Recent Advances in the Analysis of Divergence and Singularities

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Abstract. *We survey a number of related advances that have taken place over the last decade in the field of singularities, normal forms, ODEs, etc, as well as the analytic tools for tackling these problems, namely: resummation, resurgence, transseries, analysable functions. One such advance – the notion of well-behaved convolution average – has led to a simplification of the celebrated finiteness theorem for limit-cycles. Another one has clarified the (continuous) prenormalisation and (discontinuous) normalisation of local objects. Yet another – the notion of twisted resurgence monomials – has yielded a truly general method for canonical-explicit object synthesis (ie constructing local objects with prescribed analytic invariants). A fourth advance has shed new light on the classical KAM theorem about the survival of invariant tori. Lastly, a fifth development, which is arguably the most promising of all – the introduction of the new Lie algebra ARI – has led to a far-going elucidation of the arithmetics of MZV or "multiple zeta values". – Part of the results surveyed in this paper are joint work with F. Menous or B. Vallet, and mention is made of independent contributions by J. van der Hoeven.*

Contents

GENERAL CHAPTER-BY-CHAPTER OVERVIEW.

1. The Finiteness Theorem for limit-cycles and its resummation-theoretic proof twelve years on : review; simplification; aftermath. The proof in question, which relied on a constructive correspondence between a formal object (the *transseries* $\tilde{F}$) and its geometric counterpart (the *return map* F), has been significantly simplified, owing to the discovery of special *convolution-respecting averages* (in joint work with F.Menous). At the same time, the general theory of Transseries and Analysable Functions has grown and matured considerably, thanks mainly to the work of an Orsay-based mathematician, J. van der Hoeven. We shall attempt to cover some of the most exciting novelties in this fast-developing field.

2. Normal and Prenormal Forms.
After reviewing the main classical results about *normal forms*, we shall introduce an altogether new type of *prenormalisation* (developed jointly with B.Vallet) and explain what it is good for.

3. Canonical Object Synthesis.
We shall construct a new class of special functions that make possible a unified treatment of the *reverse classification problem* : finding 'canonical' local objects (differential systems, vector fields, diffeomorphisms, etc) with prescribed invariants (formal or holomorphic); studying their iso-monodromic, or iso-Stokes, iso-Galois, etc, deformations.

4. An unexpected development in KAM theory: the non-existence of Super-multiple Small Denominators.
Supermultiple Small Denominators or SSD for short (ie diophantine small denominators with an abnormally high multiplicity) were thought to be a central difficulty in KAM theory, and elaborate strategies had been evolved to account for their somewhat mysterious *'mutual compensation'*. But in 1994 it was discovered (jointly with B.Vallet) that, contrary to appearances, these much-dreaded SSD *simply do not exist* – they are a mirage conjured up by awkwardly conducted calculations. This startling claim, which at first was met with considerable skepticism, has now been fully vindicated. We shall explain the why and wherefore of this 'non-existence', and show what simplifications it brings to KAM theory, Floquet theory, etc.

5. A Tale of Three Structures.
We shall try to present the outlines of an overarching structure which unifies three separate theories that *prima facie* would seem to be worlds apart :
(i) the Lie algebra ARI (which includes as sub-cases the Ihara braid algebra and the so-called renormalisation algebra)
(ii) the analysis of *'parametric divergence'* (an important type of divergence frequently encountered in physics)
(iii) the arithmetics of the *Multiple Zeta Values* or MZV : these constants tend to pop up everywhere (in Holomorphic Dynamics, Number Theory, Feynman Diagrams, Galois Theory, Knot Theory. . .) and have remained a focus of intense activity for more than a decade, but the main MZV-related conjectures stubbornly defied proving – that is, until the recent introduction of ARI.

1 Lesson One: The finiteness theorem for limit-cycles and its resummation-theoretic proof twelve years on : review; simplification; aftermath.

1.1 Overall scheme.

The finiteness theorem for limit cycles of analytic planar vector fields

$$X = P(x,y)\,\partial_x + Q(x,y)\,\partial_y \qquad\qquad (x,y \in \mathbb{R}) \tag{1}$$

states that *on compact subsets of the plane any such field has at most a finite number of limit-cycles, ie isolated periodic trajectories. In particular, a polynomial vector field has at most a finite number $N(X)$ of limit-cycles in the whole plane.*

By itself, the theorem doesn't solve Hilbert's celebrated 16th problem : it doesn't tell us what this number $N(X)$ is. It doesn't even settle the issue of whether this number is bounded for polynomials of a given degree d. At the moment, the question is still pending – even for degree 2 – although in that case a (positive) answer appears to be close at hand.

Way back in the 1920s, Dulac gave a proof of the finiteness theorem, which alas was flawed and yet went unchallenged for more than fifty years. Eventually, two modern proofs appeared at the close of 1980s – one [Il1],[Il2] due to Yu. S. Ilyashenko, geometric in spirit and non-constructive in nature; and another one [E6],[E7] by myself. That latter proof relied on recent advances in resummation theory and fitted into a general program of "constructive formalisation" of local geometry, which stands on its own (it predated the current infatuation with polycycles and will hopefully outlive it).

1.2 Polycycles; local transit maps; global return map.

Let $\mathcal{C}$ be a simple, closed curve on a real-analytic surface $\mathcal{S}$. Let X be a real-analytic vector field, defined on a neighbourhood of $\mathcal{C}$ and such that $\mathcal{C}$ be a finite union of closed trajectories of X, with summits $S_1, S_2, \ldots, S_r$ corresponding to singular points of X, and r analytic arcs $\mathcal{C}_i$ linking S_{i-1} to S_i. On each $\mathcal{C}_i$ we fix an interior point P_i and draw an analytic curve Γ_i crossing $\mathcal{C}_i$ at P_i at a right angle. We equip Γ_i with an analytic abscissa x_i that vanishes at P_i and assumes positive values on the inside of the polycycle. Every interior orbit of X close to $\mathcal{C}$ intersects Γ_i at a point of abscissa $x_i = 1/z_i$ ($x_i \sim 0, z_i \sim +\infty$) and Γ_{i+1} at a point of abscissa $x_{i+1} = 1/z_{i+1}$ ($x_{i+1} \sim 0, z_{i+1} \sim +\infty$). By setting $z_{i+1} = G_i(z_i)$ we define a self-mapping of $[\ldots, +\infty]$. The successive G_i will be referred to as *transit maps* and their composition product $F := G_r \circ \cdots \circ G_2 \circ G_1$ as *return map* .

To sum up:

$$\begin{array}{rlr} i\text{-}th\ \ transit\ \ map\ : & G_i : z_i = 1/x_i \mapsto z_{i+1} = 1/x_{i+1} & (2) \\ (first)\ return\ \ map\ : & F : z_1 = 1/x_1 \mapsto z_{r+1} = 1/x_{r+1} & (3) \\ & F = G_r \circ \ldots G_2 \circ G_1 & (4) \end{array}$$

By a classical result on the reduction of planar singularities by means of successive blow-ups (see [Sei]) it suffices to consider the case when each summit S_i is either of *hyperbolic type* (X has there two non-zero eigenvalues) or *semi-hyperbolic type* (X has there one zero and one non-zero eigenvalue).

Actually, from the point of view of formalisation, three types of summits have to be distinguished:

$$\begin{array}{rlr} Type\ \ I\ : & (\lambda', \lambda'')\ \ \ with\ \ \lambda := -\lambda'/\lambda'' \in \mathbb{R}^+ - \mathbb{Q}^+ & (5) \\ Type\ \ II\ : & (\lambda', \lambda'')\ \ \ with\ \ \lambda := -\lambda'/\lambda'' \in \mathbb{Q}^+ & (6) \\ Type\ \ III\ : & (\lambda', 0)\ \ \ or\ \ (0, \lambda'')\ \ with\ \ \lambda', \lambda'' \in \mathbb{R}^+ & (7) \end{array}$$

For summits of type I or II (resp III) the transit maps G have formal counterparts $\tilde{G}$ which are either power series or elementary transseries (made up of two elementary series $\tilde{H}$ and $\tilde{K}$ at most) and of the form:

$$\begin{array}{rrcll} Type\ \ I\ : & \tilde{G}(z) & = & c\, z^\lambda \, \{1 + \sum_{\sigma \in \mathbb{N} + \lambda \mathbb{N}} c_\sigma \, z^{-\sigma}\} & \\ & & = & \tilde{K} \circ P_\lambda \circ \tilde{H} & (8) \\ Type\ \ II\ : & \tilde{G}(z) & = & c\, z^\lambda \{1 + \sum_{m/\sigma \leq const} c_{\sigma,m} \, z^{-\sigma} (\log z)^m\} & (9) \\ Type\ \ III^+\ : & \tilde{G}(z) & = & e^{\lambda z}.\{c_0 + \sum_{m \geq 0, n \geq 0} c_{m,n} \, e^{-\lambda m z} z^{-n}\} & \\ & & = & \tilde{K} \circ \exp \circ \tilde{H} & (10) \\ Type\ \ III^-\ : & \tilde{G}(z) & = & (\log z) \, . \, \{c_0 + \sum_{m \geq 0, n \geq 0} c_{m,n} \, z^{-m} (\log z)^{-n}\} & \\ & & = & \tilde{H} \circ \log \circ \tilde{K} & (11) \end{array}$$

Only for summits of type I with a diophantine eigenvalue ratio λ are the formal transit maps $\tilde{G}$ guaranteed to be convergent. In all other cases they are generically divergent, but with a quite specific type of divergence: *seriable* for type I with Liouvillian λ and *resurgent* for types II and III. But they are always resummable under the standard Borel-Laplace procedure relative to a single "critical time class" $\{z_\star\}$[1]:

$$\begin{aligned} Type\ \ I\ and\ II: &\quad critical\ \ time\ \ class\ \ \{z_\star\} = \{\log z\} \\ Type\ \ III^+: &\quad critical\ \ time\ \ class\ \ \{z_\star\} = \{z^p\} \\ Type\ \ III^-: &\quad critical\ \ time\ \ class\ \ \{z_\star\} = \{\log z\} \end{aligned}$$

1.3 The philosophy of "complete formalisation".

The methodological ideal – not only in this, but in a host of other problems – is to replace a local geometric object F (say, a function germ) by a formal one $\tilde{F}$ (say, a series or transseries – the tilda signals formalness) without any loss of information (F should be constructively recoverable from its idealisation $\tilde{F}$) and in such a way that all manipulations performed on F (differentiation, integration, composition, etc) should translate into purely formal – and therefore "mechanical", entirely perspicuous, and far simpler – operations on $\tilde{F}$. Graphically:

$$\begin{array}{ccc} \cdots\cdots\cdots & \cdots\cdots\cdots & \cdots\cdots\cdots \\ Formal\ \ objects & biconstructive\ \ passage & Functions\ \ germs \\ \tilde{F} & \longleftrightarrow & F \\ \{+,\times,\partial,\circ,\prec\} & \longleftrightarrow & \{+,\times,\partial,\circ,\prec\} \\ \cdots\cdots\cdots & \cdots\cdots\cdots & \cdots\cdots\cdots \end{array}$$

[1]defined up to ordinary equivalence $\sim$

Here is a table that suggests the four levels of increasing difficulty that can be encountered in this process of biconstructive formalisation:

convergent power series	*straightforward summation*	*Weierstrass element defined in a full neighbourhood of* ∞
$\tilde{F}(z) = \sum a_n z^{-n}$	$\longmapsto$	$F(z)$
divergent, monocritical power series	*Borel-Laplace summation*	*analytic germ defined in a sectorial neighbourhood of* $+\infty$
$\tilde{F}(z) = \sum a_n z^{-n}$	$\longmapsto$	$F(z)$ *asymptotic to* $\tilde{F}(z)$
divergent, polycritical power series	*accelero-summation*	*analytic germ defined in a sectorial neighbourhood of* $+\infty$
$\tilde{F}(z) = \sum a_n z^{-n}$	$\longmapsto$	$F(z)$ *asymptotic to* $\tilde{F}(z)$
divergent, polycritical transseries	*accelero-synthesis*	*analysable germ defined in a tapering neighbourhood of* $+\infty$
$\tilde{F}(z) = \sum a_\sigma A_\sigma(z)$	$\longmapsto$	$F(z)$ *transasymptotic to* $\tilde{F}(z)$

1.4 Transseries and analysable germs: some heuristics.

On the formal side, since we demand complete closure under the direct and reverse operations $+, \times, \circ, \partial_z$, we should be prepared to tackle not only creatures like:

$$\begin{array}{lllll}
z & \overset{inversion}{\longrightarrow} z^{-1} & \overset{integration}{\longrightarrow} & \log z & \overset{iteration}{\longrightarrow} \log_n z := \log^{\circ n}(z) \\
\uparrow {\scriptstyle inte\atop gration.} & & & \downarrow {\scriptstyle recipro\atop cation.} & \downarrow {\scriptstyle recipro\atop cation.} \\
1 & & & \exp z & \overset{iteration}{\longrightarrow} \exp_n z := \exp^{\circ n}(z)
\end{array}$$

but also hugely generalised "series" assembled from such elements. In other words, we should expect having to deal with *"transseries"* consisting of *well-ordered* sums of *"transmonomials"*, with a natural *transfinite indexation* on Cantor's interval $[0, \omega^\omega[$. But let's be a wee bit more specific.

(i) "well-orderedness" means that any two transmonomials should be comparable (as large or small infinitesimals) and that each subtransseries $\tilde{H}$ obtained by truncating a given transseries $\tilde{F}$ should start with a "first" or "dominant" transmonomial $\tilde{A}_0$. This in turn is indispensible to ensure multiplicative invertibility:

$$\tilde{H} = a_0\,\tilde{A}_0 + \tilde{K} \quad \to \quad \tilde{H}^{-1} = \sum_{n\geq 1} \frac{(-1)^{n-1}}{a_0^n}\,\frac{\tilde{K}^{n-1}}{\tilde{A}_0{}^n} \qquad (a_0 \in \mathbb{R}^\star)$$

(ii) transseries will consist of an *asymptotic part*, with indexation on $[0,\omega[$, that can be derived elementarily from the corresponding germ , and a *trans-asymptotic part*, with indexation on $[\omega,\omega^\omega[$, whose geometric interpretation is going be more elusive, presupposing as it does a prior *resummation* of the asymptotic part.

(iii) the transmonomials themselves are "atomic" in the sense that they cannot be split into *sums* of simpler objects, but that does not prevent them from possessing a highly intricate arborescent *substructure* nor from often carrying a (countable) infinity of coefficients.

On the analysis side, we should be prepared to encounter *omnipresent divergence, of a severe but resummable sort.* Why so? Neither multiplication, nor composition, nor inversion, nor reciprocation, nor differentiation can of themselves cause divergence, though they may aggravate it. The primary source of divergence is *integration*, and of course the solving of functional equations (ODEs, PDEs, difference equations, etc). Indeed, even quite simple transmonomials have divergent indefinite integral. For instance:

$$\int e^z z^\sigma dz \quad \to \quad e^z \sum_{m\geq 0} a_m\, z^{1+\sigma-m} \qquad (\sigma \in \mathbb{R} \dot{-} \mathbb{N})$$

This actually holds true for most other transmonomials, including elementary ones[2] like:

$$e^{P(z)}\,z^\alpha \quad , \quad z^\alpha\,(\log z)^\beta \quad , \quad e^{P(z)}\,z^\alpha \prod_{i=1}^{i=r} (\log_i z)^{\beta_i} \quad , \quad etc$$

In fact the only exceptions are z^σ , $e^{a\,z}\,z^n$, $(\log z)^n$ $(n \in \mathbb{N})$.

[2]elementary in the sense of carrying only a finite number of parameters or "internal coefficients"

Thus, integration will produce divergence at every step, while the other operations, which on their own cannot generate divergence, must be counted upon to make it more severe when they encounter it in their way. Indeed, if A, B, C be (large) transmonomials from different Archimedean scales[3], their indefinite integrals will generically exhibit monocritical divergence [4], but any operation that superposes or intermingles them, like $(\int A + \int B + \int C)^n$ or $\log_n(\int A + \int B + \int C)$, will produce *polycritical transseries* or, more baffling still, *polycritical transmonomials*, like $\exp_n(\int A + \int B + \int C)$ for n large enough.

1.5 The algebra $\mathbb{R}[[[z]]]$ of transseries.

We shall use the following suggestive notations :

	infinitely large	*infinitely small*	*unspecified*
transmonomials	$\sqcap$	$\sqcup$	$\square$
of exponential depth m	${}^m\sqcap$	${}^m\sqcup$	${}^m\square$
transseries	$\sqcap\!\sqcap$	$\sqcup\!\sqcup$	$\square\!\square$
of exponential depth m	${}^m\sqcap\!\sqcap$	${}^m\sqcup\!\sqcup$	${}^m\square\!\square$

After naming our objects, let us proceed with their inductive definition.

Alogarithmic transmonomials and transseries of depth 0:
Alogarithmic transmonomials of exponentiality 0 are ordinary monomials and *alogarithmic transseries* of exponentiality 0 are ordinary series

$$\begin{array}{lll} {}^0\sqcap(z) = z^{\sigma} & {}^0\sqcup(z) = z^{-\sigma} & (\sigma > 0) \\ {}^0\sqcap\!\sqcap(z) = \sum c_\alpha z^{\sigma_\alpha} & {}^0\sqcup\!\sqcup(z) = \sum c_\alpha z^{-\sigma_\alpha} & (\sigma_\alpha > 0) \end{array}$$

with the finite generation axiom :

$${}^0\sqcap\!\sqcap = \mathit{finite\ \ sum}\ ; \qquad {}^0\sqcup\!\sqcup\ \ \mathit{in\ \ some}\ \mathbb{R}[[z^{-\sigma_1}, \ldots, z^{-\sigma_r}]]$$

and the natural order.

Alogarithmic transmonomials and transseries of depth $m \geq 1$:
An *alogarithmic transmonomial* of exponentiality m is any expression :

$${}^m\sqcap(z) = z^{\sigma}\exp(\sqcap\!\sqcap(z)) \qquad \mathit{or} \qquad {}^m\sqcup(z) = z^{\sigma}\exp(-\sqcap\!\sqcap(z))$$

[3] meaning that their logarithms are non-equivalent

[4] $\int A, \int B, \int C$ are going to be resurgent and resummable, each with a single critical time class (or more, if A, B, C already carry internal divergence).

with $\sigma \in \mathbb{R}$ and a $\sqcap\!\sqcap$ of the form:

$$\sqcap\!\sqcap \;=\; {}^{m-1}\sqcap\!\sqcap +^{m-2} \sqcap\!\sqcap + \cdots +^{1} \sqcap\!\sqcap +^{0} \sqcap\!\sqcap > 0 \quad (formally)$$

The (total) order on such transmonomials is defined inductively by:

$$\{\ \square_1 < \square_2\ \} \quad \Leftrightarrow \quad \{\ \square_1/\square_2 = \sqcup_3\ \}$$

A *homogeneous alogarithmic transseries* ${}^m\square\!\square$ of exponentiality m is any expression of the form:

$$\sqcap\!\sqcap = \sum c_\alpha \;\; {}^m\sqcap_\alpha \quad or \quad \sqcup\!\sqcup = \sum c_\alpha \;\; {}^m\sqcup_\alpha$$

with a possibly transfinite, order-reflecting indexation α, but with the finite generation axiom:

$${}^m\sqcap\!\sqcap/{}^m\sqcap_0 \quad and \quad {}^m\sqcup\!\sqcup \quad in \;\; some \;\; \mathbb{R}[[\sqcup_1, \ldots, \sqcup_r]]$$

Lastly, a *general alogarithmic transseries* $\square\!\square$ is any finite sum of homogeneous transseries.

Unique additive decomposition:
It is of the form:

$$\begin{aligned}
\square\!\square &= \sqcap\!\sqcap + const + \sqcup\!\sqcup \\
&= {}^m\sqcap\!\sqcap + \cdots +^{1} \sqcap\!\sqcap +^{0} \sqcap\!\sqcap + const +^{0} \sqcup\!\sqcup +^{1} \sqcup\!\sqcup + \cdots +^{n} \sqcup\!\sqcup \\
&= \sum_{0 \le \alpha < \omega_0 < \omega^\omega} c_\alpha \; \square_\alpha
\end{aligned}$$

and can be obtained by repeated application of the "pull-down":

$$\exp(\sqcap\!\sqcap + const + \sqcup\!\sqcup) = e^{const}.(\exp \sqcap\!\sqcap).(1 + \sum_{n \ge 1} \frac{1}{n!} \sqcup\!\sqcup^n)$$

under which all infinitesimals are expelled or "pulled down" from the exponentials, wherever (at whichever height) they may be found.

General transmonomials and transseries of depth $m > -\infty$:
They are obtained by postcomposing *alogarithmic* transmonomials or transsseries by some finite iterate $\log_n := \log^{\circ n}$ of the logarithm. General transseries clearly inherit the unique additive expansion of their alogarithmic models.

Example of general transseries:

$$
\begin{array}{rcl}
\square\!\square & = & \\
(larger) & + & \sum \mathbf{a}.\, e^{e^{e^{3\,z}\,\sum \mathbf{c}\,\sum \mathbf{b}\, e^{-\mathbf{d}\,z}\,z^{n}}}.z^{-n} \\
(large) & + & \sum \mathbf{f}.\, e^{e^{z^2}\sum \mathbf{g}\, e^{-mz}\,z^{-n}\,(\log z)^{p}}.z^{-n^3} \\
(small) & + & \sum \mathbf{h}.\, e^{-z^2}.e^{mz}.z^{-m} \\
(smaller) & + & \sum \mathbf{k}.\, e^{-mz^3}.z^{-p}.(\log z)^{q}
\end{array}
$$

1.6 The algebra $\mathbb{R}\{\{\{z\}\}\}$ of analysable germs.

To define the notion of *analysable germ*, which is the geometric counterpart of a (resummable) transseries, we require four main ingredients:
(1) the notion of *resurgent function* and *alien derivation*
(2) the so-called *acceleration transforms*, which in some sense generalise the classical Laplace transform
(3) the notion of *cohesive function* and cohesive continuation, which considerably extends that of analytic function and analytic continuation
(4) the notion of *well-behaved average*, which makes it possible to turn multiform functions on $\mathbb{R}^+$ into uniform ones, in a way that is both agreeable to the convolution product and the acceleration transforms.

All these notions shall be briefly recalled in the next paras (and in somewhat greater detail in the Appendix §6), but in order to convince ourselves of their necessity, and to show roughly where they fit into our scheme of things, we shall venture right away a (very sketchy) definition of *analysability.*

A real function germ $\varphi(z)$ at $+\infty$ is said to be analysable *if it can be obtained from some transseries $\tilde{\varphi}(z)$ under a process known as* accelero-summation*, which involves a finite number of integral transforms (Borel, then several accelerations, then Laplace) and a finite number of critical* time classes *$\{z_i\} = \{h_i(z)\}$, with link-up functions $h_i(z)$ that are themselves required to be analysable, with their own finitely many secondary critical times $\{z_{i,j}\} = \{h_{i,j}(z)\}$, and so on, leading to a finite critical tree $\{z_{i_1,i_2,\ldots,i_r}\}$.*

Moreover, we may and often do demand (though this is by no means necessary) stability under alien differentiation*: in other words, we may and* *often do ask that all alien derivatives* $\mathbf{\Delta}_{\omega_s}^{\{z_{i_s}\}} \dots \mathbf{\Delta}_{\omega_2}^{\{z_{i_2}\}} \mathbf{\Delta}_{\omega_1}^{\{z_{i_1}\}} \varphi(z)$ *be themselves analysable.*

Here is a pictorial representation of this process of accelero-summation, relative to a sequence of r critical times:

$$z_1 \prec z_2 \prec z_3 \prec \cdots \prec z_{r-1} \prec z_r \qquad\qquad (z_i \equiv h_i(z) \succ 1)$$

$$\begin{array}{lllcllll}
 & & & \textit{Laplace} & & & \\
\tilde{\varphi}_r(z_r) & z_r & \bullet & \longleftarrow & \bullet & \zeta_r & \overset{\diamond}{\varphi}_r\,(\zeta_r) \\
 & & \downarrow & & \uparrow & \textit{acceleration} & \\
\varphi(z) & z & \bullet & & \bullet & \zeta_{r-1} & \overset{\diamond}{\varphi}_{r-1}\,(\zeta_{r-1}) \\
 & & & & \uparrow & \textit{acceleration} & \\
 & & & & \dots & \dots & \dots \\
 & & & & \uparrow & \textit{acceleration} & \\
\tilde{\varphi}(z) & z & \bullet & & \bullet & \zeta_2 & \overset{\diamond}{\varphi}_2\,(\zeta_2) \\
 & & \downarrow & & \uparrow & \textit{acceleration} & \\
\tilde{\varphi}_1(z_1) & z_1 & \bullet & \longrightarrow & \bullet & \zeta_1 & \overset{\diamond}{\varphi}_1\,(\zeta_1) \\
 & & & \textit{Borel} & & &
\end{array}$$

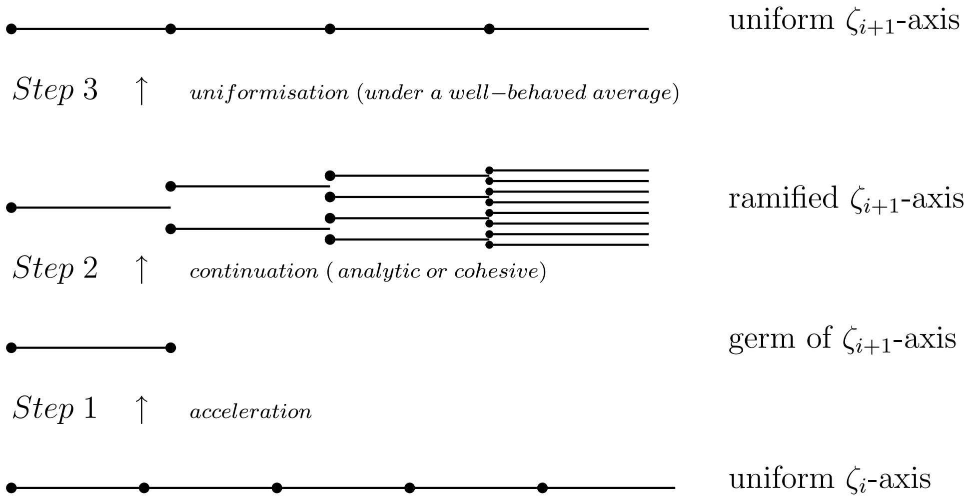
Each acceleration $\hat{\varphi}_i(\zeta_i) \mapsto \hat{\varphi}_{i+1}(\zeta_{i+1})$ is actually three steps in one.

In step 1, the acceleration transform turns a uniform function $\hat{\varphi}_i(\zeta_i)$ defined *on* the whole of $\mathbb{R}^+$ into a function germ $\hat{\varphi}_{i+1}(\zeta_{i+1})$ defined at $+0$.

In step 2, the germ in question in extended, by analytic or cohesive continuation, into a global but multiform function, defined *over* the whole of $\mathbb{R}^+$.

In step 3, under a suitable process of averaging, this multiform $\hat{\varphi}_{i+1}(\zeta_{i+1})$ is turned once again into a uniform function defined *on* the whole of $\mathbb{R}^+$, thus paving the way for the next three-stepped acceleration process.

Pictorially:



1.7 Resurgent functions and alien derivations.

The algebra of *real* resurgent fonctions (this is the one that is relevant to the construction of the trigebra of analysable germs) consists of all (analytic or cohesive) function germs at $+0$ that possess an *endless* (analytic or cohesive, but usually ramified) *forward* continuation *over* the whole of $\mathbb{R}^+$. They are subject to the convolution product, and there acts upon them an incredibly rich system of alien derivations.

More precisely, *locally* at 0, resurgent functions may be thought of as microfunctions: they are pairs $\overset{\diamond}{\varphi}(\zeta) = \{\check{\varphi}(\zeta)\,,\,\hat{\varphi}(\zeta)\}$ consisting of a major $\check{\varphi}$ defined upto a regular germ, and of a minor $\hat{\varphi}$, which is the "variation" of the major : see §6.2. The minor is exactly defined , but does not encapsulate the whole information about $\overset{\diamond}{\varphi}$, except in the important case of "integrable resurgent functions". For these the convolution product is defined by

$$(\hat{\varphi}_1 \star \hat{\varphi}_2)(\zeta) := \int_0^\zeta \hat{\varphi}_1(\zeta_1)\,\hat{\varphi}_2(\zeta-\zeta_1)\,d\zeta_1 \tag{12}$$

for ζ close to $+0$ and by analytic or cohesive continuation in the large. The convolution for general germs $\overset{\diamond}{\varphi}$ is defined in §6.2.

What really matters, however, is not the local or microfunction aspect, but the *global* properties of resurgent functions $\overset{\diamond}{\varphi}$, which come from their having endlessly continuable, but usually highly ramified minors $\hat{\varphi}$. This a source of many fascinating developments, chief amongst which is the existence of a system $\{\Delta_\omega\,,\,\omega\in\mathbb{R}^+\}$ of so-called alien derivations[5], which measure the singularities of the minor $\hat{\varphi}$ *over* the points ω and which, together, freely generate an infinite dimensional Lie algebra with a non-countable basis – despite their acting on functions of one single variable!

1.8 Basic transforms: Borel, Laplace, Acceleration.

The Borel transform:
It turns germs with subexponential growth at ∞ in the z-plane into germs at $+0$ in the ζ-plane:

$$\varphi(z)\longmapsto\hat{\varphi}(\zeta):=\frac{1}{2\pi i}\int_{c-i\infty}^{c+i\infty}\varphi(z)\,e^{z\zeta}\,dz \tag{13}$$

The Laplace transform:
It reverses the Borel transform and turns uniform functions on $\mathbb{R}^+$ with (at most) exponential growth into germs at ∞ in the z-plane or half-plane.

$$\hat{\varphi}(\zeta)\longmapsto\varphi(z):=\int_{+0}^{+\infty}\hat{\varphi}(\zeta)\,e^{-z\zeta}\,d\zeta \tag{14}$$

The acceleration transform:
It turns global, uniform functions that are defined on a ζ_1-axis and do not exceed a certain critical growth regime (the so-called *accelerable growth*, which depends on the acceleration but is always strictly faster-than-exponential) into germs that are defined at the origin of a ζ_2-axis.

The acceleration of index $F=F_{1,2}=o(z)$ transmutes under Borel-Laplace the variable change $\varphi_1\mapsto\varphi_2:=\varphi_1\circ F_{1,2}$, with the "slower time" expressed in terms of the "faster time" : $z_1=F_{1,2}(z_2)$ and $z_1\prec z_2$

$$\begin{array}{ccccc}
 & & \textit{Borel} & & \\
 & \varphi_2(z_2) & \longmapsto & \hat{\varphi}_2(\zeta_2) & \\
\textit{variable change} & \uparrow & & \uparrow & \textit{acceleration} \\
 & \varphi_1(z_1) & \longmapsto & \hat{\varphi}_1(\zeta_1) & \\
 & & \textit{Borel} & &
\end{array}$$

It is an integral transform similar in form to the Laplace transform:

$$\hat{\varphi}_1(\zeta_1)\longmapsto\hat{\varphi}_2(\zeta_2):=\int_{+0}^{+\infty}C_F(\zeta_2,\zeta_1)\,\hat{\varphi}_1(\zeta_1)\,d\zeta \tag{15}$$

[5] because they are indeed derivations, relative to the convolution product.

but with a kernel

$$C_F(\zeta_2, \zeta_1) := \frac{1}{2\pi i} \int_{c-i\infty}^{c+i\infty} e^{\,z_2\zeta_2 - z_1\zeta_1}\, d\zeta_1 \qquad \textit{with} \;\; z_1 = F(z_2) \prec z_2 \tag{16}$$

that has always strictly faster-than-exponential decrease when $\zeta_1 \to \infty$ with fixed $\zeta_2 > 0$. This explains why all acceleration operators have larger domains of definition than the Laplace transform.

Moreover, for F fixed, the closer ζ_2 is to 0, the faster that rate of ζ_1-decrease. This explains why accelerations usually yield only germs rather than (directly) global functions.

Lastly, and somewhat paradoxically, when F gets closer and closer to the identity (ie when the gap between the two times becomes less), the kernel's rate of decrease as $\zeta_1 \to +\infty$ becomes fiercer. When $\log z_1 \prec \log z_2$, ie when the two critical times are not too close, the corresponding acceleration always yields analytic [6] germs $\hat{\varphi}_2(\zeta_2)$, irrespective of the nature of the accelerand $\hat{\varphi}_1(\zeta_1)$. However, for very close, logarithmically equivalent times (ie $\log z_1 \sim \log z_2$), the domain of definition of the corresponding aceleleration operator increases hugely, while the accelerates $\hat{\varphi}_2(\zeta_2)$ usually cease to be *analytic* and become merely *cohesive* – again, irrespective of the nature of the accelerands $\hat{\varphi}_1(\zeta_1)$.

