\,x\,L_1(x))\right]$$
$$b \in \mathcal{F}_4 \;\Leftrightarrow\; b(x) \in \bigcup_{r=1}^{\infty} \left[\exp(-x\,L_1(x)\,L_r(x) \;,\; \exp(-e^{r\,x})\right]$$
$$b \in \mathcal{F}_5 \;\Leftrightarrow\; b(x) \in \bigcup_{r=1}^{\infty} \left[\exp(-e^{x\,L_r(x)}) \;,\; \tfrac{1}{E_r(x)}\right]$$

So let us examine the difference equation:

$$B(z) - B(1+z) = b(z) \qquad (b \in \mathcal{F}_k \;\;,\;\; 0 \leqslant k \leqslant 4) \tag{4.15}$$

for convergent[44] transmonomials b successively taken in each of the five fundamental intervals. But before proceeding with the discussion, let us start with three general remarks.

Remark 1: transserial solutions vs germ solutions.

In each case, the constant-free transserial solution $\widetilde{B}$ of (4.15) is given by

$$\widetilde{B}(z) = \sum_{0 \leqslant n} \tilde{b}^{\text{expand}}(z+n) \tag{4.16}$$

where $\tilde{b}^{expand}(z+n)$ denotes the natural transserial expansion of $b(z+n)$. Moreover, the resulting $\widetilde{B}$, whether convergent or not, always resums to a natural, unpolarised sum B given by

$$B(z) = \sum_{0 \leqslant n} b(z+n) \qquad (b \notin \mathcal{F}_{00} \;,\; x \textit{ large}) \tag{4.17}$$

at least when b does not belong to the subinterval $\mathcal{F}_{1*} \subset \mathcal{F}_1$:

$$b \in \mathcal{F}_{1*} \;\Leftrightarrow\; b(x) \in \bigcup_{r=1}^{\infty} \left[\frac{1}{L_r(x)} \;,\; \frac{1}{x\,L_1(x)\ldots L_r(x)}\right] \tag{4.18}$$

But despite the simplicity of the convergent germ expansion, we cannot rest satisfied with it, for three reasons:
(i) it fails for b in $\mathcal{F}_{1*}$

[44] meaning that all their nested transseries are convergent. That restriction will be dropped later on.

(ii) it does not yield the (often non trivial) display $Dpl\,B$ with the essential Stokes constant carried by it.
(iii) for the general purpose of *analysability*,[45] we cannot be content with the germ $B(z)$; we also require the underlying transseries.[46]

That said, two dichotomies dominate the picture: *convergence/resurgence* and *analytic/cohesive.*

Remark 2: Resurgence vs convergence.

For b in the range $\mathcal{F}_1 \cup \mathcal{F}_2 \cup \mathcal{F}_3$, the monomials resulting from the transserial expansions of $\tilde{b}^{expand}(z+n)$ 'interdigitate' and contribute helter skelter to a transseries $\widetilde{B}$ that is usually divergent and always resurgent. On the contrary, for b in the range $\mathcal{F}_4 \cup \mathcal{F}_5$, for $n_1 < n_2$, each transmonomial from $\tilde{b}^{expand}(z+n_1)$ neatly precedes[47] each transmonomial from $\tilde{b}^{expand}(z+n_2)$, so that $\widetilde{B}$ is now a (gradedly) convergent transseries.

Remark 3: Analyticity vs cohesiveness.

For any transmonomial b in the range $\mathcal{F}_1 \cup \mathcal{F}_2 \cup \mathcal{F}_3 \cup \mathcal{F}_4$, there always exist a right strip $S = \{\Re(z) > x_0\,,\, |\Im(z)| < y_0\}$ such that each germ $b(z+n)$ extends analytically to S, decreases there uniformly as $\Re z$ increases, yielding a germ series $\sum b(z+n)$ that converges to a sum $B(z)$ analytic on the whole of S. For b in the range $\mathcal{F}_5$, on the other hand, there is no strip, not even a tapering neighbourhood of $[x_0, +\infty[$, slim enough to ensure the convergence of $\sum b(z+n)$: that germ series does converge, but only on a neighbourhood of $+\infty$ on the real axis, and the sum B so defined is *cohesive* rather than analytic.

4.2 The general transserial difference equation.

The case $b \in \mathcal{F}_1$:

The transseries $\widetilde{B}$ solution of (4.15) is generically divergent but always resurgent with critical time z, resurgence support $\Omega_0 := 2\pi i\,\mathbb{Z}^*$, and elementary resurgence equations:

$$\Delta_{\omega_1} B(z) \;=\; A_{\omega_1} \qquad (\forall \dot{\omega}_1 \in \Omega_1) \tag{4.19}$$

$$\boldsymbol{\Delta}_{\omega_1} B(z) \;=\; A_{\omega_1}\, e^{-2\pi i k\, z} \qquad \textit{if} \quad \dot{\omega}_1 = 2\pi i k \tag{4.20}$$

[45] i.e. for the biconstructive shuttle $\widetilde{B} \leftrightarrow B$.
[46] For only thus can we *compare* and *resum* new transseries constructed from B, such as the direct iterator f_2^* in (4.11) and its reciprocal ${}^*f_2$.
[47] i.e. decreases more slowly.

The case $b \in \mathcal{F}_2$:

Let us separate the strictly subexponential parts c (transmonomial) and C (transserial) by writing:

$$b(z) = e^{-\sigma_2 z}\, c(z) \quad ; \quad B(z) = e^{-\sigma_2 z}\, C(z) \qquad (\sigma_2 > 0) \tag{4.21}$$

The equation (4.15) becomes $C(z) - e^{-\sigma_2 z}\, C(z+1) = c(z)$. Its constant-free transserial solution $\widetilde{C}$ is generically divergent but always resurgent, with critical time z, resurgence support $\Omega_2 = -\sigma_2 + 2\pi i \mathbb{Z}^*$, and elementary resurgence equations:

$$\Delta_{\omega_2} B(z) \;=\; A_{\omega_2} \qquad (\forall \omega_2 \in \Omega_2) \tag{4.22}$$
$$\boldsymbol{\Delta}_{\dot{\omega}_2} B(z) \;=\; A_{\omega_2}\, e^{-2\pi i k\, z} \qquad \textit{if} \quad \dot{\omega}_2 = -\sigma_2 + 2\pi i k \tag{4.23}$$

The case $b \in \mathcal{F}_3$:

Here again, we must isolate the dominant and subdominant parts by decomposing the transseries $\log(1/b(z)) = \sigma_3\, \beta(z) + \gamma(z)$ into a transseries $\gamma(z)$ of transmonomials all smaller than z and a supplementary transseries $\sigma_3\, \beta(z)$ normalised so as to give its leading term the form $\sigma_3\, \beta_0(z)$, with $\beta_0(z)$ a pure transmonomial, with no scalar in front of it.[48] We can then write

$$b(z) = e^{-\sigma_3\, \beta(z)}\, c(z)\; ;\; B(z) = e^{-\sigma_3\, \beta(z)}\, C(z) \quad \big(\sigma_3 > 0\;,\; c(z) = e^{-\gamma(z)}\big) \tag{4.24}$$

With respect to the subexponential parts, the difference equation becomes:

$$C(z) - d(z)\, C(z+1) = c(z) \qquad \textit{with} \quad d(z) = e^{\sigma_3 (\beta(z) - \beta(z+1))} \tag{4.25}$$

Since in this special case, we shall have to deal with *two* critical times, namely z itself and the slightly 'faster' time $z_+ := \beta(z)$, we must express our transseries relative to both variables:

$$C(z) \equiv C_+(z_+)\; ;\;\; B(x) \equiv B_+(z_+) \equiv e^{-\sigma_3 z_+}\, C_+(z_+) \;\; \big(z_+ \equiv \beta(z)\big) \tag{4.26}$$

(i) The Borel transform $\widetilde{C}(\zeta)$ with respect to the *slower* critical time x is convergent at $0_\bullet$ (the ramified origin of the ramified Borel plane $\mathbb{C}_\bullet$) and extends analytically to the whole of $\mathbb{C}_\bullet$, but without encountering other singularities than $0_\bullet$ and, consequently, without giving rise to any z-related resurgence equation. However, $\widehat{C}(\zeta)$ exhibits overexponential growth in certain directions, especially for $Arg(\dot{\zeta}) = -\pi$.
(ii) The Borel transform $\widehat{C}_+(\zeta_+)$ with respect to the *faster* critical time z_+

[48]For definiteness, we may think of $\beta_0(z)$ as being $z\,(\log z)^\alpha$ with $0 < \alpha \leqslant 1$.

does not converge near $0_\bullet$, but can nonetheless be resummed via the usual acceleration integral applied to $\widehat{C}(\zeta)$. The only singularities of $\widehat{C}_+(\zeta_+)$ other than $0_\bullet$ lie over the point $-\sigma_2$ and are described by the following resurgence equations:

$$\Delta_{\omega_3}^{(z_+)} C_+(z_+) = A_{\omega_3}(z) \quad \textit{with } \dot{\omega}_3 = -\sigma_3 \textit{ and } z \textit{ (not } z_+\textit{) in } A_{\omega_3} \tag{4.27}$$

$$\boldsymbol{\Delta}_{\omega_3}^{(z_+)} B_+(z_+) = A_{\omega_3}(z) \quad \textit{with } \dot{\omega}_3 = -\sigma_3 \textit{ and } z \textit{ (not } z_+\textit{) in } A_{\omega_3} \tag{4.28}$$

where $A_{\omega_3}(z)$ is 1-periodic Fourier series in z:

$$A_{\omega_3}(z) = \sum_{k\in\mathbb{Z}} A_{\omega_3,k}\, e^{-2\pi i k z} \tag{4.29}$$

Pay attention to the upper index z_+ that denotes the critical time in the above alien derivations. Pay even closer attention to the variable inside $A_{\omega_3}(z)$: that variable is not z_+, which would be inacceptable (since the germ produced by an alien derivation has to be subexponential relative to the critical time under consideration) but z, which is all right, since $A_{\omega_3}(z)$, being exponential in z, is subexponential in z_+.

The case $b \in \mathcal{F}_4$:

Here, the relevant decomposition of b is:

$$b(z) = e^{-\sigma_4 z \lambda(z) - \gamma(z)} = e^{-\sigma_4 z \lambda(z)}\, c(z) \qquad (\sigma_4 > 0) \tag{4.30}$$

(i) with $\lambda(z)$ starting with a leading transmonomial of the form $\lambda_0(z) \succ \log z$
(ii) with all transmonomials in $\lambda(z)$ being $\succ 1$.
(iii) with all transmonomials in $\gamma(z)$ being either z or $o(z)$.
The sum (4.16) then becomes

$$\widetilde{B}(z) = \sum_{0\leqslant n} \widetilde{b}^{\text{expand}}(z) = \sum_{0\leqslant n} e^{-\sigma_4 z \lambda(z)}\, e^{-n\sigma_4 \lambda(z)}\, \widetilde{C}_n(z) \tag{4.31}$$

with a middle factor consisting of powers of a transmonomial $e^{-\lambda(z)}$ that decreases faster than any transmonomial in any of the transseries $\widetilde{C}_n(z)$. Thus, inside the global transeries $\widetilde{B}(z)$ as given on the right-hand side of (4.31), the contributions of the various $\widetilde{C}_n(z)$ do not mingle, but keep neatly apart. The convergence abscissa x_n of each $\widetilde{C}_n(z)$ goes to $+\infty$ as n increases[49], so that we cannot have *absolute* convergence in $\widetilde{B}(z)$. But we have, unproblematically, *graded* convergence, since each $\widetilde{C}_n(z)$ converges to an analytic germ

[49] In fact, it is easily seen that $z_n \sim \mathit{Const}\, n$.

$C_n(z)$ which can be continued to a *common* interval $[x_0, +\infty[$ and since, on that interval or at least on a subinterval $[x_0', +\infty[$, the 'tildeless' germ series

$$B(z) = \sum_{0 \leqslant n} b^{\text{expand}}(z) = \sum_{0 \leqslant n} e^{-\sigma_3\, z\, \lambda(z)}\, e^{-n\, \sigma_3\, \lambda(z)}\, C_n(z) \tag{4.32}$$

converges (*absolutely* this time) to a germ $B(z)$. The only pending question is whether $B(z)$ is analytic, and that is where the upper bound of the interval $\mathcal{F}_3$ comes into play. As the prospects for analyticity worsen for faster decreasing transmonomials $b(x)$, it suffices to examine the case $b(z) = e^{-e^{r\,z}}$ for large values of r. But it is immediate that the series $\sum_{0 \leqslant n} e^{-e^{r\,(z+n)}}$ converges absolutely and normally on the halfstrip $\Re(z) > 0\,,\, |\Im(z)| \leqslant \frac{\pi}{2\,r} - \epsilon$. So, for $b(z)$ in $\mathcal{F}_3$, $B(z)$ is always analytic on a real neighbourhood of $+\infty$ and extends analytically either to a half-strip, or more often to a whole right half-plane, or even to a wider sector of $\mathbb{C}_\bullet$.

The case $b \in \mathcal{F}_5$:

The argument which clinched the graded convergence of (4.16) for b in $\mathcal{F}_4$ works *a fortiori* for b in $\mathcal{F}_5$. What we must show now is that B, though generically failing to be real-analytic, nonetheless retains a high degree of smoothness, enough to guarantee the form of quasi-analyticity known as *cohesiveness* and to ensure the property of unique continuation. The generic failure of analyticity is already obvious from the fact that, even for b at the lower end of the interval $\mathcal{F}_5$, that is to say of the form $b(x) = e^{-e^{x\, L_r(x)}}$ (r large), the translates $b(z+n)$ cannot remain bounded on any rectangle, however narrow, that straddles $\mathbb{R}^+$. To establish cohesiveness is slightly harder. The argument goes like this: for any b in $\mathcal{F}_5$ and any interval $[x_1, x_2] \subset \mathbb{R}^+$ close enough to $+\infty$, one can always find two real sequences $\{\epsilon_n\}$ and $\{\eta_n\}$ with $\epsilon_n \downarrow 0$, $\eta_n \downarrow 0$, $\sum \eta_n < +\infty$ and such that $|b(z+n)|$ be bounded by η_n on the rectangle of width $2\,\epsilon_n$ bisected by $[x_1, x_2]$. One then optimises the pair (ϵ_n, η_n) and, using the Dyn'kin criterion[50], finds the exact regularity type of the sum B – which paradoxically[51] get weaker and weaker for faster

[50]Meant is an extremely useful and flexible criterion due to Moisevich Dyn'kin, which relates the degree of smoothness (such as quasi-analytic, $\mathcal{C}^\infty$, Hölderian etc) of a real function defined on a real interval I to that function's *pseudo-analyticity* modulus, i.e. to the speed with which it can be approximated by complex functions with a small $\bar{\partial}$ defined on smaller and smaller rectangles straddling I. See [E8] §3, pp 72-74.

[51]paradoxically, but only at first sight: indeed, for faster decreasing transmonomials b, the convergence of the sum $\sum b(z+n)$ may increase on the real axis, but the moment we leave the real axis, the absolute values explode, especially for b at the upper end of the interval $\mathcal{F}_5$, i.e. of the form $1/E_r$ for r large.

decreasing transmonomials b, but always remains strong enough to ensure *cohesiveness.*

Overview.

Before winding up this long but necessary aside on difference equations, let us recast B's elementary resurgence equations and displays in a uniform mould, by resorting to the convenient ϖ-notations of §2.12, which alone can bring clarity to the sort of multicritical situation that we shall encounter in an instant, when returning to the iterators f^* and ${}^*\!f$. In all three resurgence-generating cases, we have:

$$\mathbf{\Delta}_{\varpi} B(z) = \mathbf{A}(x) \qquad ; \qquad \mathrm{Dpl}\, B = B + \mathbf{Z}^{\varpi}\, \mathcal{A}_{\varpi}(z) \tag{4.33}$$

with

$$\begin{array}{llll} b \in \mathcal{F}_1 \,\|\, \varpi(z) = \omega_1\, z & \|\, \dot{\omega}_1 \in 2\pi i\mathbb{Z} & \|\, \mathcal{A}_{\varpi}(z) = A_{\varpi}\, e^{-\omega_1 z} \\ b \in \mathcal{F}_2 \,\|\, \varpi(z) = \omega_2\, z & \|\, \dot{\omega}_2 \in -\sigma_2 + 2\pi i\mathbb{Z} & \|\, \mathcal{A}_{\varpi}(z) = A_{\varpi}\, e^{-\Im(\omega_2)\, z} \\ b \in \mathcal{F}_3 \,\|\, \varpi(z) = \omega_3\, \beta(z) \succ z & \|\, \dot{\varpi}_3 \in -\sigma_3 & \|\, \mathcal{A}_{\omega}(z) = \sum A_{\varpi,k}\, e^{-2\pi i\, k\, z} \end{array}$$

Thus, altough the resurgence-describing term $A_{\varpi}(z)$ is always 1-periodic in z – and cannot be anything else, since *Dpl* B, like B itself, has to verify the difference equation (4.15) – the natural indexation of its Fourier coefficients and that of the accompanying pseudovariable $\mathbf{Z}^{\varpi}$ varies widely from case to case.

4.3 The general transserial conjugation equation.

Now, returning to our general analysable germ f of exponentiality 1 (see 4.1) and the three associated objects – the direct iterator f^*, its functional inverse ${}^*\!f$ and the real iterates $f^{\circ t}$ – let us state the general result, first in the auxiliary z_2-chart, then in the original z-chart.

We begin with the case when f is itself (gradedly) convergent and examine the specific contributions of each of the five components $\varphi^{[k]}$ in (4.13):

- The initial transseries $\varphi^{[1]}, \varphi^{[2]}, \varphi^{[3]}$, on their own, create no other complication than generic divergence[52] associated with multicritical, non-polarising, Borel-summable resurgence in f_2^*, ${}^*\!f_2$, $f_2^{\circ t}$. The resurgent equations and the displays are as in (4.33). The corresponding germs f_2^*, ${}^*\!f_2$, $f_2^{\circ t_2}$ are always analytic on $[\ldots, +\infty[$.

[52] Except of course when $\varphi^{[1]} = \varphi^{[3]} = 0$ and when $\varphi^{[2]}$ reduces to a series $\sum c_\sigma\, e^{-\sigma\, z_2}$ of pure exponential monomials.

- The last transseries $\varphi^{[5]}$, on its own, creates no divergence in the transseries f_2^*, ${}^*f_2$, $f_2^{\circ t}$, but it generically prevents the corresponding germs from being analytic. Instead, it causes them to be *cohesive* (a special form of quasi-analyticity) on $[\ldots,+\infty[$, of class at most $\mathbb{D}\mathrm{en}_\omega$.

- The intermediate transseries $\varphi^{[4]}$, on its own, causes none of these complications – neither *divergence-resurgence* in the transseries nor strict (i.e. non-analytic) *cohesiveness* in the germs.

- In the general case, when all five components $\varphi^{[k]}$ are present, their effects combine unproblematically: 'analysability' survives; we still have the resurgence regime described in Proposition 6 below; and our transseries still resum to cohesive germs f_2^*, ${}^*f_2$, $f_2^{\circ t_2}$

Proposition 4.1 (Iterators of convergent analysable germs f)
For any (absolutely or gradedly) convergent transseries f of exponentiality 1, the iterators and real iterates verify the following resurgence equations:

$$f_2^* \circ f_2 = T \circ f_2^* \quad ; \quad f \circ {}^*f_2 = {}^*f_2 \circ T \quad ; \quad f_2^{\circ t} = {}^*f_2 \circ T^{\circ t} \circ f_2^* \tag{4.34}$$

$$\boldsymbol{\Delta}_\varpi f^* = -\mathcal{A}_\varpi(f^*) \tag{4.35}$$

$$\boldsymbol{\Delta}_\varpi {}^*f = +\mathcal{A}_\varpi\, \partial\, {}^*f \qquad \textit{with} \quad \partial := \partial_x \tag{4.36}$$

$$\frac{\boldsymbol{\Delta}_\varpi f^{\circ t}}{\partial f^{\circ t}} = +\frac{\mathcal{A}_\varpi(t+f^*) - \mathcal{A}_\varpi(f^*)}{\partial f^*} \tag{4.37}$$

with ϖ running through the set $\underline{\Omega} = \underline{\Omega}_1 \bigcup \underline{\Omega}_2 \bigcup \underline{\Omega}_3$. Of course, due to the 1-periodicity of the functions $\mathcal{A}_\varpi$, the right-hand side of (4.37) vanishes, as indeed it should, for any entire iteration order t. The resurgence equations, as usual, completely determine the displays, but here a unique simplification occurs – namely the neat separation in the displays (noted Dpl as usual) of some composition factors that consist purely of transseries and of other factors (noted Psd for pseudo) that consist purely of pseudovariables (accompanied by periodic exponentials in z). Indeed:

$$\mathrm{Dpl}\, f^* = (\mathrm{Psd}\, f^*) \circ f^* \tag{4.38}$$

$$\mathrm{Dpl}\, {}^*f = {}^*f \circ (\mathrm{Psd}\, {}^*f) \tag{4.39}$$

$$\mathrm{Dpl}\, f^{\circ t} = {}^*f \circ (\mathrm{Psd}\, f^{\circ t}) \circ f^* \tag{4.40}$$

with the Psd-part transparently defined from the differential operators

$$\mathbf{A}_\varpi := \mathcal{A}_\varpi(z)\, \partial_z \tag{4.41}$$

by the mutually inverse expansion (4.42) and (4.43):

$$\mathrm{Psd}(f^*) \;:=\; z + \sum_{1\leqslant r}\sum_{\varpi_j\in\Omega}(-1)^r\,(\mathbf{A}_{\varpi_r}\ldots\mathbf{A}_{\varpi_1}.z)\;\mathbf{Z}^{\varpi_1,\ldots,\varpi_r} \tag{4.42}$$

$$\mathrm{Psd}({}^*\!f) \;:=\; z + \sum_{1\leqslant r}\sum_{\varpi_j\in\Omega}(\mathbf{A}_{\varpi_1}\ldots\mathbf{A}_{\varpi_r}.z)\;\mathbf{Z}^{\varpi_1,\ldots,\varpi_r} \tag{4.43}$$

$$\mathrm{Psd}(f^{\circ t}) \;:=\; \mathrm{Psd}({}^*\!f)\circ T^{\circ t}\circ \mathrm{Psd}(f^*) \tag{4.44}$$

Notice that in (4.43) the indexation of the pseudovariables $\mathbf{Z}$ and that of the invariant operators $\mathbf{A}$ go in the same direction, whereas in (4.42) or indeed in the general definition of the display (2.60), the indexation of the pseudovariables $\mathbf{Z}$ and that of the alien derivations $\mathbf{\Delta}$ go in opposite directions.[53]

The identity $f^*\circ {}^*\!f = id$ implies not only $\mathrm{Dpl}(f^*)\circ\mathrm{Dpl}({}^*\!f) = id$ but also $\mathrm{Psd}(f^*)\circ\mathrm{Psd}({}^*\!f) = id$. This latter fact can also be verified directly by observing
(i) that the pseudovariables behave like constants under derivation and composition
(ii) that they obey the shuffle rule (2.30) under multiplication
(iii) that the identities (4.42) and (4.43) are equivalent to

$$\mathrm{Psd}(f^*) = \mathrm{PSD}(f^*).z \qquad \mathrm{Psd}({}^*\!f) = \mathrm{PSD}({}^*\!f).z \tag{4.45}$$

with infinite order differential operators $\mathrm{PSD}(f^*)$ and $\mathrm{PSD}({}^*\!f)$:

$$\mathrm{PSD}(f^*) \;:=\; 1 + \sum_{1\leqslant r}\sum_{\varpi_j\in\Omega}(-1)^r\;\mathbf{Z}^{\varpi_1,\ldots,\varpi_r}\;\mathbf{A}_{\varpi_r}\ldots\mathbf{A}_{\varpi_1} \tag{4.46}$$

$$\mathrm{PSD}({}^*\!f) \;:=\; 1 + \sum_{1\leqslant r}\sum_{\varpi_j\in\Omega}\;\mathbf{Z}^{\varpi_1,\ldots,\varpi_r}\;\mathbf{A}_{\varpi_1}\ldots\mathbf{A}_{\varpi_r} \tag{4.47}$$

that formally verify $\mathrm{PSD}(f^*).\mathrm{PSD}({}^*\!f)\equiv 1$.

Remark 4: Disappearance of the parasitical resurgence of step two.

To see what becomes of the elementary but (weakly) polarising resurgence (4.9) that resulted from the change of variable $z_1\to z_2$ of 'step two', we must look at the complete multicritical display :

$$\mathrm{Dpl}(f_2^*)(h_{2,1})(z_1) = \mathrm{Psd}(f_2^*)\circ f_2^*(z_2 - c_0\,\mathbf{Z}^{\varpi_0}) = \mathrm{Psd}(f^*)\circ f_2^*(z_2 - c_0\,\mathbf{Z}^{\varpi_0}) \tag{4.48}$$

[53]The reason is quite simply that, since each $\mathbf{\Delta}_{\omega_1}$ commutes with ∂ and therefore with $\mathbf{A}_{\omega_2}$ as given in (4.41), we have $\mathbf{\Delta}_{\omega_2}\mathbf{\Delta}_{\omega_1}{}^*\!f_2 = \mathbf{\Delta}_{\omega_2}\mathbf{A}_{\omega_1}{}^*\!f_2 = \mathbf{A}_{\omega_1}\mathbf{\Delta}_{\omega_2}{}^*\!f_2 = \mathbf{A}_{\omega_1}\mathbf{A}_{\omega_2}{}^*\!f_2$.

with c_0 and ϖ_0 as in (4.9). On the other hand we have the implications:

$$\begin{aligned} f_1 &= h_{1,2} \circ f_2 \circ h_{2,1} && \Longrightarrow \\ f_1^* &= f_2^* \circ h_{2,1} && \Longrightarrow \\ \mathrm{Dpl}(f_1^*) &= \mathrm{Dpl}(f_2^*) \circ \mathrm{Dpl}(h_{2,1}) \end{aligned} \tag{4.49}$$

Replacing in (4.49) the displays $Dpl(f_2^*)$ and $Dpl(h_{2,1})$ by their expression (4.38) and (4.39), we see at once that the polarising resurgence of critical time ω_0 that appeared at step 2 (see (4.9)-(4.10)) automatically disappears as soon as we revert to the variable z_1 of step 1, or a fortiori to the original variable z.

Remark 5: Geometric solution.

Not only is $f^{\circ t}(z)$ directly characterised by

$$\lim_{n\to+\infty} \frac{f^{\circ n} \circ f^{\circ t}(z) - f^{\circ n}(z)}{f^{\circ(n+1)}(z) - f^{\circ n}(z)} = t \qquad (\forall\, t \in \mathbb{R}\ ,\ \forall\, z \textit{ positive large}) \tag{4.50}$$

but in any z_3-chart such that

$$f_3(z_3) = (h_{3,2} \circ f_2 \circ h_{2,3})(z_3) = z_3 + 1 + d(z_3) + o(d(z_3)) \quad \textit{with} \quad \int d(z_3) \prec 1$$

with a small third monomial d whose constant-free indefinite integral is also small, the iterator $f_3^*(z_3)$, as a germ, is directly calculable as the limit

$$f_3^*(z_3) = \lim_{n->+\infty} (f_3^{\circ n}(z_3) - z_3 - n) \tag{4.51}$$

The disappearance of the 'earlier resurgence'[54] was predictable in a sense, because a singularity on $\mathbb{R}^+$ would create a polarisation, albeit of a very elementary sort (one for which *all* real convolution averages coincide), and that would not sit well with the existence of a privileged geometric solution.

Proposition 4.2 (Iterators of general analysable germs f)
If, instead of starting from a (gradedly) convergent analysable germ f as in Proposition 4.1, we start from a general analysable germ f (but still of exponentiality 0), little changes, except that the preexisting resurgence of f (whatever the type of that resurgence) gets superimposed, in an orderly manner, to the very specific resurgence generated by the passage $f \mapsto (f^*\ ,\ {}^*\!f)$.

[54] i.e. the one that appeared in step 2. See Remark 4.

The neatest way to describe the resulting situation is by writing down the displays[55]. *The earlier factorisations (4.38)-(4.40) now assume the form:*

$$\mathrm{Dpl}\, f^{*} = (\mathrm{Psd}\, f^{*}) \circ (\mathrm{Dpl}\, f)^{*} \tag{4.52}$$

$$\mathrm{Dpl}\, {}^{*}\!f = {}^{*}(\mathrm{Dpl}\, f) \circ (\mathrm{Psd}\, {}^{*}\!f) \tag{4.53}$$

$$\mathrm{Dpl}\, f^{\circ t} = {}^{*}(\mathrm{Dpl}\, f) \circ (\mathrm{Psd}\, f^{\circ t}) \circ (\mathrm{Dpl}\, f)^{*} \tag{4.54}$$

(i) with 'elementary factors' $(\mathrm{Dpl}\, f)^{*}$, ${}^{*}(\mathrm{Dpl}\, f)$ *that carries the 'old resurgence' and can obtained directly by solving the conjugation equations:*[56]

$$(\mathrm{Dpl}\, f)^{*} \circ (\mathrm{Dpl}\, f) = T \circ (\mathrm{Dpl}\, f)^{*} \tag{4.55}$$

$$(\mathrm{Dpl}\, f) \circ {}^{*}(\mathrm{Dpl}\, f) = {}^{*}(\mathrm{Dpl}\, f) \circ T \tag{4.56}$$

(ii) and with 'non-elementary factors' $\mathrm{Psd}\, {}^{*}\!f$, $\mathrm{Psd}\, f^{*}$ *that carry the 'new resurgence', commute with the unit shift* T*, and of course verify:*

$$id = (\mathrm{Psd}\, {}^{*}\!f) \circ (\mathrm{Psd}\, f^{*}) \tag{4.57}$$

$$\mathrm{Psd}\, f^{\circ t} = (\mathrm{Psd}\, {}^{*}\!f) \circ T \circ (\mathrm{Psd}\, f^{*}) \tag{4.58}$$

Observe that here, *'elementary'* simply means *'obtainable by purely formal manipulations on transseries'*.[57] But as far as the general shape is concerned, the factors (ii), being 1-periodic[58], are often more 'elementary' than the factors (i), since there is no a priori bound on the complexity of f, let alone on that of its display and its display's iterators.

A striking illustration of Proposition 4.2 shall be given in §13.5 with the 'continued conjugation' decomposition of a germ f.

4.4 Some examples.

Example 1: iteration of power series.

For germs f given by a power series $f(z) = c\,z + \sum_{1 \leqslant n} a_n\, z^{1-n}$ with $c > 0$, the iteration pattern is well-known,[59] but let us see how these results fit into the general transserial framework.

[55] from which the resurgence equations can be easily derived.

[56] Take care to distinguish the present *'iterators of displays'* $(\mathrm{Dpl}\, f)^{*}$, ${}^{*}(\mathrm{Dpl}\, f)$ from the earlier *'displays of iterators'* $(\mathrm{Dpl}\, f^{*})$, $(\mathrm{Dpl}\, {}^{*}\!f)$. See Proposition 4.1.

[57] and thus, without recourse to analysis in the Borel plane.

[58] more precisely : commuting with the unit shift T.

[59] In the case $c \neq 1$, this is a classical result due to Schroeder. In the identity-tangent case ($c = 1$), the geometric theory goes back to Fatou and the resurgence-resummation treatment to Ecalle.

In the non-identity-tangent case, we may assume $c > 1$. Then step 1 with $n = 1$ followed by step 2 with $h_{2,1}(z_1) = z_1/\log c$ immediately take us to the form (4.13), but with simple decreasing exponentials on the right-hand side. This is the simplest of all possible cases: the iterators are convergent.

In the identity-tangent case, $f(z) = z + a_p z^{1-p} + \ldots$, there is no need for step 1, so that $z_1 = z$, and steps 2 with $z_2 = \frac{1}{p\,a_p} z_1^p + \cdots + c_1 z_1 + \rho \log z_1$ takes us to the form (4.13), with a right-hand side in the interval $\mathcal{F}_1$ but consisting essentially of decreasing powers.[60] The only complication here is a resurgence that is governed by the general equations (4.35),(4.36),(4.37), but with invariant operators $\mathbf{A}_\omega$ that depend only on the projections $\dot{\omega}$.

Example 2: fractionnal iteration of monic polynomials.

Let f be a real monic polynomial of degree $d \geqslant 2$:

$$f(z) := z^d + \sum_{0 \leqslant k < d} a_k\, z^k \qquad (2 \leqslant d) \tag{4.59}$$

Step 1 takes f to the form

$$(L_2 \circ f \circ E_2)(z_1) \;:=\; z_1 + \log d + \alpha_2(z_1) \qquad \big(\alpha_2 \in \mathrm{Expo}_2^-\big) \tag{4.60}$$

$$= \; z_1 + \log d + \frac{a_{d-1}}{d\, e^{z_1 + (d-1)\, e_1^z}} + \ldots \tag{4.61}$$

with $f_1(z_1) - z_1$ in the interval $\mathcal{F}_4$, and step 2 (a simple dilation) keeps $f_2(z_2) - z_2$ in $\mathcal{F}_4$. As a consequence, the iterators and fractional iterates are guaranteed to be convergent. But let us for a change look directly at the fractional iterates, which are easily explicitable. Indeed, $g := f^{\circ \frac{1}{p}}$ may be sought of the form:

$$g(z) \;:=\; z^\sigma \Big(1 + S(z^{-1}, z^{-\sigma}, z^{-\sigma^2}, \ldots, z^{\sigma^{p-1}})\Big) \tag{4.62}$$

$$= \; z^\sigma \Big(1 + \sum_{n_j \geqslant 0}^{\sum n_j > 1} b_{n_0, n_1, \ldots, n_{p-1}}\, z^{-\sum_{0 \leqslant j < p} n_j\, \sigma^j}\Big) \tag{4.63}$$

with $\sigma := d^{1/p}$. Rather than directly iterating g and setting it equal to f, it is advantageous to replace g first by $\gamma(t) := 1/g(t^{-1})$ and then by a multidimensional mapping $\underline{\gamma} : \mathbb{C}^p_{,0} \mapsto \mathbb{C}^p_{,0}$ defined by:

$$\underline{\gamma} : \quad t_j \;\mapsto\; t_{j+1} \,.\, \Big(1 + S(t_0, t_1, \ldots, t_{p-1})\Big)^{-\sigma^j} \qquad (0 \leqslant j \leqslant p-2) \tag{4.64}$$

$$t_{p-1} \mapsto \; t_0^d \,.\, \Big(1 + S(t_0, t_1, \ldots, t_{p-1})\Big)^{-\sigma^{p-1}} \qquad (j = p-1) \tag{4.65}$$

[60] It is either in $\mathbb{C}[[z_2^{-1/p}]]$ if $\rho = 0$ or else in $\mathbb{C}[[z_2^{-1/p}, z_2^{-1/p} \log z_2]]$.

and lastly to write that the p^{th} iterate of $\underline{\gamma}$ is equal to:

$$\underline{\gamma}^{\circ p}: \quad t_j \quad \mapsto \quad t_j^d . \Big(1 + \sum_{0 \leqslant k < p} a_k t_{d-k}\Big)^{-\sigma^j} \qquad (0 \leqslant j \leqslant p-1) \tag{4.66}$$

Example 3: iteration of transmonomials.

As an exercise, the reader may examine the following examples

$$f(z) = e^{(\log z)^{c_1} \, P_1(\log\log z)} \qquad (P_1 \textit{ real polynomial }, c_1 > 0)$$
$$f(z) = e^{(\log z)^{c_2} \, P_2(1/\log\log z)} \qquad (P_2 \textit{ real polynomial }, c_2 > 0)$$
$$f(z) = e^{P_1((\log z)^{c_1})\, e^{(\log\log z)^{c_2} \, P_2(\log\log\log z)}}$$

where f, despite reducing to a single transmonomial, gives rise, after normalisation by the steps 1, 2, 3, to full-fledged transseries, with some or all of the possible attendant complications.

5 Conjugation/iteration of nonzero-exponentiality germs.

Since in the coming five sections most germs are defined, and most constuctions make sense, only in *real* neighbourhoods of $+\infty$, we shall throughout call the variable x rather than z, and its conjugate variable ξ rather than ζ.

5.1 Conjugation of germs with the same exponentiality.

Conjugating two analysable germs of unequal exponentiality[61], as we shall see in §6, is not possible in the relatively orderly framework of ordinary transseries, but requires the introduction of *ultraexponentials* and *ultralogarithms*. At the opposite end, conjugating two analysable germs f and g each of exponentiality 0 is always feasible[62], via a germ $h := {}^*f \circ g^*$ itself of exponentiality 0, as we just saw in §4. That leaves only the case when f and g have the same exponentiality $k \in \mathbb{Z}^*$. By considering if need be the reciprocal germs, we may assume k to be in $\mathbb{N}^*$ and it is enough to treat the case when g is the standard k-exponentialy germ. In other words, it suffices

[61]A germ f is said to be of exponentiality k if its leading transmonomial a is itself of exponentiality k, i.e. if $L_n \circ f \circ E_n \sim E_k$ for n large enough.

[62]Provided of course they are of the same type, i.e. both ultimatily contracting or expanding.

to consider pairs (f, g) with $expo(f) = k$, $g = E_k$ and then calculate the direct normaliser $f^{\lozenge}$ or its inverse ${}^{\lozenge}f$ by solving either of the equations (5.1), (5.2):

$$f^{\lozenge} \circ f = E_k \circ f^{\lozenge} \qquad (\text{expo}(f^{\lozenge}) = 0) \tag{5.1}$$

$$f \circ {}^{\lozenge}f = {}^{\lozenge}f \circ E_k \qquad (\text{expo}({}^{\lozenge}f) = 0) \tag{5.2}$$

with $f(x) = a(x) + o(a(x))$ and $expo(a) = k$. The solution $f^{\lozenge}$ (resp. ${}^{\lozenge}f$) are defined up to pre-composition by E_n (resp. post-composition by L_n) with $n \in \mathbb{Z}$ [63] and the *normalisers* are by definition the unique pair $(f^{\lozenge}, {}^{\lozenge}f)$ of reciprocal germs with exponentiality zero.

Let us focus on the (slightly simpler) direct normaliser $f^{\lozenge}$. We can always find n large enough to 'normalise' to E_k the leading transmonomial of f by a conjugation (5.3) and to make the whole transserial remainder α as small as we wish. We may for example bring f to the form f_1:

$$f_1(x_1) = (L_n \circ f \circ E_n)(x_1) = E_k(x_1) + \alpha(x) \quad \textit{with} \quad \alpha(x_1) = o(1) \tag{5.3}$$

The conjugation equation then becomes $f_1^{\lozenge} \circ (E_k + \alpha) = E_k \circ f_1^{\lozenge}$ and its unique solution (both as a transseries and an analysable germ) can easily be expanded into a sum of transserial blocks $\epsilon_{kn}(x_1)$. Their exact expression is given at the end of the section, in Example 4, after a graded series of easier cases. This expansion $f_1^{\lozenge}(x_1) = x_1 + \sum_n \epsilon_{kn}(x_1)$ converges unproblematically in the space of transseries. It also converges incredibly fast in the space of analysable germs on a suitable real neighbourhood of $+\infty$. The sum is generically non-analytic, but always *cohesive*, in the transfinite Denjoy class ${}^{\omega}DEN$ (and usually in no smaller class), irrespective of the exponentiality k.

But before tackling the general situation (Example 6, *infra*), let us examine five simpler examples, all of them directly relevant to the numerical investigation of §15. As in the preceding section (when investigating zero-exponentiality germs), we shall first assume that our analysable f has a (gradedly) convergent transseries $\widetilde{f}$ and then examine in §5.3 what changes for a general analysable f.

5.2 Graded examples.

Example 1. General f of exponentiality 1.

$$f(x) = e^x + \alpha(x) \;\; \textit{with} \;\; \alpha(x) = o(1) \; \textit{or} \; O(1) \tag{5.4}$$

[63] In the framework of transseries; in that of ultraseries, n may range through $\mathbb{R}$.

The direct normaliser $f^{\Diamond}$ admits a fast converging expansion

$$f^{\Diamond}(x) \;=\; x + \sum_{1\leqslant n} \epsilon_n(x) \tag{5.5}$$

with summands ϵ_n given by the induction

$$\epsilon_1(x) \;=\; \log\Big(1 + e^{-x}\,\alpha(x)\Big) \tag{5.6}$$

$$\epsilon_n(x) \;=\; \log\Big(\frac{id + \epsilon_1 + \epsilon_2 + \ldots \epsilon_{n-1}}{id + \epsilon_1 + \epsilon_2 + \ldots \epsilon_{n-2}}\Big) \circ f(x) \quad (\forall n \geqslant 2) \tag{5.7}$$

They are uniformly bounded by $|\epsilon_n(x)| < Const/\partial_x E_n(x)$ on a real neigbourhood of $+\infty$ and their derivatives too admit similar bounds. The sum (5.4) converges to a cohesive germ $f^{\Diamond}$ in the Denjoy class ${}^{\omega}DEN$.

Example 2. Special f of exponentiality 1.

$$f(x) \;:=\; c\,(e^x - 1) \;\; (c > 0)\;; \;\; f^{\Diamond} = id + \sum_{0\leqslant n} \epsilon_n \;\;;\;\; {}^{\Diamond}\!f = id + \sum_{0\leqslant n} \eta_n \tag{5.8}$$

Here, f and its reciprocal $f^{\circ-1}$ being equally simple, the direct and reciprocal normalisers have equally explicit expansions:

$$\begin{aligned}
\epsilon_0(x) \;&=\; \log(c) \\
\epsilon_1(x) \;&=\; \log\Big(1 + (\frac{\log(c) - c}{id + c}) \circ f(x)\Big) \\
\epsilon_n(x) \;&=\; \log\Big(1 + (\frac{\epsilon_{n-1}}{id + \epsilon_0 + \ldots \epsilon_{n-2}}) \circ f(x)\Big) \qquad (\forall n \geqslant 2)
\end{aligned} \tag{5.9}$$

$$\begin{aligned}
\eta_0(x) \;&=\; -\log(c) \\
\eta_1(x) \;&=\; \log\Big(1 + (\frac{c - \log(c)}{id}) \circ \exp(x)\Big) \\
\eta_n(x) \;&=\; \log\Big(1 + (\frac{\eta_{n-1}}{c + id + \eta_0 + \ldots \eta_{n-2}}) \circ \exp(x)\Big) \quad (\forall n \geqslant 2)
\end{aligned} \tag{5.10}$$

Example 3. Special f of exponentiality 1.

$$f(x) \;:=\; c\,x\,e^x \;\; (c > 0)\;; \;\; f^{\Diamond} = id + \sum_{0\leqslant n} \epsilon_n \tag{5.11}$$

$$\begin{aligned}
\epsilon_0(x) \;&=\; \log(c\,x) \\
\epsilon_1(x) \;&=\; \log\Big(1 + (\frac{\epsilon_0}{id}) \circ f(x)\Big) \\
\epsilon_n(x) \;&=\; \log\Big(1 + (\frac{\epsilon_{n-1}}{id + \epsilon_0 + \ldots \epsilon_{n-2}}) \circ f(x)\Big) \qquad (\forall n \geqslant 2)
\end{aligned} \tag{5.12}$$

Example 4. Special f of exponentiality 1.

$$f(x) \;:=\; c\,\sinh(x) \;=\; \frac{c}{2}\,e^{x} - \frac{c}{2}\,e^{-x} \;\; (c>0)\;;\;\; f^{\lozenge} = id + \sum_{0\leqslant n} \epsilon_n \qquad (5.13)$$

$$\epsilon_0(x) \;=\; \log(c/2)$$

$$\epsilon_1(x) \;=\; \log\Big(1 + \big(\frac{\log(c/2)}{(c/2)}\,e^{-x} - e^{-2x}\big)\Big)$$

$$\epsilon_n(x) \;=\; \log\Big(1 + \big(\frac{\epsilon_{n-1}}{id + \epsilon_0 + \ldots \epsilon_{n-2}}\big) \circ f(x)\Big) \qquad (\forall n \geqslant 2)\;(5.14)$$

Example 5. General f of exponentiality 2.

$$f(x) = e^{e^{x}} + \alpha(x) \;\; \textit{with} \;\; \alpha(x) = o(1) \;\; \textit{or} \;\; O(1) \qquad (5.15)$$

Here, the direct normaliser admits an expansion

$$f^{\lozenge}(x) \;=\; x + \sum_{1\leqslant n} \epsilon_{2\,n}(x) \qquad (5.16)$$

with summands ϵ_{2n} whose leading terms are small of exponentiality $2n$ for $\alpha = O(1)$, and of exponentiality $2n + n_0$ for α small of exponentiality n_0 :

$$\epsilon_2(x) \;=\; \log\Big(\frac{\log(f(x)}{e^{x}})\Big) \qquad (5.17)$$

$$\epsilon_{2\,n}(x) \;=\; \log\Big(\frac{\log(id + \epsilon_2 + \epsilon_4 \ldots \epsilon_{2\,(n-1)})}{\log(id + \epsilon_2 + \epsilon_4 \ldots \epsilon_{2\,(n-2)})}\Big) \circ f(x) \quad (\forall n \geqslant 2) \qquad (5.18)$$

Using the induction, these identities may also be re-written in a form better suited for majorising the (exceedingly small) terms ϵ_{2n}:

$$\epsilon_2(x) \;=\; \log\Big(1 + e^{-x}\,\log\big(1 + \frac{\alpha(x)}{e^{e^{x}}}\big)\Big) \qquad (5.19)$$

$$\epsilon_{2\,n}(x) \;=\; \log\Big(1 + \frac{\log\big(1 + \frac{\epsilon_{2\,n-2}}{id+\epsilon_2+\ldots\epsilon_{2\,n-4}}\big)}{\log(id + \epsilon_2 + \ldots \epsilon_{2\,n-4})}\Big) \circ f(x) \quad (\forall n \geqslant 3) \qquad (5.20)$$

Example 6. General f of exponentiality $k \geqslant 1$.

$$f(x) = E_k(x) + \alpha(x) \;\; \textit{with} \;\; \alpha(x) = o(1) \;\; \textit{or} \;\; O(1) \quad (k \geqslant 1) \qquad (5.21)$$

Here, the direct normaliser admits an expansion

$$f^{\lozenge}(x) \;=\; x + \sum_{1\leqslant n} \epsilon_{k\,n}(x) \qquad (5.22)$$

with summands ϵ_{kn} whose leading terms are small of exponentiality kn for $\alpha = O(1)$, and of exponentiality $kn + n_0$ for α small of exponentiality n_0:

$$\epsilon_k(x) \;=\; \log\Big(\frac{L_{k-1}(f(x)}{E_{k-1}(x)})\Big) \tag{5.23}$$

$$\epsilon_{k\,n}(x) \;=\; \log\Big(\frac{L_{k-1}(id + \epsilon_k + \epsilon_{2\,k} \ldots \epsilon_{(n-1)\,k})}{L_{k-1}(id + \epsilon_k + \epsilon_{2\,k} \ldots \epsilon_{(n-2)\,k})}\Big) \circ f(x) \quad (\forall n \geqslant 2) \tag{5.24}$$

with alternative, easier-to-majorise expressions analogous to (5.19), (5.20).

5.3 Resurgence and displays.

The mapping $f \mapsto (f^{\Diamond}, {}^{\Diamond}\!f)$, as just seen, creates no resurgence in case of a resurgent-free f; and when f is resurgent, it creates no *new* resurgence. As usual, this is best seen at the level of the displays: the displays of the normalisers coincide with the normalisers of the displays, and as such, are directly obtainable from the following composition identities:

$$(\mathrm{Dpl}\, f^{\Diamond}) = (\mathrm{Dpl}\, f)^{\Diamond} \implies (\mathrm{Dpl}\, f^{\Diamond}) \circ (\mathrm{Dpl}\, f) = E_k \circ (\mathrm{Dpl}\, f^{\Diamond}) \tag{5.25}$$

$$(\mathrm{Dpl}\, {}^{\Diamond}\!f) = {}^{\Diamond}(\mathrm{Dpl}\, f) \implies (\mathrm{Dpl}\, f) \circ (\mathrm{Dpl}\, {}^{\Diamond}\!f) = (\mathrm{Dpl}\, {}^{\Diamond}\!f) \circ E_k \tag{5.26}$$

Thus, $\mathrm{Dpl}\, f^{\Diamond}$ may be calculated simply by replacing $f^{\Diamond}$ and ϵ_{kn} by their respective displays in (5.22), (5.23), (5.24), and then formally expanding everything in series of pseudovariables.

We may note in passing that there is no contradiction between the fact that $f^{\Diamond}, {}^{\Diamond}\!f$ are generically non-analytic (merely cohesive) and the presence of resurgence, for the resurgence in question always attaches to specific *subtransseries* of $f^{\Diamond}, {}^{\Diamond}\!f$ which, when separately re-summed, *are* analytic.

6 Universal asymptotics of ultra-slow germs.

6.1 The bialgebra of iso-differentiations.[64]

An *iso-differential operator* or *iso-differentiation* of iso-degree n is a non-linear operator of the form:

$$Df := \sum_{1 \leqslant r \leqslant n} \sum_{1 \leqslant n_i}^{n_1 + \ldots n_r = n} a_{n_1,\ldots,n_r}\, H^{(n_1)} \ldots H^{(n_r)} \quad \textit{with} \quad H = \log(1/f') \tag{6.1}$$

$$:= \sum_{1 \leqslant r \leqslant n} \sum_{1 \leqslant n_i}^{n_1 + \ldots n_r = n} b_{n_1,\ldots,n_r}\, \frac{f^{(1+n_1)}}{f'} \ldots \frac{f^{(1+n_r)}}{f'} \tag{6.2}$$

[64] This algebra was first introduced by us in 1991 (see [E5]), under a different label ("post-homogeneous operators") but already in connection with ultra-slow germs.

These operators are uniquely adapted to the description of "universal asymptotics" since, as we shall see in a moment, they always produce the same asymptotic series when made to act on ultra-slow germs.

Due to their double homogeneousness (– the *iso* part of their name alludes to that –) they are essentially invariant under pre- and post-composition by simulitudes S:

$$D(S \circ f) \equiv Df \qquad ; \qquad D(f \circ S) \equiv \alpha^n \qquad (S(z) = \alpha z + \beta) \tag{6.3}$$

They also generate an interesting bialgebra, since they possess
(i) a commutative product $\times$, *distinct* from the non-commutative operator composition and additive with respect to the iso-degree:

$$(D_1 \times D_2)\, f \;:=\; (D_1\, f).(D_2\, f) \tag{6.4}$$

$$\mathrm{ideg}(D_1 \times D_2) \;=\; \mathrm{ideg}(D_1) + \mathrm{ideg}(D_2) \tag{6.5}$$

(ii) a non-commutative coproduct $D \mapsto \sigma(D)$:

$$\sigma(D) \;:=\; \sum_{\deg D = \deg D_1 + \deg D_2} a_D^{D_1,D_2}\, D_1 \otimes D_2 = D \otimes \mathbf{1} + \mathbf{1} \otimes D + \ldots \tag{6.6}$$

that reflects the action of iso-differentiations on composition products:

$$D(f_2 \circ f_1) \;:=\; \sum_{\mathrm{ideg} D = \mathrm{ideg} D_1 + \mathrm{ideg} D_2} a_D^{D_1,D_2}\, (D_1 f_1)\, (D_2 f_2) \circ f_1 \,.\, (f_1')^{n_2} \quad (n_2 := \mathrm{ideg} D_2) \tag{6.7}$$

(iii) an involution $D \mapsto \widetilde{D}$:

$$Dg \;\equiv\; (\widetilde{D} f) \circ g \,.\, (g')^n \qquad\qquad (n = \mathrm{ideg} D \,,\, f \circ g = id) \tag{6.8}$$

that reflects the action of iso-differentiations on functional inverses.

All three operations verify the predictable rules, namely:

$$\widetilde{D_1 \times D_2} = \widetilde{D}_1 \times \widetilde{D}_2 \qquad \textit{and} \qquad \sigma(D_1 \times D_2) = \sigma(D_1) \times \sigma(D_2) \tag{6.9}$$

with

$$(D_i \otimes D_j) \times (D_{i'} \otimes D_{j'}) \equiv (D_i \times D_{i'}) \otimes (D_j \times D_{j'}) \tag{6.10}$$

The resulting bialgebra *ISO* differs advantageously from the so-called Faa di Bruno bialgebra ($\times$-multiplicatively generated by all powers ∂^n) in that the latter lacks a "degree" with nice stability properties under both *product* and *co-product*.[65]

[65] It differs even more from the co-commutative Leibniz bialgebra that simply reflects the Leibniz rule.

6.2 The first two main bases $\mathrm{Dn}^{\{\bullet\}}$ and $\mathrm{Ds}^{\{\bullet\}}$ of *ISO*.

The operators $D^{\{n_1\}} : f \mapsto f^{(1+n_1)}/f'$ clearly constitute a $\times$-multiplicative basis of *ISO*, but their simplicity is deceptive.

The operators $\mathrm{Dn}^{\{n_1\}} : f \mapsto (-\partial)^{n_1} \log(1/f')$ lead to far simpler formulae for all operations: co-product, involution etc. They constitute the so-called *natural* generators of *ISO*, to which there answers the additive basis:

$$\mathrm{Dn}^{\{n_1,\dots,n_r\}} := \mathrm{Dn}^{\{n_1\}} \times \cdots \times \mathrm{Dn}^{\{n_r\}} : \quad f \mapsto \prod_i \Big((-\partial)^{n_i}. \log(1/f')\Big) \tag{6.11}$$

Here and throughout the sequel, the brackets $\{\boldsymbol{n}\}$ signal that the sequence $\boldsymbol{n}$ inside is non ordered (defined only up to order). Ordered sequences $\boldsymbol{n}$ will be within sharp brackets $<\boldsymbol{n}>$ or remain unbracketed.

We also require the *symmetric* generators $\mathrm{Ds}^{\{n_1\}}$, so-called because they react to involution in the simplest way possible:

$$\widetilde{\mathrm{Ds}}^{\{n_1\}} = -\mathrm{Ds}^{\{n_1\}} \quad ; \quad \widetilde{\mathrm{Ds}}^{\{n_1,\dots,n_r\}} = (-1)^r\, \mathrm{Ds}^{\{n_1,\dots,n_r\}} \tag{6.12}$$

Although the half-sums $1/2\ (\mathrm{Dn}^{\{n_1\}} - \widetilde{\mathrm{Dn}}^{\{n_1\}})$ would also produce such a symmetric basis, the following definition (6.13) of $\mathrm{Ds}^{\{\bullet\}}$ in terms of $\mathrm{Dn}^{\{\bullet\}}$ is to be preferred, not least because it admits an almost identical inverse (6.14), expressing $\mathrm{Dn}^{\{\bullet\}}$ in terms of $\mathrm{Ds}^{\{\bullet\}}$:

$$\mathrm{Ds}^{\{1+n_0\}} = \nabla\, \mathrm{Ds}^{\{n_0\}} - \frac{n_0}{2}\, \mathrm{Dn}^{\{1\}} \times \mathrm{Ds}^{\{n_0\}} \qquad \textit{with} \qquad \mathrm{Ds}^{\{1\}} = \mathrm{Dn}^{\{1\}} \tag{6.13}$$

$$\mathrm{Dn}^{\{1+n_0\}} = \nabla_*\, \mathrm{Dn}^{\{n_0\}} + \frac{n_0}{2}\, \mathrm{Ds}^{\{1\}} \times \mathrm{Dn}^{\{n_0\}} \qquad \textit{with} \qquad \mathrm{Dn}^{\{1\}} = \mathrm{Ds}^{\{1\}} \tag{6.14}$$

Here, ∇ and ∇_* denote operators acting as derivations on *ISO* relative to the natural product $\times$:

$$\nabla.\mathrm{Dn}^{\{n_1,\dots,n_r\}} := \sum_j \mathrm{Dn}^{\{n_1,\dots,1+n_j,\dots,n_r\}} \qquad (\nabla := -\partial) \tag{6.15}$$

$$\nabla_*.\mathrm{Ds}^{\{n_1,\dots,n_r\}} := \sum_j \mathrm{Ds}^{\{n_1,\dots,1+n_j,\dots,n_r\}} \tag{6.16}$$

The equivalence of the identities (6.13), (6.14), as well as the "symmetry" relations (6.12), follow from the formula:

$$\widetilde{\mathrm{Dn}}^{\{1+n_0\}} = \nabla\, \widetilde{\mathrm{Dn}}^{\{n_0\}} - n_0\, \mathrm{Dn}^{\{1\}} \times \widetilde{\mathrm{Dn}}^{\{n_0\}} \qquad \textit{with} \qquad \widetilde{\mathrm{Dn}}^{\{1\}} = -\mathrm{Dn}^{\{1\}} \tag{6.17}$$

which is itself a direct consequence of (6.8).