Algebra morphisms :
– Borel turns multiplication into convolution.
– Accelerations respect convolution.
– Laplace turns convolution into multiplication.
Thus, each step of the accelero-summation process is an algebra morphism – which is absolutely essential, because we want the process to apply not just to linear, but also and above all to non-linear situations. In other words, if we calculate (by purely formal manipulations – inductive coefficient identification and the like) a formal transserial solution $\tilde{\varphi}$ to some non-linear problem (say, a differential equation E), we don't want to merely "resum" $\tilde{\varphi}$. We also insist on getting a sum φ that actually verifies the original equation E. This compels us to work with algebra morphisms only.

1.9 Cohesive functions and cohesive continuation.

Analytic functions on a closed real interval I are characterised by the existence of uniform bounds on their derivatives:

$$|f^{(n)}(t)| \leq \; c_0\; c_1^n\; n^n \qquad\qquad (\forall n \in \mathbb{N}\;,\; \forall t \in I) \tag{17}$$

[6] analytic, that is, right of 0, on some internal $]0, \ldots]$, but not necessarily at 0 itself.

with constants c_0, c_1 which of course depend on f.

Similarly, the classical quasianalytic Denjoy classes ${}^{\alpha}DEN$ of finite order $\alpha \in \mathbb{N}$ are characterised by the uniform bounds:

$$
\begin{aligned}
|f^{(n)}(t)| &\leq c_0\,(c_1/\log'_{1+\alpha}(n))^n \quad \Longleftrightarrow &(18)\\
|f^{(n)}(t)| &\leq c_0\,(c_1.n.\log n\log_2 n\ldots\log_\alpha n)^n &(19)
\end{aligned}
$$

But these classes are insufficient for our purposes. We also require "Denjoy classes" of transfinite order $\alpha < \omega^\omega$, defined by the same type of bounds:

$$
|f^{(n)}(t)| \leq c_0\,(c_1/\log'_{1+\alpha} n)^n \tag{20}
$$

but relatively to suitably defined *transfinite* iterates $\log_\alpha$ of the logarithm.

The importance of the algebra *COHES* of cohesive fonctions:

$$
\mathrm{COHES} = \bigcup_{\alpha<\omega^\omega} {}^{\alpha}\mathrm{DEN} = \lim_{\alpha\to\omega^\omega} {}^{\alpha}\mathrm{DEN} \tag{21}
$$

comes:

(1) from the property of unique continuation, also known as quasianalyticity: a cohesive function defined on an interval J vanishes on the whole of J as soon as it vanishes on a given subinterval $I \subset J$ (this is I-quasianalyticity) or as soon as all its derivatives vanish at some point $t \in J$ (this is D-quasianalyticity).

(2) from its being vaster than the finite Denjoy classes: thus important functions like $\varphi(t) := \sum_{n\geq 1} \frac{1}{\exp_n(t)}$ are in ${}^{\omega}DEN$ but in no finite ${}^{n}DEN$.

(3) from its stability: whereas the *union* of all quasianalytic Carleman classes is not quasianalytic, *COHES* is closed under $+, \times, \partial$ etc

(4) from its close connection with the theory of acceleration: indeed, weak accelerates[7] are cohesive and, conversely, any cohesive function is a weak accelerate.

To fully appreciate the significance of the last point, let us recall that the whole process of recovering an analysable function φ from its transseries $\tilde{\varphi}$ resolves into a succession of accelerations, each one of which decomposes into

[7]ie *weak* means that $\log z_1 \sim \log z_2$

three substeps:

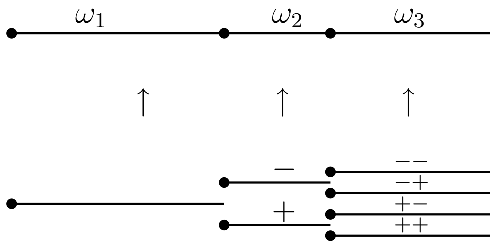
(1) the first substep turns a global, uniform function *on* $\mathbb{R}^+$ into a germ at the origin.

(2) the providential fact that this germ is either analytic or cohesive makes the second substep – unique forward continuation – possible and yields a global, but usually multiform function[8] *over* $\mathbb{R}^+$

(3) the last substep consists in averaging this multiform function to make it uniform, so that it may be subjected to the next acceleration[9]. But serious difficulties lie in wait here, which we must now address.

1.10 The problem of uniformisation.

1.10.1 The notion of "convolution-respecting average".



uniform function $(\mathbf{m}.\hat{\varphi})(\zeta)$

multiform function $\hat{\varphi}(\zeta)$

A uniformising average $\mathbf{m} : \hat{\varphi} \mapsto \mathbf{m}.\hat{\varphi}$ is defined by a system of weights $\mathbf{m}^{\left(\begin{smallmatrix}\epsilon_1 & ,\dots, & \epsilon_r\\ \omega_1 & ,\dots, & \omega_r\end{smallmatrix}\right)}$ subject to the self-consistency relations:

$$\sum_{\epsilon_i \in\{+,-\}} \mathbf{m}^{\left(\begin{smallmatrix}\epsilon_1 & ,\dots, & \epsilon_i & ,\dots, & \epsilon_r\\ \omega_1 & ,\dots, & \omega_i & ,\dots, & \omega_r\end{smallmatrix}\right)} = \mathbf{m}^{\left(\begin{smallmatrix}\epsilon_1 & ,\dots, & \epsilon_{i+1} & ,\dots, & \epsilon_r\\ \omega_1 & ,\dots, & \omega_i+\omega_{i+1} & ,\dots, & \omega_r\end{smallmatrix}\right)} \qquad \forall i \tag{22}$$

and its action is as follows:

$$(\mathbf{m}.\hat{\varphi})(\zeta) := \sum_{\epsilon_i=\pm} \mathbf{m}^{\left(\begin{smallmatrix}\epsilon_1 & ,\dots, & \epsilon_r\\ \omega_1 & ,\dots, & \omega_r\end{smallmatrix}\right)} \ \hat{\varphi}^{\left(\begin{smallmatrix}\epsilon_1 & ,\dots, & \epsilon_r\\ \omega_1 & ,\dots, & \omega_r\end{smallmatrix}\right)}(\zeta) \tag{23}$$

with $\hat{\varphi}^{\left(\begin{smallmatrix}\epsilon_1 & ,\dots, & \epsilon_r\\ \omega_1 & ,\dots, & \omega_r\end{smallmatrix}\right)}$ denoting the determination of $\hat{\varphi}$ on the branch of address $(\epsilon_1, \dots, \epsilon_r)$ over the interval $\zeta \in\,]\omega_1 + \dots \omega_r \ , \ \omega_1 + \dots \omega_{r+1}[$ between two consecutive singularities.

[8]Strange though it may seem, *cohesive functions*, despite their being defined only on the real axis and nowhere else, may possess *cohesive singularities* and these may, in some non-obvious but precise and *constructive* sense, be "by-passed", either to the right or to the left, just like analytic singularities, thus giving rise to two different determinations beyond the singularity. See [E7],[E11].

[9]or to the Laplace transform, if we are at the very end of the resummation process.

To be really useful in the present context of accelero-summation, a uniformising average must fulfill three main conditions:

(C1) It must respect convolution[10] , ie $\mathbf{m}.(\hat\varphi_1 \star \hat\varphi_2) = (\mathbf{m}.\hat\varphi_1) \star (\mathbf{m}.\hat\varphi_2)$

(C2) It must respect realness, ie $\mathbf{m}.\hat\varphi(\zeta)$, as a global function, must be real whenever $\hat\varphi(\zeta)$, as a germ at $+0$, is real.

(C3) It must respect lateral growth, that is to say, $\mathbf{m}.\hat\varphi(\zeta)$ must not grow significantly faster than the two lateral determinations (right or left) of $\hat\varphi(\zeta)$ along the positive real axis.

Such averages will be declared "well-behaved".

C1 is essential to get algebra morphisms.
C2 is natural and, in many instances, indispensible.
C3 ensures the convergence of the acceleration (or Laplace) integrals.

C2's translation in terms of weights is straightforward: the weights should change into their complex conjugates when all signs ϵ_i are changed (if real, the weights should remain unchanged). As for C1 and C3 we shall see in a moment what they imply in terms of weights. But right now we must comment, however briefly, on this vexed issue of "lateral growth".

1.10.2 Central difficulty: faster-than-lateral growth.

As pointed out, accelero-summation yields at every step i a function $\hat\varphi_i(\zeta_i)$ which has precisely the right rate of growth, ie the one that makes the next acceleration (or Laplace) possible. But this applies only to singularity-free axes or, on singularity-carrying axes, to the two lateral determinations.[11] Most other determinations of $\hat\varphi_i(\zeta_i)$, especially the ones that correspond to oft-crossing paths, tend to display slightly faster-than-lateral growth.

For instance, if the lateral growth of $\hat\varphi(\zeta)$ is exponential (this is the growth that permits Laplace), the other determinations generally admit no better uniform bounds than $|\hat\varphi(\zeta)| \le \gamma_0\ e^{\gamma_1 |\zeta| |\log \zeta|}$ (so that they cannot be subjected to Laplace). Therefore, unless we resort to carefully honed averages

[10] observe that we are dealing here with two slightly different interpretations of the convolution product: in $\mathbf{m}.(\hat\varphi_1 \star \hat\varphi_2)$ we convolute two function germs near the origin, then use analytic (or cohesive) forward continuation to get a global ramified function, and lastly we uniformise it by means of $\mathbf{m}$, whereas in $(\mathbf{m}.\hat\varphi_1) \star (\mathbf{m}.\hat\varphi_2)$ we directly convolute two global, uniform functions.

[11] and that too even if we take care of choosing in the critical time class $\{z_i\}$ a suitably slow time z_i , which precaution has the effect of smoothing the singularities of $\hat\varphi_i(\zeta_i)$.

$\mathbf{m}$, the averaged function $\mathbf{m}.\hat{\varphi}(\zeta)$ itself is going to display this slightly super-exponential growth.

This nuisance of faster-than-lateral growth is extremely common, generic almost. In the Dulac problem, it affects nearly all transit maps G_i attached to summits of type III.

In order to show just how prevalent and inescapable the phenomenon of faster-than-lateral growth is, let us adduce the simplest conceivable instance of its occurence :

$$\tilde{A}(z) \;:=\; \sum_{n\geq 1} n!\, z^{-n} \tag{24}$$

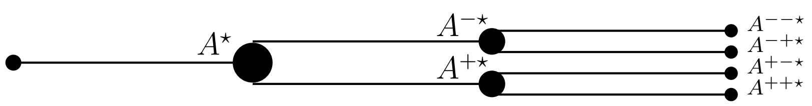
$$\hat{A}(\zeta) \;:=\; \sum_{n\geq 0} \zeta^n \;=\; 1/(1-\zeta) \tag{25}$$

$$\tilde{B}(z) \;:=\; \sum_{n\geq 1} A^n(z) \;=\; A(z)/(1-A(z)) \tag{26}$$

$$\hat{B}(\zeta) \;:=\; \sum_{n\geq 1} \hat{A}^{\star n}(\zeta) \tag{27}$$

The divergent series $\tilde{A}(z)$ verifies the Euler equation $(1+\partial_z)\tilde{A}(z) = z^{-1}$ and its Borel transform $\hat{A}(\zeta)$, with its single pole at $\zeta = 1$, is a most elementary resurgent function. Yet a simple Möbius transform turns $\tilde{A}(z)$ into a series $\tilde{B}(z)$, also solution of a first-order differential equation, but with a Borel transform $\hat{B}(\zeta)$ that has singularities at every point $\zeta = n \in \mathbb{N}$, with simple poles as leading terms and logarithmic singularities as corrections :

$$\begin{array}{rcll} \hat{B}(\zeta) & = & +R_B^{\epsilon_1,\dots,\epsilon_{n-1},\star}.(\zeta-n)^{-1} & (\mathit{simple\ pole}) \\ & & +Reg_1^{\epsilon_1,\dots,\epsilon_{n-1},\star}(\zeta-n).\log(\zeta-n) & (\mathit{logarithmic\ singularity}) \\ & & +Reg_0^{\epsilon_1,\dots,\epsilon_{n-1},\star}(\zeta-n) & (\mathit{regular\ part}) \end{array}$$



This even provides us with a discretised model of the phenomenon of faster-than-lateral growth. Indeed, the residues $R_B^{\epsilon_1,\dots,\epsilon_{n-1},\star}$ of address $\{\epsilon_i\}$ are calculable by a simple induction (see [E12]) which readily shows that they verify no better bounds than

$$|R_B^{\epsilon_1,\dots,\epsilon_{n-1},\star}| \leq c_0\, c_1^n\, n! \tag{28}$$

Therefore, although $\hat{B}$ has exponential growth (at most) on each singularity-free axis $\arg(\zeta) = \theta \neq 0$, on paths that criss-cross with the axis $\arg(\zeta) = 0$ (especially for constantly alternating ϵ_i's) it admits no better bounds than $|\hat{B}(\zeta)| \leq \gamma_0\, e^{\gamma_1 |\zeta| |\log \zeta|}$.

1.10.3 Criteria for well-behaved convolution averages.

To any uniformising average $\mathbf{m}$ we may attach the moulds[12] :

$$\mathbf{rem}^{\omega_1,\dots,\omega_r} := (-1)^r\, \mathbf{m}^{\left(\begin{smallmatrix} + & ,\dots, & + \\ \omega_1 & ,\dots, & \omega_r \end{smallmatrix}\right)} \qquad (\text{ “right-lateral mould” }) \tag{29}$$

$$\mathbf{lem}^{\omega_1,\dots,\omega_r} := (-1)^r\, \mathbf{m}^{\left(\begin{smallmatrix} - & ,\dots, & - \\ \omega_1 & ,\dots, & \omega_r \end{smallmatrix}\right)} \qquad (\text{ “left-lateral mould” }) \tag{30}$$

Due to the self-consistency relations, both the right- and left-lateral moulds encapsulate all the information about the entire weight system $\{\mathbf{m}^{\left(\begin{smallmatrix} \epsilon_1 & ,\dots, & \epsilon_r \\ \omega_1 & ,\dots, & \omega_r \end{smallmatrix}\right)}\}$, and each one can be deduced from the other in a simple manner.

The three following conditions are equivalent:
(a) the uniformising average $\mathbf{m}$ *respects convolution* [13]
(b) its right-lateral mould $\mathbf{rem}^\bullet$ is *symmetrel* [14]
(c) its left-lateral mould $\mathbf{lem}^\bullet$ is *symmetrel*

The two following conditions are equivalent:
(a) the uniformising average $\mathbf{m}$ *respects realness* [15]
(b) its moulds $\mathbf{rem}^\bullet$ and $\mathbf{lem}^\bullet$ are complex conjugate

The five following conditions are equivalent:
(d) the uniformising average $\mathbf{m}$ *respects lateral growth* [16]
(e) we have universal bounds $|\,\mathbf{rem}^{\boldsymbol{\omega}^{\prec}}\,| \leq C_1\,(D_1)^{r(\boldsymbol{\omega}^{\prec})}$
(f) we have universal bounds $|\,\mathbf{rem}^{\boldsymbol{\omega}^{\succ}}\,| \leq C_2\,(D_2)^{r(\boldsymbol{\omega}^{\succ})}$
(g) we have universal bounds $|\,\mathbf{lem}^{\boldsymbol{\omega}^{\prec}}\,| \leq C_3\,(D_3)^{r(\boldsymbol{\omega}^{\prec})}$
(h) we have universal bounds $|\,\mathbf{lem}^{\boldsymbol{\omega}^{\succ}}\,| \leq C_4\,(D_4)^{r(\boldsymbol{\omega}^{\succ})}$
for some constants C_i, D_i.

Here $M^{\bullet^{\prec}}$ (resp $M^{\bullet^{\succ}}$) denotes the forward (resp backward) *contracting arborification* of the mould $M^\bullet$, whose definition is as follows:

$$M^{\boldsymbol{\omega}^{\prec}} := \sum_{\boldsymbol{\omega^1} \gg \boldsymbol{\omega}^{\prec}} M^{\boldsymbol{\omega^1}} \qquad ; \qquad M^{\boldsymbol{\omega}^{\succ}} := \sum_{\boldsymbol{\omega^2} \gg \boldsymbol{\omega}^{\succ}} M^{\boldsymbol{\omega^2}} \tag{31}$$

[12] see §6.1
[13] this is condition C1
[14] see §6.1
[15] this is condition C2
[16] this is condition C3

The symbol $\boldsymbol{\omega}^{\prec}$ (resp $\boldsymbol{\omega}^{\succ}$) denotes any sequence $\{\omega_i\}$ with an *arborescent* (resp *anti-arborescent*) order on it, ie an order such that each element ω_i has at most one predecessor ω_{i_-} (resp one successor ω_{i_+}), whereas the sums on the right-hand side extend to all *totally ordered* sequences $\boldsymbol{\omega}^{\mathbf{1}}$ (resp $\boldsymbol{\omega}^{\mathbf{2}}$) that can be obtained from $\boldsymbol{\omega}^{\prec}$ (resp $\boldsymbol{\omega}^{\succ}$) with the possible *contraction* $\omega_i, \dots, \omega_j \mapsto \omega_i + \cdots + \omega_j$ of several consecutive elements. For some details see §6.1 and for more details go to [E5][E12][EV3].

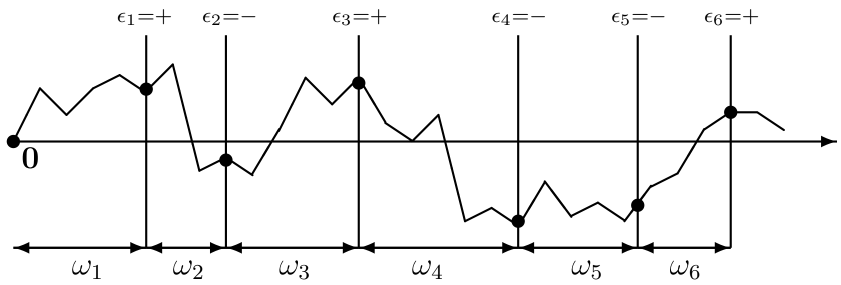
1.10.4 Examples of well-behaved averages.

For any fixed $\tau \geq 0$, consider the multiplicative semigroup $\{{}^\tau g_\omega(\bullet)\,,\,\omega > 0\}$ and its Fourier transform, the convolution semigroup $\{{}^\tau f_\omega(\bullet)\,,\,\omega > 0\}$:

$$
\begin{aligned}
{}^\tau g_\omega(y) &:= \exp(-\omega\,|y|^\tau) && (y \in \mathbb{R}) \quad (32)\\
{}^\tau f_\omega(x) &:= \frac{1}{2\pi}\int_{\mathbb{R}} {}^\tau g_\omega(y)\, e^{ixy}\, dy && (x \in \mathbb{R}) \quad (33)\\
{}^\tau g_{\omega_1}(y)\; {}^\tau g_{\omega_2}(y) &\equiv {}^\tau g_{\omega_1+\omega_2}(y) && (34)\\
({}^\tau f_{\omega_1} * {}^\tau f_{\omega_2})(x) &:= \int_{-\infty}^{+\infty} {}^\tau f_{\omega_1}(x_1)\; {}^\tau f_{\omega_2}(x-x_1)dx_1 \equiv {}^\tau f_{\omega_1+\omega_2}(x) && (35)
\end{aligned}
$$

Each function ${}^\tau f_\omega$ may be viewed as representing the probability distribution at the time $t = \omega$, on the vertical axis $\omega + i\,\mathbb{R}$, of a particle starting from the origin of $\mathbb{C}$ at $t = 0$, moving along $\mathbb{R}^+$ at uniform horizontal speed, and diffusing randomly in the vertical direction.

We may then define ${}^\tau\mathbf{m}^{\left(\begin{smallmatrix}\epsilon_1 & ,\dots, & \epsilon_r\\ \omega_1 & ,\dots, & \omega_r\end{smallmatrix}\right)}$ as the probability of our particle's successively crossing $\omega_1 + i\,\mathbb{R}^{\epsilon_1}$, $\omega_1 + \omega_2 + i\,\mathbb{R}^{\epsilon_2}, \dots, \omega_1 + \omega_2 + \dots \omega_r + i\,\mathbb{R}^{\epsilon_r}$.



Since these numbers ${}^\tau\mathbf{m}^{\left(\begin{smallmatrix}\epsilon_1 & ,\dots, & \epsilon_r\\ \omega_1 & ,\dots, & \omega_r\end{smallmatrix}\right)}$ verify the self-consistency relations (22), they may be regarded as *weights* defining a uniformising average ${}^\tau\mathbf{m}$. That average clearly respects *realness* (condition C2) but also, less obviously, *convolution* and *lateral growth* (condition C1 and C3). It is therefore well-behaved. Moreover, since the two parameters τ and ω essentially coalesce into one [17], ${}^{\tau}\mathbf{m}$ has the added distinction of being *scale-invariant*, ie invariant under a simultaneous, uniform dilation of all gaps ω_i.

1.10.5 Limit-cases: the "standard" and 'organic" averages.

For $\tau \to +\infty$ the average ${}^{\tau}\mathbf{m}$ tends to the so-called *standard* (or *median*, or *uniform*) average, with weights :

$$\mathbf{m}^{\left(\substack{\epsilon_1\ ,\dots,\ \epsilon_r\\ \omega_1\ ,\dots,\ \omega_r}\right)} := \frac{(2p)!\,(2q)!}{4^{p+q}\,(p+q)!\,p!\,q!} \qquad \begin{array}{c} p := nb\ of\ + \\ q := nb\ of\ - \end{array} \tag{36}$$

that do not depend on the gaps ω_i but only on the number p, q of $\pm$ signs in the sequence $\{\epsilon_i\}$. It is actually the only convolution- and realness-respecting average with that property, but its simplicity is deceptive, since it does not respect lateral growth.

For $\tau \to +0$ the average ${}^{\tau}\mathbf{m}$ tends to the so-called *organic* average, [18] whose weights are given by the following induction:

$$\mathbf{m}^{\left(\substack{\epsilon_1\ ,\dots,\ \epsilon_r\\ \omega_1\ ,\dots,\ \omega_r}\right)} := \mathbf{m}^{\left(\substack{\epsilon_1\ ,\dots,\ \epsilon_{r-1}\\ \omega_1\ ,\dots,\ \omega_{r-1}}\right)}.\Big(1 - \frac{1}{2}\frac{\omega_r}{\omega_1 + \dots + \omega_r}\Big) \qquad if\ \epsilon_{r-1} = \epsilon_r \tag{37}$$

$$\mathbf{m}^{\left(\substack{\epsilon_1\ ,\dots,\ \epsilon_r\\ \omega_1\ ,\dots,\ \omega_r}\right)} := \mathbf{m}^{\left(\substack{\epsilon_1\ ,\dots,\ \epsilon_{r-1}\\ \omega_1\ ,\dots,\ \omega_{r-1}}\right)}.\frac{1}{2}\frac{\omega_r}{\omega_1 + \dots + \omega_r} \qquad if\ \epsilon_{r-1} \neq \epsilon_r \tag{38}$$

and which, unlike the standard average, is *well-behaved.* A closer investigation reveals that it is, in some precise sense, the simplest of all such averages.

1.11 Application to Dulac's transit maps G_i.

Let us now return to Dulac's problem and examine what simplifications the recourse to well-behaved averages brings about. For all three types of summit S_i the formal maps $\tilde{G}_i$ may be divergent, but they are always resummable under the general resummation scheme of §1.6, with at most one critical time class for each summit [19]. Summits of type I or II (hyperbolic) offer no special difficulty, but those of type III (semi-hyperbolic) do, because for them the formal map is not a *bona fide* asymptotic series (with only one order of infinitesimals) but a strict *transseries* that mixes several orders of infinitesimals – in the present instance only two, namely powers and exponentials, or powers and logarithms. Indeed, depending on the transit direction (either

[17] indeed ${}^{\tau}f_{\omega}(x) \equiv \omega^{-1/\tau}\ {}^{\tau}f_1(x/\omega^{1/\tau})$

[18] don't ask why it is dubbed "organic": it had to be given some name !

[19] so resummation here reduces to a simple Borel-Laplace shuttle.

expanding or contracting) at a given type III summit, any formalisation $\tilde{G}_i$ of G_i that encodes all the information needed to recover G_i , must necessarily be of the form:

$$\begin{array}{rrcl} \textit{Type } III^{+} \ \ (\textit{expanding}): & \tilde{G}_i & := & \tilde{K}_i \circ \exp \circ \tilde{U}_i^{\star} \\ \textit{Type } III^{-} \ \ (\textit{contracting}): & \tilde{G}_i & := & {}^{\star}\tilde{U}_i \circ \log \circ \tilde{H}_i \end{array}$$

with ordinary formal power series (of z^{-1}) at both ends and an exponential or logarithm as mid factor. In the standard transseries expansion we get:

$$\tilde{G}_i(z) = \sum_{0\leq\alpha<\omega}^{\star} c_{i,\alpha}(z)\ A_{i,\alpha}(z) + \sum_{\omega\leq\alpha<\omega^2}^{\star\star} c_{i,\alpha}(z)\ A_{i,\alpha}(z) \tag{39}$$

ie we have a combination of an asymptotic part $\sum^{\star}$ with finite ordinals α as indices, and a transfinite part $\sum^{\star\star}$ with transfinite indices α. As a consequence, *the very definition of the formal map $\tilde{G}_i$ becomes a non-trivial affair*, and involves three distinct steps, which we shall detail, for definiteness, in the expanding case (Type III^{+}):

Step one: we calculate the formal map $\tilde{U}_i^{\star}$ as the *asymptotic part* of the geometric map $\log \circ G_i$. This formal map $\tilde{U}_i^{\star}$ turns out to be the normalising map of an identity-tangent, unitary map[20] U_i, which is none other than the holonomy map of the field X at S_i. Therefore the Borel transform of $\tilde{U}_i^{\star}$ is convergent, with singularities over $\mathbb{Z}$.

Step two: we resum $\tilde{U}_i^{\star}$ to a true map germ $U_i^{\star}$ by Borel-Laplace, *relative to some convolution-preserving average* $\mathbf{m}$ *of our choosing, but sticking with the same* $\mathbf{m}$ *for all summits.*

Step three: we calculate $\tilde{K}_i$ as the asymptotic part of the germ $G_i \circ {}^{\star}U_i \circ \log$, where ${}^{\star}U_i$ is of course the germ reciprocal to $U_i^{\star}$.

1.12 Application to Dulac's return map F.

If we now turn to the global Dulac problem, ie to the investigation of the return map F and its formalisation $\tilde{F}$:

$$\tilde{F} := \tilde{G}_r \circ \ldots \tilde{G}_2 \circ \tilde{G}_1\,(z) = \sum_{0\leq\alpha<\gamma<\omega^{\omega}} c_{\alpha}\, A_{\alpha}(z) \tag{40}$$

(c_α scalar, $A_\alpha(z)$ transmonomial) we find that this latitude in the choice of the convolution-respecting average $\mathbf{m}$ can lead to three different methods:

[20] ie a map of the form $z \mapsto U_i(z) = z(1 + \sum a_{i,n} z^{-n})$ with $\bar{U}_i \circ U_i = id$

crude ; smarter ; smartest.

First method (crude) : with the lateral average.
We may select the trivial *right-* or *left-lateral* average (either). Then of course we have no problem preserving lateral growth, but we are saddled with sums $U_i^\star$ that carry *imaginary parts.* The other factor, namely $\tilde{K}_i$, will be convergent. Both $\tilde{K}_i$ and its trivial sum K_i will have their own imaginary parts, which will cancel out the imaginary parts in $U_i^\star$, so that the composition product $K_i \circ \exp \circ U_i^\star$ will indeed yield the *real germ* G_i.

Still, the procedure introduces imaginary parts in the (strictly) trans-asymptotic coefficients c_α of the transseries $\tilde{F}$ and, even worse, inside some of its transmonomials $\tilde{A}$ – namely "upstairs", inside the towers of piled-up exponentials. This is a severe drawback for two reasons. First, imaginary parts are *personae non gratae* in the formalisation of an inherently *real object* such as F. Second, the imaginary numbers tucked away upstairs inside the exponential towers might create oscillations in the *sums* A_α of some of the transmonomials and so in $F(z) - z$ itself. By a careful inductive reasoning, we can show *that this in fact is not the case*, because the imaginary parts sitting upstairs are always neutralised by larger infinitesimals which are *purely real.* Nonetheless, the presence of imaginary parts is an aesthetic irritant and a practical nuisance. It robs the non-oscillation of $F(z) - z$ of the intuitive obviousness which it ought to possess, and which is restored with the second and third methods.

Second method (smarter) : with the standard average.
We now select the *standard* or *median* average [21]. This does away with all imaginary parts, but at the cost of introducing faster-than-lateral growth in the *uniformised* or *averaged* Borel transform $\mathbf{m}\,\hat{U}_i^\star$. This is offset, fortunately, by the phenomenon of *emanated resurgence* which has been analysed at great length in [E7] and which induces both *divergence* and *resurgence* inside the factor $\tilde{K}_i$. This time, both $\tilde{U}_i^\star$ and $\tilde{K}_i$ are *real* and *divergent-resurgent*, and the faster-than-exponential growth *cancels out* in the (uniformised) *joint Borel transform* of $\tilde{K}_i \circ \exp \circ \tilde{U}_i^\star$, so that all $\tilde{G}_1, \ldots, \tilde{G}_r$ and $\tilde{F}$ can be *exactly* accelero-summed to $G_1, \ldots, G_r$ and F.

There does, however, remain a slight flaw : unlike the transseries $\tilde{G}_i$ and $\tilde{F}$ taken as a whole, *some partial sums of these transseries may not always be resummed exactly, but only up to arbitrarily small germ ideals.* Due once again to *emanation resurgence*[22] these ideals can be rendered as small as $1/\exp_n(z)$ for any given iteration order n, that is to say, *smaller than any term present in a given transseries.* This is sufficient for all intents and purposes, and in

[21]see §1.10.5 supra. In [E.7] it was denoted by **med** and in [EM] by **mun**

[22]see [E7], pp 78-82

particular more than sufficient for proving the non-oscillation of F. But the impossibility of resumming *exactly* (rather than modulo smaller and smaller ideals) certain sub-transseries of our transseries is slightly irksome. This last remaining imperfection vanishes with the third method.

Third method (smartest) : with the organic average.
We select a *well-behaved average* $\mathbf{m}$, like the organic average of §1.10.5 or the diffusion-induced averages of §1.10.4 Then the formal factors $\tilde{K}_i$ will automatically be *convergent*, like in the first method, but also *real*, like in the second method. There will be no faster-than-lateral growth to worry about, nor any need for any compensation of any sort. And not only will all our transseries be exactly resummable to their correct sums, but so too will all their sub-transseries.

Let us summarise the main differences between the three methods in the following table. There $\tilde{G}_i^{\#}$ denotes a strict sub-transseries of $\tilde{G}_i$, while $G_i^{\#}$ and G_i stand for the corresponding sums. Whereas $G_i^{\#}$ depends on the method, G_i obvioully doesn't. Lastly, when applied to sums, *nearly real* means *real upto an exponentially small part* $\exp(-c_0 z)$, and *almost exact* means *defined up to transexponential accuracy*, ie up to infinitesimals of the form $1/\exp_n(z)$ for any finite n.

	Method I	*Method II*	*Method III*
.....	....	....	....
$\tilde{U}_i^\star$	*real* *divergent, resurgent*	*real* *divergent, resurgent*	*real* *divergent, resurgent*
$\tilde{K}_i$	*complex* *convergent*	*real* *divergent, resurgent*	*real* *convergent*
$\tilde{G}_i$	*complex* *divergent, resurgent*	*real* *divergent, resurgent*	*real* *divergent, resurgent*
$\tilde{G}_i^{\#}$	*complex* *divergent, resurgent*	*real* *divergent, resurgent*	*real* *divergent, resurgent*
.....	....	....	....
$U_i^\star$	*nearly real* *exact*	*real* *almost exact*	*real* *exact*
K_i	*nearly real* *exact*	*real* *almost exact*	*real* *exact*
G_i	*real* *exact*	*real* *exact*	*real* *exact*
$G_i^{\#}$	*nearly real* *exact*	*real* *almost exact*	*real* *exact*

1.13 Recapitulation. What has been gained?

Let us enumerate some of the main gains that accrue come from working with a *well-behaved average* – preferably the *organic average* –, first in the general theory of analysabilty, then in the particular application most relevant to this Summer School : Dulac's problem and the non-oscillation of the return map.

Exact resummation of all sub-transseries and sub-transmonomials.
This applies not only to the ones 'downstairs' but also to those 'upstairs'.

Neat representation of the tree of all "critical time classes" .
Each critical time class can now be represented by an (infinitely large) transmonomial $A(z)$, which is the common leading term shared by all representatives of that class.

Neat, canonical choice of slow critical times.
Namely $B(z) := A(z) - A(z)/\log_n(A(z))$ for n large enough.