The corresponding analytical expressions read

$$\widetilde{\mathrm{Dn}}^{\{n_0\}} \equiv \sum_{1\leqslant r} \sum_{n_0=n_1+\ldots+n_r}^{n_1\leqslant n_2\ldots\leqslant n_r} (-1)^r\, H^{n_0}_{n_1,\ldots,n_r}\; \mathrm{Dn}^{\{n_1,\ldots,n_r\}} \tag{6.18}$$

$$\mathrm{Ds}^{\{n_0\}} \equiv \sum_{1\leqslant r} \sum_{n_0=n_1+\ldots+n_r}^{n_1\leqslant n_2\ldots\leqslant n_r} (-2)^{1-r}\, H^{n_0}_{n_1,\ldots,n_r}\; \mathrm{Dn}^{\{n_1,\ldots,n_r\}} \tag{6.19}$$

$$\mathrm{Dn}^{\{n_0\}} \equiv \sum_{1\leqslant r} \sum_{n_0=n_1+\ldots+n_r}^{n_1\leqslant n_2\ldots\leqslant n_r} (+2)^{1-r}\, H^{n_0}_{n_1,\ldots,n_r}\; \mathrm{Ds}^{\{n_1,\ldots,n_r\}} \tag{6.20}$$

with the same positive, integer-valued structure constants $H^{n_0}_{n_1,\ldots,n_r}$ in all three formulae. We may remark in passing that if we set

$$A(t) := +t + \sum_{1\leqslant n} \alpha_n \frac{t^{n+1}}{(n+1)!} \quad \textit{with} \quad \alpha_{n_0} := \sum_{n_1+\ldots n_r=n_0}^{r\geqslant 1} H^{n_0}_{n_1,\ldots,n_r} \tag{6.21}$$

$$B(t) := -t + \sum_{1\leqslant n} \beta_n \frac{t^{n+1}}{(n+1)!} \quad \textit{with} \quad \beta_{n_0} := \sum_{n_1+\ldots n_r=n_0}^{r\geqslant 1} (-1)^r\, H^{n_0}_{n_1,\ldots,n_r} \tag{6.22}$$

the integers α_n, β_n possess tree-theoretical interpretations[66] and the generating series A and B verify first order ODEs:

$$1 - A'(t) = \log(1-A(t)) \quad ; \quad -1 - B'(t) = \log(1-B(t)) \tag{6.23}$$

As for the co-product σ, the identities (6.6), (6.7) lead to the induction[67]

$$\sigma(\mathrm{Dn}^{\{1+n_1\}}) = \Big\{\nabla\otimes id + id\otimes\nabla + \mathrm{Dn}^{\{1\}}\otimes Ideg\Big\}.\sigma(\mathrm{Dn}^{\{n_1\}}) \tag{6.24}$$

$$\sigma(\mathrm{Ds}^{\{1+n_1\}}) = \Big\{\nabla_*\otimes id + id\otimes\nabla_* + \frac{1}{2}\mathrm{Ds}^{\{1\}}\otimes Ideg - \frac{1}{2}Ideg\otimes \mathrm{Ds}^{\{1\}}\Big\}.\sigma(\mathrm{Ds}^{\{n_1\}})$$

which in turn yields the analytical expression

$$\sigma(\mathrm{Dn}^{\{n_0\}}) = \sum_{|\boldsymbol{n^1}|+n_2=n_0} K^{\boldsymbol{n^1},n_2}\mathrm{Dn}^{\{\boldsymbol{n^1}\}}\otimes D^{\{n_2\}} \tag{6.25}$$

$$\sigma(\mathrm{Ds}^{\{n_0\}}) = \sum_{|\boldsymbol{n^1}||\boldsymbol{n^2}|=n_0} K^{\boldsymbol{n^1},\boldsymbol{n^2}}\Big(\mathrm{Ds}^{\{\boldsymbol{n^1}\}}\otimes\mathrm{Ds}^{\{\boldsymbol{n^2}\}} - \widetilde{\mathrm{Ds}}^{\{\boldsymbol{n^2}\}}\otimes\widetilde{\mathrm{Ds}}^{\{\boldsymbol{n^1}\}}\Big) \tag{6.26}$$

$$= \sum_{|\boldsymbol{n^1}||\boldsymbol{n^2}|=n_0} K^{\boldsymbol{n^1},\boldsymbol{n^2}}\Big(\mathrm{Ds}^{\{\boldsymbol{n^1}\}}\otimes\mathrm{Ds}^{\{\boldsymbol{n^2}\}} - (-1)^{r(\boldsymbol{n^1}.\boldsymbol{n^2})}\mathrm{Ds}^{\{\boldsymbol{n^2}\}}\otimes\mathrm{Ds}^{\{\boldsymbol{n^1}\}}\Big)$$

[66]Thus the integer α_{n-1} is the number of *increasing trees* with n nodes and cyclically ordered branches. An *increasing tree* is a rooted tree whose n nodes carry distinct labels ranging over $\{1,...,n\}$, with the labels increasing along any branch starting from the root.

[67]with the notation $\big(Op_1\otimes Op_2\big).\big(D_1\otimes D_2\big) := (Op_1\, D_1)\otimes(Op_2\, D_2)$ for any two linear operators on *ISO*, and with *Ideg* denoting the scalar multiplication of any D in *ISO* by its iso-degree $ideg(D)$.

For the symmetric basis, the right-hand side is (unsurprisingly) alternately symmetric or antisymmetric. For the natural basis, it is linear in the second argument,[68] although semi-linearity, by itself, does not suffice to characterise the natural basis.[69]

6.4 Universal asymptotics. The algebras *Isolog* $\subset$ $^\sharp$*Isolog*.

Let us consider the algebras[70] $^\sharp Isolog_k$ resp. $^\sharp Isolog$ spanned by the formal series

$$\ell e_k^{\langle n_1,\ldots,n_r\rangle}(x) \;=\; \sum_{1\leqslant p_1<p_2<\ldots<p_r\leqslant k} (L'_{p_1}(x))^{n_1}\ldots(L'_{p_r}(x))^{n_r} \quad \text{if} \quad r\leqslant k \tag{6.27}$$

$$\;=\; 0 \qquad \text{if} \quad r>k \tag{6.28}$$

$$\ell e^{\langle n_1,\ldots,n_r\rangle}(x) \;=\; \sum_{1\leqslant p_1<p_2<\ldots<p_r<+\infty} (L'_{p_1}(x))^{n_1}\ldots(L'_{p_r}(x))^{n_r} \tag{6.29}$$

These series consist of monomials of the form:

$$\lambda_\sigma = L'_{q_1}\ldots L'_{q_r} \quad (1\leqslant q_1\leqslant q_2\leqslant \ldots \leqslant q_n\ ,\ n=n_1+\ldots+n_r) \tag{6.30}$$

with an alternative indexation by transfinite ordinals

$$\tau = \omega^{n-1}q_1+\omega^{n-2}(q_2-q_1)+\omega^{n-3}(q_3-q_2)+\cdots+(q_n-q_{n-1}) \tag{6.31}$$

that reflects the natural ordering of the monomials: the larger τ as an ordinal, the faster the rate of decrease of λ_σ as a germ.

The significance of the series $\ell e_k^{\langle\bullet\rangle}$ (resp. $\ell e^{\langle\bullet\rangle}$) comes from the fact that the iso-operators, acting on finite iterates of the logarithm (resp. on transfinite iterates or more generally on ultra-slow germs) always produce germs expressible as *particular* combinations of $\ell e_k^{\langle\bullet\rangle}$ (resp. admitting asymptotic series given by particular combinations of $\ell e^{\langle\bullet\rangle}$). But whereas each $\ell e_k^{\langle\bullet\rangle}$ converges on a suitable real neighborhood of $+\infty$, the question does not even arise for the $\ell e^{\langle\bullet\rangle}$, since their summands cannot be simultaneously defined on a common neighborhood of $+\infty$. This, however, does not prevent these $\ell e^{\langle\bullet\rangle}$ from making perfect sense as well-ordered sums of λ_τ and as formal series, consisting each of an *asymptotic part*, starting with $\lambda_{\tau_1},\lambda_{\tau_2}$ etc, and a (sometimes vanishing) *transasymptotic part*, starting with $\lambda_{\tau_\omega},\lambda_{\tau_{\omega+1}}$ etc.

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[68]it involves the single-indexed $\mathrm{Dn}^{\{n_2\}}$ rather than the multi-indexed $\mathrm{Ds}^{\{\boldsymbol{n}^2\}}$ of (6.26).

[69]The pseudo-natural operators $D^{\{n_1\}}: f\mapsto f^{1+n_1}/f'$, mentioned and then dismissed at the beginning of this section, also possess right semi-linearity with respect to σ.

[70]relative to the ordinary product of formal series

For reasons that shall soon become obvious, it is convenient to consider, alongside the bases $\ell e_k^{\langle\bullet\rangle}$ and $\ell e^{\langle\bullet\rangle}$ of $^\sharp Isolog_k$ and $^\sharp Isolog$, two new bases $\ell a_k^{\langle\bullet\rangle}$ and $\ell a^{\langle\bullet\rangle}$ derived from the former through post-composition[71] by the symmetr**a**l mould $sa^\bullet$ and its composition inverse, the altern**e**l mould $cosa^\bullet$:

$$\begin{aligned} sa^{n_1,\dots,n_r} &:= \frac{1}{(n_1+\dots+n_r)(n_2+\dots+n_r)\dots n_r} && (6.32)\\ cosa^{n_1,\dots,n_r} &:= (-1)^{r-1}\, n_1 && (6.33)\\ sa^\bullet \circ cosa^\bullet &:= I^\bullet && (6.34)\end{aligned}$$

The conversion formulae read:

$$\ell a_k^{\langle\bullet\rangle}(x) = \ell e_k^{\langle\bullet\rangle}(x)\circ sa^\bullet \quad ; \quad \ell e_k^{\langle\bullet\rangle}(x) = \ell a_k^{\langle\bullet\rangle}(x)\circ cosa^\bullet \tag{6.35}$$

$$\ell a^{\langle\bullet\rangle}(x) = \ell e^{\langle\bullet\rangle}(x)\circ sa^\bullet \quad ; \quad \ell e^{\langle\bullet\rangle}(x) = \ell a^{\langle\bullet\rangle}(x)\circ cosa^\bullet \tag{6.36}$$

The product rules:

$$\ell e^{\langle \boldsymbol{n}'\rangle} . \ell e^{\langle \boldsymbol{n}''\rangle} = \sum_{\boldsymbol{n}\in \boldsymbol{she}(\boldsymbol{n}',\boldsymbol{n}'')} \ell e^{\langle \boldsymbol{n}\rangle} \tag{6.37}$$

$$\ell a^{\langle \boldsymbol{n}'\rangle} . \ell a^{\langle \boldsymbol{n}''\rangle} = \sum_{\boldsymbol{n}\in \boldsymbol{sha}(\boldsymbol{n}',\boldsymbol{n}'')} \ell a^{\langle \boldsymbol{n}\rangle} \tag{6.38}$$

which also hold for the k-truncated equivalents, simply mean that $\ell e^{\langle\bullet\rangle}, \ell e_k^{\langle\bullet\rangle}$ (resp. $\ell a^{\langle\bullet\rangle}, \ell a_k^{\langle\bullet\rangle}$) are symmetr**e**l (resp. symmetr**a**l).
The rules for post-composition by iterates of log are the same in both cases

$$\ell e_{k_1+k_2}^{\langle \boldsymbol{n}\rangle} = \sum_{\boldsymbol{n}'.\boldsymbol{n}''=\boldsymbol{n}} \ell e_{k_1}^{\langle \boldsymbol{n^1}\rangle} . \ell e_{k_2}^{\langle \boldsymbol{n^2}\rangle}\circ L_{k_1} . (L'_{k_1})^{|\boldsymbol{n^2}|} \qquad \forall\, k_1,k_2 \tag{6.39}$$

$$\ell a_{k_1+k_2}^{\langle \boldsymbol{n}\rangle} = \sum_{\boldsymbol{n}'.\boldsymbol{n}''=\boldsymbol{n}} \ell a_{k_1}^{\langle \boldsymbol{n^1}\rangle} . \ell a_{k_2}^{\langle \boldsymbol{n^2}\rangle}\circ L_{k_1} . (L'_{k_1})^{|\boldsymbol{n^2}|} \qquad \forall\, k_1,k_2 \tag{6.40}$$

but there is a significant difference in the rules for ordinary derivation[72]

$$\partial\, \ell e^{\langle \boldsymbol{n}\rangle} = -\sum_{\boldsymbol{n}' n_j \boldsymbol{n}''=\boldsymbol{n}} (n_j+|\boldsymbol{n}''|)\left(\ell e^{\langle \boldsymbol{n}',1,n_j,\boldsymbol{n}''\rangle} + \ell e^{\langle \boldsymbol{n}',1+n_j,\boldsymbol{n}''\rangle}\right) \tag{6.41}$$

$$\partial\, \ell a^{\langle \boldsymbol{n}\rangle} = -\sum_{\boldsymbol{n}' n_j \boldsymbol{n}''=\boldsymbol{n}} (n_j+|\boldsymbol{n}''|)\ell a^{\langle \boldsymbol{n}',1,n_j,\boldsymbol{n}''\rangle} - \sum_{\boldsymbol{n}' n_j \boldsymbol{n}''=\boldsymbol{n}} n_j\, \ell a^{\langle \boldsymbol{n}',1+n_j,\boldsymbol{n}''\rangle} \tag{6.42}$$

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[71]Mould composition operates likes this:
$\{C^\bullet = A^\bullet \circ B^\bullet\} \Leftrightarrow \{C^{\boldsymbol{n}} = \sum_{1\leqslant s}\sum_{\boldsymbol{n^1}\dots\boldsymbol{n^s}=\boldsymbol{n}} A^{|\boldsymbol{n^1}|,\dots,|\boldsymbol{n^s}|} B^{\boldsymbol{n^1}}\dots B^{\boldsymbol{n^s}}\}$.
The unit for mould composition is $I^\bullet$ with $I^{n_1}\equiv 1\ \forall n_1$ and $I^{n_1,\dots,n_r}\equiv 0\ \forall r \neq 1$.
[72]$\partial := d/dz$

Short proofs:
(i) The symmetr**e**lity relation (6.37) is a direct consequence of the construction (6.27)-(6.29).
(ii) The symmetr**a**lity relation (6.38) results from the conversion formulae (6.35), (6.36), the specific symmetries of $sa^\bullet$, $cosa^\bullet$, and the general symmetry conversion formulae:

$$\text{symmetr}\mathbf{el}^\bullet \circ \text{symmetr}\mathbf{al}^\bullet \;=\; \text{symmetr}\mathbf{al}^\bullet \tag{6.43}$$
$$\text{symmetr}\mathbf{al}^\bullet \circ \text{altern}\mathbf{el}^\bullet \;=\; \text{symmetr}\mathbf{el}^\bullet \tag{6.44}$$

These in turn make sense (are contradiction free) because mould composition respects both symmetr**e**lity and altern**a**lity, and because the composition inverse of symmetr**al** is altern**el**.
(iii) The composition rule (6.39) is a straightforward consequence of (6.7) and (6.40) follows under (6.36).
(iv) The first derivation rule (6.41) results from repeated applications of the identity

$$\frac{L''_p}{L'_p} = - \sum_{1 \leqslant q \leqslant p} L'_q \tag{6.45}$$

(v) The second derivation rule (6.42) results, inductively on index length, from (6.41) applied to the second identity (6.35).

Remark 6.1. First reasons behind the choice of $sa^\bullet$.

Post-composition in (6.36) by *any* symmetral mould other than $sa^\bullet$ [73] would produce symmetral series $\ell a^{\langle \bullet \rangle}$ and also ensure the composition rule (6.40), but it would inevitably introduce non-integer coefficients in the derivation rule (6.42). The disappearance of the denominators in (6.42) for $sa^\bullet$ defined as in (6.32) is a striking piece of luck, which by itself would be justification enough for this particular choice. Other, even more compelling justifications will emerge in the coming section.

Remark 6.2. Description of the sub-algebra *Isolog* of $^\sharp$*Isolog*.

The algebra *Isolog* generated by the formal limits:

$$D\,\mathcal{L} \;:=\; \lim_{k->\infty} DL_k \qquad (D \in Iso) \tag{6.46}$$

[73]such as the moulds $varsa^{n_1,\ldots,n_r} := 1/r!$ or $varsa^{n_1,\ldots,n_r} := sa^{n_r,\ldots,n_1}$.

is but a small sub-algebra of ${}^\sharp\mathit{Isolog}$, since for any given isodegree $n \geqslant 2$:

$$\dim(\mathit{Isolog}^{(n)}) = p(n) \quad < \quad \dim({}^\sharp\mathit{Isolog}^{(n)}) = 2^{n-1} \tag{6.47}$$

with $p(n)$ denoting the number of partitions of n.

6.5 The bialgebra ${}^\sharp\mathit{Iso}$ and its two bases $\mathbb{D}\mathrm{e}^{\langle\bullet\rangle}$ and $\mathbb{D}\mathrm{a}^{\langle\bullet\rangle}$.

The embedding $\mathit{Isolog} \subset {}^\sharp\mathit{Isolog}$ prompts us to similarly embed the bigebra *Iso* in a larger ${}^\sharp\mathit{Iso}$ endowed with two bases $\mathbb{D}\mathrm{e}^{\langle\bullet\rangle}$, $\mathbb{D}\mathrm{a}^{\langle\bullet\rangle}$ analogous to $\ell e^{\langle\bullet\rangle}$, $\ell a^{\langle\bullet\rangle}$. Here again, the relation between the 'natural' system $\{\mathbb{D}\mathrm{e}^{\langle\bullet\rangle}\}$ and the actually more practical system $\{\mathbb{D}\mathrm{a}^{\langle\bullet\rangle}\}$ is via the familiar moulds $sa^\bullet$, $cosa^\bullet$:

$$\mathbb{D}\mathrm{a}^{\langle\bullet\rangle} \;=\; \mathbb{D}\mathrm{e}^{\langle\bullet\rangle} \circ sa^\bullet \tag{6.48}$$

$$\mathbb{D}\mathrm{e}^{\langle\bullet\rangle} \;=\; \mathbb{D}\mathrm{a}^{\langle\bullet\rangle} \circ cosa^\bullet \tag{6.49}$$

To $\mathbb{D}\mathrm{e}^{\langle \boldsymbol{n}\rangle}$ and $\mathbb{D}\mathrm{a}^{\langle \boldsymbol{n}\rangle}$ we assign the iso-degree $|\boldsymbol{n}| := n_1 + \cdots + n_r$.
The product rule is still symmetr**e**lity resp. symmetr**a**lity :

$$\mathbb{D}\mathrm{e}^{\langle \boldsymbol{n}'\rangle} \times \mathbb{D}\mathrm{e}^{\langle \boldsymbol{n}''\rangle} \;=\; \sum_{\boldsymbol{n}\in \boldsymbol{she}(\boldsymbol{n}',\boldsymbol{n}'')} \mathbb{D}\mathrm{e}^{\langle \boldsymbol{n}\rangle} \tag{6.50}$$

$$\mathbb{D}\mathrm{a}^{\langle \boldsymbol{n}'\rangle} \times \mathbb{D}\mathrm{a}^{\langle \boldsymbol{n}''\rangle} \;=\; \sum_{\boldsymbol{n}\in \boldsymbol{sha}(\boldsymbol{n}',\boldsymbol{n}'')} \mathbb{D}\mathrm{a}^{\langle \boldsymbol{n}\rangle} \tag{6.51}$$

but the rules for the co-product σ and the involution $\sim$ undergo a drastic simplification:

$$\sigma(\mathbb{D}\mathrm{e}^{\langle \boldsymbol{n}\rangle}) \;=\; \sum_{\boldsymbol{n}'.\boldsymbol{n}''\,=\,\boldsymbol{n}} \mathbb{D}\mathrm{e}^{\langle \boldsymbol{n}'\rangle} \otimes \mathbb{D}\mathrm{e}^{\langle \boldsymbol{n}''\rangle} \tag{6.52}$$

$$\sigma(\mathbb{D}\mathrm{a}^{\langle \boldsymbol{n}\rangle}) \;=\; \sum_{\boldsymbol{n}'.\boldsymbol{n}''\,=\,\boldsymbol{n}} \mathbb{D}\mathrm{a}^{\langle \boldsymbol{n}'\rangle} \otimes \mathbb{D}\mathrm{a}^{\langle \boldsymbol{n}''\rangle} \tag{6.53}$$

$$\widetilde{\mathbb{D}\mathrm{e}}^{\langle \boldsymbol{n}\rangle} \;=\; (-1)^{r(\boldsymbol{n})} \sum_{1\leqslant s} \sum_{\boldsymbol{n}^1 \ldots \boldsymbol{n}^s\,=\,\boldsymbol{n}} \mathbb{D}\mathrm{e}^{\langle |\boldsymbol{n}^s|,\,\ldots\,,\,|\boldsymbol{n}^2|,\,|\boldsymbol{n}^1|\rangle} \tag{6.54}$$

$$\widetilde{\mathbb{D}\mathrm{a}}^{\langle \boldsymbol{n}\rangle} \;=\; (-1)^{r(\boldsymbol{n})}\;\; \mathbb{D}\mathrm{a}^{\langle \tilde{\boldsymbol{n}}\rangle} \;\; \mathit{with}\;\; \widetilde{(n_1,...,n_r)} = (n_r,...,n_1) \tag{6.55}$$

The action de ∂ on ${}^\sharp\mathit{Iso}$ is patterned on its action (6.41),(6.42) on ${}^\sharp\mathit{Isolog}$:

$$-\partial\, \mathbb{D}\mathrm{e}^{\langle \boldsymbol{n}\rangle} \;=\; \sum_{\boldsymbol{n}' n_j \boldsymbol{n}''=\boldsymbol{n}} (n_j + |\boldsymbol{n}''|) \left(\mathbb{D}\mathrm{e}^{\langle \boldsymbol{n}',1,n_j,\boldsymbol{n}''\rangle} + \mathbb{D}\mathrm{e}^{\langle \boldsymbol{n}',1+n_j,\boldsymbol{n}''\rangle}\right) \tag{6.56}$$

$$-\partial\, \mathbb{D}\mathrm{a}^{\langle \boldsymbol{n}\rangle} \;=\; \sum_{\boldsymbol{n}' n_j \boldsymbol{n}''=\boldsymbol{n}} (n_j + |\boldsymbol{n}''|) \mathbb{D}\mathrm{a}^{\langle \boldsymbol{n}',1,n_j,\boldsymbol{n}''\rangle} + \sum_{\boldsymbol{n}' n_j \boldsymbol{n}''=\boldsymbol{n}} n_j\, \mathbb{D}\mathrm{a}^{\langle \boldsymbol{n}',1+n_j,\boldsymbol{n}''\rangle} \tag{6.57}$$

To actually embed *Iso* in $^\sharp Iso$ we must indicate a *self-consistent* way of calculating the expansion $praj(D)$ resp. $prej(D)$ of any $D \in Iso$ in the basis $\mathbb{D}\mathrm{a}^{\langle\bullet\rangle}$ resp. $\mathbb{D}\mathrm{e}^{\langle\bullet\rangle}$. This is done by the induction formulae

$$praj(\mathrm{Dn}^{\{1+n_0\}}) = \nabla.praj(\mathrm{Dn}^{\{n_0\}}) \quad ; \quad prej(\mathrm{Dn}^{\{1+n_0\}}) = \nabla.prej(\mathrm{Dn}^{\{n_0\}}) \qquad (6.58)$$

$$praj(\mathrm{Ds}^{\{1+n_0\}}) = \nabla_*.praj(\mathrm{Ds}^{\{n_0\}}) \quad ; \quad prej(\mathrm{Ds}^{\{1+n_0\}}) = \nabla_*.prej(\mathrm{Ds}^{\{n_0\}}) \qquad (6.59)$$

supplemented by the initial conventions:

$$praj(\mathrm{Dn}^{\{1\}}) := praj(\mathrm{Ds}^{\{1\}}) := \mathbb{D}\mathrm{a}^{\langle 1\rangle} = \mathbb{D}\mathrm{e}^{\langle 1\rangle} =: prej(\mathrm{Ds}^{\{1\}}) =: prej(\mathrm{Dn}^{\{1\}})$$

In the above induction formulae, the derivation ∇ acts like $-\partial$ on the elements of both bases, i.e. according to the formulae (6.41),(6.42), while the derivation ∇_* acts like $-\partial + \frac{1}{2}\mathbb{D}\mathrm{a}^{\langle 1\rangle} Ideg$ on the $\mathbb{D}\mathrm{a}^{\langle\bullet\rangle}$-basis and like $-\partial + \frac{1}{2}\mathbb{D}\mathrm{e}^{\langle 1\rangle} Ideg$ on the $\mathbb{D}\mathrm{e}^{\langle\bullet\rangle}$-basis. In particular

$$\nabla_*\,\mathbb{D}\mathrm{a}^{\langle \boldsymbol{n}\rangle} = \sum_{\boldsymbol{n}'\boldsymbol{n}''=\boldsymbol{n}} \frac{|\boldsymbol{n}''|-|\boldsymbol{n}'|}{2}\,\mathbb{D}\mathrm{a}^{\langle \boldsymbol{n}',1,\boldsymbol{n}''\rangle} + \sum_{\boldsymbol{n}',n_j,\boldsymbol{n}''=\boldsymbol{n}} n_j\,\mathbb{D}\mathrm{a}^{\langle \boldsymbol{n}',1+n_j,\boldsymbol{n}''\rangle}$$

We may note that the operator $\mathbb{D}\mathrm{a}^{\langle 1\rangle} = \mathbb{D}\mathrm{e}^{\langle 1\rangle} = \mathrm{Dn}^{\{1\}}$ and the operator $\mathbb{D}\mathrm{a}^{\langle 2\rangle} = 1/2\,\mathbb{D}\mathrm{e}^{\langle 2\rangle} = -\mathrm{Dn}^{\{2\}} - 1/2\,\mathrm{Dn}^{\{1\}}\mathrm{Dn}^{\{1\}}$, which coincides with the Schwarzian derivative[74] , are true differential operators, but, starting from $n = 3$, none of the 'hyperschwarzians' $\mathbb{D}\mathrm{a}^{\langle n\rangle} = 1/n\,\mathbb{D}\mathrm{e}^{\langle n\rangle}$ are.

6.6 Action of $^\sharp Iso$ on the group $\mathbb{G}_{<T,E>}$.

In the next section, the embedding $Iso \subset {}^\sharp Iso$ is going to yield a new notion of convexity, uniquely adapted to germ composition. For the moment, however, the elements of $^\sharp Iso$ are only convenient symbols, meant in the first place to simplify the expression of σ and $\sim$. The first step in turning these symbols into genuine operators is to define, in a consistent manner, their action on E, L and T_α:

$$\mathbb{D}\mathrm{e}^{\langle n_1,\dots,n_r\rangle}(x+\alpha) \equiv 0 \qquad (6.60)$$

$$\mathbb{D}\mathrm{e}^{\langle n_1,\dots,n_r\rangle}(\exp x) \equiv (-1)^r \quad \forall\, r \geqslant 1 \qquad (6.61)$$

$$\mathbb{D}\mathrm{e}^{\langle n_1,\dots,n_r\rangle}(\log x) \equiv 0 \quad \forall\, r \geqslant 2 \quad \textit{and} \quad \mathbb{D}\mathrm{e}^{\langle n_1\rangle}(\log x) \equiv x^{-n_1} \qquad (6.62)$$

$$\mathbb{D}\mathrm{a}^{\langle n_1,\dots,n_r\rangle}(x+\alpha) \equiv 0 \qquad (6.63)$$

$$\mathbb{D}\mathrm{a}^{\langle n_1,\dots,n_r\rangle}(\exp x) \equiv (-1)^r\, sa^{n_r,\dots,n_1} \equiv \frac{(-1)^r}{n_1\,(n_1+n_2)\dots(n_1+\dots+n_r)}$$

$$\mathbb{D}\mathrm{a}^{\langle n_1,\dots,n_r\rangle}(\log x) \equiv x^{-(n_1+\dots+n_r)}\, sa^{n_1,\dots,n_r} \equiv \frac{x^{-(n_1+\dots+n_r)}}{n_r\,(n_{r-1}+n_r)\dots(n_1+\dots+n_r)}$$

[74]Indeed, $\mathbb{D}\mathrm{a}^{\langle 1\rangle} f \equiv -\frac{f''}{f'}$ and $\mathbb{D}\mathrm{a}^{\langle 2\rangle} f \equiv \frac{f'''}{f'} - \frac{3}{2}\,(\frac{f''}{f'})^2$.

and then to extend the action of $^\sharp\mathit{Iso}$ to the whole group $\mathbb{G}_{<T,E>} = \mathbb{G}_{<T,L>}$ by using the composition rule derived from the co-product (6.50),(6.51):

$$\mathbb{D}\mathrm{e}^{\langle n_1,\ldots,n_r\rangle}(f_2 \circ f_1) \equiv \sum_j (\mathbb{D}\mathrm{e}^{\langle n1,\ldots,n_j\rangle}.f_1)\ (\mathbb{D}\mathrm{e}^{\langle n_{j+1},\ldots,n_r\rangle}.f_2)\ (f_1')^{n_1+\ldots+n_j} \tag{6.64}$$

$$\mathbb{D}\mathrm{a}^{\langle n_1,\ldots,n_r\rangle}(f_2 \circ f_1) \equiv \sum_j (\mathbb{D}\mathrm{a}^{\langle n1,\ldots,n_j\rangle}.f_1)\ (\mathbb{D}\mathrm{a}^{\langle n_{j+1},\ldots,n_r\rangle}.f_2)\ (f_1')^{n_1+\ldots+n_j} \tag{6.65}$$

This leads in particular to

$$\mathbb{D}\mathrm{e}^{\langle n_1,\ldots,n_r\rangle}\sigma_{a,b}(x) = 0 \tag{6.66}$$

$$\mathbb{D}\mathrm{e}^{\langle n_1,\ldots,n_r\rangle}\pi_p(x) = (-1)^r\,(p^{n_{1,\ldots,r}} - p^{n_{2,\ldots,r}})\,x^{-n_{1,\ldots,r}} \tag{6.67}$$

$$\begin{aligned}\mathbb{D}\mathrm{e}^{\langle n_1,\ldots,n_r\rangle}\theta_{p,c}(x) &= (-1)^r\,(p^{n_{1,\ldots,r}} - p^{n_{2,\ldots,r}})\,x^{-n_{1,\ldots,r}} \\ &\quad\times\Big(1 - \frac{p^{n_1}}{\big(p\,(1+c\,x^{-p})\big)^{n_{1,\ldots,r}}} + \sum_{j=2}^{j=r}\frac{1-p^{n_j}}{\big(p\,(1+c\,x^{-p})\big)^{n_{j,\ldots,r}}}\Big)\end{aligned} \tag{6.68}$$

with the usual definitions

$$\sigma_{a,b}(x) := a\,x + b\ ,\ \pi_p(x) := x^p\ ,\ \theta_{p,c}(x) := (c + x^p)^{\frac{1}{p}} = x\,(1 + \frac{c}{p}x^{-p} + \ldots)$$

However, the action so defined on $\mathbb{G}_{<T,E>}$ is not continuous, for the formal topology of $\mathbb{G}_{<T,E>}$. It does not even become continuous when restricted to the subgroup consisting of power series, as one can easily show based on (6.68).

So, to make sure that the action defined by the above rules is *consistent*, we need to show that all identity relations in the group $\mathbb{G}_{<T,E>}$ are generated by the identity relations between similitudes $\sigma_{a,b}$. Now, when the first draft of this paper was completed, in January 2016, we knew of no such result. But in the meantime a remarkable result by D. Panazzolo has appeared [P] which, if we are not mistaken, implies exactly that.[75] So, at least on the group $\mathbb{G}_{<T,E>}$, we have a consistent (albeit non-continuous) action, not just of the iso-differatial operators, but also of the far more numerous iso-differential symbols.

[75] Panazzolo actually establishes the existence of a normal form for elements of the groupoid generated by the exponential and all complex similitudes $z \mapsto a\,z + b$. He has to work in a *groupoid* in order to accommodate all complex similitudes with their distinct fixed points, but since we are interested here only in the real similitudes and $+\infty$ as their common fixed point, we may rephrase his results in the more familiar setting of *groups*.

7 Iso-convexity and the extremal basis $\mathrm{Da}^{\{\bullet\}}$.

7.1 The positive cones $^{\sharp}ISO^{+} \subset {}^{\sharp}ISO$.

The positive cone $^{\sharp}ISO^{+} \subset {}^{\sharp}ISO$, consisting of all elements of the form $\sum c_{\boldsymbol{n}}\, \mathbb{D}\mathrm{a}^{\langle \boldsymbol{n}\rangle}$ with non-negative coefficients $c_{\boldsymbol{n}}$, is trivially stable under both the product $\times$ and the co-product σ, and this double stability automatically carries over to the positive cone $ISO^{+} := {}^{\sharp}ISO^{+} \cap ISO$ consisting of *differential* iso-operators. The real surprises begin when we start looking for a natural basis of ISO^{+}. Such a basis — the *"extremal basis"* — not only exists, of unimpugnable naturalness, but it enjoys a whole string of improbable properties that find their reflection in remarkably explicit formulae — none of which would survive if we tinkered ever so slightly with the definition of ISO^{+}, for instance by replacing $sa^{\bullet}$ in (6.48) by any other symmetral, positive valued mould.

7.2 The extremal basis. Main statements.

For any non-ordered sequence of the form

$$\{\boldsymbol{n}\} \;=\; \{n_1, n_2, \ldots, n_r\} \;=\; \{m_1^{(r_1)}, m_2^{(r_2)}, \ldots, m_s^{(r_s)}\} \tag{7.1}$$

$$\textit{with} \qquad n_1 \leqslant n_2 \leqslant \cdots \leqslant n_r \quad \textit{and} \quad m_1 < m_2 < \cdots < m_s \tag{7.2}$$

the multiplicity correction $\mu^{\{\boldsymbol{n}\}}$ is defined as

$$\mu^{\{\boldsymbol{n}\}} \;=\; \prod_{1 \leqslant j \leqslant s} \frac{1}{(1+r_j)!} \tag{7.3}$$

and we denote $\overrightarrow{\boldsymbol{n}}$ resp. $\overleftarrow{\boldsymbol{n}}$ the ordered sequence obtained by arranging the elements of $\{\boldsymbol{n}\}$ in increasing resp. decreasing order. If $\boldsymbol{n}$ is an ordered sequence, $\widetilde{\boldsymbol{n}}$ denotes the same sequence with its order reversed. Lastly, for any $t \in \mathbb{R}$ we set:

$$(t)^{+} := |t| \quad \textit{if} \quad t > 0 \qquad \textit{and} \qquad (t)^{+} := 0 \quad \textit{if} \quad t \leqslant 0 \tag{7.4}$$

$$(t)^{-} := |t| \quad \textit{if} \quad t < 0 \qquad \textit{and} \qquad (t)^{-} := 0 \quad \textit{if} \quad t \geqslant 0 \tag{7.5}$$

Alongside $\mathrm{Da}^{\{\boldsymbol{n}\}}$ and $\mathbb{D}\mathrm{a}^{\langle \boldsymbol{n}\rangle}$ we also require the variants:

$$\underline{\mathrm{Da}}^{\{n_1,\ldots,n_r\}} := \frac{\mathrm{Da}^{\{n_1,\ldots,n_r\}}}{\prod(1+n_j)!} \quad ; \quad \underline{\mathbb{D}\mathrm{a}}^{\langle n_1,\ldots,n_r\rangle} := \frac{\mathbb{D}\mathrm{a}^{\langle n_1,\ldots,n_r\rangle}}{\prod n_j\,(1+n_j)} \tag{7.6}$$

Proposition 7.1 (The extremal basis.)
For each $n_1 \geqslant 1$ *there exists a unique iso-operator* $\mathrm{Da}^{\{n_1\}} = (n_1+1)!\,\underline{\mathrm{Da}}^{\{n_1\}}$ *in the positive cone* $\mathrm{Da}^+ \subset \mathbb{D}\mathrm{a}^+$ *verifying the normalisation condition*

$$\mathrm{Da}^{\{n_1\}} = (n_1-1)!\,\mathbb{D}\mathrm{a}^{\langle n_1\rangle} + \ldots \quad \Leftrightarrow \quad \underline{\mathrm{Da}}^{\{n_1\}} = \underline{\mathbb{D}\mathrm{a}}^{\langle n_1\rangle} + \ldots \tag{7.7}$$

and characterised by either of the following properties:
(i) among all iso-operators so normalised, $\mathrm{Da}^{\{n_1\}}$ *and* $\underline{\mathrm{Da}}^{\{n_1\}}$ *are* least elements *in the cone* Da^+
(ii) the expression of $\mathrm{Da}^{\{n_1\}}$ *resp.* $\underline{\mathrm{Da}}^{\{n_1\}}$ *in the basis* $\mathbb{D}\mathrm{a}^{\langle \boldsymbol{n}\rangle}$ *resp.* $\underline{\mathbb{D}\mathrm{a}}^{\langle \boldsymbol{n}\rangle}$ *involves no weakly decreasing sequences* $\boldsymbol{n} = (n_1, ..., n_r)$ *of length* $r \geqslant 2$.
The system $\mathrm{Da}^{\{\bullet\}}$ *or* $\underline{\mathrm{Da}}^{\{\bullet\}}$ *constitutes the so-called* extremal basis *of* ISO.

Proposition 7.2 (Analytical properties of the extremal basis.)
The elements of the positive basis are given by $\mathrm{Da}^{\{1\}} = 2\,\underline{\mathrm{Da}}^{\{1\}} := \mathrm{Dn}^{\{1\}}$ *and by an induction rule*

$$\mathrm{Da}^{\{n\}} = -\partial\,\mathrm{Da}^{\{n-1\}} - \sum_{|\boldsymbol{n}|=n} \mathrm{ka}_{\{\boldsymbol{n}\}} \quad \mu^{\{\boldsymbol{n}\}}\,\mathrm{Da}^{\{\boldsymbol{n}\}} \tag{7.8}$$

$$(1+n)\,\underline{\mathrm{Da}}^{\{n\}} = -\partial\,\underline{\mathrm{Da}}^{\{n-1\}} - \sum_{|\boldsymbol{n}|=n} \underline{\mathrm{ka}}_{\{\boldsymbol{n}\}} \quad \mu^{\{\boldsymbol{n}\}}\,\underline{\mathrm{Da}}^{\{\boldsymbol{n}\}} \tag{7.9}$$

involving non-negative coefficients $\underline{\mathrm{ka}}_{\{\boldsymbol{n}\}}$ *of simple multiplicative structure:*

$$\underline{\mathrm{ka}}_{\{\boldsymbol{n}\}} = (r_1-1)\,(-1+n_1)^-\,(-1+n_1-n_2)^- \prod_{2\leqslant j\leqslant r} (-1+n_1+\ldots+n_{j-1}-n_j)^+ \tag{7.10}$$

Here r_1 *denotes the multiplicity of the smallest element in the non-ordered sequence* $\boldsymbol{n}$. *As a consequence,* $\underline{\mathrm{ka}}_{\{\boldsymbol{n}\}}$ *is* > 0 *if and only if*

$$n_1 = n_2 \geqslant 2 \quad \textit{and} \quad n_1 + n_2 + \cdots + n_{j-1} \geqslant 2 + n_j \tag{7.11}$$

The expansion of the $\mathrm{Da}^{\{n\}}, \underline{\mathrm{Da}}^{\{n\}}$ *in the* $\mathbb{D}\mathrm{a}^{\langle \boldsymbol{n}\rangle}$ *basis:*

$$\mathrm{Da}^{\{n\}} = \sum_{1\leqslant r} \sum_{n=n_1+\ldots+n_r} \mathrm{ta}_{n_1,\ldots,n_r}\,\mathbb{D}\mathrm{a}^{\langle n_1,\ldots,n_r\rangle} \tag{7.12}$$

$$\underline{\mathrm{Da}}^{\{n\}} = \sum_{1\leqslant r} \sum_{n=n_1+\ldots+n_r} \underline{\mathrm{ta}}_{n_1,\ldots,n_r}\,\underline{\mathbb{D}\mathrm{a}}^{\langle n_1,\ldots,n_r\rangle} \tag{7.13}$$

as well as the expression of the involution $\sim : \mathrm{Da}^{\{n\}} \mapsto \widetilde{\mathrm{Da}}^{\{n\}}$

$$\widetilde{\mathrm{Da}}^{\{n\}} = \mathrm{fa}_{\{\boldsymbol{n}\}} \quad \mu^{\{\boldsymbol{n}\}}\,\mathrm{Da}^{\{\boldsymbol{n}\}} \tag{7.14}$$

$$\widetilde{\underline{\mathrm{Da}}}^{\{n\}} = \underline{\mathrm{fa}}_{\{\boldsymbol{n}\}} \quad \mu^{\{\boldsymbol{n}\}}\,\underline{\mathrm{Da}}^{\{\boldsymbol{n}\}} \qquad (\textit{with} \quad \underline{\mathrm{fa}}_{\{\boldsymbol{n}\}} = (-1)^{r(\boldsymbol{n})}\underline{\mathrm{ta}}_{\vec{\boldsymbol{n}}}) \tag{7.15}$$

or again the formula for the co-product[76] $\sigma : \mathrm{Da}^{\{n\}} \mapsto \sigma(\mathrm{Da}^{\{n\}})$

$$\sigma(\mathrm{Da}^{\{n\}}) = \sum_{|\boldsymbol{p}|+|\boldsymbol{q}|=n} \mathrm{ha}_{\{\boldsymbol{p}\},\{\boldsymbol{q}\}}\ \mu^{\{\boldsymbol{p}\}}\,\mathrm{Da}^{\{\boldsymbol{p}\}} \otimes \mu^{\{\boldsymbol{q}\}}\,\mathrm{Da}^{\{\boldsymbol{q}\}} \tag{7.16}$$

$$\sigma(\underline{\mathrm{Da}}^{\{n\}}) = \sum_{|\boldsymbol{p}|+|\boldsymbol{q}|=n} \underline{\mathrm{ha}}_{\{\boldsymbol{p}\},\{\boldsymbol{q}\}}\ \mu^{\{\boldsymbol{p}\}}\,\underline{\mathrm{Da}}^{\{\boldsymbol{p}\}} \otimes \mu^{\{\boldsymbol{q}\}}\,\underline{\mathrm{Da}}^{\{\boldsymbol{q}\}} \qquad (\textit{with}\ \ \underline{\mathrm{ha}}_{\{\boldsymbol{p}\},\{\boldsymbol{q}\}} = \underline{\mathrm{ta}}_{\overleftarrow{\boldsymbol{p}},\overleftarrow{\boldsymbol{q}}})$$

also involve non-negative integers[77] $\underline{\mathrm{ta}}_{\boldsymbol{n}}$, $\underline{\mathrm{fa}}_{\{\boldsymbol{n}\}}$, $\underline{\mathrm{ha}}_{\{\boldsymbol{p}\},\{\boldsymbol{q}\}}$, *but the only structure constants with a transparent factorization are those coefficients* $\underline{\mathrm{ha}}_{\{\boldsymbol{p}\},\{\boldsymbol{q}\}}$ *for which one of the sequences* $\{\boldsymbol{p}\} = \{p_1 \leqslant p_2 \leqslant \ldots\}$ *or* $\{\boldsymbol{q}\} = \{q_1 \leqslant q_2 \leqslant \ldots\}$ *is of length one:*

$$\underline{\mathrm{ha}}_{\{\boldsymbol{p}\},\{q_1\}} = (q_1 - p_1)^+ \prod_{2\leqslant j\leqslant r} (q_1 + p_1 + p_2 + \ldots + p_{j-1} - p_j)^+ \tag{7.17}$$

$$\underline{\mathrm{ha}}_{\{p_1\},\{\boldsymbol{q}\}} = (p_1 - q_1)^- \prod_{2\leqslant j\leqslant r} (p_1 + q_1 + q_2 + \ldots + q_{j-1} - q_j)^+ \tag{7.18}$$

Proposition 7.3 (Expression of the general structure constants.)
Due to the normalisation rule, $\underline{\mathrm{ta}}_{n_1} = 1$. *For* $r \geqslant 2$ *there are two logically consistent ways of calculating* $\underline{\mathrm{ta}}_{\boldsymbol{n}}$. *One is the* rightward induction

$$\underline{\mathrm{ta}}_{\boldsymbol{n}} := \sum_{1\leqslant s<r}\ \sum_{\{\boldsymbol{n^1}\ldots\boldsymbol{n^s}.n_r\}=\{\boldsymbol{n}\}} \underline{\mathrm{ha}}_{\{|\boldsymbol{n^1}|,\ldots,|\boldsymbol{n^s}|\},\{n_r\}}\ \underline{\mathrm{ta}}_{\boldsymbol{n^1}} \ldots \underline{\mathrm{ta}}_{\boldsymbol{n^s}}\ \underline{\mathrm{ta}}_{n_r} \tag{7.19}$$

which expresses $\underline{\mathrm{ta}}_{\boldsymbol{n}}$ *as a superposition of the special coefficients* $\underline{\mathrm{ha}}_{\{p_1,\ldots,p_r\},\{q_1\}}$ *as factorised in (7.17). The other is the* leftward induction

$$\underline{\mathrm{ta}}_{\boldsymbol{n}} = \sum_{1\leqslant s<r}\ \sum_{\{n_1.\boldsymbol{n^1}\ldots\boldsymbol{n^s}\}=\{\boldsymbol{n}\}} \underline{\mathrm{ha}}_{\{n_1\},\{|\boldsymbol{n^1}|,\ldots,|\boldsymbol{n^s}|\}}\ \underline{\mathrm{ta}}_{n_1}\ \underline{\mathrm{ta}}_{\boldsymbol{n^1}} \ldots \underline{\mathrm{ta}}_{\boldsymbol{n^s}} \tag{7.20}$$

which expresses $\underline{\mathrm{ta}}_{\boldsymbol{n}}$ *as a superposition of the special coefficients* $\underline{\mathrm{ha}}_{\{p_1,\ldots,p_r\},\{q_1\}}$ *as factorised in (7.18).*

The coefficients $\mathrm{ta}_{\boldsymbol{n}}$ *in turn yield direct expressions for the* general *structure constants* $\underline{\mathrm{ha}}_{\{\boldsymbol{p}\},\{\boldsymbol{q}\}}$ *and* $\underline{\mathrm{fa}}_{\{\boldsymbol{n}\}}$ *:*

$$\underline{\mathrm{ha}}_{\{\boldsymbol{p}\},\{\boldsymbol{q}\}} = \underline{\mathrm{ta}}_{\overleftarrow{\boldsymbol{p}},\overleftarrow{\boldsymbol{q}}} \tag{7.21}$$

$$\underline{\mathrm{fa}}_{\{\boldsymbol{n}\}} = (-1)^{r(\boldsymbol{n})}\ \underline{\mathrm{ta}}_{\overrightarrow{\boldsymbol{n}}} \tag{7.22}$$

with ordered sequences $\overleftarrow{\boldsymbol{p}}$, $\overleftarrow{\boldsymbol{q}}$ *made up of the elements of* $\{\boldsymbol{p}\}$, $\{\boldsymbol{q}\}$ *arranged in decreasing order.*

[76] Recall that σ is co-associative, but not co-commutative

[77] except $\underline{\mathrm{fa}}_{\{\boldsymbol{n}\}}$ whose sign is that of $(-1)^{r(\boldsymbol{n})}$

7.3 Complements.

Remark 7.1 Multiple co-product.

The preceding Propositions amount to the calculation sequence:

$$\textit{special}\ \underline{\mathrm{ha}}_{\{\bullet\},\{\bullet\}} \implies \textit{general}\ \underline{\mathrm{ta}}_{\bullet} \implies \textit{general}\ \underline{\mathrm{ha}}_{\{\bullet\},\{\bullet\}} \tag{7.23}$$

In fact, the general $\underline{\mathrm{ta}}_{\bullet}$ also yield the structure coefficients

$$\underline{\mathrm{ha}}_{\{\boldsymbol{n^1}\},\{\boldsymbol{n^2}\},\dots,\{\boldsymbol{n^s}\}} = \underline{\mathrm{ta}}_{\overleftarrow{\boldsymbol{n^1}},\overleftarrow{\boldsymbol{n^2}},\dots,\overleftarrow{\boldsymbol{n^s}}} \tag{7.24}$$

attached to the multiple co-product

$$\sigma^{s-1}(\underline{\mathrm{Da}}^{\{n\}}) = \sum_{|\boldsymbol{n^1}|+\dots|\boldsymbol{n^s}|=n} \underline{\mathrm{ha}}_{\{\boldsymbol{n^1}\},\{\boldsymbol{n^2}\},\dots,\{\boldsymbol{n^s}\}}\ \mu^{\boldsymbol{n^1}}\ \underline{\mathrm{Da}}^{\{\boldsymbol{n^1}\}} \dots \mu^{\boldsymbol{n^s}}\ \underline{\mathrm{Da}}^{\{\boldsymbol{n^s}\}} \tag{7.25}$$

Remark 7.2 Inductive calculation of the structure constants.

Let us examine how the induction scheme (7.23) works up to $r = 4$:

$$\begin{aligned} \underline{\mathrm{ta}}_{n_1} &= 1 \qquad (\forall\ n_1) \\ \underline{\mathrm{ta}}_{n_1,n_2} &= \underline{\mathrm{ha}}_{\{n_1\},\{n_2\}}\ \underline{\mathrm{ta}}_{n_1}\ \underline{\mathrm{ta}}_{n_2} = (n_2-n_1)^+ = (n_1-n_2)^- \end{aligned}$$

$$\begin{array}{lcl} \underline{\mathrm{ta}}_{n_1,n_2,n_3} = & \| & \underline{\mathrm{ta}}_{n_1,n_2,n_3} = \\ +\underline{\mathrm{ha}}_{\{n_1,n_2\},\{n_3\}}\ \underline{\mathrm{ta}}_{n_1}\ \underline{\mathrm{ta}}_{n_2}\ \underline{\mathrm{ta}}_{n_3} & \| & +\underline{\mathrm{ha}}_{\{n_1\},\{n_2,n_3\}}\ \underline{\mathrm{ta}}_{n_1}\ \underline{\mathrm{ta}}_{n_2}\ \underline{\mathrm{ta}}_{n_3} \\ +\underline{\mathrm{ha}}_{\{n_1+n_2\},\{n_3\}}\ \underline{\mathrm{ta}}_{n_1,n_2}\ \underline{\mathrm{ta}}_{n_3} & \| & +\underline{\mathrm{ha}}_{\{n_1\},\{n_2+n_3\}}\ \underline{\mathrm{ta}}_{n_1}\ \underline{\mathrm{ta}}_{n_2,n_3} \end{array}$$

$$\begin{array}{lcl} \underline{\mathrm{ta}}_{n_1,n_2,n_3,n_4} = & \| & \underline{\mathrm{ta}}_{n_1,n_2,n_3,n_4} = \\ +\underline{\mathrm{ha}}_{\{n_1,n_2,n_3\},\{n_4\}}\ \underline{\mathrm{ta}}_{n_1}\ \underline{\mathrm{ta}}_{n_2}\ \underline{\mathrm{ta}}_{n_3}\ \underline{\mathrm{ta}}_{n_4} & \| & +\underline{\mathrm{ha}}_{\{n_1\},\{n_2,n_3,n_4\}}\ \underline{\mathrm{ta}}_{n_1}\ \underline{\mathrm{ta}}_{n_2}\ \underline{\mathrm{ta}}_{n_3}\ \underline{\mathrm{ta}}_{n_4} \\ +\underline{\mathrm{ha}}_{\{n_1+n_2,n_3\},\{n_4\}}\ \underline{\mathrm{ta}}_{n_1,n_2}\ \underline{\mathrm{ta}}_{n_3}\ \underline{\mathrm{ta}}_{n_4} & \| & +\underline{\mathrm{ha}}_{\{n_1\},\{n_2+n_3,n_4\}}\ \underline{\mathrm{ta}}_{n_1}\ \underline{\mathrm{ta}}_{n_2,n_3}\ \underline{\mathrm{ta}}_{n_4} \\ +\underline{\mathrm{ha}}_{\{n_1+n_3,n_2\},\{n_4\}}\ \underline{\mathrm{ta}}_{n_1,n_3}\ \underline{\mathrm{ta}}_{n_2}\ \underline{\mathrm{ta}}_{n_4} & \| & +\underline{\mathrm{ha}}_{\{n_1\},\{n_2+n_4,n_3\}}\ \underline{\mathrm{ta}}_{n_1}\ \underline{\mathrm{ta}}_{n_2,n_4}\ \underline{\mathrm{ta}}_{n_3} \\ +\underline{\mathrm{ha}}_{\{n_2+n_3,n_1\},\{n_4\}}\ \underline{\mathrm{ta}}_{n_2,n_3}\ \underline{\mathrm{ta}}_{n_1}\ \underline{\mathrm{ta}}_{n_4} & \| & +\underline{\mathrm{ha}}_{\{n_1\},\{n_3+n_4,n_2\}}\ \underline{\mathrm{ta}}_{n_1}\ \underline{\mathrm{ta}}_{n_3,n_4}\ \underline{\mathrm{ta}}_{n_2} \\ +\underline{\mathrm{ha}}_{\{n_1+n_2+n_3\},\{n_4\}}\ \underline{\mathrm{ta}}_{n_1,n_2,n_3}\ \underline{\mathrm{ta}}_{n_4} & \| & +\underline{\mathrm{ha}}_{\{n_1\},\{n_2+n_3+n_4\}}\ \underline{\mathrm{ta}}_{n_1}\ \underline{\mathrm{ta}}_{n_2,n_3,n_4} \end{array}$$

In general, the vanishing terms predominate in these sums.

Exercise: write down, then directly check, the compatibility relations for the right- and leftward induction.

There also exists a more general induction that covers both the right- and leftward inductions as special cases. It goes like this:

$$\underline{\mathrm{ta}}_{\boldsymbol{p}.\boldsymbol{q}} = \sum_{\boldsymbol{p^1}\dots\boldsymbol{p^{s_1}}\ =\ \boldsymbol{p}}^{1\leqslant s_1\leqslant r_1}\ \sum_{\boldsymbol{q^1}\dots\boldsymbol{q^{s_1}}\ =\ \boldsymbol{q}}^{1\leqslant s_2\leqslant r_2} \underline{\mathrm{ha}}_{\{|\boldsymbol{p^1}|,\dots,|\boldsymbol{p^{s_1}}|\},\{|\boldsymbol{q^1}|,\dots,|\boldsymbol{q^{s_2}}|\}} \prod_{i=1}^{i=s_1} \underline{\mathrm{ta}}_{\boldsymbol{p^i}} \prod_{j=1}^{j=s_2} \underline{\mathrm{ta}}_{\boldsymbol{q^j}} \tag{7.26}$$

Remark 7.3 Recurrent patterns in the structure constants.

Let us introduce the auxiliary expressions:

$$\theta_{\omega_0,\omega_1,\dots,\omega_r} := (\omega_0-\omega_1)(\omega_0+\omega_1-\omega_2)\dots(\omega_0+\dots+\omega_{r-1}-\omega_r) \quad (7.27)$$

$$\theta_{\binom{*}{\omega_0},\binom{\epsilon_1}{\omega_1},\dots,\binom{\epsilon_r}{\omega_r}} := (\omega_0-\omega_1)^{\epsilon_1}(\omega_0+\omega_1-\omega_2)^{\epsilon_2}\dots(\omega_0+\dots+\omega_{r-1}-\omega_r)^{\epsilon_r} \quad (7.28)$$

with $\omega_j \in \mathbb{R}$ and $t^{\pm}$ defined as in (7.4)-(7.5). The plain coefficients (7.27) have an obvious Lie algebra interpretation:

$$\theta_{\omega_0,\omega_1,\dots,\omega_r}\, e^{(\omega_0+\dots+\omega_r).z}\,\partial \;\equiv\; [e^{\omega_r.z}\,\partial,\dots[e^{\omega_2.z}\,\partial,[e^{\omega_1.z}\,\partial,e^{\omega_0.z}\,\partial]]\dots] \quad (7.29)$$

but it takes the sign-modified coefficients (7.28) to express the basic structure constants of the bialgebras $ISO \subset {}^{\sharp}ISO$. The formulae read:

$\underline{\mathrm{ta}}_{n_1,n_2,\dots,n_r}$	$\equiv$	$0 \quad \textit{if} \quad n_r=1$	$\textit{or} \quad n_1 \geqslant n_2 \dots \geqslant n_r$
$\underline{\mathrm{ta}}_{n_1,n_2,\dots,n_r}$	$\equiv$	$\theta_{\binom{*}{n_r},\binom{-}{n_{r-1}},\binom{+}{\dots},\binom{+}{n_2},\dots,\binom{+}{n_1}}$	$\textit{if} \quad n_1+\dots n_{r-1} \leqslant n_r$
$\underline{\mathrm{ha}}_{\{p_1,\dots,p_r\},\{q_1\}}$	$\equiv$	$\theta_{\binom{*}{q_1},\binom{+}{p_1},\binom{+}{p_2},\binom{+}{p_3},\dots,\binom{+}{p_r}}$	$(p_1 \leqslant p_2 \dots \leqslant p_r)$
$\underline{\mathrm{ha}}_{\{p_1\},\{q_1,\dots,q_r\}}$	$\equiv$	$\theta_{\binom{*}{p_1},\binom{-}{q_1},\binom{+}{q_2},\binom{+}{q_3},\dots,\binom{+}{q_r}}$	$(q_1 \leqslant q_2 \dots \leqslant q_r)$
$\underline{\mathrm{ka}}_{\{n_1,n_2,\dots,n_r\}}$	$\equiv$	$(r_1-1)\,\theta_{\binom{*}{-1},\binom{-}{n_1},\binom{-}{n_2},\binom{+}{n_3},\dots,\binom{+}{n_r}}$	$(n_1 \leqslant n_2 \dots \leqslant n_r)$

These are in effect the only fully factorable structure constants, but from them all others can be recovered under the induction rules (7.19),(7.20),(7.21). Regarding the first identity, we may note that, due to the condition $n_1 + \dots n_{r-1} \leqslant n_r$, the summands in the expansion (7.19) of $\underline{\mathrm{ta}}_{\boldsymbol{n}}$ ("rightward induction") have non-vanishing $\underline{\mathrm{ha}}_{\{\bullet\},\{n_r\}}$-factors and, depending on the relative sizes of $n_1,\dots,n_{r-1}$, varying mixtures of vanishing or non-vanishing "earlier" $\underline{\mathrm{ta}}_{\bullet}$-factors. Remarkably, this composite make-up does not prevent the global $\underline{\mathrm{ta}}_{\boldsymbol{n}}$ (i.e. the one on the left-hand side of (7.19)) from admitting a full and uniform (i.e. case independent) factorisation.

Remark 7.4 Relations between $\underline{\mathrm{ta}}_{\bullet}$ and $\underline{\mathrm{fa}}_{\{\bullet\}}$.

The involutive nature of the transform $Da \mapsto \widetilde{Da}$ has for analytical expression the following [78] relation (7.30):

$$I_{\{\bullet\}} \;=\; \underline{\mathrm{fa}}_{\{\bullet\}} \circ \underline{\mathrm{fa}}_{\{\bullet\}} \quad (7.30)$$

$$(-1)^{r(\bullet)}\, \underline{\mathrm{ta}}_{\tilde{\bullet}} \;=\; \underline{\mathrm{fa}}_{\{\bullet\}} \circ \underline{\mathrm{ta}}_{\bullet} \quad (7.31)$$

[78] with $I_{\{n_1\}} := 1$ and $I_{\{n_1,\dots,n_r\}} := 0$ if $r \geqslant 2$.

The relation (7.31), on the other hand, reflects the expansion (7.13) of $\underline{\mathrm{Da}}^{\{\bullet\}}$ in the basis $\underline{\mathbb{D}\mathrm{a}}^{\langle\bullet\rangle}$, combined with the expression $\underline{\mathbb{D}\mathrm{a}}^{\langle\bullet\rangle} \mapsto (-1)^{r(\bullet)}\, \underline{\mathbb{D}\mathrm{a}}^{\langle\tilde{\bullet}\rangle}$ of the involution $\sim$ in that basis (see (6.55)).

Here, $\circ$ denotes as usual the mould (or comould) composition,[79] but since in both cases the first composition factor, namely $\underline{\mathrm{fa}}_{\{\bullet\}}$, carries non ordered sequences $\{\bullet\}$, the sums on the right-hand side of (7.30) or (7.31) should extend to all partitions[80] of $\{\bullet\}$. Thus, for $r = 3$, the identities (7.30) and (7.31) read:

$$\begin{array}{l|l}
0 = & (-1)^3\, \underline{\mathrm{ta}}_{n_3,n_2,n_1} \;= \\
+\underline{\mathrm{fa}}_{\{n_1+n_2+n_3\}}\, \underline{\mathrm{fa}}_{\{n_1,n_2,n_3\}} & +\underline{\mathrm{fa}}_{\{n_1+n_2+n_3\}}\, \underline{\mathrm{ta}}_{n_1,n_2,n_3} \\
+\underline{\mathrm{fa}}_{\{n_1,n_2+n_3\}}\, \underline{\mathrm{fa}}_{\{n_1\}}\, \underline{\mathrm{fa}}_{\{n_2,n_3\}} & +\underline{\mathrm{fa}}_{\{n_1,n_2+n_3\}}\, \underline{\mathrm{ta}}_{n_1}\, \underline{\mathrm{ta}}_{n_2,n_3} \\
+\underline{\mathrm{fa}}_{\{n_2,n_1+n_3\}}\, \underline{\mathrm{fa}}_{\{n_2\}}\, \underline{\mathrm{fa}}_{\{n_1,n_3\}} & +\underline{\mathrm{fa}}_{\{n_2,n_1+n_3\}}\, \underline{\mathrm{ta}}_{n_2}\, \underline{\mathrm{ta}}_{n_1,n_3} \\
+\underline{\mathrm{fa}}_{\{n_3,n_1+n_2\}}\, \underline{\mathrm{fa}}_{\{n_3\}}\, \underline{\mathrm{fa}}_{\{n_1,n_2\}} & +\underline{\mathrm{fa}}_{\{n_3,n_1+n_2\}}\, \underline{\mathrm{ta}}_{n_3}\, \underline{\mathrm{ta}}_{n_1,n_2} \\
+\underline{\mathrm{fa}}_{\{n_1,n_2,n_3\}}\, \underline{\mathrm{fa}}_{\{n_1\}}\, \underline{\mathrm{fa}}_{\{n_2\}}\, \underline{\mathrm{fa}}_{\{n_3\}} & +\underline{\mathrm{fa}}_{\{n_1,n_2,n_3\}}\, \underline{\mathrm{ta}}_{n_1}\, \underline{\mathrm{ta}}_{n_2}\, \underline{\mathrm{ta}}_{n_3}
\end{array}$$

One may note the absence of the multiplicity correction $\mu^\bullet$ in (7.30) and (7.31). Moreover, for $\boldsymbol{n}$ weakly decreasing, all partial $\underline{\mathrm{ta}}_{\boldsymbol{n}^i}$ factors on the right-hand side of (7.31) also carry weakly decreasing sequences $\boldsymbol{n^i}$ and therefore vanish unless $r(\boldsymbol{n^i}) = 1$. That leaves only a single non-vanishing summand, namely $\underline{\mathrm{fa}}_{\{n_1,\ldots,n_r\}}\underline{\mathrm{ta}}_{n_1} \ldots \underline{\mathrm{ta}}_{n_r}$, so that in this case (7.31) reduces to (7.22).

Remark 7.5 Mould inversion and sign change.

The relations (7.30), (7.31) are vaguely evocative of other relations verified by an important pair ($sofo^\bullet$, $musofo^\bullet$) of mutually inverse, symmetrel[81] moulds that, just like $\underline{\mathrm{fa}}_{\{\bullet\}}, \underline{\mathrm{ta}}_{\,\bullet}$, are also 'product- and sign-based' :

$$\mathrm{sofo}^{x_1,\ldots,x_r} := (-1)^r \prod_{1\leqslant j\leqslant r} \sigma_+(x_1 + \cdots + x_j) \tag{7.32}$$

$$\mathrm{musofo}^{x_1,\ldots,x_r} := (-1)^{r-1}\, \sigma_-(x_1 + \cdots + x_r) \prod_{2\leqslant j\leqslant r} \sigma_+(x_j + \cdots + x_r) \tag{7.33}$$

$$1^\bullet := \mathrm{sofo}^\bullet \times \mathrm{musofo}^\bullet \tag{7.34}$$

with $\sigma_\pm(t) := 1$ *if* $\pm\, t > 0$ and $\sigma_\pm(t) := 0$ *if* $\pm\, t < 0$.

[79] see §6.3.

[80] The number of all partitions of a set of r labelled elements is known as the Bell number. The first Bell numbers are 1, 2, 5, 15, 52, 203 etc. Ordinary mould composition involves fewer summand, namely 2^{r-1}.

[81] Symmetrelity holds only if we regard the moulds as distributions on $\mathbb{R}$. If we view them as defined on $\mathbb{Z}$, symmetrelity fails on certain negligible subsets $x_i + \ldots + x_j = 0$.

Remark 7.6 Amplification-specialisation.

The very special role played by the indices $n_i \equiv 1$ in many formulae[82] makes it tempting to look whether we might not gain in simplicity by replacing the $\mathrm{Da}^{\{\bullet\}}$ basis of $^{\sharp}ISO$ by the basis $\mathbb{D}\mathrm{aa}^{\langle\bullet\rangle}$ resulting from the amplification-specialisation transform

$$\mathrm{amp}\mathbb{D}\mathrm{a}^{\left\langle {a_1 \atop n_1} {\dots,\dots \atop \dots,\dots} {a_r \atop n_r} \right\rangle} = \sum_{2\leqslant n_j\,,\,0\leqslant d_j} \mathbb{D}\mathrm{a}^{\langle 1^{(\boldsymbol{d_1})},n_1,\dots,1^{(\boldsymbol{d_r})},n_r\rangle} \prod_{1\leqslant j\leqslant r} (a_j+\dots+a_r)^{d_j} \tag{7.35}$$

$$= \sum_{2\leqslant n_j\,,\,0\leqslant m_j} \mathbb{D}\mathrm{aa}^{\left\langle {m_1 \atop n_1} {\dots,\dots \atop \dots,\dots} {m_r \atop n_r} \right\rangle} \prod_{1\leqslant j\leqslant r} (a_j)^{m_j} \tag{7.36}$$

which turns the symmetral $\mathbb{D}\mathrm{a}^{\langle\bullet\rangle}$ into a symmetral $\mathbb{D}\mathrm{aa}^{\langle\bullet\rangle}$ via the equally symmetral $\mathrm{amp}\mathbb{D}\mathrm{a}^{\langle\bullet\rangle}$. On closer examination, however, it turns out that we would gain nothing from switching to $\mathbb{D}\mathrm{aa}^{\langle\bullet\rangle}$.

Conclusion.

All the lemmas in this and the preceding section can be established, roughly in the order in which they are enunciated, by resorting to the standard methods of mould calculus (conservation/transformation of the main symmetry types etc) and, in nearly all cases, by reasoning inductively on the length r of the mould components. The main surprise, once again, resides in the highly improbable properties of the extremal basis, such as the prevalence of integer structure coefficients where one would expect rational ones. Regarding applications, those pertaining to the universal asymptotics of slow-growing germs are summarized in §6 and treated at greater length in [E5], chapter 7, pp 287-303, though with slightly different notations.[83] As for the vast subject of iso-convex functions and iso-differential equations (as an alternative means of enlarging our composition groups), we leave it open for now.

8 Up to ω^ω : the ultra-exponential scale.

This section is devoted to constructing the *minimal systems* or *'towers'* of fast/slow growing germs necessary for *compositional closure*[84] – notably for the conjugation of germs of unequal exponentiality and for the continuous iteration of germs of non-zero exponentiality.

[82] for instance in the derivation rule (6.57) or in the induction rule (7.11) or again in the fact that $\underline{\mathrm{ta}}_{n_1,\dots,n_r} \equiv 0$ whenever $n_r = 1$.

[83] with *Post* instead of *Iso*, *post-homogeneous* instead of *iso-differential* etc.

[84] which of course does not mean *closure under composition*, but closure under the *solving of all (meaningful) composition equations.*

8.1 Towers of ultraexponentials and ultralogarithms.

A tower $\boldsymbol{\mathcal{L}}$ (resp. $\boldsymbol{\mathcal{E}}$) of ultralogarithms (resp. ultraexponentials) is a sequence of slow-growing (resp. fast-growing) germs

$$\boldsymbol{\mathcal{L}} := \big(\mathcal{L}_1, \mathcal{L}_2, \mathcal{L}_3, \dots\big) \tag{8.1}$$
$$\boldsymbol{\mathcal{E}} := \big(\mathcal{E}_1, \mathcal{E}_2, \mathcal{E}_3, \dots\big) \qquad \textit{with} \qquad \mathcal{L}_n \circ \mathcal{E}_n \equiv id \tag{8.2}$$

such that

$$\mathcal{L}_1 \circ E \circ \mathcal{E}_1 \equiv T \quad ; \quad \mathcal{L}_1 \circ L \circ \mathcal{E}_1 \equiv T^{\circ(-1)} \tag{8.3}$$
$$\mathcal{L}_r \circ \mathcal{E}_{r-1} \circ \mathcal{E}_r \equiv T \quad ; \quad \mathcal{L}_r \circ \mathcal{L}_{r-1} \circ \mathcal{E}_r \equiv T^{\circ(-1)} \qquad (\forall r \geqslant 2) \tag{8.4}$$

The induction automatically implies that each $\mathcal{E}_n$ (resp. $\mathcal{L}_n$) grows at a faster (resp. slower) rate than any finite iterates of $\mathcal{E}_{n-1}$ (resp. $\mathcal{L}_{n-1}$).

To alleviate notations, we often write:

$$\mathcal{E}_{n,k} := \mathcal{E}_n^{\circ\, k} \quad ; \quad \mathcal{L}_{n,k} := \mathcal{L}_n^{\circ\, k} \quad ; \quad \mathcal{E}_{n,-k} := \mathcal{L}_{n,k} \qquad (\forall n \in \mathbb{N}\ ,\ k \in \mathbb{Z}) \tag{8.5}$$

8.2 Central indeterminacy. Growth types.

The functional equations (8.3),(8.4) determine each succesive pair $\{\mathcal{E}_n, \mathcal{L}_n\}$ in terms of the preceding one, but only up to pre-compostion of $\mathcal{L}_n$ by a smooth 1-periodic germ P (i.e. such that $T \circ P = P \circ T$) and post-composition of $\mathcal{E}_n$ by $P^{\circ(-1)}$.

To be able to look on all these competing determinations as one single object, we quotient the semi-group of *smooth, slow germs* (i.e. germs of exponentiality k in $-\mathbb{N}^*$) by the equivalence relations $\overset{(i)}{\sim}$:

$$f_1 \overset{(i)}{\sim} f_2 \iff \exists\, c_1, c_2 \ \ s.t.\ \ 0 < c_1 < \frac{f_1^{(i)}(x)}{f_2^{(i)}(x)} < c_2 < +\infty \quad (x \gg 1) \tag{8.6}$$

Each $\overset{(i)}{\sim}$ may seem to be stronger than $\overset{(i-1)}{\sim}$, but in fact only $\overset{(1)}{\sim}$ is stronger than $\overset{(0)}{\sim}$. From $i = 1$ onwards, all $\overset{(i)}{\sim}$ are of equal strength. That readily follows from the universal asymptotics of slow functions (see §6).

For slow germs the following implications hold:

$$f_1 \overset{(i)}{\sim} f_2\ ,\ g_1 \overset{(i)}{\sim} g_2 \implies f_1 \circ g_1 \overset{(i)}{\sim} f_2 \circ g_2 \tag{8.7}$$
$$f_1 \overset{(i)}{\sim} f_2 \implies f_1^* \overset{(i)}{\sim} f_2^* \tag{8.8}$$

The first implication means that the composition $\circ$ carries over to the classes $[f]_i$ of slow germs. The second implication actually means two things: first, that the class $[f^*]_i$ of the iterator f^* does not depend on the particular *solution* of the equation $f^* \circ f = -1 + f^*$ that we select; and second, that it depends only on the *class* $[f]_i$ of the original germ. Thus, the operation $f \mapsto f^*$ carries over to the classes $[f]_i$. These classes are known as *growth types* of order i.

By reciprocation, the semi-group of *slow* growth types induces a semi-group of *fast* growth types, but the latter notion is entirely derivative on the former. In particular, even for equivalent fast germs $f_1 \overset{i}{\sim} f_2$ and $g_1 \overset{i}{\sim} g_2$, the ratios $(f_1 \circ g_1)^{(i)}/(f_2 \circ g_2)^{(i)}$ and $(f_1^*)^{(i)}/(f_2^*)^{(i)}$ often vary too wildly to remain within fixed bounds, even for $i = 0$.

8.3 Geometric incarnation of the semi-ring $[1, \omega^\omega[$.

For any growth type t *with successive iterators* t*, t**... *and for any transfinite ordinal* $\alpha < \omega^\omega$ *of expression:*

$$\alpha = \omega^r\, n_r + \omega^{r-1}\, n_{r-1} + \cdots + \omega_1\, n_1 + n_0 \qquad (n_i \in \mathbb{N}) \tag{8.9}$$

we define the transfinite iterate t^α *by*

$$t^\alpha := (t)^{n_0} \circ (t^*)^{n_1} \circ (t^{**})^{n_2} \circ \ldots (t^{*\cdots*})^{n_r} \qquad (\textit{order inversion}) \tag{8.10}$$

This actually defines on the whole semi-group of growth types a tranfinite iteration that obeys the rules

$$(t^\alpha) \circ (t^\beta) = t^{(\beta+\alpha)} \qquad (\textit{order inversion}) \qquad (\forall\, \alpha, \beta < \omega^\omega) \tag{8.11}$$

$$(t^\alpha)^\beta = t^{\alpha\beta} \qquad (\textit{no order inversion}) \qquad (\forall\, \alpha, \beta < \omega^\omega) \tag{8.12}$$

which (up to the order reversal in (8.10)-(8.11)) exactly reproduce the non-commutative arithmetics of the semi-ring $[1, \omega^\omega[$

This transfinite iteration carries over from the semi-group of *slow* to that of *fast* germs[85], with restoration of the 'correct' order in (8.10) and (8.11) and preservation of the already 'correct' order in (8.12). But these slight formal advantages count for little when weighed against the entirely derivative character of the classes of *fast* germs.[86]

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[85]but of course *not* to the total group consisting of slow, moderate, or fast germs.

[86]In other words, the rules for *fast* germs read:

$$\begin{aligned} t^\alpha &:= (t^{*\cdots*})^{n_r} \circ \ldots (t^{**})^{n_2} \circ (t^*)^{n_1} \circ (t)^{n_0} \\ (t^\alpha) \circ (t^\beta) &= t^{(\alpha+\beta)} \qquad (\forall\, \alpha, \beta < \omega^\omega) \\ (t^\alpha)^\beta &= t^{\alpha\beta} \qquad (\forall\, \alpha, \beta < \omega^\omega) \end{aligned}$$

8.4 Some useful notations. Iterators and connectors.

This subsection is mainly for settling notations. Given a ultraexponential $\mathcal{E}_n$ and a germ f that resembles $\mathcal{E}_n$ in the sense that:

$$\text{stat.lim.}_{k\to+\infty}\mathcal{L}_n^{\circ k}\circ f\circ\mathcal{E}_n^{\circ k}=\mathcal{E}_n \qquad (\Longleftrightarrow \text{expo}_{\mathcal{E}_n} f=1)$$

we produce a 'f-based' ultraexponential $\mathcal{E}_{n+1}$ of the next order by suitably combining the normalisers of f at its fixed points $+\infty$ and x_0 (which is often taken to be 0). Here are the definitions:
Normalisation at $x=+\infty$:

$$f^*\circ f=T\circ f^* \quad ; \quad f\circ {}^*\!f={}^*\!f\circ T \qquad (\text{expo}\, f=0) \tag{8.13}$$

$$f^\Diamond\circ f=E\circ f^\Diamond \quad ; \quad f\circ {}^\Diamond\!f={}^\Diamond\!f\circ E \qquad (\text{expo}\, f=1) \tag{8.14}$$

$$f^{\Diamond_n}\circ f=\mathcal{E}_n\circ f^{\Diamond_n} \quad ; \quad f\circ {}^{\Diamond_n}\!f={}^{\Diamond_n}\!f\circ \mathcal{E}_n \qquad (\text{expo}_{\mathcal{E}_n} f=1) \tag{8.15}$$

$$id \;=\; {}^*\!f\circ f^* \;=\; {}^\Diamond\!f\circ f^\Diamond \;=\; {}^{\Diamond_n}\!f\circ f^{\Diamond_n} \tag{8.16}$$

Normalisation at $x=0^+$:

$$f^\ddagger\circ f=\delta_c\circ f^\ddagger \quad ; \quad f\circ {}^\ddagger\!f={}^\ddagger\!f\circ \delta_c \qquad \big(c:=f'(0)>1\big) \tag{8.17}$$

$$f^\dagger\circ f=T\circ f^\dagger \quad ; \quad f\circ {}^\dagger\!f={}^\dagger\!f\circ T \qquad \Big(f^\dagger(x):=\frac{\log(f^\ddagger(x))}{\log c}\Big) \tag{8.18}$$

$$id \;=\; {}^\dagger\!f\circ f^\dagger \;=\; {}^\ddagger\!f\circ f^\ddagger \;=\; {}^\Diamond\!f\circ f^\Diamond \tag{8.19}$$

Normalisation at $x=x_0^+$:

$$f^\ddagger\circ f=\delta_{c,x_0}\circ f^\ddagger \quad ; \quad f\circ {}^\ddagger\!f={}^\ddagger\!f\circ \delta_{c,x_0} \qquad \big(\delta_{c,x_0}:=T_{x_0}\circ\delta_c\circ T_{-x_0}\big) \tag{8.20}$$

$$f^\dagger\circ f=T\circ f^\dagger \quad ; \quad f\circ {}^\dagger\!f={}^\dagger\!f\circ T \qquad \Big(f^\dagger(x):=\frac{\log(f^\ddagger(x)-x_0)}{\log c}\Big) \tag{8.21}$$

Notion of f-based ultraexponential:

$$\mathcal{E}_1^{[f]}:=f^\dagger\circ {}^\Diamond\!f \quad , \quad \mathcal{L}_1^{[f]}:=f^\Diamond\circ {}^\dagger\!f \tag{8.22}$$

$$E\circ\mathcal{E}_1^{[f]}=\mathcal{E}_1^{[f]}\circ T \quad , \quad \mathcal{L}_1^{[f]}\circ L=T^{\circ(-1)}\circ\mathcal{L}_1^{[f]} \tag{8.23}$$

$$\mathcal{E}_{n+1}^{[f]}:=f^\dagger\circ {}^{\Diamond_n}\!f \quad , \quad \mathcal{L}_{n+1}^{[f]}:=f^{\Diamond_n}\circ {}^\dagger\!f \tag{8.24}$$

$$\mathcal{E}_n\circ\mathcal{E}_{n+1}^{[f]}=\mathcal{E}_{n+1}^{[f]}\circ T \quad , \quad \mathcal{L}_{n+1}^{[f]}\circ L_n=T^{\circ(-1)}\circ\mathcal{L}_{n+1}^{[f]} \tag{8.25}$$

Periodic connectors: They are periodic mappings that measure the closeness of two f_i-based ultra-exponentials. Here is the definiton for n=1:

$$P_1^{[f_1,f_2]} \;:=\; f_1^\dagger\circ {}^\Diamond\!f_1\circ f_2^\Diamond\circ {}^\dagger\!f_2 \tag{8.26}$$

$$:=\delta_{\gamma_1}^{-1}\circ L\circ f_1^\ddagger\circ {}^\Diamond\!f_1\circ f_2^\Diamond\circ {}^\ddagger\!f_2\circ E\circ\delta_{\gamma_2} \tag{8.27}$$

$$\mathcal{E}_1^{[f_1]} \circ P_1^{[f_1,f_2]} = \mathcal{E}_1^{[f_2]} \qquad \textit{with} \qquad P_1^{[f_1,f_2]} \circ T = T \circ P_1^{[f_1,f_2]} \tag{8.28}$$

For $n > 1$ this becomes:

$$P_{n+1}^{[f_1,f_2]} := f_1^{\dagger} \circ {}^{\Diamond_n}f_1 \circ f_2^{\Diamond_n} \circ {}^{\dagger}f_2 \tag{8.29}$$

$$:= \delta_{\gamma_1}^{-1} \circ \mathcal{L}_n \circ f_1^{\ddagger} \circ {}^{\Diamond_n}f_1 \circ f_2^{\Diamond_n} \circ {}^{\ddagger}f_2 \circ \mathcal{E}_n \circ \delta_{\gamma_2} \tag{8.30}$$

$$\mathcal{E}_{n+1}^{[f_1]} \circ P_{n+1}^{[f_1,f_2]} = \mathcal{E}_{n+1}^{[f_2]} \qquad \textit{with} \qquad P_{n+1}^{[f_1,f_2]} \circ T = T \circ P_{n+1}^{[f_1,f_2]} \tag{8.31}$$

8.5 Construction of analytic ultraexponential towers.

We start with the ultraexponential $\mathcal{E}_1^{pre} := \mathcal{E}_1^{kneser}$ constructed by H. Kneser (see §8.7 below) and choose a_1 large enough for the shifted ultraexponential $\mathcal{E}_1 := \mathcal{E}_1^{pre} \circ T_{-a_1}$ to have a largest fixed point x_1. We then take the local iterator $\mathcal{E}_1^{\dagger}$ relative to that fixed point, as in (8.18), call the resulting germ $\mathcal{E}_2^{pre}$, and repeat the process with a_2 large enough, leading to $\mathcal{E}_2$. And so on:

$$\begin{array}{lllll}
\mathcal{E}_1^{\mathrm{pre}} := \mathcal{E}_1^{\mathrm{kneser}} & \to & \mathcal{E}_1 := \mathcal{E}_1^{\mathrm{pre}} \circ T_{-a_1} & \to & \\
\mathcal{E}_2^{\mathrm{pre}} := \mathcal{E}_1^{\dagger} & \to & \mathcal{E}_2 := \mathcal{E}_2^{\mathrm{pre}} \circ T_{-a_2} & \to & \\
\mathcal{E}_3^{\mathrm{pre}} := \mathcal{E}_2^{\dagger} & \to & \mathcal{E}_3 := \mathcal{E}_3^{\mathrm{pre}} \circ T_{-a_3} & \to & \dots\dots\dots
\end{array}$$

Instead of postcomposing by simple shifts, we may use analytic, 1-periodic mappings $\mathcal{P}_r$ with large built-in shifts[87], leading to the modified construction:

$$\begin{array}{lllll}
\mathcal{E}_1^{\mathrm{pre}} := \mathcal{E}_1^{\mathrm{kneser}} & \to & \mathcal{E}_1 := \mathcal{E}_1^{\mathrm{pre}} \circ \mathcal{P}_1^{\circ(-1)} & \to & \\
\mathcal{E}_2^{\mathrm{pre}} := \mathcal{E}_1^{\dagger} & \to & \mathcal{E}_2 := \mathcal{E}_2^{\mathrm{pre}} \circ \mathcal{P}_2^{\circ(-1)} & \to & \\
\mathcal{E}_3^{\mathrm{pre}} := \mathcal{E}_2^{\dagger} & \to & \mathcal{E}_3 := \mathcal{E}_3^{\mathrm{pre}} \circ \mathcal{P}_3^{\circ(-1)} & \to & \dots\dots\dots
\end{array}$$

8.6 Action of the periodic towers on ultraexponential towers.

Lemma (Conjugation averages).
For $\mathcal{L}$ slow and A, B identity-tangent (at infinity), the equation

$$A \circ \mathcal{L} \circ B^{\circ(-1)} = H \circ \mathcal{L} \circ H^{\circ(-1)} \tag{8.32}$$

admits a unique identity-tangent solution $H = \mathrm{Jugav}_{\mathcal{L}}(A, B)$ with

$$\mathrm{Jugav}_{\mathcal{L}}(A, B) := \lim_{n\to+\infty} A \circ \mathcal{L} \circ B^{\circ(-1)} \circ \mathcal{E} \circ \big(\mathcal{L} \circ A \circ \mathcal{L} \circ B^{\circ(-1)} \circ \mathcal{E}\big)^{\circ n} \circ \mathcal{E}^n \tag{8.33}$$

[87] i.e. such that $sup_x|\mathcal{P}_r(x) - x| = a_r \gg 1$

Since $\mathrm{Jugav}_{\mathcal{L}}(A,A) \equiv A$, *we may view* $\mathrm{Jugav}_{\mathcal{L}}(A,B)$ *as the 'conjugation average' of* A, B *respective to* $\mathcal{L}$.
Proof: Since $A \circ \mathcal{L} \circ B^{\circ(-1)} = \mathcal{A} \circ \mathcal{L}$ with $\mathcal{A} = A \circ \mathcal{L} \circ B^{\circ(-1)} \circ \mathcal{E} \sim id$, it suffices to show that the equation

$$A \circ \mathcal{L} = H \circ \mathcal{L} \circ H^{\circ(-1)} \tag{8.34}$$

admits a unique identity-tangent solution $H = \mathrm{Jugav}_{\mathcal{L}}(\mathcal{A}, id)$ with

$$\mathrm{Jugav}_{\mathcal{L}}(\mathcal{A}, id) := \lim_{n\to+\infty} \mathcal{A}\big(\mathcal{L} \circ \mathcal{A}\big)^{\circ n} \circ \mathcal{E}^{n} \tag{8.35}$$

If the limit exists in (8.33), it clearly verifies (8.32): replace H and $H^{\circ(-1)}$ in (8.32) by their expression (8.33) with n changed to $m+1$ and m respectively, and go to the limit. Moreover, if $H := \mathrm{Jugav}_{\mathcal{L}}(\mathcal{A}, id)$ is $\sim id$, no other solution of (8.32) can be identity-tangent. Indeed, we get the general solution H_{gen} by post-composing the particular solution H by real iterates of $\mathcal{L}$:

$$H_{gen} = H \circ \mathcal{L}^{\circ t} := H \circ \mathcal{E}^{*} \circ P^{\circ(-1)} \circ T^{\circ(-t)} \circ P \circ \mathcal{L}^{*}$$

For $t \neq 0$, H_{gen} fails to be $\sim id$, and for $t = 0$, the right-hand side above reduces to H irrespective of the choice of the periodic mapping P.

H can be formally expanded as

$$H := id + \sum_{0\leqslant n} a_n \qquad \textit{with} \tag{8.36}$$

$$id + a_0 = \mathcal{A} \qquad \textit{and} \tag{8.37}$$

$$\begin{aligned} id + a_0 + \dots a_n &= \mathcal{A} \circ (\mathcal{L} \circ \mathcal{A})^{\circ n} \circ \mathcal{E}^{\circ n} \qquad (8.38) \\ &= \mathcal{A}\circ(\mathcal{L}\circ\mathcal{A}\circ\mathcal{E})\circ(\mathcal{L}^{\circ 2}\circ\mathcal{A}\circ\mathcal{E}^{\circ 2})\circ\dots(\mathcal{L}^{\circ n}\circ\mathcal{A}\circ\mathcal{E}^{\circ n}) \end{aligned}$$

But for any n, any ϵ and any x large enough ($x \geqslant Const(\epsilon)$) we have

$$(1-\epsilon)\,\frac{a_0 \circ \mathcal{E}^{\circ n}(x)}{\partial_x \mathcal{E}^{\circ n}(x)} \geqslant \big|(\mathcal{L}^{\circ n}\circ\mathcal{A}\circ\mathcal{E}^{\circ n})(x) - x\big| \geqslant (1+\epsilon)\,\frac{a_0 \circ \mathcal{E}^{\circ n}(x)}{\partial_x \mathcal{E}^{\circ n}(x)} \tag{8.39}$$

$$\textit{with} \qquad \frac{a_0 \circ \mathcal{E}^{\circ n}(x)}{\partial_x \mathcal{E}^{\circ n}(x)} \prec \frac{1}{\partial_x \mathcal{E}^{\circ(n-1)}(x)} \prec \frac{1}{\mathcal{E}^{\circ(n-1)}(x)} \tag{8.40}$$

So the composition product in (8.33) clearly converges to an identity-tangent germ.

Let us also mention, for future use, the infinitesimal variant of (8.32)-(8.36), valid for any pair $(a(x) = o(x), b(x) = o(x))$:

$$\mathrm{jugav}_{\mathcal{L}}(a,b) := \lim_{\epsilon\to 0} \frac{1}{\epsilon}\, \mathrm{Jugav}_{\mathcal{L}}(id + \epsilon\, a, id + \epsilon\, b) \tag{8.41}$$

$$= \sum_{0\leqslant n} \frac{a \circ \mathcal{E}^{\circ n}}{\partial_x\, \mathcal{E}^{\circ n}} - \sum_{1\leqslant n} \frac{b \circ \mathcal{E}^{\circ n}}{\partial_x\, \mathcal{E}^{\circ n}} \tag{8.42}$$

Here again, the summation rules ensure that $jugav_{\mathcal{L}}(a,a) \equiv a$.

Proposition.(The general ultralogarithmic tower.)
Let towers $\boldsymbol{P}$ of periodic self-mappings of $\mathbb{R}$

$$\boldsymbol{P} := \big(P_1, P_2, P_3, \dots\big) \qquad \textit{with} \qquad P_r \circ T \equiv T \circ P_r \tag{8.43}$$

act on ultraexponential or, what amounts to the same, on ultralogarithmic towers:

$$H_{\boldsymbol{P}} : \quad \boldsymbol{\mathcal{L}} = \big(\mathcal{L}_1, \mathcal{L}_2, \mathcal{L}_3, \dots\big) \mapsto \underline{\boldsymbol{\mathcal{L}}} = \big(\underline{\mathcal{L}}_1, \underline{\mathcal{L}}_2, \underline{\mathcal{L}}_3, \dots\big) \tag{8.44}$$

via the following induction

$$\underline{\mathcal{L}}_1 := H_1 \circ \mathcal{L}_1 \circ H_1^{\circ(-1)} \quad \textit{with} \quad H_1 := \mathrm{Jugav}_{\mathcal{L}_1}(P_1, id) \tag{8.45}$$

$$\underline{\mathcal{L}}_r := H_r \circ \mathcal{L}_r \circ H_r^{\circ(-1)} \quad \textit{with} \quad H_r := \mathrm{Jugav}_{\mathcal{L}_r}(P_r, H_{r-1}) \quad (\forall r \geqslant 2) \tag{8.46}$$

or, more tellingly:

$$\underline{\mathcal{L}}_1 := \quad P_1 \circ \mathcal{L}_1 \qquad =: \quad H_1 \circ \mathcal{L}_1 \circ H_1^{\circ(-1)} \tag{8.47}$$

$$\underline{\mathcal{L}}_r := \quad P_r \circ \mathcal{L}_r \circ H_{r-1}^{-1} =: \quad H_r \circ \mathcal{L}_r \circ H_r^{\circ(-1)} \quad (\forall r \geqslant 2) \tag{8.48}$$

The sequence $\underline{\boldsymbol{\mathcal{L}}}$ so defined is actually a new ultralogarithmic tower, and conversely, any ultralogarithmic tower $\underline{\boldsymbol{\mathcal{L}}}$ is of the form $H_{\boldsymbol{P}}(\boldsymbol{\mathcal{L}})$ for some periodic germ tower $\boldsymbol{P}$.

The operations $H_{\boldsymbol{P}}$ are stable under composition, and obey the rule

$$H_{\boldsymbol{P}^{3,2}} \circ H_{\boldsymbol{P}^{2,1}} \equiv H_{\boldsymbol{P}^{3,2} \circ \boldsymbol{P}^{2,1}} \tag{8.49}$$

with the second "$\circ$" denoting the component-wise composition of periodic germ towers:

$$\{\boldsymbol{P}^{3,1} = \boldsymbol{P}^{3,2} \circ \boldsymbol{P}^{2,1}\} \Longleftrightarrow \{P_r^{3,1} = P_r^{3,2} \circ P_r^{2,1} \ , \ \forall r\} \tag{8.50}$$

If $P_r = id + \epsilon\, p_r$ with $p_r(x+1) \equiv p_r(x)$, then $H_r = id + \epsilon\, h_r + o(\epsilon)$ with:

$$h_1 \ = \sum_{0 \leqslant n_1} \frac{p_1 \circ \mathcal{E}_1^{\circ n_1}}{\partial_x\, \mathcal{E}_1^{\circ n_1}} \tag{8.51}$$

$$h_2 \ = \sum_{0 \leqslant n_2} \frac{p_2 \circ \mathcal{E}_2^{\circ n_2}}{\partial_x\, \mathcal{E}_2^{\circ n_2}} - \sum_{0 \leqslant n_1}^{1 \leqslant n_2} \frac{p_1 \circ \mathcal{E}_1^{\circ n_1} \circ \mathcal{E}_2^{\circ n_2}}{\partial_x(\mathcal{E}_1^{\circ n_1} \circ \mathcal{E}_2^{\circ n_2})} \tag{8.52}$$

$\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots$

$$h_r \ = \sum_{1 \leqslant s \leqslant r} (-1)^{r-s} \sum_{0 \leqslant n_s}^{1 \leqslant n_{s+1},\, \dots,\, 1 \leqslant n_r} \frac{p_s \circ \mathcal{E}_s^{\circ n_s} \circ \mathcal{E}_{s+1}^{\circ n_{s+1}} \circ \dots \mathcal{E}_r^{\circ n_r}}{\partial_x(\mathcal{E}_s^{\circ n_s} \circ \mathcal{E}_{s+1}^{\circ n_{s+1}} \circ \dots \mathcal{E}_r^{\circ n_r})} \tag{8.53}$$

Remark 1: Up to ultraexponentially small, non-periodic perturbations, each H_r coincides with the periodic mapping P_r. This is particularly apparent at the infinitesimal level: in view of the summation rules in (8.53), these relations reduce to $h_r = p_r + small$, with p_r periodic and *small* ultraexponentially small.

Remark 2: The operator $H_{\boldsymbol{P}}$ associated with a periodic tower respects the cohesiveness of ultraexponential towers, but usually (always?) destroys their analyticity. Conversely, the Witt tower connecting two analytic ultraexponential towers is usually (always?) non-analytic, but merely cohesive of Denjoy class ${}^{\omega}DEN$.

Remark 3: Appearances to the contrary, the periodic mappings $\mathcal{P}_r$ used in §8.5 to construct analytic ultraexponential towers have nothing to do with the periodic mappings P_r that go into the making of the operators $H_{\boldsymbol{P}}$. For their action to be defined, the $\mathcal{P}_r$ must have large enough shifts but, when analytic, they always result in analytic towers. The P_r , on the other hand, can be any periodic mappings but, even when analytic, they do not respect the analyticity of towers. But the real difference is this: while the $\mathcal{P}_r$ acts on the ultraexponentials as global functions on $\mathbb{R}$ (as soon as $r > 1$), the operator $H_{\boldsymbol{P}}$ acts on the ultraexponentials as germs at $+\infty$.

8.7 Kneser's analytic iteration of exp.

In [K], H. Kneser constructed not just a square root of iteration for the exponential (as the paper's title announces), but also an analytic solution $\mathcal{E}_1$ (real-analytic on $\mathbb{R}$) of the equation $E \circ \mathcal{E}_1 = \mathcal{E}_1 \circ T$, that is to say an analytic first-order ultraexponential. His elegant construction relies on classical Schröder iteration at the two complex fixed points of E closest to the real axis, combined with a realness-restoring conformal transform. It can be duplicated in numerous other situations. But it is numerically costly, and in any case, getting hold of analytic ultaexponentials $\mathcal{E}_k$ is of little consequence in our perspective: it does not alter the fact that the immense majority of germs with non-zero exponentialy will have, even *relatively to this analytic system* of ultraexponentials, *cohesive* rather than *analytic* fractional iterates.

8.8 Analytic ultra-quasiexponential towers.

The ultraexponential towers constructed in §8.5, though exact solutions of the system (8.3)-(8.4) of conjugation identities, have two drawbacks:
(i) The germs $\mathcal{E}_r$, $\mathcal{L}_r$ extend to full isomorphisms of $\mathbb{R}$ only for $r \geqslant 2$, but

neither for $r = 0$, since $\mathcal{E}_0 = E$, $\mathcal{L}_0 = L$, nor for $r = 1$, since the Kneser pair $\mathcal{E}_1^{kneser}$, $\mathcal{L}_1^{kneser}$ does not map $\mathbb{R}$ onto itself.
(ii) They ultimately rely on the construction of the Kneser pair, which involves a complicated conformal mapping and is computationally very costly.

So it is often convenient to consider instead *ultra-quasiexponentials*, which exactly reproduce the asymptotic behaviour of the ultraexponentials; are much easier to construct; and, despite verifying slightly different conjugation identities, can advantageously replace the ultraexponentials both as milestones of the ultrafast growth scale and as builting blocks of the trigebra of ultraseries.

The aim is to construct pairs $\{\mathcal{E}v_n, \mathcal{L}v_n\}$ of reciprocal ultrafast/ultraslow germs that verify (i), (ii) and, optionally, (iii):
(i) $\mathcal{L}v_n$ belongs to the growth type $[L]^{\omega^n}$
(ii) $\mathcal{L}v_n$ and $\mathcal{E}v_n$ are analytic close to $+\infty$ and extend to reciprocal analytic isomorphisms of $\mathbb{R}^+$ with no other fixed point than 0 and $+\infty$
(iii) $\mathcal{L}v_n$ and $\mathcal{E}v_n$ are given at 0^+ by convergent, odd power series and extend therefore to odd analytic isomorphisms of $\mathbb{R}$ with no other fixed points than 0 and $\pm\infty$.

The simplest way to produce such pairs $\{\mathcal{E}v_n, \mathcal{L}v_n\}$ is to start from a pair $\{Ev, Lv\}$ of reciprocal analytic isomorphisms of $\mathbb{R}^+$ that behave like $\{E, L\}$ at infinity. More precisely, we demand that:

$$\begin{array}{lllll} Ev(x) \sim x \;\; at \;\; 0^+ & , & Ev(x) \sim \mathrm{Const}\, e^x \;\; at \;\; +\infty & , & Ev(x) > x \;\; \forall x > 0 \\ Lv(x) \sim x \;\; at \;\; 0^+ & , & Lv(x) \sim \log(x) \;\; at \;\; +\infty & , & Lv(x) < x \;\; \forall x > 0 \end{array}$$

Moreover, Ev and Lv should both admit simple, explicit expressions and Ev should preferably be infinitely convex. This practically narrows down the choice to

$$\text{Choice 1}: \quad Ev(x) = e^x - 1 \quad , \quad Lv(x) = \log(1+x) \tag{8.54}$$

$$\text{Choice 2}: \quad Ev(x) = x\, e^x \quad , \quad Lv(x) = x \sum_{0 \leqslant n} \frac{(n+1)^{n-1}}{n!}(-x)^n \tag{8.55}$$

$$\text{Choice 3}: \quad Ev(x) = 2\sinh(x) \quad , \quad Lv(x) = \mathrm{arcsinh}(x/2) \tag{8.56}$$

For any given series of scalars $c_n > 1$ we define two series $\mathcal{L}w_n$ and $\mathcal{L}v_n$ of slow-growing analytic isomorphisms of $\mathbb{R}^+$ by the following induction:

$$\mathcal{L}w_0 := \mathcal{L}v_0 := Lv \qquad and \;\; for \;\; n \geqslant 1: \tag{8.57}$$

$$\mathcal{L}w_n := \left(\mathcal{L}w_{n-1} \circ \delta_{c_n^{-1}}\right)^{\ddagger} = \lim_{k \to +\infty} c_n^k \, \left(\mathcal{L}w_{n-1} \circ \delta_{c_n^{-1}}\right)^{\circ k} \tag{8.58}$$

$$\mathcal{L}v_n := Lv \circ \mathcal{L}w_n \tag{8.59}$$

with $\delta_c(x) := c\,x$ as usual. Our germs verify:

$$\mathcal{L}v_n(x) \sim \mathcal{L}w_n(x) \sim x \tag{8.60}$$
$$\mathcal{L}v_n(x) \leqslant \mathcal{L}w_n(x) < x \quad \forall x > 0 \tag{8.61}$$
$$\big[\mathcal{L}v_n\big]_i \;=\; \big[\mathcal{L}v_n\big]_i \;=\; \big[L\big]_i^{\omega^n} \qquad\qquad (n \geqslant 1, i \geqslant 0) \tag{8.62}$$

The local behaviour (8.60) at 0 is obvious. The global behaviour (8.61) follows from the identities

$$x > (\mathcal{L}w_n \circ \delta_{c_n^{-1}})(x)\,c_n > (\mathcal{L}w_n \circ \delta_{c_n^{-1}})^{\circ 2}(x)\,c_n^2 > \cdots > (\mathcal{L}w_n \circ \delta_{c_n^{-1}})^{\circ k}(x)\,c_n^k$$

which easily result from a double induction, first on n, then on k. Regarding the local behaviour (8.62) at $+\infty$, we may start from the relations:

$$\mathcal{L}w_{n-1} \,=\, \mathcal{E}w_n \circ \delta_{c_n^{-1}} \circ \mathcal{L}w_n \tag{8.63}$$
$$\mathcal{L}w_{n-1} \,=\, \big(\mathcal{E}w_n \circ E \circ \delta_{\gamma_n^{-1}}\big) \circ T^{\circ(-1)} \circ \big(\delta_{\gamma_n} \circ L \circ \mathcal{L}w_n\big) \quad (\gamma_n := \log c_n > 0) \tag{8.64}$$

which merely reflect the definition (8.58) of $\mathcal{L}w_n$, and reason by induction on n. The relations (8.62) with $i = 0$ clearly hold for $n = 1$. If they do for some larger n, the growth types $\big[\mathcal{L}w_n\big]$ and $\big[\mathcal{L}v_n\big] = \big[L \circ \mathcal{L}w_n\big]$ may differ, but the growth types $\big[\mathcal{L}w_n^*\big]$ and $\big[\mathcal{L}v_n^*\big]$ of the iterators at $+\infty$ do coincide [88] and in view of §8.3 this growth type is exactly $\big[L\big]_0^{\omega^{n+1}}$. This completes the inductive proof of (8.62) for $i = 0$; the case $i = 1$ is proven along the same lines; and the case $i \geqslant 2$ follow on the strength of the 'universal asymptotics' of slow functions and the regularity of Ev, Lv and their successive derivatives for all three choices (8.56)-(8.58).

8.9 Concluding remarks.

Universal asymptotics.

The fascinating subject of universal asymptotics for slow-growing germs, briefly touched upon in §6.4 *supra*, is dealt with at greater length in [E5], chapters 6 and 7. It has also, strangely enough, model-theoretical aspects, some of which are discussed in [JvdH1], [JvdH2].

[88] Indeed, for any slow germ $\mathcal{L}$, though the growth types of $\mathcal{L}^*$ and $(L \circ \mathcal{L})^*$ differ, the growth types of the respective iterators coincide. Cf the related identity $\Big[\big[L\big]_i^{\circ(\omega^n+1)}\Big]_i^{\omega} = \big[L\big]_i^{\omega^{n+1}}$, which simply reflects the identity $(\omega^n + 1)\,\omega = \omega^{n+1}$.

The scope of ultraexponential indeterminacy.

The section §15 *infra* takes a closer look at the indeterminacy inherent in the choice of the ultraexponentials $\mathcal{E}_n$. It shows, based on extensive numerical data, that all the main *natural* choices, especially for $\mathcal{E}_1$, are astonishingly close to one another. This suggests that, their theoretical equivalence non-withstanding, it would be somehow rash to look upon all the possible choices within the equivalence class $[\mathcal{E}_1]$ as being *exactly* on the same footing.

The curse of the stair-case phenomenon.

In section §15 we also examine the *staircase* phenomenon. It says, roughly, that the *connector* relating two condidates for $\mathcal{E}_1$, constructed from two auxiliary functions such as $exp(x)-a$ and $exp(x)-b$, tends to a staircase function when b goes to $+\infty$ while a remains fixed. This dashes all hope of selecting a privileged $\mathcal{E}_1$ based purely on real-asymptotic criteria. But on the other hand, we also show how *slowly* the connector tends to the staircase regime. This confirms the above remark about not all candidates $\mathcal{E}_1$ being on the same footing.

9 Beyond ω^ω : the meta-exponential scale.

In view of the sweeping closure properties that ultraseries appear to possess (see towards the end of §10), the question as to what lies beyond the ultra-exponential scale seems largely academic. So we shall be content here with a few (unfortunately rather inconclusive) remarks:

9.1 Iterates of order $\alpha \geqslant \omega^\omega$.

Let $\{\mathfrak{E}_n, n \in \mathbb{N}^*\}$ be a strictly increasing sequence of self-mappings of $\mathbb{R}^+$, each of which grows ultimately faster than any finite iterate of its predecessor:

$$\mathfrak{E}_n(x) < \mathfrak{E}_{n+1}(x) \qquad \forall n \geqslant 1, \forall x > 0 \tag{9.1}$$

$$\lim_{x\to+\infty} \mathfrak{E}_n^{\circ k}(x)/\mathfrak{E}_{n+1}(x) = 0 \qquad \forall n \geqslant 1, \forall k \geqslant 1 \tag{9.2}$$

Any series of the form

$$\mathfrak{E}(x) := \sum_{n\geqslant 1} \epsilon_n \frac{\mathfrak{E}_n(x)}{\mathfrak{E}_{n'}(n'')} \qquad (n < n' \, , \, n < n'' \, , \, \sum \epsilon_n < \infty) \tag{9.3}$$

defines a self-mapping $\mathfrak{E}$ of $\mathbb{R}^+$ that ultimately grows faster than any $\mathfrak{E}_n$ or indeed any element in the semi-group generated by the $\mathfrak{E}_n$. Moreover, if the

$\mathfrak{E}_n$ are real-analytic or real-cohesive and generate a non-oscillating differential trigebra[89], they still do so after adjunction of $\mathfrak{E}$.

Consider now the reciprocal slow mappings $\mathfrak{L}_n$ and $\mathfrak{L}$. Recall that on slow germs, equivalence classes $[...]_i$ have been defined[90], which are stable under composition and possess natural, unambiguous transfinite iterates $[...]_i^{\circ\alpha}$ for all orders $\alpha < \omega^\omega$. Using the construction (9.3) with $\mathfrak{E}_n := \mathcal{E}_n$ for some coherent system $\mathcal{E}_n := E_{\omega^n}$, we get, at the cost of the huge indeterminacy implicit in the choice of $\{n', n'', \epsilon_n\}$, a pair $(\mathfrak{E}, \mathfrak{L})$ that lies outside the previous scale. So there is no incoherence in decreeing that $(\mathfrak{E}, \mathfrak{L}) := (E_{\omega^\omega}, L_{\omega^\omega})$. That decision once made, the intrinsic notion of transfinite iteration of all orders $\alpha < \omega^\omega$ gives us, without further indeterminacy, all classes L_β for $\beta < \omega^{\omega^2}$. Repeating the process, we can reach any reasonable ordinal ω^*, but at the cost of a new $\{n', n'', \epsilon_n\}$-indeterminacy for each *non-approachable* ordinal, i.e. for each limit ordinal β not of the form $\beta'\alpha$ with $\alpha < \omega^\omega$. So, even to reach such a small ordinal as ω^{ω^2} we cannot avoid indeterminacies, even countably many – one for each $\omega^{\omega.n}$ $(n \in \mathbb{N}^*)$.

Thus we can partially[91] incarnate the aritmetics of, say, $[1, \omega^{\omega^\omega}[$, in the sense that:

$$[L_\gamma]_i \circ [L_\beta]_i = [L_{\beta+\gamma}]_i \quad (\textit{inversion}\,!) \quad \forall\beta < \omega^{\omega^\omega}, \forall\gamma < \omega^{\omega^\omega} \tag{9.4}$$

$$[L_\beta]_i^{\circ\alpha} = [L_{\beta.\alpha}]_i \qquad \forall\beta < \omega^{\omega^\omega} \quad \textit{but} \ \ \forall\alpha < \omega^\omega \tag{9.5}$$

However, here is a simple fact, not difficult to prove, that brings home the hugeness of the indeterminacy:

For each coherent system $\{[L_\beta]_i \,,\, \beta < \omega^{\omega^\omega}\}$, there exists an equally coherent system $\{[L^*_\beta]_i \,,\, \beta < \omega^{\omega^\omega}\}$ such that

$$[L^*_\beta]_i = [L_\beta]_i \qquad \forall\beta < \omega^\omega \qquad (\textit{of course}) \tag{9.6}$$

$$[L^*_\beta]_i \prec [L_{\omega^\omega}]_i \qquad \forall\beta < \omega^{\omega^\omega} \qquad !!! \tag{9.7}$$

One obvious way of narrowing down the indeterminacy would be to bring our mappings $E_\alpha : \mathbb{R}^+ \to \mathbb{R}^+$ into close correspondance with one of the classical hierarchies of mappings $\underline{E}_\alpha : \mathbb{N}^* \to \mathbb{N}^*$

$$\underline{E}_0(n) = n + 1 \tag{9.8}$$

$$\underline{E}_{\alpha+1}(n) = (\underline{E}_\alpha)^{\circ n}(n) \tag{9.9}$$

$$\underline{E}_\alpha(n) = \underline{E}_{\alpha_n}(n) \qquad (\textit{for } \alpha \textit{ limit ordinal}\,,\ \lim\uparrow \alpha_n = \alpha) \tag{9.10}$$

These hierarchies of integer mappings entail a much lesser degree of arbitrariness, since they only depend on the choice of a fundamendal approaching

[89] or $\lessdot$-ordered: see §.

[90] see §8

[91] *partially*, because in (9.5) α cannot exceed ω^ω.

sequence α_n for each limit ordinal α and since, up to fairly large ordinals, there exists only *one* 'natural' choice.

For each limit$^\times$ ordinal β (i.e. for each limit ordinal β not of the form $\beta'\alpha$ with $\alpha < \omega^\omega$), one would then define the real-to-real mapping E_β by series of the form

$$E_\beta(x) := \sum_{n\geqslant 1} \epsilon_n \frac{E_{\beta_n}(x)}{E_{\beta'_n}(n'')} \qquad (\beta_n \leqslant \beta'_n\,,\ \lim\uparrow\beta_n = \lim\uparrow\beta'_n = \beta) \quad (9.11)$$

with triplets $(\beta'_n, n'', \epsilon_n)$ defined 'β-uniformly' in a way that
(i) would conforms with the chosen hierarchy on integers: $E_\beta(n) = \underline{E}_\beta(n)$
(ii) and that would be *coherent* in the sense of being compatible with the direct definition of E_β for approachable ordinals β.[92]

The trouble, though, is that so far no natural uniform choice for the triplet $(\beta'_n, n'', \epsilon_n)$ *has been found that would determine privileged i-classes* $[\mathrm{L}_\beta]_\mathrm{i}$ *for non-approachable ordinals* β*. Indeed, all uniform choices seem to select interpolations of* $\underline{E}_\beta$ *between* n *and* $n+1$ *that asymptotically tend to the good-for-nothing 'stair-case' interpolation.*

9.2 No all-inclusive quasi-analyticity class.

The existence of privileged classes $[L_\beta]_1$ on a huge transfinite interval would correspondingly extend the range of canonical, increasing quasi-analytic Denjoy classes ${}^\beta DEN$ far beyond the class *COHES* (sufficient for all purposes of analysis, but still not 'ultimate') and take us closer to some notion of 'all-inclusive' quasi-analyticity class [93]– probably a chimerical hope. So, all considered, this would seem to be one more reason for *doubting* the existence of a canonical system of classes $[L_\beta]_1$ for $\beta \geqslant \omega^\omega$.

9.3 Non-oscillating extensions beyond the ultraexponential range.

Enlarging any group $\mathbb{G}$ contained in the ultraexponential range by adjunction of a coherent system $\{E_\alpha, L_\alpha\}$ with $\omega^\omega < \alpha < \omega^*$, seems to be the only way of guaranteeing a 'non-oscillating' extension $\mathbb{G}^{\mathrm{ext}}$, i.e. an extension where the order $\prec$ still holds.[94]

[92]since E_β can then be directly defined, up to postcomposition by a periodic mapping, from an earlier $E_{\beta*}$, (β^* the largest non-approachable ordinal $< \beta$) by the universal ω^n-iteration, first of slow growth classes, then of the fast reciprocals.

[93]see (2.48) in §2.7.

[94]See §1.2.