Accelero-synthesis without parasitical divergence.
The process, which leads to the gradual actualisation of all subtransseries of a given transseries (in each model ζ_i only those parts formally exponential or subexponential in z_i can be actualised), has been described in [E7],[E10]. It still applies, but now we get rid of the parasitical resurgence, since the latter stemmed from the so-called emanation resurgence, which in turn was a consequence of working with the median average (not well-behaved).

Decelero-analysis and the "trans-formalisation" algorithm .
This is the reverse process: going from F to $\tilde{F}$. In classical asymptotics it is essentially trivial, but in transasymptotics it ceases to be so. There are indeed several ways, such as ${}^{\tau_1}\tilde{F}$ and ${}^{\tau_2}\tilde{F}$, of formalising one and the same analysable germ, depending on the w.-b. average we select, say ${}^{\tau_1}\mathbf{m}$ or ${}^{\tau_2}\mathbf{m}$, but there also exists an object even more general than the transseries, namely the *display* (it carries the *pseudo-variables* attached to *all critical times*, see [E13],ch.1) which contains at once, in explicit fashion, *all* the information about *all* the possible formalisations, and also shows how to "trans-formalise", ie how to go from one formalisation to the other

Dulac's problem: the all-convergent case .
In the all-convergent case (ie when all $\tilde{G}_i$'s are convergent) we have the commutative diagram:

$$\begin{array}{ccccc} & \{G_r,\ldots,G_2,G_1\} & \overset{compose}{\longrightarrow} & G_r\circ\cdots\circ G_2\circ G_1 =: F & \\ sum & \uparrow \quad \ldots \quad \uparrow \ \ \uparrow & & \uparrow & sum \\ & \{\tilde{G}_r,\ldots,\tilde{G}_2,\tilde{G}_1\} & \overset{compose}{\longrightarrow} & \tilde{G}_r\circ\cdots\circ \tilde{G}_2\circ \tilde{G}_1 =: \tilde{F} & \end{array}$$

which tranforms the formal dichotomy :

$$Either\ \ \tilde{F}(z)\equiv z\ \ or\ \ \tilde{F}(z)\equiv z+c_0\,\tilde{\Box}_0(z)\ +\tilde{o}\,(\tilde{\Box}_0(z)\,) \qquad with \quad c_0\neq 0$$

into the geometric or 'effective' dichotomy :

$$Either\ \ F(z)\equiv z\ \ or\ \ F(z)\equiv z+c_0\,\Box_0(z)\ +o\,(\Box_0(z)\,) \qquad with \quad c_0\neq 0$$

Dulac's problem: the general case .

$$z_1\prec z_2\prec z_3\prec\cdots\prec z_{r-1}\prec z_r \qquad\qquad (z_i\equiv h_i(z)\succ 1)$$

$$
\begin{array}{ccccccc}
 & & & \textit{Laplace} & & & \\
\tilde{F}_r(z_r) & z_r & \bullet & \longleftarrow & \bullet & \zeta_r & \overset{\diamond}{F}_r\,(\zeta_r) \\
 & & \downarrow & & \uparrow & \textit{acceleration} & \\
F(z) & z & \bullet & & \bullet & \zeta_{r-1} & \overset{\diamond}{F}_{r-1}\,(\zeta_{r-1}) \\
 & & & & \uparrow & \textit{acceleration} & \\
 & & & & \dots & \dots & \dots \\
 & & & & \uparrow & \textit{acceleration} & \\
\tilde{F}(z) & z & \bullet & & \bullet & \zeta_2 & \overset{\diamond}{F}_2\,(\zeta_2) \\
 & & \downarrow & & \uparrow & \textit{acceleration} & \\
\tilde{F}_1(z_1) & z_1 & \bullet & \longrightarrow & \bullet & \zeta_1 & \overset{\diamond}{F}_1\,(\zeta_1) \\
 & & & \textit{Borel} & & &
\end{array}
$$

Here, we have as many commutative diagrams as there are steps (even substeps) in the accelero-summation process, but the conclusion is the same as in the all-convergent case. We still have a formal dichotomy :

$$\textit{Either}\ \ \tilde{F}(z) \equiv z\ \ \textit{or}\ \ \tilde{F}(z) \equiv z + c_0\, \tilde{\square}_0(z)\ + \tilde{o}\,(\,\tilde{\square}_0(z)\,) \qquad \textit{with}\quad c_0 \neq 0$$

that automatically induces a geometric or 'effective' one :

$$\textit{Either}\ \ F(z) \equiv z\ \ \textit{or}\ \ F(z) \equiv z + c_0\, \square_0(z)\ + o\,(\,\square_0(z)\,) \qquad \textit{with}\quad c_0 \neq 0$$

An aside : the case when $F = id$.
In that case there seem to be extremely strong constraints that imply the "pairing" of divergent summits. But results here are still far from complete.

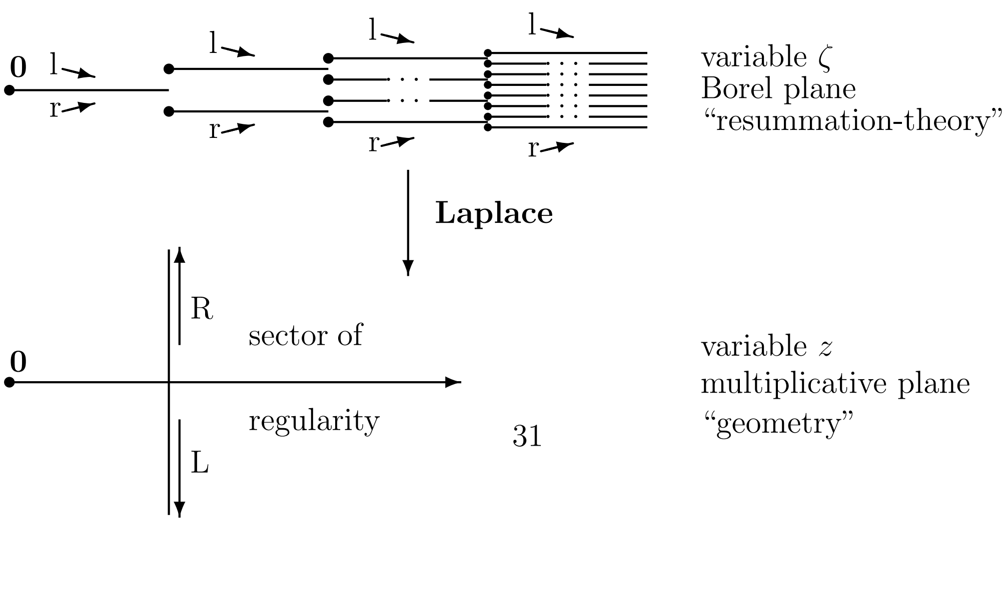
1.14 Is the finiteness proof credible ? Q & A.

Q : *Formalisation – in the present instance, replacing F by $\tilde{F}$ – may be highly efficient, but does it not also bring about an impoverishment ? Does it not rob problem-solving of all spontaneity and personal initiative, turning everything into a drab, mechanical, mindless routine ?*
A : In a sense, maybe. But that's only part of the picture. I agree : the excitement of having to grope in the dark, the attractions of trial-and-error, etc, may get eroded, but the "beauty" and "substance" are not gone : rather, *they*

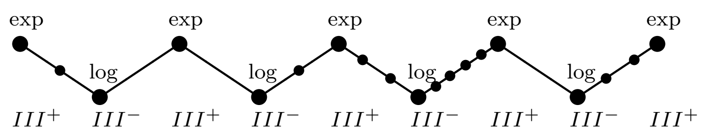
have been absorbed into the apparatus on which formalisation relies. Think of all these enchanting structures – resurgence, alien calculus, mould calculus, accelero-summation, cohesiveness, transseries, well-behaved averages, arborification, etc etc – which 'formalisation' has unearthed, polished, furbished, and brought under the clear light of day ! Can you dispute that the result is a net gain, not only in terms of problem-cracking power, but even from a purely aesthetic point of view ?

Q : *Still, you must be aware that Yu. S. Ilyashenko has come up with an alternative proof ot the finiteness theorem, which is almost entirely geometric in nature. It sticks to what you would call the 'multiplicative plane', never bothers with the 'Borel plane', and makes practically no use of resummation theory, at least not in your sense. Does it not undermine your case ? Does it not weaken your claims about the merits of 'formalisation' ?*

A : It does not. For one thing, there exist degrees in the accuracy of the various descriptions of F, and non-oscillation is but one feature among many. Then we should not forget that the return map F is a very special type of analysable germ – namely a composition product of fairly elementary transit maps G_i. It is a far cry from "general analysability". Due to its idiosyncracies, F may indeed yield to several alternative approaches, including purely geometric ones. But faced with a truly generic instance of analysability, the most that 'geometry' may hope to achieve, by going to the fringes of the z-sectors of regularity, is to capture *something* of the *lateral* behaviour (right and left of a singularity-carrying axis) in the ζ-plane (see picture), whereas the real difficulties (faster-than-lateral growth, etc) arise precisely on oftcrossing paths, which are completely off-limits for 'geometry' (it cannot even *see* them). Really, I cannot even begin to imagine how 'geometry' could possibly duplicate such things as *well-behaved derivations, w.-b. averages, w.-b. resurgence monomials*, etc, which, at some stage, become simply indispensible for getting a full grasp of analysability.



Q : *Did not a gentleman from Poland, Professor Zołądek, pass disparaging comments on your proof, while declaring himself "convinced" by Ilyashenko's ?*
A : He did. A colleague of mine has brought Zołądek's ten-line 'assessment' to my attention. What can I say ? Z. does not give the slightest indication of having read, let alone understood, even a fraction of what he quaintly calls the "French proof". So please forgive me for not commenting on his comments. But I must say I was slightly puzzled by the enthusiasm he expressed there for the "Russian proof". So I asked my colleague what she made of this, and she gave me a quite interesting answer : in her estimate, what Z. did really absorb was not the complete finiteness proof as set forth in [Il2], but the 100 page long proof in [Il1] of the special case when summits of type III^+ alternate with summits of type III^- (with any number of type I or II summits in between – they make no difference). Now, this is an *interesting, non-trivial, but still rather elementary* special case. And if one goes for comparisons, then one should compare comparable things. In my scheme of things, that case offers few difficulties (it presents no full-blown transmonomials, no exponential towers, no cohesive steps, no critical time tree, and requires only elementary accelerations) and it can be dealt with quite expeditiously ([E7], p 202).



Q : *Going back to the general case and its two long-winded proofs : What trust can we repose in them ? Have they been checked from A to Z by serious mathematicians? Is there going to be certainty/agreement any time soon ?*
A : I can only speak for myself. I don't think my book has been read and checked, from cover to cover, by anyone really intent on verifying the proof. But I do know that its *methods* have come under close scrutiny, and that too from people who desired to *apply* them, including in limit situations, and also to *extend* them. *These methods have not been found wanting.* And let us not forget that we are speaking here about a *constructive* proof. Experience teaches that such proofs have a self-righting quality, a resilience all their own. The notion that, due to some gap or mistake lurking somewhere in the meanders of the proofs, the whole thing might suddenly collapse – that notion, I think, can be discounted rightaway. But you asked : when will there be general acceptance ? Well, I would imagine that it will come in due

course, but gradually, and not as a result of *someone actually checking all the minutiae of the proof* (a thankless undertaking!), but due to *a growing familiarity with, and confidence in, the methods on which it rests.* Possibly, 15 or 20 years from now, *analysability* and all the wherewithal will have become so much part of the mathematical landscape that people, when faced with an object like the return map, will say, after a brief mental evaluation: "Hm, hm! This F is a composition product of clearly analysable factors G_i. But everyone knows that composition preserves analysability. QED!" In the meantime, however, there are bound to remain doubters alongside the believers, and that's how it should be!

1.15 Complement 1: the trans-Lagrange inversion formula.

To convey something of the flavour of the theory of transseries, let us mention the analogue of the classical Lagrange formula for calculating the functional inverse. Everything being formal here, we drop all tildas.

Framework. We consider mutually reciprocal transseries $f, g\ :\ f\circ g = id$ with bounded logarithmic depth (only those make full transasymptotic sense). To further simplify, we apply a "flattening" change of variables:

$$(f\ ,\ g)\quad\mapsto\quad(\log_p\circ f\circ\exp_q\ ,\ \log_q\circ g\circ\exp_p)$$

with p, q large enough to ensure that $f(x)=x+o(1)$, $g(x)=x+o(1)$.

Notations. For any transseries φ and any transmonomial A, let $Proj_A(\varphi)$ denote the scalar coefficient $c_A\in\mathbb{R}$ of A in the canonical expansion $\varphi = \sum c_A\, A$. In particular, let $Proj_1(\varphi)$ denote the constant term in that same expansion (it separates the infinitely large transmonomials to its left from those infinitely small to its right.) Lastly, let φ' (resp $'\varphi$) denote the derivative of φ (resp its indefinite integral *without constant term*).

Matrix elements. For any two transseries A, B and any $f\succ 1$, we set

$$f_{[A,B]}\ :=\ Proj_1(A\circ f\,.\,B)\qquad(\in\mathbb{R})\tag{41}$$

In the special case when A, B (and so too $1/B$) are transmonomials, the coefficients $f_{[A,1/B]}$ may be viewed as matrix elements of the f-postcomposition operator. Indeed we have:

$$A\circ f\ \equiv\ \sum_B f_{[A,1/B]}\,.\,B\qquad\text{(transfinite sum)}\tag{42}$$

$$(f_1\circ f_2)_{[A,1/C]}\ \equiv\ \sum_B f_{1[A,1/B]}\cdot f_{2[B,1/C]}\qquad\text{(finite sum)}\tag{43}$$

$$(f_1, f_2\ \ \textit{transseries}\succ 1\quad,\quad A,B,C\quad \textit{transmonomials})\tag{44}$$

The trans-Lagrange inversion formula.
Lemma: *Mutually reciprocal transseries f, g, flattened so as to verify $f(x) = x + o(1)$, $g(x) = x + o(1)$, always have their matrix elements linked by the simple involutive formula:*

$$g_{[A,B]} = -f_{[A^\star,B^\star]} \qquad \textit{with} \qquad A^\star = {}'(B \,.\, \log'_q) \quad , \quad B^\star = A' \,/\, \log'_q \tag{45}$$

valid for any q larger than the logarithmic depth of A, B, f, g.

The Lagrange inversion formula. If f, g are ordinary power series:

$$f(x) \;=\; x\,(1 + \sum a_n\, x^{-n}) \qquad\qquad (x \sim +\infty) \tag{46}$$

$$g(x) \;=\; x\,(1 + \sum b_n\, x^{-n}) \qquad\qquad (x \sim +\infty) \tag{47}$$

and if we take (A, B) to be ordinary monomials (x^m, x^n), we may choose $q = 1$ in the above formula, and then we get:

$$f_{[x^m,x^n]} \;=\; \frac{m}{n}\, g_{[x^n,x^m]} \qquad\qquad (m, n \in \mathbb{Z}^\star) \tag{48}$$

which is just an unusual presentation of the classical Lagrange inversion formula.

1.16 Complement 2: beyond the exponential range.

The *indiscernibility theorem*, very roughly, asserts that *from a purely asymptotic viewpoint there exists no canonical notion of fractional or transfinite iteration for the maps* log *or* exp . More precisely, *for any strictly fractional $\alpha \in \mathbb{Q}^+ - \mathbb{N}$ or any strictly transfinite ordinal $\alpha \in [\omega, \omega^\omega[$, no purely asymptotic criterion at $+\infty$, based on the behaviour (sign) of expressions finitely constructed from $\log_\alpha$ or $\exp_\alpha$, plus the direct/inverse operations $+, \times, \partial, \circ$, plus a finite number of analysable germs φ_i, can enable us to isolate a privileged representative in the class of all possible iterates of the same order α – no matter what degree of regularity* [23] *we choose to impose.*

This and a few other results of the same ilk suggest that, in some sense, the trigebras in the pair:

$$\mathbb{R}[[[z]]]\ (\textit{transseries}) \quad \textit{and} \quad \mathbb{R}\{\{\{z\}\}\}\ (\textit{analysable germs})$$

[23] whether cohesivity, analyticity, softness (see [E7], p 287) or any combination of these.

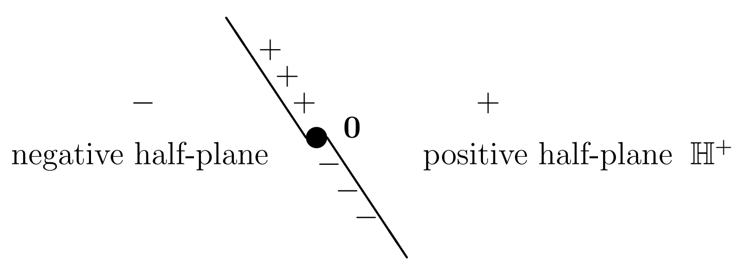
are the "largest of their kind". If we insist on enlarging them, some essential properties are bound to give way. Fortunately, these two provide a framework capacious and flexible enough for most problems of smooth asymptotics. It is worth noting, in particular, that differential equations can never generate an asymptotic behaviour even remotely like that of $\log_\alpha$ or $\exp_\alpha$ for a fractional or transfinite α (unless this behaviour is already there in the data).

1.17 Complement 3 : van der Hoeven's intermediate value theorem.

Theorem (J. van der Hoeven):
Let P be a differential polynomial with transserial coefficients, ie with coefficients in the trigebra $\mathbb{R}[[[z]]]$. Then, given any pair of transseries $\tilde{F} < \tilde{G}$ in $\mathbb{R}[[[z]]]$ such that $P(\tilde{F}) < 0 < P(\tilde{G})$ there exists $\tilde{H}$ in $\mathbb{R}[[[z]]]$ such that $\tilde{F} < \tilde{H} < \tilde{G}$ and $P(\tilde{H}) = 0$. Furthermore, there exists a fully algorithmic procedure for constructing such an "intermediate solution" $\tilde{H}$.

1.18 Complement 4 : van der Hoeven's complex transseries as a tool for solving algebraic differential equations.

The difficulty with complexifying transseries is that the presence of imaginary parts "high up" in the transmonomials (inside exponentials) creates oscillations, so that we can no longer compare, much less totally order, these complex transmonomials. But there is a way round this difficulty : we may equip $\mathbb{C}$ with an addition-respecting (total) order, regard our complex transmonomials as pure symbols, and then proceed with the inductive construction of complex transseries exactly as in the real case. There are as many addition-respecting orders on $\mathbb{C}$ as there are "positive half-planes" $\mathbb{H}^+$ containing $\mathbb{R}^+$ and excluding $\mathbb{R}^-$, with an additional convention for including half the boundary of $\mathbb{H}^+$ and excluding the other half. Different choices lead to different constructions $\mathbb{C}^{\mathbb{H}^+}[[[z]]]$, but these are all isomorphic. So it is permissible to speak of *the* trigebra $\mathbb{C}[[[z]]]$ of complex transseries.



Theorem (J. van der Hoeven):
Any algebraic differential equation $P(y) = 0$ *polynomial in* $y, y', y'', \dots, y^{(d)}$ *and with arbitrary transserial coefficients in* $\mathbb{C}[[[x]]]$ *admits solutions in* $\mathbb{C}[[[x]]]$.

However, there doesn't always exist a full d-parameter family of solutions, so that not *all* solutions are caught in our net – which is another way of saying that $\mathbb{C}[[[x]]]$ is not algebraically-differentially closed.

The true challenge, however, would be to resum as many complex transseries as possible – beginning with those of natural origin, like the algebraic-differential elements of $\mathbb{C}[[[x]]]$. Encouraging forays have been made in this direction, but there are serious obstacles, mainly because it is not at all clear what should take the place of accelero-summation in the new context.

1.19 Complement 5 : finitary transseries and the dimorphic core $\mathbb{N}\mathrm{a}$ of $\mathbb{R}$.

Let us now return to the real transseries. If in our inductive construction of $\mathbb{R}[[[x]]]$ in §1.5 we replace $\mathbb{R}$ by the field $\mathbb{A}$ of algebraic numbers, and the rings of power series $\mathbb{R}[[z^{-\sigma_1}, \dots, z^{-\sigma_r}]]$ and $\mathbb{R}[[\sqcup_1, \dots, \sqcup_r]]$ by the polynomial rings $\mathbb{A}[z^{-\sigma_1}, \dots, z^{-\sigma_r}]$ and $\mathbb{A}[\sqcup_1, \dots, \sqcup_r]$, we get the trigebra $\mathbb{A}^{\mathrm{fin}}[[[x]]]$ of *finitary transseries*, so-called because their definition (or construction) draws only on a finite number of parameters.

Two huge sets of exotic derivations – the *alien derivations* $\Delta_\omega^{\{z_i\}}$ and the *foreign derivations* $\nabla_\omega^{\{z_i\}}$, each relative to a specific index $\omega \in \mathbb{R}^+$ and a specific time class $\{z_i\}$ – act on these finitary transseries and produce transcendental constants, the so-called *naturals*, which (unlike the over-publicised *periods*) come equipped with a natural indexation. The *naturals* order themselves spontaneously into a hierarchy of *dimorphic rings* $\mathbb{D}_1, \mathbb{D}_2, \mathbb{D}_3, \mathbb{D}_4, \dots$:

$$
\begin{array}{lcll}
\mathbb{Q} & = & \textit{rationals} & \\
\mathbb{A} & = & \textit{algebraic numbers} & \Longleftarrow \ \textit{Galois theory} \\
\mathbb{D}_1 & = & \textit{multizetas} & \Longleftarrow \ \textit{ARI/GARI} \\
\mathbb{D}_2 & = & \textit{hyperlogarithmic constants} & \Longleftarrow \ \textit{ARI/GARI} \ ? \\
\mathbb{D}_3 & = & \textit{linear alg. differ. numbers} & \Longleftarrow \ ?\,? \\
\mathbb{D}_4 & = & \textit{nonlinear alg. differ. numbers} & \Longleftarrow \ ?\,?\,?
\end{array}
$$

that is to say, rings that possess two natural bases, each with its own multiplication rule, and, half-way between these two bases, a *canonical and highly* *non-trivial system of irreducibles.* That at least is the case with $\mathbb{D}_1$. See §5. The rings that lie beyond are still very much uncharted territory :

$$\begin{array}{ccccc} \textit{first natural basis with} & & & & \textit{second natural basis with} \\ \textit{multiplication table} & & & & \textit{multiplication table} \\ \{\alpha_m\} & & \textit{hidden canonical} & & \{\beta_n\} \\ & \searrow & \textit{irreducibles} & \swarrow & \\ & & \{\gamma_s\} & & \end{array}$$

2 Lesson Two: Normal and Prenormal Forms.

2.1 Local objects and their homogeneous components. Resonance, quasiresonance, nihilence.

By *local analytic object* we shall mean, primarily:
(1) germs of singular analytic vector fields at 0 on $\mathbb{C}^\nu$, often referred to as just *fields* for short
(2) germs of analytic diffeomorphisms of $\mathbb{C}^\nu$ into itself, with 0 as fixed point, or *diffeos* for short
and, secondarily, all those equations or systems (differential, difference, functional, etc) which may, in a standard manner, be rephrased in terms of fields or diffeos.

Fields will be noted

$$X = \sum_{1 \le i \le \nu} X_i(x)\,\partial_{x_i} \qquad ; \qquad X_i(x) \in \mathbb{C}\{x_1, \ldots, x_\nu\} \tag{49}$$

but instead of diffeos proper:

$$f : (x_i) \mapsto (f_i(x)) \qquad i = 1, \ldots, \nu\ ; \quad f_i(x) \in \mathbb{C}\{x_1, \ldots, x_\nu\} \tag{50}$$

it will be more convenient to handle the corresponding substitution operators F (same symbols, but capitalised):

$$(F\varphi)(x) := \varphi(f(x)) \quad ; \qquad \forall \varphi(x) \in \mathbb{C}\{x_1, \ldots, x_\nu\} \tag{51}$$

Working with *operators* makes it easier to isolate a local object's *homogeneous components* $\mathbb{B}_n$:

$$
\begin{aligned}
X &= X^{\text{lin}} + \sum_n \mathbb{B}_n \qquad \text{(for a field)} && (52)\\
F &= (1 + \sum_n \mathbb{B}_n)\, F^{\text{lin}} \qquad \text{(for a diffeo)} && (53)\\
\textit{with} \qquad \mathbb{B}_n &: \quad x^m.\mathbb{C} \to x^{m+n}.\mathbb{C} \qquad (m, n \ \textit{multiintegers}) && (54)
\end{aligned}
$$

For simplicity, we shall assume that the linear part is diagonalisable, and work in an analytic chart where it is actually diagonal:

$$
\begin{aligned}
X^{\text{lin}} &= \sum_i \lambda_i\, x_i\, \partial_{x_i} && (55)\\
F^{\text{lin}} &: \ \varphi(x_1, \dots, x_r) \mapsto \varphi(l_1\, x_1, \dots, l_r\, x_r) \qquad \textit{with} \ \ l_i \in \mathbb{C}^\star && (56)
\end{aligned}
$$

The discussion hinges on the nature of the object's spectrum, ie the eigenvalues of its linear part: $\lambda = (\lambda_1, \dots, \lambda_\nu)$ for a field; $l = (l_1, \dots, l_\nu)$ for a diffeo. If the spectrum is "generic", then the object is analytically conjugate to its linear part – and that ends the matter, at least from the local point of view.

Difficulties arise only in the case of
– *resonance*: relations of type $0 = \lambda_i - \sum m_i\, \lambda_j$ or $0 = l_i - \prod l_j^{m_j}$ with non-negative integers m_j.
– *quasiresonance*: ie when Bryuno's well-known diophantine condition, which minorises the above expressions in terms of $\|m\|$, is not fulfilled [24].
– *nihilence*: this complication, which presupposes resonance but bears on coefficients of all orders, occurs mostly, though not only, in symplectic or volume-preserving objects[25].

The more 'complicated' an object, the larger its set of *invariants* tends to be. Alongside the *formal* and *analytic invariants* (ie relative to formal or analytic coordinate changes) we have the notion of *holomorphic invariants* – ie invariants that depend holomorphically on the object **Ob** (or, in practical terms, its Taylor coefficients), at least when **Ob** remains within a fixed formal conjugacy class. [26]

Resonance generates *formal invariants* (other than the spectrum itself), of which there may be an infinite number[27] if the resonance degree is ≥ 2.

[24] see conditions (171)+(172) in Lesson 4.
[25] see Lesson 4.
[26] thus, when applied to *invariants*, the words *analytic* and *holomorphic* assume quite different meanings: analytic invariants are not necessarily holomorphic.
[27] more precisely, an infinite number of *independent* formal invariants.

Each of the aforementioned complications – resonance, quasiresonance, nihilence – whether in isolation or in combination, gives rise to analytic invariants (strictly analytic, ie non formal). Moreover, when resonance alone is at work, there tend to exist [28] complete systems of analytic-cum-holomorphic invariants $\{\mathbb{A}_\omega\}$.

2.2 Continuous prenormalisation versus discontinuous normalisation.

A resonant object X or F usually cannot be fully linearised – not even formally – but, under a formal change of coordinates corresponding to a substitution operator Θ_{pre} , it can always be brought into a so-called *prenormal form* X^{pre} or F^{pre}

$$X = \Theta_{\text{pre}}\, X^{\text{pre}}\, \Theta_{\text{pre}}^{-1} \tag{57}$$

$$F = \Theta_{\text{pre}}\, F^{\text{pre}}\, \Theta_{\text{pre}}^{-1} \tag{58}$$

that is to say, a form which commutes with the linear part of X:

$$[X^{\text{lin}}, X^{\text{pre}}] = 0 \qquad ; \qquad [F^{\text{lin}}, F^{\text{pre}}] = 0 \tag{59}$$

and which therefore involves only *resonant* homogeneous components $B_n^\star$:

$$X^{\text{pre}} = X^{\text{lin}} + \sum \mathbb{B}_n^\star \qquad \textit{with} \quad < n, \lambda >= 0 \tag{60}$$

$$F^{\text{pre}} = (\,1 + \sum \mathbb{B}_n^\star\,)\, F^{\text{lin}} \qquad \textit{with} \quad l^n = 1 \tag{61}$$

If the number of (non-vanishing) resonant components $\mathbb{B}_n^\star$ is minimal (which implies that their coefficients are formal invariants of the object), then X^{pre} or F^{pre} deserves to be regarded as a *normal form* and will be noted X^{nor} or F^{nor}. For objects with one (resp several) degrees of resonance, there exist finitely (resp infinitely) many independent formal invariants.

Though simplest in terms of outward shape, the normal forms X^{nor} or F^{nor} have quite a few drawbacks. One is the unavoidably non-continuous nature of the maps $X \mapsto X^{\text{nor}}$ or $F \mapsto F^{\text{nor}}$, even when we keep the linear parts X^{lin} or F^{lin} fixed. Another is the absence, thus far, of general and truly algorithmic procedures for determining them, especially in the case of highly resonant spectra. A third drawback (manifest even in the case of *simple* resonance) is their unsuitability for mechanical computation: the

[28] with slight qualifications, see eg [E5].

exact shape of the normal forms always depends on one or several discrete invariants (such as the "levels", see for ex. [E3]), whose exact values, in turn, depend on whether certain real or complex numbers (which depend polynomially on the Taylor coefficients of the object) do or do not vanish – a matter which a computer usually cannot decide, for want of algorithmic tests.

So, for many purposes, it is preferable to work with *continuous prenormal forms* or rather, to be quite precise, with *prenormal forms that depend continuously on the homogeneous components* $\mathbb{B}_n$ *of the object, while the linear part is kept fixed.*[29]

In concrete terms, a *continuous prenormalisation* is a universal correspondence of the form:

$$\begin{aligned} X = X^{\rm lin} + \sum \mathbb{B}_n \quad &\mapsto \quad X^{\rm pre} = X^{\rm lin} + \sum {\rm pran}^\bullet\, \mathbb{B}_\bullet && (62)\\ &= \quad X^{\rm lin} + \sum {\rm pran}^{\omega_1,\dots,\omega_r}\, \mathbb{B}_{n_1,\dots,n_r} && (63)\\ F = (1 + \sum \mathbb{B}_n)\, F^{\rm lin} \quad &\mapsto \quad F^{\rm pre} = (1 + \sum {\rm pren}^\bullet\, \mathbb{B}_\bullet)\ F^{\rm lin} && (64)\\ &= \quad (1 + \sum {\rm pren}^{\omega_1,\dots,\omega_r}\, \mathbb{B}_{n_1,\dots,n_r})\ F^{\rm lin} && (65)\\ \textit{with} \qquad \mathbb{B}_{n_1,n_2,\dots,n_r} \quad &:= \quad \mathbb{B}_{n_r} \dots \mathbb{B}_{n_2} \mathbb{B}_{n_1} && (66) \end{aligned}$$

which is entirely determined by a given system $\{pran^\bullet\}$ or $\{pren^\bullet\}$ of scalar coefficients[30] independent of the object and indexed by sequences $\boldsymbol{\omega}$ constructed from the 'frequencies' $\omega_i :=< n_i, \lambda >$.

2.3 Examples of prenormalisations : 'regal' and 'royal'.

Let us concentrate on *vector fields* for simplicity. In practice, the construction of the (alternal) *prenormal* mould $pran^\bullet$ goes hand in hand with the construction of the (symmetral) *prenormalising* mould $spran^\bullet$ that yields the prenormalising substitution operator:

$$\Theta_{\rm pre} \ := \ 1 + \sum {\rm spran}^\bullet\, \mathbb{B}_\bullet = 1 + \sum {\rm spran}^{\omega_1,\dots,\omega_r}\, \mathbb{B}_{n_1,\dots,n_r}$$

Here are two examples – the simplest of all, as it will turn out.

Regal prenormalisation :

[29] continuous dependence on the *linear part* would be an obviously impossible demand.

[30] The vowel a (resp e) alludes to the symmetry type – symmetral or alternal (resp symmetrel or alternel) – of the moulds $pran^\bullet$, $spran^\bullet$, *etc* (resp $pren^\bullet$, $spren^\bullet$, *etc*) which we call into play in the case of fields (resp diffeos).