10 Ultraseries and their all-round completeness.

10.1 Ultraseries and ultramonomials.

Whereas transseries carry only finite exponential iterates, ultraseries are permitted to carry transfinite iterates, up to order ω^ω (not included). This brings two main complications:
(i) on the analysis side, as we saw in §8, there is an unremovable latitude in the choice of the successive ultraexponentials $\mathcal{E}_n = E^{\omega^n}$.
(ii) on the formal side, as we shall see in a moment, the clear distinction between prime and non-prime transmonomials gets blurred, and instead of one canonical representation for transseries, for ultraseries we have several.

Nonetheless, the well order survives, so does non-oscillation, and so too does (once we have fixed a coherent system of ultraexponentials) the biconstructive correspondence between summable ultraseries and the associated germs.

Fortunately, there are few natural sources of ultraseries. Unlike with the transseries, which crop up everywhere in differential calculus, there are far fewer contexts that force us to resort to ultraseries: conjugation of germs of unequal exponentiality, or again exceptionnally complex composition or functional equations, but *never* differential equations. So we shall be content with a very cursory treatment.

10.2 Simplification rules for ultramonomials.

Rule 1. Simplification inside ulraexponentials.

Whenever a large transseries $S_r = A_r + B_r$ *occurs inside a strict ultraexponential* $\mathcal{E}_r$ $(r \geqslant 1)$*, only the "r-large" part* A_r *should remain there, while the "r-small" part* B_r*, characterised by*

$$B_r \prec \frac{\mathcal{E}_r \circ S_r}{\mathcal{E}'_r \circ S_r} \sim \frac{\mathcal{E}_r \circ A_r}{\mathcal{E}'_r \circ A_r} \tag{10.1}$$

should be ejected by means of the Taylor expansion

$$\mathcal{E}_r(A_r + B_r) = \mathcal{E}_r(A_r) + \sum_{1 \leqslant n} \frac{1}{n!} \left(\mathcal{E}^{(n)} \circ A_r\right) (B_r)^n \tag{10.2}$$

which automatically converges close to $+\infty$.

Setting $\epsilon_{r,n} := \mathcal{E}_r^{(n-1)} \circ \mathcal{L}_r / \mathcal{E}_r^{(n)} \circ \mathcal{L}_r$ and using the universal asymptotics of slow germs, we find that $\epsilon_{r,n-1} \sim \epsilon_{r,n}$. As a consequence, the inequality

(10.1) implies $B_r \prec (\frac{\mathcal{E}_r \circ S_r}{\mathcal{E}'_r \circ S_r})^n$ and we easily get the right bounds to ensure the convergence in the right-hand side of (10.2).[95]

Rule 2. Simplification inside ultralogarithms.

No logarithm or ultralogarithm $\mathcal{L}_r$ ($r \geqslant 0$) should occur with an argument other than the variable x or an ultralogarithm $\mathcal{L}_s(x)$ of equal or superior strength ($r \leqslant s$).

In other words, ultralogarithms should occur only within finite sequences of the form:

$$\mathcal{L}_{r_1}^{\circ n_1} \circ \mathcal{L}_{r_2}^{\circ n_2} \circ \dots \mathcal{L}_{r_k}^{\circ n_k} \qquad \textit{with} \quad 0 \leqslant r_1 < r_2 < \dots < r_k \ , \ 1 \leqslant n_i \ (\forall i) \quad (10.3)$$

The restrictions laid upon the arguments of ultralogarithms are thus much more stringent than those imposed on the arguments of ultraexponentials, but they can be met by resorting (at most finitely many times) to the following six subrules.

Subrule 2.1: Simplification of $\mathcal{L}_r(A+B)$ with $A \succ B$:

The straightforward Taylor expansion $\mathcal{L}_r(A+B) = \mathcal{L}_r(A) + \dots$ does the trick, without any convergence difficulty, since for any ultralogarithm $\mathcal{L}_r$, analyticity on the real half axis *automatically* implies analyticity on a right half-plane.[96]

Subrule 2.2: Simplification of $\mathcal{L}_r(A.B)$ with $A \succ B$:

$$\mathcal{L}_r(A\,B) = \mathcal{L}_r(A) + \theta_r(A,B) \quad (10.4)$$
$$= +A\,\mathcal{L}'_r(A)\,\log(B) + o\big(A\,\mathcal{L}'_r(A)\,\log(B)\big) \quad (10.5)$$

with

$$\theta_0(A,B) = \log(B) \quad (10.6)$$
$$\theta_r(A,B) = \kappa_r(A, \theta_{r-1}(A,B)) \ \ \forall r \geqslant 1 \quad (10.7)$$

[95] Although the ultraexponential $\mathcal{E}_r$ remains small only within a narrow stripe around $\mathbb{R}^+$ that tapers off very fast at $+\infty$, the condition (10.1) of "r-smallness" means that $A_r(x) \pm B_r(x)\,i$ remains safely within that stripe as x grows. Unlike with the ordinary exponential ($r = 0$), where "0-small" simply means "small", with strict ultraexponentials, "r-smallness" depends not only on r but also on the leading terms of the transseries S_r.

[96] Much more than that, in fact: it implies analyticity on a whole spiralling ramified neighbourhood of ∞ on $\mathbb{C}_\bullet$.

and

$$\kappa_r(x,y) \;=\; \sum_{1\leqslant n} (\mathcal{L}_r^{(n)} \circ \mathcal{L}_{r-1})(x)\, \frac{y^n}{n!} \tag{10.8}$$

Using the basic identity $\mathcal{L}'_r \circ \mathcal{L}_{r-1}\, \mathcal{L}' = \mathcal{L}'_r$, we can rid the right-hand side of (10.8) of all composite terms $\mathcal{L}_r^{(n)} \circ \mathcal{L}_{r-1}$ by expressing these as polynomials in

$$\frac{1}{\mathcal{L}'_{r-1}} \;\;;\;\; \mathcal{L}''_{r-1}, \mathcal{L}'''_{r-1}, \ldots, \mathcal{L}_{r-1}^{(n)} \;\;;\;\; \mathcal{L}'_r; \mathcal{L}''_r, \ldots, \mathcal{L}_r^{(n)};$$

Inductive proof of (10.4): Since $\mathcal{L}_0 = L = \log$, the identity (10.4) is obvious for $r = 0$, and the following induction takes care of $r \geqslant 1$:

$$\begin{aligned}
\mathcal{L}_r(A\,B) &= 1 + \mathcal{L}_r(\mathcal{L}_{r-1}(A\,B)) && (10.9)\\
&= 1 + \mathcal{L}_r(\mathcal{L}_{r-1}(A) + \theta_{r-1}(A,B)) && (10.10)\\
&= 1 + \mathcal{L}_r(\mathcal{L}_{r-1}(A)) + \kappa_r(A, \theta_{r-1}(A,B)) && (10.11)\\
&= \mathcal{L}_r(A) + \kappa_r(A, \theta_{r-1}(A,B)) && (10.12)
\end{aligned}$$

Subrule 2.3: Simplification of $\mathcal{L}_r(A^\alpha)$:

This is not a special case of rule 2.2 (which assumed $A \succ B$) but an easy variant. The same type of induction yields, with the same θ_r as in (10.4):

$$\begin{aligned}
\mathcal{L}_r(A^\alpha) &= \mathcal{L}_r(A) + \theta_r(\log A, \alpha) && (10.13)\\
&= \mathcal{L}_r(A) + \mathcal{L}'_r(A)\, A\, \log A\, \log\alpha + o\big(\mathcal{L}'_r(A)\, A\, \log A\, \log\alpha\big)
\end{aligned}$$

Subrule 2.4: Simplification of $\mathcal{L}_{r+k} \circ \mathcal{L}_r$:

$$\begin{aligned}
\mathcal{L}_{r+k} \circ \mathcal{L}_r &= \mathcal{L}_{r+k} + \ell_{r,k} && (10.14)\\
\textit{with}\quad \ell_{r,1}(x) &= -1 && (10.15)\\
\textit{and}\quad \ell_{r,k}(x) &= \kappa_{r+k}(x, \ell_{r,k-1}(x)) \qquad \big(\kappa_r \;\; \textit{as above}\big) && (10.16)\\
&= \mathcal{L}_{r+k} - \frac{\mathcal{L}'_{r+k}}{\mathcal{L}'_{r+1}} + o\Big(\frac{\mathcal{L}'_{r+k}}{\mathcal{L}'_{r+1}}\Big) && (10.17)
\end{aligned}$$

Indeed, the identity holds for $k = 1$, and for $k \geqslant 2$ we have the induction:

$$\begin{aligned}
\mathcal{L}_{r+k} \circ \mathcal{L}_r &= (\mathcal{L}_{r+k} \circ \mathcal{E}_{r+k-1}) \circ (\mathcal{L}_{r+k-1} \circ \mathcal{L}_r) && (10.18)\\
&= 1 + \mathcal{L}_{r+k}(\mathcal{L}_{r+k-1} + \ell_{r,k-1}) && (10.19)\\
&= 1 + \mathcal{L}_{r+k} \circ \mathcal{L}_{r+k-1} + \kappa_{r+k}(\ell_{r,k-1}) && (10.20)\\
&= \mathcal{L}_{r+k} + \kappa_{r+k}(id, \ell_{r,k-1}) && (10.21)
\end{aligned}$$

Subrule 2.5: Simplification of $\mathcal{L}_{r+k} \circ \mathcal{E}_r$:

$$\begin{aligned}
\mathcal{L}_{r+k} \circ \mathcal{E}_r &= \mathcal{L}_{r+k} \circ \mathcal{E}_{r+k-1} \circ \mathcal{L}_{r+k-1} \circ \mathcal{E}_{r+k-2} \circ \cdots \circ \mathcal{L}_{r+1} \circ \mathcal{E}_r \\
&= T \circ \mathcal{L}_{r+k} \circ T \circ \mathcal{L}_{r+k-1} \circ \cdots \circ T \circ \mathcal{L}_{r+1} && (10.22) \\
&= \mathcal{L}_{r+k} + \frac{\mathcal{L}'_{r+k}}{\mathcal{L}'_{r+1}} + o\big(\frac{\mathcal{L}'_{r+k}}{\mathcal{L}'_{r+1}}\big) && (10.23)
\end{aligned}$$

Compare (10.17) and (10.23).

Subrule 2.6: Simplification of $\mathcal{L}_r \circ \mathcal{E}_{r+k}$:

$$\begin{aligned}
\mathcal{L}_r \circ \mathcal{E}_{r+k} &= \mathcal{L}_r \circ \mathcal{E}_{r+1} \circ \mathcal{L}_{r+1} \circ \mathcal{E}_{r+2} \circ \cdots \circ \mathcal{L}_{r+k-1} \circ \mathcal{E}_{r+k} \\
&= \mathcal{E}_{r+1} \circ T_{-1} \circ \mathcal{E}_{r+2} \circ T_{-1} \circ \cdots \circ \mathcal{E}_{r+k} \circ T_{-1} && (10.24)
\end{aligned}$$

10.3 Several competing presentations, but one well order on ultraseries.

The main difference with transseries is Rule 1, which permits the ejection, not of all small terms, but only of the r-small terms, thereby removing the clear dichotomy between prime and non-prime ultramonomials. The above rules, however, taken together, are sufficient to compare ultramonomials pairwise and thus to ensure a well order, and this is what really matters.

With some extra work, these reduction rules also lead to canonical ulramonomial expansions for ultraseries, but there are several competing choices here, and for each of them the full reduction procedure is pretty clumsy. In practice, there is no need *to fully reduce our ultraseries* (taken in the form which they naturally assume as algorithmic solutions of functional or composition equations), *but only so far as necessary for mutual comparison.*

10.4 Integration of ultramonomials.

Like with transmonomial integration (see §3.5), ultramonomial integration $a \mapsto A = \partial^{-1} a$ generates resurgence relative to a critical time x_0 given by

$$x_0 = \text{stat.lim}_{r \to +\infty} \Big(\Big| \log \frac{a(x)}{\mathcal{L}'_r(x)} \Big| \Big) \qquad (10.25)$$

The relation resembles (3.23), with $\mathcal{L}_r$ in place of L_r, and here again the limit is 'stationary': for two large enough values of r, the germs on the right-hand side of (10.25) become equivalent at ∞. Concretely, this yields:

$$x_0 \;=\; \big|\log a(x)\big| \qquad \textit{if} \qquad 1 < \lim \frac{\log a(x)}{\log x} \leqslant +\infty \tag{10.26}$$

$$x_0 \;=\; \log x \qquad \textit{if} \qquad 0 \leqslant \lim \frac{\log a(x)}{\log x} < 1 \tag{10.27}$$

Here is an example analogous to the one of §3.5:

$$A'(x) = a(x) = \mathcal{L}'(x)\, b(x) \quad \textit{with} \quad \mathrm{expo}(b) \leqslant -4 \tag{10.28}$$

with a very slow germ $\mathcal{L}$ in the growth type of $[L]^\gamma$, with $\gamma = \omega^{r_s}\, n_s + \cdots + \omega^{r_1}\, n_1$.

$$\mathcal{L} = \mathcal{L}_{r_1}^{\circ n_1} \circ \mathcal{L}_{r_2}^{\circ n_2} \circ \ldots \mathcal{L}_{r_k}^{\circ n_k} \qquad (0 \leqslant r_1 < r_2 < \ldots < r_s \;\; , \;\; n_i > 0) \tag{10.29}$$

As for b, it may be large or small, but must have an even slower rate (of growth or decrease, as the case may be) than $\mathcal{L}$. For instance:

$$b(x) \;:=\; \big(\mathcal{L}_{r'_1}(x)\big)^{\alpha_1}\, \big(\mathcal{L}_{r'_2}(x)\big)^{\alpha_2}\, \big(\mathcal{L}_{r'_3}(x)\big)^{\alpha_3} \quad (r_s < r'_1 < r'_2 \ldots) \quad \textit{or}$$

$$b(x) \;:=\; \mathcal{E}_{r'_0}\Big(\big(\mathcal{L}_{r'_0+r'_1}(x)\big)^{\beta_1}\, \big(\mathcal{L}_{r'_0+r'_2}(x)\big)^{\beta_2}\, \big(\mathcal{L}_{r'_0+r'_3}(x)\big)^{\beta_3}\Big) \qquad (\beta_1 > 0)$$

The critical time here is $x_0 = L(\mathcal{L}(x))$: it belong to the growth type of $[\text{Ł}]^{\gamma+1}$. Relative to this critical variable, (10.28) becomes

$$A'_0(x_0) = e^{x_0}\; b_0(x_0) \qquad \big(A_0(x_0) \equiv A(x)\;,\; b_0(x_0) \equiv b(x)\big) \tag{10.30}$$

The formal solution is given by

$$A_0(x_0) = e^{x_0}\; B_0(x_0) \;=\; e^{x_0}\; (1 + \partial_{x_0})^{-1}\, \big(b_0(x_0)\big) \tag{10.31}$$

with $(1 + \partial)^{-1}$ expanded straightforwardly in positive powers of ∂, and the sum is given by the Laplace transform

$$\widehat{B}_0(\xi_0) = (1 - \xi_0)^{-1}\; \widehat{b}_0(\xi_0) \tag{10.32}$$

The definition of $\widehat{b}_0$ here is unproblematic, since the monomial $b_0(x_0) = b(\mathcal{E}(E(x_0)))$ is automatically subexponential in x_0, and Laplace summation too is unproblematic, since there is only one singularity on the positive real axis in the Borel plane.

10.6 Further remarks.

Resurgent ultraseries and their displays.

The above example (10.31)-(10.32) shows that the display $DplA$ is going to depend on the initial choice of ultraexponential tower, since the real residue $\widehat{b}_0(1)$ in (10.32) clearly depends on it. This is a general feature: *the displays of resurgence-carrying ultraseries depend on the choice of ultraexponential tower underlying the construction.*[97]

However, any two exponential towers are connected by a periodic Witt tower (see §8.6) and, based on these, one can produce conversion formulae for the corresponding displays. These formulae show in particular that the independence relations implied by the displays *do not*, unlike the displays themselves, depend on the choice of ultraexponential tower.

Sweeping closure properties.

The variable and/or the unknown germ could even be allowed to sit *inside the iteration orders* of our functional equations – and we still would have closure! In fact, it is hard to think of meaningful problems in analysis that would take us beyond the range of ultraseries, with ω^ω as natural upper limit for the iteration orders.

11 Composition equations: resurgence and displays.

This and the next two sections take up the subject of general *composition equations* and also, occasionally, *composition systems.* Though these problems make sense in the general transserial setting, we shall restrict ourselves mostly to germs expressible as power series (often identity-tangent ones), not only to avoid unnecessary – and on the whole notational rather than substantial – complications, but also because this more familiar setting already presents us with the typical difficulties inherent in the subject and with the main methods required for overcoming them. Moreover, since our data and unknowns, though still *real* germs and defined as usual on $[..., +\infty[$, will extend to sectorial neighbourhoods of $+\infty$ in $\mathbb{C}_\bullet$, we shall revert to calling z

[97] to the extent that one and the same ultraseries may be convergent or divergent depending on that choice: think again of A in (10.28), which will be convergent or divergent according as the residue $\widehat{b}_0(1)$ vanishes or not, which again depends on the choice of ultraexponential tower.

the variable (in multplicative plane) and ζ the conjugate variable (in the convolutive or Borel plane).

11.1 Composition equations: alternance.

For the most general composition equations, i.e. for equations of type $\mathcal{T}_4$ (see at the beginning of §1.1), there exist various notions of k-alternance, which roughly measure the number of free parameters present in the general (oscillation-free) solution. But all useful definitions agree in assigning 0-alternance to the "positive" composition equations (i.e. those of type $\mathcal{T}_3$), 1-alternance to the conjugation equations (type $\mathcal{T}_2$), and k-alternance to those very special equations involving k imbricated commutators:

$$\{...\{\{f, f_1\}, f_2\}, \ldots, f_k\} = f_0 \qquad \textit{with} \quad \{f, g\} := f \circ g \circ f^{-1} \circ g^{-}1$$

For a thorough discussion in the case of *"twin"* equations, see [EV]. In any case, the present section is devoted to 0-alternance equations, whose definition is entirely unproblematic.

11.2 Composition equations: resurgence and displays.

Let us examine the general 0-alternance composition equation with data g_i real-analytic at $+\infty$:

$$\begin{aligned} W(f) &= id && (11.1)\\ W(f) &:= f^{\circ m_r} \circ g_r \ldots f^{\circ m_1} \circ g_1 && (m_i \in \mathbb{Z}\,, \textstyle\sum m_i \neq 0) \end{aligned}$$

The factors g_i are given, and the unknown f is sought, of the form:

$$g_i(z) = z + \sigma_i + \psi_i(z) = z + \sigma_i + \tau_i\, z^{-1} + \ldots \quad \big(\psi_i(z) \in O(z^{-1})\big) \qquad (11.2)$$

$$f(z) = z + \sigma + \varphi(z) = z + \sigma + \tau\, z^{-1} + \ldots \quad \big(\varphi(z) \in O(z^{-1})\big) \qquad (11.3)$$

with a real shift $\sigma := -\sum \sigma_i / \sum m_i$ and a real residue $\tau := -\sum \tau_i / \sum m_i$.

Crucial to the discussion are these two exponential polynomials:

$$S_W(\lambda) = \sum_{1 \leqslant i \leqslant r}^{m_i^*} \operatorname{sgn}(m_i) e^{\sigma_{i,m_i^*}\lambda} = \sum_{1 \leqslant i \leqslant m_*}^{\alpha_i \uparrow} s_i\, e^{\alpha_i \lambda} \qquad (s_i \in \mathbb{Z}, \alpha_i \in \mathbb{R}) \qquad (11.4)$$

$$T_W(\lambda) = \sum_{1 \leqslant i \leqslant r}^{m_i^*} \operatorname{sgn}(m_i) e^{\sigma_{i,m_i^*}\lambda}\, \tau_{i,m_i^*} = \sum_{1 \leqslant i \leqslant m_*}^{\alpha_i \uparrow} t_i\, e^{\alpha_i \lambda} \quad (t_i \in \mathbb{R}, \alpha_i \in \mathbb{R}) \qquad (11.5)$$

with

$$
\begin{aligned}
S_W(0) &= s_1 + \cdots + s_{m*} = m_1 + \cdots + m_r \neq 0 \quad (m_* \leqslant \sum |m_i|) && (11.6)\\
\tau_{i,m_i^*} &= (\tau_1 + ... + \tau_i) + (m_1 + ... + m_{i-1} + m_i^*)\,\tau && (11.7)\\
\sigma_{i,m_i^*} &= (\sigma_1 + ... + \sigma_i) + (m_1 + ... + m_{i-1} + m_i^*)\,\sigma && (11.8)
\end{aligned}
$$

Here, each m_i^* ranges through $[0, m_i - 1]$ or $[m_i, -1]$ depending on the sign of m_i, and m^* is the number of distinct frequencies σ_{i,m_i^*}

The second exponential polynomial $T_W(\lambda)$ is also second in importance. It merely determines the ramification factors $z^{\rho_{\boldsymbol{n}}}$ in the parameter saturated solution $\widetilde{f}(z,\boldsymbol{u})$ of $W(f) = id$. It vanishes when all residues τ_i vanish, in which case there is no ramification.

The first exponential polynomial $S_W(\lambda)$ is the one that really matters, because its roots λ_j determine
(i) the nature of the exponentials in the saturated solution of $W(f) = id$
(ii) the location of the singularities in the Borel plane
(iii) the set of active alien derivations.[98]
The roots λ_j of $S_W(\lambda)$, with j running through an enumerable set $\mathcal{J}$, are all $\neq 0$ (due to (11.6)) and located within a vertical strip $\lambda_- \leqslant \Re\lambda \leqslant \lambda_+$ (due to $\alpha_i \in \mathbb{R}$).

Let us at first make the (generically fulfilled) assumption that the λ_j are linearly independent, or rather the weaker assumption that they are non-resonant, i.e. verify no finite identity of the form

$$
\lambda_{j_0} = \sum_j n_j\,\lambda_j \quad \textit{with} \quad 0 \leqslant n_j \quad \textit{and} \quad \sum_j n_j < +\infty \quad (n_j \in \mathbb{N}) \quad (11.9)
$$

Let us also assume, for now, that each λ_j is a simple zero of $S_W(\lambda) = 0$.

Proposition 11.1 (Generic composition equation of 0-alternance) *Under the above genericity assumptions, the composition equation (11.1) has a unique parameter-saturated, normal* [99] *solution of the form*

$$
\widetilde{f}(z,u) = \widetilde{f}(z) + \sum \boldsymbol{u}^{\boldsymbol{n}}\, e^{\omega\, z}\, \widetilde{f}_{\boldsymbol{n}}(z) \;\; \textit{with} \;\; \boldsymbol{u}^{\boldsymbol{n}} = \prod u_j^{n_j} \;,\; \omega = <\boldsymbol{n}, \boldsymbol{\lambda}> (11.10)
$$

with $\boldsymbol{n}$ *running through the set* $\mathcal{J}_0^{\mathbb{N}}$ *of all* $\mathcal{J}$*-indexed, finitely supported, integer-valued sequences of the form* $\boldsymbol{n} = \{n_j \,|\, j \in \mathcal{J}, n_j \in \mathbb{N}, \sum n_j < \infty\}$ *and with generically divergent, but always resurgent power series*

$$
\widetilde{f}_n(z) \in z^{<\boldsymbol{n},\boldsymbol{\rho}>}\, \mathbb{C}[[z^{-1}]] \quad \Big(<\boldsymbol{n},\boldsymbol{\rho}> = \sum n_j\,\rho_j \;\;,\;\; \rho_j := -\lambda_j\, \frac{R_W(\lambda_j)}{S'_W(\lambda_j)} \Big) \;(11.11)
$$

[98] That is to say, the set of all $\boldsymbol{\Delta}_\omega$ liable to act (with a non-vanishing result) either on f or on some of its successive alien derivatives.

[99] The normalisation condition is $\widetilde{f}_{\boldsymbol{n}^j}(z) = z^{\rho_j} + o(z^{\rho_j})$. It bears on the *pilot* components $\widetilde{f}_{\boldsymbol{n}^j}(z)$ preceded by the factors $u_j\, e^{\lambda_j z}$.

whose ramification factor $z^{\rho_{\boldsymbol n}} = z^{<\boldsymbol{n},\boldsymbol{\rho}>}$ *is always* $\equiv 1$ *when the residues* τ_j *vanish.*[100]

The resurgence support Ω *contains the additive semi-group* Ω_* *generated by the* λ_j*, but is larger than* Ω_**: it also contains all elements of the form* $\lambda_* - \lambda_j$*, with* $\lambda_* \in \Omega_*$*.*

As usual, all resurgence properties are accounted for by the Bridge Equation, which here asumes the form:

$$\boldsymbol{\Delta}_\omega \widetilde{f}(z,u) \;=\; \mathbf{A}_\omega \widetilde{f}(z,u) \qquad\qquad (\forall\, \dot\omega \in \Omega) \tag{11.12}$$

$$\textit{with}\;\; \boldsymbol{\Delta}_\omega \;=\; e^{-\omega\, z}\, \Delta_\omega \qquad\qquad (\textit{as always})$$

$$\textit{and}\;\; \mathbf{A}_\omega \;=\; \boldsymbol{u}^{\boldsymbol{n}} \sum A^j_\omega\, u_j\, \partial_{u_j} \qquad (\forall\, \dot\omega = <\boldsymbol{n},\boldsymbol{\lambda}> \in \Omega_*) \tag{11.13}$$

$$\textit{or}\;\; \mathbf{A}_\omega \;=\; \boldsymbol{u}^{\boldsymbol{n}}\, A^j_\omega\, \partial_{u_j} \qquad (\forall\, \dot\omega = <\boldsymbol{n},\boldsymbol{\lambda}> -\lambda_j \in \Omega - \Omega_*) \tag{11.14}$$

The component-by-component interpretation of the Bridge Equation yields

$$\boldsymbol{u}^{\boldsymbol{n}'} \Delta_\omega\, \widetilde{f}_{\boldsymbol{n}'}(z) = \boldsymbol{u}^{\boldsymbol{n}} \big(\sum A^j_\omega\, u_j\, \partial_{u_j}\big)\, \boldsymbol{u}^{\boldsymbol{n}''}\, \widetilde{f}_{\boldsymbol{n}''}(z) \tag{11.15}$$

with $\boldsymbol{n}' = \boldsymbol{n} + \boldsymbol{n}''$ *to ensure the simultaneous elimination of the exponential terms and* $\boldsymbol{u}$*-factors. Eventually,* 11.15 *reduces to the identities*

$$\Delta_\omega\, \widetilde{f}_{\boldsymbol{n}'}(z) = \Big(\sum (n'_j - n_j)\, A^j_\omega\Big)\, \widetilde{f}_{\boldsymbol{n}'-\boldsymbol{n}}(z) \quad \textit{with} \quad \dot\omega = <\boldsymbol{n},\boldsymbol{\lambda}> \tag{11.16}$$

with only a finite number of terms on the right-hand side.
In the special case $\boldsymbol{n}' = \mathbf{0}$*, we have the identities*

$$\Delta_{\nu_j}\, \widetilde{f}(z) = A^j_{\nu_j}\, \widetilde{f}_{\boldsymbol{n}^{\boldsymbol{j}}}(z) \quad \textit{with} \quad \dot\nu_j = -\lambda_j \;,\; \boldsymbol{n}^{\boldsymbol{j}} = \{n^j_i | n^j_i = \delta^j_i,\, i \in \mathcal{J}\} \tag{11.17}$$

with a single term on the right-hand side.[101] *Lastly the displays of* $\widetilde{f}(z,\boldsymbol{u})$ *and* $\widetilde{f}(z)$ *are given by:*

$$\mathrm{Dpl}\, \widetilde{f}(z,\boldsymbol{u}) \;=\; \widetilde{f}(z,\boldsymbol{u}) + \sum_r \sum_{\omega_i \in \Omega} \mathbf{Z}^{\omega_1,\ldots,\omega_r}\; \mathbf{A}_{\omega_1} \ldots \mathbf{A}_{\omega_r}\, \widetilde{f}(z,\boldsymbol{u}) \tag{11.18}$$

$$\mathrm{Dpl}\, \widetilde{f}(z) \;=\; \widetilde{f}(z) + \sum_r \sum_{\omega_i \in \Omega} \mathbf{Z}^{\omega_1,\ldots,\omega_r}\; \big[\mathbf{A}_{\omega_1} \ldots \mathbf{A}_{\omega_r}\, \widetilde{f}(z,\boldsymbol{u})\big]_{\boldsymbol{u}=\mathbf{0}} \tag{11.19}$$

Sketch of the proof:
First, a few words about the interpretation of the Bridge equation. Although (11.16) and (11.17) show that only alien derivations $\boldsymbol{\Delta}_\omega$ of a very special sort

[100] The scalars ρ_j in (11.11) are well defined. Indeed, $S'_W(\lambda_j) \neq 0$ since we assumed all zeros of S_W to be simple.

[101] $\boldsymbol{n}^{\boldsymbol{j_0}}$ is the sequence $\{n_j;\, n_{j_0} = 1,\, n_j = 0 \textit{ if } j \neq j_0\}$.

can act (effectively) on any given $\widetilde{f}_{\boldsymbol{n}}$, yet for any $\omega_* \in \Omega_{\boldsymbol{n}} - \Omega_*$ a derivation chains $\boldsymbol{\Delta}_{\omega_r} \ldots \boldsymbol{\Delta}_{\omega_1}$ with $\sum \omega_i = \omega_*$ can always be found that will act effectively on $\widetilde{f}$ or $\widetilde{f}_{\boldsymbol{n}}$. As a consequence, in the Borel plane the functions $\widehat{f}_{\boldsymbol{n}}(\zeta)$ or $\widehat{f}(\zeta)$ generally possess singularities over all points of $\Omega_{\boldsymbol{n}} - \Omega_*$ resp. $-\Omega_*$. Moreover, barring the exceptional cases when some of the special (and pairwise commuting) operators $\mathbf{A}_{\nu_j} = A^j_{\nu_j}\, \partial_{u_j}$ (with $\dot{\nu}_j = -\lambda_j$) have vanishing coefficients $A^j_{\nu_j}$, the identity[102]

$$\Big(\prod_j (A^j_{\nu_j})^{n_j}\Big)\, \widetilde{f}_{\boldsymbol{n}}(z) = \Big(\prod_j \frac{(\Delta_{\nu_j})^{n_j}}{n_j!}\Big)\, \widetilde{f}(z) \qquad (\dot{\nu}_j = -\lambda_j) \tag{11.20}$$

makes it possible to recover all components $\widetilde{f}_{\boldsymbol{n}}$ from the sole knowledge of $\widetilde{f}$, via some analysis in the Borel plane. Of course, if one *knows* the composition equation $W(f) = id$ of which $\widetilde{f}$ is the solution, it is far more economical to get these $\widetilde{f}_{\boldsymbol{n}}$ by formally calculating its saturated solution $\widetilde{f}(z,\boldsymbol{u})$. But if one *does not* know $W(f)$, the identity (11.20) shows how to retrieve all components $\widetilde{f}_{\boldsymbol{n}}$ from $\widetilde{f}$, which of course would be impossible if $\widetilde{f}$ were convergent. In other words, composition equations with strictly resurgent solutions exhibit a far greater 'inner cohesion': knowing even a small part of the saturated solution, one can retrieve everything, including (modulo some hard work) the original equation $W(f) = id$ itself.

As for proving Proposition 11.1, calculating the formal integral $\widetilde{f}(z,\boldsymbol{u})$ offers no difficulty, since the coefficients of $\widetilde{f}$ and $\widetilde{f}_{\boldsymbol{n}}$:

$$\widetilde{f}(z) \;=\; z + \sigma + \sum a_k z^{1-k} \tag{11.21}$$

$$\widetilde{f}_{\boldsymbol{n}}(z) \;=\; z^{<\boldsymbol{n},\boldsymbol{\rho}>} \sum a_{\boldsymbol{n},k}\, z^{-k} \tag{11.22}$$

are given by inductions of the form[103]

$$S_W(0)\, a_k \;=\; \textit{earlier terms} \tag{11.23}$$

$$S_W(\omega)\, a_{\boldsymbol{n},k} \;=\; \textit{earlier terms} \qquad (\dot{\omega} = <\boldsymbol{n},\boldsymbol{\lambda}>) \tag{11.24}$$

$$(\rho_j - k)\, S'_W(\lambda_j)\, a_{\boldsymbol{n^j},k} \;=\; \textit{earlier terms} \qquad (\lambda_j = <\boldsymbol{n^j},\boldsymbol{\lambda}>) \tag{11.25}$$

As for the *analysis part* of Proposition 11.1, the shortest way is to solve the perturbed composition equation $W_\epsilon(f_\epsilon) = id$ derived from $W(f) = id$ by

[102] which results from a repeated application of the Bridge equation.

[103] The "earlier terms" in (11.24) cover all coefficients $a_{\boldsymbol{n}',k'}$ such that $\boldsymbol{n'} \leqslant \boldsymbol{n}, k' \leqslant k$ and $|\boldsymbol{n}'| + k' < |\boldsymbol{n}| + k$. In (11.25), like in (11.17), $\boldsymbol{n^j}$ denotes the sequence $\{n^j_i \,|\, n^j_i := \delta^j_i\}$. If some $\rho_j - k$ exceptionally vanishes, that simply introduce a logarithmic terms in $f_{\boldsymbol{n}^j}$.

viewing both its data and unknown as perturbations of simple shifts:

$$g_i \to g_{i,\epsilon} \quad with \quad g_{i,\epsilon}(z) = z + \sigma_i + \epsilon\, \psi_i(z) \qquad (g_{i,1} = g_i) \tag{11.26}$$

$$f \to f_\epsilon \quad with \quad f_\epsilon(z) = z + \tau + \textstyle\sum_{1\leqslant k} \epsilon^k\, \varphi_k(z) \quad (f_1 = f) \tag{11.27}$$

Expanding $W_\epsilon(f_\epsilon)$ in powers of ϵ, we get as coefficient of ϵ^k the identity (11.28), which translates to (11.29) in the Borel plane:

$$S_W(\partial)\, \widetilde{\varphi}_k(z) = \textit{polyn. in earlier terms} \quad \partial^p\, \widetilde{\varphi}_{k'}(z+\alpha_q) \quad (k' < k) \tag{11.28}$$

$$S_W(-\zeta)\, \widehat{\varphi}_k(\zeta) = \textit{convol. polyn. in earlier terms} \quad (-\zeta)^p\, e^{-\alpha_q \zeta}\, \widehat{\varphi}_{k'}(\zeta) \tag{11.29}$$

Repeated division by the exponential polynomial $S_W(-\zeta)$ and repeated convolutions make clear where the singular points of each $\widehat{\varphi}_k(\zeta)$ are going to be. Moreover, the right-hand side of (11.29), though more complicated than in the case of pure iteration equations (type $\mathcal{T}_1$, see §1.1), are essentially similar, and surprisingly easy to majorize, especially above $\mathbb{R}^+$. By duplicating the argument used for iteration equations,[104] one sees that each ϵ and in particular for $\epsilon = 1$, the function $\widehat{f}_\epsilon(\zeta)$ is endlessly continuable, with only isolated singularities[105] and (at most) exponential growth along any non-vertical[106] axis $\arg\zeta = \theta$. And this is *all* the analysis we need in order to establish Proposition 11.1 : the algebraic machinery of resurgence takes care of the rest, and leads straightaway to the Bridge equation (11.12)-(11.13).

Proposition 11.2 (General composition equation of 0-alternance)
If, retaining 0-alternance, we drop both the non-resonance and simplicity assumption for the countably many zeros λ_j of $S_W(\lambda)$, the preceding results remain in force after a number of modifications.

The indices $\boldsymbol{n}$ now range in the space $\mathscr{J}_0^{\mathbb{N}}$ of double-indexed sequences $\boldsymbol{n} = \{n_{j,\mu} \mid j \in \mathcal{J}, 0 \leqslant k < \mu_j,\ n_{j,\mu} \in \mathbb{N}, \sum n_{j,\mu} < \infty\}$, where μ_j denotes the multiplicity of the zero λ_j of $S_W(\lambda)$. These multiplicities are bounded $\sup \mu_j < \infty$ and therefore possess a finite smallest common multiple μ_*.

The saturated solution broadly retains its form (11.10), but with frequencies ω no longer in one-to-one correspondance with the indices $\boldsymbol{n}$:

$$\widetilde{f}(z,u) = \widetilde{f}(z) + \sum \boldsymbol{u}^{\boldsymbol{n}}\, e^{\omega z}\, \widetilde{f}_{\boldsymbol{n}}(z) \quad with \quad \boldsymbol{u}^{\boldsymbol{n}} = \prod u_{j,\mu}^{n_{j,\mu}}\ ,\ \omega = \langle \boldsymbol{n}, \boldsymbol{\lambda} \rangle \tag{11.30}$$

[104]See for example $[E_2]$, pp 310-318.

[105]at least on each Riemann sheet; their projection on $\mathbb{C}$ may be, and often is, dense.

[106]This is true also along vertical axes, but harder to prove. This latter fact, however, is not required here. It would be required only if we were to investigate the growth of the invariants $|A^j_\omega|$ as ω grows.

and with subexponential ramification factors flanked by ramified powers $z^{<\boldsymbol{n},\boldsymbol{\rho}>}$*:*

$$\widetilde{f}_{\boldsymbol{n}}(z) \;\in\; z^{<\boldsymbol{n},\boldsymbol{\rho}(0)>}\, e^{\left(\sum_{0<\kappa<1} <\boldsymbol{n},\boldsymbol{\rho}(\kappa)>\, z^{\kappa}\right)}\, \mathbb{C}[[z^{-\frac{1}{\mu_*}}]] \tag{11.31}$$

$$\widetilde{f}_{\boldsymbol{n}^{j,\mu}}(z) \in \quad z^{\rho_j(0)} \quad e^{\left(\sum_{0<\kappa<1}\; \rho_{j,\mu}(\kappa)\; z^{\kappa}\right)}\, \mathbb{C}[[z^{-\frac{1}{\mu_j}}]] \qquad \left(\boldsymbol{u}^{\boldsymbol{n}^{j,\mu}} = u_{j,\mu}\right) \tag{11.32}$$

The finite sum in the exponential factor of (11.31) is over all rational numbers κ *in* $\frac{1}{\mu_*}\mathbb{Q} \cup [0,1[$*. The frequencies* $<\boldsymbol{n},\boldsymbol{\rho}> := \sum n_{j,\mu}\rho_{j,\mu}$ *depend on scalars* $\rho_{j,\mu}(\kappa)$*, which appear in pure form in (11.32), but due to*

$$\rho_{j,\mu}(\kappa) := \rho_j(\kappa) e^{(2\pi i)\mu\,\kappa} \qquad \text{if} \quad \kappa \in \left\{0, \frac{1}{\mu_j}, \frac{2}{\mu_j}, ..., \frac{\mu_j - 1}{\mu_j}\right\} \tag{11.33}$$

$$\rho_{j,\mu}(\kappa) := 0 \qquad\qquad \text{otherwise} \tag{11.34}$$

reduce to the sequences:

$$\rho_j(0), \rho_j(\frac{1}{\mu_j}), \rho_j(\frac{2}{\mu_j}), \ldots, \rho_j(\frac{\mu_j - 1}{\mu_j}) \tag{11.35}$$

For each j*, the leading term corresponds to* $\kappa = \kappa_j := 1 - 1/\mu_j$*. It is directly defined, up to a unit root of order* μ_j*, by*

$$\left(\rho_j(\kappa_j)\right)^{\mu_j} \;=\; -\frac{\lambda_j\, R_W(\lambda_j)}{S_W^{(\mu_j)}(\lambda_j)} \frac{\mu_j!}{(\kappa_j)^{\mu_j}} \quad \text{with} \quad \kappa_j := 1 - \frac{1}{\mu_j} \tag{11.36}$$

Once a determination of $\rho_j(\kappa_j)$ *has been fixed, all other coefficients in the (11.35) are given, without ambiguity, by similar formulae. For the pilot components*[107] $\widetilde{f}_{\boldsymbol{n}^{j,\mu}}$ *of (11.32) ., the* μ*-dependence is elementary, since:*

$$\widetilde{f}_{\boldsymbol{n}^{j,\mu}}(z) \equiv \widetilde{f}_{\boldsymbol{n}^{j,0}}(e^{(2\pi i)\,\mu}\, z) \tag{11.37}$$

but no such relations apply for the general components $\widetilde{f}_{\boldsymbol{n}}$*.*

The invariant operators $\mathbf{A}_\omega$ *now assume the form*

$$\mathbf{A}_\omega \;:=\; \sum_{\dot{\omega} = <\boldsymbol{n},\boldsymbol{\lambda}> - \lambda_j} \boldsymbol{u}^{\boldsymbol{n}}\, A^{j,\mu}_{\omega,\boldsymbol{n}}\, \partial_{u_{j,\mu}} \qquad \boldsymbol{n} \in \mathscr{J}_0^{\mathbb{N}} \tag{11.38}$$

Note that, despite being slightly redundant, the double lower indexation of the scalars $A^{j,k}_{\omega,\boldsymbol{n}}$ *cannot be dispensed with since* $\boldsymbol{n}$ *and* ω *no longer determine each other.*[108] *This also compels us to write the monomial* $\boldsymbol{u}^{\boldsymbol{n}}$ *to the right of the first* $\sum$ *in (11.38), whereas in (11.13) it could be factored and moved to*

[107] they are preceded by the linear factors $\boldsymbol{u}^{\boldsymbol{n}^{j,\mu}} = u_{j,\mu}$ and verify the normalisation condition $\widetilde{f}_{\boldsymbol{n}^{j,\mu}} = z^{\rho_j(0)} \quad e^{\sum_{0<\kappa<1}\; \rho_{j,\mu}(\kappa)\; z^{\kappa}}.(1 + o(1))$

[108] ω clearly does not determine $\boldsymbol{n}$, and $\boldsymbol{n}$ determines only $\dot{\omega}$, but not ω.

the left of $\sum$. *The Bridge Equation of course retains its form (11.12)-(11.13) but its component-by-component interpretation undergoes a slight change:*

$$\Delta_\omega \widetilde{f}_{\boldsymbol{n}'}(z) = \Big(\sum (\boldsymbol{n}'_{\boldsymbol{j},\boldsymbol{k}} - \boldsymbol{n}_{\boldsymbol{j},\boldsymbol{k}})\, A^{j,k}_{\omega,\boldsymbol{n}} \Big)\ \widetilde{f}_{\boldsymbol{n}'-\boldsymbol{n}}(z) \ \textit{with}\ \dot{\omega} = <\boldsymbol{n}' - \boldsymbol{n}, \boldsymbol{\lambda}> - \lambda_j \tag{11.39}$$

Lastly, the formula (11.18)-(11.19) for the displays remains valid, without our having to change anything.

The *analysis* part of the proof is exactly the same as for Proposition 11.1. It still relies entirely on the repeated use of (11.18). What changes is the *formal part*, i.e. the way of calculating $\widetilde{f}(z, \boldsymbol{u})$ as a formal object.

For the basic component $\widetilde{f}$, the same induction holds as in (11.23).

For all components $\widetilde{f}_{\boldsymbol{n}}$ whose frequencies $\omega := <\boldsymbol{n}, \boldsymbol{\lambda}>$ are not zeros of S_W, the same induction holds as in (11.25), except that now $\boldsymbol{\mathcal{J}}$ replaces $\mathcal{J}$.

For the pilot components $\widetilde{f}_{\boldsymbol{n}^{j,\mu}}$, which due to (11.37) reduce to $\widetilde{f}_{\boldsymbol{n}^{j,0}}$:

$$\widetilde{f}_{\boldsymbol{n}^{j,0}}(z) = z^{\rho_j(0)}\, e^{(\sum_{0<\kappa<1} \rho_j(\kappa)\ z^\kappa)} \Big(1 + \sum_{k\in(1/\mu_j)\,\mathbb{N}^*} a_{\boldsymbol{n}^{j,\boldsymbol{0}},k}\, z^{-k}\Big) \tag{11.40}$$

the induction rule becomes:

$$(\rho_j(0) - k)\, \frac{(\kappa_j\, \rho_j(\kappa_j))^{\mu_j}}{\mu_j!}\, S_W^{(\mu_j)}(\lambda_j)\, a_{\boldsymbol{n}^{j,\boldsymbol{0}},k} \ = \ \textit{earlier terms} \tag{11.41}$$

The same type of induction (with a non-vanishing factor $S_W^{(\mu_j)}(\lambda_j)$) also applies to all components $\widetilde{f}_{\boldsymbol{n}}$ whose frequency $\omega := <\boldsymbol{n}, \boldsymbol{\lambda}>$ is of the form λ_j (due to the resonances, this may happen even if $\boldsymbol{n}$ is not of type $\boldsymbol{n}^{j,\mu}$).

$$(<\boldsymbol{n}, \boldsymbol{\rho}_j(0)> - k)\ \textit{Const}_{\boldsymbol{n}}\, S_W^{(\mu_j)}(\lambda_j)\, a_{\boldsymbol{n},k} \ = \ \textit{earlier terms} \tag{11.42}$$

Lastly, when some of the factors $(\rho_j(0) - k)$ or $(<\boldsymbol{n}, \boldsymbol{\rho}_j(0)> -k)$ in (11.41) or (11.42) vanish, the essential part[109] of $f_{\boldsymbol{n}^{j,\mu}}$ or $f_{\boldsymbol{n}}$, instead of living in $\mathbb{C}[[z^{-\frac{1}{\mu *}}]]$, now lives in $\mathbb{C}[[z^{-\frac{1}{\mu *}}]] \otimes \mathbb{C}[\log z]$.

11.3 Some remarks.

Remark 1: Display and saturated solution.

There is a vague kinship between the saturated solution $f(z, \boldsymbol{u})$ and the display *Dpl* f: both verify the composition equation[110] $W(f) = id$ and both

[109] i.e. the *series part*, as opposed to the *subexponential factor* that precedes it.

[110] As noted in the introduction, the pseudovariables behave like *constants* under ordinary differentiation or composition, and multiply according to the shuffle product: see §2.4.

involve a mixture of power series and exponential terms.[111]. But the display is a far richer and more complex object. For one thing, the saturated solution $f(z,\boldsymbol{u})$ has its power series indexed by elements $\boldsymbol{n}$ of $\mathcal{J}_0^{\mathbb{N}}$ or $\mathcal{JK}_0^{\mathbb{N}}$, whereas the power series in the display $Dpl\, f$ are indexed by the incomparably more numerous *sequences* of ω_i in Ω. Secondly, whereas $f(z,\boldsymbol{u})$ can be derived from the composition equation by purely formal manipulations on power series, $Dpl\, f$ carries scalars like A^j_ω or $A^{j,k}_{\omega,n}$, which, being generically transcendental Stokes constants, are beyond the reach of formal deduction: their calculation necessarily involve some (and often a good deal) of analysis, be it analytic continuation in the Borel plane[112] or the recourse to closed (but highly multiple) expansions involving two ingredients: *universal monics* on the one hand, and the Taylor coefficients of the data g_i in the composition equation.

Remark 2: A priori constraints on the holomorphic invariants.

If, like for the 0-alternance composition equation (11.1) when all the residues τ_i vanish, the components $f_{\boldsymbol{n}}(z)$ of the saturated solution $f(z,\boldsymbol{u})$ are ordinary power series of z^{-1}, the action of the derivations $\boldsymbol{\Delta}_\omega$, and by way of consequence the values of the invariant operators $\mathbf{A}_\omega$, will not depend on ω as an element of $\mathbb{C}_\bullet$, but only on the projection $\dot{\omega}$ on $\mathbb{C}^*$. This entails a drastic simplification of the display $Dpl\, f(z,\boldsymbol{u})$, whose pseudovariables $\mathbf{Z}^{\omega_1,\dots,\omega_r}$ may themselves be indexed by projections $\dot{\omega}_i$.

Even when the $f_{\boldsymbol{n}}(z)$ are not themselves power series, they are often simply related to power series $h_{\boldsymbol{n}}(z)$, via an elementary monomial factor

$$f_{\boldsymbol{n}}(z) = z^{<\boldsymbol{n},\boldsymbol{\rho}>}\, h_{\boldsymbol{n}}(z) \quad \textit{with} \quad h_{\boldsymbol{n}}(z) \in \mathbb{C}[[z^{-1}]] \tag{11.43}$$

This in turn implies

$$\mathbf{A}_{\epsilon\,\omega} \equiv \epsilon^{<\boldsymbol{n},\boldsymbol{\rho}>}\, \mathbf{A}_\omega \qquad (\forall \omega \in \mathbb{C}_\bullet\,,\ \forall\, \epsilon = e^{2\pi i k} \in \mathbb{C}_\bullet\,,\ k \in \mathbb{Z}) \tag{11.44}$$

so that, here again, it suffices to know the operators $\mathbf{A}_\omega$ for ω ranging through a single sheet of $\mathbb{C}_\bullet$.

If, instead, the $h_{\boldsymbol{n}}(z)$ are ramified power series[113] in $\mathbb{C}[[z^{-1/p}]]$, what is required is the knowledge of operators $\mathbf{A}_\omega$ for ω ranging through p consecutive sheets of $\mathbb{C}_\bullet$.

More complex situations may arise, but it is exceedingly rare for *all* values of $\mathbf{A}_\omega$ (with $\dot{\omega}$ fixed) to be truly independent, unless of course one starts from fully ramified data g_i, e.g. $g_i(z) \in \mathbb{R}\{z^{-1}, z^{-1}\log z\}$.

[111] in the display, the exponential terms enter via the alien derivations $\boldsymbol{\Delta}_\omega = e^{-\omega\, z}\, \Delta_\omega$.

[112] via the (wholly constructive) definition of the alien derivations: see §2.3.

[113] That would be the case if in the composition equation (11.1) we were to consider factors g_i of the form $g_i(z) = z + \sigma_i\, z^{1-p} + \dots$

Remark 3: Composition equations with resurgent data.

If we now consider composition equations $W(f, g_1, \ldots, g_r) = id$ whose data g_i are themselves resurgent (with one or several critical times), the earlier argument[114] shows that resurgence will survive, with the specific resurgence generated by the composition equation simply getting 'grafted' onto the pre-existing resurgence carried by the data. But how exactly will the two combine? This is where the *displays* come in handy, since we can calculate $Dpl\, f$, which exhaustively describes *both* resurgences, *old* and *new*, in their exact combination, by formally solving the 'displayed' composition equation:

$$W(Dpl\, f, Dpl\, g_1, \ldots, Dpl\, g_r) = id \tag{11.45}$$

with $Dpl\, f$ as unknown and the $Dpl\, g_i$ as data.

Remark 4: Twin-related composition equations.

When all shifts σ_i in (11.3), and so too all frequencies α_i in (11.4), are commensurate (this is always the case for twin-related composition equations – see §13), the sum $S_W(\lambda)$ is a polynomial of degree d in $e^{\alpha_* \lambda}$ for some maximal α_*. The roots λ_j of $S_W(\lambda)$ are therefore of the form

$$\lambda_{*1} + \frac{2\pi i}{\alpha_*} m_1, \ldots, \lambda_{*d} + \frac{2\pi i}{\alpha_*} m_d \qquad (m_j \in \mathbb{Z}) \tag{11.46}$$

If the λ_{*j} are non-resonant,[115] the λ_j are not resonant either; but if the λ_{*j} are, then the relations (11.46) massively amplify that resonance. Iteration and conjugation equations are a striking case in point. So let us have a closer look at them.

11.4 Iteration and conjugation equations: what is so special about them.

For the purpose of comparison, let us write the resurgence formulae for the solutions of iteration and composition equations, first in the standard form, then based on parameter-saturated solutions.

Let f, f_1, f_2 be real-analytic germs of the form $z \mapsto z + 1 + O(z^{-1})$, with their invariant operators $\mathbf{A}_\omega, \mathbf{A}_{1,\omega}, \mathbf{A}_{2,\omega}$, and let $h_{2,1} := {}^*f_2 \circ f_1^*$ be the conjugator of f_2 to f_1, normalised by the condition $h_{2,1}(z) = z + O(z^{-1})$. For simplicity, we drop the tildas everywhere.

[114] See the proof of Proposition 11.1, towards the end.
[115] in the sense of (11.9.

Iteration related resurgence: the standard form.

$$f^* \circ f = T \circ f^* \quad ; \quad f \circ {}^*f = {}^*f \circ T \quad ; \quad f^{\circ t} = {}^*f \circ T^{\circ t} \circ f^* \tag{11.47}$$

Resurgence support: $\Omega := 2\pi i\, \mathbb{Z}^*$.
Complete system of invariants: $\{A_\omega \mid \dot{\omega} \in \Omega\}$
Resurgence equations:

$$\mathbf{\Delta}_\omega f^*(z) \;=\; -A_\omega\, e^{-\omega\, f^*(z)} \tag{11.48}$$

$$\mathbf{\Delta}_\omega\, {}^*f(z) \;=\; +A_\omega\, e^{-\omega\, z}\, \partial\, {}^*f(z) \qquad \Big(\partial := \frac{d}{dz}\Big) \tag{11.49}$$

$$\frac{\mathbf{\Delta}_\omega f^{\circ t}(z)}{\partial f^{\circ t}(z)} \;=\; +A_\omega \left(e^{-\omega\, t} - 1\right) \frac{e^{-\omega\, f^*(z)}}{\partial f^*(z)} \qquad \big(\equiv 0 \;\; if \;\; t \in \mathbb{Z}\big) \tag{11.50}$$

Conjugation related resurgence: the standard form.

$$h_{2,1} = {}^*f_2 \circ f_1^* \qquad h_{2,1} \circ f_1 = f_2 \circ h_{2,1} \tag{11.51}$$

$$\frac{\mathbf{\Delta}_\omega\, h_{2,1}}{\partial\, h_{2,1}} \;=\; (A_{2,\omega} - A_{1,\omega})\, \frac{e^{-\omega\, f_1^*}}{\partial f_1^*} \tag{11.52}$$

$$\mathbf{\Delta}_{\omega_r} ... \mathbf{\Delta}_{\omega_1} \Big(\frac{\mathbf{\Delta}_{\omega_0}\, h_{2,1}}{\partial\, h_{2,1}}\Big) \;=\; \Big(\prod_{i=1}^{i=r} (\omega_i^* - \omega_i)\, A_{1,\omega_i}\Big) (A_{2,\omega_0} - A_{1,\omega_0})\, \frac{e^{-\omega^*\, f_1^*}}{\partial f_1^*} \tag{11.53}$$

with $\omega_i^* := \omega_0 + ... + \omega_{i-1}$ and $\omega^* := \omega_{r+1}^* = \omega_0 + ... + \omega_r$.

Iteration related resurgence: the parameter-saturated form.