The moulds are characterised by

$$\square\,\mathrm{spran}^\bullet \;=\; \mathrm{pran}^\bullet\times\mathrm{spran}^\bullet-\mathrm{spran}^\bullet\times\mathrm{I}^\bullet \tag{67}$$

$$\mathrm{lang}\,.\,\mathrm{spran}^\bullet \;=\; \mathrm{copran}^\bullet\times\mathrm{spran}^\bullet \tag{68}$$

together with the initial conditions

$$\mathrm{spran}^\emptyset = 1 \qquad (\textit{due to symmetrality}) \tag{69}$$

$$\mathrm{pran}^\emptyset = \mathrm{copran}^\emptyset = 0 \qquad (\textit{due to alternality}) \tag{70}$$

and the "complementarity" conditions

$$\mathrm{pran}^{\boldsymbol{\omega}} = 0 \qquad (if\ \ \|\boldsymbol{\omega}\|\neq 0) \tag{71}$$

$$\mathrm{copran}^{\boldsymbol{\omega}} = 0 \qquad (if\ \ \|\boldsymbol{\omega}\| = 0) \tag{72}$$

The two mould operators $\square$ and *lang* occuring in the above system multiply moulds by, respectively, the sum $\|\boldsymbol{\omega}\|$ or the length $r(\boldsymbol{\omega})$ of their indexing sequences $\boldsymbol{\omega}$, and the obvious fact of their being *mould derivations*:

$$\square\,(A^\bullet\times B^\bullet)\equiv(\square\,A^\bullet)\times B^\bullet+A^\bullet\times(\square\,B^\bullet)$$

$$\mathrm{lang}\,(A^\bullet\times B^\bullet)\equiv(\mathrm{lang}\,A^\bullet)\times B^\bullet+A^\bullet\times(\mathrm{lang}\,B^\bullet)$$

ensures that the (clearly unique) solution of the above system consists of a mould pair *spran*$^\bullet$/*pran*$^\bullet$ of *symmetral/alternal* type.

Royal prenormalisation :

It is defined by the same system of mould equations as the *regal form*, except that we replace the mould derivation *lang* by the mould derivation *lan*, which mutiplies a mould, not by its length $r(\boldsymbol{\omega})$ (total number of indices), but its reduced length $r^\star(\boldsymbol{\omega})$, ie the number of its non-zero indices ω_i. So the new system reads :

$$\square\,\mathrm{spran}^\bullet \;=\; \mathrm{pran}^\bullet\times\mathrm{spran}^\bullet-\mathrm{spran}^\bullet\times\mathrm{I}^\bullet \tag{73}$$

$$\mathrm{lan}\,.\,\mathrm{spran}^\bullet \;=\; \mathrm{copran}^\bullet\times\mathrm{spran}^\bullet \tag{74}$$

with the same initial conditions as above and also the same "complementarity" conditions.

This innocuous-looking change brings with it considerable simplifications, and in fact yields what must be adjudged the *simplest* and most regular of all prenormalisations – hence its name *royal*.

2.4 Mould amplification.

To show that the *royal prenormalisation* cannot be simplified or improved upon, and also to establish its analytic properties (namely, generic divergence + resurgence), we require a mould transform – the so-called amplification – that preserves a mould's symmetry type but isolates the contribution of its vanishing indices $\omega_i = 0$. Indeed, if a mould $M^\bullet$ is alternal (resp symmetral), then the amplified mould $\underline{M}^\bullet$, which is defined by

$$\underline{M}^{\varpi_1,\dots,\varpi_r} \quad := \quad \sum_{n_i \geq 0} M^{\,\omega_1,0^{[n_1]},\dots,\omega_r,0^{[n_r]}}\,(\check{a}_1)^{n_1}(\check{a}_2)^{n_2}\dots(\check{a}_r)^{n_r} \quad (75)$$

$$\varpi_i := \binom{\omega_i}{a_i} \in \binom{\mathbb{C}^\star}{\mathbb{C}}; \qquad \check{a}_i := a_1 + a_2 + \cdots + a_i\,; \qquad 0^{[n_i]} := \overbrace{(0,\dots,0)}^{n_i}$$

is also alternal (resp. symmetral). Amplification "almost" respects mould multiplication (see [EV1], p16), and has a simple effect on most other mould operations. For instance, the three mould derivations that enter the definition of the *regal* and *royal* prenormalisations become after amplification :

$$\begin{aligned}
\underline{\square}\,.\,\underline{M}^{\boldsymbol{\varpi}} \;&:=\; \|\boldsymbol{\varpi}\|\;\underline{M}^{\boldsymbol{\varpi}} \;:=\; (\|\boldsymbol{a}\| + \|\boldsymbol{\omega}\|)\;\underline{M}^{\boldsymbol{\varpi}} && (76)\\
\underline{\mathrm{lang}}\,.\,\underline{\mathrm{M}}^{\boldsymbol{\varpi}} \;&=\; (r(\boldsymbol{\omega}) + \sum a_i\,\partial_{a_i})\;\underline{\mathrm{M}}^{\boldsymbol{\varpi}} && (77)\\
\underline{\mathrm{lan}}\,.\,\underline{\mathrm{M}}^{\boldsymbol{\varpi}} \;&=\; r^\star(\boldsymbol{\omega})\;\underline{\mathrm{M}}^{\boldsymbol{\varpi}} && (78)
\end{aligned}$$

Thus, one easily finds that the first non-trivial values of $pran^{\boldsymbol{\varpi}}$, corresponding to a length $r = 2$ and $\omega_1 + \omega_2 = 0, \omega_i \neq 0$, are as follows :

$$\begin{aligned}
\underline{\mathrm{pran}}^{\varpi_1,\varpi_2} \;&=\; \frac{1}{a_1+a_2}\,\log\left(\frac{1+\frac{a_2}{\omega_2}}{1+\frac{a_1}{\omega_1}}\right) \quad ; \qquad \text{"regal"} && (79)\\
\underline{\mathrm{pran}}^{\varpi_1,\varpi_2} \;&=\; \frac{1}{2}\left(\frac{1}{\omega_2+a_2} - \frac{1}{\omega_1+a_1}\right) \quad ; \qquad \text{"royal"} && (80)
\end{aligned}$$

More generally, the moulds $pran^\bullet$ and $spran^\bullet$ associated with the royal (resp. regal) prenormalisation are rational (resp hyperlogarithmic) functions of their two sets of indices (a_i and ω_i).

Let us ponder a moment the above formulas.
(1) Though the regal and royal $pran^\bullet$ differ, they are seen to coincide for $a_1 + a_2 = 0$ (recall that $\omega_1 + \omega_2 = 0$ anyhow).
(2) For fixed indices ω_i both are regular functions of the a_i-variables at the origin, but with a (common) singular locus away from the origin, and no natural analytic boundaries.

Those two observations, still valid beyond $r = 2$, are highly significant, since the presence of singularities in the a_i variables stands in close relation to the *divergence* of the corresponding prenormalisation, while the absence of obstacles to endless analytic continuation (around these singularities) is related to the *resurgence* properties of that same prenormalisation.

This immediately raises the question: are these singularities in the a_i variables peculiar to the two prenormalisations just examined, or are they absolutely unavoidable? As it happens, the latter is the case, but to establish this we require a mould $hard^\bullet$ that describes "all that is common" (singularities included) to *all* (not just the *regal* or *royal*) *prenormalisations.*

2.5 The 'hard core' common to all prenormalisations.

Taking our clue from the above remark, we define $\underline{prann}^{\boldsymbol{\varpi}}$ as being the restriction of $\underline{pran}^{\boldsymbol{\varpi}}$ on the hyperplane $\|\boldsymbol{a}\| = 0$. [31]
We then define the 'hard core' $hard^\bullet$ in this way: for any set $\boldsymbol{\varpi^1, \varpi^2, \ldots, \varpi^m}$ of unbreakable, zero-sum sequences,[32] and any natural integer n no greater than m we set

$$\mathrm{hard}_n^{\boldsymbol{\varpi^1};\ldots;\boldsymbol{\varpi^m}} := \sum_{\boldsymbol{\varpi^{(1)}}\ldots\boldsymbol{\varpi^{(n)}}\in\mathbf{circ}(\boldsymbol{\varpi^1};\ldots;\boldsymbol{\varpi^m})} \underline{\mathrm{Prann}}^{\boldsymbol{\varpi^{(1)}}}\ldots\underline{\mathrm{Prann}}^{\boldsymbol{\varpi^{(n)}}} \qquad (81)$$

with sequences $\boldsymbol{\varpi^{(i)}}$ made up of one or several sequences $\boldsymbol{\varpi^j}$ and with a sum $\sum$ ranging over all decompositions of the form:

$$\boldsymbol{\varpi^{(1)}\varpi^{(2)}}\ldots\boldsymbol{\varpi^{(n)}} \equiv \boldsymbol{\varpi^i\varpi^{i+1}}\ldots\boldsymbol{\varpi^m\varpi^1}\ldots\boldsymbol{\varpi^{(i-2)}\varpi^{(i-1)}} \qquad (for\ some\ i)$$

Thus for $m = 1, 2, 3$ we get:

$$\begin{aligned}
\mathrm{hard}_1^{\boldsymbol{\varpi^1}} &:= \underline{\mathrm{prann}}^{\boldsymbol{\varpi^1}}\\
\mathrm{hard}_1^{\boldsymbol{\varpi^1};\boldsymbol{\varpi^2}} &:= \underline{\mathrm{prann}}^{\boldsymbol{\varpi^1\varpi^2}} + \underline{\mathrm{prann}}^{\boldsymbol{\varpi^2\varpi^1}}\\
\mathrm{hard}_2^{\boldsymbol{\varpi^1};\boldsymbol{\varpi^2}} &:= \underline{\mathrm{prann}}^{\boldsymbol{\varpi^1}}\,\underline{\mathrm{prann}}^{\boldsymbol{\varpi^2}}\\
\mathrm{hard}_1^{\boldsymbol{\varpi^1};\boldsymbol{\varpi^2};\boldsymbol{\varpi^3}} &:= \underline{\mathrm{prann}}^{\boldsymbol{\varpi^1\varpi^2\varpi^3}} + \underline{\mathrm{prann}}^{\boldsymbol{\varpi^2\varpi^3\varpi^1}} + \underline{\mathrm{prann}}^{\boldsymbol{\varpi^3\varpi^1\varpi^2}}\\
\mathrm{hard}_2^{\boldsymbol{\varpi^1};\boldsymbol{\varpi^2};\boldsymbol{\varpi^3}} &:= \underline{\mathrm{prann}}^{\boldsymbol{\varpi^1\varpi^2}}\,\underline{\mathrm{prann}}^{\boldsymbol{\varpi^3}} + \underline{\mathrm{prann}}^{\boldsymbol{\varpi^2\varpi^3}}\,\underline{\mathrm{prann}}^{\boldsymbol{\varpi^1}} + \underline{\mathrm{prann}}^{\boldsymbol{\varpi^3\varpi^1}}\,\underline{\mathrm{prann}}^{\boldsymbol{\varpi^2}}\\
\mathrm{hard}_3^{\boldsymbol{\varpi^1};\boldsymbol{\varpi^2};\boldsymbol{\varpi^3}} &:= \underline{\mathrm{prann}}^{\boldsymbol{\varpi^1}}\,\underline{\mathrm{prann}}^{\boldsymbol{\varpi^2}}\,\underline{\mathrm{prann}}^{\boldsymbol{\varpi^3}}
\end{aligned}$$

Note that we must take care to avoid repetitions in the permutations: thus, for $n = 2, m = 3$ we get three terms on the right-hand side, not six. More

[31] Of course, for $\underline{pran}^{\boldsymbol{\varpi}}$ or $\underline{prann}^{\boldsymbol{\varpi}}$ to be $\neq 0$, we must also have $\|\boldsymbol{\omega}\| = 0$.
[32] ie not a product of zero-sum subsequences.

generally, the number of terms on the right-hand side is exactly $\sum \frac{m!}{m_1!...m_n!}$, with a sum ranging over $1 \leq m_1 \leq m_2 \cdots \leq m_n$ and $m_1 + \cdots + m_n = m$.

Universality of the 'hard core' *$hard^\bullet$*.
For any fixed set of zero-sum, unbreakable sequences $\boldsymbol{\varpi^1}, \boldsymbol{\varpi^2}, \ldots \boldsymbol{\varpi^m}$ and any integer $\mathrm{n} \leq \mathrm{m}$*, the complex number* $\mathrm{hard}_n^{\boldsymbol{\varpi^1};...;\boldsymbol{\varpi^m}}$ *does not depend on the actual choice of prenormalisation.*

Analytic expression of the 'hard core' *$hard^\bullet$*.
Again, for any set of m *zero-sum unbreakable sequences* $\boldsymbol{\varpi^i}$ *and* $\mathrm{n} \leq \mathrm{m}$*, the universal 'hard core' is given by the formula:*

$$
\begin{aligned}
\mathrm{hard}_n^{\boldsymbol{\varpi^1};...;\boldsymbol{\varpi^m}} &= \frac{\mathrm{daa}^{\mathrm{m-n}}}{(m-n)!}\,(\mathrm{Taa}^{\boldsymbol{\varpi^1}} \ldots \mathrm{Taa}^{\boldsymbol{\varpi^m}}) \qquad \textit{with} \qquad (82)\\
\mathrm{Taa}^{\boldsymbol{\varpi}} &:= \frac{(-1)^{r-1}}{\check{\varpi}_1 \ldots \check{\varpi}_{r-1}} = \frac{1}{\hat{\varpi}_2 \ldots \hat{\varpi}_r} \qquad \textit{and}\\
\frac{\mathrm{daa}^s}{s!}\,\mathrm{Taa}^{\boldsymbol{\varpi}} &:= \mathrm{Taa}^{\boldsymbol{\varpi}} \sum_{1 \leq i_1 \cdots \leq i_s \leq r-1} \frac{1}{\check{\varpi}_{i_1} \ldots \check{\varpi}_{i_s}} = \mathrm{Taa}^{\boldsymbol{\varpi}} \sum_{2 \leq i_1 \cdots \leq i_s \leq r} \frac{(-1)^s}{\hat{\varpi}_{i_1} \ldots \hat{\varpi}_{i_s}}
\end{aligned}
$$

with the predictable abbreviations $\check{\varpi}_i := \check{\omega}_i + \check{a}_i$, $\hat{\varpi}_i := \hat{\omega}_i + \hat{a}_i$ *and with* daa *viewed as a symbolic derivation that acts according to the Leibniz rule on the product on the right-hand side of (82).* The proofs, as well as the parallel formulae for diffeos (with alternel/symmetrel moulds in place of alternal/ symmetral ones for vector fields) may be found in [EV1].

The above expression of the "hard core" and the presence in it of unavoidable singularities on the hyperplanes

$$(\omega_i + \omega_{i+1} + \cdots + \omega_j) + (a_i + a_{i+1} + \cdots + a_j) \;=\; 0$$

confirms what we were suggesting a moment ago, namely:

– any universal prenormalisation necessarily involves divergence (which is unfortunate) but of resurgent type (which is fortunate), not only in the prenormalising transformations (which was predictable enough) but also in the prenormal form itself (which is more surprising)

– of all prenormalisations, the "royal" one stands out as the simplest choice:
(i) it displays no parasitical singularities and is acted upon non-trivially by a minimal number of alien derivations (see §2.6 *infra*)
(ii) its resurgence equations, though complex enough, are 'simplest'

(iii) the associated moulds $pran^\bullet$ and $spran^\bullet$ are simplest too : namely, rational in the $\boldsymbol{\omega}$ variables, and with amplifications $\underline{pran}^\bullet, \underline{spran}^\bullet$ also rational in both the $\boldsymbol{\omega}$ and $\boldsymbol{a}$ variables.

2.6 Normalisation and prenormalisation resurgence.

We first recall the normalisation and prenormalisation equations for a resonant (for definiteness, simply resonant) vector field:

$$\begin{array}{lll} X &=& \Theta_{\rm nor}\, X^{\rm nor}\, \Theta_{\rm nor}^{-1} \qquad\qquad (X^{\rm nor} = \textit{normal form}) \\ X &=& \Theta_{\rm pre}\, X^{\rm pre}\, \Theta_{\rm pre}^{-1} \qquad\qquad (X^{\rm pre} = \textit{prenormal form}) \end{array}$$

Both transformations give rise to resurgence, but of a rather different sort – even when we choose, as we shall in the sequel, the *royal prenormalisation*, which is closest to normalisation.

This difference merely reflects the one that exists between the moulds $va^\bullet(z)$ and $wa^\bullet(z)$ which enter the expansions of the normaliser and prenormaliser:

$$\begin{array}{lll} \Theta_{\rm nor} &=& \sum \mathrm{va}^\bullet(z)\ \mathbb{B}_\bullet \qquad\qquad (\textit{normaliser}) \\ \Theta_{\rm pre} &=& \sum \mathrm{wa}^\bullet(z)\ \mathbb{B}_\bullet \qquad\qquad (\textit{royal prenormaliser}) \end{array}$$

As usual, we work in a chart $z, u_1, \ldots, u_{\nu-1}$ so chosen as to 'unload' the whole divergence and resurgence on one single variable z, the remaining u_i being inert parameters. As for the $\mathbb{B}_\bullet$, they are ordinary differential operators elementarily constructed from the Taylor coefficients of the field X. For numerous examples, see eg [E3],[E5]. Both moulds $va^\bullet(z)$ and $wa^\bullet(z)$ are symmetral, and both have components that are divergent-resurgent in z. But there is a marked difference in complexity. Whereas $va^\bullet(z)$ is given by the simple induction :

$$(\partial_z + \|\boldsymbol{\omega}\|)\ \mathrm{va}^{\eta_1,\ldots,\eta_r}(z) \ = \ -\,\mathrm{va}^{\eta_1,\ldots,\eta_{r-1}}(z)\ z^{-\sigma_r - 1} \qquad \textit{with} \quad \eta_i := \binom{\omega_i}{\sigma_i}$$

or more explicitely :

$$(\partial_z + \omega_1 + \cdots + \omega_r)\ \mathrm{va}^{\binom{\omega_1\ ,\ldots,\ \omega_r}{\sigma_1\ ,\ldots,\ \sigma_r}}(z) \ = \ -\,\mathrm{va}^{\binom{\omega_1\ ,\ldots,\ \omega_{r-1}}{\sigma_1\ ,\ldots,\ \sigma_{r-1}}}(z)\ z^{-\sigma_r - 1} \tag{83}$$

the definition of $wa^\bullet(z)$ is more involved:

$$\begin{aligned}
\mathrm{wa}^{\boldsymbol{\eta}}(z) \;&:=\; \sum_{\substack{\boldsymbol{\eta}=\boldsymbol{\eta}^1\,\boldsymbol{\eta}^2\dots\boldsymbol{\eta}^{s-1}\,\boldsymbol{\eta}^s\,\boldsymbol{\eta}^\star\\ \|\boldsymbol{\omega}^1\|=\dots=\|\boldsymbol{\omega}^s\|=0,\|\boldsymbol{\omega}^\star\|\neq 0}} (-1)^s\,\frac{\mathrm{lava}^{\boldsymbol{\eta}^1}(z)\,\mathrm{lava}^{\boldsymbol{\eta}^2}(z)\dots\mathrm{lava}^{\boldsymbol{\eta}^s}(z)\;\;\mathrm{va}^{\boldsymbol{\eta}^\star}(z)}{r_1.(r_1+r_2)\dots(r_1+\dots+r_s)}\\
\textit{with}\qquad & r_i := \textit{length of } \boldsymbol{\eta}^i \qquad \textit{and}\\
\mathrm{lava}^\bullet(z) \;&:=\; (\mathrm{lan.va}^\bullet(z))\times(\mathrm{va}^\bullet(z))^{-1} \qquad\qquad (\mathrm{lava}^\bullet\ \ \textit{alternal})\\
\mathrm{lan.va}^{\boldsymbol{\eta}}(z) \;&:=\; (\sum_{\omega_i\neq 0} 1)\,.\,\mathrm{va}^{\boldsymbol{\eta}}(z) \qquad\qquad (\mathrm{lan}^\bullet\ =\ \textit{mould derivation})
\end{aligned}$$

The main difference, however, is in the Borel transforms $\hat{va}^\bullet(\zeta)$ and $\hat{wa}^\bullet(\zeta)$. Assume for simplicity that all σ_i are 0. Then $\hat{va}^\bullet(\zeta)$ is regular *at* the origin $\zeta = 0$, but usually not *over* the origin, ie at the points that lie over 0 on the other Riemann leaves, whereas $\hat{wa}^\bullet(\zeta)$ is always regular both *at* and *over* the origin.

The shortest way to check this is via the resurgence equations of $va^\bullet$ and those of the alternal mould $lava^\bullet$ which, alongside $va^\bullet$, is the main building block of $wa^\bullet$. These resurgence equations involve one and the same constant (ie z-independent) and alternal mould $ka^\bullet$, but again there is a significant gap in complexity:

$$\Delta\ \mathrm{va}^\bullet(z) \;=\; \mathrm{ka}^\bullet \times \mathrm{va}^\bullet(z) \tag{84}$$

$$\Delta\ \mathrm{lava}^\bullet(z) \;=\; \mathrm{lan.ka}^\bullet + \mathrm{ka}^\bullet \times \mathrm{lava}^\bullet(z) - \mathrm{lava}^\bullet(z)\times \mathrm{ka}^\bullet \tag{85}$$

From here it is but a short step to derive the resurgence equations for our main objects. For the (direct/inverse) normaliser they read:

$$[\boldsymbol{\Delta}_\omega, \Theta_{\mathrm{nor}}] \;=\; -\,\Theta_{\mathrm{nor}}\,\mathbb{A}_\omega \qquad \text{(B.E. for direct normaliser)} \tag{86}$$

$$[\boldsymbol{\Delta}_\omega, \Theta_{\mathrm{nor}}^{-1}] \;=\; +\,\mathbb{A}_\omega\,\Theta_{\mathrm{nor}}^{-1} \qquad \text{(B.E. for inverse normaliser)} \tag{87}$$

and are none other than the Bridge Equation, which is an amazingly general and flexible tool for extracting the object's *analytic invariants* $\mathbb{A}_\omega$ and even, if we so wish, for expressing these $\mathbb{A}_\omega$ in terms of two basic ingredients:
– the operators $\mathbb{B}_n$ which represent the field X via its Taylor coefficients.
– the mould $ka^\bullet$ which represents the "universal-transcendental" part.

Similar, but more complex, resurgence equations hold for the prenormalising transformation Θ_{pre} as well as the prenormalised field X^{pre} (which, unlike X^{nor}, is itself resurgent). Writing down these equations – they are entirely deducible from the system (84),(85) – is a rewarding exercise, which we highly recommend.

3 Lesson Three: Canonical Object Synthesis.

3.1 Local objects and analytic invariants.

We revert to the notations and conventions of Lesson Two. Here too, by *local analytic object* we shall mean essentially :

(1) germs of singular analytic vector fields at 0 on $\mathbb{C}^\nu$ or *fields* for short :

$$X = \sum_{1 \le i \le \nu} X_i(x)\, \partial_{x_i} \qquad ; \qquad X_i(x) \in \mathbb{C}\{x_1, \ldots, x_\nu\} \tag{88}$$

(2) germs of analytic diffeomorphisms of $\mathbb{C}^\nu$ into itself or *diffeos* for short :

$$f : x_i \mapsto f_i(x) \quad ; \qquad i = 1, \ldots, \nu \ , \quad f_i(x) \in \mathbb{C}\{x_1, \ldots, x_\nu\} \tag{89}$$

with the corresponding substitution operators F denoted by a capital letter :

$$(F\varphi)(x) := \varphi(f(x)) \quad ; \qquad \forall \varphi(x) \in \mathbb{C}\{x_1, \ldots, x_\nu\} \tag{90}$$

We recall that everything depends on the nature of the object's spectrum, ie the eigenvalues of its linear part: $\lambda = (\lambda_1, \ldots, \lambda_\nu)$ for a field or $l = (l_1, \ldots, l_\nu)$ for a diffeo. If the spectrum is 'generic', then the object is analytically conjugate to its linear part. Difficulties arise only in case of
– *resonance* of the spectrum (see §2.1)
– *quasiresonance* of the spectrum (see §2.1)
– *nihilence* of the object (see §2.1)

As already pointed out, the more 'complicated' an object, the larger its set of *invariants* tends to be. Alongside the *formal* and *analytic invariants* (ie relative to formal or analytic coordinate changes) we have the notion of *holomorphic invariants* – ie those invariants that depend holomorphically on the object **Ob** (or, in practical terms, its Taylor coefficients), at least when **Ob** remains within a fixed formal conjugacy class.

Resonance generates *formal invariants*: finitely many for one degree of resonance, infinitely many for several degrees.

Each of the afore-mentioned complications – resonance, quasiresonance, nihilence – also gives rise to analytic invariants [33]. These complications may combine, but when resonance alone is at work, there tend to exist [34] complete systems of analytic-cum-holomorphic invariants $\{\mathbb{A}_\omega\}$.

[33] strictly analytic, ie not deducible from formal invariants.
[34] with slight qualifications, see eg [E3],[E5].

We shall in this Third Lesson deal with the purely resonant case. This covers in particular such important objects as:
– identity-tangent diffeomorphisms,
– vector fields with (one or several) vanishing eigenvalues,
– most singular differential equations or systems [35]

"Object analysis" starts from some object **Ob** and is concerned with finding its invariants. For resonant objects, which alone matter to us here, there is a method of sweeping generality – the Bridge Equation (see *infra*) – for conctructing complete systems $\{\mathbb{A}_\omega\}$ of analytic-cum-holomorphic invariants, in the form of specific differential operators $\mathbb{A}_\omega$, with indices ω running through a countable set Ω generated by the object's spectrum. Moreover, relatively to a special class of "nice" bases $\{\Delta_\omega^{\mathrm{nice}}\}$ of the algebra ALIEN of alien derivations, technically known as *well-behaved* bases, the Bridge Equation yields systems $\{\mathbb{A}_\omega^{\mathrm{nice}}\}$ that can be characterised by means of simple, transparent growth conditions on the invariants $\mathbb{A}_\omega^{\mathrm{nice}}$ as ω increases.

"Object synthesis" is the converse problem: starting from a prescribed system $\{\mathbb{A}_\omega\}$ with the proper growth pattern, find an object **Ob** whose invariants coincide with that system. There are actually four degrees:
– *existence:* showing that such an object **Ob** does exist.
– *constructiveness:* producing an effective procedure for constructing it.
– *expliciteness:* expanding the object **Ob**, in a manner both explicit and universal, by means of elementary special functions, the so-called resurgence monomials, that are not constructed *ad hoc*, but given once and for all.
– *canonicity:* examining whether perchance there exists a "canonical" choice for **Ob** and also (since we don't want to forego *expliciteness*) a corresponding system of "canonical" resurgence monomials.

We won't recall (see [E2],[E3],[E15]) the basic facts about *existential, constructive, explicit* synthesis, nor shall we bother with the various strategies, some going back to the late 70s, for establishing the related results. Our concern here is with the dream-goal of *explicit-canonical* synthesis, particularly for non-linear problems. The earliest attempts in this direction were based on the notion of *hyperlogarithmic monomials*. We shall show why these attempts, while interesting in their own way and insightful, were doomed to partial failure. We shall then mention the existence of a whole new class

[35] after their standard rephrasing as time-autonomous systems or, equivalently, as vector fields.

of *resurgence monomials*, based on “prodiffusions” and which on account of their nice growth properties, *do permit explicit synthesis* in all cases. Lastly, we shall show that there exists a particular subclass, the so-called *paralogarithmic* or *spherical monomials*, which unquestionably stand out as ‘canonical’ and which can be harnessed to synthesise objects **Ob** that inherit their ‘canonicity’.

3.2 Object Analysis: the Bridge Equation.

Let **Ob** be some (purely) resonant object – field or diffeo – expressed in a particular analytic chart $x = \{x_1, \ldots, x_\nu\}$ that diagonalises the object’s linear part. The object’s complete linearisation is usually impossible, even formally, and what takes its place is *formal normalisation*, which removes all but a few resonant monomials, or the more radical step of *formal trivialisation*, which forfeits *entireness* [36] but reduces the object to the simplest conceivable form, namely ∂_z for as field and $z \mapsto z+1$ for a diffeo.

Let $y = \{y_1, \ldots, y_\nu\}$ be a formal normal chart, and consider the formal-entire coordinate changes $y_i = \theta_i(x)$ and $x_i = \theta_i^{-1}(y)$ with the substitution operators Θ and Θ^{-1} that go with them : $\Theta^{\pm 1}\varphi := \varphi \circ \theta^{\pm 1}$

Also consider the “trivial chart” $\{z, u\} = \{z, u_1, \ldots, u_{\nu-1}\}$. Expressing the given coordinates $x = \{x_i\}$ in terms of the trivial coordinates $\{z, u_i\}$, we get the so-called *formal integral* $x(z,u) = \{x_1(z,u), \ldots, x_\nu(z,u)\}$, which verifies:

$$\partial_z\, x_i(z,u) \;\equiv\; X_i(x(z,u)) \qquad \forall\, i \qquad \text{for a field} \tag{91}$$

$$f_i(x(z,u)) \;\equiv\; x_i(z+1,u) \qquad \forall\, i \qquad \text{for a diffeo} \tag{92}$$

The Bridge Equation (B.E.) is a powerful and versatile tool for extracting the object’s *invariants* from the divergence-resurgence of the *trivialising* or (direct/inverse) *normalising* transformations. Here are its three main forms :

$$\boldsymbol{\Delta}_\omega\, x(z,u) \;=\; \mathbb{A}_\omega\, x(z,u) \qquad \text{(B.E. for the formal integral)} \tag{93}$$

$$[\boldsymbol{\Delta}_\omega, \Theta] \;=\; -\,\Theta\, \mathbb{A}_\omega \qquad \text{(B.E. for the direct normaliser)} \tag{94}$$

$$[\boldsymbol{\Delta}_\omega, \Theta^{-1}] \;=\; +\,\mathbb{A}_\omega\, \Theta^{-1} \qquad \text{(B.E. for the inverse normaliser)} \tag{95}$$

The indices ω on both sides of the Bridge Equation range through a countable set Ω spanned by the object’s multipliers, ie the λ_j in the case of a field, and the $\log l_j$ (to which one must add the universal multiplier $\lambda_0 := 2\pi i$) in the case of a diffeo.

[36]it no longer relies on entire power series.

The $\mathbb{\Delta}_\omega$ on the left-hand side denotes the alien derivation relative to the variable z and index ω but with a built-in exponential factor $\mathbb{\Delta}_\omega := e^{-\omega\, z}\Delta_\omega$ that makes it commute with ∂_z and ensures the invariance rule under non-divergent changes of equivalent variables $z \mapsto z_\star$ with $z \sim z_\star$[37].

$$\mathbb{\Delta}_\omega^{(z)}\, \varphi(z) \equiv \mathbb{\Delta}_\omega^{(z_\star)}\, \varphi_\star(z_\star) \qquad \textit{if} \quad \varphi(z) \equiv \varphi_\star(z_\star) \quad \textit{and} \quad z \sim z_\star \tag{96}$$

The alien-differentiation variable z, also known as *critical variable*[38], is always $\sim \infty$. In (93) it is simply the z inside the formal integral. In (94) or (95) it is the inverse of some *resonant monomial*, ie $z := 1/x^m$ or $z := 1/y^m$. Due to the afore-mentioned invariance property of alien differentiation,[39] the *critical variable* is actually defined up to equivalence $\sim$ and so the proper intrinsic notion is in fact that of *critical class*.

The $\mathbb{A}_\omega$ on the right-hand side are ordinary differential operators – in the variables (z, u_i) or (x_i) or (y_i) respectively. They are constructively determined, even overdetermined, by the requirement of equality in the Bridge Equation – whichever of its variants we choose to work with, and whichever critical variable we pick (within the critical class) for alien differentiation. Each single $\mathbb{A}_\omega$ is an invariant of the object **Ob**, and the total collection $\{\mathbb{A}_\omega, \omega \in \Omega\}$ constitutes a set, both complete and free, of analytic-cum-holomorphic invariants.

All these claims, as sketchy as they are sweeping, clearly cry for explanations and qualifications, which cannot be supplied here but are available in the literature ([E2][E3][E5][E10]). We recalled these statements simply as a general backdrop for the twin problems of object analysis and synthesis – but to illustrate the method we shall focus on just four typical examples.