The above formulae give the complete resurgence picture with all the Stokes constants, and cannot be bettered for simplicity. However, to get a real grasp of the difference with generic composition equations, we must re-write these results in the general, necessarily clumsier form, based on the parameter-saturated solutions. If we introduce formal periodic functions $P_{\boldsymbol{u}}^*$, ${}^*P_{\boldsymbol{u}}$, $P_{\boldsymbol{u}}^{\circ t}$ of the form:

$$P_{\boldsymbol{u}}^*(z) \;=\; z - \sum_{j \in \mathbb{Z}^*} u_j\, e^{(2\pi i) j z} \tag{11.54}$$

$${}^*P_{\boldsymbol{u}}(z) \;=\; z + \sum_{j \in \mathbb{Z}} v_j(\boldsymbol{u})\, e^{(2\pi i) j z} \qquad with \qquad P_{\boldsymbol{u}}^* \circ {}^*P_{\boldsymbol{u}} = id \tag{11.55}$$

$$P_{\boldsymbol{u}}^{\circ t}(z) \;=\; z + \sum_{j \in \mathbb{Z}} w_j(t; \boldsymbol{u})\, e^{(2\pi i) j z} = ({}^*P_{\boldsymbol{u}} \circ T^{\circ\, t} \circ P_{\boldsymbol{u}}^*)(z) \tag{11.56}$$

there is no difficulty in expressing the coefficients $v_j(\boldsymbol{u})$ and $w^t_j(\boldsymbol{u})$ as formal power series of $\boldsymbol{u}$, and we find that the saturated solutions attached to $f^*, {}^*f, f^{\circ t}$ admit factorisations of the form

$$f^*(z,\boldsymbol{u}) = (P^*_{\boldsymbol{u}} \circ f^*)(z) \tag{11.57}$$
$${}^*f(z,\boldsymbol{u}) = ({}^*f \circ {}^*P_{\boldsymbol{u}})(z) \tag{11.58}$$
$$f^{\circ t}(z,\boldsymbol{u}) = ({}^*f \circ P^{\circ t}_{\boldsymbol{u}} \circ f^*)(z) \tag{11.59}$$

and analytical expressions of the form

$$f^*(z,\boldsymbol{u}) = f^*(z) - \sum u_j\, e^{2\pi i j f^*(z)} \tag{11.60}$$
$${}^*f(z,\boldsymbol{u}) = {}^*f(z) + \sum \frac{v_{j_1}^{n_{j_1}}}{n_{j_r}!} \cdots \frac{v_{j_r}^{n_{j_r}}}{n_{j_1}!}\; e^{2\pi i <\boldsymbol{n},\boldsymbol{j}> z}\; \partial_z^{|\boldsymbol{n}|}\; {}^*f(z) \tag{11.61}$$
$$f^{\circ t}(z,\boldsymbol{u}) = f^{\circ t}(z) + \sum \frac{w_{j_1}^{n_{j_1}}}{n_{j_r}!} \cdots \frac{w_{j_r}^{n_{j_r}}}{n_{j_1}!}\; e^{2\pi i <\boldsymbol{n},\boldsymbol{j}> z}\; H^t_{\boldsymbol{n}}(f^*, {}^*f)(z) \tag{11.62}$$

In view of the factorisations (11.57), (11.58), (11.59), we find that all three saturated solutions verify the same Bridge Equation

$$\boldsymbol{\Delta}_\omega\, f^*(z,u) \;=\; \mathbb{A}_\omega\, f^*(z,u) \qquad (\dot\omega \in 2\pi i\,\mathbb{Z}^*) \tag{11.63}$$
$$\boldsymbol{\Delta}_\omega\, {}^*f(z,u) \;=\; \mathbb{A}_\omega\, {}^*f(z,u) \qquad (\dot\omega \in 2\pi i\,\mathbb{Z}^*) \tag{11.64}$$
$$\boldsymbol{\Delta}_\omega\, f^{\circ t}(z,u) \;=\; \mathbb{A}_\omega\, f^{\circ t}(z,u) \qquad (\dot\omega \in 2\pi i\,\mathbb{Z}^*) \tag{11.65}$$

but with invariant operators $\mathbb{A}_\omega$ of the form

$$\mathbb{A}_\omega = 2\pi i A_\omega \sum_{k\in\mathbb{Z}^*} (j+k)\, u_{j+k}\, \partial_{u_j} \qquad \textit{if} \quad \dot\omega = (2\pi i)k \;\; (k \in \mathbb{Z}^*) \tag{11.66}$$

These $\mathbb{A}_\omega$ are much simpler than the $\mathbf{A}_\omega$ predicted by the general theory (see (11.38)). In the present instance, the general $\mathbf{A}_\omega$ would be of the form:

$$\mathbf{A}_\omega = \sum_{<\boldsymbol{n},\boldsymbol{j}>-j=k} u_{j_1}^{n_{j_r}} \ldots u_{j_1}^{n_{j_r}}\; A^j_{\omega,\boldsymbol{n}}\, \partial_{u_j} \qquad \big(\dot\omega = (2\pi i)\, k)\big) \tag{11.67}$$

Comparison with generic composition equations.

To grasp the scope of the simplification, let us start from r analytic germs:

$$g_1(z) \;=\; z + 1 + \epsilon_1 \sum_{2\leqslant n} a_{1,n+1}\, z^{-n} \tag{11.68}$$
$$g_i(z) \;=\; z + \epsilon_i \sum_{2\leqslant n} a_{i,n+1}\, z^{-n} \qquad (2 \leqslant i < r) \tag{11.69}$$

and consider these similar-looking composition equations:

$$id = W(f) = f^{\circ r} \circ g_1 \tag{11.70}$$
$$id = \underline{W}(f) = f \circ g_r \circ \cdots \circ f \circ g_1 \circ \tag{11.71}$$

Both equations admit a unique, generically divergent, and always resurgent solution f, with the same resurgence support $\Omega := (2\pi i)\,(\mathbb{Z}^* - r\,\mathbb{Z}^*)$, the same exponential polynomials

$$S_W(\lambda) = S_{\underline{W}}(\lambda) = e^{\lambda/r}\frac{1-e^{\lambda}}{1-e^{\lambda/r}} \quad , \quad R_W(\lambda) = R_{\underline{W}}(\lambda) \equiv 0 \tag{11.72}$$

In both cases the saturated solutions $f(z, \boldsymbol{u})$ have unramified[116] components $f_{\boldsymbol{n}}(z) \in \mathbb{C}[[z^{-1}]]$ and verify the Bridge Equation. But whereas for the iteration equation (11.70) the corresponding invariant operators are of the elementary form (11.66), in the case of the mixed equation (11.71) they are (as soon as $r \geqslant 3$) of the general form (11.67), without any universally valid a priori relations[117] between the various scalars $A^j_{\omega,\boldsymbol{n}}$.

11.5 Stokes constants and coefficient asymptotics.

Let $\widetilde{\varphi}(z)$ be a resurgent power series and $\widehat{\varphi}(\zeta)$ its Borel transform. Let Ω^{prox} the finite set of its 'closest singular points' ω in the Borel plane, i.e. those lying on the boundary of the convergence disk of $\hat{\varphi}(\zeta)$, and let $\Delta_\omega\,\widetilde{\varphi}(z)$ be the corresonding alien derivatives.

$$\Delta_\omega\,\widetilde{\varphi}(z) = A_\omega\,\widetilde{\varphi}_\omega(z) = A_\omega\,\widetilde{\phi}_\omega(\omega\, z) \qquad (\omega \in \Omega^{\mathrm{prox}}\ ,\ z \sim \infty) \tag{11.73}$$

$$\widetilde{\varphi}(z) = \sum a_n\, z^{-n} \qquad ; \qquad \widetilde{\phi}_\omega(z) = A_\omega \sum b_{\omega,m}\, z^{-m} \tag{11.74}$$
$$\hat{\varphi}(\zeta) = \sum a_n\, \frac{\zeta^{n-1}}{(n-1)!} \qquad ; \qquad \hat{\phi}_\omega(\zeta) = A_\omega \sum b_{\omega,m}\, \frac{\zeta^{m-1}}{(m-1)!} \tag{11.75}$$

In all instances of 'equational resurgence', in particular in all cases of resurgence resulting for composition equations, the coefficients a_n of $\widetilde{\varphi}(z)$ as well as the coefficients $b_{\omega,m}$ of the alien derivatives are easily accessible (by formal calculations) – the former exactly, the latter up to multiplication by the

[116]because $R_W(\lambda) = R_{\underline{W}}(\lambda) \equiv 0$.

[117]Other than the trivial relations pointed out in Remark 3 above (in this case: dependence on $\dot{\omega}$ alone). The shortest way to prove this is to expand each $A^j_{\omega,n}$ as an entire function of $\boldsymbol{\epsilon} := (\epsilon_1, \ldots, \epsilon_r)$ and to push the Taylor expansion in $\boldsymbol{\epsilon}$ far enough to disprove the possibility of any given a priori constraints between the scalars $A^j_{\omega,\boldsymbol{n}}$.

invariants (Stokes constants) A_ω in front of them. This leads, for the calculation of the *dominant* Stokes constants A_ω (those with 'closest' indices ω), to a method which relies solely on the asymptotics of the a_n's and the knowledge of a few first coefficients $b_{\omega,m}$, and is not just simpler, but numerically more efficient than analytic continution in the Borel plane.

Simple contour integration in the Borel plane shows that

$$\frac{a_n}{(n-1)!} = A_\omega \sum_{\omega\in\Omega^{\text{prox}}} \omega^{-n} \int_1^{1+\epsilon} \widehat{\phi}_\omega(t-1)\, t^{-n}\, dt + \mathcal{O}(|\,\omega\,(1+\epsilon)|^{-n}) \qquad (11.76)$$

$$\frac{a_n}{(n-1)!} = A_\omega \sum_{\omega\in\Omega^{\text{prox}}} \frac{\omega^{-n}}{2\pi i} \int_{\Gamma_\epsilon} \check{\phi}_\omega(1-t)\, t^{-n}\, dt + \mathcal{O}(|\,\omega\,(1+\epsilon)|^{-n}) \qquad (11.77)$$

The second variant, which relies on the *majors*[118] $\check{\phi}_\omega(\zeta)$ and on a contour integration Γ_ϵ that avoids the origin, applies even when the minor $\widehat{\phi}_\omega(\zeta)$ fails to be integrable there, due to positive powers of z in $\widetilde{\phi}_\omega(z)$. The same recourse to majors makes it possible to extend the identity

$$\frac{(n-m-1)!}{(n-1)!} = \int_1^{+\infty} \frac{(t-1)^{m-1}}{(m-1)!}\, t^{-n}\, dt \qquad \textit{if} \quad n > m > 1 \qquad (11.78)$$

to all real pairs m, n with $n > m$. Assuming that sole condition, the contibution of $b_{\omega,m}$ to a_n is thus $\omega^{-n}\,(n-m-1)!$. Therefore, for any fixed $m_0 > 0$, as n goes to $+\infty$, we have

$$a_n = \sum_{\omega\in\Omega_1^{\text{prox}}} \omega^{-n} A_\omega \sum_{m<m_0} (n-m-1)!\, b_{\omega,m} \;+\; \text{Rem}(n, m_0) \qquad (11.79)$$

with a remainder $Rem(n, m_0)$ bounded by $Const.(n-m_0-1)!$ and negligible compared with the preceding terms.

So far, the indices n (resp. m) were assumed to form increasing sequences in $\mathbb{Z}$ (resp. $\mathbb{R}$), but we may also, and often must, allow n to range over $\frac{1}{q}\mathbb{N}$. The relation (11.79) still holds, provided we divide its right-hand side by q and replace Ω_1^{prox} by Ω_q^{prox} (defined as containing q consecutive copies, on q consecutive Riemann sheets, of each *closest* $\dot{\omega}$). The identity (11.78) remains in force, and so does the asymptotic formula (11.79), again with Ω_q^{prox} in place of Ω_1^{prox}.

These results still hold in the not infrequent case[119] when some of the $\phi_\omega(z)$ are no longer of the form (11.74) but of the form

$$\widetilde{\phi}_\omega(z) = e^{c\, z^\alpha}\, z_0^m \sum_{m\in\frac{1}{q}\mathbb{N}} b_{\omega,m}\, z^{-m} \qquad \textit{with} \quad 0 < \alpha = \frac{p}{q} < 1 \qquad (11.80)$$

[118]See (2.10).
[119]see the Proposition 11.2 *supra*.

Expanding the exponential $e^{c\,z^\alpha}$ and multiplying it with the power series, one gets a bilateral power series $\widetilde{\psi}_\omega(z)$, each coefficient of which is expressible as an infinite but *convergent*[120] sum (with contributions from the two factor series). One may then apply the asymptotic formula (11.77) without compunction to this bilateral $\widetilde{\psi}_\omega(z)$, or rather to the major $\check{\psi}_\omega(\zeta)$ of its Borel transform.

The method also applies when some of the alien derivatives $\Delta_\omega\,\widetilde{\varphi}(z)$ involve not *one* Stokes constant A_ω, as in (11.74), but several of them, as in the situation of Proposition 11.2 *supra* or in the case of *intertwined germs*: see the example §13.2 *infra* and in particular (13.11).

12 Some examples of composition equations.

To illustrate the general results of §11 and in particular the method for the calculation of the dominant Stokes constants, we shall now examine a series of simple composition equations $W(f)=id$ of 0-alternance.[121] To avoid the (inessential) complications that come from the ramification factors $z^{<\boldsymbol{n},\boldsymbol{\rho}>}$, we will plump for data $g_i(z) = z + \sigma_i + \tau_i/z + ...$ with vanishing residues τ_i. This way, we shall have only unramified power series to handle, and operators $\mathbf{\Delta}_\omega$ or $\mathbf{A}_\omega$ with indices ω in $\mathbb{C}^*$ rather than $\mathbb{C}_\bullet := \widetilde{\mathbb{C}-\{0\}}$. On the other hand, to spice up matters a bit, we shall impose additional symmetries on our composition equations (like invariance under $z \mapsto -z$) and examine how these symmetries impact the resurgence pattern.

12.1 Example of non-polarising composition equation.

Let our first composition equation be:

$$g_1 \circ f \circ g_1 = f \circ g_2 \circ f \qquad \text{with} \qquad g_1 := T_1 \;\; ; \;\; g_2 := T_1 \circ T_3 \circ T_1 \qquad (12.1)$$

and with $T_1(z) := z+1$, $T_3(z) := (z^3+1)^{1/3}$. The symmetries in the equation ensure that the formal solution will verify:

$$\widetilde{f} \circ \tau \circ \widetilde{f} \circ \tau = id \quad \textit{with} \quad \tau(z) := -z$$

The main (exponential-free) component of the full solution $\widetilde{f}(z,\boldsymbol{u})$ is of the form:

$$\widetilde{f}(z) = z + \sum_{2\leqslant n} a_n z^{-n} = z - \frac{1}{3}\,z^{-2} + 2\,z^{-4} - \frac{1}{9}\,z^{-5} - \frac{110}{3}\,z^{-6} + 2\,z^{-7} + \frac{113719}{81}\,z^{-8}\dots$$

[120] The convergence comes from α being < 1 and from ϕ_ω being Gevey-1.

[121] Whenever convenient, we shall spread the various composition factors $f^{\circ m_i}$ and g_i of (11.1) on both sides of the equation.

The second exponential polynomial R_W is $\equiv 0$ (since g_1 and g_2 have no residues). The first polynomial $S_W(\lambda) = e^{2\lambda} - e^{\lambda} + 1 = 0$ has simple roots

$$e^{\lambda} = \frac{1}{2} \pm \frac{\sqrt{3}}{2}\, i = e^{\pm \pi i/3} \quad ; \quad \lambda \in \pm\frac{\pi i}{3} + 2\pi i\,\mathbb{Z} \tag{12.2}$$

which all lie on the imaginary axis (so that we have a non-polarising situation) but verify many resonance relations (so that we must apply Proposition 11.1). The resurgence support is $\Omega := (\pi i/3)\,\mathbb{Z}^*$. All alien derivatives $\delta\widetilde{f} = \mathbf{\Delta}_\omega \widetilde{f}$ satisfy the same linear homogeneous equation:

$$(\delta\widetilde{f}) \circ g_1 - (\delta\widetilde{f}) \circ g_2 \circ \widetilde{f} - (\delta\widetilde{f})\,.\,(f \circ g_2)' \circ \widetilde{f} = 0$$

but its solutions depend on the exponential factor $e^{-\omega z}$ implicit in $\mathbf{\Delta}_\omega$:

$$(\mathbf{\Delta}_\omega \widetilde{f})(z) \;=\; A_\omega\, e^{-\omega x} \sum_{0 \leqslant n} b_n^+(\omega)\; z^{-n} \quad \textit{if} \quad \omega \in +\frac{\pi\, i}{3} + 2\pi i\,\mathbb{Z} \tag{12.3}$$

$$(\mathbf{\Delta}_\omega \widetilde{f})(z) \;=\; A_\omega\, e^{-\omega z} \sum_{0 \leqslant n} b_n^-(\omega)\; z^{-n} \quad \textit{if} \quad \omega \in -\frac{\pi\, i}{3} + 2\pi i\,\mathbb{Z} \tag{12.4}$$

The coefficients $b_n^{\pm}(\omega) \in \mathbb{Q}[\omega] + i\sqrt{3}\,\mathbb{Q}[\omega]$ are polynomials of degree $[\frac{n}{2}]$ in ω. One goes from $b_n^+(\omega)$ to $b_n^-(\omega)$ by complex conjugation. Thus:

$$b_0^+(\omega) \;=\; 1 \;,\; b_1^+(\omega) = 0 \;,\; b_2^+(\omega) = \frac{1}{6}\,(1 + \frac{\sqrt{3}}{3}\, i)\;\omega \;,\; b_3^+(\omega) = \frac{1}{3} - \frac{\sqrt{3}}{9}\, i - \frac{2}{9}\;\omega \;\;,$$
$$b_4^+(\omega) \;=\; \frac{2}{3} - (1 + \frac{4\sqrt{3}}{9}\, i)\;\omega + \frac{1}{108}\,(1 + \sqrt{3}\, i)\;\omega^2$$

The only non-elementary part in the expansions (12.3)-(12.4) are the Stokes constants A_ω. For the dominant pair (corresponding to $\omega_0 := \pm\pi i/3$), the method of coefficient asymptotics quickly yields more than 50 exact digits:

$$A_{\omega_0} = 0.20118243445592428494859682763527358656660758988420307679 63\ldots$$

12.2 Example of polarising composition equations.

Consider now the simplest polarising composition equation

$$f \circ f \circ g_1 = g_2 \circ f \quad \textit{with} \quad g_1(z) = z + 1 \;\; ; \;\; g_2(z) = z + 1 + z^{-2} \tag{12.5}$$

and its power series solution

$$\widetilde{f}(z) = z + \sum_{2 \leqslant n} a_n z^{-n} = z + z^{-2} + 4\, z^{-3} + 18\, z^{-4} + 104\, z^{-5} + \ldots \tag{12.6}$$

The first exponential polynomial $S_W(\lambda) := 2\,e^{\lambda} - 1$ again has only simple, but strongly resonating zeros $\lambda_j := -\log 2 + (2\pi i)\,j$, giving rise to a resurgence support

$$\Omega = \big(\log 2 + (2\pi i)\,\mathbb{Z}\big) \bigcup_{1\leqslant k} \big(-k\,\log 2 + (2\pi i)\,\mathbb{Z}\big)\big) \tag{12.7}$$

All alien derivatives $\delta\widetilde{f} = \mathbf{\Delta}_\omega \widetilde{f}$ verify the same linear homogeneous equation

$$(\delta\widetilde{f}) \circ \widetilde{f} \circ g_1 + (\widetilde{f}' \circ \widetilde{f} \circ g_1).(\delta\widetilde{f}) \circ g_1 = (g_2' \circ \widetilde{f}).(\delta\widetilde{f}) \tag{12.8}$$

and are of the form

$$(\mathbf{\Delta}_\omega \widetilde{f})(z) = A_\omega\, e^{-\omega z} \sum_{0\leqslant n} b_n(\omega) z^{-n} \quad \textit{with} \quad \omega \in \log 2 + 2\pi i\,\mathbb{Z} \tag{12.9}$$

The coefficients $b_n(\omega)$ are polynomials of degree n in ω:

$$b_0 = 1\ ,\ b_1 = -\frac{1}{2}\,\omega\ ,\ b_2 = \frac{1}{2} - \frac{3}{4}\,\omega + \frac{1}{8}\,\omega^2\ ,\ b_3 = -\frac{1}{2} - \frac{9}{4}\,\omega + \frac{5}{12}\,\omega^2 - \frac{1}{48}\,\omega^3$$

The dominant Stokes constant A_{ω_0} (with $\omega_0 = \log 2$) is

$$A_{\omega_0} := 1.36772857448473051598441729438316567750642 69\ldots \tag{12.10}$$

12.3 Example of polarising composition equation with additional symmetry.

Let us again consider a polarising composition equation, but with an added built-in symmetry:

$$g_1 \circ f \circ f \circ f \circ g_1 = f \circ g_2 \circ f \quad \text{with} \quad g_1 = T_1\ ;\ g_2 := T_1 \circ T_3 \circ T_1 \tag{12.11}$$

and with $T_1(z) := z + 1\ ,\ T_3(z) := (z^3 + 1)^{1/3}$. Since

$$T_k^{\circ(-1)} = \tau \circ T_k \circ \tau \qquad \forall k\ \ \textit{odd} \quad \textit{with} \quad \tau(z) := -z \tag{12.12}$$

the symmetries in the equation (12.11) ensure that

$$f^{\circ(-1)} = \tau \circ f \circ \tau \tag{12.13}$$

The first exponential polynomial $S_W(\lambda) := e^{2\lambda} - 3\,e^{\lambda} + 1$ has only simple, highly resonant zeros $\lambda_j^{\pm}$ symmetrical with respect to the origin:

$$\lambda_j^{\pm} := \log\big((3 \pm \sqrt{5})/2\big) \pm (2\pi i)\,j = \pm\log\big((3 + \sqrt{5})/2\big) \pm (2\pi i)\,j \tag{12.14}$$

giving rise to the resurgent support

$$\Omega := \log(\frac{3}{2} + \frac{\sqrt{5}}{2})\,\mathbb{Z}^* + (2\pi i)\,\mathbb{Z} \tag{12.15}$$

The power series solution of the composition equation is of the form:

$$f(z) = z + \sum_{2\leqslant n} a_n z^{-n} = z - \frac{1}{3}\,z^{-2} + 2\,z^{-4} - \frac{1}{9}\,z^{-5} - \frac{394}{9}\,z^{-6} - 2\,z^{-7} + \frac{162005}{81}\,z^{-8} + \dots$$

The associated linear homogeneous equation for $\delta\widetilde{f} = \mathbf{\Delta}_\omega \widetilde{f}$ reads

$$(\delta f)\circ f^2 \circ g_1 \,.\, g_1' \circ f^3 \circ g_1 + (\delta f)\circ f \circ g_1 \,.\, (g_1 \circ f)' \circ f^2 \circ g_1$$
$$+ (\delta f)\circ g_1 \,.\, (g_1 \circ f^2)' \circ f \circ g_1 - (\delta f)\circ g_2 \circ f - (\delta f)\,.\,(f\circ g_2)'\circ f = 0$$

leading to alien derivatives of the form:

$$(\mathbf{\Delta}_\omega f)(z) = A_\omega\, e^{-\omega z} \sum_{0\leqslant n} b_n^+(\omega)\; z^{-n} \quad \textit{if} \quad \omega \in \omega_+ + 2\pi i\,\mathbb{Z} \tag{12.16}$$

$$= A_\omega\, e^{-\omega z} \sum_{0\leqslant n} b_n^-(\omega)\; z^{-n} \quad \textit{if} \quad \omega \in \omega_- + 2\pi i\,\mathbb{Z} \tag{12.17}$$

$$\textit{with} \qquad \omega = \omega_\pm \mod 2\pi i\,\mathbb{Z} \quad \big(\omega_\pm = \log(\frac{3}{2} \pm \frac{\sqrt{5}}{2}) = \pm\, 2.6180339...\big)$$

The coefficients $b_n^\pm(\omega) \in \mathbb{Q}[\omega] + \sqrt{5}\,\mathbb{Q}[\omega]$ are polynomials of degree n in ω, with $b_n^+(\omega)$ and $b_n^-(\omega)$ exchanged under rational conjugation $\sqrt{5} \mapsto -\sqrt{5}$.

$$b_0^+(\omega) = 1 \;\;,\;\; b_1^+(\omega) = -\frac{1}{3}\,\omega \;\;,\;\; b_2^+(\omega) = \frac{1}{3} + (-\frac{1}{6} + \frac{\sqrt{5}}{5})\,\omega + \frac{1}{18}\,\omega^2 \;\;,$$
$$b_3^+(\omega) = -\frac{1}{3} - \frac{2\sqrt{5}}{5} - \frac{287}{270}\,\omega + (\frac{1}{18} + \frac{8\sqrt{5}}{135})\,\omega^2 - \frac{1}{162}\,\omega^3$$

The dominant Stokes constant is

$$A_{\omega_\pm} := 0.15078974841062388571094727 2\dots \big(\omega_\pm = \pm\log(\frac{3}{2}+\frac{\sqrt{5}}{2})\big) \tag{12.18}$$

12.4 Parity separation.

To any partition $\mathbb{N}^* = \coprod_{1\leqslant i\leqslant r} A_i$ there clearly corresponds a unique factorisation of every identity-tangent germ

$$f = f_1 \circ f_2 \circ \dots f_r \qquad \textit{with} \quad A_k\textit{-supported factors} \;\; f_k \tag{12.19}$$

whereby "A-supported" may refer to the germ itself

$$f(z) = z\,\big(1 + \sum_{n\in A} a_n\, z^{-n}\big) \tag{12.20}$$

or, what is dynamically more meaningful,[122] to its infinitesimal generator f_*:

$$f_*(z) = z\,\Big(\sum_{n\in A} \alpha_n\, z^{-n}\Big) \qquad \textit{with} \quad f(z) = \Big(\exp\big(f_*(z)\,\partial_z\big)\Big).z. \tag{12.21}$$

For $r = 2$ in particular, and with $\mathbb{N}_e$ resp. $\mathbb{N}_o$ standing for the sets of *even* and *odd* integers, this leads to a parity separation

$$f = f_e \circ f_o \quad \textit{with} \quad f_e = \tau \circ f_e \circ \tau \quad \textit{and} \quad f_o^{\circ -1} = \tau \circ f_o \circ \tau \tag{12.22}$$

Here τ stands as usual for the reflection $\tau := z \mapsto -z$. The solutions f_e, f_o are clearly resurgent, since finding f_o reduces to extracting the iteration square root of $\tau \circ f^{\circ -1} \circ \tau \circ f$. If for instance $f(z) = z + 1 + o(1)$, the resurgence support Ω is $2\pi i\,\mathbb{Z}^*$, and f_o verifies the resurgence equation (11.50) with $t = 1/2$. This in turn determines the resurgence pattern of f_e as well as the displays $Dpl\, f_e$, $Dpl\, f_o$. The latter are subject to no constraints other than those flowing from the three relations (12.20) re-written in terms of the displays.

For $r \geqslant 3$ and $\tau_r := z \mapsto e_r\, z$ with $e_r := exp(2\pi i/r)$, there are two equally natural generalistions of the parity factorisation (12.22). One is

$$f \;=\; f_{r,e} \circ f_{r,o} \qquad \textit{with} \tag{12.23}$$

$$f_{r,e} \;=\; \tau_r \circ f_{r,e} \circ \tau_r^{\circ -1} \qquad \textit{and} \qquad (f_{r,o} \circ \tau_r)^{\circ\, r} = id \tag{12.24}$$

and the other is

$$f \;=\; f_{r,0} \circ f_{r,1} \circ \cdots \circ f_{r,r-1} \qquad \textit{with} \tag{12.25}$$

$$f_{r,k}^{\circ\, e_r^k} \;=\; \tau_r \circ f_{r,k} \circ \tau_r^{\circ -1} \qquad \textit{and} \qquad e_r^k = exp(2\pi i\, \frac{k}{r}) \tag{12.26}$$

The condition (12.25) amounts to asking that the infinitesimal generator of $f_{r,k}$ be supported by the set $\mathbb{N}_{r,k}$ of all n such that $n \equiv k \mod r$. Here again, the factors on the right-hand side of (12.23) or (12.25) are always resurgent, with a single critical time z but rather complex resurgent supports Ω as soon as $r \geqslant 3$.

13 More examples: twins and continued conjugation.

13.1 Reminders about formal, identity-tangent twins.

Intertwined formal[123] germs f, g, or *twins* for short, are non-commuting for-

[122]The two conditions are never equivalent.

[123]To alleviate notations in this introductury paragraph, we omit all tildes even on formal objects.

mal germs (power-serial or transserial) related by some non-trivial composition identity $W(f,g)=id$. It was long thought that no formal, identity-tangent, power-serial twins (f,g) existed, until in [EV] we came up with a series of examples and developped a detailed typology of twins. Whether a given equation W admits twin solutions, and how many of them, essentially depends on the series $w(a,b):=\log W(e^a,e^b)$ viewed as an element of the (closure of the) Lie algebra $Lie[a,b]$ freely generated by a,b, and on its non-vanishing bi-homogeneous components $w_{p,q}(a,b)$. Let $\mathcal{M}_W:=\{(f,g)/conj\}$ be the set of all formal identity-tangent solutions of $W(f,g)=id$, quotiented by all formal identity-tangent conjugations $h:(f,g)\mapsto(h\circ f\circ h^{\circ-1},h\circ g\circ h^{\circ-1})$. When $\mathcal{M}_W$ is a discrete (necessarily finite) set, we speak of *rigid twins*. When not, $\mathcal{M}_W$ is a discrete collection of manifolds $\mathcal{M}_{W,k}$ and the key index – the degree of freedom – is $\sup\dim(\mathcal{M}_{W,k})$.

Being rather thin on the ground, twins have something of the power of fascination proper to sporadic objects. For a start, it appears that twins (f,g) can always be rendered *resurgent* after simultaneous conjugation by some suitable h. Two questions then arise. *First*, what are their *non-removable* resurgence invariants (Stokes constants), i.e. those invariants that cannot be eliminated under any (common) resurgent conjugation h? *Second*, do there exist *analytic* twins (f,g)? Regarding the first question, we shall show on an example how to isolate the removable invariants, get rid of them, and isolate the non-removable core. As for the second question, the answer is either *no* (most likely) or *very very few*, but the matter appears extremely hard to settle.

Reminder: normal forms of formal identity-tangent germs.

Let $\theta_{p,\sigma,\rho}$ be the identity-tangent germ of tangency order p defined by the power series in z^{-1}:

$$\theta_{p,\sigma,\rho}(z):=z+\sum_{1\leqslant n}\frac{1}{n!}\Big[\frac{\sigma\,z}{z^p+\rho}\,\frac{\partial_z}{p}\Big]^n.z=z+\frac{\sigma}{p}z^{1-p}+\Big(\frac{(1-p)\sigma^2}{2\,p^2}-\frac{\rho\sigma}{p}\Big)z^{1-2p}+\dots$$

or equivatently by the relation

$$\theta^*_{p,\sigma,\rho}\circ\theta_{p,\sigma,\rho}=1+\theta^*_{p,\sigma,\rho}\qquad \textit{with}\qquad \theta^*_{p,\sigma,\rho}(z):=\frac{1}{\sigma}\big[z^p+\rho\log(z^p)\big]\tag{13.1}$$

All $\theta_{p,\sigma,\rho}$ essentially reduce to $\theta_{1,1,\rho}$, as evidenced by the relations:

$$\theta^{\circ t}_{p,\sigma,\rho}\equiv\theta_{p,t\,\sigma,\rho}\tag{13.2}$$

$$\delta_c\circ\theta_{p,\sigma,\rho}\circ\delta_{c^{-1}}\equiv\theta_{p,c^p\sigma,c^p\rho}\qquad \textit{with}\qquad \delta_c(z):=c.z\tag{13.3}$$

$$\pi_{q^{-1}}\circ\theta_{p,\sigma,\rho}\circ\pi_q\equiv\theta_{pq,\sigma,\rho}\qquad \textit{with}\qquad \pi_q(z):=z^q\tag{13.4}$$

and any identity-tangent, power-serial $\widetilde{f}$ is conjugate to a well-defined $\theta_{p,\sigma,\rho}$ under some identity-tangent, power-serial $\widetilde{f}^{\#}$, which is itself determined up to pre-compostion by any iterate $\theta^{\circ t}_{p,\sigma,\rho}$:

$$\widetilde{f}^{\#}\circ\widetilde{f}=\theta_{p,\sigma,\rho}\circ\widetilde{f}^{\#}\qquad\qquad (\widetilde{f}(z)\sim z,\widetilde{f}^{\#}(z)\sim z)\tag{13.5}$$

13.2 Simplest instance of rigid twins.

The simplest instance of rigid twin equation is:

$$W(f,g)=id\quad\text{with}\quad W(f,g):=g^{\circ-1}\circ f^{\circ-3}\circ g\circ f\circ g^{\circ-3}\circ f\circ g^{\circ3}\circ f\tag{13.6}$$

The identity

$$W(f^{\circ-1},g)=(g^{\circ-1}\circ f^{\circ-3}\circ g)\circ\big(W(f,g)\big)^{\circ-1}\circ(g^{\circ-1}\circ f^{\circ3}\circ g)\tag{13.7}$$

shows that the solutions of (13.6) go by pairs (f,g) and $(f^{\circ-1},g)$. Moreover, if one restricts oneself to power-series solutions at infinity, these are automatically identity-tangent. More precisely, using the normal forms $\theta_{p,\sigma,\rho}$ of §13.1 *supra*, one checks that f and g are necessarily conjugate to $\theta_{1,c,0}$ and $\theta_{2,-4c^2,-4c^2}$ respectively, under *different* conjugations. But a *common* dilation $z\mapsto c'\,z$ makes it possible to fix c arbitrarily, so that we may normalise f to the unit shift $\theta_{1,1,0}:z\mapsto z+1$. The twin g is then defined up to conjugation by iterates of f, i.e. by shifts, and we may focus on the (unique) determination that commutes with $\tau:z\mapsto -z$. We thus get twins $(\underline{f},\underline{g})$ of the form:[124]

$$\underline{f}(z)\;=\;z+1\quad\textit{and}\quad\underline{g}(z)=z+\sum_{1\leqslant n}a_{2n-1}\,z^{-2n+1}\quad\textit{with}\tag{13.8}$$

$$\underline{g}(z)\;=\;z-2\,z^{-1}-6\,z^{-3}-\frac{1522}{39}\,z^{-5}-\frac{21659}{65}\,z^{-7}-\frac{2279405017}{692055}\,z^{-9}\dots\tag{13.9}$$

Our composition equation $W(\underline{f},\underline{g})=id$ being highly alternate, the propositions of §11.2 do not apply, even after the first twin has been fixed (normalised). Nevertheless, a direct investigation shows that the series $\underline{g}$ in (13.8), (13.9) is not only Gevrey 1 but also resurgent with critical time z and resurgence support $\Omega=2\pi i\,\mathbb{Z}^*$. The linearised equation verified by all alien derivatives $\delta\underline{g}=\mathbf{\Delta}_\omega\underline{g}$ reads:

$$\sum_{1\leqslant i\leqslant 8}\epsilon_i\,\frac{\delta\underline{g}\circ\underline{h}_{p_i}(z)}{\partial\underline{h}_{q_i}(z)}=0\tag{13.10}$$

[124]for simplicity we drop the tildae.

with upper factors $\underline{h}_{p_i}$, lower factors $\partial\underline{h}_{q_i}$, and coefficients ϵ_i defined as follows:

$$\begin{array}{lcl|lcl|lcl|}
\underline{h}_1 &=& \underline{f} & \underline{h}_5 &=& \underline{f}\circ\underline{h}_4 & \underline{h}_9 &=& \underline{f}\circ\underline{h}_8 \\
\underline{h}_2 &=& \underline{g}\circ\underline{h}_1 & \underline{h}_6 &=& \underline{g}^{-1}\circ\underline{h}_5 & \underline{h}_{10} &=& \underline{g}\circ\underline{h}_9 \\
\underline{h}_3 &=& \underline{g}\circ\underline{h}_2 & \underline{h}_7 &=& \underline{g}^{-1}\circ\underline{h}_6 & \underline{h}_{11} &=& \underline{f}^{-3}\circ\underline{h}_{10} \\
\underline{h}_4 &=& \underline{g}\circ\underline{h}_3 & \underline{h}_8 &=& \underline{g}^{-1}\circ\underline{h}_7 & \underline{h}_{12} &=& \underline{g}^{-1}\circ\underline{h}_{11}
\end{array}$$

$$\begin{array}{lcl|lcl|lcl|lcl|}
(p_1,q_1) &=& (1,2) & (p_5,q_5) &=& (7,6) & \epsilon_1 &=& + & \epsilon_5 &=& - \\
(p_2,q_2) &=& (2,3) & (p_6,q_6) &=& (8,7) & \epsilon_2 &=& + & \epsilon_6 &=& - \\
(p_3,q_3) &=& (3,4) & (p_7,q_7) &=& (9,10) & \epsilon_3 &=& + & \epsilon_7 &=& + \\
(p_4,q_4) &=& (6,5) & (p_8,q_8) &=& (12,11) & \epsilon_4 &=& - & \epsilon_8 &=& -
\end{array}$$

In agreement with the fact that the first exponential polynomial $S_W(\lambda) = (e^\lambda - 1)^3$ has only zeros of order three, the linearised equation (13.10) yields alien derivatives with three *a priori* free Stokes constants $A_\omega, A_\omega^\pm$ in them:[125]

$$\begin{aligned}
(\boldsymbol{\Delta}_\omega\,\underline{g})(z) \;=\; & +A_\omega\, e^{-\omega z}\,(\omega z)^{-1} \sum_{0\leqslant n} b_n(\omega)\,(\omega z)^{-n} \\
& +A_\omega^+\, e^{-(\omega z+\sqrt{24\,\omega z})}\,(24\,\omega z)^{-\frac{7}{4}} \sum_{0\leqslant n} b^+_{\frac{n}{2}}(\omega)\,(24\,\omega z)^{-\frac{n}{2}} \\
& +A_\omega^-\, e^{-(\omega z-\sqrt{24\,\omega z})}\,(24\,\omega z)^{-\frac{7}{4}} \sum_{0\leqslant n} b^-_{\frac{n}{2}}(\omega)\,(24\,\omega z)^{-\frac{n}{2}}
\end{aligned} \tag{13.11}$$

Each $b_n(\omega)$ and each $b^\pm_{n/2}(\omega)$ is an *even* polynomial of degree $2\,n$ in ω. Unlike the $b^\pm_{n/2}(\omega)$, the $b_n(\omega)$ carry no terms of degree less than $n-1$. Moreover, the double series $b^\pm_{n/2}(\omega)$ reduces to one, since $b^+_{n/2}(\omega)/b^-_{n/2}(\omega) \equiv (-1)^n$. Here are the first polynomials:

$$\begin{aligned}
b_0(\omega) &= 1 \;\;,\;\; b_1(\omega) = -1+\omega^2 \;\;,\;\; b_2(\omega) = 3\,\omega^2 + \frac{2}{3}\,\omega^4 \\
b_3(\omega) &= -9\,\omega^2 + 6\,\omega^4 + \frac{1}{3}\,\omega^6 \;\;,\;\; b_4(\omega) = \frac{761}{39}\,\omega^4 + 6\,\omega^6 + \frac{2}{15}\,\omega^8 \\
b_5(\omega) &= -\frac{3805}{39}\,\omega^4 + \frac{1873}{39}\,\omega^6 + 6\,\omega^8 + \frac{2}{45}\,\omega^{10} \\
b_6(\omega) &= \frac{21659}{130}\,\omega^6 + \frac{2224}{39}\,\omega^8 + 2\,\omega^{10} + \frac{4}{315}\,\omega^{12}
\end{aligned}$$

[125] If we had to do with a composition equation $W(\underline{f},\underline{g}) = id$ of 0-alternance (in $\underline{g}$), the Proposition 11.1 would predict subexpontial factors of the form $e^{\rho(2/3)z^{2/3}+\rho(1/3)z^{1/3}}$. Here, however, we do not have 0-alternance, and so there is no contradiction in finding subexpontial factors of the form $e^{\rho(1/2)z^{1/2}}$.

$$
\begin{aligned}
b_0^+(\omega) &= 1\\
b_{\frac{1}{2}}^+(\omega) &= -\frac{93}{8} + 14\,\omega^2\\
b_1^+(\omega) &= \frac{12465}{128} + \frac{105}{4}\omega^2 + 98\,\omega^4\\
b_{\frac{3}{2}}^+(\omega) &= -\frac{750327}{1024} + \frac{28383}{64}\,\omega^2 + \frac{92669}{60}\,\omega^4 + \frac{1372}{3}\,\omega^6\\
b_2^+(\omega) &= \frac{178961931}{32768} - \frac{2143389}{512}\,\omega^2 + \frac{4952933}{320}\,\omega^4 + \frac{412013}{30}\,\omega^6 + \frac{4802}{3}\,\omega^8
\end{aligned}
$$

Although the indices ω of simple alien derivations have to be $\neq 0$, the existence of composite derivations with zero-sum indices, such as $[\boldsymbol{\Delta}_{\omega_2}, \boldsymbol{\Delta}_{\omega_1}]$ with $\dot{\omega}_1 + \dot{\omega}_2 = 0$, forces us to consider the solutions of the linearised equation (13.10) for $\omega = 0$ also. These exponential-free solutions assume a somewhat special form:

$$
\begin{aligned}
\delta\underline{g}(z) \;=\; &+ C \qquad z^{-2} \sum_{0\leqslant n} c_{2\,n}\,(z)^{-2\,n}\\
&+ C^+ \;\; z^{-(\frac{1}{2}+\frac{\sqrt{23}}{2}\,i)} \sum_{0\leqslant n} c^+_{2\,n}\, z^{-2\,n}\\
&+ C^- \;\; z^{-(\frac{1}{2}-\frac{\sqrt{23}}{2}\,i)} \sum_{0\leqslant n} c^-_{2\,n}\, z^{-2\,n}
\end{aligned}
\tag{13.12}
$$

with only even-indexed coefficients $c_{2n}, c^+_{2n}, c^-_{2n}$ (the latter two complex-conjugate):

$$
\begin{array}{lcc|lcr}
c_0 &=& 1 & c_0^\pm &=& 1\\
c_2 &=& 9 & c_2^\pm &=& -\frac{28}{9} \mp \frac{17\sqrt{23}}{9}\,i\\
c_4 &=& \frac{3805}{39} & c_4^\pm &=& -\frac{3682829}{45630} \mp \frac{131851\sqrt{23}}{45630}\,i\\
c_6 &=& \frac{151613}{130} & c_6^\pm &=& -\frac{27042544817}{37690380} \pm \frac{3136135007\sqrt{23}}{37690380}\,i\\
c_8 &=& \frac{2279405017}{153790} & c_8^\pm &=& -\frac{1119884000708633}{213139098900} \pm \frac{49083606237437\sqrt{23}}{30448442700}\,i
\end{array}
$$

Due to the presence of the subexponential factors $e^{\pm\sqrt{24\omega z}}$, the Stokes constants A_ω and $A^\pm_\omega$ in (13.11) are more difficult to calculate with high accuracy than in the examples of §12. Nonetheless, the dominant ones (for $\omega = \pm 2\pi i$) have been computed to 12 exact digits – enough to make sure that our twin equation admits no analytic solution, only resurgent ones.

13.3 Removable and non-removable invariants.

To fully describe the resurgence properties of g, we must form the general solution $\underline{g}(z,\boldsymbol{u})$ of $W(\underline{f},\underline{g}) = id$, with $\underline{f} = T =$ the unit shift. This we do by introducing parameters $u_{j,\mu}$ with $j \in \mathbb{Z}$ and $\mu \in \{-1,0,1\}$ and by adding:

(i) the *basic component* $\underline{g}(z)$ as in (13.8)-(13.9).

(ii) the *pilot components* $e^{\omega_j z}\sum_{-1\leqslant\mu\leqslant 1} u_{j,\mu}\, \underline{g}_{\boldsymbol{n}^{j,\mu}}(z)$ $\big(\omega_j := (2\pi i)j, j\neq 0\big)$, obtained by replacing $A_\omega, A_\omega^{\pm}$ in (13.11) by $u_{j,0}, u_{j,\pm 1}$ with $\omega = -\omega_j$.

(iii) the *pilot component* $\sum_{-1\leqslant\mu\leqslant 1} u_{0,\mu}\, \underline{g}_{\boldsymbol{n}^{0,\mu}}(z)$, obtained by replacing $C, C^{\pm}$ in (13.12) by $u_{0,0}, u_{0,\pm 1}$.

(iv) the *general components* $e^{(2\pi i)<\boldsymbol{n},\boldsymbol{j}>z}\sum_{-1\leqslant\mu\leqslant 1} u^{\boldsymbol{n}}\, \underline{g}_{\boldsymbol{n}}(z)$ $\big(\, |\boldsymbol{n}| > 1\big)$, inductively calculable from the pilot components.

Although $W(\underline{f},\underline{g}) = id$, viewed as a composition equation in $\underline{g}$, does not have 0-alternance, the Bridge equation

$$\boldsymbol{\Delta}_\omega\, \underline{g}(z,\boldsymbol{u}) = \mathbf{A}_\omega\, \underline{g}(z,\boldsymbol{u}) \qquad \forall\omega \in \Omega := (2\pi i)\,\mathbb{Z}^* \tag{13.13}$$

still applies, with differential operators $\mathbf{A}_\omega$ of the type indicated in Proposition 11.2. Moreover, purely formal considerations show that the displays of $\underline{g}_{\boldsymbol{u}}(z) := \underline{g}(z,\boldsymbol{u})$ and $\underline{g}(z)$ must be of the form:

$$\mathrm{Dpl}\, \underline{g}_{\boldsymbol{u}} = ({}^*\!P_{\boldsymbol{u}}) \circ (\mathfrak{Dpl}\, \underline{g}_{\boldsymbol{u}}) \circ (P^*_{\boldsymbol{u}}) \tag{13.14}$$

$$\mathrm{Dpl}\, \underline{g} \;=\; ({}^*\!P) \circ (\mathfrak{Dpl}\, \underline{g}) \circ (P^*) \qquad \Big(\underline{g} = \underline{g}_{\mathbf{0}},\ P^* = P^*_{\mathbf{0}}\,,\ {}^*\!P = {}^*\!P_{\mathbf{0}}\Big) \tag{13.15}$$

with *maximal* and mutually reciprocal factors $P^*_{\boldsymbol{u}}(z)$ and ${}^*\!P_{\boldsymbol{u}}(z)$ that depend only on the variables $u_{j,0}$, since only these variables are accompanied by pure exponentials $e^{\omega z}$ (without perturbating subexponential factors $e^{\pm\sqrt{24\omega\, z}}$) and therefore lead to factors $P^*_{\boldsymbol{u}}(z)$ and ${}^*\!P_{\boldsymbol{u}}(z)$ that commute with the unit shift T. But $\underline{f} = T$. There must therefore exist an analytic germ f, of tangency orders 1 and with resurgent iterators f^*, ${}^*\!f$ such that

$$\mathrm{Dpl}\, f^* = P^* \circ f^* \quad , \quad \mathrm{Dpl}\, {}^*\!f = {}^*\!f \circ {}^*\!P \tag{13.16}$$

with the very same $P^*, {}^*\!P$ as in (13.15) above. We can therefore jointly conjugate the semi-normalised pair $(\underline{f},\underline{g})$ to a new pair (f,g):

$$(\underline{f},\underline{g}) \mapsto (f,g) := ({}^*\!f \circ \underline{f} \circ f^*, {}^*\!f \circ \underline{g} \circ f^*) \tag{13.17}$$

whose displays will be

$$\mathrm{Dpl}\, f = f \quad , \quad \mathrm{Dpl}\, g = {}^*\!f \circ (\mathfrak{Dpl}\, \underline{g}) \circ f^* \tag{13.18}$$

The loss of semi-normalisation in (f, g) is more than made up by the concommitant simplification in the displays. Indeed, unlike $Dpl\,\underline{g}$, the new display $Dpl\,g$ no longer carries the *removable* invariants (Stokes constants) attached to $P^*, {}^*P$ and accompanied by the variables $u_{j,0}$. It retains only the *non-removable* invariants (Stokes constants) attached to the core $\mathfrak{Dpl}\,\underline{g}$ of $Dpl\,\underline{g}$ and accompanied by the variables $u_{j,\pm 1}$.

The 'cleansed' pair (f, g) is of course defined only up to a joint analytic conjugation. Its construction depends on the procedure known as 'synthesis' (constructing f from its analytic invariants, carried here by P^*) and, although there exist privileged solutions (the so-called 'spherical synthesis', see $[E_4]$), the construction is computationally very costly. It is not known whether there exist more direct ways of arriving as such pairs (f, g) cleansed of all *removable* invariants. It should be noted, moreover, that in any such pair it is f, not g, that has to be analytic. This dissymmetry stems from the fact that, whereas f and g have tangency orders 1 and 2 respectively, we have only *one* intrinsic critical time, namely z, not z^2.

13.4 Simplest instance of non-rigid twins.

Let $W(A, B)$ be an element of the group $< A, B >$ freely generated by the symbols A, B. Let $w(a, b)$ be its formal infinitesimal generator, of components $w_{p,q}(a, b)$ in the Lie algebra freely generated by a, b:

$$\log(W(e^a, e^b)) = \sum_{0 \leqslant p,q,\ 1 \leqslant p+q} w_{p,q}(a, b) \qquad \big(w_{p,q}(a, b) \in \mathrm{Lie}_{p,q}(a, b)\big) \quad (13.19)$$

Assume that there is a point $(p_0, q_0) \in \mathbb{N}^* \times \mathbb{N}^*$ and two lines L_j passing through (p_0, q_0), of equations $L_j(p, q) = (p - p_0) + l_j\,(q - q_0) = 0$, with positive anti-slopes $0 < l_1 < l_2$, and such that:

(i) L_1, L_2 are *contiguous*, in the sense that there exist no points in $\mathbb{N}^* \times \mathbb{N}^*$ lying strictly between L_1 and L_2

(ii) for all points (p, q) *below* L_1 or L_2, i.e. such that $L_1(p, q) + L_2(p, q) < 0$, the corresponding component of $w(a, b)$ vanishes: $w_{p,q}(a, b) = 0$.

(iii) $w_{p_0,q_0}(a, b) \neq 0$ but $w_{p_0,q_0}(z^{1-p}\partial, z^{1-q}\partial) \equiv 0$, $\forall p, q$.

Then for almost all integers (p, q) such that $l_1 < \frac{p}{q} < l_2$, the composition equation $W(f, g) = id$ admits identity-tangent solutions of the form

$$f(z) = z\left(1 + a_p z^{-p} + \ldots\right) \quad , \quad g(z) = z\left(1 + b_q z^{-q} + \ldots\right) \qquad (13.20)$$

More generally, for all but a finite number of real numbers $\alpha \in]l_1, l_2[$, there exist identity-tangent, ramified twins of the form

$$f(z) \;=\; z\big(1+\sum_{1\leqslant m+n} a_{m,n} z^{-m-n\alpha}\big) \qquad (a_{0,1}\neq 0) \tag{13.21}$$

$$g(z) \;=\; z\big(1+\sum_{1\leqslant m+n} b_{m,n} z^{-m-n\alpha}\big) \qquad (b_{1,0}\neq 0) \tag{13.22}$$

This construction becomes possible for all $(p_0,q_0)\geqslant(3,3)$ but $\neq(3,3)$. Here is the simplest example, with $(p_0,q_0)=(4,3)$ and a Lie component $w_{4,3}(a,b)$ of the form (with the notation $\bar{x}y := [x,y]$):

$$\begin{aligned} w_{4,3} &= [[b,\bar{a}^3b],\bar{a}b]+6[[b,\bar{a}b],\bar{a}^3b]-3[[b,\bar{a}^2b],\bar{a}^2b]-3[[\bar{a}b,\bar{a}^2b],\bar{a}b] \\ &= [[[a,\bar{b}^2a],a],\bar{b}a]-6[[[a,\bar{b}a],a],\bar{b}^2a]-3[[a,\bar{b}a],[a,\bar{b}^2a]]+4[[[a,\bar{b}a],\bar{b}a],\bar{b}a] \end{aligned} \tag{13.23}$$

One can easily construct words $W(A,B)$ that verify all three conditions (i)-(ii)-(iii) relative to the contiguous lines L_1, L_2 or L_2, L_3, of equations

$$L_1(p,q):=p+q-7\ ,\ \ L_2(p,q):=2p+3q-17\ ,\ \ L_3(p,q):=p+2q-10 \tag{13.24}$$

and of slopes $l_1 := 1, l_2 = \frac{2}{3}, l_3 = \frac{1}{2}$.

In [EV], §7.5, pp 77-81, instances of equations $W(f,g) = id$ are even constructed that admit twin solutions of the above type for *any* tangency ratio $p/q > 0$ or *any* real $\alpha > 0$/ This has the advantage of permitting expansions in the free parameter α, in particular for α near 0 or ∞, leading, for each finite derivative in α, to unusual but rather tractable resurgence patterns, all linked to linear equations of a mixed, *difference-cum-differential* type.

13.5 An analogue of continued fractions: continued conjugation.

A 'p-approximant' is any anlytic germ ϑ of tangency order p and of the form:

$$\vartheta^*\circ\vartheta = 1+\vartheta^* \quad \textit{with} \quad \vartheta^*(z) := \frac{1}{\sigma}\big[z^p+\sum_{1<k<p}\tau_k\, z^k+\rho\log(z^p)\big] \tag{13.25}$$

For any power-serial $\widetilde{f}$ of tangency order p, there is clearly a unique power-serial $\widetilde{f}^\sharp$ of tangency order $p^\sharp > p$ and a unique p-approximant ϑ such that:

$$\widetilde{f}^\sharp\circ\widetilde{f} = \vartheta\circ\widetilde{f}^\sharp \qquad \Big(\widetilde{f}(z)=z+O(z^{1-p})\,,\ \widetilde{f}^\sharp(z)=z+O(z^{1-p^\sharp})\Big) \tag{13.26}$$

This opens the way to a continuous conjugation:[126]

$$\begin{array}{ccccccccccc} f & \to & f^{\sharp} & \to & f^{\sharp 2} & \to & \dots & \to & f^{\sharp n} & \to & \dots \\ \vartheta_0 & \to & \vartheta_1 & \to & \vartheta_2 & \to & \dots & \to & \vartheta_n & \to & \dots \end{array}$$

with p_n-approximants ϑ_n of strictly increasing tangency orders p_n.

Resurgence and display of the n^{th} conjugator $f^{\sharp\, n}$.

If we start from an analytic f, the successive conjugators $f^{\sharp n}$ are of course polycritically resurgent, with critical times $z_1 := z^{p_1}, \dots, z_n := z^{p_n}$. The corresponding resurgence is best captured by the polycritical displays $Dpl\, f^{\sharp n}$. Their *general form*[127] is easily found, inductively on n, by solving the equations:

$$(\mathrm{Dpl}\, f^{\sharp n}) \circ (\mathrm{Dpl}\, f^{\sharp (n-1)}) = \vartheta_{p_{n-1}} \circ (\mathrm{Dpl}\, f^{\sharp n}) \tag{13.27}$$

with $Dpl\, f^{\sharp(n-1)}$ regarded as given and $Dpl\, f^{\sharp n}$ as unknown. However, not only does the exact numerical determination of the n^{th}-order Stokes constants pose formidable problems[128], but even their theoretical properties, growth patterns etc, are far from clear.[129] It is not even obvious what a priori restrictions[130] constrain the successive approximants ϑ_n when the initial germ submitted to 'continuous conjugation' is, say, real-analytic at ∞.

14 Tables: iso-derivations and iso-operators.

We use throughout the notations of §6 and §7.

[126] It has to rely on the approximants ϑ. The normal forms $\theta_{p,\sigma,\rho}$ of §13.1 would be ill-suited for the purpose, if only because they would not lead to *increasing* sequences $\{p_n\}$ of tangency orders.

[127] i.e. the form which it assumes when we regard the Stokes constants as free parameters.

[128] despite their definition being in principle fully constructive.

[129] Still, at each induction order n, one would expect an at-most-exponential growth pattern in ω for A_ω whenever the display is defined relative to a *well-behaved* system of alien derivations, like the *organic* system.

[130] one thing at any rate is clear: there are no *growth* restrictions on their coefficients, not even the key coefficients σ_n or ρ_n.

14.1 The three bases $\mathrm{Dn}^{\{\bullet\}}$, $\mathrm{Ds}^{\{\bullet\}}$, $\mathrm{Da}^{\{\bullet\}}$ of ISO.

Table 1: $\mathrm{Ds}^{\{\bullet\}}$ **in terms of** $\mathrm{Dn}^{\{\bullet\}}$ **and** ***vice versa*.**

All conversion formulae basically involve the same structure constants

$$\widetilde{\mathrm{Dn}}^{\{n_0\}} \equiv \sum_{1\leqslant r} \sum_{n_0=n_1+\ldots+n_r}^{n_1\leqslant n_2\ldots\leqslant n_r} (-1)^r\, H^{n_0}_{n_1,\ldots,n_r}\ \ \mathrm{Dn}^{\{n_1,\ldots,n_r\}} \qquad (14.1)$$

$$\mathrm{Ds}^{\{n_0\}} \equiv \sum_{1\leqslant r} \sum_{n_0=n_1+\ldots+n_r}^{n_1\leqslant n_2\ldots\leqslant n_r} (-2)^{1-r}\, H^{n_0}_{n_1,\ldots,n_r}\ \ \mathrm{Dn}^{\{n_1,\ldots,n_r\}} \qquad (14.2)$$

$$\mathrm{Dn}^{\{n_0\}} \equiv \sum_{1\leqslant r} \sum_{n_0=n_1+\ldots+n_r}^{n_1\leqslant n_2\ldots\leqslant n_r} (+2)^{1-r}\, H^{n_0}_{n_1,\ldots,n_r}\ \ \mathrm{Ds}^{\{n_1,\ldots,n_r\}} \qquad (14.3)$$

with positive integers $H^{n_0}_{n_1,\ldots,n_r}$.

$$\mathrm{Ds}^{\{1\}} = \mathrm{Dn}^{\{1\}} \qquad \mathrm{Dn}^{\{1\}} = \mathrm{Ds}^{\{1\}}$$

$$\mathrm{Ds}^{\{2\}} = \mathrm{Dn}^{\{2\}} - \tfrac{1}{2}\mathrm{Dn}^{\{1^2\}} \qquad \mathrm{Dn}^{\{2\}} = \mathrm{Ds}^{\{2\}} + \tfrac{1}{2}\mathrm{Ds}^{\{1^2\}}$$

$$\mathrm{Ds}^{\{3\}} = \mathrm{Dn}^{\{3\}} - 2\mathrm{Dn}^{\{1,2\}} + \tfrac{1}{2}\mathrm{Dn}^{\{1^3\}} \qquad \mathrm{Dn}^{\{3\}} = \mathrm{Dn}^{\{3\}} + 2\mathrm{Ds}^{\{1,2\}} + \tfrac{1}{2}\mathrm{Ds}^{\{1^3\}}$$

$$\mathrm{Ds}^{\{4\}} = \mathrm{Dn}^{\{4\}} - \tfrac{7}{2}\mathrm{Dn}^{\{1,3\}} - 2\mathrm{Dn}^{\{2,2\}} + \tfrac{9}{2}\mathrm{Dn}^{\{1^2,2\}} - \tfrac{3}{4}\mathrm{Dn}^{\{1^4\}}$$

$$\mathrm{Ds}^{\{4\}} = \mathrm{Dn}^{\{4\}} + \tfrac{7}{2}\mathrm{Dn}^{\{1,3\}} + 2\mathrm{Dn}^{\{2,2\}} + \tfrac{9}{2}\mathrm{Dn}^{\{1^2,2\}} + \tfrac{3}{4}\mathrm{Dn}^{\{1^4\}}$$

Table 2: $\mathrm{Da}^{\{\bullet\}}$ **in terms of** $\mathrm{Dn}^{\{\bullet\}}$**.**

$$\mathrm{Da}^{\{1\}} = +\mathrm{Dn}^{\{1\}}$$

$$\mathrm{Da}^{\{2\}} = +\mathrm{Dn}^{\{2\}} - 1/2\,\mathrm{Dn}^{\{1^2\}};$$

$$\mathrm{Da}^{\{3\}} = +\mathrm{Dn}^{\{3\}} - \mathrm{Dn}^{\{2,1\}}$$

$$\mathrm{Da}^{\{4\}} = +\mathrm{Dn}^{\{4\}} - \mathrm{Dn}^{\{3,1\}} - 2\,\mathrm{Dn}^{\{2^2\}} + \mathrm{Dn}^{\{2,1^2\}} - 1/4\,\mathrm{Dn}^{\{1^4\}}$$

$$\mathrm{Da}^{\{5\}} = +\mathrm{Dn}^{\{5\}} - \mathrm{Dn}^{\{4,1\}} - 5\,\mathrm{Dn}^{\{3,2\}} + \mathrm{Dn}^{\{3,1^2\}} + 2\,\mathrm{Dn}^{\{2^2,1\}} - \mathrm{Dn}^{\{2,1^3\}}$$

$$\begin{aligned}\mathrm{Da}^{\{6\}} = &+\mathrm{Dn}^{\{6\}} - \mathrm{Dn}^{\{5,1\}} - 6\,\mathrm{Dn}^{\{4,2\}} - 15/2\,\mathrm{Dn}^{\{3^2\}} + \mathrm{Dn}^{\{4,1^2\}} + 11\,\mathrm{Dn}^{\{3,2,1\}}\\ &-4/3\,\mathrm{Dn}^{\{2^3\}} - \mathrm{Dn}^{\{3,1^3\}} - 1/2\,\mathrm{Dn}^{\{2^2,1^2\}} - 5/2\,\mathrm{Dn}^{\{2,1^4\}} + 5/12\,\mathrm{Dn}^{\{1^6\}}\end{aligned}$$

$$\begin{aligned}\mathrm{Da}^{\{7\}} = &+\mathrm{Dn}^{\{7\}} - \mathrm{Dn}^{\{6,1\}} - 7\,\mathrm{Dn}^{\{5,2\}} - 21\,\mathrm{Dn}^{\{4,3\}} + \mathrm{Dn}^{\{5,1^2\}} + 13\,\mathrm{Dn}^{\{4,2,1\}}\\ &+11\,\mathrm{Dn}^{\{3^2,1\}} + 7\,\mathrm{Dn}^{\{3,2^2\}} - \mathrm{Dn}^{\{4,1^3\}} - 4\,\mathrm{Dn}^{\{3,2,1^2\}} - \mathrm{Dn}^{\{2^3,1\}}\\ &-5/2\,\mathrm{Dn}^{\{3,1^4\}} - 10\,\mathrm{Dn}^{\{2^2,1^3\}} + 5/2\,\mathrm{Dn}^{\{2,1^5\}}\end{aligned}$$

$$
\begin{aligned}
\mathrm{Da}^{\{8\}} = {} & +\mathrm{Dn}^{\{8\}} - \mathrm{Dn}^{\{[\}}7,1] - 8\,\mathrm{Dn}^{\{6,2\}} - 28\,\mathrm{Dn}^{\{5,3\}} - 28\,\mathrm{Dn}^{\{4^2\}} + \mathrm{Dn}^{\{6,1^2\}} \\
& +15\,\mathrm{Dn}^{\{5,2,1\}} + 49\,\mathrm{Dn}^{\{4,3,1\}} + 48\,\mathrm{Dn}^{\{4,2^2\}} + 25\,\mathrm{Dn}^{\{3^2,2\}} - \mathrm{Dn}^{\{5,1^3\}} \\
& -21\,\mathrm{Dn}^{\{4,2,1^2\}} - 11\,\mathrm{Dn}^{\{3^2,1^2\}} - 39\,\mathrm{Dn}^{\{3,2^2,1\}} - 64\,\mathrm{Dn}^{\{2^4\}} + \mathrm{Dn}^{\{4,1^4\}} \\
& -16\,\mathrm{Dn}^{\{3,2,1^3\}} + 68\,\mathrm{Dn}^{\{2^3,1^2\}} - \mathrm{Dn}^{\{3,1^5\}} - 54\,\mathrm{Dn}^{\{2^2,1^4\}} + 21\,\mathrm{Dn}^{\{2,1^6\}} \\
& -21/8\,\mathrm{Dn}^{\{1^8\}}
\end{aligned}
$$

$$
\begin{aligned}
\mathrm{Da}^{\{9\}} = {} & +\mathrm{Dn}^{\{9\}} - \mathrm{Dn}^{\{8,1\}} - 9\,\mathrm{Dn}^{\{7,2\}} - 36\,\mathrm{Dn}^{\{6,3\}} - 84\,\mathrm{Dn}^{\{5,4\}} + \mathrm{Dn}^{\{7,1^2\}} \\
& +17\,\mathrm{Dn}^{\{6,2,1\}} + 63\,\mathrm{Dn}^{\{5,2^2\}} + 64\,\mathrm{Dn}^{\{5,3,1\}} + 49\,\mathrm{Dn}^{\{4^2,1\}} + 195\,\mathrm{Dn}^{\{4,3,2\}} \\
& -45\,\mathrm{Dn}^{\{3^3\}} - \mathrm{Dn}^{\{6,1^3\}} - 24\,\mathrm{Dn}^{\{5,2,1^2\}} - 43\,\mathrm{Dn}^{\{4,3,1^2\}} - 81\,\mathrm{Dn}^{\{4,2^2,1\}} \\
& +110\,\mathrm{Dn}^{\{3^2,2,1\}} - 435\,\mathrm{Dn}^{\{3,2^3\}} + \mathrm{Dn}^{\{5,1^4\}} - 12\,\mathrm{Dn}^{\{4,2,1^3\}} + 156\,\mathrm{Dn}^{\{3,2^2,1^2\}} \\
& +276\,\mathrm{Dn}^{\{2^4,1\}} - 16\,\mathrm{Dn}^{\{3^2,1^3\}} - \mathrm{Dn}^{\{4,1^5\}} - 218\,\mathrm{Dn}^{\{3,2,1^4\}} - 356\,\mathrm{Dn}^{\{2^3,1^3\}} \\
& +77/2\,\mathrm{Dn}^{\{3,1^6\}} + 231\,\mathrm{Dn}^{\{2^2,1^5\}} - 77/2\,\mathrm{Dn}^{\{2,1^7\}}
\end{aligned}
$$

$$
\begin{aligned}
\mathrm{Da}^{\{10\}} = {} & +\mathrm{Dn}^{\{10\}} - \mathrm{Dn}^{\{9,1\}} - 10\,\mathrm{Dn}^{\{8,2\}} - 45\,\mathrm{Dn}^{\{7,3\}} - 120\,\mathrm{Dn}^{\{6,4\}} - 105\,\mathrm{Dn}^{\{5^2\}} \\
& +\mathrm{Dn}^{\{8,1^2\}} + 19\,\mathrm{Dn}^{\{7,2,1\}} + 80\,\mathrm{Dn}^{\{6,2^2\}} + 595\,\mathrm{Dn}^{\{5,3,2\}} + 204\,\mathrm{Dn}^{\{5,4,1\}} \\
& +244\,\mathrm{Dn}^{\{4^2,2\}} + 81\,\mathrm{Dn}^{\{6,3,1\}} - 45\,\mathrm{Dn}^{\{4,3^2\}} - \mathrm{Dn}^{\{7,1^3\}} - 27\,\mathrm{Dn}^{\{6,2,1^2\}} \\
& -109\,\mathrm{Dn}^{\{5,3,1^2\}} - 213\,\mathrm{Dn}^{\{5,2^2,1\}} - 64\,\mathrm{Dn}^{\{4^2,1^2\}} - 28\,\mathrm{Dn}^{\{4,3,2,1\}} - 656\,\mathrm{Dn}^{\{4,2^3\}} \\
& +215\,\mathrm{Dn}^{\{3^3,1\}} - 1510\,\mathrm{Dn}^{\{3^2,2^2\}} + 34\,\mathrm{Dn}^{\{5,2,1^3\}} + \mathrm{Dn}^{\{6,1^4\}} - 2\,\mathrm{Dn}^{\{4,3,1^3\}} \\
& +309\,\mathrm{Dn}^{\{4,2^2,1^2\}} + 159\,\mathrm{Dn}^{\{3^2,2,1^2\}} + 1556\,\mathrm{Dn}^{\{3,2^3,1\}} - 144\,\mathrm{Dn}^{\{2^5\}} - \mathrm{Dn}^{\{5,1^5\}} \\
& +248\,\mathrm{Dn}^{\{2^4,1^2\}} - 370\,\mathrm{Dn}^{\{4,2,1^4\}} - 851/4\,\mathrm{Dn}^{\{3^2,1^4\}} - 2129\,\mathrm{Dn}^{\{3,2^2,1^3\}} \\
& +56\,\mathrm{Dn}^{\{4,1^6\}} - 161\,\mathrm{Dn}^{\{2^3,1^4\}} + 1575/2\,\mathrm{Dn}^{\{3,2,1^5\}} - 56 * \mathrm{Dn}^{\{3,1^7\}} \\
& +1673/4\,\mathrm{Dn}^{\{2^2,1^6\}} - 175\,\mathrm{Dn}^{\{2,1^8\}} + 35/2\,\mathrm{Dn}^{\{1^{10}\}}
\end{aligned}
$$

14.2 The involution $D \mapsto \widetilde{D}$ in the three bases of ISO.

Table 3: $\widetilde{\mathrm{Dn}}^{\{\bullet\}}$ in terms of $\mathrm{Dn}^{\{\bullet\}}$.

$$
\begin{aligned}
\widetilde{\mathrm{Dn}}^{\{1\}} &= -\mathrm{Dn}^{\{1\}}\\
\widetilde{\mathrm{Dn}}^{\{2\}} &= -\mathrm{Dn}^{\{2\}} + \mathrm{Dn}^{\{1^2\}}\\
\widetilde{\mathrm{Dn}}^{\{3\}} &= -\mathrm{Dn}^{\{3\}} + 4\mathrm{Dn}^{\{1,2\}} - 2\mathrm{Dn}^{\{1^3\}}\\
\widetilde{\mathrm{Dn}}^{\{4\}} &= -\mathrm{Dn}^{\{4\}} + 7\mathrm{Dn}^{\{1,3\}} + 4\mathrm{Dn}^{\{2^2\}} - 18\mathrm{Dn}^{\{1^2,2\}} + 6\mathrm{Dn}^{\{1^4\}}\\
\widetilde{\mathrm{Dn}}^{\{5\}} &= -\mathrm{Dn}^{\{5\}} + 11\mathrm{Dn}^{\{1,4\}} + 15\mathrm{Dn}^{\{2,3\}} - 46\mathrm{Dn}^{\{1^2,3\}} - 52\mathrm{Dn}^{\{1,2^2\}}\\
&\quad +96\mathrm{Dn}^{\{1^3,2\}} - 24\mathrm{Dn}^{\{1^5\}}\\
\widetilde{\mathrm{Dn}}^{\{6\}} &= -\mathrm{Dn}^{\{6\}} + 16\mathrm{Dn}^{\{1,5\}} + 26\mathrm{Dn}^{\{2,4\}} + 15\mathrm{Dn}^{\{3^2\}} - 101\mathrm{Dn}^{\{1^2,4\}} - 271\mathrm{Dn}^{\{1,2,3\}}\\
&\quad -52\mathrm{Dn}^{\{2^3\}} + 326\mathrm{Dn}^{\{1^3,3\}} + 548\mathrm{Dn}^{\{1^2,2^2\}} - 600\mathrm{Dn}^{\{1^4,2\}} + 120\mathrm{Dn}^{\{1^6\}}\\
\widetilde{\mathrm{Dn}}^{\{7\}} &= -\mathrm{Dn}^{\{7\}} + 22\mathrm{Dn}^{\{1,6\}} + 42\mathrm{Dn}^{\{2,5\}} + 56\mathrm{Dn}^{\{3,4\}} - 197\mathrm{Dn}^{\{1^2,5\}} - 629\mathrm{Dn}^{\{1,2,4\}}\\
&\quad -361\mathrm{Dn}^{\{1,3^2\}} - 427\mathrm{Dn}^{\{2^2,3\}} + 932\mathrm{Dn}^{\{1^3,4\}} + 3700\mathrm{Dn}^{\{1^2,2,3\}} + 1408\mathrm{Dn}^{\{1,2^3\}}\\
&\quad -2556\mathrm{Dn}^{\{1^4,3\}} - 5688\mathrm{Dn}^{\{1^3,2^2\}} + 4320\mathrm{Dn}^{\{1^5,2\}} - 720\mathrm{Dn}^{\{1^7\}}\\
\widetilde{\mathrm{Dn}}^{\{8\}} &= -\mathrm{Dn}^{\{8\}} + 29\mathrm{Dn}^{\{1,7\}} + 64\mathrm{Dn}^{\{2,6\}} + 98\mathrm{Dn}^{\{3,5\}} + 56\mathrm{Dn}^{\{4^2\}} - 351\mathrm{Dn}^{\{1^2,6\}}\\
&\quad -1317\mathrm{Dn}^{\{1,2,5\}} - 1743\mathrm{Dn}^{\{1,3,4\}} - 1056\mathrm{Dn}^{\{2^2,4\}} - 1215\mathrm{Dn}^{\{2,3^2\}}\\
&\quad +2311\mathrm{Dn}^{\{1^3,5\}} + 6227\mathrm{Dn}^{\{1^2,3^2\}} + 14613\mathrm{Dn}^{\{1,2^2,3\}} + 10899\mathrm{Dn}^{\{1^2,2,4\}}\\
&\quad +1408\mathrm{Dn}^{\{2^4\}} - 9080\mathrm{Dn}^{\{1^4,4\}} - 47500\mathrm{Dn}^{\{1^3,2,3\}} - 26920\mathrm{Dn}^{\{1^2,2^3\}}\\
&\quad +22212\mathrm{Dn}^{\{1^5,3\}} + 61416\mathrm{Dn}^{\{1^4,2,2\}} - 35280\mathrm{Dn}^{\{1^6,2\}} + 5040\mathrm{Dn}^{\{1^8\}}
\end{aligned}
$$

Table 4: $\widetilde{\mathrm{Da}}^{\{\bullet\}}$ in terms of $\mathrm{Da}^{\{\bullet\}}$.

$$
\begin{aligned}
\widetilde{\mathrm{Da}}^{\{1\}} &= -\mathrm{Da}^{\{1\}}\\
\widetilde{\mathrm{Da}}^{\{2\}} &= -\mathrm{Da}^{\{2\}}\\
\widetilde{\mathrm{Da}}^{\{3\}} &= -\mathrm{Da}^{\{3\}} + 2\,\mathrm{Da}^{\{2,1\}}\\
\widetilde{\mathrm{Da}}^{\{4\}} &= -\mathrm{Da}^{\{4\}} + 5\,\mathrm{Da}^{\{3,1\}} - 5\,\mathrm{Da}^{\{2,1^2\}}\\
\widetilde{\mathrm{Da}}^{\{5\}} &= -\mathrm{Da}^{\{5\}} + 9\,\mathrm{Da}^{\{4,1\}} + 5\,\mathrm{Da}^{\{3,2\}} - 45/2\,\mathrm{Da}^{\{3,1^2\}} - 5\,\mathrm{Da}^{\{2^2,1\}} + 15\,\mathrm{Da}^{\{2,1^3\}}\\
\widetilde{\mathrm{Da}}^{\{6\}} &= -\mathrm{Da}^{\{6\}} + 14\,\mathrm{Da}^{\{5,1\}} + 14\,\mathrm{Da}^{\{4,2\}} - 63\,\mathrm{Da}^{\{4,1^2\}} - 70\,\mathrm{Da}^{\{3,2,1\}} + 105\,\mathrm{Da}^{\{3,1^3\}}\\
&\quad +35\,\mathrm{Da}^{\{2^2,1^2\}} - 105/2\,\mathrm{Da}^{\{2,1^4\}}\\
\widetilde{\mathrm{Da}}^{\{7\}} &= -\mathrm{Da}^{\{7\}} + 20\,\mathrm{Da}^{\{6,1\}} + 28\,\mathrm{Da}^{\{5,2\}} + 14\,\mathrm{Da}^{\{4,3\}} - 140\,\mathrm{Da}^{\{5,1^2\}} - 280\,\mathrm{Da}^{\{4,2,1\}}\\
&\quad -35\,\mathrm{Da}^{\{3^2,1\}} - 70\,\mathrm{Da}^{\{3,2^2\}} + 420\,\mathrm{Da}^{\{4,1^3\}} + 770\,\mathrm{Da}^{\{3,2,1^2\}} - 525\,\mathrm{Da}^{\{3,1^4\}}\\
&\quad +140/3\,\mathrm{Da}^{\{2^3,1\}} - 280\,\mathrm{Da}^{\{2^2,1^3\}} + 210\,\mathrm{Da}^{\{2,1^5\}}
\end{aligned}
$$

$$
\begin{aligned}
\widetilde{\mathrm{Da}}^{\{8\}} = {} & -\mathrm{Da}^{\{8\}} + 27\,\mathrm{Da}^{\{7,1\}} + 48\,\mathrm{Da}^{\{6,2\}} + 42\,\mathrm{Da}^{\{5,3\}} - 270\,\mathrm{Da}^{\{6,1^{\mathbf{2}}\}} \\
& -756\,\mathrm{Da}^{\{5,2,1\}} - 378\,\mathrm{Da}^{\{4,3,1\}} - 336\,\mathrm{Da}^{\{4,2^{\mathbf{2}}\}} - 105\,\mathrm{Da}^{\{3^{\mathbf{2}},2\}} \\
& +1260\,\mathrm{Da}^{\{5,1^{\mathbf{3}}\}} + 3780\,\mathrm{Da}^{\{4,2,1^{\mathbf{2}}\}} + 945/2\,\mathrm{Da}^{\{3^{\mathbf{2}},1^{\mathbf{2}}\}} + 2100\,\mathrm{Da}^{\{3,2^{\mathbf{2}},1\}} \\
& -2835\,\mathrm{Da}^{\{4,1^{\mathbf{4}}\}} - 6930\,\mathrm{Da}^{\{3,2,1^{\mathbf{3}}\}} - 840\,\mathrm{Da}^{\{2^{\mathbf{3}},1^{\mathbf{2}}\}} + 2835\,\mathrm{Da}^{\{3,1^{\mathbf{5}}\}} \\
& +1890\,\mathrm{Da}^{\{2^{\mathbf{2}},1^{\mathbf{4}}\}} - 945\,\mathrm{Da}^{\{2,1^{\mathbf{6}}\}}
\end{aligned}
$$

Table 4 bis: $\widetilde{\underline{\mathrm{Da}}}^{\{\bullet\}}$ in terms of $\underline{\mathrm{Da}}^{\{\bullet\}}$.

$$
\begin{aligned}
\widetilde{\underline{\mathrm{Da}}}^{\{1\}} &= -\underline{\mathrm{Da}}^{\{1\}} \\
\widetilde{\underline{\mathrm{Da}}}^{\{2\}} &= -\underline{\mathrm{Da}}^{\{2\}} \\
\widetilde{\underline{\mathrm{Da}}}^{\{3\}} &= -\underline{\mathrm{Da}}^{\{3\}} + \underline{\mathrm{Da}}^{\{2,1\}} \\
\widetilde{\underline{\mathrm{Da}}}^{\{4\}} &= -\underline{\mathrm{Da}}^{\{4\}} + 2\,\underline{\mathrm{Da}}^{\{3,1\}} - \underline{\mathrm{Da}}^{\{2,1^{\mathbf{2}}\}} \\
\widetilde{\underline{\mathrm{Da}}}^{\{5\}} &= -\underline{\mathrm{Da}}^{\{5\}} + 3\,\underline{\mathrm{Da}}^{\{4,1\}} + \underline{\mathrm{Da}}^{\{3,2\}} - 3\,\underline{\mathrm{Da}}^{\{3,1^{\mathbf{2}}\}} - 1/2\,\underline{\mathrm{Da}}^{\{2^{\mathbf{2}},1\}} + \underline{\mathrm{Da}}^{\{2,1^{\mathbf{3}}\}} \\
\widetilde{\underline{\mathrm{Da}}}^{\{6\}} &= -\underline{\mathrm{Da}}^{\{6\}} + 4\,\underline{\mathrm{Da}}^{\{5,1\}} + 2\,\underline{\mathrm{Da}}^{\{4,2\}} - 6\,\underline{\mathrm{Da}}^{\{4,1^{\mathbf{2}}\}} - 4\,\underline{\mathrm{Da}}^{\{3,2,1\}} + 4\,\underline{\mathrm{Da}}^{\{3,1^{\mathbf{3}}\}} \\
&\quad +\underline{\mathrm{Da}}^{\{2^2,1^{\mathbf{2}}\}} - \underline{\mathrm{Da}}^{\{2,1^{\mathbf{4}}\}}
\end{aligned}
$$

$$
\begin{aligned}
\widetilde{\underline{\mathrm{Da}}}^{\{7\}} = {} & -\underline{\mathrm{Da}}^{\{7\}} + 5\,\underline{\mathrm{Da}}^{\{6,1\}} + 3\,\underline{\mathrm{Da}}^{\{5,2\}} + \underline{\mathrm{Da}}^{\{4,3\}} - 10\,\underline{\mathrm{Da}}^{\{5,1^{\mathbf{2}}\}} - 10\,\underline{\mathrm{Da}}^{\{4,2,1\}} \\
& -\underline{\mathrm{Da}}^{\{3^{\mathbf{2}},1\}} - 3/2\,\underline{\mathrm{Da}}^{\{3,2^{\mathbf{2}}\}} + 10\,\underline{\mathrm{Da}}^{\{4,1^{\mathbf{3}}\}} + 11\,\underline{\mathrm{Da}}^{\{3,2,1^{\mathbf{2}}\}} + 1/2\,\underline{\mathrm{Da}}^{\{2^{\mathbf{3}},1\}} \\
& -5\,\underline{\mathrm{Da}}^{\{3,1^{\mathbf{4}}\}} - 2\,\underline{\mathrm{Da}}^{\{2^{\mathbf{2}},1^{\mathbf{3}}\}} + \underline{\mathrm{Da}}^{\{2,1^{\mathbf{5}}\}}
\end{aligned}
$$

$$
\begin{aligned}
\widetilde{\underline{\mathrm{Da}}}^{\{8\}} = {} & -\underline{\mathrm{Da}}^{\{8\}} + 6\,\underline{\mathrm{Da}}^{\{7,1\}} + 4\,\underline{\mathrm{Da}}^{\{6,2\}} + 2\,\underline{\mathrm{Da}}^{\{5,3\}} - 15\,\underline{\mathrm{Da}}^{\{6,1^{\mathbf{2}}\}} - 18\,\underline{\mathrm{Da}}^{\{5,2,1\}} \\
& -6\,\underline{\mathrm{Da}}^{\{4,3,1\}} - 4\,\underline{\mathrm{Da}}^{\{4,2^{\mathbf{2}}\}} - \underline{\mathrm{Da}}^{\{3^{\mathbf{2}},2\}} + 20\,\underline{\mathrm{Da}}^{\{5,1^{\mathbf{3}}\}} + 30\,\underline{\mathrm{Da}}^{\{4,2,1^{\mathbf{2}}\}} \\
& +3\,\underline{\mathrm{Da}}^{\{3^{\mathbf{2}},1^{\mathbf{2}}\}} + 10\,\underline{\mathrm{Da}}^{\{3,2^{\mathbf{2}},1\}} - 15\,\underline{\mathrm{Da}}^{\{4,1^{\mathbf{4}}\}} - 22\,\underline{\mathrm{Da}}^{\{3,2,1^{\mathbf{3}}\}} - 2\,\underline{\mathrm{Da}}^{\{2^{\mathbf{3}},1^{\mathbf{2}}\}} \\
& +6\,\underline{\mathrm{Da}}^{\{3,1^{\mathbf{5}}\}} + 3\,\underline{\mathrm{Da}}^{\{2^{\mathbf{2}},1^{\mathbf{4}}\}} - \underline{\mathrm{Da}}^{\{2,1^{\mathbf{6}}\}}
\end{aligned}
$$

14.3 The co-product $D \mapsto \sigma(D)$ in the three bases of ISO.

Table 5: $\sigma(\mathrm{Dn}^{\{\bullet\}})$ in terms of $\mathrm{Dn}^{\{\bullet\}}$.

$$
\begin{aligned}
\mathrm{Dn}^{\{1\}} &\to +\mathbf{1} \otimes \mathrm{Dn}^{\{1\}} + \mathrm{Dn}^{\{1\}} \otimes \mathbf{1} \\
\mathrm{Dn}^{\{2\}} &\to +\mathbf{1} \otimes \mathrm{Dn}^{\{2\}} + \mathrm{Dn}^{\{1\}} \otimes \mathrm{Dn}^{\{1\}} + \mathrm{Dn}^{\{2\}} \otimes \mathbf{1} \\
\mathrm{Dn}^{\{3\}} &\to +\mathbf{1}\otimes\mathrm{Dn}^{\{3\}} + 3\mathrm{Dn}^{\{1\}}\otimes\mathrm{Dn}^{\{2\}} + \mathrm{Dn}^{\{2\}}\otimes\mathrm{Dn}^{\{1\}} + \mathrm{Dn}^{\{1^{\mathbf{2}}\}}\otimes\mathrm{Dn}^{\{1\}} + \mathrm{Dn}^{\{3\}}\otimes\mathbf{1} \\
\mathrm{Dn}^{\{4\}} &\to +\mathbf{1} \otimes \mathrm{Dn}^{\{4\}} + 6\mathrm{Dn}^{\{1\}} \otimes \mathrm{Dn}^{\{3\}} + 4\mathrm{Dn}^{\{2\}} \otimes \mathrm{Dn}^{\{2\}} + 7\mathrm{Dn}^{\{1^{\mathbf{2}}\}} \otimes \mathrm{Dn}^{\{2\}} \\
&\quad +\mathrm{Dn}^{\{3\}} \otimes \mathrm{Dn}^{\{1\}} + 3\mathrm{Dn}^{\{1,2\}} \otimes \mathrm{Dn}^{\{1\}} + \mathrm{Dn}^{\{1^{\mathbf{3}}\}} \otimes \mathrm{Dn}^{\{1\}} + \mathrm{Dn}^{\{4\}} \otimes \mathbf{1}
\end{aligned}
$$

$$
\begin{aligned}
\mathrm{Dn}^{\{5\}} \to\ & +\mathbf{1}\otimes \mathrm{Dn}^{\{5\}} + 10\mathrm{Dn}^{\{1\}}\otimes \mathrm{Dn}^{\{4\}} + 10\mathrm{Dn}^{\{2\}}\otimes \mathrm{Dn}^{\{3\}} + 25\mathrm{Dn}^{\{1^{\mathbf{2}}\}}\otimes \mathrm{Dn}^{\{3\}}\\
& +5\mathrm{Dn}^{\{3\}}\otimes \mathrm{Dn}^{\{2\}} + 25\mathrm{Dn}^{\{1,2\}}\otimes \mathrm{Dn}^{\{2\}} + 15\mathrm{Dn}^{\{1^{\mathbf{3}}\}}\otimes \mathrm{Dn}^{\{2\}}\\
& +\mathrm{Dn}^{\{4\}}\otimes \mathrm{Dn}^{\{1\}} + 4\mathrm{Dn}^{\{1,3\}}\otimes \mathrm{Dn}^{\{1\}} + 3\mathrm{Dn}^{\{2^{\mathbf{2}}\}}\otimes \mathrm{Dn}^{\{1\}}\\
& +6\mathrm{Dn}^{\{1^{\mathbf{2}},2\}}\otimes \mathrm{Dn}^{\{1\}} + \mathrm{Dn}^{\{1^{\mathbf{4}}\}}\otimes \mathrm{Dn}^{\{1\}} + \mathrm{Dn}^{\{5\}}\otimes \mathbf{1}
\end{aligned}
$$

$$
\begin{aligned}
\mathrm{Dn}^{\{6\}} \to\ & +\mathbf{1}\otimes \mathrm{Dn}^{\{6\}} + 15\mathrm{Dn}^{\{1\}}\otimes \mathrm{Dn}^{\{5\}} + 20\mathrm{Dn}^{\{2\}}\otimes \mathrm{Dn}^{\{4\}} + 65\mathrm{Dn}^{\{1^{\mathbf{2}}\}}\otimes \mathrm{Dn}^{\{4\}}\\
& +15\mathrm{Dn}^{\{3\}}\otimes \mathrm{Dn}^{\{3\}} + 105\mathrm{Dn}^{\{1,2\}}\otimes \mathrm{Dn}^{\{3\}} + 90\mathrm{Dn}^{\{1^{\mathbf{3}}\}}\otimes \mathrm{Dn}^{\{3\}}\\
& +6\mathrm{Dn}^{\{4\}}\otimes \mathrm{Dn}^{\{2\}} + 28\mathrm{Dn}^{\{2^{\mathbf{2}}\}}\otimes \mathrm{Dn}^{\{2\}} + 39\mathrm{Dn}^{\{1,3\}}\otimes \mathrm{Dn}^{\{2\}}\\
& +101\mathrm{Dn}^{\{1^{\mathbf{2}},2\}}\otimes \mathrm{Dn}^{\{2\}} + 31\mathrm{Dn}^{\{1^{\mathbf{4}}\}}\otimes \mathrm{Dn}^{\{2\}} + \mathrm{Dn}^{\{5\}}\otimes \mathrm{Dn}^{\{1\}}\\
& +5\mathrm{Dn}^{\{1,4\}}\otimes \mathrm{Dn}^{\{1\}} + 10\mathrm{Dn}^{\{2,3\}}\otimes \mathrm{Dn}^{\{1\}} + 10\mathrm{Dn}^{\{1^{\mathbf{2}},3\}}\otimes \mathrm{Dn}^{\{1\}}\\
& +15\mathrm{Dn}^{\{1,2^{\mathbf{2}}\}}\otimes \mathrm{Dn}^{\{1\}} + 10\mathrm{Dn}^{\{1^{\mathbf{3}},2\}}\otimes \mathrm{Dn}^{\{1\}} + \mathrm{Dn}^{\{1^{\mathbf{5}}\}}\otimes \mathrm{Dn}^{\{1\}} + \mathrm{Dn}^{\{6\}}\otimes \mathbf{1}
\end{aligned}
$$

$$
\begin{aligned}
\mathrm{Dn}^{\{7\}} \to\ & +\mathbf{1}\otimes \mathrm{Dn}^{\{7\}} + 21\mathrm{Dn}^{\{1\}}\otimes \mathrm{Dn}^{\{6\}} + 35\mathrm{Dn}^{\{2\}}\otimes \mathrm{Dn}^{\{5\}} + 140\mathrm{Dn}^{\{1^{\mathbf{2}}\}}\otimes \mathrm{Dn}^{\{5\}}\\
& +35\mathrm{Dn}^{\{3\}}\otimes \mathrm{Dn}^{\{4\}} + 315\mathrm{Dn}^{\{1,2\}}\otimes \mathrm{Dn}^{\{4\}} + 350\mathrm{Dn}^{\{1^{\mathbf{3}}\}}\otimes \mathrm{Dn}^{\{4\}}\\
& +21\mathrm{Dn}^{\{4\}}\otimes \mathrm{Dn}^{\{3\}} + 189\mathrm{Dn}^{\{1,3\}}\otimes \mathrm{Dn}^{\{3\}} + 133\mathrm{Dn}^{\{2^{\mathbf{2}}\}}\otimes \mathrm{Dn}^{\{3\}}\\
& +686\mathrm{Dn}^{\{1^{\mathbf{2}},2\}}\otimes \mathrm{Dn}^{\{3\}} + 301\mathrm{Dn}^{\{1^{\mathbf{4}}\}}\otimes \mathrm{Dn}^{\{3\}} + 7\mathrm{Dn}^{\{5\}}\otimes \mathrm{Dn}^{\{2\}}\\
& +56\mathrm{Dn}^{\{1,4\}}\otimes \mathrm{Dn}^{\{2\}} + 105\mathrm{Dn}^{\{2,3\}}\otimes \mathrm{Dn}^{\{2\}} + 273\mathrm{Dn}^{\{1,2^{\mathbf{2}}\}}\otimes \mathrm{Dn}^{\{2\}}\\
& +189\mathrm{Dn}^{\{1^{\mathbf{2}},3\}}\otimes \mathrm{Dn}^{\{2\}} + 336\mathrm{Dn}^{\{1^{\mathbf{3}},2\}}\otimes \mathrm{Dn}^{\{2\}} + 63\mathrm{Dn}^{\{1^{\mathbf{5}}\}}\otimes \mathrm{Dn}^{\{2\}}\\
& +\mathrm{Dn}^{\{6\}}\otimes \mathrm{Dn}^{\{1\}} + 6\mathrm{Dn}^{\{1,5\}}\otimes \mathrm{Dn}^{\{1\}} + 15\mathrm{Dn}^{\{2,4\}}\otimes \mathrm{Dn}^{\{1\}}\\
& +10\mathrm{Dn}^{\{3^{\mathbf{2}}\}}\otimes \mathrm{Dn}^{\{1\}} + 15\mathrm{Dn}^{\{1^{\mathbf{2}},4\}}\otimes \mathrm{Dn}^{\{1\}} + 60\mathrm{Dn}^{\{1,2,3\}}\otimes \mathrm{Dn}^{\{1\}}\\
& +15\mathrm{Dn}^{\{2^{\mathbf{3}}\}}\otimes \mathrm{Dn}^{\{1\}} + 20\mathrm{Dn}^{\{1^{\mathbf{3}},3\}}\otimes \mathrm{Dn}^{\{1\}} + 45\mathrm{Dn}^{\{1^{\mathbf{2}},2^{\mathbf{2}}\}}\otimes \mathrm{Dn}^{\{1\}}\\
& +15\mathrm{Dn}^{\{1^{\mathbf{4}},2\}}\otimes \mathrm{Dn}^{\{1\}} + \mathrm{Dn}^{\{1^{\mathbf{6}}\}}\otimes \mathrm{Dn}^{\{1\}} + \mathrm{Dn}^{\{7\}}\otimes \mathbf{1}
\end{aligned}
$$

Table 6: $\sigma(\mathrm{Ds}^{\{\bullet\}})$ **in terms of** $\mathrm{Ds}^{\{\bullet\}}$**.**

$$
\begin{aligned}
\mathrm{Ds}^{\{1\}} &\to (\mathbf{1}\otimes \mathrm{Ds}^{\{1\}} + \mathrm{Ds}^{\{1\}}\otimes \mathbf{1})\\
\mathrm{Ds}^{\{2\}} &\to (\mathbf{1}\otimes \mathrm{Ds}^{\{2\}} + \mathrm{Ds}^{\{2\}}\otimes \mathbf{1})\\
\mathrm{Ds}^{\{3\}} &\to (\mathbf{1}\otimes \mathrm{Ds}^{\{3\}} + \mathrm{Ds}^{\{3\}}\otimes \mathbf{1}) + (\mathrm{Ds}^{\{1\}}\otimes \mathrm{Ds}^{\{2\}} - \mathrm{Ds}^{\{2\}}\otimes D_*^{\{1\}})\\
\mathrm{Ds}^{\{4\}} &\to (\mathbf{1}\otimes \mathrm{Ds}^{\{4\}} + \mathrm{Ds}^{\{4\}}\otimes \mathbf{1}) + \frac{5}{2}(\mathrm{Ds}^{\{1\}}\otimes \mathrm{Ds}^{\{3\}} - \mathrm{Ds}^{\{3\}}\otimes \mathrm{Ds}^{\{1\}})\\
&\quad -\frac{1}{2}(\mathrm{Ds}^{\{1\}}\otimes\mathrm{Ds}^{\{1,2\}} + \mathrm{Ds}^{\{1,2\}}\otimes\mathrm{Ds}^{\{1\}}) + (\mathrm{Ds}^{\{2\}}\otimes\mathrm{Ds}^{\{1^{\mathbf{2}}\}} + \mathrm{Ds}^{\{1^{\mathbf{2}}\}}\otimes\mathrm{Ds}^{\{2\}})
\end{aligned}
$$

$$
\begin{aligned}
\mathrm{Ds}^{\{5\}} \to\ & (\mathbf{1}\otimes\mathrm{Ds}^{\{5\}} + \mathrm{Ds}^{\{5\}}\otimes\mathbf{1}) + \frac{9}{2}\,(\mathrm{Ds}^{\{1\}}\otimes\mathrm{Ds}^{\{4\}} - \mathrm{Ds}^{\{4\}}\otimes\mathrm{Ds}^{\{1\}}) \\
& + \frac{5}{2}\,(\mathrm{Ds}^{\{2\}}\otimes\mathrm{Ds}^{\{3\}} - \mathrm{Ds}^{\{3\}}\otimes\mathrm{Ds}^{\{2\}}) - \frac{7}{4}\,(\mathrm{Ds}^{\{1\}}\otimes D_*^{\{1,3\}} + \mathrm{Ds}^{\{1,3\}}\otimes\mathrm{Ds}^{\{1\}}) \\
& - \frac{1}{2}\,(\mathrm{Ds}^{\{1\}}\otimes\mathrm{Ds}^{\{2^{\mathbf{2}}\}} + \mathrm{Ds}^{\{2^{\mathbf{2}}\}}\otimes\mathrm{Ds}^{\{1\}}) + \frac{3}{2}\,(\mathrm{Ds}^{\{2\}}\otimes\mathrm{Ds}^{\{1,2\}} + \mathrm{Ds}^{\{1,2\}}\otimes\mathrm{Ds}^{\{2\}}) \\
& + \frac{19}{4}\,(\mathrm{Ds}^{\{3\}}\otimes\mathrm{Ds}^{\{1^{\mathbf{2}}\}} + \mathrm{Ds}^{\{1^{\mathbf{2}}\}}\otimes\mathrm{Ds}^{\{3\}}) + \frac{1}{4}\,(\mathrm{Ds}^{\{1\}}\otimes\mathrm{Ds}^{\{1^{\mathbf{2}},2\}} - \mathrm{Ds}^{\{1^{\mathbf{2}},2\}}\otimes\mathrm{Ds}^{\{1\}}) \\
& - (\mathrm{Ds}^{\{2\}}\otimes\mathrm{Ds}^{\{1^{\mathbf{3}}\}} - \mathrm{Ds}^{\{1^{\mathbf{3}}\}}\otimes\mathrm{Ds}^{\{2\}}) + \frac{7}{4}\,(\mathrm{Ds}^{\{1,2\}}\otimes\mathrm{Ds}^{\{1^{\mathbf{2}}\}} - \mathrm{Ds}^{\{1^{\mathbf{2}}\}}\otimes\mathrm{Ds}^{\{1,2\}})
\end{aligned}
$$

$$
\begin{aligned}
\mathrm{Ds}^{\{6\}} \to\ & (\mathbf{1}\otimes\mathrm{Ds}^{\{6\}} + \mathrm{Ds}^{\{6\}}\otimes\mathbf{1}) + 7\,(\mathrm{Ds}^{\{1\}}\otimes\mathrm{Ds}^{\{5\}} - D_*^{\{5\}}\otimes\mathrm{Ds}^{\{5\}}) \\
& + 7\,(\mathrm{Ds}^{\{2\}}\otimes\mathrm{Ds}^{\{4\}} - \mathrm{Ds}^{\{4\}}\otimes\mathrm{Ds}^{\{2\}}) - 4\,(\mathrm{Ds}^{\{1\}}\otimes\mathrm{Ds}^{\{1,4\}} + \mathrm{Ds}^{\{1,4\}}\otimes\mathrm{Ds}^{\{1\}}) \\
& - \frac{11}{4}\,(\mathrm{Ds}^{\{1\}}\otimes\mathrm{Ds}^{\{2,3\}} + \mathrm{Ds}^{\{2,3\}}\otimes\mathrm{Ds}^{\{1\}}) - \frac{11}{4}\,(\mathrm{Ds}^{\{2\}}\otimes\mathrm{Ds}^{\{1,3\}} + \mathrm{Ds}^{\{1,3\}}\otimes\mathrm{Ds}^{\{2\}}) \\
& + (\mathrm{Ds}^{\{2\}}\otimes\mathrm{Ds}^{\{2^{\mathbf{2}}\}} + \mathrm{Ds}^{\{2^{\mathbf{2}}\}}\otimes\mathrm{Ds}^{\{2\}}) + \frac{59}{4}\,(\mathrm{Ds}^{\{3\}}\otimes\mathrm{Ds}^{\{1,2\}} + \mathrm{Ds}^{\{1,2\}}\otimes\mathrm{Ds}^{\{3\}}) \\
& + \frac{55}{4}\,(\mathrm{Ds}^{\{4\}}\otimes\mathrm{Ds}^{\{1^{\mathbf{2}}\}} + \mathrm{Ds}^{\{1^{\mathbf{2}}\}}\otimes\mathrm{Ds}^{\{4\}}) + \frac{9}{8}\,(\mathrm{Ds}^{\{1\}}\otimes\mathrm{Ds}^{\{1^{\mathbf{2}},3\}} - \mathrm{Ds}^{\{1^{\mathbf{2}},3\}}\otimes\mathrm{Ds}^{\{1\}}) \\
& + \frac{3}{4}\,(\mathrm{Ds}^{\{1\}}\otimes\mathrm{Ds}^{\{1,2^{\mathbf{2}}\}} - \mathrm{Ds}^{\{1,2^{\mathbf{2}}\}}\otimes\mathrm{Ds}^{\{1\}}) - \frac{17}{4}\,(\mathrm{Ds}^{\{2\}}\otimes\mathrm{Ds}^{\{1^{\mathbf{2}},2\}} - \mathrm{Ds}^{\{1^{\mathbf{2}},2\}}\otimes\mathrm{Ds}^{\{2\}}) \\
& - \frac{65}{8}\,(\mathrm{Ds}^{\{3\}}\otimes\mathrm{Ds}^{\{1^{\mathbf{3}}\}} - \mathrm{Ds}^{\{1^{\mathbf{3}}\}}\otimes\mathrm{Ds}^{\{3\}}) - 10\,(\mathrm{Ds}^{\{1^{\mathbf{2}}\}}\otimes\mathrm{Ds}^{\{1,3\}} - \mathrm{Ds}^{\{1,3\}}\otimes\mathrm{Ds}^{\{1^{\mathbf{2}}\}}) \\
& - \frac{11}{4}\,(\mathrm{Ds}^{\{1^{\mathbf{2}}\}}\otimes\mathrm{Ds}^{\{2^{\mathbf{2}}\}} - \mathrm{Ds}^{\{2^{\mathbf{2}}\}}\otimes\mathrm{Ds}^{\{1^{\mathbf{2}}\}}) - \frac{1}{8}\,(\mathrm{Ds}^{\{1\}}\otimes\mathrm{Ds}^{\{1^{\mathbf{3}},2\}} + D_*^{\{1^{\mathbf{3}},2\}}\otimes\mathrm{Ds}^{\{1\}}) \\
& + (D_*^{\{2\}}\otimes\mathrm{Ds}^{\{1^{\mathbf{4}}\}} + \mathrm{Ds}^{\{1^{\mathbf{4}}\}}\otimes D_*^{\{2\}}) + \frac{9}{4}\,(\mathrm{Ds}^{\{1^{\mathbf{2}}\}}\otimes\mathrm{Ds}^{\{1^{\mathbf{2}},2\}} + \mathrm{Ds}^{\{1^{\mathbf{2}},2\}}\otimes\mathrm{Ds}^{\{1^{\mathbf{2}}\}}) \\
& - \frac{33}{8}\,(\mathrm{Ds}^{\{1,2\}}\otimes\mathrm{Ds}^{\{1^{\mathbf{3}}\}} + \mathrm{Ds}^{\{1^{\mathbf{3}}\}}\otimes\mathrm{Ds}^{\{1,2\}})
\end{aligned}
$$

Table 7: $\sigma(\mathrm{Da}^{\{\bullet\}})$ **in terms of** $\mathrm{Da}^{\{\bullet\}}$**.**

$$
\begin{aligned}
\sigma(\mathrm{Da}^{\{1\}}) &= +\mathbf{1}\otimes\mathrm{Da}^{\{1\}} + \mathrm{Da}^{\{1\}}\otimes\mathbf{1} \\
\sigma(\underline{\mathrm{Da}}^{\{2\}}) &= +\mathbf{1}\otimes\mathrm{Da}^{\{2\}} + \mathrm{Da}^{\{2\}}\otimes\mathbf{1} \\
\sigma(\mathrm{Da}^{\{3\}}) &= +\mathbf{1}\otimes\mathrm{Da}^{\{3\}} + 2\,\mathrm{Da}^{\{1\}}\otimes\mathrm{Da}^{\{2\}} + \mathrm{Da}^{\{3\}}\otimes\mathbf{1} \\
\sigma(\mathrm{Da}^{\{4\}}) &= +\mathbf{1}\otimes\mathrm{Da}^{\{4\}} + 5\,\mathrm{Da}^{\{1\}}\otimes\mathrm{Da}^{\{3\}} + 5\,\mathrm{Da}^{\{1,1\}}\otimes\mathrm{Da}^{\{2\}} + \mathrm{Da}^{\{4\}}\otimes\mathbf{1} \\
\sigma(\mathrm{Da}^{\{5\}}) &= +\mathbf{1}\otimes\mathrm{Da}^{\{5\}} + 9\,\mathrm{Da}^{\{1\}}\otimes\mathrm{Da}^{\{4\}} + 5\,\mathrm{Da}^{\{1\}}\otimes\mathrm{Da}^{\{2^{\mathbf{2}}\}} + 5\,\mathrm{Da}^{\{2\}}\otimes\mathrm{Da}^{\{3\}} \\
&\quad +45/2\mathrm{Da}^{\{1^{\mathbf{2}}\}}\otimes\mathrm{Da}^{\{3\}} + 10\mathrm{Da}^{\{2,1\}}\otimes\mathrm{Da}^{\{2\}} + 15\mathrm{Da}^{\{1^{\mathbf{3}}\}}\otimes\mathrm{Da}^{\{2\}} + \mathrm{Da}^{\{5\}}\otimes\mathbf{1}
\end{aligned}
$$

$$\begin{aligned}\sigma(\mathrm{Da}^{\{6\}}) = {} & +\mathbf{1}\otimes \mathrm{Da}^{\{6\}} + 14\,\mathrm{Da}^{\{1\}}\otimes \mathrm{Da}^{\{5\}} + 14\,\mathrm{Da}^{\{2\}}\otimes \mathrm{Da}^{\{4\}} + 63\,\mathrm{Da}^{\{1^{\mathbf{2}}\}}\otimes \mathrm{Da}^{\{4\}} \\ & +35\,\mathrm{Da}^{\{1^{\mathbf{2}}\}}\otimes \mathrm{Da}^{\{2^{\mathbf{2}}\}} + 70\,\mathrm{Da}^{\{2,1\}}\otimes \mathrm{Da}^{\{3\}} + 105\,\mathrm{Da}^{\{1^{\mathbf{3}}\}}\otimes \mathrm{Da}^{\{3\}} \\ & +70\,\mathrm{Da}^{\{2,1^{\mathbf{2}}\}}\otimes \mathrm{Da}^{\{2\}} + 105/2\,\mathrm{Da}^{\{1^{\mathbf{4}}\}}\otimes \mathrm{Da}^{\{2\}} + \mathrm{Da}^{\{6\}}\otimes \mathbf{1}\end{aligned}$$

$$\begin{aligned}\sigma(\mathrm{Da}^{\{7\}}) = {} & +\mathbf{1}\otimes \mathrm{Da}^{\{7\}} + 20\,\mathrm{Da}^{\{1\}}\otimes \mathrm{Da}^{\{6\}} + 35\,\mathrm{Da}^{\{1\}}\otimes \mathrm{Da}^{\{3^{\mathbf{2}}\}} \\ & +140/3\,\mathrm{Da}^{\{1\}}\otimes \mathrm{Da}^{\{2^{\mathbf{3}}\}} + 28\,\mathrm{Da}^{\{2\}}\otimes \mathrm{Da}^{\{5\}} + 140\,\mathrm{Da}^{\{1^{\mathbf{2}}\}}\otimes \mathrm{Da}^{\{5\}} \\ & +70\,\mathrm{Da}^{\{1^{\mathbf{2}}\}}\otimes \mathrm{Da}^{\{3,2\}} + 14\,\mathrm{Da}^{\{3\}}\otimes \mathrm{Da}^{\{4\}} + 252\,\mathrm{Da}^{\{2,1\}}\otimes \mathrm{Da}^{\{4\}} \\ & +140\,\mathrm{Da}^{\{2,1\}}\otimes \mathrm{Da}^{\{2^{\mathbf{2}}\}} + 420\,\mathrm{Da}^{\{1^{\mathbf{3}}\}}\otimes \mathrm{Da}^{\{4\}} + 280\,\mathrm{Da}^{\{1^{\mathbf{3}}\}}\otimes \mathrm{Da}^{\{2^{\mathbf{2}}\}} \\ & +70\,\mathrm{Da}^{\{3,1\}}\otimes \mathrm{Da}^{\{3\}} + 70\,\mathrm{Da}^{\{2^{\mathbf{2}}\}}\otimes \mathrm{Da}^{\{3\}} + 630\,\mathrm{Da}^{\{2,1^{\mathbf{2}}\}}\otimes \mathrm{Da}^{\{3\}} \\ & +525\,\mathrm{Da}^{\{1^{\mathbf{4}}\}}\otimes \mathrm{Da}^{\{3\}} + 70\,\mathrm{Da}^{\{3,1^{\mathbf{2}}\}}\otimes \mathrm{Da}^{\{2\}} + 140\,\mathrm{Da}^{\{2^{\mathbf{2}},1\}}\otimes \mathrm{Da}^{\{2\}} \\ & +420\,\mathrm{Da}^{\{2,1^{\mathbf{3}}\}}\otimes \mathrm{Da}^{\{2\}} + 210\,\mathrm{Da}^{\{1^{\mathbf{5}}\}}\otimes \mathrm{Da}^{\{2\}} + \mathrm{Da}^{\{7\}}\otimes \mathbf{1}\end{aligned}$$

Table 7 bis: $\sigma(\underline{\mathrm{Da}}^{\{\bullet\}})$ in terms of $\underline{\mathrm{Da}}^{\{\bullet\}}$.

$$\begin{aligned}\sigma(\underline{\mathrm{Da}}^{\{1\}}) &= +\mathbf{1}\otimes \underline{\mathrm{Da}}^{\{1\}} + \underline{\mathrm{Da}}^{\{1\}}\otimes \mathbf{1} \\ \sigma(\underline{\mathrm{Da}}^{\{2\}}) &= +\mathbf{1}\otimes \underline{\mathrm{Da}}^{\{2\}} + \underline{\mathrm{Da}}^{\{2\}}\otimes \mathbf{1} \\ \sigma(\underline{\mathrm{Da}}^{\{3\}}) &= +\mathbf{1}\otimes \underline{\mathrm{Da}}^{\{3\}} + \underline{\mathrm{Da}}^{\{3\}}\otimes \mathbf{1} + \underline{\mathrm{Da}}^{\{1\}}\otimes \underline{\mathrm{Da}}^{\{2\}} \\ \sigma(\underline{\mathrm{Da}}^{\{4\}}) &= +\mathbf{1}\otimes \underline{\mathrm{Da}}^{\{4\}} + 2\,\underline{\mathrm{Da}}^{\{1\}}\otimes \underline{\mathrm{Da}}^{\{3\}} + \underline{\mathrm{Da}}^{\{1^{\mathbf{2}}\}}\otimes \underline{\mathrm{Da}}^{\{2\}} + \underline{\mathrm{Da}}^{\{4\}}\otimes \mathbf{1} \\ \sigma(\underline{\mathrm{Da}}^{\{5\}}) &= +\mathbf{1}\otimes \underline{\mathrm{Da}}^{\{5\}} + 3\,\underline{\mathrm{Da}}^{\{1\}}\otimes \underline{\mathrm{Da}}^{\{4\}} + 1/2\,\underline{\mathrm{Da}}^{\{1\}}\otimes \underline{\mathrm{Da}}^{\{2^{\mathbf{2}}\}} + \underline{\mathrm{Da}}^{\{2\}}\otimes \underline{\mathrm{Da}}^{\{3\}} \\ & \quad +3\,\underline{\mathrm{Da}}^{\{1^{\mathbf{2}}\}}\otimes \underline{\mathrm{Da}}^{\{3\}} + \underline{\mathrm{Da}}^{\{2,1\}}\otimes \underline{\mathrm{Da}}^{\{2\}} + \underline{\mathrm{Da}}^{\{1^{\mathbf{3}}\}}\otimes \underline{\mathrm{Da}}^{\{2\}} + \underline{\mathrm{Da}}^{\{5\}}\otimes \mathbf{1}\end{aligned}$$

$$\begin{aligned}\sigma(\underline{\mathrm{Da}}^{\{6\}}) = {} & +\mathbf{1}\otimes \underline{\mathrm{Da}}^{\{6\}} + 4\,\underline{\mathrm{Da}}^{\{1\}}\otimes \underline{\mathrm{Da}}^{\{5\}} + 2\,\underline{\mathrm{Da}}^{\{2\}}\otimes \underline{\mathrm{Da}}^{\{4\}} + 6\,\underline{\mathrm{Da}}^{\{1^{\mathbf{2}}\}}\otimes \underline{\mathrm{Da}}^{\{4\}} \\ & +\underline{\mathrm{Da}}^{\{1^{\mathbf{2}}\}}\otimes \underline{\mathrm{Da}}^{\{2^{\mathbf{2}}\}} + 4\,\underline{\mathrm{Da}}^{\{2,1\}}\otimes \underline{\mathrm{Da}}^{\{3\}} + 4\,\underline{\mathrm{Da}}^{\{1^{\mathbf{3}}\}}\otimes \underline{\mathrm{Da}}^{\{3\}} \\ & +2\,\underline{\mathrm{Da}}^{\{2,1^{\mathbf{2}}\}}\otimes \underline{\mathrm{Da}}^{\{2\}} + \underline{\mathrm{Da}}^{\{1^{\mathbf{4}}\}}\otimes \underline{\mathrm{Da}}^{\{2\}} + \underline{\mathrm{Da}}^{\{6\}}\otimes \mathbf{1}\end{aligned}$$

$$\begin{aligned}\sigma(\underline{\mathrm{Da}}^{\{7\}}) = {} & +\mathbf{1}\otimes \underline{\mathrm{Da}}^{\{7\}} + 5\,\underline{\mathrm{Da}}^{\{1\}}\otimes \underline{\mathrm{Da}}^{\{6\}} + \underline{\mathrm{Da}}^{\{1\}}\otimes \underline{\mathrm{Da}}^{\{3^{\mathbf{2}}\}} + 1/2\,\underline{\mathrm{Da}}^{\{1\}}\otimes \underline{\mathrm{Da}}^{\{2^{\mathbf{3}}\}} \\ & +3\,\underline{\mathrm{Da}}^{\{2\}}\otimes \underline{\mathrm{Da}}^{\{5\}} + 10\,\underline{\mathrm{Da}}^{\{1^{\mathbf{2}}\}}\otimes \underline{\mathrm{Da}}^{\{5\}} + \underline{\mathrm{Da}}^{\{1^{\mathbf{2}}\}}\otimes \underline{\mathrm{Da}}^{\{3,2\}} + \underline{\mathrm{Da}}^{\{3\}}\otimes \underline{\mathrm{Da}}^{\{4\}} \\ & +9\,\underline{\mathrm{Da}}^{\{2,1\}}\otimes \underline{\mathrm{Da}}^{\{4\}} + 3/2\,\underline{\mathrm{Da}}^{\{2,1\}}\otimes \underline{\mathrm{Da}}^{\{2^{\mathbf{2}}\}} + 10\,\underline{\mathrm{Da}}^{\{1^{\mathbf{3}}\}}\otimes \underline{\mathrm{Da}}^{\{4\}} \\ & +2\,\underline{\mathrm{Da}}^{\{1^{\mathbf{3}}\}}\otimes \underline{\mathrm{Da}}^{\{2^{\mathbf{2}}\}} + 2\,\underline{\mathrm{Da}}^{\{3,1\}}\otimes \underline{\mathrm{Da}}^{\{3\}} + 3/2\,\underline{\mathrm{Da}}^{\{2^{\mathbf{2}}\}}\otimes \underline{\mathrm{Da}}^{\{3\}} \\ & +9\,\underline{\mathrm{Da}}^{\{2,1^{\mathbf{2}}\}}\otimes \underline{\mathrm{Da}}^{\{3\}} + 5\,\underline{\mathrm{Da}}^{\{1^{\mathbf{4}}\}}\otimes \underline{\mathrm{Da}}^{\{3\}} + \underline{\mathrm{Da}}^{\{3,1^{\mathbf{2}}\}}\otimes \underline{\mathrm{Da}}^{\{2\}} \\ & +3/2\,\underline{\mathrm{Da}}^{\{2^{\mathbf{2}},1\}}\otimes \underline{\mathrm{Da}}^{\{2\}} + 3\,\underline{\mathrm{Da}}^{\{2,1^{\mathbf{3}}\}}\otimes \underline{\mathrm{Da}}^{\{2\}} + \underline{\mathrm{Da}}^{\{1^{\mathbf{5}}\}}\otimes \underline{\mathrm{Da}}^{\{2\}} + \underline{\mathrm{Da}}^{\{7\}}\otimes \mathbf{1}\end{aligned}$$