3.3 Examples of local objects.

Example 1: one-dimensional identity-tangent diffeo.

$$f^{\mathrm{nor}} \quad : \quad z \longrightarrow z + 2\pi i \tag{97}$$

$$f \quad : \quad z \longrightarrow z + 2\pi i + \sum_{2 \le n} a_n\, z^{-n} \tag{98}$$

Remark : we might of course have chosen the *unit shift* as our normal form, but choosing the $2\pi i$-shift has the advantage of placing the singularities over $\mathbb{Z}$ in the Borel-plane, and of rendering the parallel with Example 2 (*infra*)

[37] eg $z_\star := z\,(1 + \sum a_n\, z^{-n})$ or $z_\star := z\,(1 + \sum a_n\, (\log z)^{-n})$ with convergent series $\sum$.
[38] or *critical time*.
[39] it mirrors the invariance rule $\varphi'(z)\, dz \equiv \varphi'_\star(z_\star)\, dz_\star$ for ordinary differentiation.

more obvious.
Example 2: singular, non-linear differential equation.[40]

$$d_z\, y^{\mathrm{nor}} = y^{\mathrm{nor}} \tag{99}$$

$$d_z\, y = y + \sum_{1+n\geq 0} b_n(z)\, y^{1+n} \qquad \in\; y + \mathbb{C}\{y, z^{-1}\} \tag{100}$$

Example 3: singular linear differential system.

$$d_z\, y_i^{\mathrm{nor}} = \lambda_i\, y_i^{\mathrm{nor}} \qquad (1 \leq i \leq \nu\ ;\ \lambda_i \neq \lambda_j\ \ if\ \ i \neq j) \tag{101}$$

$$d_z\, y_i = \lambda_i\, y_i + \sum_{1\leq j\leq \nu} b_{i,j}(z)\, y_j \qquad b_{i,j}(z) \in \mathbb{C}\{z^{-1}\} \tag{102}$$

Example 4: singular non-linear differential system.

$$d_z\, y_i^{\mathrm{nor}} = \lambda_i\, y_i^{\mathrm{nor}} \qquad (1 \leq i \leq \nu\ ;\ \lambda\ \ not\ res.\ nor\ quasi.res.) \tag{103}$$

$$d_z\, y_i = \lambda_i\, y_i + \sum_{\substack{1+n_i\geq 0 \\ n_j\geq 0\, if\, j\neq i}} b_{i,n}(z)\, y_i\, y^n \ \in \lambda_i\, y_i + \mathbb{C}\{y_1, \ldots, y_\nu, z^{-1}\} \tag{104}$$

3.4 Resurgence equations and analytic invariants.

Example 1: one-dimensional identity-tangent diffeo.
We may work with the formal normalising map $f^\star$ or its inverse ${}^\star\!f$:

$$f^\star \circ f \equiv f^{\mathrm{nor}} \circ f^\star \qquad ie \qquad f^\star(f(z)) \equiv 2\pi i + f^\star(z) \tag{105}$$

$$f \circ {}^\star\!f \equiv {}^\star\!f \circ f^{\mathrm{nor}} \qquad ie \qquad f({}^\star\!f(z)) \equiv {}^\star\!f(z + 2\pi i) \tag{106}$$

Both are generically divergent but always resurgent. They verify the resurgence equations:

$$\mathbf{\Delta}_n\, f^\star(z) \;\equiv\; -A_n\, \exp(-n\, f^\star(z)) \qquad (\forall n \in \mathbb{Z}^\star) \tag{107}$$

$$\mathbf{\Delta}_n\, {}^\star\!f(z) \;\equiv\; +A_n\, e^{-nz}\partial_z\, {}^\star\!f(z) \;\;=:\; \mathbb{A}_n\, {}^\star\!f(z) \qquad (\forall n \in \mathbb{Z}^\star) \tag{108}$$

which in turn yield the complete and free system of analytic invariants:

$$\mathbb{A} = \{\mathbb{A}_n := A_n\, e^{-nz}\partial_z\ ;\quad n \in \mathbb{Z}^\star\ \ ,\ \ A_n \in \mathbb{C}\} \tag{109}$$

Example 2: singular, non-linear differential equation.
We may work with the formal integral

$$y(z, u) \in \mathbb{C}[[z^{-1},\, u\, e^z]] \qquad (u\; =\; integration\;\; parameter) \tag{110}$$

[40]under addition of a variable $x := z^{-1}$ it translates into a two-dimensional vector field, local (at $0 \in \mathbb{C}^2$), singular, and resonant (with one vanishing and one non-vanishing multiplier)

which is generically divergent (in z) but always resurgent (again, in z) and verifies the Bridge equation :

$$\Delta_n\, y(z,u) \equiv A_n\, u^{n+1}\, \partial_u\, y(z,u) =: \mathbb{A}_n\, y(z,u) \qquad (n=-1,1,2,3,\dots) \tag{111}$$

yielding the complete and free system of analytic invariants:

$$\mathbb{A} = \{\mathbb{A}_n := A_n\, u^{n+1}\, \partial_u \quad ; \quad n \in \{-1\} \cup \mathbb{N}^\star \ , \ \ A_n \in \mathbb{C}\} \tag{112}$$

Example 3: singular linear differential system.
Here the formal integral reduces to

$$y(z,u) = \sum_{1\le i\le \nu} b_i(z)\, e^{\lambda_i z}\, u_i \qquad \textit{with} \quad b_i(z) \in \mathbb{C}[[z^{-1}]] \tag{113}$$

The Bridge equation reads :

$$\Delta_{\lambda_i-\lambda_j}\, y(z,u) \ \ \equiv \ \ A_{\lambda_i-\lambda_j}\, u_j\, \partial_{u_i}\, y(z,u) =: \mathbb{A}_{\lambda_i-\lambda_j}\, y(z,u) \quad (i \neq j) \tag{114}$$

and once again yields a complete and free, but this time *finite*, system of analytic invariants:

$$\{\mathbb{A}_{\lambda_i-\lambda_j} := A_{\lambda_i-\lambda_j}\, u_j\, \partial_{u_i} \quad ; \quad 1 \le i \neq j \le \nu\} \tag{115}$$

Example 4: singular non-linear differential system.
The formal integral involves ν integration parameters u_i, each with its accompanying exponential factor :

$$y(z,u) \in \mathbb{C}[[z^{-1},\, u_1\, e^{\lambda_1 z}, \dots, u_\nu\, e^{\lambda_\nu z}]] \qquad \text{(with } \mathbb{Q}\text{-independent } \lambda_i\text{'s)} \tag{116}$$

The Bridge equation reads :

$$\Delta_\omega\, y(z,u) \equiv \mathbb{A}_\omega\, y(z,u) \qquad (\forall \omega = \mathbf{\Omega}) \tag{117}$$

with indices ω running through a set :

$$\mathbf{\Omega} = \{\omega \ ; \ \omega = \sum_{1\le i\le\nu} m_i\, \lambda_i \ , \ m_i \ge -1 \ , \ \sum_{m_i=-1} 1 = 0 \ \textit{or} \ 1\} \tag{118}$$

and with differential operators of the form :

$$\mathbb{A}_\omega := u_1^{m_1} \dots u_\nu^{m_\nu} \sum_{1\le i\le\nu} A^i_\omega\, u_i\, \partial_{u_i} \qquad \textit{if} \ \ \omega = \sum m_i\, \lambda_i \qquad (A^i_\omega \in \mathbb{C}) \tag{119}$$

which, together, constitute a complete and free system $\{\mathbb{A}_\omega \ ; \ \omega = \mathbf{\Omega}\}$ of analytic invariants.

Caveat: Of course, in all these example, the systems of invariants $\{\mathbb{A}_\omega\}$ are *"free"* only in the sense of being subject to no *finite constraints* (ie constraints bearing on finite subsets) but they are subject to an *infinite constraint* which, relative to a nice ("well-behaved", see *infra*) basis of ALIEN, reduces to the existence of *exponential bounds in* ω.

3.5 Alien derivations: lateral, standard or organic.

Alien derivations Δ_ω are determined by systems of weights $\mathbf{d}^{\binom{\epsilon_1\ ,\ldots,\ \epsilon_i\ ,\ldots,\ \epsilon_r}{\omega_1\ ,\ldots,\ \omega_i\ ,\ldots,\ \omega_r}}$ subject to the self-consistency relations:

$$\sum_{\epsilon_i \in \{+,-\}} \mathbf{d}^{\binom{\epsilon_1\ ,\ldots,\ \epsilon_i\ ,\ldots,\ \epsilon_r}{\omega_1\ ,\ldots,\ \omega_i\ ,\ldots,\ \omega_r}} := \mathbf{d}^{\binom{\epsilon_1\ ,\ldots,\ \epsilon_{i+1}\ ,\ldots,\ \epsilon_r}{\omega_1\ ,\ldots,\ \omega_i+\omega_{i+1}\ ,\ldots,\ \omega_r}} \qquad \forall i \tag{120}$$

and their action in the *convolutive model* is given by:

$$\Delta_\omega\ \hat{\varphi}(\zeta) := \sum_{\epsilon_i=\pm} \mathbf{d}^{\binom{\epsilon_1\ ,\ldots,\ \epsilon_r}{\omega_1\ ,\ldots,\ \omega_r}}\ \ \hat{\varphi}^{\binom{\epsilon_1\ ,\ldots,\ \epsilon_r}{\omega_1\ ,\ldots,\ \omega_r}}(\zeta+\omega) \tag{121}$$

for ζ close to $+0$ and by analytic continuation in the large. There being no scope for confusion, we also use the same symbols to denote the alien derivations acting in the *multiplicative models (formal or geometric)*, ie the pull-backs by Borel-Laplace of the operators Δ_ω as defined by (121).

3.5.1 Lateral alien operators Δ_ω^+ and Δ_ω^-

The right-lateral operators Δ_ω^+ are defined by:

$$\begin{aligned} \mathbf{d}^{\binom{\epsilon_1\ ,\ldots,\ \epsilon_r}{\omega_r\ ,\ldots,\ \omega_r}} &:= \epsilon_r\, 1 \qquad \mathit{if}\ \ (\epsilon_1,\ldots,\epsilon_{r-1}) = (+,\ldots,+) \\ &:= \quad 0 \qquad \mathit{otherwise} \end{aligned}$$

and the left-lateral operators Δ_ω^- are defined by:

$$\begin{aligned} \mathbf{d}^{\binom{\epsilon_1\ ,\ldots,\ \epsilon_r}{\omega_r\ ,\ldots,\ \omega_r}} &:= -\,\epsilon_r\, 1 \qquad \mathit{if}\ \ (\epsilon_1,\ldots,\epsilon_{r-1}) = (-,\ldots,-) \\ &:= \quad 0 \qquad \mathit{otherwise} \end{aligned}$$

Their simplicity is deceptive because they are not *first-order* derivations. Instead of verifying (in the multiplicative models) the Leibniz rule:

$$\Delta_\omega(\varphi_1\ \varphi_2) \equiv (\Delta_\omega\varphi_1)\ \varphi_2 + \varphi_1\ (\Delta_\omega\varphi_2)$$

they verify messier relations:

$$\Delta_\omega^\pm(\varphi_1\ \varphi_2) \equiv (\Delta_\omega^\pm\varphi_1)\ \varphi_2 + \varphi_1\ (\Delta_\omega^\pm\varphi_2) + \sum_{\omega_1+\omega_2=\omega} (\Delta_{\omega_1}^\pm\varphi_1)\ (\Delta_{\omega_2}^\pm\varphi_2)$$

with a sum extending to all ω_1, ω_2 colinear with ω. For any given pair of test functions φ_1, φ_2 the above sum makes sense, since it can never involve more than a finite number of non-zero terms.

3.5.2 Standard alien derivations $\Delta^{\mathrm{stan}}_{\omega}$.

They correspond to the weights:

$$\mathbf{d}^{\binom{\epsilon_1\ ,\dots,\ \epsilon_r}{\omega_1\ ,\dots,\ \omega_r}} := \epsilon_r\ \frac{p!\,q!}{(p+q+1)!} = \epsilon_r\ \frac{p!\,q!}{r!} \qquad \begin{array}{l} p := \#\{1 \le i < r\,;\ \epsilon_i = +\} \\ q := \#\{1 \le i < r\,;\ \epsilon_i = -\} \end{array}$$

They are first-order alien derivations (and thus an impovement on the $\Delta^{\pm}_{\omega}$) but they are not *well-behaved.*

3.5.3 Organic alien derivations $\Delta^{\mathrm{org}}_{\omega}$.

They correspond to the weights:

$$\begin{array}{rcll} \mathbf{d}^{\binom{\epsilon_1\ ,\dots,\ \epsilon_r}{\omega_1\ ,\dots,\ \omega_r}} & := & \frac{\epsilon_r}{2}\,\frac{\omega_{p+1}}{\omega_1+\dots+\omega_r} & if \quad (\epsilon_1,\dots,\epsilon_r) = ((+)^p,(-)^q,\epsilon_r) \\ & := & \frac{\epsilon_r}{2}\,\frac{\omega_{q+1}}{\omega_1+\dots+\omega_r} & if \quad (\epsilon_1,\dots,\epsilon_r) = ((-)^q,(+)^p,\epsilon_r) \\ & := & 0 & \quad otherwise \end{array}$$

They are *first-order and well-behaved* alien derivations. In fact, at the moment, they constitute the simplest extant system of such derivations.

3.5.4 Conversion rule.

Each system $\{\Delta^{\mathrm{org}}_{\omega}\}$ or $\{\Delta^{\mathrm{stan}}_{\omega}\}$ freely generates the algebra ALIEN of alien derivations, and we have the simple conversion rules:

$$\Delta^{\mathrm{org}}_{\omega_0} \ \equiv\ \Delta^{\mathrm{stan}}_{\omega_0} + \sum_{3\le r\ odd} H^{\omega_1,\dots,\omega_r}_{\omega_0}\ [[\Delta^{\mathrm{stan}}_{\omega_r},\dots,[\Delta^{\mathrm{stan}}_{\omega_2},\Delta^{\mathrm{stan}}_{\omega_1}]] \tag{122}$$

$$\Delta^{\mathrm{stan}}_{\omega_0} \ \equiv\ \Delta^{\mathrm{org}}_{\omega_0} + \sum_{3\le r\ odd} K^{\omega_1,\dots,\omega_r}_{\omega_0}\ [[\Delta^{\mathrm{org}}_{\omega_r},\dots,[\Delta^{\mathrm{org}}_{\omega_2},\Delta^{\mathrm{org}}_{\omega_1}]] \tag{123}$$

with scalar coefficients $H^{\omega_1,\dots,\omega_r}_{\omega_0}$ and $K^{\omega_1,\dots,\omega_r}_{\omega_0}$ which of course vanish unless $\omega_0 \neq \sum \omega_i$. For instance:

$$H^{\omega_1,\dots,\omega_{2r+1}}_{\omega_0} := \frac{1}{2r+1} \sum_{0\le i\le 2r} \frac{(-1)^i\,(2r)!}{i!\,(2r-i)!}\ \frac{\omega_{i+1}}{\omega_0} \qquad if\ \ \omega_0 = \sum \omega_i \tag{124}$$

3.5.5 Characterisation of well-behaved alien derivations.

To each system of alien derivations we may associate the moulds:

$$\mathbf{red}^{\,\omega_1,\dots,\omega_r} \ :=\ (-1)^r\ \mathbf{d}^{\binom{+\ ,\dots,\ +}{\omega_1\ ,\dots,\ \omega_r}} \qquad \text{(right-lateral mould)} \tag{125}$$

$$\mathbf{led}^{\,\omega_1,\dots,\omega_r} \ :=\ (-1)^r\ \mathbf{d}^{\binom{-\ ,\dots,\ -}{\omega_1\ ,\dots,\ \omega_r}} \qquad \text{(left-lateral mould)} \tag{126}$$

Due to the self-consistency relations, both the right- and left-lateral moulds encapsulate all the information about the entire weight system $\{\mathbf{d}^{\binom{\epsilon_1\ ,\dots,\ \epsilon_r}{\omega_1\ ,\dots,\ \omega_r}}\}$, and each one can be deduced from the other in a simple manner.

The three following conditions are equivalent:
(a) the alien operators Δ_ω in (121) are (first-order) *alien derivations*
(b) the right-lateral mould $\mathbf{red}^\bullet$ is *alternel*
(c) the left-lateral mould $\mathbf{led}^\bullet$ is *alternel*

The five following conditions are equivalent:
(d) the system Δ_ω in (121) of (first-order) alien derivations is *well-behaved*
(e) we have universal bounds $|\,\mathbf{red}^{\boldsymbol{\omega}^{\twoheadleftarrow}}\,| \leq C_1\,(D_1)^{r(\boldsymbol{\omega}^{\twoheadleftarrow})}$
(f) we have universal bounds $|\,\mathbf{red}^{\boldsymbol{\omega}^{\twoheadrightarrow}}\,| \leq C_2\,(D_2)^{r(\boldsymbol{\omega}^{\twoheadrightarrow})}$
(g) we have universal bounds $|\,\mathbf{led}^{\boldsymbol{\omega}^{\twoheadleftarrow}}\,| \leq C_3\,(D_3)^{r(\boldsymbol{\omega}^{\twoheadleftarrow})}$
(h) we have universal bounds $|\,\mathbf{led}^{\boldsymbol{\omega}^{\twoheadrightarrow}}\,| \leq C_4\,(D_4)^{r(\boldsymbol{\omega}^{\twoheadrightarrow})}$

3.6 Well-behaved resurgence monomials.

3.6.1 Multiplicative systems of resurgence monomials.

They are basically systems $\{\mathcal{U}^{\boldsymbol{\omega}}(z) = \mathcal{U}^{\omega_1,\dots,\omega_r}(z)\}$ of elementary resurgent functions which:
(1) behave simply under multiplication[41]
(2) behave simply under alien differentiation
(3) are "complete" in the sense that they should enable us to expand (or approximate) any given resurgent function φ :

$$\varphi(z) \quad "=" \quad \sum_{\boldsymbol{\omega}} c_{\boldsymbol{\omega}}(z)\ \mathcal{U}^{\boldsymbol{\omega}}(z) \quad := \quad \sum_{r\geq 0}\sum_{\omega_i} c_{\omega_1,\dots,\omega_r}(z)\ \mathcal{U}^{\omega_1,\dots,\omega_r}(z) \tag{127}$$

with coefficients $c_{\boldsymbol{\omega}}(z)$ that are either ordinary constants or "resurgence constants", that is to say functions with only vanishing alien derivatives: $\Delta_{\omega_0} c_{\boldsymbol{\omega}}(z) \equiv 0\ ,\forall \omega_0$.

Condition (2) is relative to a given basis $\{\Delta_\omega\}$ of the algebra ALIEN of alien derivations. In concrete terms the condition stipulates that:

$$\begin{aligned} \Delta_{\omega_0}\mathcal{U}^{\omega_1,\dots,\omega_r} \quad &\equiv \quad \mathcal{U}^{\omega_2,\dots,\omega_r} && if\ \ \omega_0 = \omega_1 \\ &\equiv \quad 0 && if\ \ \omega_0 \neq \omega_1 \end{aligned} \tag{128}$$

[41] or convolution, depending on the model.

Condition (1) means that:

$$\mathcal{U}^{\boldsymbol{\omega}'}\,\mathcal{U}^{\boldsymbol{\omega}''} \equiv \sum_{\boldsymbol{\omega}\in\mathbf{sha}(\boldsymbol{\omega}',\,\boldsymbol{\omega}'')} \mathcal{U}^{\boldsymbol{\omega}} \tag{129}$$

with a sum extending to all sequences $\boldsymbol{\omega}$ obtained by *shuffling* the two factor sequences $\boldsymbol{\omega}'$, $\boldsymbol{\omega}''$. In other words, the mould $\mathcal{U}^\bullet$ should be *symmetral.*[42]

Clearly, there exist infinitely many multiplicative systems of resurgence monomials. Indeed, if $\mathcal{U}^\bullet(z)$ is one such system, so will be the system $\mathcal{U}^\bullet_{\mathcal{C}}(z) := \mathcal{U}^\bullet(z) \times \mathcal{C}^\bullet(z)$ derived therefrom by postmultiplication[43] by any symmetral, resurgence-constant mould $\mathcal{C}^\bullet(z)$.

3.6.2 Integral alien calculus.

Multiplicative systems of resurgence monomials are extremely useful to solve resurgence equations, or systems of such equations, and to express their solutions in the form of expansions of type (127), often with constant coefficients $c_{\boldsymbol{\omega}}$. Thus, if we revert to Example 1 and try to solve the system of resurgence equations (107) that characterise the normalising transformation $f^\star$, we find:

$$\begin{aligned} f^\star(z) \;&:=\; z - \sum_r \sum_{n_i} A_{n_1}\dots A_{n_1}\;\Gamma_{n_1,\dots,n_r}\;\mathcal{U}^{n_1,\dots,n_r}(z) && (130)\\ \textit{with}\quad \Gamma_{n_1,\dots,n_r} \;&:=\; (n_1)\,(n_1+n_2)\dots(n_1+n_2+\dots+n_{r-1}) && (131)\end{aligned}$$

But the real issue of course is convergence. We might try to solve it on an *ad hoc* basis, ie by choosing our resurgence monomials differently for each problem. But we are more ambitious: we want resurgence monomials that work in *all* cases. [44] That may seem a tall order, but it is feasibble! The answer lies in the notion of *well-behaved* systems of resurgence monomials. And not only do such systems exist, but there is a canonical choice!

3.6.3 Characterisation of well-behaved resurgence monomials.

To any given system of resurgence monomials we may associate a *right-lateral* mould ${}^{re}\mathcal{U}^\bullet(z)$ and a *left-lateral* mould ${}^{le}\mathcal{U}^\bullet(z)$ characterised by the orthogonality conditions:

$$\{{}^{re}\mathcal{U}^{\omega_1,\dots,\omega_r}\}\quad \textit{orthogonal to}\quad \{\Delta^+_{\omega_1,\dots,\omega_r} := \Delta^+_{\omega_r}\dots\Delta^+_{\omega_1}\} \tag{132}$$

$$\{{}^{le}\mathcal{U}^{\omega_1,\dots,\omega_r}\}\quad \textit{orthogonal to}\quad \{\Delta^-_{\omega_1,\dots,\omega_r} := \Delta^-_{\omega_r}\dots\Delta^-_{\omega_1}\} \tag{133}$$

[42] see §6.1

[43] in the sense of moulds.

[44] ie for all systems of resurgence equations that do admit solutions.

The three following conditions are equivalent:
(a) a system of resurgence monomials is *multiplicative*
(b) its right-lateral mould ${}^{re}\mathcal{U}^\bullet$ is *symmetrel*[45]
(c) its left-lateral mould ${}^{le}\mathcal{U}^\bullet$ is *symmetrel*

Observe that the criterion here is *symmetrel* and not *symmetral* as in §3.6.1 This is because the lateral alien operators $\Delta^{\pm}_{\omega}$ are not first-order alien derivations (see §3.5.1).

The five following conditions are equivalent:
(d) a system of multiplicative resurgence monomials is *well-behaved*
(e) we have universal bounds $\| {}^{re}\mathcal{U}^{\boldsymbol{\omega}^{\prec}} \| \leq C_1\,(D_1)^{r(\boldsymbol{\omega}^{\prec})}$
(f) we have universal bounds $\| {}^{re}\mathcal{U}^{\boldsymbol{\omega}^{\succ}} \| \leq C_2\,(D_2)^{r(\boldsymbol{\omega}^{\succ})}$
(g) we have universal bounds $\| {}^{le}\mathcal{U}^{\boldsymbol{\omega}^{\prec}} \| \leq C_3\,(D_3)^{r(\boldsymbol{\omega}^{\prec})}$
(h) we have universal bounds $\| {}^{le}\mathcal{U}^{\boldsymbol{\omega}^{\succ}} \| \leq C_4\,(D_4)^{r(\boldsymbol{\omega}^{\succ})}$
for a suitable norm $\|.\|$ (see [E15]) and with the arborification rule (31).

Resorting to so-called ‘prodiffusion integrals’ (rather similar to the diffusion integrals which we used in Lesson One, §1.10.4, to construct our well-behaved averages, but with Borel-Laplace replacing the Fourier transform) one may produce a large variety of well-behaved systems of resurgence monomials. Furthermore, there exits a canonical choice, the only remaining latitude being in the determination of a single real parameter c.

3.6.4 The canonical choice: “spherical” or “twisted” monomials.

As just mentioned, they depend on a positive parameter $c > 0$ (the “twist”) and are defined by the absolutely convergent integrals:

$$\mathcal{U}a_c^{\omega_1,\dots,\omega_r}(z) \;:=\; \frac{S.P.A.}{(2\pi i)^r}\int_0^\infty \frac{e^{-\sum_1^r(\omega_i\,y_i+c^2\bar{\omega}_i\,y_i^{-1})}\;dy_1\dots dy_r}{(y_r-y_{r-1})\dots(y_2-y_1)(y_1-z)} \tag{134}$$

$$\mathcal{U}e_c^{\omega_1,\dots,\omega_r}(z) \;:=\; \mathcal{U}a_c^{\omega_1,\dots,\omega_r}(z)\; e^{(\sum_1^r\omega_i)\,z+(\sum_1^r\bar{\omega}_i)\,c^2\,z^{-1}} \tag{135}$$

$$\mathcal{U}_c^{\omega_1,\dots,\omega_r}(z) \;:=\; \mathcal{U}a_c^{\omega_1,\dots,\omega_r}(z)\; e^{(\sum_1^r\bar{\omega}_i)\,c^2\,z^{-1}} \tag{136}$$

with integration along the rays $\arg(\omega_i\,y_i) = \arg(\bar{\omega}_i/y_i) = 0$.

The $\mathcal{U}a_c^{\boldsymbol{\omega}}$ are auxiliary expressions. The resurgence monomials proper are the $\mathcal{U}_c^{\boldsymbol{\omega}}$ (orthogonal to the ordinary alien derivations Δ_ω) and the exponential-carrying $\mathcal{U}e_c^{\boldsymbol{\omega}}$ (orthogonal to the exponential-carrying alien derivations $\mathbb{\Delta}_\omega$).

[45]see §6.1.

Interpretation of *S.P.A.*

S.P.A. in front of the integral means *suitable path average*. If we integrate first in y_1, then y_2, etc, the question arises[46] as to how (ie on which side) y_i should bypass the next (yet unused) variable y_{i+1}. If to the right, we set $\epsilon_i := +$. If to the left, we set $\epsilon_i := -$. To each choice $\{\epsilon_1, \epsilon_2, \dots\}$ there corresponds a different integration path, and *S.P.A.* means that one should take a precise average of all such paths, depending on which system Δ_ω of alien derivations one wishes the $\mathcal{U}^{\boldsymbol{\omega}}$ to be orthogonal to. But for the right- or left-lateral moulds (characterised by orthogonality to $\Delta^{\pm}_\omega$) the *S.P.A.* average reduces to one single path, with all ϵ_i identical (either $+$ or $-$).

Interpretation of $1/(y_1 - z)$

The integral (134) defines $\mathcal{U}^{\boldsymbol{\omega}}$ in all three models (formal, geometric, convolutive) at one stroke, depending on how we construe $1/(y_1 - z)$:
– as a power series in z^{-1},
– or as a function germ at ∞,
– or again as its own Borel tranform.

$$
\begin{array}{lll}
\textit{Formal model}: & \frac{1}{y_1 - z} \to & -\sum_0^\infty z^{-n-1}\, y_1^n \\
\qquad\qquad \Longrightarrow & \tilde{\mathcal{U}}_c^{\boldsymbol{\omega}}(z) & \textit{as a formal power series} \\
\textit{Geometric model}: & \frac{1}{y_1 - z} \to & z\textit{-germ at } \infty \\
\qquad\qquad \Longrightarrow & \mathcal{U}_c^{\boldsymbol{\omega}}(z) & \textit{as a sectorial } z\textit{-germ at } \infty \\
\textit{Convolutive model}: & \frac{1}{y_1 - z} \to & -\exp(y_1\, \zeta) \\
\qquad\qquad \Longrightarrow & \hat{\mathcal{U}}_c^{\boldsymbol{\omega}}(\zeta) & \textit{as a full } \zeta\textit{-germ at } 0
\end{array}
$$

Main result:

For positive values $c > 0$ of the twist, $\mathcal{U}_c^\bullet$ constitutes a well-behaved, multiplicative system of resurgence monomials.

Gist of the proof: The difficult bit of course is *well-behaved*. We use the criteria of §3.6.3 and the fact (see §6.1) that the moulds $tas^\bullet_{a,\infty}$, $tas^\bullet_{\infty,b}$ as well as the moulds $sofo^\bullet$, $sefo^\bullet$ (which are essentially their "Fourier transforms") *conserve their form* under either arborification or anti-arborification, and as a consequence do not register any significant increase in component size [47].

The limit-case $c = 0$**.**

[46] at least when two consecutive integration axes coincide, ie when $\arg \omega_i = \arg \omega_{i+1}$.
[47] see towards the end of §6.1 and also [E15],[EV3].

In the limit-case $c = 0$ the integrals (134) remain well-defined and we still have a multiplicative system $\mathcal{U}_0^\bullet$ of resurgence monomials, but it is no longer *well-behaved*. In fact, it coincides with the much more ancient system $\mathcal{U}^\bullet$ of hyperlogarithmic monomials, so-called because their dependence in the ω_i's is indeed of hyperlogarithmic type. In contradistinction, the $\mathcal{U}_c^\bullet$ and the host of special functions attached to them (see §6.7) are called *paralogarithmic.*

Why "twisted" and why "spherical" ?
The presence of a free parameter c slightly detracts from the "canonicity" of our system, but this cannot be helped: no system of well-behaved resurgence monomials can suffice for all problems unless there is at least one free parameter that can be adjusted from case to case. The miracle is rather that *one* parameter should be enough! So much for the twist. As for "spherical", it refers to the striking symmetry of behaviour which our monomials $\mathcal{U}_c^\bullet$ exhibit at the antipodes 0 and ∞ of the Riemann sphere when $c > 0$, and which, remarkably enough, disappears when we "untwist" them, ie when $c = 0$.

3.7 Canonical object synthesis at work.

Basically, with the twisted monomials at our disposal, Object Synthesis becomes a purely mechanical affair. This is precisely what we had set out to achieve: to reduce the whole process to a succession of formal manipulations. But here we must be content with outlining the six main steps:

Step 1: select a formal class of local analytic objects, characterised by a formal normal form $\mathbf{Ob}^{\mathrm{nor}}$, and start from any given *admissible* system of analytic invariants $\mathbb{A} = \{\mathbb{A}_\omega\,;\ \omega \in \mathbf{\Omega}\}$

Step 2: choose a *well-behaved system of alien derivations*, preferably the *organic* system $\Delta^{\mathrm{org}} = \{\Delta^{\mathrm{org}}_\omega\,;\ \omega \in \mathbb{C}^\star\}$, and express the analytic invariants in the corresponding basis $\mathbb{A}^{\mathrm{org}} = \{\mathbb{A}^{\mathrm{org}}_\omega\,;\ \omega \in \mathbf{\Omega}\}$

Step 3: solve *"mechanically"*[48] the system of resurgence equations that characterise the direct or inverse normaliser $\Theta^{\pm 1}$. For instance, in the case

[48]ie without worrying about convergence. Mark the choice of words: *mechanically*, ie by means of expansions into series (127) of abstract resurgence monomials, rather than *formally*, which would suggest solving the problem in the ring of formal power series.

of simply resonant fields we find these expansions[49] :

$$\Theta \overset{always}{:=} 1 + \sum_{1\le r} \sum_{\omega_i \in \Omega} (-1)^r\, \mathcal{U}\!e^{\,\omega_1,\dots,\omega_r}(z)\ \mathbb{A}_{\omega_r} \dots \mathbb{A}_{\omega_1} \tag{137}$$

$$\Theta^{-1} \overset{conditionally}{:=} 1 + \sum_{1\le r} \sum_{\omega_i \in \Omega} \mathcal{U}\!e^{\,\omega_r,\dots,\omega_1}(z)\ \mathbb{A}_{\omega_r} \dots \mathbb{A}_{\omega_1} \tag{138}$$

Step 4: replace in that "mechanical" solution the abstract monomials $\mathcal{U}^{\boldsymbol{\omega}}_{\mathrm{org}}(z)$ by the *twisted* or *spherical* monomials $\mathcal{U}^{\boldsymbol{\omega}}_{c,\,\mathrm{org}}(z)$ for a large enough twist c.

Step 5: *arborify* the expansions according to the rules given in §6.1 so as to render the previously obtained expansions for $\Theta^{\pm 1}$ *convergent* in the space of resurgent functions.

Step 6: Construct the sought-after analytic object **Ob** from its normaliser by using $\mathbf{Ob} = \Theta\ \mathbf{Ob}^{\mathrm{nor}}\ \Theta^{-1}$

The reader may easily work this out in the case of our four Examples. For details and more examples, he may turn to [E.15].

3.7.1 Remarks and complements.

Remark 1: Antipodal involution.
As already pointed out, our twisted monomials have much the same behaviour at both poles of the Riemann sphere. The exact correspondence is described in §6.7 using the so-called *antipodal involution*:

$$\mathrm{pod}\ :\qquad z \mapsto c^2\, z^{-1} \quad \textit{and} \quad (\omega_i, c^2\,\bar{\omega}_i) \mapsto (\bar{\omega}_i, c^2\,\omega_i) \tag{139}$$

In terms of the objects being produced, this means that canonical object synthesis automatically generates two objects for the price of one: the 'true' object, local at ∞ and with exactly the prescribed invariants, and a 'mirror reflection', local at 0 and with closely related invariants. Depending on the nature of the problem (linear/non-linear, etc) and of the invariants, these two objects may or may not link up under analytic continuation on the Riemann sphere.

Remark 2: Analogy with q-equations.

[49]The second expansion, for the reverse normaliser, is valid only if all the invariants $\mathbb{A}_\omega$ have no ∂_z-component and so commute with the resurgence monomials $\mathcal{U}\!e^{\boldsymbol{\omega}}(z)$. When this is not the case, the expansion (138) should be slightly modified, but one can also be content to work with (137), which is always valid, and then derive Θ^{-1} by straightforward inversion of Θ.

Authors like Sauloy recently observed that q-difference equations are in some sense easier to tackle than difference or differential equations, due to dilations $z \mapsto q\,z$ having two fixed points 0 and ∞, whereas shifts $z \mapsto a + z$ have only one, namely ∞. It is certainly no coincidence that the simplest resurgence monomials that permit object synthesis are precisely the twisted ones ($c \neq 0$), for whom the antipodal symmetry is restored, and that the twistless monomials ($c = 0$), though apparently more simple, turn out to be inadequate for this particular purpose.