14.4 Embedding of ISO into ${}^\sharp ISO$.

Table 8: $\mathrm{Dn}^{\{\bullet\}}$ **or rather** $\mathrm{dn}^{\{\bullet\}} := \frac{1}{\bullet !}\mathrm{Dn}^{\{\bullet\}}$ **in terms of** $\mathbb{D}\mathrm{e}^{\langle\bullet\rangle}$ **and** $\mathbb{D}\mathrm{a}^{\langle\bullet\rangle}$.

$$\mathrm{dn}^{\{1\}} = +\mathbb{D}\mathrm{e}^{\langle 1\rangle} = +\mathbb{D}\mathrm{a}^{\langle 1\rangle}$$

$$\mathrm{dn}^{\{2\}} = +\frac{1}{2}\mathbb{D}\mathrm{e}^{\langle 2\rangle} + \frac{1}{2}\mathbb{D}\mathrm{e}^{\langle 1,1\rangle} = +\frac{1}{2}\mathbb{D}\mathrm{a}^{\langle 2\rangle} + \frac{1}{2}\mathbb{D}\mathrm{a}^{\langle 1,1\rangle}$$

$$\mathrm{dn}^{\{3\}} = +\frac{1}{3}\mathbb{D}\mathrm{e}^{\langle 3\rangle} + \frac{1}{3}\mathbb{D}\mathrm{e}^{\langle 2,1\rangle} + \frac{1}{2}\mathbb{D}\mathrm{e}^{\langle 1,2\rangle} + \frac{1}{2}\mathbb{D}\mathrm{e}^{\langle 1,1,1\rangle}$$

$$\mathrm{dn}^{\{3\}} = +\frac{1}{3}\mathbb{D}\mathrm{a}^{\langle 3\rangle} + \frac{1}{6}\mathbb{D}\mathrm{a}^{\langle 2,1\rangle} + \frac{1}{2}\mathbb{D}\mathrm{a}^{\langle 1,2\rangle} + \frac{1}{2}\mathbb{D}\mathrm{a}^{\langle 1,1,1\rangle}$$

$$\begin{aligned}\mathrm{dn}^{\{4\}} = &+\frac{1}{4}\mathbb{D}\mathrm{e}^{\langle 4\rangle} + \frac{1}{4}\mathbb{D}\mathrm{e}^{\langle 3,1\rangle} + \frac{11}{24}\mathbb{D}\mathrm{e}^{\langle 2,2\rangle} + \frac{1}{2}\mathbb{D}\mathrm{e}^{\langle 1,3\rangle} + \frac{11}{24}\mathbb{D}\mathrm{e}^{\langle 2,1,1\rangle}\\ &+\frac{1}{2}\mathbb{D}\mathrm{e}^{\langle 1,2,1\rangle} + \frac{3}{4}\mathbb{D}\mathrm{e}^{\langle 1,1,2\rangle} + \frac{3}{4}\mathbb{D}\mathrm{e}^{\langle 1,1,1,1\rangle}\end{aligned}$$

$$\begin{aligned}\mathrm{dn}^{\{4\}} = &+\frac{1}{4}\mathbb{D}\mathrm{a}^{\langle 4\rangle} + \frac{1}{12}\mathbb{D}\mathrm{a}^{\langle 3,1\rangle} + \frac{1}{6}\mathbb{D}\mathrm{a}^{\langle 2,2\rangle} + \frac{1}{2}\mathbb{D}\mathrm{a}^{\langle 1,3\rangle} + \frac{1}{6}\mathbb{D}\mathrm{a}^{\langle 2,1,1\rangle}\\ &+\frac{1}{4}\mathbb{D}\mathrm{a}^{\langle 1,2,1\rangle} + \frac{3}{4}\mathbb{D}\mathrm{a}^{\langle 1,1,2\rangle} + \frac{3}{4}\mathbb{D}\mathrm{a}^{\langle 1,1,1,1\rangle}\end{aligned}$$

$$\begin{aligned}\mathrm{dn}^{\{5\}} = &+\frac{1}{5}\mathbb{D}\mathrm{e}^{\langle 5\rangle} + \frac{1}{5}\mathbb{D}\mathrm{e}^{\langle 4,1\rangle} + \frac{5}{12}\mathbb{D}\mathrm{e}^{\langle 3,2\rangle} + \frac{7}{12}\mathbb{D}\mathrm{e}^{\langle 2,3\rangle} + \frac{1}{2}\mathbb{D}\mathrm{e}^{\langle 1,4\rangle} + \frac{5}{12}\mathbb{D}\mathrm{e}^{\langle 3,1,1\rangle}\\ &+\frac{7}{12}\mathbb{D}\mathrm{e}^{\langle 2,2,1\rangle} + \frac{7}{8}\mathbb{D}\mathrm{e}^{\langle 2,1,2\rangle} + \frac{1}{2}\mathbb{D}\mathrm{e}^{\langle 1,3,1\rangle} + \frac{11}{12}\mathbb{D}\mathrm{e}^{\langle 1,2,2\rangle} + \mathbb{D}\mathrm{e}^{\langle 1,1,3\rangle}\\ &+\frac{7}{8}\mathbb{D}\mathrm{e}^{\langle 2,1,1,1\rangle} + \frac{11}{12}\mathbb{D}\mathrm{e}^{\langle 1,2,1,1\rangle} + \mathbb{D}\mathrm{e}^{\langle 1,1,2,1\rangle} + \frac{3}{2}\mathbb{D}\mathrm{e}^{\langle 1,1,1,2\rangle} + \frac{3}{2}\mathbb{D}\mathrm{e}^{\langle 1,1,1,1,1\rangle}\end{aligned}$$

$$\begin{aligned}\mathrm{dn}^{\{5\}} = &+\frac{1}{5}\mathbb{D}\mathrm{a}^{\langle 5\rangle} + \frac{1}{20}\mathbb{D}\mathrm{a}^{\langle 4,1\rangle} + \frac{1}{12}\mathbb{D}\mathrm{a}^{\langle 3,2\rangle} + \frac{1}{6}\mathbb{D}\mathrm{a}^{\langle 2,3\rangle} + \frac{1}{2}\mathbb{D}\mathrm{a}^{\langle 1,4\rangle} + \frac{1}{12}\mathbb{D}\mathrm{a}^{\langle 3,1,1\rangle}\\ &+\frac{1}{12}\mathbb{D}\mathrm{a}^{\langle 2,2,1\rangle} + \frac{1}{4}\mathbb{D}\mathrm{a}^{\langle 2,1,2\rangle} + \frac{1}{6}\mathbb{D}\mathrm{a}^{\langle 1,3,1\rangle} + \frac{1}{3}\mathbb{D}\mathrm{a}^{\langle 1,2,2\rangle} + \mathbb{D}\mathrm{a}^{\langle 1,1,3\rangle}\\ &+\frac{1}{4}\mathbb{D}\mathrm{a}^{\langle 2,1,1,1\rangle} + \frac{1}{3}\mathbb{D}\mathrm{a}^{\langle 1,2,1,1\rangle} + \frac{1}{2}\mathbb{D}\mathrm{a}^{\langle 1,1,2,1\rangle} + \frac{3}{2}\mathbb{D}\mathrm{a}^{\langle 1,1,1,2\rangle} + \frac{3}{2}\mathbb{D}\mathrm{a}^{\langle 1,1,1,1,1\rangle}\end{aligned}$$

$$
\begin{aligned}
\mathrm{dn}^{\{6\}} = &+\frac{1}{6}\,\mathbb{D}\mathrm{e}^{\langle 6\rangle} + \frac{1}{6}\,\mathbb{D}\mathrm{e}^{\langle 5,1\rangle} + \frac{137}{360}\,\mathbb{D}\mathrm{e}^{\langle 4,2\rangle} + \frac{5}{8}\,\mathbb{D}\mathrm{e}^{\langle 3,3\rangle} + \frac{17}{24}\,\mathbb{D}\mathrm{e}^{\langle 2,4\rangle} + \frac{1}{2}\,\mathbb{D}\mathrm{e}^{\langle 1,5\rangle} \\
&+\frac{137}{360}\,\mathbb{D}\mathrm{e}^{\langle 4,1,1\rangle} + \frac{5}{8}\,\mathbb{D}\mathrm{e}^{\langle 3,2,1\rangle} + \frac{15}{16}\,\mathbb{D}\mathrm{e}^{\langle 3,1,2\rangle} + \frac{17}{24}\,\mathbb{D}\mathrm{e}^{\langle 2,3,1\rangle} + \frac{187}{144}\,\mathbb{D}\mathrm{e}^{\langle 2,2,2\rangle} \\
&+\frac{17}{12}\,\mathbb{D}\mathrm{e}^{\langle 2,1,3\rangle} + \frac{1}{2}\,\mathbb{D}\mathrm{e}^{\langle 1,4,1\rangle} + \frac{25}{24}\,\mathbb{D}\mathrm{e}^{\langle 1,3,2\rangle} + \frac{35}{24}\,\mathbb{D}\mathrm{e}^{\langle 1,2,3\rangle} + \frac{5}{4}\,\mathbb{D}\mathrm{e}^{\langle 1,1,4\rangle} \\
&+\frac{15}{16}\mathbb{D}\mathrm{e}^{\langle 3,1,1,1\rangle} + \frac{187}{144}\mathbb{D}\mathrm{e}^{\langle 2,2,1,1\rangle} + \frac{17}{12}\mathbb{D}\mathrm{e}^{\langle 2,1,2,1\rangle} + \frac{17}{8}\mathbb{D}\mathrm{e}^{\langle 2,1,1,2\rangle} + \frac{25}{24}\mathbb{D}\mathrm{e}^{\langle 1,3,1,1\rangle} \\
&+\frac{35}{24}\,\mathbb{D}\mathrm{e}^{\langle 1,2,2,1\rangle} + \frac{35}{16}\,\mathbb{D}\mathrm{e}^{\langle 1,2,1,2\rangle} + \frac{5}{4}\,\mathbb{D}\mathrm{e}^{\langle 1,1,3,1\rangle} + \frac{55}{24}\,\mathbb{D}\mathrm{e}^{\langle 1,1,2,2\rangle} + \frac{5}{2}\,\mathbb{D}\mathrm{e}^{\langle 1,1,1,3\rangle} \\
&+\frac{17}{8}\,\mathbb{D}\mathrm{e}^{\langle 2,1,1,1,1\rangle} + \frac{35}{16}\,\mathbb{D}\mathrm{e}^{\langle 1,2,1,1,1\rangle} + \frac{55}{24}\,\mathbb{D}\mathrm{e}^{\langle 1,1,2,1,1\rangle} + \frac{5}{2}\,\mathbb{D}\mathrm{e}^{\langle 1,1,1,2,1\rangle} \\
&+\frac{15}{4}\,\mathbb{D}\mathrm{e}^{\langle 1,1,1,1,2\rangle} + \frac{15}{4}\,\mathbb{D}\mathrm{e}^{\langle 1,1,1,1,1,1\rangle}
\end{aligned}
$$

$$
\begin{aligned}
\mathrm{dn}^{\{6\}} = &+\frac{1}{6}\,\mathbb{D}\mathrm{a}^{\langle 6\rangle} + \frac{1}{30}\,\mathbb{D}\mathrm{a}^{\langle 5,1\rangle} + \frac{1}{20}\,\mathbb{D}\mathrm{a}^{\langle 4,2\rangle} + \frac{1}{12}\,\mathbb{D}\mathrm{a}^{\langle 3,3\rangle} + \frac{1}{6}\,\mathbb{D}\mathrm{a}^{\langle 2,4\rangle} + \frac{1}{2}\,\mathbb{D}\mathrm{a}^{\langle 1,5\rangle} \\
&+\frac{1}{20}\,\mathbb{D}\mathrm{a}^{\langle 4,1,1\rangle} + \frac{1}{24}\,\mathbb{D}\mathrm{a}^{\langle 3,2,1\rangle} + \frac{1}{8}\,\mathbb{D}\mathrm{a}^{\langle 3,1,2\rangle} + \frac{1}{18}\,\mathbb{D}\mathrm{a}^{\langle 2,3,1\rangle} + \frac{1}{9}\,\mathbb{D}\mathrm{a}^{\langle 2,2,2\rangle} \\
&+\frac{1}{3}\,\mathbb{D}\mathrm{a}^{\langle 2,1,3\rangle} + \frac{1}{8}\,\mathbb{D}\mathrm{a}^{\langle 1,4,1\rangle} + \frac{5}{24}\,\mathbb{D}\mathrm{a}^{\langle 1,3,2\rangle} + \frac{5}{12}\,\mathbb{D}\mathrm{a}^{\langle 1,2,3\rangle} + \frac{5}{4}\,\mathbb{D}\mathrm{a}^{\langle 1,1,4\rangle} \\
&+\frac{1}{8}\,\mathbb{D}\mathrm{a}^{\langle 3,1,1,1\rangle} + \frac{1}{9}\,\mathbb{D}\mathrm{a}^{\langle 2,2,1,1\rangle} + \frac{1}{6}\,\mathbb{D}\mathrm{a}^{\langle 2,1,2,1\rangle} + \frac{1}{2}\,\mathbb{D}\mathrm{a}^{\langle 2,1,1,2\rangle} + \frac{5}{24}\,\mathbb{D}\mathrm{a}^{\langle 1,3,1,1\rangle} \\
&+\frac{5}{24}\,\mathbb{D}\mathrm{a}^{\langle 1,2,2,1\rangle} + \frac{5}{8}\,\mathbb{D}\mathrm{a}^{\langle 1,2,1,2\rangle} + \frac{5}{12}\,\mathbb{D}\mathrm{a}^{\langle 1,1,3,1\rangle} + \frac{5}{6}\,\mathbb{D}\mathrm{a}^{\langle 1,1,2,2\rangle} + \frac{5}{2}\,\mathbb{D}\mathrm{a}^{\langle 1,1,1,3\rangle} \\
&+\frac{1}{2}\,\mathbb{D}\mathrm{a}^{\langle 2,1,1,1,1\rangle} + \frac{5}{8}\,\mathbb{D}\mathrm{a}^{\langle 1,2,1,1,1\rangle} + \frac{5}{6}\,\mathbb{D}\mathrm{a}^{\langle 1,1,2,1,1\rangle} + \frac{5}{4}\,\mathbb{D}\mathrm{a}^{\langle 1,1,1,2,1\rangle} \\
&+\frac{15}{4}\,\mathbb{D}\mathrm{a}^{\langle 1,1,1,1,2\rangle} + \frac{15}{4}\,\mathbb{D}\mathrm{a}^{\langle 1,1,1,1,1,1\rangle}
\end{aligned}
$$

$$
\begin{aligned}
\mathrm{dn}^{\{7\}} = & +\frac{1}{7}\,\mathbb{D}\mathrm{e}^{\langle 7\rangle} + \frac{1}{7}\,\mathbb{D}\mathrm{e}^{\langle 6,1\rangle} + \frac{7}{20}\,\mathbb{D}\mathrm{e}^{\langle 5,2\rangle} + \frac{29}{45}\,\mathbb{D}\mathrm{e}^{\langle 4,3\rangle} + \frac{7}{8}\,\mathbb{D}\mathrm{e}^{\langle 3,4\rangle} + \frac{5}{6}\,\mathbb{D}\mathrm{e}^{\langle 2,5\rangle} \\
& +\frac{1}{2}\,\mathbb{D}\mathrm{e}^{\langle 1,6\rangle} + \frac{7}{20}\,\mathbb{D}\mathrm{e}^{\langle 5,1,1\rangle} + \frac{29}{45}\,\mathbb{D}\mathrm{e}^{\langle 4,2,1\rangle} + \frac{29}{30}\,\mathbb{D}\mathrm{e}^{\langle 4,1,2\rangle} + \frac{7}{8}\,\mathbb{D}\mathrm{e}^{\langle 3,3,1\rangle} \\
& +\frac{77}{48}\,\mathbb{D}\mathrm{e}^{\langle 3,2,2\rangle} + \frac{7}{4}\,\mathbb{D}\mathrm{e}^{\langle 3,1,3\rangle} + \frac{5}{6}\,\mathbb{D}\mathrm{e}^{\langle 2,4,1\rangle} + \frac{125}{72}\,\mathbb{D}\mathrm{e}^{\langle 2,3,2\rangle} + \frac{175}{72}\,\mathbb{D}\mathrm{e}^{\langle 2,2,3\rangle} \\
& +\frac{25}{12}\,\mathbb{D}\mathrm{e}^{\langle 2,1,4\rangle} + \frac{1}{2}\,\mathbb{D}\mathrm{e}^{\langle 1,5,1\rangle} + \frac{137}{120}\,\mathbb{D}\mathrm{e}^{\langle 1,4,2\rangle} + \frac{15}{8}\,\mathbb{D}\mathrm{e}^{\langle 1,3,3\rangle} + \frac{17}{8}\,\mathbb{D}\mathrm{e}^{\langle 1,2,4\rangle} \\
& +\frac{3}{2}\,\mathbb{D}\mathrm{e}^{\langle 1,1,5\rangle} + \frac{29}{30}\,\mathbb{D}\mathrm{e}^{\langle 4,1,1,1\rangle} + \frac{77}{48}\,\mathbb{D}\mathrm{e}^{\langle 3,2,1,1\rangle} + \frac{7}{4}\,\mathbb{D}\mathrm{e}^{\langle 3,1,2,1\rangle} + \frac{21}{8}\,\mathbb{D}\mathrm{e}^{\langle 3,1,1,2\rangle} \\
& +\frac{125}{72}\,\mathbb{D}\mathrm{e}^{\langle 2,3,1,1\rangle} + \frac{175}{72}\,\mathbb{D}\mathrm{e}^{\langle 2,2,2,1\rangle} + \frac{175}{48}\,\mathbb{D}\mathrm{e}^{\langle 2,2,1,2\rangle} + \frac{25}{12}\,\mathbb{D}\mathrm{e}^{\langle 2,1,3,1\rangle} \\
& +\frac{275}{72}\,\mathbb{D}\mathrm{e}^{\langle 2,1,2,2\rangle} + \frac{25}{6}\,\mathbb{D}\mathrm{e}^{\langle 2,1,1,3\rangle} + \frac{137}{120}\,\mathbb{D}\mathrm{e}^{\langle 1,4,1,1\rangle} + \frac{15}{8}\,\mathbb{D}\mathrm{e}^{\langle 1,3,2,1\rangle} \\
& +\frac{45}{16}\mathbb{D}\mathrm{e}^{\langle 1,3,1,2\rangle} + \frac{17}{8}\mathbb{D}\mathrm{e}^{\langle 1,2,3,1\rangle} + \frac{187}{48}\mathbb{D}\mathrm{e}^{\langle 1,2,2,2\rangle} + \frac{17}{4}\mathbb{D}\mathrm{e}^{\langle 1,2,1,3\rangle} \\
& +\frac{3}{2}\mathbb{D}\mathrm{e}^{\langle 1,1,4,1\rangle} + \frac{25}{8}\,\mathbb{D}\mathrm{e}^{\langle 1,1,3,2\rangle} + \frac{35}{8}\,\mathbb{D}\mathrm{e}^{\langle 1,1,2,3\rangle} + \frac{15}{4}\,\mathbb{D}\mathrm{e}^{\langle 1,1,1,4\rangle} \\
& +\frac{21}{8}\,\mathbb{D}\mathrm{e}^{\langle 3,1,1,1,1\rangle} + \frac{175}{48}\,\mathbb{D}\mathrm{e}^{\langle 2,2,1,1,1\rangle} + \frac{275}{72}\,\mathbb{D}\mathrm{e}^{\langle 2,1,2,1,1\rangle} + \frac{25}{6}\,\mathbb{D}\mathrm{e}^{\langle 2,1,1,2,1\rangle} \\
& +\frac{25}{4}\,\mathbb{D}\mathrm{e}^{\langle 2,1,1,1,2\rangle} + \frac{45}{16}\,\mathbb{D}\mathrm{e}^{\langle 1,3,1,1,1\rangle} + \frac{187}{48}\,\mathbb{D}\mathrm{e}^{\langle 1,2,2,1,1\rangle} + \frac{17}{4}\,\mathbb{D}\mathrm{e}^{\langle 1,2,1,2,1\rangle} \\
& +\frac{51}{8}\,\mathbb{D}\mathrm{e}^{\langle 1,2,1,1,2\rangle} + \frac{25}{8}\,\mathbb{D}\mathrm{e}^{\langle 1,1,3,1,1\rangle} + \frac{35}{8}\,\mathbb{D}\mathrm{e}^{\langle 1,1,2,2,1\rangle} + \frac{105}{16}\,\mathbb{D}\mathrm{e}^{\langle 1,1,2,1,2\rangle} \\
& +\frac{15}{4}\,\mathbb{D}\mathrm{e}^{\langle 1,1,1,3,1\rangle} + \frac{55}{8}\,\mathbb{D}\mathrm{e}^{\langle 1,1,1,2,2\rangle} + \frac{15}{2}\,\mathbb{D}\mathrm{e}^{\langle 1,1,1,1,3\rangle} + \frac{25}{4}\,\mathbb{D}\mathrm{e}^{\langle 2,1,1,1,1,1\rangle} \\
& +\frac{51}{8}\,\mathbb{D}\mathrm{e}^{\langle 1,2,1,1,1,1\rangle} + \frac{105}{16}\,\mathbb{D}\mathrm{e}^{\langle 1,1,2,1,1,1\rangle} + \frac{55}{8}\,\mathbb{D}\mathrm{e}^{\langle 1,1,1,2,1,1\rangle} \\
& +\frac{15}{2}\,\mathbb{D}\mathrm{e}^{\langle 1,1,1,1,2,1\rangle} + \frac{45}{4}\,\mathbb{D}\mathrm{e}^{\langle 1,1,1,1,1,2\rangle} + \frac{45}{4}\,\mathbb{D}\mathrm{e}^{\langle 1,1,1,1,1,1,1\rangle}
\end{aligned}
$$

$$
\begin{aligned}
\mathrm{dn}^{\{7\}} = {} & +\frac{1}{7}\,\mathbb{D}\mathrm{a}^{\langle 7\rangle} + \frac{1}{42}\,\mathbb{D}\mathrm{a}^{\langle 6,1\rangle} + \frac{1}{30}\,\mathbb{D}\mathrm{a}^{\langle 5,2\rangle} + \frac{1}{20}\,\mathbb{D}\mathrm{a}^{\langle 4,3\rangle} + \frac{1}{12}\,\mathbb{D}\mathrm{a}^{\langle 3,4\rangle} + \frac{1}{6}\,\mathbb{D}\mathrm{a}^{\langle 2,5\rangle} \\
& +\frac{1}{2}\,\mathbb{D}\mathrm{a}^{\langle 1,6\rangle} + \frac{1}{30}\,\mathbb{D}\mathrm{a}^{\langle 5,1,1\rangle} + \frac{1}{40}\,\mathbb{D}\mathrm{a}^{\langle 4,2,1\rangle} + \frac{3}{40}\,\mathbb{D}\mathrm{a}^{\langle 4,1,2\rangle} + \frac{1}{36}\,\mathbb{D}\mathrm{a}^{\langle 3,3,1\rangle} \\
& +\frac{1}{18}\,\mathbb{D}\mathrm{a}^{\langle 3,2,2\rangle} + \frac{1}{6}\,\mathbb{D}\mathrm{a}^{\langle 3,1,3\rangle} + \frac{1}{24}\,\mathbb{D}\mathrm{a}^{\langle 2,4,1\rangle} + \frac{5}{72}\,\mathbb{D}\mathrm{a}^{\langle 2,3,2\rangle} + \frac{5}{36}\,\mathbb{D}\mathrm{a}^{\langle 2,2,3\rangle} \\
& +\frac{5}{12}\,\mathbb{D}\mathrm{a}^{\langle 2,1,4\rangle} + \frac{1}{10}\,\mathbb{D}\mathrm{a}^{\langle 1,5,1\rangle} + \frac{3}{20}\,\mathbb{D}\mathrm{a}^{\langle 1,4,2\rangle} + \frac{1}{4}\,\mathbb{D}\mathrm{a}^{\langle 1,3,3\rangle} + \frac{1}{2}\,\mathbb{D}\mathrm{a}^{\langle 1,2,4\rangle} \\
& +\frac{3}{2}\,\mathbb{D}\mathrm{a}^{\langle 1,1,5\rangle} + \frac{3}{40}\,\mathbb{D}\mathrm{a}^{\langle 4,1,1,1\rangle} + \frac{1}{18}\,\mathbb{D}\mathrm{a}^{\langle 3,2,1,1\rangle} + \frac{1}{12}\,\mathbb{D}\mathrm{a}^{\langle 3,1,2,1\rangle} + \frac{1}{4}\,\mathbb{D}\mathrm{a}^{\langle 3,1,1,2\rangle} \\
& +\frac{5}{72}\,\mathbb{D}\mathrm{a}^{\langle 2,3,1,1\rangle} + \frac{5}{72}\,\mathbb{D}\mathrm{a}^{\langle 2,2,2,1\rangle} + \frac{5}{24}\,\mathbb{D}\mathrm{a}^{\langle 2,2,1,2\rangle} + \frac{5}{36}\,\mathbb{D}\mathrm{a}^{\langle 2,1,3,1\rangle} + \frac{5}{18}\,\mathbb{D}\mathrm{a}^{\langle 2,1,2,2\rangle} \\
& +\frac{5}{6}\,\mathbb{D}\mathrm{a}^{\langle 2,1,1,3\rangle} + \frac{3}{20}\,\mathbb{D}\mathrm{a}^{\langle 1,4,1,1\rangle} + \frac{1}{8}\,\mathbb{D}\mathrm{a}^{\langle 1,3,2,1\rangle} + \frac{3}{8}\,\mathbb{D}\mathrm{a}^{\langle 1,3,1,2\rangle} + \frac{1}{6}\,\mathbb{D}\mathrm{a}^{\langle 1,2,3,1\rangle} \\
& +\frac{1}{3}\,\mathbb{D}\mathrm{a}^{\langle 1,2,2,2\rangle} + \mathbb{D}\mathrm{a}^{\langle 1,2,1,3\rangle} + \frac{3}{8}\,\mathbb{D}\mathrm{a}^{\langle 1,1,4,1\rangle} + \frac{5}{8}\,\mathbb{D}\mathrm{a}^{\langle 1,1,3,2\rangle} + \frac{5}{4}\,\mathbb{D}\mathrm{a}^{\langle 1,1,2,3\rangle} \\
& +\frac{15}{4}\,\mathbb{D}\mathrm{a}^{\langle 1,1,1,4\rangle} + \frac{1}{4}\,\mathbb{D}\mathrm{a}^{\langle 3,1,1,1,1\rangle} + \frac{5}{24}\,\mathbb{D}\mathrm{a}^{\langle 2,2,1,1,1\rangle} + \frac{5}{18}\,\mathbb{D}\mathrm{a}^{\langle 2,1,2,1,1\rangle} \\
& +\frac{5}{12}\,\mathbb{D}\mathrm{a}^{\langle 2,1,1,2,1\rangle} + \frac{5}{4}\,\mathbb{D}\mathrm{a}^{\langle 2,1,1,1,2\rangle} + \frac{3}{8}\,\mathbb{D}\mathrm{a}^{\langle 1,3,1,1,1\rangle} + \frac{1}{3}\,\mathbb{D}\mathrm{a}^{\langle 1,2,2,1,1\rangle} \\
& +\frac{1}{2}\,\mathbb{D}\mathrm{a}^{\langle 1,2,1,2,1\rangle} + \frac{3}{2}\,\mathbb{D}\mathrm{a}^{\langle 1,2,1,1,2\rangle} + \frac{5}{8}\,\mathbb{D}\mathrm{a}^{\langle 1,1,3,1,1\rangle} + \frac{5}{8}\,\mathbb{D}\mathrm{a}^{\langle 1,1,2,2,1\rangle} \\
& +\frac{15}{8}\,\mathbb{D}\mathrm{a}^{\langle 1,1,2,1,2\rangle} + \frac{5}{4}\,\mathbb{D}\mathrm{a}^{\langle 1,1,1,3,1\rangle} + \frac{5}{2}\,\mathbb{D}\mathrm{a}^{\langle 1,1,1,2,2\rangle} + \frac{15}{2}\,\mathbb{D}\mathrm{a}^{\langle 1,1,1,1,3\rangle} \\
& +\frac{5}{4}\,\mathbb{D}\mathrm{a}^{\langle 2,1,1,1,1,1\rangle} + \frac{3}{2}\,\mathbb{D}\mathrm{a}^{\langle 1,2,1,1,1,1\rangle} + \frac{15}{8}\,\mathbb{D}\mathrm{a}^{\langle 1,1,2,1,1,1\rangle} + \frac{5}{2}\,\mathbb{D}\mathrm{a}^{\langle 1,1,1,2,1,1\rangle} \\
& +\frac{15}{4}\,\mathbb{D}\mathrm{a}^{\langle 1,1,1,1,2,1\rangle} + \frac{45}{4}\,\mathbb{D}\mathrm{a}^{\langle 1,1,1,1,1,2\rangle} + \frac{45}{4}\,\mathbb{D}\mathrm{a}^{\langle 1,1,1,1,1,1,1\rangle}
\end{aligned}
$$

Table 9: $\mathrm{Ds}^{\{\bullet\}}$ or rather $\mathrm{ds}^{\{\bullet\}} := \frac{1}{\bullet!}\,\mathrm{Ds}^{\{\bullet\}}$ in terms of $\mathbb{D}\mathrm{a}^{\langle\bullet\rangle}$.

$$
\begin{aligned}
\mathrm{ds}^{\{1\}} &= +\mathbb{D}\mathrm{a}^{\langle 1\rangle} \\
\mathrm{ds}^{\{2\}} &= +\frac{1}{2}\,\mathbb{D}\mathrm{a}^{\langle 2\rangle} \\
\mathrm{ds}^{\{3\}} &= +\frac{1}{3}\,\mathbb{D}\mathrm{a}^{\langle 3\rangle} + \frac{1}{6}\,(\mathbb{D}\mathrm{a}^{\langle 1,2\rangle} - \mathbb{D}\mathrm{a}^{\langle 2,1\rangle}) \\
\mathrm{ds}^{\{4\}} &= +\frac{1}{4}\,\mathbb{D}\mathrm{a}^{\langle 4\rangle} + \frac{5}{24}\,(\mathbb{D}\mathrm{a}^{\langle 1,3\rangle} - \mathbb{D}\mathrm{a}^{\langle 3,1\rangle}) + \frac{1}{12}\,(\mathbb{D}\mathrm{a}^{\langle 1,1,2\rangle} + \mathbb{D}\mathrm{a}^{\langle 2,1,1\rangle}) - \frac{1}{8}\,\mathbb{D}\mathrm{a}^{\langle 1,2,1\rangle}
\end{aligned}
$$

$$\begin{aligned}
\mathrm{ds}^{\{5\}} = &+\frac{1}{5}\,\mathbb{D}\mathrm{a}^{\langle 5\rangle}+\frac{9}{40}\,(\mathbb{D}\mathrm{a}^{\langle 1,4\rangle}-\mathbb{D}\mathrm{a}^{\langle 4,1\rangle})+\frac{1}{24}\,(\mathbb{D}\mathrm{a}^{\langle 2,3\rangle}-\mathbb{D}\mathrm{a}^{\langle 3,2\rangle})\\
&+\frac{19}{120}\,(\mathbb{D}\mathrm{a}^{\langle 1,1,3\rangle}+\mathbb{D}\mathrm{a}^{\langle 3,1,1\rangle})-\frac{1}{120}\,(\mathbb{D}\mathrm{a}^{\langle 1,2,2\rangle}+\mathbb{D}\mathrm{a}^{\langle 2,2,1\rangle})\\
&-\frac{13}{60}\,\mathbb{D}\mathrm{a}^{\langle 1,3,1\rangle}+\frac{1}{30}\,\mathbb{D}\mathrm{a}^{\langle 2,1,2\rangle}+\frac{1}{20}\,(\mathbb{D}\mathrm{a}^{\langle 1,1,1,2\rangle}-\mathbb{D}\mathrm{a}^{\langle 2,1,1,1\rangle})\\
&-\frac{13}{120}\,(\mathbb{D}\mathrm{a}^{\langle 1,1,2,1\rangle}-\mathbb{D}\mathrm{a}^{\langle 1,2,1,1\rangle})
\end{aligned}$$

$$\begin{aligned}
\mathrm{ds}^{\{6\}} = &+\frac{1}{6}\,\mathbb{D}\mathrm{a}^{\langle 6\rangle}+\frac{7}{30}\,(\mathbb{D}\mathrm{a}^{\langle 1,5\rangle}-\mathbb{D}\mathrm{a}^{\langle 5,1\rangle})+\frac{7}{120}\,(\mathbb{D}\mathrm{a}^{\langle 2,4\rangle}-\mathbb{D}\mathrm{a}^{\langle 4,2\rangle})\\
&+\frac{11}{48}\,(\mathbb{D}\mathrm{a}^{\langle 1,1,4\rangle}+\mathbb{D}\mathrm{a}^{\langle 4,1,1\rangle})+\frac{59}{1440}\,(\mathbb{D}\mathrm{a}^{\langle 1,2,3\rangle}+\mathbb{D}\mathrm{a}^{\langle 3,2,1\rangle})\\
&-\frac{9}{160}\,(\mathbb{D}\mathrm{a}^{\langle 1,3,2\rangle}+\mathbb{D}\mathrm{a}^{\langle 2,3,1\rangle})+\frac{59}{1440}\,(\mathbb{D}\mathrm{a}^{\langle 2,1,3\rangle}+\mathbb{D}\mathrm{a}^{\langle 3,1,2\rangle})\\
&+\frac{1}{360}\,\mathbb{D}\mathrm{a}^{\langle 2,2,2\rangle}-\frac{71}{240}\,\mathbb{D}\mathrm{a}^{\langle 1,4,1\rangle}+\frac{13}{96}\,(\mathbb{D}\mathrm{a}^{\langle 1,1,1,3\rangle}-\mathbb{D}\mathrm{a}^{\langle 3,1,1,1\rangle})\\
&-\frac{11}{720}\,(\mathbb{D}\mathrm{a}^{\langle 1,1,2,2\rangle}-\mathbb{D}\mathrm{a}^{\langle 2,2,1,1\rangle})+\frac{59}{1440}\,(\mathbb{D}\mathrm{a}^{\langle 1,2,1,2\rangle}-\mathbb{D}\mathrm{a}^{\langle 2,1,2,1\rangle})\\
&-\frac{71}{288}\,(\mathbb{D}\mathrm{a}^{\langle 1,1,3,1\rangle}-\mathbb{D}\mathrm{a}^{\langle 1,3,1,1\rangle})+\frac{1}{30}\,(\mathbb{D}\mathrm{a}^{\langle 1,1,1,1,2\rangle}+\mathbb{D}\mathrm{a}^{\langle 2,1,1,1,1\rangle})\\
&-\frac{49}{480}\,(\mathbb{D}\mathrm{a}^{\langle 1,1,1,2,1\rangle}+\mathbb{D}\mathrm{a}^{\langle 1,2,1,1,1\rangle})+\frac{13}{90}\,\mathbb{D}\mathrm{a}^{\langle 1,1,2,1,1\rangle}
\end{aligned}$$

Table 10: $\mathrm{Da}^{\{\bullet\}}$ **or rather** $\mathrm{da}^{\{\bullet\}} := \frac{1}{\bullet!}\,\mathrm{Da}^{\{\bullet\}}$ **in terms of** $\mathbb{D}\mathrm{a}^{\langle\bullet\rangle}$**.**

$$\begin{aligned}
\mathrm{da}^{\{1\}} &= \mathbb{D}\mathrm{a}^{\langle 1\rangle}\\
\mathrm{da}^{\{2\}} &= +\frac{1}{2}\,\mathbb{D}\mathrm{a}^{\langle 2\rangle}\\
\mathrm{da}^{\{3\}} &= +\frac{1}{3}\,\mathbb{D}\mathrm{a}^{\langle 3\rangle}+\frac{1}{3}\,\mathbb{D}\mathrm{a}^{\langle 1,2\rangle}\\
\mathrm{da}^{\{4\}} &= +\frac{1}{4}\,\mathbb{D}\mathrm{a}^{\langle 4\rangle}+\frac{5}{12}\,\mathbb{D}\mathrm{a}^{\langle 1,3\rangle}+\frac{5}{12}\,\mathbb{D}\mathrm{a}^{\langle 1,1,2\rangle}
\end{aligned}$$

$$\begin{aligned}
\mathrm{da}^{\{5\}} = &+\frac{1}{5}\,\mathbb{D}\mathrm{a}^{\langle 5\rangle}+\frac{9}{20}\,\mathbb{D}\mathrm{a}^{\langle 1,4\rangle}+\frac{1}{12}\,\mathbb{D}\mathrm{a}^{\langle 2,3\rangle}+\frac{3}{4}\,\mathbb{D}\mathrm{a}^{\langle 1,1,3\rangle}+\frac{1}{12}\,\mathbb{D}\mathrm{a}^{\langle 1,2,2\rangle}\\
&+\frac{1}{12}\,\mathbb{D}\mathrm{a}^{\langle 2,1,2\rangle}+\frac{3}{4}\,\mathbb{D}\mathrm{a}^{\langle 1,1,1,2\rangle}\\
\mathrm{da}^{\{6\}} = &+\frac{1}{6}\,\mathbb{D}\mathrm{a}^{\langle 6\rangle}+\frac{7}{15}\,\mathbb{D}\mathrm{a}^{\langle 1,5\rangle}+\frac{7}{60}\,\mathbb{D}\mathrm{a}^{\langle 2,4\rangle}+\frac{21}{20}\,\mathbb{D}\mathrm{a}^{\langle 1,1,4\rangle}+\frac{7}{36}\,\mathbb{D}\mathrm{a}^{\langle 1,2,3\rangle}+\frac{7}{36}\,\mathbb{D}\mathrm{a}^{\langle 2,1,3\rangle}\\
&+\frac{7}{4}\,\mathbb{D}\mathrm{a}^{\langle 1,1,1,3\rangle}+\frac{7}{36}\,\mathbb{D}\mathrm{a}^{\langle 1,1,2,2\rangle}+\frac{7}{36}\,\mathbb{D}\mathrm{a}^{\langle 1,2,1,2\rangle}+\frac{7}{36}\,\mathbb{D}\mathrm{a}^{\langle 2,1,1,2\rangle}+\frac{7}{4}\,\mathbb{D}\mathrm{a}^{\langle 1,1,1,1,2\rangle}
\end{aligned}$$

$$\begin{aligned}\mathrm{da}^{\{7\}} = {} & +\frac{1}{7}\,\mathbb{D}\mathrm{a}^{\langle 7\rangle} + \frac{10}{21}\,\mathbb{D}\mathrm{a}^{\langle 1,6\rangle} + \frac{2}{15}\,\mathbb{D}\mathrm{a}^{\langle 2,5\rangle} + \frac{1}{30}\,\mathbb{D}\mathrm{a}^{\langle 3,4\rangle} + \frac{4}{3}\,\mathbb{D}\mathrm{a}^{\langle 1,1,5\rangle} + \frac{1}{3}\,\mathbb{D}\mathrm{a}^{\langle 1,2,4\rangle}\\
& +\frac{3}{10}\,\mathbb{D}\mathrm{a}^{\langle 2,1,4\rangle} + \frac{1}{18}\,\mathbb{D}\mathrm{a}^{\langle 1,3,3\rangle} + \frac{1}{18}\,\mathbb{D}\mathrm{a}^{\langle 3,1,3\rangle} + \frac{1}{18}\,\mathbb{D}\mathrm{a}^{\langle 2,2,3\rangle} + 3\,\mathbb{D}\mathrm{a}^{\langle 1,1,1,4\rangle}\\
& +\frac{11}{18}\,\mathbb{D}\mathrm{a}^{\langle 1,1,2,3\rangle} + \frac{5}{9}\,\mathbb{D}\mathrm{a}^{\langle 1,2,1,3\rangle} + \frac{1}{2}\,\mathbb{D}\mathrm{a}^{\langle 2,1,1,3\rangle} + \frac{1}{18}\,\mathbb{D}\mathrm{a}^{\langle 1,1,3,2\rangle} + \frac{1}{18}\,\mathbb{D}\mathrm{a}^{\langle 1,3,1,2\rangle}\\
& +\frac{1}{18}\,\mathbb{D}\mathrm{a}^{\langle 3,1,1,2\rangle} + \frac{1}{18}\,\mathbb{D}\mathrm{a}^{\langle 1,2,2,2\rangle} + \frac{1}{18}\,\mathbb{D}\mathrm{a}^{\langle 2,1,2,2\rangle} + \frac{1}{18}\,\mathbb{D}\mathrm{a}^{\langle 2,2,1,2\rangle} + 5\,\mathbb{D}\mathrm{a}^{\langle 1,1,1,1,3\rangle}\\
& +\frac{2}{3}\,\mathbb{D}\mathrm{a}^{\langle 1,1,1,2,2\rangle} + \frac{11}{18}\,\mathbb{D}\mathrm{a}^{\langle 1,1,2,1,2\rangle} + \frac{5}{9}\,\mathbb{D}\mathrm{a}^{\langle 1,2,1,1,2\rangle} + \frac{1}{2}\,\mathbb{D}\mathrm{a}^{\langle 2,1,1,1,2\rangle} + 5\,\mathbb{D}\mathrm{a}^{\langle 1,1,1,1,1,2\rangle}\end{aligned}$$

Table 10 bis: $\mathrm{Da}^{\{\bullet\}}$ or rather $\underline{\mathrm{Da}}^{\{\bullet\}} := \frac{1}{(1+\bullet)!}\,\mathrm{Da}^{\{\bullet\}}$ in terms of $\underline{\mathbb{D}\mathrm{a}}^{\langle\bullet\rangle}$.

$$\begin{aligned}
\underline{\mathrm{Da}}^{\{1\}} &= +\underline{\mathbb{D}\mathrm{a}}^{\langle 1\rangle}\\
\underline{\mathrm{Da}}^{\{2\}} &= +\underline{\mathbb{D}\mathrm{a}}^{\langle 2\rangle}\\
\underline{\mathrm{Da}}^{\{3\}} &= +\underline{\mathbb{D}\mathrm{a}}^{\langle 3\rangle} + \underline{\mathbb{D}\mathrm{a}}^{\langle 1,2\rangle}\\
\underline{\mathrm{Da}}^{\{4\}} &= +\underline{\mathbb{D}\mathrm{a}}^{\langle 4\rangle} + 2\,\underline{\mathbb{D}\mathrm{a}}^{\langle 1,3\rangle} + 2\,\underline{\mathbb{D}\mathrm{a}}^{\langle 1,1,2\rangle}\\
\underline{\mathrm{Da}}^{\{5\}} &= +\underline{\mathbb{D}\mathrm{a}}^{\langle 5\rangle} + 3\,\underline{\mathbb{D}\mathrm{a}}^{\langle 1,4\rangle} + \underline{\mathbb{D}\mathrm{a}}^{\langle 2,3\rangle} + 6\,\underline{\mathbb{D}\mathrm{a}}^{\langle 1,1,3\rangle} + \underline{\mathbb{D}\mathrm{a}}^{\langle 1,2,2\rangle} + \underline{\mathbb{D}\mathrm{a}}^{\langle 2,1,2\rangle} + 6\,\underline{\mathbb{D}\mathrm{a}}^{\langle 1,1,1,2\rangle}\\
\underline{\mathrm{Da}}^{\{6\}} &= +\underline{\mathbb{D}\mathrm{a}}^{\langle 6\rangle} + 4\,\underline{\mathbb{D}\mathrm{a}}^{\langle 1,5\rangle} + 2\,\underline{\mathbb{D}\mathrm{a}}^{\langle 2,4\rangle} + 12\,\underline{\mathbb{D}\mathrm{a}}^{\langle 1,1,4\rangle} + 4\,\underline{\mathbb{D}\mathrm{a}}^{\langle 1,2,3\rangle} + 4\,\underline{\mathbb{D}\mathrm{a}}^{\langle 2,1,3\rangle}\\
&\quad +24\,\underline{\mathbb{D}\mathrm{a}}^{\langle 1,1,1,3\rangle} + 4\,\underline{\mathbb{D}\mathrm{a}}^{\langle 1,1,2,2\rangle} + 4\,\underline{\mathbb{D}\mathrm{a}}^{\langle 1,2,1,2\rangle} + 4\,\underline{\mathbb{D}\mathrm{a}}^{\langle 2,1,1,2\rangle} + 24\,\underline{\mathbb{D}\mathrm{a}}^{\langle 1,1,1,1,2\rangle}\\
\underline{\mathrm{Da}}^{\{7\}} &= +\underline{\mathbb{D}\mathrm{a}}^{\langle 7\rangle} + 5\,\underline{\mathbb{D}\mathrm{a}}^{\langle 1,6\rangle} + 3\,\underline{\mathbb{D}\mathrm{a}}^{\langle 2,5\rangle} + \underline{\mathbb{D}\mathrm{a}}^{\langle 3,4\rangle} + 20\,\underline{\mathbb{D}\mathrm{a}}^{\langle 1,1,5\rangle} + 10\,\underline{\mathbb{D}\mathrm{a}}^{\langle 1,2,4\rangle} + 9\,\underline{\mathbb{D}\mathrm{a}}^{\langle 2,1,4\rangle}\\
&\quad +2\,\underline{\mathbb{D}\mathrm{a}}^{\langle 1,3,3\rangle} + 2\,\underline{\mathbb{D}\mathrm{a}}^{\langle 3,1,3\rangle} + 3\,\underline{\mathbb{D}\mathrm{a}}^{\langle 2,2,3\rangle} + 60\,\underline{\mathbb{D}\mathrm{a}}^{\langle 1,1,1,4\rangle} + 22\,\underline{\mathbb{D}\mathrm{a}}^{\langle 1,1,2,3\rangle}\\
&\quad +20\,\underline{\mathbb{D}\mathrm{a}}^{\langle 1,2,1,3\rangle} + 18\,\underline{\mathbb{D}\mathrm{a}}^{\langle 2,1,1,3\rangle} + 2\,\underline{\mathbb{D}\mathrm{a}}^{\langle 1,3,1,2\rangle} + 2\,\underline{\mathbb{D}\mathrm{a}}^{\langle 3,1,1,2\rangle} + 2\,\underline{\mathbb{D}\mathrm{a}}^{\langle 1,1,3,2\rangle}\\
&\quad +3\,\underline{\mathbb{D}\mathrm{a}}^{\langle 1,2,2,2\rangle} + 3\,\underline{\mathbb{D}\mathrm{a}}^{\langle 2,1,2,2\rangle} + 3\,\underline{\mathbb{D}\mathrm{a}}^{\langle 2,2,1,2\rangle} + 120\,\underline{\mathbb{D}\mathrm{a}}^{\langle 1,1,1,1,3\rangle} + 24\,\underline{\mathbb{D}\mathrm{a}}^{\langle 1,1,1,2,2\rangle}\\
&\quad +22\,\underline{\mathbb{D}\mathrm{a}}^{\langle 1,1,2,1,2\rangle} + 20\,\underline{\mathbb{D}\mathrm{a}}^{\langle 1,2,1,1,2\rangle} + 18\,\underline{\mathbb{D}\mathrm{a}}^{\langle 2,1,1,1,2\rangle} + 120\,\underline{\mathbb{D}\mathrm{a}}^{\langle 1,1,1,1,1,2\rangle}
\end{aligned}$$

Table 11: Inductive construction of $\mathrm{Da}^{\{\bullet\}}$ or rather $\underline{\mathrm{Da}}^{\{\bullet\}} := \frac{1}{(1+r(\bullet))!}\,\mathrm{Da}^{\{\bullet\}}$.

$$\begin{aligned}
2\,\underline{\mathrm{Da}}^{\{1\}} &:= \mathrm{Da}^{\{1\}} = \mathrm{Dn}^{\{1\}} \qquad (\textit{recall that } \mathrm{Dn}^{\{1\}}.f := -\frac{f''}{f'})\\
(n+1)\,\underline{\mathrm{Da}}^{\{n\}} &:= -\partial\,\underline{\mathrm{Da}}^{\{n-1\}} - \underline{\mathrm{coDa}}^{\{n\}} \qquad (\forall n \geqslant 2)
\end{aligned}$$

$$
\begin{aligned}
\underline{\mathrm{coDa}}^{\{1\}} &= 0\\
\underline{\mathrm{coDa}}^{\{2\}} &= +\underline{\mathrm{Da}}^{\{1^{2}\}}\\
\underline{\mathrm{coDa}}^{\{3\}} &= 0\\
\underline{\mathrm{coDa}}^{\{4\}} &= +3/2\,\underline{\mathrm{Da}}^{\{2^{2}\}}\\
\underline{\mathrm{coDa}}^{\{5\}} &= 0\\
\underline{\mathrm{coDa}}^{\{6\}} &= +2\,\underline{\mathrm{Da}}^{\{3^{2}\}} + \underline{\mathrm{Da}}^{\{2^{3}\}}\\
\underline{\mathrm{coDa}}^{\{7\}} &= 0\\
\underline{\mathrm{coDa}}^{\{8\}} &= +5/2\,\underline{\mathrm{Da}}^{\{4^{2}\}} + 9/8\,\underline{\mathrm{Da}}^{\{2^{4}\}}\\
\underline{\mathrm{coDa}}^{\{9\}} &= +8/3\,\underline{\mathrm{Da}}^{\{3^{3}\}} + 2\,\underline{\mathrm{Da}}^{\{3,2^{3}\}}\\
\underline{\mathrm{coDa}}^{\{10\}} &= +3\,\underline{\mathrm{Da}}^{\{5^{2}\}} + 2\,\underline{\mathrm{Da}}^{\{4,3^{2}\}} + \underline{\mathrm{Da}}^{\{4,2^{3}\}} + 3/2\,\underline{\mathrm{Da}}^{\{2^{5}\}}\\
\underline{\mathrm{coDa}}^{\{11\}} &= +9/2\,\underline{\mathrm{Da}}^{\{3,2^{4}\}}\\
\underline{\mathrm{coDa}}^{\{12\}} &= +7/2\,\underline{\mathrm{Da}}^{\{6^{2}\}} + 5\,\underline{\mathrm{Da}}^{\{4^{3}\}} + 5\,\underline{\mathrm{Da}}^{\{3^{4}\}} + 27/8\,\underline{\mathrm{Da}}^{\{4,2^{4}\}} + 5\,\underline{\mathrm{Da}}^{\{3^{2},2^{3}\}} + 35/16\,\underline{\mathrm{Da}}^{\{2^{6}\}}\\
\underline{\mathrm{coDa}}^{\{13\}} &= +5\,\underline{\mathrm{Da}}^{\{5,4^{2}\}} + 32/3\,\underline{\mathrm{Da}}^{\{4,3^{3}\}} + 9/4\,\underline{\mathrm{Da}}^{\{5,2^{4}\}} + 8\,\underline{\mathrm{Da}}^{\{4,3,2^{3}\}} + 9\,\underline{\mathrm{Da}}^{\{3,2^{5}\}}\\
\underline{\mathrm{coDa}}^{\{14\}} &= +4\,\underline{\mathrm{Da}}^{\{7^{2}\}} + 5/2\,\underline{\mathrm{Da}}^{\{6,4^{2}\}} + 8\,\underline{\mathrm{Da}}^{\{5,3^{3}\}} + 5\,\underline{\mathrm{Da}}^{\{4^{2},3^{2}\}} + 9/8\,\underline{\mathrm{Da}}^{\{6,2^{4}\}}\\
&\quad +6\,\underline{\mathrm{Da}}^{\{5,3,2^{3}\}} + 5/2\,\underline{\mathrm{Da}}^{\{4^{2},2^{3}\}} + 15/2\,\underline{\mathrm{Da}}^{\{4,2^{5}\}} + 63/4\,\underline{\mathrm{Da}}^{\{3^{2},2^{4}\}} + 27/8\,\underline{\mathrm{Da}}^{\{2^{7}\}}\\
\underline{\mathrm{coDa}}^{\{15\}} &= +8\,\underline{\mathrm{Da}}^{\{5^{3}\}} + 16/3\,\underline{\mathrm{Da}}^{\{6,3^{3}\}} + 8\,\underline{\mathrm{Da}}^{\{5,4,3^{2}\}} + 4\,\underline{\mathrm{Da}}^{\{6,3,2^{3}\}} + 4\,\underline{\mathrm{Da}}^{\{5,4,2^{3}\}}\\
&\quad +32/3\,\underline{\mathrm{Da}}^{\{3^{5}\}} + 27\,\underline{\mathrm{Da}}^{\{4,3,2^{4}\}} + 6\,\underline{\mathrm{Da}}^{\{5,2^{5}\}} + 40/3\,\underline{\mathrm{Da}}^{\{3^{3},2^{3}\}} + 35/2\,\underline{\mathrm{Da}}^{\{3,2^{6}\}}\\
\underline{\mathrm{coDa}}^{\{16\}} &= +9/2\,\underline{\mathrm{Da}}^{\{8^{2}\}} + 9\,\underline{\mathrm{Da}}^{\{6,5^{2}\}} + 8/3\,\underline{\mathrm{Da}}^{\{7,3^{3}\}} + 6\,\underline{\mathrm{Da}}^{\{6,4,3^{2}\}} + 105/8\,\underline{\mathrm{Da}}^{\{4^{4}\}}\\
&\quad +2\,\underline{\mathrm{Da}}^{\{7,3,2^{3}\}} + 3\,\underline{\mathrm{Da}}^{\{6,4,2^{3}\}} + 35\,\underline{\mathrm{Da}}^{\{4,3^{4}\}} + 9/2\,\underline{\mathrm{Da}}^{\{6,2^{5}\}} + 45/2\,\underline{\mathrm{Da}}^{\{5,3,2^{4}\}}\\
&\quad +189/16\,\underline{\mathrm{Da}}^{\{4^{2},2^{4}\}} + 35\,\underline{\mathrm{Da}}^{\{4,3^{2},2^{3}\}} + 245/16\,\underline{\mathrm{Da}}^{\{4,2^{6}\}} + 81/2\,\underline{\mathrm{Da}}^{\{3^{2},2^{5}\}}\\
&\quad +693/128\,\underline{\mathrm{Da}}^{\{2^{8}\}}\\
\underline{\mathrm{coDa}}^{\{17\}} &= +6\,\underline{\mathrm{Da}}^{\{7,5^{2}\}} + 4\,\underline{\mathrm{Da}}^{\{7,4,3^{2}\}} + 30\,\underline{\mathrm{Da}}^{\{5,4^{3}\}} + 2\,\underline{\mathrm{Da}}^{\{7,4,2^{3}\}} + 30\,\underline{\mathrm{Da}}^{\{5,3^{4}\}}\\
&\quad +128/3\,\underline{\mathrm{Da}}^{\{4^{2},3^{3}\}} + 18\,\underline{\mathrm{Da}}^{\{6,3,2^{4}\}} + 81/4\,\underline{\mathrm{Da}}^{\{5,4,2^{4}\}} + 30\,\underline{\mathrm{Da}}^{\{5,3^{2},2^{3}\}}\\
&\quad +32\,\underline{\mathrm{Da}}^{\{4^{2},3,2^{3}\}} + 3\,\underline{\mathrm{Da}}^{\{7,2^{5}\}} + 105/8\,\underline{\mathrm{Da}}^{\{5,2^{6}\}} + 72\,\underline{\mathrm{Da}}^{\{4,3,2^{5}\}}\\
&\quad +105/2\,\underline{\mathrm{Da}}^{\{3^{3},2^{4}\}} + 135/4\,\underline{\mathrm{Da}}^{\{3,2^{7}\}}
\end{aligned}
$$

$$\begin{aligned}
\underline{\mathrm{coDa}}^{\{18\}} = {} & +5\,\underline{\mathrm{Da}}^{\{9^2\}} + 35/3\,\underline{\mathrm{Da}}^{\{6^3\}} + 3\,\underline{\mathrm{Da}}^{\{8,5^2\}} + 35/2\,\underline{\mathrm{Da}}^{\{5^2,4^2\}} + 25\,\underline{\mathrm{Da}}^{\{6,4^3\}} \\
& +2\,\underline{\mathrm{Da}}^{\{8,4,3^2\}} + \underline{\mathrm{Da}}^{\{8,4,2^3\}} + 15\,\underline{\mathrm{Da}}^{\{4^3,3^2\}} + 224/3\,\underline{\mathrm{Da}}^{\{5,4,3^3\}} + 25\,\underline{\mathrm{Da}}^{\{6,3^4\}} \\
& +15/2\,\underline{\mathrm{Da}}^{\{4^3,2^3\}} + 135/8\,\underline{\mathrm{Da}}^{\{6,4,2^4\}} + 220/9\,\underline{\mathrm{Da}}^{\{3^6\}} + 63/8\,\underline{\mathrm{Da}}^{\{5^2,2^4\}} \\
& +56\,\underline{\mathrm{Da}}^{\{5,4,3,2^3\}} + 27/2\,\underline{\mathrm{Da}}^{\{7,3,2^4\}} + 25\,\underline{\mathrm{Da}}^{\{6,3^2,2^3\}} + 3/2\,\underline{\mathrm{Da}}^{\{8,2^5\}} \\
& +135/4\,\underline{\mathrm{Da}}^{\{4^2,2^5\}} + 175/16\,\underline{\mathrm{Da}}^{\{6,2^6\}} + 567/4\,\underline{\mathrm{Da}}^{\{4,3^2,2^4\}} + 110/3\,\underline{\mathrm{Da}}^{\{3^4,2^3\}} \\
& +63\,\underline{\mathrm{Da}}^{\{5,3,2^5\}} + 243/8\,\underline{\mathrm{Da}}^{\{4,2^7\}} + 385/4\,\underline{\mathrm{Da}}^{\{3,3,2^6\}} + 143/16\,\underline{\mathrm{Da}}^{\{2^9\}}
\end{aligned}$$

15 Tables: how construction-sensitive is $\mathcal{E}_1$?

The main point of this section is to find out how much, or how little, the first-order ultraexponentials $\mathcal{E}_1^{[f]}$ (i.e. those that verify $\mathcal{E}_1^{[f]} \circ T = E \circ \mathcal{E}_1^{[f]}$) depend on the auxiliary germ f used to construct them. The answer will turn out to be: *surpisingly little.* We also propose to illustrate the *extremely slow* onset of the stair-case phenomenon, which says that, when f_1 and f_2 drift far apart, the corresponding $\mathcal{E}_1^{[f_1]}$ and $\mathcal{E}_1^{[f_2]}$ tend to differ by post-composition by a stair-case function.