Remark 3: Necessity of a one-parameter freedom.
The necessity of having at least one degree of freedom in object synthesis has been known since the 1980s at least. This holds even for such elementary objects as linear systems (Example 3). Indeed, in most cases, the twist c must exceed a certain lower bound c_{min} that depends on the invariants $\{\mathbb{A}_\omega\}$. There exists, however, an important exception: the so-called *unilateral classes*, when for instance all non-vanishing $\mathbb{A}_\omega$ have their indices on the same half-line. There *any choice* $c > 0$ *will do!* This applies in particular to Example 2 when $\mathbb{A}_{-1} = 0$

Remark 4: Iso-invariant deformations.
There exists a closed system of formulae (see §6.7 and [E15]) to describe the exact dependence (partial derivatives, asymptotics, etc) of our canonical resurgence monomials as functions of their variable z, twist c and indices $\omega_i, \bar{\omega}_i$. As a result, one may write down the – often unexpectedly simple – partial differential equations which govern the sundry deformations (iso-invariant, iso-monodromic, iso-resurgent, iso-Galoisian, etc) of our synthesised objects.

Remark 5: WB derivations and WB monomials: unequal status.
Working with well-behaved alien derivations is merely convenient, whereas the recourse to well-behaved resurgence monomials is truly indispensible. There is a subtle difference here, which should be well understood. Indeed, the choice of this or that system of WB alien derivations *does not affect the result*: it simply gives us a comfortable basis of ALIEN to work with. Besides, there is always the lazy option of working with the lateral alien operators $\Delta^{\pm}_{\omega}$, the only drawback being that the corresponding invariants $\mathbb{A}^{\pm}_{\omega}$ cease to be first-order differential operators. In complete contrast, the synthesised object very much depends on the choice of the system of WB monomials. And in the absence of well-behaved monomials, canonical synthesis would founder altogether.

Remark 6: Non-canonical synthesis.
Linear object synthesis (Example 3) is of course a very old subject. As for

non-linear object synthesis, adequate methods (especially for situations like in our examples 1,2,4) were evolved in the late 1970s (see [E2],[E3],[Ma]) but they were not always explicit, much less canonical, except in a few special instances.

4 Lesson Four: A surprise development in KAM theory: the non-existence of Supermultiple Small Denominators.

4.1 A few landmarks.

– **Poincaré** leaned (ca 1880) towards the view that the small denominators occuring in Hamiltonian systems must generically create unsurmountable divergence in the so-called *Lindstedt series* (the formally quasiperiodic Fourier series which describe the motion).

– The first truly general convergence results in the presence of small denominators were established ca 1940 by **Siegel** [Sie1][Sie2], but under unnecessarily strong diophantine assumptions, and moreover in non-hamiltonian situations, where the risk of SSD (*supermultiple small denominators* , ie small denominators aggravated by *resonance*) does not arise.

– Using successive approximations, **Kolmogorov** proved in 1954 his epochmaking result about the survival of invariant tori near integrability (under Siegel's condition), thus indirectly establishing the convergence of the Lindstedt series – and proving Poincaré wrong. This set off an extremely active line of research (**Arnold, Moser**, etc), known as KAM theory[50].

– **Cherry** before World War II and (far more systematically) **Bryuno** and **Rüssmann** in the 70s began proving small denominator results under a diophantine condition weaker than Siegel's and probably optimal (ie minimal), known as Bryuno's condition.

– towards the close of the 80s **Yoccoz** and in his wake **Perez-Marco**, using an original renormalisation approach, proved the optimality of Bryuno's condition for a number of non-linear problems (in low dimensions).

– about the same time **Eliasson** re-proved the convergence result for the Lindstedt series near integrability, but *directly*, by a super-meticulous examination of their coefficients. This was a *premiere* of sorts, but it used a

[50]an acronym for the three main contributors : Kolmogorov, Arnold, Moser

forbidding amount of analysis (Eliasson's famed method of "sign compensations") and it somehow entrenched the impression that the SSD (supermultiple small multipliers) were "really there". Numerous epigones (mainly Italians, **Gallavotti** etc, also **Krikorian**) took up and developed the method, which came to be known as the "direct method" in KAM theory.

In the course of this Fourth Lesson, I shall draw on joint work by B. Vallet and myself, and show, by purely algebraic or rather combinatorial arguments, that the SSD actually *do not exist*. We shall focus on three closely related problems, of slightly increasing difficulty :

– the so-called "correction" of resonant vector fields
– Floquet theory (differential equations with quasi-periodic coefficients)
– the survival of tori for near-integrable Hamitonian systems

4.2 *Cor* and *Pre* : differences/similarities.

Let X be a local, singular, analytic vector field with diagonal linear part X^{lin} and homogeneous components $\mathbb{B}_{\mathbf{n}}$:

$$\begin{aligned} X &= X^{\mathrm{lin}} + \mathbb{B} = X^{\mathrm{lin}} + \sum_{\mathbf{n}} \mathbb{B}_{\mathbf{n}} \qquad\qquad \textit{with} && (140)\\ X^{\mathrm{lin}} &= \sum_{1 \le i \le \nu} \lambda_i\, x_i\, \partial_{x_i} \qquad \textit{and} \qquad \mathbb{B}_{\mathbf{n}} = x^{\mathbf{n}} \sum_{1 \le i \le \nu} b_{\mathbf{n},i}\, x_i\, \partial_{x_i} \end{aligned}$$

In the non-resonant case, X is formally linearisable, even analytically so if there is no quasiresonance. In the resonant case, however, the simplification cannot be so thorough. The most we can hope for is to remove all non-resonant terms, and allow resonant terms on one side of the conjugation equation only. If we allow them on the left-hand side, we get familiar objects, the so-called normal or prenormal forms $X^{\mathrm{nor}}, X^{\mathrm{pre}}$, which we studied at length in Lesson Two. But if we allow them on the right-hand side, we encounter another type of object, the so-called *correction* X^{cor}, with altogether different properties, both formal and analytic.

$$\begin{aligned} X &\overset{\mathrm{conj.}}{\sim} X^{\mathrm{lin}} && \text{(non-resonant case)} && (141)\\ X &\overset{\mathrm{conj.}}{\sim} X^{\mathrm{lin}} + X^{\mathrm{pre}} && (\text{resonant case },\ X^{\mathrm{pre}} \in \mathrm{Reson}) && (142)\\ X - X^{\mathrm{cor}} &\overset{\mathrm{conj.}}{\sim} X^{\mathrm{lin}} && (\text{resonant case },\ X^{\mathrm{cor}} \in \mathrm{Reson}) && (143) \end{aligned}$$

The main differences are these:

X^{cor} ("*correction*")	X^{pre} ("*prenormal form*")
unique (in any given chart)	non-unique (even after chart choice) but unique *mod* [Reson, Reson]
generically convergent	generically divergent-resurgent
interpretation: recondite (Gallavotti's 'Wick Invariant')	interpretation: simple (geometrical-dynamical)

4.3 The 'correction' of resonant vector fields.

The corrected form $X - X^{\text{cor}}$ and its linearising transformation Θ_{cor}:

$$X \mapsto X^{\text{corrd}} = X - X^{\text{cor}} \qquad (X^{\text{cor}}\ \ \textit{unique}) \tag{144}$$

$$X - X^{\text{cor}} = \Theta_{cor}\, X^{\text{lin}}\, \Theta_{cor}^{-1} \qquad (\Theta_{\text{cor}}\ \ \textit{non-unique}) \tag{145}$$

$$[X^{\text{lin}}, X^{\text{corrd}}] = [X^{\text{lin}}, X^{\text{cor}}] = 0 \tag{146}$$

both admit mould expansions:

$$X^{\text{cor}} = \sum \text{Carr}^{\bullet}\, \mathbb{B}_{\bullet} = \sum_{r\geq 1}\sum \text{Carr}^{\omega_1,\dots,\omega_r}\, \mathbb{B}_{n_r}\dots\mathbb{B}_{n_1} \tag{147}$$

$$\Theta_{\text{cor}} = \sum \text{Scarr}^{\bullet}\, \mathbb{B}_{\bullet} = \sum_{r\geq 1}\sum \text{Scarr}^{\omega_1,\dots,\omega_r}\, \mathbb{B}_{n_r}\dots\mathbb{B}_{n_1} \tag{148}$$

with "frequencies" $\omega_i := <\lambda, n_i>$ and with moulds $Carr^{\bullet}$ (alternal) and $Scarr^{\bullet}$ (symmetral) that depend rationally on these frequencies. Here, we shall leave $Scarr^{\bullet}$ alone and concentrate on the mould $Carr^{\bullet}$. It is calculable from the following induction:

$$\text{var}_{\text{i}}\, \text{Carr}^{\boldsymbol{\omega}} = \sum_{\mathbf{a}\omega_i\mathbf{bc}=\boldsymbol{\omega}} \text{Carr}^{\mathbf{a}\omega_i\mathbf{c}}\text{Carr}^{\mathbf{b}} - \sum_{\mathbf{ab}\omega_i\mathbf{c}=\boldsymbol{\omega}} \text{Carr}^{\mathbf{b}}\text{Carr}^{\mathbf{a}\omega_i\mathbf{c}} \tag{149}$$

with the initial conditions:

$$\text{Carr}^{\emptyset} = 0\,;\quad \text{Carr}^{0} = 1\,;\quad \text{Carr}^{\omega_1} = 0 \quad if \quad \omega_1 \neq 0 \tag{150}$$

and with 'variation operators'[51] var_i that act like this:

$$\text{var}_{\text{i}}\, M^{\omega_1,\dots,\omega_r} := \omega_i\, M^{\omega_1,\dots,\omega_r} + M^{\omega_1,\dots,\omega_i+\omega_{i+1},\dots,\omega_r} - M^{\omega_1,\dots,\omega_{i-1}+\omega_i,\dots,\omega_r} \tag{151}$$

[51] for their geometric interpretation, see [EV2]

Key lemma: Non-repetition of denominators in $Carr^{\boldsymbol{\omega}}$**.**
For sequences $\boldsymbol{\omega}$ *of a given length* r *and a fixed degeneracy pattern of order* d *(see below) the correction coefficient* $Carr^{\boldsymbol{\omega}}$*, as a rational function of its* r-d *independent variables* ω_i*, has only poles of the form* $\eta^{-\mu}$*, with linear combinations* η *of the* ω_i *obtained by splitting unbreakable zero-sum sub-sequences* $\boldsymbol{\omega}^\star$ *of* $\boldsymbol{\omega}$*:*

$$\pm\eta = \|\boldsymbol{\omega}'\| = -\|\boldsymbol{\omega}''\| \qquad (\boldsymbol{\omega}'\boldsymbol{\omega}'' = \boldsymbol{\omega}^\star = (\omega_i, \ldots, \omega_j)) \tag{152}$$

and with a multiplicity μ *no larger than the number of unbreakable, zero-sum sequences* $\boldsymbol{\omega}^\star$ *which, when split, can produce* η*. Moreover, although there is in general no canonical "best way" of decomposing* $Carr^{\boldsymbol{\omega}}$*, there always exist decompositions of the form*

$$\mathrm{Carr}^{\boldsymbol{\omega}} = \sum m_P \prod_{i=1\ldots s_P} (\eta_{P,i})^{-\mu_{P,i}} \qquad \textit{with} \qquad \sum_i \mu_{P,i} \equiv r-1 \tag{153}$$

which involve only effective poles η*, with a multiplicity never exceeding the intrinsic multiplicity* μ *and with integral coefficients* m_P *bounded by:*

$$\sum_P |m_P| \le \frac{(2r-2)!}{(r-1)!\,r!} \le 4^r \qquad (r = r(\boldsymbol{\omega});\ m_p \in \mathbb{Z}^\star) \tag{154}$$

Comments and proof.
A fixed *degeneracy pattern* of order d is a set of d pairs (i,j) verifying $1 \le i \le j \le r$ and such that the sequence $\boldsymbol{\omega}^\star = (\omega_i, \ldots, \omega_j)$ be unbreakable with zero-sum. "Unbreakability", we recall, rules out non-trivial factorisations $\boldsymbol{\omega}^\star = \boldsymbol{\omega}'\boldsymbol{\omega}''$ with either $\|\boldsymbol{\omega}'\| = 0$ or $\|\boldsymbol{\omega}''\| = 0$, but it does not rule out non-trivial factorisations of the form $\boldsymbol{\omega}^\star = \boldsymbol{\omega}'\boldsymbol{\omega}''\boldsymbol{\omega}'''$ with a zero-sum middle factor $\|\boldsymbol{\omega}''\| = 0$.
What the above lemma tells us about the multiplicities μ of the poles $\eta^{-\mu}$ amounts to this: *the denominators* η *of* $Carr^{\boldsymbol{\omega}}$ *undergo no repetitions unless they are already repeated within the sequence* $\boldsymbol{\omega}$*, these repetitions being induced by the degeneracy pattern itself.* This is in sharp contrast to the behaviour of most other moulds, such as $sa^\bullet$, $musa^\bullet$, etc, and all prenormalisation moulds $pran^\bullet$.
Thus, if $\boldsymbol{\omega}$ has even length $r = 2r'$ and the following degeneracy pattern:

$$0 = \omega_1 + \omega_r; \qquad 0 = \omega_2 + \omega_3 = \omega_4 + \omega_5 = \cdots = \omega_{r-2} + \omega_{r-1} \tag{155}$$

(which leaves as independent variables ω_1 and $\omega_2, \omega_4, \omega_6, \omega_{r-2}$) we have on the one hand, for all coefficients $pran^{\boldsymbol{\omega}}$ [52]:

$$pran^{\boldsymbol{\omega}} = -(\omega_1)^{-r'} \prod_{i=1}^{r'-1} (\omega_1 + \omega_{2i})^{-1} \tag{156}$$

with a "huge" multiplicity (either r' or $1+r'$) for ω_1 ; and on the other hand, calculating $Carr^{\boldsymbol{\omega}}$ by the procedure which we shall spell out in a moment, we find:

$$Carr^{\boldsymbol{\omega}} = (-1)^{r'-1}(\omega_1)^{-1} \prod_{1 \le i \le r'} (\omega_{2i})^{-1}(\omega_1 + \omega_{2i})^{-1} \tag{157}$$

in full agreement with the above lemma, and with no unwarranted repetition of poles.

We say that a ZUS (short for "zero-sum, unbreakable sequence") $\boldsymbol{\omega}^\star$ is adjacent to a component ω_i if ω_i either *initiates* or *terminates* $\boldsymbol{\omega}^\star$ (as its first or last element) or if it immediately *precedes* or *follows* $\boldsymbol{\omega}^\star$. Thus, any ω_i has at most four distinct adjacent ZUS. If ω_i initiates a (necessarily unique) ZUS $\boldsymbol{\omega}^+$, we set $r_i^+ = r(\boldsymbol{\omega}^+) =$ length of $\boldsymbol{\omega}^+$ and, if not, we set $r_i^+ = 0$. Similarly, if ω_i terminates a (necessarily unique) ZUS $\boldsymbol{\omega}^-$, we set $r_i^- = r(\boldsymbol{\omega}^-) =$ length of $\boldsymbol{\omega}^-$ and, if not, we set $r_i^- = 0$.
We can now state the two selection rules for the index i of var_i. They read:

$[C_1]$ $\{0 < r_i^+ + r_i^-\}$
$[C_2]$ $\{0 < r_i^+ + r_{i+1}^+ \ \ or \ \ 0 = r_{i+1}^+\} \quad and \quad \{0 < r_i^- + r_{i-1}^- \ \ or \ \ 0 = r_{i-1}^-\}$

and can be interpreted as follows:
$[C_1]$ says that ω_i should be the first or last component of some ZUS, or both.
$[C_2]$ says that ω_i should not be squeezed between two adjacent ZUS $\boldsymbol{\omega}^\star$ and $\boldsymbol{\omega}^{\star\star}$ such that $\boldsymbol{\omega}^\star \subset \boldsymbol{\omega}^{\star\star}$. It says, too, that if ω_i is *externally adjacent* to some ZUS $\boldsymbol{\omega}^\star$, it should also be *internally adjacent* to some other ZUS $\boldsymbol{\omega}^{\star\star}$ that overlaps with $\boldsymbol{\omega}^\star$, but doesn't contain it.

There clearly exist $25 = 5^5$ distinct *adjacency types* $T_1, T_2, \ldots, T_{25}$ but a careful check shows that only $8 = 3^2 - 1$ of them, namely $T_1, T_2, \ldots, T_8$ are allowed under the selection rule $[C_1] + [C_2]$. All licit and illicit adjacency types are listed in the table below, where all sub-sequences $\lceil \ldots \rceil$ or $\lfloor \ldots \rfloor$ squeezed between two opposite upper or lower brackets (facing one another) are assumed to be of ZUS type.

[52] more accurately, for all prenormalisations that are free of parasitical singularities, like the *royal* or *regal* one, but for the others the phenomenon of "supermultiplicity" is either just as bad or worse.

	licit		illicit		illicit
T_1	$\lfloor \omega_i\, \mathbf{c} \rfloor$	T_9	ω_i	T_{17}	$\lfloor \mathbf{b} \rfloor \lfloor \omega_i \lceil \mathbf{c} \rfloor \mathbf{d} \rceil$
T_2	$\lceil \mathbf{b}\, \omega_i \rceil$	T_{10}	$\omega_i \lceil \mathbf{c} \rceil$	T_{18}	$\lfloor \mathbf{b} \rfloor \lfloor \omega_i \lceil \mathbf{c} \rceil \mathbf{d} \rfloor$
T_3	$\lceil \mathbf{b} \lfloor \omega_i \rceil \mathbf{c} \rfloor$	T_{11}	$\lfloor \mathbf{b} \rfloor \omega_i$	T_{19}	$\lceil \mathbf{b} \lfloor \omega_i \rceil \lceil \mathbf{c} \rceil \mathbf{d} \rfloor$
T_4	$\lfloor \omega_i \lceil \mathbf{c} \rfloor \mathbf{d} \rceil$	T_{12}	$\lfloor \mathbf{b} \rfloor \omega_i \lceil \mathbf{c} \rceil$	T_{20}	$\lfloor \mathbf{a} \lceil \mathbf{b} \rfloor \omega_i \rceil \lceil \mathbf{c} \rceil$
T_5	$\lfloor \mathbf{a} \lceil \mathbf{b} \rfloor \omega_i \rceil$	T_{13}	$\lceil \mathbf{b}\, \omega_i \rceil \lfloor \mathbf{c} \rfloor$	T_{21}	$\lceil \mathbf{a} \lfloor \mathbf{b} \rfloor \omega_i \rceil \lceil \mathbf{c} \rceil$
T_6	$\lceil \mathbf{b} \lfloor \omega_i \rceil \lceil \mathbf{c} \rfloor \mathbf{d} \rceil$	T_{14}	$\lfloor \mathbf{b} \rfloor \lfloor \omega_i\, \mathbf{c} \rfloor$	T_{22}	$\lceil \mathbf{a} \lfloor \mathbf{b} \rfloor \lfloor \omega_i \rceil \mathbf{c} \rfloor$
T_7	$\lfloor \mathbf{a} \lceil \mathbf{b} \rfloor \lfloor \omega_i \rceil \mathbf{c} \rfloor$	T_{15}	$\lfloor \omega_i \lceil \mathbf{c} \rceil \mathbf{d} \rfloor$	T_{23}	$\lfloor \mathbf{a} \lceil \mathbf{b} \rfloor \lfloor \omega_i \rceil \lceil \mathbf{c} \rceil \mathbf{d} \rfloor$
T_8	$\lfloor \mathbf{a} \lceil \mathbf{b} \rfloor \lfloor \omega_i \rceil \lceil \mathbf{c} \rfloor \mathbf{d} \rceil$	T_{16}	$\lceil \mathbf{a} \lfloor \mathbf{b} \rfloor \omega_i \rceil$	T_{24}	$\lceil \mathbf{a} \lfloor \mathbf{b} \rfloor \lfloor \omega_i \rceil \lceil \mathbf{c} \rfloor \mathbf{d} \rceil$
				T_{25}	$\lceil \mathbf{a} \lfloor \mathbf{b} \rfloor \lfloor \omega_i \rceil \lceil \mathbf{c} \rceil \mathbf{d} \rfloor$,

The next step is to prove that any $\boldsymbol{\omega}$, whatever its degeneracy type, has at least *one* licit component ω_i . In fact, it always has *two*, and often much more : see [EV2].

A linear combination $\eta = \omega_p + \cdots + \omega_q$ is said to be a *formal pole* of the sequence $\boldsymbol{\omega}$ if it can be obtained by breaking up some ZUS. The *formal multiplicity* of $\left[{\eta \atop \boldsymbol{\omega}} \right]$ of η in $\boldsymbol{\omega}$ is defined as being the number of ZUS which, when split, can produce η.

Collecting all the above results, rewriting the induction rule as follows :

$$\mathrm{Carr}^{\boldsymbol{\omega}} = \frac{1}{\omega_i} \{+\mathrm{Carr}^{\boldsymbol{\omega^1}} - \mathrm{Carr}^{\boldsymbol{\omega^2}} + \sum \mathrm{Carr}^{\boldsymbol{\omega^3}}\mathrm{Carr}^{\boldsymbol{\omega^4}} - \sum \mathrm{Carr}^{\boldsymbol{\omega^5}}\mathrm{Carr}^{\boldsymbol{\omega^6}}\}$$

with

$$\begin{array}{lcl}
\boldsymbol{\omega^1} = (\dots, \omega_{i-1} + \omega_i, \dots) & ; & \boldsymbol{\omega^2} = (\dots, \omega_i + \omega_{i+1}, \dots) \\
\boldsymbol{\omega^3} = \boldsymbol{\omega^{34}} \omega_i\, \boldsymbol{\omega^{43}} & \mathit{for} & \boldsymbol{\omega} = \boldsymbol{\omega^{34}} \omega_i\, \boldsymbol{\omega^4} \boldsymbol{\omega^{43}} \\
\boldsymbol{\omega^6} = \boldsymbol{\omega^{65}} \omega_i\, \boldsymbol{\omega^{56}} & \mathit{for} & \boldsymbol{\omega} = \boldsymbol{\omega^{65}} \boldsymbol{\omega^5} \omega_i\, \boldsymbol{\omega^{56}}
\end{array}$$

and taking care of applying this rewritten induction rule for licit ω_i's only, we can easily verify, by going through all the 8 licit cases, that the *actual multiplicity of* η *as pole of the rational function* $\mathrm{Carr}^{\boldsymbol{\omega}}$ *is no larger than the formal multiplicity of* η *in the sequence* $\boldsymbol{\omega}$. *QED*

Arborification.
The above argument shows the absence of SSD (supermultiple small denominators) but is not enough to establish the convergence of X^{cor} under Bryuno's condition. Indeed, the straightforward mould expansion (147) is usually divergent. But it can easily be rendered convergent by resorting to the usual trick of arborification-coarborification (see §6.1) :

$$X^{\mathrm{cor}} \;=\; \sum \mathrm{Carr}^{\bullet^{\prec}}\, \mathbb{B}_{\bullet^{\prec}} \;=\; \sum_{r\geq 1}\sum \mathrm{Carr}^{\omega_1,\dots,\omega_r{}^{\prec}}\, \mathbb{B}_{n_1,\dots,n_r{}^{\prec}} \tag{158}$$

and by dutifully 'arborifying' the whole argument that we went through. Thus the induction rule becomes :

$$\mathrm{var_i}\, \mathrm{Carr}^{\boldsymbol{\omega}^{\prec}} = \sum_{\substack{\omega_i\in\mathbf{a}^{\prec}\\ \omega_i<\mathbf{b}^{\prec}}} \mathrm{Carr}^{\mathbf{a}^{\prec}}\mathrm{Carr}^{\mathbf{b}^{\prec}} - \sum_{\substack{\omega_i\in\mathbf{c}^{\prec}\\ \omega_i>\mathbf{b}^{\prec}}} \mathrm{Carr}^{\mathbf{b}^{\prec}}\mathrm{Carr}^{\mathbf{c}^{\prec}} \tag{159}$$

The sums now extend to all connected sub-trees $\mathbf{b}^{\prec}$ of $\boldsymbol{\omega}^{\prec}$ either directly following or preceding ω_i and $\mathbf{a}^{\prec}$ (resp. $\mathbf{c}^{\prec}$) denotes the remaining part of $\boldsymbol{\omega}^{\prec}$, with the arborescent order inherited from the parent sequence $\boldsymbol{\omega}^{\prec}$. The variance operator var_i also must be arborified. The transposition goes like this:

$$\mathrm{var}_i\, \mathrm{Carr}^{\boldsymbol{\omega}^{\prec}} = \omega_i\, \mathrm{Carr}^{\boldsymbol{\omega}^{\prec}} - \mathrm{Carr}^{\boldsymbol{\eta}^{\prec}} + \sum \mathrm{Carr}^{\boldsymbol{\sigma}^{\prec}} \tag{160}$$

where $\boldsymbol{\eta}^{\prec}$ denotes the unique arborified sequence obtained by contracting ω_i with its (necessarily unique) immediate predecessor ω_{i_-}; and the sum $\sum$ extends to all arborescent sequences $\boldsymbol{\sigma}^{\prec}$ obtainable by contracting ω_i with any one of its immediate successors ω_j (ie all ω_j such that $\omega_{j_-}=\omega_i$). For details and numerous examples see [EV2].

The upshot is that not only *is the correction X^{cor} always analytic under Bryuno's diophantine condition* but also that *its analyticity owes nothing to any so-called compensation of supermultiple small denominators, given that these SSD quite simply do not exist.*

4.4 Application to Floquet theory.

Floquet theory deals with differential equations whose coefficients are q.p. (quasiperiodic) functions of the variable. The central difficulty, as with the correction or KAM theory, comes from the *small denominators*, which integration creates at every step. Let us consider a simple but fairly typical case, that of a homogeneous linear differential system :

$$\partial_t X(t) = U(t)\, X(t) \tag{161}$$

for a q.p. matrix $U(\bullet)$ with frequencies in a fixed set Ω:

$$
\begin{aligned}
U(t) &:= \sum_{\omega\in\Omega} e^{i\,\omega\,t}\,U_\omega && (U_\omega = const\ ,\ t\in\mathbb{R}) \quad (162)\\
\omega\in\Omega &:= \lambda_1\mathbb{Z}+\lambda_2\mathbb{Z}+\cdots+\lambda_\nu\,\mathbb{Z} && \quad (163)
\end{aligned}
$$

An important, very popular sub-case is:

$$
\begin{aligned}
& U(t) := l\,A + \epsilon\,B(t) = l\,A + \epsilon\,(\sum e^{i\,\omega\,t}\,B_\omega) && (164)\\
& (A\ ,\ B_\omega\ \ constant\ matrices\ ,\ 1 << l\ ,\ \epsilon << 1) && (165)
\end{aligned}
$$

Some notations.
The method is to reduce (161) to a trivial equation

$$
\partial_t\,Y(t) = V\,Y(t) \qquad \text{with} \quad V = Const \tag{166}
$$

by means of a change of unknown

$$
X(t) = \Theta(t)\,Y(t) \tag{167}
$$

The problem thus reduces to finding a q.p. matrix $\Theta(\bullet)$ verifying:

$$
V + \Theta^{-1}(t)\ \partial_t\,\Theta(t) = \Theta^{-1}(t)\,U(t)\,\Theta(t) \tag{168}
$$

On the space of q.p. matrices with frequencies in Ω, i.e. of the form (162), let there be defined the following operators:

$$
\overline{V} := (\partial_t - ad(V)) \quad ; \qquad \underline{V} := (\partial_t - ad(V))^{-1} \tag{169}
$$

The 'derivation' $\overline{V}$ acts on all components $e^{i\,\omega\,t}\,U_\omega$. The 'integration' $\underline{V}$ acts unambiguously only on those components that are non-constant ($\omega \neq 0$). For instance, if we take for U_ω the general matrix $[a_{j,k}]$ and for V the diagonal matrix with spectrum $\{iv_j\}$, we get for each $\omega \neq 0$:

$$
\underline{V}(e^{i\omega t}[a_{j,k}]) = e^{i\omega t}[b_{j,k}] \quad with \quad b_{j,k} = \frac{a_{j,k}}{i(\omega - v_{j,k})}; v_{j,k} := v_j - v_k \tag{170}
$$

A similar formula holds for a general input V and, in all cases, the only denominators to appear are the $(\omega - v_{j,k})$, which of course includes the ω themselves (for $j = k$). We call such denominators *intrinsic*[53].

[53]as opposed to the *extrinsic* denominators which bedevil all approximation methods.

The occurrence of (potentially small) denominators other that the ω makes it necessary to impose a modified form of Bryuno's classical diophantine condition:

$$S(\lambda) := \sum_k 2^{-k} \log(\frac{1}{\varpi(2^k)}) < \infty \tag{171}$$

with

$$\varpi(K) := \inf |\sum m_i \lambda_i| \quad for \sum m_i \lambda_i \neq 0 \ \ and \ \ \sum |m_i| \leq K \tag{172}$$

Here, we must replace $S(\lambda)$ by $S_{\boldsymbol{v}}(\lambda)$, with the same definition as in (171), but relative the diophantine gauge $\varpi_{\boldsymbol{v}}(K)$:

$$\varpi_{\boldsymbol{v}}(K) := inf|(\sum m_i \lambda_i) - (v_j - v_k)| \tag{173}$$

Everything hinges on the arithmetics of the spectrum $i\boldsymbol{v} = \{iv_1, \ldots, iv_n\}$ of the constant matrix V. The parameters v_j are the so-called Floquet exponents.[54] They are defined only modulo Ω (or, exceptionnaly, modulo a slightly larger group), but they possess a "principal determination" [55], and the way to calculate the latter is as follows: start from the fundamental *matricial* solution $X(t)$ of (161), with initial conditions $X(0) = I$ and of course $det X(t) \neq 0$; then follow a continuous determination of $(1/t) \log X(t)$; lastly take the limit of its spectrum as $t \to \pm\infty$.

Since the frequencies λ_i that span Ω were assumed to verify the Bryuno-type condition (171)+(173), we can pick $\epsilon \leq \epsilon_0(B)$ small enough to ensure the convergence not only of the familiar-looking series:

$$\begin{aligned} C &= \sum_r \epsilon^r \sum_{\omega_i} \ Carr^{\omega_1,\ldots,\omega_r} B_{\omega_r} \ldots B_{\omega_2} B_{\omega_1} && (174)\\ \Theta &= 1 + \sum_r \epsilon^r \sum_{\omega_i} e^{i\,(\sum \omega_j)\,t} \ Scarr^{\omega_1,\ldots,\omega_r} B_{\omega_r} \ldots B_{\omega_2} B_{\omega_1} && (175) \end{aligned}$$

but also of these two new series:

$$\begin{aligned} C &= \sum_r \epsilon^r \sum_{\omega_i} Carr^{B_{\omega_1},\ldots,B_{\omega_r}} && (176)\\ \Theta &= 1 + \sum_r \epsilon^r \sum_{\omega_i} e^{i\,(\sum \omega_j)\,t} \ Scarr^{B_{\omega_1},\ldots,B_{\omega_r}} && (177) \end{aligned}$$

[54] For the *cognoscenti*: in the 'hyperbolic' case, the v_j may possess an imaginary part, which creates in the iv_j a real part that corresponds to the so-called Lyapunov exponents.

[55] which in the case of $sl(2, \mathbf{R})$ reduces to a classical (but generally complex) 'rotation number'.