We take over the notations of §8.4. We consider auxiliary real-analytic self-mappings f of $\mathbb{R}^+$. We denote $f^{\Diamond}, {}^{\Diamond}f$ their normalisers at $+\infty$ (they conjugate f with E) and $f^{\ddagger}, {}^{\ddagger}f$ their normalisers at 0^+ (they conjugate f with the dilation $\delta_c : x \mapsto c\,x = f'(0)\,x\ (c > 1)$)

The most convenient tools for comparing two ultraexponentials $\mathcal{E}_1^{[f]}$ is the periodic *connector* $P_1^{[f_1,f_2]}$ and its Fourier coefficients. The connector is characterised by

$$\mathcal{E}_1^{[f_1]} \circ P_1^{[f_1,f_2]} = \mathcal{E}_1^{[f_2]} \qquad \big(\, P_1^{[f_1,f_2]} \circ T = T \circ P_1^{[f_1,f_2]} \,\big) \tag{15.1}$$

Since we shall be dealing mostly with first-order ultraexponentials, we shall most of the time drop the lower index 1.

Practically , we must express the *connector* in terms of the two kinds of *normalisers*. The formula which does this reads:

$$\begin{aligned}
P^{[f_1,f_2]} &:= \delta_{\gamma_1}^{-1} \circ L \circ f_1^{\ddagger} \circ {}^{\Diamond}f_1 \circ f_2^{\Diamond} \circ {}^{\ddagger}f_2 \circ E \circ \delta_{\gamma_2} &(15.2)\\
&:= T^{n_1} \circ \delta_{\gamma_1}^{-1} \circ L \circ f_1^{\ddagger} \circ f_1^{-n_1-n_0} \circ {}^{\Diamond}f_1 \circ f_2^{\Diamond} \circ f_2^{n_0+n_2} \circ {}^{\ddagger}f_2 \circ E \circ \delta_{\gamma_2} \circ T^{-n_2}
\end{aligned}$$

Due to the defining properties of the normalisers, the second line is just a tautological re-writing of the first, but it contains three free parameters (the integers n_0, n_1, n_2), which are extremely useful for optimising computational efficiency.

15.1 The $c\,(e^x-1)$-based ultraexponentials for c near 1.

We set $P := P^{[f_1,f_2]}$ with $f_1(x) := 2\,(e^x-1)$, $f_2(x) := (1+2^{-p})\,(e^x-1)$. When p increases, f_2 goes to the identity-tangent (at 0^+) germ e^x-1, but rather than the plain connector

$$P^{[f_1,f_2]}(x) = x + a_0 + \sum_{n\in\mathbb{Z}^*} a_n\, e^{2\pi i n x} \qquad (a_{-n} = \bar{a}_n) \tag{15.3}$$

it is the shift-corrected, or *shift-free*, connector

$$\widetilde{P}^{[f_1,f_2]}(x) = P^{[f_1,f_2]}(x-a_0) = x + \sum_{n\in\mathbb{Z}^*} \tilde{a}_n\, e^{2\pi i n x} \qquad (\tilde{a}_{-n} = \bar{\tilde{a}}_n) \tag{15.4}$$

that goes to a limit. So, alongside the shift a_0 of P and its variation h :

$$h := \sup(P(x)-x) - \min(P(x)-x))$$

we tabulate its shift-corrected coefficients $\tilde{a}_n$ up to $k=6$.

p	:	h	a_0
1	:	0.000173	0.812416342070693331
2	:	0.000228	3.714151639563557264
3	:	0.000244	12.161685214161111968
4	:	0.000248	34.469041109222426463
5	:	0.000249	90.043900828527882086
6	:	0.000250	223.248440548965845345

p	:	$10^5\,\tilde{a}_1$	$10^7\,\tilde{a}_2$
1	:	$-2.59184923 + 3.48082750\,i$	$-3.93628643 - 1.02092641\,i$
2	:	$-3.42656414 + 4.57522456\,i$	$-5.15710254 - 1.35080762\,i$
3	:	$-3.66952951 + 4.89142090\,i$	$-5.50831912 - 1.44687006\,i$
4	:	$-3.73566860 + 4.97731255\,i$	$-5.60360689 - 1.47302175\,i$
5	:	$-3.75297060 + 4.99976900\,i$	$-5.62851173 - 1.47986314\,i$
6	:	$-3.75741801 + 5.00551564\,i$	$-5.63494008 - 1.48065193\,i$

p	:	$10^8\,\tilde{a}_3$	$10^{10}\,\tilde{a}_4$	$10^{11}\,\tilde{a}_5$
1	:	$-0.96152 - 0.61508\,i$	$-4.945 - 2.051\,i$	$-3.38 + 0.41\,i$
2	:	$-1.26263 - 0.80316\,i$	$-6.512 - 2.664\,i$	$-4.45 + 0.56\,i$
3	:	$-1.34955 - 0.85702\,i$	$-6.966 - 2.838\,i$	$-4.76 + 0.61\,i$
4	:	$-1.37315 - 0.87162\,i$	$-7.090 - 2.885\,i$	$-4.84 + 0.62\,i$
5	:	$-1.37932 - 0.87543\,i$	$-7.122 - 2.898\,i$	$-4.87 + 0.62\,i$
6	:	$-1.37451 - 0.87571\,i$	$-7.057 - 2.901\,i$	$-8.67 + 0.62\,i$

15.2 The $c\,(e^x – 1)$-based ultraexponentials for c moderate.

We are now to compare the ultraexponentials constructed from:

$$f_1(x) := 2\,(e^x - 1) \quad , \quad f_2(x) := 2^p\,(e^x - 1) \;\; for \;\; p \leqslant 10 \tag{15.5}$$

There is no point here in making a shift-correction, so we revert to the normal definition of the connector:

$$P^{[f_1,f_2]}(x) = x + a_0 + \sum_{n \in \mathbb{Z}^*} a_n\, e^{2\pi i n x} \qquad (a_{-n} = \bar{a}_n) \tag{15.6}$$

The following tables show how small the connector remains even when c ranges over the interval $[2, 2^{10}]$. Thus, the main index of smallness, the variation h, remains under 1/100 (resp. 1/10) as long as c remains within the interval $[2, 17]$ (resp. $[2, 1045]$).

p	h	a_0
2	0.0010	−0.039730218808207703
3	0.0039	0.140206086332004488
4	0.0097	0.298993371423610350
5	0.0191	0.428478926333908364
6	0.0317	0.534974918731011484
7	0.0467	0.624226819271341737
8	0.0628	0.700373568094414955
9	0.0794	0.766327669818064974
10	0.0958	0.824172830948231323

p	$10^2\,\lvert a_1\rvert$	$10^3\,\lvert a_2\rvert$	$10^4\,\lvert a_3\rvert$	$10^5\,\lvert a_4\rvert$	$10^6\,\lvert a_5\rvert$	$10^7\,\lvert a_6\rvert$	$10^8\,\lvert a_7\rvert$	$10^9\,\lvert a_8\rvert$
2	0.0273	0.0028	0.0008	0.0003	0.0002	0.0001	0.0001	0.0001
3	0.0980	0.0123	0.0038	0.0016	0.0009	0.0006	0.0005	0.0005
4	0.2428	0.0409	0.0160	0.0081	0.0044	0.0024	0.0016	0.0016
5	0.4777	0.1103	0.0567	0.0389	0.0285	0.0197	0.0113	0.0054
6	0.7928	0.2448	0.1572	0.1425	0.1448	0.1475	0.1394	0.1100
7	1.1636	0.4639	0.3521	0.3930	0.5155	0.7085	0.9601	1.2221
8	1.5643	0.7798	0.6730	0.8664	1.3727	2.3544	4.1118	7.0499
9	1.9735	1.1956	1.1511	1.6246	2.9451	5.9621	12.5587	26.5791
10	2.3761	1.7072	1.8187	2.7219	5.4000	12.3830	30.0981	74.5903

15.3 The $c\,(e^x – 1)$-based ultraexponentials for c large and the tardy onset of the staircase regime.

We still compare $f_1(x) := 2\,(e^x - 1)$ and $f_2(x) := 2^p\,(e^x - 1)$ but for very large values of p, with a view to showing how *slowly* the connector (minus the shift a_0) converges to the odd step function *ent*:

$$\begin{aligned} ent(x) &:= n+\frac{1}{2} \quad if \quad n < x < n+1 \quad (n \in \mathbb{Z}) \\ ent(x) &:= n \quad if \quad x = n \qquad\qquad (n \in \mathbb{Z}) \end{aligned}$$

So we expand the limit connector P^∞ as a Fourier series:

$$P^\infty(x) = a_0^\infty + \mathrm{ent}(x) = a_0^\infty + \sum_{n \in \mathbb{Z}*} a_n^\infty\, e^{2\,\pi\,i\,n\,x} \quad with \quad a_n^\infty := \frac{1}{2\,\pi\,i\,n} \tag{15.7}$$

and examine how the Fourier coefficients of the current connector converge to their limit values. We separate their arguments and absolute values, as follows:

$$P^{[f_1,f_2]}(x) = x + a_0 + \sum_{n \in \mathbb{Z}*} |a_n|\, e^{-2\,\pi\,i\,(\theta_n + \frac{1}{4})}\, e^{2\,\pi\,i\,n\,x} \qquad (a_{-n} = \bar{a}_n) \tag{15.8}$$

so as to have $|a_n/a_n^\infty| \nearrow 1$ and $\theta_n \searrow 0$ as p goes to $+\infty$. For greater clarity, we do not take the main determination of the argument θ_n $(mod\ 2\pi)$, but an exact determination in $\mathbb{R}^+$, followed by continuity, backwards from the limit value 0. We also tabulate the variation h and minimal slope s of the connector $P^{[f_1,f_2]}$ as well as the corresponding data $h^\sharp$ and $s^\sharp$ for the reverse connector $P^{[f_1,f_2]}$. We may notice that the reverse pair $(h^\sharp, s^\sharp)$ goes to its limit $(1,0)$ slightly less sluggishly than the direct pair (h,s).

p	h	$h^\sharp$	s	$s^\sharp$	a_0
10	0.096	0.537	0.728	0.750	0.824172830948231323
20	0.218	0.600	0.462	0.516	1.169249345157300399
30	0.285	0.621	0.353	0.403	1.338801550912898204
40	0.328	0.662	0.298	0.336	1.445091229720390133
50	0.360	0.678	0.262	0.289	1.520177351544718161
60	0.384	0.686	0.237	0.256	1.577114227264200653
70	0.404	0.705	0.219	0.230	1.622362811018353058
80	0.420	0.715	0.205	0.210	1.659543136186266516
90	0.434	0.725	0.194	0.193	1.690866157220920835
100	0.446	0.736	0.185	0.179	1.717770910145828546
110	0.457	0.743	0.178	0.167	1.741240989738605260
120	0.467	0.749	0.171	0.157	1.761975329007507570
130	0.475	0.755	0.165	0.148	1.780486537845824490
140	0.483	0.760	0.160	0.140	1.797160570587154804
150	0.490	0.764	0.156	0.133	1.812294524905533289
160	0.497	0.768	0.152	0.127	1.826121469319542517
170	0.503	0.772	0.148	0.121	1.838827260907611583
180	0.508	0.776	0.145	0.116	1.850562243985180026
190	0.513	0.779	0.142	0.112	1.861449578437993691
200	0.518	0.782	0.140	0.108	1.871591290595732904
250	0.539	0.794	0.129	0.091	1.91368817224511269.
300	0.555	0.805	0.121	0.079	1.9457952784130651..
350	0.568	0.812	0.115	0.070	1.971459590821952...
400	0.578	0.818	0.110	0.063	1.99266499489222....
450	0.587	0.824	0.106	0.058	2.0106234087384.....
500	0.595	0.829	0.103	0.053	2.026124575139......
600	0.608	0.837	0.098	0.046	2.051762383852......
700	0.619	0.843	0.093	0.041	2.07233675865.......
800	0.628	0.848	0.090	0.037	2.0893952176.........
900	0.636	0.852	0.087	0.034	2.1038857620.........
1000	0.642	0.856	0.085	0.031	2.116427741..........
1500	0.666	0.869	0.077	0.022	2.16129606...........
2000	0.682	0.878	0.072	0.018	2.1901946............
∞	1	1	0	0	a_0^∞

p	θ_1	θ_2	θ_3	θ_4	θ_5	θ_6	θ_7	θ_8	θ_9	θ_{10}
10	0.415	0.704	0.946	1.160	1.378	1.613	1.863	1.124	1.394	1.672
20	0.309	0.512	0.693	0.856	0.998	1.111	1.184	1.247	1.334	1.441
30	0.259	0.423	0.571	0.709	0.840	0.963	1.078	1.180	1.251	1.202
40	0.227	0.369	0.495	0.614	0.728	0.838	0.944	1.049	1.152	1.256
50	0.205	0.331	0.443	0.547	0.648	0.745	0.841	0.935	1.028	1.121
60	0.189	0.303	0.404	0.498	0.588	0.676	0.762	0.846	0.930	1.013
70	0.176	0.281	0.373	0.460	0.542	0.622	0.700	0.777	0.853	0.929
80	0.165	0.263	0.348	0.428	0.504	0.578	0.650	0.721	0.791	0.860
90	0.157	0.248	0.328	0.402	0.473	0.542	0.608	0.674	0.739	0.803
100	0.149	0.236	0.311	0.381	0.447	0.511	0.574	0.635	0.696	0.756
110	0.143	0.225	0.296	0.362	0.425	0.485	0.544	0.602	0.659	0.715
120	0.137	0.216	0.283	0.346	0.405	0.462	0.518	0.573	0.626	0.680
130	0.132	0.207	0.272	0.331	0.388	0.442	0.495	0.547	0.598	0.649
140	0.128	0.200	0.262	0.319	0.373	0.425	0.475	0.525	0.573	0.621
150	0.124	0.193	0.253	0.307	0.359	0.409	0.457	0.505	0.551	0.597
160	0.120	0.187	0.245	0.297	0.347	0.395	0.441	0.487	0.531	0.575
170	0.117	0.182	0.237	0.288	0.336	0.382	0.427	0.470	0.513	0.555
180	0.114	0.177	0.230	0.279	0.326	0.370	0.413	0.455	0.496	0.537
190	0.111	0.172	0.224	0.272	0.316	0.359	0.401	0.442	0.481	0.520
200	0.109	0.168	0.218	0.264	0.308	0.349	0.390	0.429	0.467	0.505
250	0.099	0.151	0.195	0.235	0.273	0.310	0.345	0.379	0.412	0.444
300	0.091	0.138	0.178	0.214	0.248	0.281	0.312	0.342	0.371	0.400
350	0.085	0.129	0.165	0.198	0.229	0.258	0.287	0.314	0.341	0.367
400	0.080	0.121	0.155	0.185	0.214	0.241	0.267	0.292	0.316	0.340
450	0.076	0.114	0.146	0.174	0.201	0.226	0.250	0.274	0.296	0.319
500	0.073	0.109	0.139	0.165	0.190	0.214	0.237	0.258	0.280	0.300
600	0.068	0.100	0.127	0.151	0.173	0.195	0.215	0.234	0.253	0.272
700	0.065	0.093	0.118	0.140	0.160	0.180	0.198	0.216	0.233	0.250
800	0.060	0.088	0.111	0.131	0.150	0.168	0.185	0.201	0.217	0.232
900	0.057	0.084	0.105	0.124	0.142	0.158	0.174	0.189	0.204	0.218
1000	0.055	0.080	0.100	0.118	0.134	0.150	0.165	0.179	0.193	0.206
1500	0.047	0.067	0.083	0.098	0.111	0.123	0.135	0.146	0.156	0.167
2000	0.042	0.060	0.074	0.086	0.097	0.107	0.117	0.126	0.135	0.144
∞	0	0	0	0	0	0	0	0	0	0

p	$\lvert a_1/a_1^\infty\rvert$	$\lvert a_2/a_2^\infty\rvert$	$\lvert a_3/a_3^\infty\rvert$	$\lvert a_4/a_4^\infty\rvert$	$\lvert a_5/a_5^\infty\rvert$
10	0.149296837	0.021453706	0.003428333	0.000684112	0.000169649
20	0.329302890	0.117523401	0.040360172	0.013272026	0.004168146
30	0.416717927	0.205025907	0.099772889	0.047178160	0.021503420
40	0.467646656	0.267320358	0.154851766	0.088523369	0.049567335
50	0.502022810	0.312267954	0.199914399	0.127726793	0.080696615
60	0.527421403	0.346248776	0.236180707	0.162052274	0.110668066
70	0.547313467	0.373063558	0.265739021	0.191461410	0.137945316
80	0.563528612	0.394963283	0.290298293	0.216651171	0.162251474
90	0.577134122	0.413337163	0.311097215	0.238391932	0.183791883
100	0.588801263	0.429083602	0.329014185	0.257346891	0.202913290
110	0.598977252	0.442809139	0.344676732	0.274046130	0.219970016
120	0.607973836	0.454938910	0.358540854	0.288903536	0.235278286
130	0.616016376	0.465780653	0.370944468	0.302241106	0.249105813
140	0.623272720	0.475563356	0.382142940	0.314310602	0.261674356
150	0.629871025	0.484461495	0.392332835	0.325310604	0.273166209
160	0.635911224	0.492610765	0.401668051	0.335399373	0.283731147
170	0.641472650	0.500118598	0.410270948	0.344704393	0.293492627
180	0.646619276	0.507071387	0.418240181	0.353329448	0.302552939
190	0.651403390	0.513539570	0.425656301	0.361359890	0.310997359
200	0.655868247	0.519581277	0.432585833	0.368866583	0.318897448
250	0.674493354	0.544852007	0.461604506	0.400333347	0.352064175
300	0.688802683	0.564360793	0.484061392	0.424724290	0.377817421
350	0.700304938	0.580114526	0.502244186	0.444506937	0.398733355
400	0.709849414	0.593241284	0.517435215	0.461064154	0.416262213
450	0.717958579	0.604435487	0.530422594	0.475244851	0.431295002
500	0.724975215	0.614153538	0.541724009	0.487606100	0.444416255
600	0.736608533	0.630333048	0.560598616	0.508300206	0.466423770
700	0.745962428	0.643404191	0.575903846	0.525130397	0.484364679
800	0.753724012	0.654292064	0.588692809	0.539229958	0.49942697.
900	0.760317429	0.663570663	0.59962076.	0.55130503.	0.5123512..
1000	0.76602147.	0.67161906.	0.60912173.	0.5618241..	0.5236294..
1500	0.78636529.	0.7004769..	0.6433523..	0.599885...	0.564591...
2000	0.7993700..	0.719037...	0.66549....	0.62464....	0.59136....
∞	1	1	1	1	1

p	$\lvert a_6/a_6^\infty\rvert$	$\lvert a_7/a_7^\infty\rvert$	$\lvert a_8/a_8^\infty\rvert$	$\lvert a_9/a_9^\infty\rvert$	$\lvert a_{10}/a_{10}^\infty\rvert$
10	0.000046683	0.000013238	0.000003749	0.000001043	0.000000282
20	0.001283575	0.000445047	0.000208345	0.000116199	0.000065900
30	0.009340991	0.003791385	0.001381776	0.000414870	0.000120156
40	0.027047815	0.014294459	0.007248175	0.003470368	0.001520955
50	0.050211698	0.030677830	0.018344810	0.010693432	0.006043869
60	0.074892433	0.050102507	0.033075126	0.021508905	0.013754013
70	0.098901558	0.070387963	0.049652546	0.034677737	0.023956263
80	0.121290785	0.090264151	0.066771801	0.049050118	0.035756545
90	0.141773365	0.109105539	0.083633365	0.063791915	0.048386692
100	0.160371726	0.126665038	0.099803498	0.078368323	0.061284842
110	0.177234877	0.142897512	0.115079534	0.092466642	0.074076160
120	0.192550889	0.157857017	0.129394374	0.105922686	0.086528013
130	0.206507693	0.171640660	0.142755916	0.118665855	0.098507291
140	0.219277279	0.184359279	0.155210684	0.130681905	0.109947179
150	0.231010797	0.196122880	0.166822761	0.141988939	0.120823425
160	0.241838505	0.207033969	0.177661973	0.152622401	0.131138156
170	0.251871641	0.217185008	0.187797519	0.162625925	0.140909146
180	0.261204901	0.226657948	0.197294756	0.172045888	0.150162844
190	0.269918911	0.235524744	0.206213766	0.180928278	0.158929932
200	0.278082467	0.243848292	0.214608922	0.189316977	0.167242550
250	0.312442041	0.279013554	0.250259078	0.225172567	0.203053745
300	0.339186815	0.306482580	0.278242892	0.253496234	0.231564468
350	0.360938616	0.328862449	0.301094401	0.276694079	0.255003099
400	0.379188669	0.347661176	0.320314928	0.296237755	0.274789444
450	0.394856629	0.363815608	0.336847756	0.313066457	0.291848223
500	0.408546553	0.377943064	0.351317893	0.327807567	0.306803955
600	0.431542271	0.401703269	0.375680865	0.352651694	0.332034670
700	0.450325236	0.42114217.	0.39564061.	0.37303069.	0.35275422.
800	0.46612281.	0.43751635.	0.41247557.	0.3902389..	0.3702679..
900	0.4797002..	0.4516092..	0.4269828..	0.4050839..	0.385390...
1000	0.4915659..	0.4639414..	0.439692...	0.418102...	0.398665...
1500	0.534807...	0.509018...	0.486273...	0.46593....	0.44754....
2000	0.56319....	0.538733...	0.51709....	0.4976.....	0.4801.....
∞	1	1	1	1	1

15.4 Ultraexponentials based on $c\,(e^x–1)$, $c\,x\,e^x$, $2c\sinh(x)$.

Here, we simply tabulate the variations $h^{i,j}$ and the shifts $a_0^{i,j}$ of the connectors $P_{i,j} := P^{[f_i,f_j]}$ linking the ultraexponentials based on the three germs

$$f_1(x) := c\,(e^x - 1) \quad ; \quad f_2(x) := c\,x\,e^x \qquad f_3(x) := 2\,c\,\sinh(x) \tag{15.9}$$

for $c = 1 + 2^{-p}$ (first table) and $c = 2^p$ (second table). As the tables show, these connectors remain very small for all values of c and even tend to the identity as c goes to $+\infty$. This latter fact holds, and can be rigorously established, for a large class of germs of exponentiality 1.

Connectors with $c = 1 + 2^{-p}$:

p	$h^{1,2}$	$a_0^{1,2}$	$h^{1,3}$	$a_0^{1,3}$
1	0.00422	1.395126423236	0.00076	−2.127251395721
2	0.00407	2.714522127338	0.00052	−4.905079538365
3	0.00400	5.403319183800	0.00040	−11.578143679197
4	0.00397	10.852572866710	0.00035	−27.395815918790
5	0.00395	21.839053523673	0.00032	−64.253570081964
6	0.00394	43.910979716185	0.00031	−148.722696491661

Connectors with $c = 2^p$:

p	$h^{1,2}$	$a_0^{1,2}$	$h^{1,3}$	$a_0^{1,3}$
1	0.00452	0.748324404654	0.00125	−0.972320429299
2	0.00591	0.318250096038	0.00425	−0.333303223359
3	0.00904	0.186108126535	0.01342	−0.171056248431
4	0.01428	0.124282176384	0.02406	−0.103958740427
5	0.02045	0.089235571607	0.03172	−0.069491792962
6	0.02601	0.067035593825	0.03584	−0.049467356752
7	0.03016	0.051952216575	0.03732	−0.036858573573
8	0.03278	0.041221071920	0.03716	−0.028445448367
9	0.03408	0.033339609126	0.03603	−0.022576076169
10	0.03435	0.027412493771	0.03441	−0.018332145060

p	$h^{1,2}$	$a_0^{1,2}$	$h^{1,3}$	$a_0^{1,3}$
50	0.00904	0.186108126535	0.00787	−0.000735493223
100	0.00464	0.000280711837	0.00385	−0.000183872768
150	0.00306	0.000124760035	0.00243	−0.000081735922
200	0.00215	0.000070169964	0.00182	−0.000045806387
250	0.00157	0.000044887230	0.00146	−0.000029104000
300	0.00131	0.000031159126	0.00121	−0.000020245070
350	0.00112	0.000022913631	0.00104	−0.000015127559
400	0.00098	0.000017606832	0.00091	−0.000011947330
450	0.00087	0.000014015734	0.00081	−0.000009845296
500	0.00079	0.000011491570	0.00073	−0.000008382795

15.5 The $c\,(e^{e^x}-1)$-based ultraexponentials.

To wind up this numerical investigation, we now examine the connectors $P^{[f_1,f_2]}$ linking the ultraexponentials based on germs of exponentiality 2:

$$f_1(x) := c_1\left(e^{e^x-1}-1\right) \quad ; \quad f_2(x) := c_2\left(e^{e^x-1}-1\right) \quad (c_1, c_2 > 1) \tag{15.10}$$

Rather than applying a modified version of formula (15.2) with normalisers $f^\lozenge, {}^\lozenge f$ conjugating f with E_2 at $+\infty$, we directly expand the map $k_{1,2}$ that conjugates f_1 and f_2:

$$k_{1,2} \circ f_2 = f_1 \circ k_{1,2} \quad , \quad k_{1,2} := {}^\lozenge f_1 \circ f_2^\lozenge$$

into a very fast converging series:

$$\begin{aligned} k_{1,2}(x) &= x + \log\Big(1 + e^{-x}\,\log\big(\epsilon_0(x) + \epsilon_1(x) + \epsilon_2(x) + \ldots\big)\Big) && (15.11)\\ \epsilon_0 &= \frac{c_2}{c_1} = \text{Const} \\ \epsilon_1 &= \frac{c_1 - c_2}{c_1 + f_1}\,\log\Big(1 + e^{-f_2}\log(\epsilon_0)\Big) \\ \epsilon_n &= \frac{1}{c_1 + f_1}\,\log\left(\frac{1 + e^{-f_2}\log(\epsilon_0 + \epsilon_1\circ f_2 + \ldots \epsilon_{n-1}\circ f_2)}{1 + e^{-f_2}\log(\epsilon_0 + \epsilon_1\circ f_2 + \ldots \epsilon_{n-2}\circ f_2)}\right) && (\forall n \geqslant 2) \end{aligned}$$

Then, as usual, we boost numerical efficiency by plugging suitably large iterates of f_1, f_2 into the expression of the connector:

$$\begin{aligned} P^{[f_1,f_2]} &:= \delta_{\gamma_1}^{-1} \circ L \circ f_1^\ddagger \circ\, k_{1,2} \circ {}^\ddagger f_2 \circ E \circ \delta_{\gamma_2} && (15.12)\\ &:= T^{n_1} \circ \delta_{\gamma_1}^{-1} \circ L \circ f_1^\ddagger \circ f_1^{-n_1-n_0} \circ\, k_{1,2} \circ f_2^{n_0+n_2} \circ {}^\ddagger f_2 \circ E \circ \delta_{\gamma_2} \circ T^{-n_2} \end{aligned}$$

The following table gives the variations h and $h^\sharp$ of the connectors $P^{[f_1,f_2]}$ and $P^{[f_2,f_1]}$ with $c_1 = 2$ and $c_2 = 2^p$ and p ranging over the interval $[2, 10]$.

p	h	$h^\sharp$	p	h	$h^\sharp$	p	h	$h^\sharp$
2	0.0010	0.0161	20	0.2178	0.3916	200	0.5184	0.6724
3	0.0039	0.0379	30	0.2850	0.4626	300	0.5547	0.7008
4	0.0097	0.0650	40	0.3285	0.5059	400	0.5782	0.7188
5	0.0191	0.0956	50	0.3560	0.5359	500	0.5953	0.7316
6	0.0317	0.1275	60	0.3843	0.5583	600	0.6085	0.7414
7	0.0467	0.1588	70	0.4039	0.5761	700	0.6192	0.7492
8	0.0628	0.1884	80	0.4203	0.5906	800	0.6280	0.7556
9	0.0794	0.2156	90	0.4343	0.6027	900	0.6357	0.7612
10	0.0958	0.2404	100	0.4464	0.6132	1000	0.6423	0.7661

As was the case with germs f_1, f_2 of exponentiality 1, both h and $h^\sharp$ remain small – but not as small as before – for moderate values of c_1, c_2, and they also increase slightly less slowly when $|c_1 - c_2|$ grows, pointing to a slightly less sluggish convergence of the connectors to the limit staircase regime.

This trend, which gets more pronounced for germs f_1, f_2 of larger exponentiality $r \geqslant 2$ and for the connectors $P_1^{[f_1,f_2]}$ linking the ultraexponentials $\mathcal{E}_1$ based on them, becomes absolutely dominant for germs f_1, f_2 which are themselves of *ultraexponential* order r and for the connectors $P_{r+1}^{[f_1,f_2]}$ linking the ultraexponentials $\mathcal{E}_{r+1}$ based on them.

15.6 Conclusion.

To sum up, three facts stand out:

- All ultraexponential $\mathcal{E}_1$ constructed from 'reasonable' germs f of exponentiality 1 are surprisingly close to one another.
- This is no longer the case for the ultraexponential $\mathcal{E}_{r+1}$ or higher order ($r \geqslant 1$) constructed from germs f, reasonable or not, of ulraexponential order r.
- Whatever the ultraexponential order, any attempt to base the construction of $\mathcal{E}_{r+1}$ on the restriction of a given f on smaller and smaller neighbourhoods of $+\infty$, so as to get a truly 'germinal' and 'intrinsic' result — any such attempt (already doomed due to the universal differential asymptotics of fast/slow germ; see §6) must founder on the *staircase phenomenon.*

16 Conclusion: central facts, main questions.

16.1 Central facts.

$\mathbf{F_1}$: Ubiquity of resurgence.

All types of composition equations or systems, even the simplest ones (fractional iteration or conjugation) and the ones formally most unproblematic,[131] can and often do produce divergence, even when there is none in the data. That divergence, however, is always re-summable (it never involves such complications as Liouvillian small denominators) because it is resurgent, and resurgent of a very special type: it is either *non-polarising* (meaning that the singularities in the Borel planes do not lie on $\mathbb{R}^+$) or at most *weakly polarising* (i.e. with only a finite number of active alien derivation $\boldsymbol{\Delta}_\omega$ with index $\dot{\omega} \in \mathbb{R}^+$). In the non-polarising case (which includes fractional iteration and conjugation), $\widetilde{f}$ has a privileged real sum f. In the (far less common) weakly polarising case, $\widetilde{f}$ admits several sums $f_{\boldsymbol{\tau}}$, depending on the choice of convolution average(s), but the standard average always works,[132] yielding a 'privileged' sum, which we may simply denote f.

$\mathbf{F_2}$: Ubiquity of cohesiveness.

Cohesiveness is the natural and unavoidable accompaniment of iterated exponentials or ultraexponentials. On its own, it generates no extra divergence (that is to say, the transseries or ultraseries $\widetilde{f}$ remains convergent if its subseries are themselves convergent or, if not, it becomes so after these have been separately re-summed); it introduces no extra polarizations; and it results in sums f that always belong to the quasi-analytic class *COHES* on some real neighborhood $]..., +\infty[$, but usually without extension outside the real axis. It should be emphasised, however, that due to the non-polarising or weakly polarising nature of the resurgence encountered in this context, cohesiveness is notably absent from that other place where it sometimes occurs in accelero-summation, namely in the auxiliary Borel planes or axes.

[131] i.e. those that involve no jump in *formal* complexity, in the sense of admitting formal power series solutions $\widetilde{f}$ when all the data $\widetilde{f}_i$ are formal power series, etc.

[132] there being only finitely many singularities on $\mathbb{R}^+$, we are spared the complication of *faster than lateral growth on oft-crossing paths* and there is no need to resort to the fine-tuned *well-behaved averages.*

$\mathbf{F_3}$: Ubiquity of analysability.

No matter how intricate and divergence-ridden our group extensions may be, the constructive correspondance between $\widetilde{f}$ and f – between the formal and geometric sides – is never lost: our germs remain completely *'analysable'*, and all questions pertaining to them can in theory be rephrased, tackled, and often answered, on the formal side, which of course is the more tractable of the two. Moreover, the 'analysable' character of our germs holds not only at the (highly singular) point $+\infty$, but also near it, on some real neighborhood, where our germs f are always either *real-analytic* or *real-cohesive* (*cohesiveness* being a very special, regular, and stable subclass of *quasi-analyticity.*)

$\mathbf{F_4}$: Display, transcendence, trans-polarisation.

To each resurgent $\widetilde{f}$, whether mono- or poly-critical, is associated a so-called *display*, noted $Dpl.\widetilde{f}$ on the formal side and $(Dpl.f)_{\boldsymbol{\tau}}$ on the geometric side. It combines two dual things: the so-called *pseudo-variables* $\mathbf{Z}^{\boldsymbol{\varpi}}$ and the alien derivatives, of *all* orders and relative to *all* critical times:

$$\text{Dpl.}\widetilde{f} \;=\; \widetilde{f}+\sum_{1\leqslant r}\;\sum_{\varpi_1,\dots,\varpi_r}\mathbf{Z}^{\varpi_1,\dots,\varpi_r}\,\boldsymbol{\Delta}_{\varpi_r}\dots\boldsymbol{\Delta}_{\varpi_1}\,\widetilde{f} \tag{16.1}$$

$$\downarrow \mathcal{S}_{\boldsymbol{\tau}}$$

$$(\text{Dpl.}f)_{\boldsymbol{\tau}} \;=\; f_{\boldsymbol{\tau}}+\sum_{1\leqslant r}\;\sum_{\varpi_1,\dots,\varpi_r}\mathbf{Z}^{\varpi_1,\dots,\varpi_r}\,(\boldsymbol{\Delta}_{\varpi_r}\dots\boldsymbol{\Delta}_{\varpi_1}\,f)_{\boldsymbol{\tau}} \tag{16.2}$$

The multiple indexation $\varpi_i := \omega_i\,M_i$ involves all critical *time classes* $[z_i]$ through their transmonomial representatives $M_i(z)$ and, for each such class, the singularity-carrying $\omega_i \in \mathbb{C}_\bullet$.[133]

Here $\mathcal{S}_{\boldsymbol{\tau}}$ denotes accelero-summation $\widetilde{f} \mapsto f_{\boldsymbol{\tau}}$ relative to some multipolarisation $\boldsymbol{\tau}$, which in each critical Borel plane prescribes an integration axis $\arg\zeta_i = \theta_j$ and a convolution average μ_i.

True to its name, the display does indeed *display*, in ultra compact and algebraically operative form, all the information about the object - not just its Stokes constants, but also exhaustive information about the 'Borel side'. It also leads to the *trans-polarisation formulae* (2.62)-(2.63) which show how to derive any polarised sum $(Dpl.f)_{\boldsymbol{\tau}'}$ from any given sum $(Dpl.f)_{\boldsymbol{\tau}}$ by a purely formal operation performed on the sole pseudo-variables and using *universal constants* $\mathbf{P}^{\bullet}_{\boldsymbol{\tau}',\boldsymbol{\tau}}$ that depend only on the pair $(\boldsymbol{\tau}',\boldsymbol{\tau})$. Another nice feature is that any relation between resurgent objects automatically extends to their displays, which facilitates the proof of transcendence and independence results.

[133]or, in the case of the *lesser* display $dpl\widetilde{f}$, all $\omega_i \in \mathbb{R}^+$.

F_5 : Near-completeness of resummable transseries.

Composition equations $W(\widetilde{f};\widetilde{f}_1,\dots,\widetilde{f}_s) = id$ involving only transserial (resp. transserial *and* resummable) inputs $\widetilde{f}_i$ admit transserial (resp. transserial *and* resummable) solutions $\widetilde{f}$ unless *exponentiality stands in the way*, that is, unless the *exponentially shrunken*[134] equation $W(E_n;E_{n_1},\dots,E_{n_s})=id$ admits no solution E_n ($n\in\mathbb{Z}$). The price to pay for the admission of exponentials and their finite iterates is of course the frequent non-analyticity of the sum f and its replacement by cohesiveness.

F_6 : Total completeness of resummable ultraseries.

To remove this last hurdle - the exponentiality hurdle - we are compelled to enter the ultra-exponential range, i.e. to introduce a coherent system of transfinite iterates E_α and L_α of E and L (but with $\alpha < \omega^\omega$). This time at last, *as far as we can see*[135], we get full closure, but at the cost of two complications: (i) the apparent non-existence of a privileged analytic realisation of the formal system of transfinite iterates E_α, L_α and (ii) the coexistence of several competing 'canonical forms' for the formal ultraseries, or rather the transmonomials in them. Yet, in another sense, we are at the end of our travails as far as *analysis*[136] is concerned, for there exist no operations or equations that would require us to consider iteration orders larger than ω^ω or even equal to it.

F_7 : Growth types and the arithmetics of $[0,\omega^\omega[$.

As just pointed out, the special conjugation equations verified by the transfinite iterates E_α and L_α of $E := exp$ and $L := log$ (for all $\alpha < \omega^\omega$) do not entirely characterise the iterates, but that residual indeterminacy disappears when we replace the slow germs L_α by their classes $[L_\alpha]$ relative to a suitable equivalence relation.[137] These well-defined classes $[L_\alpha]$ are then found to generate a semi-group that exactly reproduces the non-commutative arithmetics of the transfinite interval $[1,\omega^\omega[$:

$$[\mathrm{L}_\beta]\circ[\mathrm{L}_\alpha] = [\mathrm{L}_{(\alpha+\beta)}] \quad , \quad [\mathrm{L}_\alpha]^{\circ\beta} = [\mathrm{L}_{(\alpha.\beta)}] \qquad (\forall\, \alpha,\beta\in[1,\omega^\omega[) \quad (16.3)$$

[134] obtained by *shrinking* in $W = id$ each $\widetilde{f}_i$ to $E_{n_i} := \mathrm{stat.lim}_{k\to+\infty}(L_k\circ\widetilde{f}_i\circ E_k)$ ($n_i\in\mathbb{Z}$).
[135] an important proviso, since we are still in the early stages of transserial analysis.
[136] as opposed to *mathematical logic*, which routinely considers equations (mostly on $\mathbb{N}$) where the variable is allowed to occur inside the iteration order.
[137] *suitable* here means: compatible with composition and iteration.

$\mathbf{F_8}$: Iso-operators and iso-convexity.

The bialgebra *ISO*[138] of *iso-differential* operators $\mathrm{Dn}^{\{\boldsymbol{n}\}}$

$$\mathrm{Dn}^{\{n_1,\ldots,n_r\}} f := \prod_i (\mathrm{Dn}^{\{n_i\}} f) \quad \textit{with} \quad \mathrm{Dn}^{\{n_i\}} f := (-1)^{n_i}\, \partial^{n_i} \log(1/f') \qquad (16.4)$$

with its commutative product $\times$, its non co-commutative co-product σ (reflecting the interaction with germ composition $\circ$) and its 'iso-degree' *ideg* compatible with $\times$ and σ, is not only better suited to the study of the germ groups $\mathbb{G}$, especially in the fast and slow growing ranges (see below) but it also possesses a remarkable positive cone ISO^+. That cone induces a notion of *iso-convexity* more relevant to germ composition than ordinary convexity. It also admits a special basis $\mathrm{Da}^{\{\boldsymbol{n}\}}$ extremely rich in improbable algebraic-combinatorial properties.

$\mathbf{F_9}$: Universal asymptotics of slow functions.

Any iso-differential operator $\mathcal{D}$ acting on any ultra-slow germ $\mathcal{L}$ (say, on any transfinite iterate of L) produces a germ $\mathcal{D}.\mathcal{L}$ whose natural asymptotic expansion depends on $\mathcal{D}$ alone, not on $\mathcal{L}$.[139] This may be taken as the foundational statement of *'universal asymptotics'* – a fascinating subject with ramifications in logic and model theory.[140]

$\mathbf{F_{10}}$: The Natural Growth Scale or 'Grand Cantor'.

Assuming the indeterminacy in the ultra-exponential/ultra-logarithmic scale to be unsurmountable, and lumping together into the same *'zones'* all germs that are 'indiscernible' (in the sense of being different geometric realisations of the same transseries), we arrive at a counter-intuitively fractal picture of the *natural growth scale*. That scale, far from being the quintessential continuum that one would imagine, turns out to be thoroughly fractal and even doubly 'Cantorian':
(i) in *the large* it resembles Cantor's transfinite interval $[1, \omega^\omega[$, with a profusion of detail round each E_α but an inter-galactic void between each E_α and its successor $E_{\alpha+1}$
(ii) and *locally* it reproduces the global picture at ever smaller scales, giving rise to patterns which this time are more reminiscent of the historical Cantor set constructed by repeated trisection of the real interval $[0, 1]$.

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[138] It is sometimes known as the Connes-Moscovici bialgebra, although it was introduced by us a decade earlier, in 1991, in our book [E5] on "Analysable Functions etc".

[139] It is only the trans-asymptotic part of $\mathcal{D}.\mathcal{L}$ that depends on $\mathcal{L}$.

[140] See [JvdH2] and also J.v.d.Hoeven's Habilitation's thesis.

16.2 Some open questions.

$\mathbf{Q_1}$: Indeterminacy in the ultraexponential towers.

The fact that the system (1.4)-(1.5) determines each pair $(\mathcal{L}_n, \mathcal{E}_n)$ in terms of $(\mathcal{L}_{n-1}, \mathcal{E}_{n-1})$ only up to pre/post-composition by some 1-periodic[141] germ P, together with the observation that all ultra slow/fast germs share a universal asymptotics, dashes all hope of selecting a privileged solution $(\mathcal{L}_n, \mathcal{E}_n)$ based purely on real-asymptotic criteria. On the other hand, we cannot discount the possibility, however remote, that *one* of these systems might possess extensions to the complex domain so regular or remarkable as to mark it out as indisputably 'optimal'. To further complicate the picture, we found in §15 that all the 'reasonable' candidates for the first non-elementary pair[142] $(\mathcal{L}_1, \mathcal{E}_1)$ are extremely close to one another. So the question is still open, and likely to remain so for quite a while.

$\mathbf{Q_2}$: Are there privileged analytic ultraexponential towers?

The question about analytic choices has two aspects. First, does Kneser's construction (with its reliance on a pair of closest fixed points etc) apply at each induction step like it does at step one? If it does, the corresponding towers $\{\mathcal{L}_n, \mathcal{E}_n\}$ would enjoy an arguably privileged position among all analytic representatives. If not, are there always analytic representatives in the conjugacy classes of each $\{\mathcal{L}_n, \mathcal{E}_n\}$? If there is one, there are infinitely many, but might there be natural criteria for removing or, more realistically, reducing this indeterminacy? *Remark:* the action of analytic Witt towers on analyticity towers $\{\mathcal{L}_n, \mathcal{E}_n\}$ usually destroys their analyticity. Conversely, the Witt tower connecting two analytic ultraexponential towers are only exceptionally analytic.

$\mathbf{Q_3}$: The choice of carriers.

Might not the ultra-quasiexponential towers $\{\mathcal{L}v_n, \mathcal{E}v_n\}$ of §8.8 with their guaranteed analyticity and their more natural, as well as computationally less costly, construction, be the best solution after all? True, once an analytic $\{\mathcal{L}v_n, \mathcal{E}v_n\}$ is chosen, it automatically determines a $\{\mathcal{L}_n, \mathcal{E}_n\}$ which will be merely cohesive. But the converse also holds, and in any case, no matter what system of ultra-exponentials or ultra-quasiexponentials we choose, the formal solutions $\widetilde{f}$ of most composition equations are bound to re-sum to cohesive rather than analytic germs f.

[141] More precisely, a germ P that commutes with the unit shift T.
[142] derived from the initial pair $(\mathcal{L}_0, \mathcal{E}_0) = (L, E)$.

$\mathbf{Q_4}$: Beyond ω^ω.

Although analysis will probably never require iterates of order $\alpha \geqslant \omega^\omega$, the fact remains that adding such iterates to a group $\mathbb{G}$ of ultra-exponentials is one of the few fault-proof means of producing a 'non-oscillating' extension $\mathbb{G}^{\text{ext}}$, i.e. an extension where the order $\lessdot$ still holds.[143]

$\mathbf{Q_5}$: Real-analytic vs real-cohesive solutions.

When the only complication is resurgence, whether mono- or polycritical, resummation always yields real-analytic solutions (outside $+\infty$, of course).

But what about equations with transserial solutions $\widetilde{f}$ of unbounded exponential depth? The sums f are always real-cohesive, but can they exceptionally be better than that, i.e. real-analytic?

In the case of ultraseries, once an ultraserial solution $\widetilde{f}$ is fixed, its sum f still depends on the choice of an ultraexponential tower. Does there always exist a (real-analytic or real-cohesive) ultraexponential tower that makes the sum of that given, particular $\widetilde{f}$ analytic?

$\mathbf{Q_6}$: Independence theorems.

Clearly, the group $< T, E >$ is not *freely* generated by T (unit shift) and E (exponential), since it contains 'similitudes' $S : x \mapsto a\,x + b$. But the question remains: are all relations in $< T, E >$ generated by 'elementary relations', i.e. by the transparent relations verified by these similitudes? A *yes* answer would mean that the group $< T, E >$ (which, contrary to appearances, contains transseries of the most general type) is acted upon not only by the iso-differentiations of ISO but also by the much more numerous, non-differential iso-operators of $^\sharp ISO$.

Another related question is this: can we have identities of type

$$id \;=\; f_1 \circ g_1 \circ f_2 \circ g_2 \circ \cdots \circ f_s \circ g_s \qquad \textit{with} \tag{16.5}$$

$$f_i(x) \;:=\; a_i\, x\, (1 + \sum a_{i,n}\, x^{-n}) \tag{16.6}$$

$$g_i(x) \;:=\; b_i\, x\, (1 + \sum b_{i,n}\, x^{-n\gamma}) \qquad (\gamma \in \mathbb{R}^+ \dot{-} \mathbb{Q}^+) \tag{16.7}$$

other than in the trivial case, when each factor f_i and g_i reduces to a similitude $x \mapsto c\,x + d$? The answer is almost certainly *no*[144], but the question appears to be still open.

[143] See §1.2.

[144] Indeed, if we write that the coefficients of all monomials $x^{1-n_1-\gamma\, n_2}$ on the right-hand side of (16.5) vanish for all indices n_1, n_2 up to N, the number of apparently independent conditions grows like N^2 whereas the number of coefficients involved grows only like $2\,s\,N$.

$\mathbf{Q_7}$: No a priori constraints on the holomorphic invariants?

A general principle holds that the only a priori constraints on the general shape of the display $Dpf\,\widetilde{f}$ of a resurgent f are *formal constraints*. Put another way, it says that as long as $Dpf\,\widetilde{f}$ verifies the defining equation of $\widetilde{f}$, it can be anything.[145] That principle, which applies to all known instances of the Bridge Equation, also predicts the correct form of the display for the composition equations examined in §11-§12. Is its validity boundless?

$\mathbf{Q_8}$: Primary representatives of identity-tangent twins.

Since identity-tangent *twins* or *siblings* generically exhibit (mono- or poly-critical) resurgence but are defined only up to conjugation by a common h, is there always a special resurgent h, conjugation by which optimally simplifies the *displays* of our twins and siblings,[146] leading to *'primary'* or *'minimally resurgent'* solutions? In the example of §13.2, the answer was *yes*, but is that always so? In the same vein: do their exist *analytic* identity-tangent twins?

$\mathbf{Q_9}$: Geometric solutions of non-polarising composition equations.

When does a composition equation $W(f; f_1, \ldots, f_s) = id$ admit a solution f capable of a convergent geometric representation of the form

$$f = \lim_{n \to +\infty} W_n(f_1, \ldots, f_s) \qquad (W_n \in \,< f_1, \ldots, f_s >) \tag{16.8}$$

with W_n an explicitable element generated by the inputs f_i ? For equations of type $\mathcal{T}_1$ or $\mathcal{T}_2$ (iteration or conjugation), a representation (16.8) does exist (barring obvious obstructions linked to 'exponentiality'), but what about the types $\mathcal{T}_3$ and $\mathcal{T}_4$? And what about *twins* or *siblings*, where the data f_i are completely missing? Then again, does 'semi-polarisation' (i.e. the presence of finitely many active derivations $\mathbf{\Delta}_\omega$ with $\dot{\omega} \in \mathbb{R}^+$ in one or several Borel planes - see §2.12 *supra*) unsurmountably precludes representations of type (16.8)? If not, do these representations necessarily 'pick' the simplest polarisation $\boldsymbol{\tau}$, i.e. the one that corresponds to the *standard convolution average*?

[145]There exist, of course, *growth constraints* in ω on the resurgence constants A_ω that the display carries, but this is another story: here, we are viewing the display simply as a formal expansion in the true variable and the pseudo-variables, leaving aside all questions of coefficient growth.

[146]or, if you prefer, reduces their stock of Stokes constants.

REFERENCES.

[ADH] M.Aschenbrenner,L.v.d.Dries,J.v.d.Hoeven, *Asymptotic Differential Algebra and Model Theory of Transseries*, ArXiv 1509.02588

[C.1] O.Costin, *On Borel summation and Stokes phenomena of nonlinear differential systems*, Duke Math. j., vol. 93, No 2, 1998.

[C.2] O.Costin, *Asymptotics and Borel Summability*, 2008, Chapman and Hall, 256 pp.

[CEF] O.Costin, P.Ehrlich, M.Friedman, *Integration on the Surreals: a conjecture of Conway, Kruskal and Norton*, ArXiv.

[CEF] O.Costin, M.Kruskal, A.Macintyre (ed.), *Analyzable Functions and Applications*, A.M.S. Contemporary Math., vol. 373.

[E_1] J.Ecalle, *Les fonctions résurgentes, Vol.1*, Pub. Math. d'Orsay, 1981.

[E_2] J.E., *Les fonctions résurgentes, Vol.2*, Pub. Math. d'Orsay, 1981.

[E_3] J.E., *Les fonctions résurgentes, Vol.3*, Pub. Math. d'Orsay, 1985.

[E_4] J.E., *The acceleration operators and their applications*, Proc. I.C.M. 1990, Kyoto, vol.2, Springer, Tokyo 1991,pp1249-1258.

[E_5] J.E., *Introduction aux fonctions analysables et preuve constructive de la conjecture de Dulac*, Actual. Math., Hermann, Paris, 1992.

[E_6] J.E., *Six Lectures on Transseries, Analysable Functions, and the Constructive Proof of Dulac's Conjecture*, in Bifurcations and Periodic Orbits of Vector Fields, D. Schlomiuk ed., pp 75-184, 1993, kluwer.

[E_7] J.E., *Recent Advances in the Analysis of Divergence and Singularities*, in Normal Forms and Hilbert's 16th Problem, Yu. S. Ilyashenko and C. Rousseau ed.,2003, Kluwer.

[E_8] J.E., *Cohesive Functions and Weak Accelerations*, J. Anal. Math., Sz. Mandelbrojt Memorial Volume, Vol. 60, 1993.

[E_9] J.E., *Twisted Resurgence Monomials and canonical-spherical synthesis of Local Objects*, Proc. of 2002 Edinburgh Conference on Asymptotics and Analysable Functions, O. Costin ed, World Scient. Pub., 105pp.

[EM] J.E. & F.Menous

[EV] J.E. & B.Vallet, *Intertwined Mappings*, Ann. Fac. Toulouse, Tome 23, no.3 ,2004,p291-376.

[$JvdH$1] J.v.d.Hoeven, *Automatic Asymptotics*, Ph.D. Thesis, Ecole Polytechnique, Fr., 1997.

[$JvdH$2] J.v.d.Hoeven, *Transseries and Real Differential Algebra*, Lec. Notes in Math.1888, Springer, 2006, 250pp.

[$JvdH2$] J.v.d.Hoeven, *A differential intermediate value theorem*, in B.L.J.Braaksma, G.K.Immink, M.v.d.Put, J.Top ed., in Differential equations and the Stokes phenomenon, p 147-170, World Scientific, 2002.

[K] H.Kneser, *Reelle analytische Lösungen der Gleichung* $\varphi(\varphi(x)) = e^x$, Journal f. reine u. angew. Math., **187**, 1950, 66-67.

[M] F.Menous, *Les bonnes moyennes uniformisantes et leurs applications à la resommation réelle*, PhD Orsay 1996.

[P] D.Panazzolo, $PSL(2,\mathbb{C})$*, the exponential and some new free group*, Quarterly Journal of Math., 20017 (DOI).