The first pair of series is clearly patterned on the expansions which we found for the "correction" of a resonant vector field (see §4.3 *supra*). The second pair is derived from the first by applying a simple transposition rule which can be easily grasped from the following example, for a length $r = 3$:

$$
\begin{aligned}
Carr^{\omega_1,\omega_2,\omega_3} B_{\omega_3} B_{\omega_2} B_{\omega_1} \quad &:= \quad \frac{1}{3}\Big(\frac{1}{\omega_1(\omega_1+\omega_2)} - \frac{1}{\omega_1\omega_3} + \frac{1}{\omega_3(\omega_2+\omega_3)}\Big) B_{\omega_3} B_{\omega_2} B_{\omega_1} \\
\longrightarrow \quad Carr^{B_{\omega_1},B_{\omega_2},B_{\omega_3}} \quad &:= \quad +\frac{(-1)^3}{3}\Big((e^{i\omega_3} B_{\omega_3})(\underline{V}((e^{i\omega_2} B_{\omega_2})(\underline{V} e^{i\omega_1} B_{\omega_1})))\Big) \\
&\qquad -\frac{(-1)^3}{3}\Big((\underline{V} e^{i\omega_3} B_{\omega_3})(e^{i\omega_2} B_{\omega_2})(\underline{V} e^{i\omega_1} B_{\omega_1}))\Big) \\
&\qquad +\frac{(-1)^3}{3}\Big((\underline{V}((\underline{V} e^{i\omega_3} B_{\omega_3})(e^{i\omega_2} B_{\omega_2})))(e^{i\omega_1} B_{\omega_1})\Big)
\end{aligned}
$$

Since the coefficients $Carr^{\bullet}$ (unlike $Scarr^{\bullet}$) vanish for $\omega_1 + \cdots + \omega_r \neq 0$, the series C (unlike $\Theta(t)$) is t-constant, as indeed it should be. Nonetheless, in order to properly carry out the above transposition, one should take care of adopting not *any* expression of $Carr^{\bullet}$ and $Scarr^{\bullet}$, but a *suitably symmetrised* one. Even so, the expression is not unique, but this does not affect the one-to-one character of the transposition, since to each algebraic identity of type:

$$
\frac{1}{\omega_1(\omega_1+\omega_2)} + \frac{1}{\omega_2(\omega_1+\omega_2)} = \frac{1}{\omega_1\omega_2} \tag{178}
$$

there corresponds an non-commutative integration-by-parts identity:

$$
\begin{aligned}
&\underline{V}((e^{i\omega_2} B_{\omega_2})(\underline{V}(e^{i\omega_1} B_{\omega_1}))) + \underline{V}(\underline{V}(e^{i\omega_2} B_{\omega_2}))(e^{i\omega_1} B_{\omega_1}))) = \\
&\qquad\qquad (\underline{V}(e^{i\omega_2} B_{\omega_2}))(\underline{V}(e^{i\omega_1} B_{\omega_1}))
\end{aligned} \tag{179}
$$

For $V = 0$, the series (176),(177) coincide with the series (174),(175). So by construction they are guaranteed to converge, since at the outset the perturbation $\epsilon\, B(\bullet)$ was assumed to be small enough.

For $V \neq 0$ on the other hand, the series (176),(177) converge only if $S_{\boldsymbol{v}}(\lambda)$ is close enough to $S(\lambda)$. If this already holds for the 'principal determination' $\boldsymbol{v} := \boldsymbol{u}$, so much the better, for that means that our construction is at an end: $U(\bullet)$ is indeed of the form $V - C + \epsilon B(\bullet)$ for some matrix V with spectrum $i\mathbf{u}$ (we may even, if we so wish, exactly calculate that matrix) and the normalising matrix $\Theta(\bullet)$ is explicitly given by (177).

When applied to a Floquet equation such as (161)+(164), the above considerations immediately establish the reducibility theorem[56] *for a l-set of*

[56] ie the existence of a q.p. non-singular matrix $\Theta(t)$ that reduces the given equation (161) to the trivial form (166).

positive Lebesgue measure. In fact, a slight refinement of the argument[57] yields reducibility for a l-set of *full measure.* The phenomenon was first discovered by Eliasson (by his own methods) and then extended by Krikorian (by much the same methods). See [El1],[EL2],[Kri].

4.5 Application to KAM theory.

We shall perforce remain extremely sketchy. We work under the classical (analytic) KAM assumptions, ie we perturb an integrable hamiltonian h :

$$h(y) \quad = \quad <\lambda, y> + <y, Q, y> \; = \; \sum \lambda_i \, y_i + \sum Q_{i,j} \, y_i \, y_j \tag{180}$$

(with $\mathbb{Q}$-independent basic frequencies λ_i) into a non-integrable H :

$$\begin{aligned} H(x,y) \quad &= \quad h(y) + \epsilon \, b(x,y) \quad (x \in \mathbb{T}^\nu \, , \, y \in \mathbb{R}_0^\nu) &(181)\\ &= \quad <\lambda, y> + \sum_{m,n} H_{m,n}(x,y) &(182)\end{aligned}$$

and we wish to establish, under Bryuno's (not Siegel's) diophantine assumption on the λ_i's, the convergence, for $y = 0$ and for small enough values of the perturbation parameter ϵ, of the "uncorrected" Lindstedt series :

$$\begin{aligned} \sum H_{m,n}(x,y) \quad &= \quad \sum c_{m,n}(\epsilon) \, e^{2\pi i <x,m>} \; y^n \; = \; \sum c_{m,n}(\epsilon) \, e^{2\pi i <\lambda,m> \, t} \, y^n &(183)\\ \omega \quad &:= \quad <m, \lambda> = \text{“frequency”} \qquad\qquad (m \in \mathbb{Z}^\nu) &(184)\\ \eta \quad &:= \quad |n| - 1 = \sum n_i - 1 = \text{“grade”} \geq -1 \;\; (n \in \mathbb{N}^\nu) &(185)\end{aligned}$$

We prefer to work with fields rather than their potentials. The thing is to partially "correct" and partially "normalise" [58] our field X^H :

$$\begin{aligned} X^H - X^{\text{cor}} \quad &\overset{conj}{\sim} \quad X^{\text{lin}} + X^{\text{nor}} &(186)\\ \text{frequency}(X^{\text{cor}}) = 0 \quad &, \quad \text{frequency}(X^{\text{nor}}) = 0 &(187)\\ \text{grade}(X^{\text{cor}}) = 0 \quad &, \quad \text{grade}(X^{\text{nor}}) \neq 0 &(188)\end{aligned}$$

by allowing only terms of zero (resp non-zero) grade on the left- (resp right-hand) side of (186). Like in §4.3 the correction still possesses a mould expansion of type :

$$X^{\text{cor}} \quad = \quad \sum_{r \geq 1} \sum \text{Bicarr}^{\binom{\omega_1 \; ,\ldots, \; \omega_r}{\eta_1 \; ,\ldots, \; \eta_r}} \; X^{H_{m_r,n_r}} \ldots X^{H_{m_2,n_2}} X^{H_{m_1,n_1}} \tag{189}$$

[57] by playing on the indeterminacy of the Floquet exponents, which enables us to change v_i into $v_i + \omega_i$ with ω_i in Ω and large enough, so as to make $S_{\boldsymbol{v}}(\lambda)$ as close to $S(\lambda)$ as necessary.

[58] in the sense of §2. Note that in this case, due to symplecticity, there is no difference between *normalising* and *prenormalising* : all prenormalisations coincide with one another and with the normalisation.

with frequencies $\omega_i :=<m_i, \lambda>$ and grades $\eta_i := -1 + \|n_i\|$. The normal part X^{nor} also has a similar mould expansion, but we need not worry about it, since it vanishes for $y = 0$ and so does not contribute to the Lindstedt series.

The alternal mould $Bicarr^\bullet$ is more complex than, but essentially similar to, the mould $Carr^\bullet$ of §4.3. In fact, $Bicarr^\bullet$ reduces to $Carr^\bullet$ when all the grades η_i are 0 or, more generally, when to each vanishing partial sum $\omega_i + \cdots + \omega_j = 0$ there corresponds a vanishing partial sum $\eta_i + \cdots + \eta_j = 0$.

To cut a long story short: we can duplicate in this case all the steps of §4.3 and prove, once again, the non-occurence of *supermultiple small denominators*, except that now the *formal multiplicity* of a divisor is exactly *twice* what it was in §4.3. That apart, precious little changes. We still must *arborify* to get the convergence of X^{cor}. This establishes, for a small enough pertubation parameter ϵ, the convergence of the Lindstedt series for the *corrected hamiltonian*. But a standard argument going back to Poincaré (known as "killing the constants" and using the possibility of changing the integration constants) readily yields the convergence of the Lindstedt series for the *given* hamiltonian itself.

4.6 KAM theory "in three theorems".

Since we are on this subject of KAM theory, I cannot resist mentioning that the formalisation approach applies with equal success to three distinct cases, which between themselves cover almost the whole spectrum of possibilities:

(1) The general case: no assumption on λ whatsoever.
The proper tool here is the (genuine) theory of compensation[59] and its *compensators*. We get special, divergent but resummable expansions of the movement, valid for nearly optimal time intervals $-t_1(\epsilon) < t < t_2(\epsilon)$.

(2)Maximal resonance: all λ_i's commensurate.
There are no small denominators at all. There is divergence, but of a perfectly manageable sort: resurgent and exactly resummable. We therefore get a good description of the movement, including the tendency towards escape (the so-called Arnold diffusion).

(3) The generic case: λ diophantine.
This is the case we have just discussed: we have a Cantor of surviving invariant tori near integrability. We cannot beat Kolmogorov's original proof for brevity or elegance, but we dispense with the non-intrinsic approximation process inherent in that proof. We also get behind the "true reasons"

[59]see §4.7 *infra* and [E9].

for the convergence of the Lindstedt series, by a direct examination of their coefficients, and unravel the mechanism that inhibits the occurence of SSD.

4.7 Genuine versus illusory compensation.

To preclude any terminological confusion, let us point out that the notion of "sign compensation", as used by the "Italo-Nordic KAM school" (Eliasson, Gallavotti, etc) bears no relation whatsoever to the genuine notion of compensation as introduced and used by us in the late 80's ([E9],[E10]). The latter belongs to an altogether different context, namely that of local objects with a Liouvillian (ie non-diophantine) spectrum. Even when such objects can be formally linearised, they generically resist analytical linearisation. However, after ramifying *one*, *two*, or at the utmost *three* coordinates (ie by considering expansions involving non-entire powers of these coordinates), it is always possible to construct a *ramified* and yet *effective* (as opposed to formal) linearisation of these Liouvillian objects [60]. The real compensation at work here takes place within the so-called "compensators", the simplest of which have the form:

$$z^{\sigma_0,\sigma_1,\ldots,\sigma_r} := \sum_{0\le i\le r} z^{\sigma_i} \prod_{j\neq i} (\sigma_i-\sigma_j)^{-1} \qquad (z\in\mathbb{C}_\bullet\ ,\ \sigma_i\in\mathbb{R}^+) \tag{190}$$

When the exponents σ_i are pair-wise different, but very close to each other, the coefficient in front of the power z^{σ_i} (which has a genuine, individualised existence) becomes very large and yet, due to a true phenomenon of compensation, the finite sum on the right-hand side of (190) remains bounded.

To recap, let us illustrate the difference on a trivial, yet telling example:

$$\begin{array}{llll} \text{Bogus compensation:} & f(\omega_1,\omega_2) & := \frac{1}{\omega_1\,(\omega_1+\omega_2)}+\frac{1}{\omega_2\,(\omega_1+\omega_2)} & \equiv \frac{1}{\omega_1\,\omega_2} \\ \text{Genuine compensation:} & g_t(\sigma_1,\sigma_2) & := \frac{t^{\sigma_1}}{\sigma_1-\sigma_2}+\frac{t^{\sigma_2}}{\sigma_2-\sigma_1} & \end{array}$$

5 Lesson Five: A Tale of Three Structures: Singular Perturbations; Multizetas; ARI/GARI

5.1 Dimorphy : functional or numerical.

Three interlinked structures.

[60] inside ramified neighbourhoods of the origin with spiral-like boundaries. Compensation (in our sense) also comes into its own in KAM theory, but in statements like §4.6, case (1), and not at all for establishing the convergence of the Lindstedt series

The first "structure" referred to in the title cannot, alas, be seriously discussed in this Survey, but it deserves at least a passing mention, because this is *the* development which in a sense sparked off everything. It is the theory of *singular perturbations* and *co-equational resurgence.* Singular peturbation parameters typically sit in front of the highest-order derivative in ODEs or PDEs. They are ubiquitous in physics, and invariably give rise to divergent expansions, which fortunately are often resurgent and resummable. Their resurgence, however, is unlike any other. We call it *co-equational* because it is loosely dual to *equational resurgence*[61]. Whereas equational resurgence is governed by one single Bridge Equation[62], co-equational resurgence requires two Bridge Equations for its complete description. Moreover, the whole edifice of co-equational resurgence is underpinned by a novel algebraic structure, which operates with two sets of parameters, the u_i's that get added and the v_i's that get subtracted. As it gradually emerged, the proper framework to deal with these operations was a new Lie algebra ARI with its group GARI. Then, roughly ten years after its cristallisation, the new-fangled algebraic apparatus was found (ca August 1999) to provide the key – some of the keys, at least – to another extremely active subject : the *arithmetics of multizetas* (or MZV, for Multiple Zeta Values) and the more general phenomenon of *dimorphy.*

"Dimorphous dimorphy".

Strange to say, but the notion of dimorphy is itself 'dimorphous' :

(i) For a space $\boldsymbol{\mathcal{D}}$ of *functions*, dimorphy means closure under two *distinct* products : usually point-wise multiplication and some form or other of convolution[63].

(ii) For a space $\mathbb{D}$ of *numbers*, dimorphy means being a countable $\mathbb{Q}$-ring and possessing two distinct, *natural* bases $\{\alpha_m\}$ and $\{\beta_n\}$, each with its countable indexation m and n [64], with finite conversion laws :

$$\alpha_m \equiv \sum_n H_m^n \, \beta_n \quad ; \quad \beta_n \equiv \sum_m K_n^m \, \alpha_m \tag{191}$$

[61] ie the sort of resurgence that is carried, not by *parameters*, but by the active *variable(s)* of a (differential, partial-differential, functional, etc) *equation.*

[62] we saw many forms of that B.E., but it is essentially one equation.

[63] see for example the algebra of *biresurgent functions* in [E13], §4

[64] m an n usually do not range through $\mathbb{N}$, but through more complex sets, like the monoid generated by $\mathbb{N}$.

and with *two distinct ways* of calculating the *one and only* product on $\mathbb{D}$, which is ordinary number multiplication :

$$\alpha_m\,\alpha_n \equiv \sum_r A^r_{m,n}\,\alpha_r \quad ; \quad \beta_m\,\beta_n \equiv \sum_r B^r_{m,n}\,\beta_r \tag{192}$$

All four sums have to be finite, with rational constants H, K, A, B.

Clearly, since one may always concoct artificial bases $\{\alpha_m\}$,$\{\beta_n\}$ to meet the above conditions, the whole emphasis in this notion of numerical dimorphy must lie on the *naturalness* of the two bases . This may seem a rather shaky foundation for a mathematical definition, but we venture to suggest that in fact it is not : in all known instances of dimorphy, there is no scope for hesitation ; the two bases $\{\alpha_m\}$,$\{\beta_m\}$ are clearly *there for all to see*, unmistakably *nature-given*, whereas it often takes a considerable amount of toil to *extract the hidden core* of $\mathbb{D}$, which usually is an algebraically free system $\{\gamma_r\}$ of irreducibles. This third set $\{\gamma_r\}$, typically, lies buried deep below the surface and, at least when taken in canonical form, tends to be found exactly mid-way between the two 'emerging' sets $\{\alpha_m\}$,$\{\beta_n\}$. So, even though it may be argued that numerical dimorphy is, ultimately, pure *māyā*, it is the sort of *māyā* that you must work hard to dispel. . .

In any case, functional and numerical dimorphy go hand in hand, and the proper framework for their joint investigation would seem to be, not the so-called 'theory of periods' [65], but the twin systems $\boldsymbol{\mathcal{N}\!a}, \mathbb{N}\mathbb{a}$, whose construction, very roughly, goes like this :

(i) We produce the function germs f (of z, at ∞) in $\boldsymbol{\mathcal{N}\!a}$ 'out of nothing', i.e. from $f(z) \equiv 1$, by taking larger and larger closures under the (direct and reverse) operations $+, \times, \partial_z, \circ$, with $\mathbb{Q}$ or $\mathbb{A}$ as scalar field. Very early on in this enlargement process, chronic *divergence* appears in the formal series or transseries $\tilde{f}$ which 'expand' at ∞ the germs f in $\boldsymbol{\mathcal{N}\!a}$.

(ii) We carefully refrain from introducing artificial derivations on $\boldsymbol{\mathcal{N}\!a}$, for fear of compromising the natural character of the construction. Rather, we ask : are there − *already, without our doing* − exotic derivations (i.e. derivations not generated by ∂_z) that act on $\boldsymbol{\mathcal{N}\!a}$ and respect its natural topology ? And we find that there is indeed a teeming profusion of them − two systems in fact, the alien derivations Δ_ω and the foreign derivations ∇_ω . The reason

[65] despite all the hullabaloo about 'periods', the constants there are given pell-mell, with no natural indexation, and all the symmetries central to dimorphy are broken, beginning with the sum/integral symmetry.

for this plethora of exotic derivations is the omnipresence of *divergence* in $\boldsymbol{\mathcal{N}a}$: to analyse this divergence, qualitatively and quantitatively, suitable operators are called for, which are precisely the exotic derivations.

(iii) To produce $\mathbb{N}\mathbb{a}$ from $\boldsymbol{\mathcal{N}a}$ (i.e. numbers from function germs, monics from monomials) we *do not* evaluate our germs at given points [66]. Rather, we let the exotic derivations act on these germs, and it turns out that the exotic derivatives of our monomials are expressible as sums of 'simpler' or 'earlier' monomials, with well-defined, generically transcendental scalar coefficients.

(iv) We harvest all these coefficients, declare them to be 'monics', and call $\mathbb{N}\mathbb{a}$ the ring generated by them.

The advantage of *exotic derivation* over *pointwise evaluation* lies not at all in the nature of the constants being produced (they are much the same with both methods) but rather in the orderliness of the procedure, which turns out monics directly in *mould form*, automatically gives them the *right type of indexation*, and tells us to which fundamental *symmetry type* they belong (as moulds).

There are at least four major 'domains of dimorphy', of increasing size, in $\mathbb{N}\mathbb{a}$. They comprise, respectively :

(i) the multizetas

(ii) the general hyperlogarithmic monics (see §4)

(iii) the monics associated with monomials that verify *affine* differential equations, with coefficients in $\mathbb{Q}[z]$ or $\mathbb{A}[z]$

(iv) the monics associated with monomials that verify *'bipolynomial'* [67] differential equations, again with coefficients in $\mathbb{Q}[z]$ or $\mathbb{A}[z]$

The third and (especially) fourth domain of dimorphy are incredibly large and would seem to encompass more or less all constants encountered in 'real life'. In fact, dimorphy appears to extend as far as he sight reaches : the whole of $\mathbb{R}$'s explorable-constructive part seems to be 'dimorphic' to the core.

Needless to say, these constructions certainly admit many variations, and their exhaustive investigation (for instance extending to Domains 2,3,4 the

[66] say, rational or algebraic points.

[67] 'bipolynomial' means that the differential equation may involve not only ordinary products of f, f',f'', etc, but also convolution products of type $\circledast$.

whole algebraic apparatus developped for Domain 1) would require huge efforts – and might not repay them. Yet, strangely, the central fact about $\mathbb{N}\mathbb{a}$, namely *numerical dimorphy*, is easy enough to establish, at least for these four domains. It directly mirrors the fact of *functional dimorphy*, which follows from the stability of $\mathcal{N}\!a$ under the generalised 'Borel-Laplace' transform, which itself is but an adaptation of the Fourier transform. So the least we can say is that dimorphy is 'well-connected' ! It is definitely no accident or freak of nature, but arguably $\mathcal{N}\!a$'s most outstanding feature.

P.S. This section owes much to discussions I had with Joris van der Hoeven.

Are there exotic derivations acting on numbers ?

Dimorphy is by no means the end of the story. After identifying the phenomenon and acknowledging its scope, we must raise another question : does dimorphy really exhaust the arithmetico-algebraic structure of $\mathbb{N}\mathbb{a}$, the 'explorable part' of $\mathbb{R}$? For instance, narrowing the focus to Domain 1 : do the 'quadratic relations' of §5.3.1 exhaust the set of algebraic constraints on multizetas ? One would assume the answer to be yes, but at the moment the tool-kit of transcendence theory seems woefully inadequate to tackle such questions[68]. That might change, however, if we had at our disposal, for *numbers*, the sort of high-powered machinery that we have for *functions*, namely : exotic derivations. We might then go about disproving the existence of 'undesirable relations' $R(\alpha, \beta, \dots) = 0$ for *numbers* by subjecting them to exotic differentiation, in search of a contradiction. The scheme works wonders with the resurgent or analysable *functions* and the *alien derivations* that operate on them. So do there exist numerical derivations, non-elementary and useful, which annihilate $\mathbb{Q}$ and $\mathbb{A}$ but act non-trivially on some countable ring $\mathbb{D}$ containing $\mathbb{A}$ (:= the field of algebraic numbers) ? Well, it is too early to say, but our canonical decomposition for the (formal) multizetas does strongly suggest a positive answer. More precisely, it *does* give us a system of non-trivial derivations on the ring of *formal* multizetas, and these derivations in all probability extend to the *true* multizetas. See §5.3.6 *infra* and [E16].

[68] despite spectacular but localised break-throughs by Apéry and, quite recently, Rivoal.

5.2 The overarching structure: ARI/GARI.

5.2.1 Bimoulds. Swap/Push. Contractions.

A few basic facts about the *mould* formalism may be found in §6.1. As for *bimoulds*, they are moulds that depend on double sequences:

$$A^\bullet = A^{\mathbf{w}} = A^{w_1,\dots,w_r} = A^{\binom{u_1\ ,\ \dots\ ,\ u_r}{v_1\ ,\ \dots\ ,\ v_r}}$$

and, more crucially, that are subjected to operations which mix up intimately the two sequences [69].

One such operation is the basic involution *swap*:

$$A_*^\bullet = \mathrm{swap}(A^\bullet) \quad \Longleftrightarrow \quad A_*^{\binom{u_1\ ,\ \dots\ ,\ u_r}{v_1\ ,\ \dots\ ,\ v_r}} = A^{\binom{v_r\ \ ,\ v_{r-1:r}\ \ ,\ \dots\ ,\ v_{2:3}\ ,\ v_{1:2}}{u_{1\dots r}\ ,\ u_{1\dots r-1}\ ,\ \dots\ ,\ u_{12}\ ,\ u_1}} \tag{193}$$

Another operation is the *push*:

$$A_*^\bullet = \mathrm{push}(A^\bullet) \quad \Longleftrightarrow \quad A_*^{\binom{u_1\ ,\ \dots\ ,\ u_r}{v_1\ ,\ \dots\ ,\ v_r}} = A^{\binom{-u_{1..r}\ ,\ u_1\ \ u_2\ ,\ \dots\ ,\ u_{r-1}}{-v_r\ \ ,\ v_{1:r}\ v_{2:r}\ ,\ \dots\ ,\ v_{r-1:r}}} \tag{194}$$

(We make constant use of the shorthand $u_{12} := u_1+u_2$, $v_{1:2} := v_1-v_2$ etc).

It is often convenient to represent bimoulds in the so-called 'augmented notation', which consists in adding to any given sequence $\mathbf{w}$ a *redundant* initial term $w_0 = \binom{u_0}{v_0}$. The $\mathbf{u}$-variables are then constrained by the condition $u_0 + u_1 + ..u_r = 0$ and, dually, the $\mathbf{v}$-variables are defined upto addition of a common constant. Thus:

$$A^{\binom{u_1,\dots,u_r}{v_1,\dots,v_r}} \equiv \mathrm{aug}A^{\binom{u_0,\ u_1,\dots,u_r}{v_0,\ v_1,\dots,v_r}} \quad \textit{with} \quad u_0 := -u_{1..r}\ ;\ v_0 := 0. \tag{195}$$

With the augmented notations, for instance, the *push* reduces to a unit shift on the sequence $\mathbf{w}$.

More operations on bimoulds shall be defined in the sequel, but nearly all of them involve four specific types of *sequence contractions*, denoted by the symbols $\rceil$, $\lceil$, $\rfloor$, $\lfloor$. These are always relative to some given *factorisation* $\mathbf{w} = \mathbf{w}^1\mathbf{w}^2\dots\mathbf{w}^s$ of the total sequence. The contraction rules are immediately apparent from the following example. Relative to the factorisation:

$$\mathbf{w} = \dots\mathbf{a}.\mathbf{b}\dots = \dots\binom{u_3\ ,\ u_4\ ,\ u_5}{v_3\ ,\ v_4\ ,\ v_5}\binom{u_6\ ,\ u_7\ ,\ u_8\ ,\ u_9}{v_6\ ,\ v_7\ ,\ v_8\ ,\ v_9}\dots$$

[69] otherwise they ought to be regarded as moulds whose indices $w_i = \binom{u_i}{v_i}$ simply happen to be in $\mathbb{C}^2$ rather than in $\mathbb{C}$.

the symbols $\rceil$, $\lceil$, $\rfloor$, $\lfloor$ signal the following changes :

$$\mathbf{a}\rceil := (\begin{smallmatrix} u_3 \,,\, u_4 \,,\, u_{56789} \\ v_3 \,,\, v_4 \,,\, v_5 \end{smallmatrix}) \qquad \lceil\mathbf{b} := (\begin{smallmatrix} u_{3456} \,,\, u_7 \,,\, u_8 \,,\, u_9 \\ v_6 \,,\, v_7 \,,\, v_8 \,,\, v_9 \end{smallmatrix}) \tag{196}$$

$$\mathbf{a}\rfloor := (\begin{smallmatrix} u_3 \,,\, u_4 \,,\, u_5 \\ v_{3:6} \,,\, v_{4:6} \,,\, v_{5:6} \end{smallmatrix}) \qquad \lfloor\mathbf{b} := (\begin{smallmatrix} u_6 \,,\, u_7 \,,\, u_8 \,,\, u_9 \\ v_{6:5} \,,\, v_{7:5} \,,\, v_{8:5} \,,\, v_{9:5} \end{smallmatrix}) \tag{197}$$

with the usual abbreviations for sums and differences.

Thus we see that the contractor $\rceil$ *adds* to *the* upper-right element of **a** *all* upper elements of neighbouring **b**, whereas the contractor $\rfloor$ *subtracts* from *all* lower elements of **a** *the* lower-left element of neighbouring **b**. And *vice versa* for $\lceil$ and $\lfloor$. Indeed, the **u**-variables are meant to be added together, and the **v**-variables to be subtracted from one another.

5.2.2 The Lie algebra ARI.

Consider the bilinear product *ari* :

$$C^\bullet = \mathrm{ari}(A^\bullet, B^\bullet) \iff C^{\mathbf{w}} = \sum_{\mathbf{w}=\mathbf{b}.\mathbf{c}} (A^{\mathbf{b}} B^{\mathbf{c}} - B^{\mathbf{b}} A^{\mathbf{c}})$$

$$+ \sum_{\mathbf{w}=\mathbf{b}.\mathbf{c}.\mathbf{d}} (A^{\lfloor\mathbf{c}} B^{\mathbf{b}\rceil\mathbf{d}} - B^{\lfloor\mathbf{c}} A^{\mathbf{b}\rceil\mathbf{d}}) + \sum_{\mathbf{w}=\mathbf{a}.\mathbf{b}.\mathbf{c}} (A^{\mathbf{a}\lceil\mathbf{c}} B^{\mathbf{b}\rfloor} - B^{\mathbf{a}\lceil\mathbf{c}} A^{\mathbf{b}\rfloor}) \tag{198}$$

with $\mathbf{b} \neq \emptyset, \mathbf{c} \neq \emptyset$ in all three sums (but **a** and **d** may be empty) .

The ari-*bracket is anti-commutative, verifies the Jacobi identity, and turns the space of all bimoulds such that* $A^\emptyset = 0$ *into a Lie algebra, known as* ARI.

5.2.3 The Lie group GARI.

Consider the binary law *gari* :

$$C^\bullet = \mathrm{gari}(A^\bullet, B^\bullet) \qquad \iff \qquad C^{\mathbf{w}} =$$

$$\sum_{\mathbf{w}=\mathbf{a^1}.\mathbf{b^1}.\mathbf{c^1}\ldots\mathbf{a^s}.\mathbf{b^s}.\mathbf{c^s}.\mathbf{a^{s+1}}} A^{\lceil\mathbf{b^1}\rceil\ldots\lceil\mathbf{b^s}\rceil} B^{\mathbf{a^1}\rfloor} \ldots B^{\mathbf{a^s}\rfloor} B^{\mathbf{a^{s+1}}\rfloor} B_\star^{\lfloor\mathbf{c^1}} \ldots B_\star^{\lfloor\mathbf{c^s}} \tag{199}$$

with summation over all $s \geq 1$ and with factor sequences subject only to $\mathbf{b^i} \neq \emptyset$ and $\mathbf{c^i}.\mathbf{a^{i+1}} \neq \emptyset$ (but consecutive factors $\mathbf{c^i}$ and $\mathbf{a^{i+1}}$ may be empty *separately* and the extreme factors $\mathbf{a^1}$,$\mathbf{c^s}$,$\mathbf{a^{s+1}}$ and even the product $\mathbf{c^s}.\mathbf{a^{s+1}}$ may also be empty, *separetly or simultaneously*). Here $B_\star^\bullet$ denotes the inverse $\mathrm{invmu}(B^\bullet)$ of $B^\bullet$ relative to the ordinary (associative, non-commutative) product (*mu* or $\times$) on moulds :

$$C^\bullet = \mathrm{mu}(A^\bullet, B^\bullet) = A^\bullet \times B^\bullet \iff C^{\mathbf{w}} = \sum_{\mathbf{w}=\mathbf{w^1}.\mathbf{w^2}} A^{\mathbf{w^1}} B^{\mathbf{w^2}} \tag{200}$$

This *gari*-product is clearly affine in $A^\bullet$ but severely non-linear in $B^\bullet$.

It is also associative, and turns the set of all bimoulds such that $A^\emptyset = 1$ *into a Lie group, known as* GARI*, whose Lie algebra is* ARI.

5.2.4 Some properties of ARI/GARI. Allied structures.

Like ordinary moulds, most interesting bimoulds fall into a few basic *symmetry types*. The definition for *symmetral/alternal* and *symmetrel/alternel* is exactly the same as for ordinary moulds, but in the case of *symmetril/alternil* the additive contraction $w_i + w_j$ changes to $w_i \otimes w_j$ with :

$$A^{\dots,w_i\otimes w_j,\dots} := P(v_{i:j})\, A^{\dots,\binom{u_{ij}}{v_i},\dots} + P(v_{j:i}) A^{\dots,\binom{u_{ij}}{v_j},\dots} \quad \textit{with} \quad P(t) := 1/t \tag{201}$$

Thus, for a *symmetral* bimould $A^\bullet$ and factor sequences of length 1 and 2 we get :

$$A^{\binom{u_1}{v_1}} A^{\binom{u_2,u_3}{v_2,v_3}} \equiv A^{\binom{u_1,\, u_2,\, u_3}{v_1,\, v_2,\, v_3}} + A^{\binom{u_2,\, u_1,\, u_3}{v_2,\, v_1,\, v_3}} + A^{\binom{u_2,\, u_3,\, u_1}{v_2,\, v_3,\, v_1}}$$

but if $A^\bullet$ is *symmetrel* (resp. *symmetril*) we get additional, 'contracted' terms on the right-hand side, namely $\quad A^{\binom{u_{12},\, u_3}{v_{12},\, v_3}} + A^{\binom{u_2,\, u_{13}}{v_2,\, v_{13}}} \quad$ resp.

$$P(v_{1:2})\, A^{\binom{u_{12},\, u_3}{v_1,\, v_3}} + P(v_{2:1})\, A^{\binom{u_{12},\, u_3}{v_2,\, v_3}} + P(v_{1:3})\, A^{\binom{u_2,\, u_{13}}{v_2,\, v_1}} + P(v_{3:1})\, A^{\binom{u_2,\, u_{13}}{v_2,\, v_3}}$$

The set of all alternal bimoulds is a subalgebra of ARI *.That of all symmetral bimoulds is a subgroup of* GARI.

These are closure properties for moulds with a simple symmetry. But ARI/GARI is specially well-suited for the study of bimoulds with a double symmetry :

The set $\mathrm{ARI}_{\underline{\mathrm{al/al}}}$ *of bialternal even bimoulds (i.e. bimoulds that are alternal and whose swappee is also alternal) constitute an important subalgebra of* ARI*, and similarly the set* $\mathrm{GARI}_{\underline{\mathrm{as/as}}}$ *of bisymmetral even bimoulds is an important subgroup of* GARI.

Here, "even" means that, for any given length r, the component $A^{w_1,\dots,w_r}$ is an even functions of $\mathbf{w}$. Actually, 'evenness' is *almost* a consequence of the double symmetry : thus, it may be shown that a bialternal bimould automatically has even components for all lengths r, except at most for $r = 1$. But to ensure stability under the ARI-bracket, length-one components also have to be even. This subsidiary parity condition is signalled by underlining : e.g. $\underline{\mathrm{al/al}}$ and $\underline{\mathrm{as/as}}$.

Even more important for our purpose is *the subalgebra* $\mathrm{ARI}_{\underline{\mathrm{al/il}}}$ *of alternal bimoulds with an alternil swappee , and the subgroup* $\mathrm{GARI}_{\underline{\mathrm{as/is}}}$ *of symmetral* *bimoulds with an symmetril swappee*. Here also, the parity condition implies the evenness of length-one components, but is slightly more technical for $r \geq 2$.

The double symmetry has other consequences: it implies invariance under some form or other of idem-potent transformation, like
(i) the *push* for $\mathrm{ARI}_{\underline{\mathrm{al/al}}}$;
(ii) the *spush* for $\mathrm{GARI}_{\underline{\mathrm{as/as}}}$;
(iii) variants of these for $\mathrm{ARI}_{\underline{\mathrm{al/il}}}$ and $\mathrm{GARI}_{\underline{\mathrm{as/is}}}$ (see[E14],[E16],[E17]).

It also ensures the existence of an *involutive (or group) automorphism*: thus, the involution *swap*, which is no algebra automorphism on ARI as a whole, becomes one when restricted to $\mathrm{ARI}_{\underline{\mathrm{al/al}}}$.

5.2.5 Some remarkable elements of ARI.

Bimoulds with a double symmetry *do matter*– and in more ways than one. But they are rather thin on the ground, and not so easy to construct. So it comes as a relief to know that *most of them*, and in some important cases *all of them*, can be derived from a small set of rather elementary bimoulds, the so-called bielementals $belam_r^\bullet/belim_r^\bullet$. These depend only on the component length r and on a two-variable function $xaxi(w_1) := xa(u_1)\, xi(v_1)$ [70], or rather the even part of $xaxi$. All components of $belam_r^\bullet/belim_r^\bullet$ are $\equiv 0$, except the component of length r, which reduces to a simple superposition:

$$\mathrm{belam}_{r,\,\mathrm{xaxi}}^{w_1,\ldots,w_r} = \mathrm{belam}_{r,\,\mathrm{xaxi}}^{\binom{u_1\ ,\ldots,\ u_r}{v_1\ ,\ldots,\ v_r}} :=$$
$$\sum_{\substack{i,j,m,n\in\mathbb{Z}_{r+1}\\ \cdots<i\leq m<j\leq n<\cdots}} \mathrm{bel}_r^{i,j;\,m,n}\ \mathrm{xa}(u_i+u_{i+1}+\cdots+u_{j-1})\ \mathrm{xi}(v_m-v_n)\ \equiv$$
$$\sum_{\substack{i,j,m,n\in\mathbb{Z}_{r+1}\\ \cdots<i\leq m<j\leq n<\cdots}} \frac{1}{2}\ \mathrm{bel}_r^{i,j;\,m,n} \Big(\mathrm{xa}(u_{i\ldots j-1})\ \mathrm{xi}(v_{m:n})\ +\mathrm{xa}(u_{j\ldots i-1})\ \mathrm{xi}(v_{n:m})\Big) \tag{202}$$

with a *swappee*

$$\mathrm{belim}_{r,\mathrm{xaxi}}^\bullet := \mathrm{swap}(\mathrm{belam}_{r,\,\mathrm{xaxi}}^\bullet) \equiv \mathrm{belam}_{r,\,\mathrm{xixa}}^\bullet \tag{203}$$

and with integer coefficients

$$\mathrm{bel}_r^{i,j;\,m,n} \equiv \mathrm{bel}_r^{j,i;\,n,m} := \frac{(-1)^{[m-i]_r+[n-j]_r}\quad [r-1]_r!}{[m-i]_r!\ [n-j]_r!\ [j-m-1]_r!\ [i-n-1]_r!} \tag{204}$$

[70] with variables u_1, v_1 in two (possibly different) abelian groups.

This calls for a few comments:

The above formulas use the cyclic *augmented notation*: we index the variable u_i, v_i of the r-th component of a bimould on $\mathbb{Z}_{r+1} := \mathbb{Z}/(r+1)\mathbb{Z}$ after adding the two 'redundant' variables $u_0 := -u_{1\ldots r}$ et $v_0 := 0$. The inequalities under the $\sum$ sign are of course relative to the cyclic order on $\mathbb{Z}_{r+1}$ and, for any $k \in \mathbb{Z}_{r+1}$, $[k]_r$ denotes the representative of k in $\{0, 1, \ldots r\}$.

Formula (203) shows that the involution *swap* leaves bielementals unchanged, apart from swapping xa and xi. But the main facts are these:

(i) all bimoulds $belam^{\bullet}_{r,\,\mathrm{xaxi}}$ are *bialternal*
(ii) they vanish for *odd* or (iff $r \geq 2$) *semi-constant* functions $xaxi$
(iii) they are non-zero for *even* functions $xaxi$ constant in neither variable
(iv) they generate *most other* bialternals under the *ari*-bracket.

5.2.6 Further remarkable elements of ARI.

As usual, we set: $P(t) := 1/t$ and $Q_c(t) := c/\tan(c\,t)$ for some $c \in \mathbb{C}$.
The identities

$$\mathrm{pa}_1^{w_1} := P(u_1) \quad ; \quad \mathrm{pa}_r^{w_1,\ldots,w_r} := P(u_{12\ldots r})\,(\mathrm{pa}_{r-1}^{w_1,\ldots,w_{r-1}} - \mathrm{pa}_{r-1}^{w_2,\ldots,w_r}) \tag{205}$$

$$\mathrm{pi}_r^{w_1,\ldots,w_r} := (v_1 + v_2 + \ldots v_r)\; P(v_1)\, P(v_{1:2})\, P(v_{2:3}) \ldots P(v_{r-1:r})\, P(v_r) \tag{206}$$

define (the former by induction, the latter directly) two series of rather peculiar bimoulds, the $pa^{\bullet}_r$ and $pi^{\bullet}_r$, which depend each on one set of variables – the u_i or v_i, and have only one non-zero component, that of length r. The $pa^{\bullet}_r$ and $pi^{\bullet}_r$ are *alternal*, and although not *bialternal*, they still possess a double symmetry of sorts, since they are exchanged, not under the involution *swap*, but under another important, if less general involution : the *slap* [71]. Moreover they self-reproduce under the ARI-bracket:

$$\mathrm{ari}(\mathrm{pa}^{\bullet}_{r_1}, \mathrm{pa}^{\bullet}_{r_2}) = (r_1 - r_2)\; \mathrm{pa}_{r_1+r_2} \tag{207}$$

$$\mathrm{ari}(\mathrm{pi}^{\bullet}_{r_1}, \mathrm{pi}^{\bullet}_{r_2}) = (r_1 - r_2)\; \mathrm{pi}_{r_1+r_2} \tag{208}$$

which means that the subalgebras $\mathrm{ARI}_{\mathrm{pa}}$ and $\mathrm{ARI}_{\mathrm{pi}}$ of ARI generated by the bimoulds $pa^{\bullet}_r$ or $pi^{\bullet}_r$ are each isomorphic to the algebra Diff_t spanned by the differential operators $t^{n+1}\partial_t$.

[71] wich acts as an automorphism, but only on the subalgebra $\mathrm{ARI}_{\mathrm{eupol}} \subset \mathrm{ARI}$ consisting of so-called *eupolar* bimoulds, which are particular rational functions of $\mathbf{u}$ or $\mathbf{v}$

5.2.7 Some remarkable elements of GARI.

By Lie exponentiation, the algebra isomorphisms just mentioned induce group isomorphisms between each of the subgroups:
(i) $\mathrm{GARI_{pa}} := \mathrm{expari}(\mathrm{ARI_{pa}})$
(ii) $\mathrm{GARI_{pi}} := \mathrm{expari}(\mathrm{ARI_{pi}})$
and the group:
(iii) $\mathrm{Diffeo}_t := \exp(\mathrm{Diff}_t)$
of formal, identity-tangent diffeomorphisms $t \mapsto t + O(t^2)$ of $\mathbb{C}_{,0}$ unto itself.

Of special interest are the images $par^\bullet \in \mathrm{GARI_{pa}}$ and $pil^\bullet \in \mathrm{GARI_{pi}}$ of the diffeomorphism $f \in \mathrm{Diffeo_t}$ defined by $f(t) := 1 - \exp(-t)$. Like all bimoulds in $\mathrm{GARI_{pa}}$ and $\mathrm{GARI_{pi}}$, $par^\bullet$ and $pil^\bullet$ are symmetral, but the remarkable and unexpected thing is that their *swappees* $pir^\bullet := swap(par^\bullet)$ and $pal^\bullet := swap(pil^\bullet)$ are symmetral too.

The bisymmetral pairs $\mathrm{pal}^\bullet/\mathrm{pil}^\bullet$ *and* $\mathrm{par}^\bullet/\mathrm{pir}^\bullet$ *thus defined are central to the theory.* They *do not* fulfill the parity condition and so *do not* belong to $\mathrm{GARI}_{\underline{\mathrm{as/as}}}$. Indeed, upto rescaling and under suitable additional conditions ("eupolarity"), they are the only bisymmetral (bi)moulds [72] that depend on one set of variables only ($\mathbf{u}$ or $\mathbf{v}$) and whose r-th component is homogeneous of degree -r.

Of the two pairs, $pal^\bullet/pil^\bullet$ is the more important by far. It has a 'eu-trigonometric' counterpart $tal^\bullet/til^\bullet$, obtained by replacing $P(t) := 1/t$ by $Q_c(t) := c/\tan(c\,t)$ and adding suitable corrective terms that involve only even powers of c . See [E14].

These bisymmetral bimoulds enjoy an incredible number of properties and sit at the hub of a galaxy of some sixty 'special bimoulds', which are investigated in [E17] and whose applications far outstrip multizeta theory.

5.2.8 Further remarkable elements of GARI.

The *scramble* is a general bimould transform defined by:

$$A^\bullet \mapsto B^\bullet \ = \ \mathrm{scramble}(A^\bullet) \quad \textit{with} \quad B^\bullet := \sum_{\mathbf{w}^* \in \mathrm{scram}(\mathbf{w})} \epsilon(\mathbf{w},\mathbf{w}^*)\ A^{\mathbf{w}^*} \tag{209}$$

where $\mathrm{scram}(\mathbf{w})$ is the set of all sequences $\mathbf{w}^* = (\begin{smallmatrix} u_1^* & ,\dots, & u_r^* \\ v_1^* & ,\dots, & v_r^* \end{smallmatrix})$ which have the same length r as $\mathbf{w} = (\begin{smallmatrix} u_1 & ,\dots, & u_r \\ v_1 & ,\dots, & v_r \end{smallmatrix})$ and are characterised by the property that

[72] being constant in one series of variables, bimoulds like $pal^\bullet/pil^\bullet$, $par^\bullet/pir^\bullet$ etc are often referred to as "moulds".

for each $j \in \{1, \ldots, r\}$:

$$u_1^* v_1^* + u_2^* v_2^* + \ldots u_j^* v_j^* = \sum_{1 \le i \le j} \Big(\sum_{p_{j,i-1} < p \le p_{j,i}} u_p \Big) v_{q_{j,i}} \tag{210}$$

for some pair $\{p_{j,k}\}, \{q_{j,k}\}$ of intertwined sequences:

$$0 = p_{j,0} < q_{j,1} \le p_{j,1} < q_{j,2} \le p_{j,2} < \cdots < q_{j,j} \le p_{j,j} < r$$

There are exactly $r!! := 1.3 \ldots (2r-1)$ such sequences $\mathbf{w}^*$. Each u_j^* is a sum of one or several consecutive u_i and each v_j^* is either of the form $v_{j_\star}$, in which case we set $\epsilon(\mathbf{w}, \mathbf{w}^*, j) := 1$, or of the form $v_{j_\star} - v_{j_{\star\star}}$, in which case we set $\epsilon(\mathbf{w}, \mathbf{w}^*, j) := \mathrm{sign}(j_{\star\star} - j_\star)$. (Mark the inversion). Multiplied together, these signs define the global sign factor $\epsilon(\mathbf{w}, \mathbf{w}^*) := \prod_{j=1}^{j=r} \epsilon(\mathbf{w}, \mathbf{w}^*, j)$ in the definition of the scramble transform.

In the above definition (209), $A^\bullet$ was assumed to be a bimould, but it could just as well be a mere mould, in which case $A^{\mathbf{w}^*}$ should be interpreted as $A^{u_1^* v_1^*, \ldots, u_r^* v_r^*}$. Thus, the *scramble* turns moulds and bimoulds alike into bimoulds.

One of the reasons behind the importance of the *scramble* is that it preserves the two basic symmetry types: *if the mould or bimould $A^\bullet$ is alternal (resp. symmetral), so is the bimould $B^\bullet = \mathrm{scramble}(A^\bullet)$* .

Remarkable (bi)moulds tend to have remarkable *'scramblees'*. Thus the symmetral mould $\mathcal{V}^\bullet(z)$, (see §6.7.7) which is central to *equational resurgence* and *Singular Systems*, yields the bimould $S^\bullet(x) := \mathrm{scramble}(\mathcal{V}^\bullet(x))$, which is central to the theory of *co-equational resurgence* and *Singularly Perturbed Systems*. This theory is *one of the three structures* referred to in the title of our Fifth Lesson. This being a hasty Survey, it must be given short shrift here, but a detailed treatment is available in [E8].

Closely linked to the symmetral resurgence monomials $\mathcal{V}^\bullet(z)$ are the alternal hyperlogarithmic monics $V^\bullet = V^\bullet_{\omega_0}$ featuring [73] in the resurgence equations:

$$\Delta_{\omega_0} \mathcal{V}^\bullet(z) = V^\bullet_{\omega_0} \times \mathcal{V}^\bullet(z) \tag{211}$$

When scrambled, that mould yields the so-called *tesselation* mould ([E8]) $tes^\bullet := scramble(V^\bullet)$, which dominates the geometry of co-equational resurgence in the Borel planes, and possesses many arresting features, like *being locally constant in its two series of variables, the u_i as well as the v_i*

[73]The lower index is actually redundant and may be dropped, since $V^{\omega_1, \ldots, \omega_r}_{\omega_0} \equiv 0$ unless $\omega_1 + \ldots \omega_r = \omega_0$

(although $tes^\bullet$ is a superposition of several highly complex functions). Thus for $r = 2$, the tesselation coefficient :

$$\mathrm{tes}^{w_1,w_2} := V^{u_1 v_1, u_2 v_2} + V^{u_{12} v_2, u_1 v_{1:2}} - V^{u_{12} v_1, u_2 v_{2:1}}$$

is, contrary to appearances, locally constant on $\mathbb{C}^4$ and assumes only three distinct values there, namely 0 and ± 1 .

5.2.9 Basic complexity of ARI/GARI.

The basic complexity of ARI/GARI (as reflected in its main operations, associated structures, fundamental bimoulds, etc) is quite high. Thus, for a given component length r, the inversion *invgari* in GARI or the Lie exponential *expari* of ARI into GARI resolves into a sum of a fast increasing number (marked # in the table below) of terms, each of which fills upto half a line, or more, of small print :

length r	1	2	3	4	5	6	7	8	...
#(invgari)	1	4	20	112	672	4 224	27 459	183 040	...
#(expari)	1	4	21	126	818	5 594	39 693	289 510	...

For $r = 8$, we already get six figure numbers, and spelling out the corresponding formulas in full would take about one hundred pages. This means that one must often rely heavily on automatic computation when *exploring* the fringes and by-paths of ARI/GARI. Fortunately, however, the whole field is so strongly structured, and so harmonious too, offering so many hints and props to intuition, that facts and formulas are easy to guess and, once guessed, quickly yield to rigorous proving. Writing down all these proofs is of course another matter, due to the sheer mass of the facts already unearthed or yet to emerge !

5.3 The arithmetics of multizetas.

5.3.1 Formal multizetas $\mathrm{za}^\bullet/\mathrm{ze}^\bullet$.

In the *first encoding* , the *generalised* or *modulated* multizetas are defined by :

$$\mathrm{Ze}^{\binom{\epsilon_1\ ,\dots,\ \epsilon_r}{s_1\ ,\dots,\ s_r}} := \sum_{n_1>\dots n_r>0} n_1^{-s_1} \dots n_r^{-s_r}\ e_1^{n_1} \dots e_r^{n_r} \tag{212}$$

with $s_j \in \mathbb{N}^*$, $\epsilon_j \in \mathbb{Q}/\mathbb{Z}$, $e_j := \exp(2\pi i \epsilon_j)$
The *second encoding* may be directly defined, via the polylogarithms, but it is more expeditious to derive it from the first encoding :

$$\mathrm{Za}^{e_1, 0^{(s_1-1)}, \dots, e_r, 0^{(s_r-1)}} \overset{essentially}{:=} \mathrm{Ze}^{\binom{\epsilon_r\ ,\ \epsilon_{r-1:r}\ ,\dots,\ \epsilon_{1:2}}{s_r\ ,\ s_{r-1}\ ,\dots,\ s_1}} \tag{213}$$

Setting $\epsilon_j \equiv 0, e_j \equiv 1$, we get the *usual* or *plain* multizetas.

On its obvious domain of convergence $\Re(s_1) > 1, \Re(s_{12}) > 2, \dots,$ the series (212) defines a holomorphic function *with a meromorphic extension to the whole of* $\mathbb{C}^r$ which in turn possesses:
(i) a remarkable singular locus,
(ii) remarkably simple multipoles described by the *Bernoulli mould*,
(iii) a *'parity property'* reminiscent of the *reflexion property* of the Riemann zeta function [74].

From the arithmetical point of view, however, the $\mathbb{Q}$-ring generated by the values of the multizeta function on $\mathbb{Z}^r$ (at regular points or even at singular ones, after canonical removal of the multipole) is no larger than the $\mathbb{Q}$-ring generated (in fact: spanned) by its values on $\mathbb{N}^r$. So we may restrict our attention to the latter.

The multizetas, whether plain or modulated, are eminently 'dimorphic' entities: they are doubly closed under multiplication, since to the two encodings there correspond two distinct ways of calculating their products. These are the two classical systems of *quadratic relations*, which can be derived in any number of ways. In pithy mould language, with the conventions of §6.1, they can be enuntiated as follows:

(i) *The mould* $\mathrm{Ze}^\bullet$*, where defined, is symmetrel* [75]*, and there is a unique extension to the divergent case that keeps it symmetrel and gives* $\mathrm{Ze}^{\binom{0}{1}} = 0$.

(ii) *The mould* $\mathrm{Za}^\bullet$*, where defined, is symmetral and there is a unique extension to the divergent case that keeps it symmetral and gives* $\mathrm{Za}^0 = \mathrm{Za}^1 = 0$.

But these two extensions do not exactly coincide. There is a slight discrepancy, which calls for some simple corrective terms ([E14]) in the conversion formula (213). Hence the mention "essentially" in the middle of (213).

All the indications, numerical and theoretical, are that the two sets of 'quadratic relations' do express the totality of algebraic constraints on multizetas. So we may confidently replace the *true* multizetas $Ze^\bullet, Za^\bullet$, which at the moment are still largely beyond the reach of arithmetics [76], by their

[74] somewhat confusingly known as 'functional equation'

[75] since the mould $Ze^\bullet$ has two-storeyed indices $\omega_i = \binom{\epsilon_i}{s_i}$, the contractions $\omega_i + \omega_j$ must of course be interpreted as $\binom{\epsilon_i+\epsilon_j}{s_i+s_j}$

[76] despite the trail-blazing work of R.Apéry and T.Rivoal.

formal or *symbolic* counterparts $ze^\bullet, za^\bullet$, written in lower-case letters and subject only to
(i) the symmetrelity of $ze^\bullet$
(ii) the symmetrality of $za^\bullet$
(iii) the conversion rules (213)
and address the problem of unravelling all the algebraic consequences.

5.3.2 Generating functions $\mathrm{zag}^\bullet/\mathrm{zig}^\bullet$.

In *scalar* form, the multizetas are rather unwieldy, and it is convenient to replace them by *generating series*, so tailored as to preserve the simplicity of the two symmetries and the transparency of the conversion rule. The proper definitions are:

$$\mathrm{zig}^{\binom{\epsilon_1\ ,\dots,\ \epsilon_r}{v_1\ ,\dots,\ v_r}} := \sum_{1\le s_j} \mathrm{ze}^{\binom{\epsilon_1\ ,\dots,\ \epsilon_r}{s_1\ ,\dots,\ s_r}}\ v_1^{s_1-1}\dots v_r^{s_r-1} \tag{214}$$

$$\mathrm{zag}^{\binom{u_1\ ,\dots,\ u_r}{\epsilon_1\ ,\dots,\ \epsilon_r}} := \sum_{1\le s_j} \mathrm{za}^{e_1,0^{(s_1-1)},\dots,e_r,0^{(s_r-1)}}\ u_1^{s_1-1}u_{12}^{s_2-1}\dots u_{12\dots r}^{s_r-1} \tag{215}$$

In the *formal* case the components of the two new moulds $zag^\bullet/zig^\bullet$ are mere power series, but in the *genuine* case, i.e. of for the moulds $Zag^\bullet/Zig^\bullet$ built from the *numerical* multizetas, these power series sum up to meromorphic functions with interesting properties, such as verifying simple difference equations ([E14]).

Moreover, we have the implications:

$$\begin{aligned} \mathrm{ze}^\bullet\ \textit{symmetrel} &\iff \mathrm{zig}^\bullet\ \textit{symmetril} \\ \mathrm{za}^\bullet\ \textit{symmetral} &\iff \mathrm{zag}^\bullet\ \textit{symmetral} \end{aligned}$$

and the conversion rule (213) translates into:

$$\mathrm{swap}(\mathrm{zig}^\bullet) \overset{exactly}{=} \mathrm{mu}(\mathrm{zag}^\bullet, \mathrm{mono}^\bullet) \tag{216}$$

Here, $mono^\bullet$ is an elementary, constant-valued mould: up to rational factors, its values are 'monozetas' $\zeta(s)$, hence its name. So for the generating functions, the conversion rule essentially reduces to the involution *swap*. Remark that formula (216) uses the primary mould product *mu* (see §6.1). But due to the elementary nature of $mono^\bullet$, the right-hand side of (216) may also be written as an (exceptionnally commutative) product in GARI. Indeed:

$$\mathrm{mu}(\mathrm{zag}^\bullet, \mathrm{mono}^\bullet) \equiv \mathrm{gari}(\mathrm{zag}^\bullet, \mathrm{mono}^\bullet) \equiv \mathrm{gari}(\mathrm{mono}^\bullet, \mathrm{zag}^\bullet)$$

Thus, studying the formal multizetas boils down to *finding and describing all the symmetral/symmetril pairs of (essential) swappees* $\mathrm{zag}^\bullet/\mathrm{zig}^\bullet$ *with values in the ring of formal power series.*

5.3.3 Immediate bipartion and arduous tripartition.

As a natural element of GARI, the mould $zag^\bullet$ splits into two and even three factors :

$$\mathrm{zag}^\bullet = \mathrm{gari}(\mathrm{zag}^\bullet_{\mathrm{I+II}}, \mathrm{zag}^\bullet_{\mathrm{III}}) \qquad (\mathrm{zag}^\bullet_{\mathrm{III}} \in \mathrm{GARI}^{\mathrm{o.l.}}_{\underline{\mathrm{as/is}}}) \tag{217}$$

$$\mathrm{zag}^\bullet = \mathrm{gari}(\mathrm{zag}^\bullet_{\mathrm{I}}, \mathrm{zag}^\bullet_{\mathrm{II}}, \mathrm{zag}^\bullet_{\mathrm{III}}) \qquad (\mathrm{zag}^\bullet_{\mathrm{II}} \in \mathrm{GARI}^{\mathrm{e.l.}}_{\underline{\mathrm{as/is}}}) \tag{218}$$

The factors $zag^\bullet_I$, $zag^\bullet_{II}$, $zag^\bullet_{I+II}$ are of type "e.l.", meaning that their components of even/odd length are even/odd functions of $\mathbf{w}$. The factor $zag^\bullet_{III}$ on the other hand is of type "o.l.", meaning that its components of even/odd length are odd/even functions of $\mathbf{w}$. Under the algebra isomorphisms of §5.3.4 *infra*, bimoulds of type "e.l." (resp. "o.l.") correspond to bimoulds that are (in both cases) even functions of $\mathbf{w}$ but whose only non-zero components have even (resp. odd) lengths [77]. Hence the abbreviations e.l. (even-lengthed) and o.l. (odd-lengthed).

But there is a major difference between (217) and (218). The first factorisation is elementary, immediate, and indisputably canonical, with the $zag^\bullet_{III}$ factor given by :

$$\mathrm{gari}(\mathrm{zag}^\bullet_{\mathrm{III}}, \mathrm{zag}^\bullet_{\mathrm{III}}) = \mathrm{gari}(\mathrm{imne}(\mathrm{invgari}(\mathrm{zag}^\bullet)), \mathrm{zag}^\bullet) \tag{219}$$

where *imne* denotes the elementary ARI/GARI automorphism:

$$\mathrm{imne} := \mathrm{impar} \circ \mathrm{neg} : \qquad A^{\binom{u_1\ ,\dots,\ u_r}{v_1\ ,\dots,\ v_r}} \mapsto (-1)^r\, A^{\binom{-u_1\ ,\dots,\ -u_r}{-v_1\ ,\dots,\ -v_r}} \tag{220}$$

Since all elements of GARI have exactly one square root, (219) determines $zag^\bullet_{III}$ and then (217) determines $zag^\bullet_{I+II}$ by division.

The difficulty with the second, more precise factorisation (218), which consists in disentangling the factors $zag^\bullet_I$ and $zag^\bullet_{II}$, is not its existence, which again is quite straightforward, but its canonicity : there are infinitely many ways of detaching the 'polar-trigonometric' factor $zag^\bullet_I$, which carries the π^2-dependence, from $zag^\bullet_{I+II}$, and the 'right' choice will be sketched in §5.3.6.

[77] this applies only to the factors $zag^\bullet_I$ and $zag^\bullet_{II}$.

These factorisations hold not only in the *formal*, but also in the *genuine* case, and thus lead to two canonical splittings of the $\mathbb{Q}$-ring of multizetas, one immediate, the other more recondite :

$$\mathbb{Z}\mathrm{eta} \;=\; \mathbb{Z}\mathrm{eta}_{\mathrm{I+II}} \otimes \mathbb{Z}\mathrm{eta}_{\mathrm{III}} \tag{221}$$

$$\mathbb{Z}\mathrm{eta} \;=\; \mathbb{Z}\mathrm{eta}_{\mathrm{I}} \otimes \mathbb{Z}\mathrm{eta}_{\mathrm{II}} \otimes \mathbb{Z}\mathrm{eta}_{\mathrm{III}} \quad \textit{with} \quad \mathbb{Z}\mathrm{eta}_{\mathrm{I}} = \mathbb{Q}[\pi^2] \tag{222}$$

The ring $\mathbb{Z}\mathrm{eta}_{\mathrm{III}}$ (resp. $\mathbb{Z}\mathrm{eta}_{\mathrm{I+II}}$) is generated by all *irreducibles* of odd length (resp. by those of even length, plus the odd man out $\zeta(2) = \pi^2/6$).

5.3.4 The free generation theorem.

The $\mathbb{Q}$*-ring* $\mathbb{Z}\mathrm{eta}$ *of formal multizetas, as well as the three factor-rings I,II,III, are polynomial rings, that is to say, they are freely generated on* $\mathbb{Q}$ *by a countable system of 'irreducibles'. This holds equally for the* plain *and the more general* modulated *multizetas.*

Although it has been open for the better part of the nineties, this *free generation theorem* is a very simple affair. The only difficulty is to establish the closure under the *ari*-bracket of the space $\mathrm{ARI}_{\underline{\mathrm{al/il}}}$ of all alternal/alternil bimoulds (with the subsidiary 'parity' condition). The neatest proof consists in observing that the space $\mathrm{ARI}_{\underline{\mathrm{al/al}}}$ of bialternal bimoulds is (trivially) a subalgebra of ARI, and in using either of the two explicit isomorphisms:

$$\mathrm{adari}(\mathrm{pal}^\bullet) \;\; \textit{or} \;\; \mathrm{adari}(\mathrm{par}^\bullet) : \qquad \mathrm{ARI}_{\underline{\mathrm{al/al}}} \;\Rightarrow\; \mathrm{ARI}_{\underline{\mathrm{al/il}}} \tag{223}$$

where *adari*$(pal^\bullet)$ (resp. *adari*$(par^\bullet)$) denotes the adjoint action in ARI of the bisymmetral mould $pal^\bullet$ (resp. $par^\bullet$) constructed in §5.2.6.

Observe that, while these isomophisms make it certain that $\mathrm{ARI}^{\mathrm{ent}}_{\underline{\mathrm{al/il}}}$ *is* a subalgebra, they do not exchange the subalgebras $\mathrm{ARI}^{\mathrm{ent}}_{\underline{\mathrm{al/al}}}$ and $\mathrm{ARI}^{\mathrm{ent}}_{\underline{\mathrm{al/il}}}$ of 'entire-valued' bimoulds (i.e. bimoulds with values in the ring of formal power series). In fact, these two subalgebras *are not* isomorphic.

The *general entire-valued*, symmetral/symmetril pair of swappees $zag^\bullet/zig^\bullet$ is then obtained by postcomposition in GARI of a *particular* $zag^\bullet$ (e.g. the 'genuine','numerical' $Zag^\bullet$ or its first factor $Zag^\bullet_I$) by the *general* element of $\mathrm{GARI}^{\mathrm{ent}/\#}_{\underline{\mathrm{as/is}}} = expari(\mathrm{ARI}^{\mathrm{ent}/\#}_{\underline{\mathrm{al/il}}})$. Here, *ent* means *entire-valued* as usual, and # denotes an additional condition which depends on the ring of multizetas that is being considered : thus for the *plain* multizetas, # simply means *constant in the* $\mathbf{v}$*-variables.* For the *modulated* multizetas, see §5.3.5 below.

The problem has thus been completely linearised, and the *free generators* of $\mathbb{Z}$eta, or *irreducibles* (other than π^2), are seen to be in one-to-one correspondence with the generators of $\mathrm{ARI}^{\mathrm{ent}/\#}_{\underline{\mathrm{al/il}}}$ *as a vector space.*

But *as a Lie algebra*, $\mathrm{ARI}^{\mathrm{ent}/\#}_{\underline{\mathrm{al/il}}}$ has far more structure on it than the $\mathbb{Q}$-ring $\mathbb{Z}$eta of formal multizetas, and its *linear generators* may be further analysed – down to *Lie generators.* This is where the exciting work on multizetas actually *begins* !

5.3.5 Generators and dimensions.

The Broadhurst-Kreimer conjectures.
Let us consider jointly the *plain multizetas* (without unit roots) and the *Eulerian multizetas* (modulated by the unit roots ± 1) – the former because of their obvious importance; the latter because, contrary to appearences, they are actually *simpler.* Let $P_{s,r}$ (resp. $E_{s,r}$) be the smallest number of irreducibles of length r and weight s $(:= s_1 + \ldots + s_r)$ needed to produce, jointly, a complete system of irreducibles for the plain (resp. Eulerian) multizetas.

Relying on extensive numerical computations and some inspired guesswork, Broadhurst and Kreimer have conjectured that the dimensions $P_{s,r}$ and $E_{s,r}$ for the 'genuine', as opposed to 'formal', multizetas could be read off the generating functions:

$$\prod_{s\geq 3, r\geq 1} (1-x^s y^r)^{P_{s,r}} \overset{?}{=} 1 - \frac{x^3 y}{1-x^2} + \frac{x^{12} y^2 (1-y^2)}{(1-x^4)(1-x^6)} \tag{224}$$

$$\prod_{s\geq 3, r\geq 1} (1-x^s y^r)^{E_{s,r}} \overset{?}{=} 1 - \frac{x^3 y}{(1-x^2)(1-xy)} \tag{225}$$

The two series of relevant algebras.
For any $p \geq 1$ let $\mathbb{Z}_p$ be the subgroup $\{0, 1/p, \ldots, (p-1)/p\}$ of $\mathbb{Q}/\mathbb{Z}$ and let $\mathrm{ARI}^{\underline{\mathrm{ent}/\mathbb{Z}_p}}$ denote the subalgebra of ARI (it is one !) which regroups all bimoulds $A^\bullet$:
(i) with $\mathbf{u}$-variables ranging through $\mathbb{C}$
(ii) with $\mathbf{v}$-variables ranging through $\mathbb{Z}_p$
(iii) with values in the ring of formal power series in $\mathbf{u}$
(iv) with the self-correlation constraints:

$$A^{\binom{u_1\ ,\ldots,\ u_r}{qv_1\ ,\ldots,\ qv_r}} \equiv \sum_{qv_i^* = qv_i} A^{\binom{qu_1\ ,\ldots,\ qu_r}{v_1^*\ ,\ldots,\ v_r^*}} \qquad (\forall\, q \mid p) \tag{226}$$

and in particular (for q=p) :

$$A^{\left(\begin{smallmatrix} u_1 & ,\dots, & u_r \\ 0 & ,\dots, & 0 \end{smallmatrix}\right)} \equiv \sum_{v_i^* \in \mathbb{Z}_p} A^{\left(\begin{smallmatrix} qu_1 & ,\dots, & qu_r \\ v_1^* & ,\dots, & v_r^* \end{smallmatrix}\right)} \tag{227}$$

The subalgebras directly relevant to the study of the multizetas modulated by unit roots of order p are $\mathrm{ARI}^{\mathrm{ent}/\mathbb{Z}_p}_{\underline{\mathrm{al/al}}}$ and $\mathrm{ARI}^{\mathrm{ent}/\mathbb{Z}_p}_{\underline{\mathrm{al/il}}}$. For $p=1$ their elements are simply $\mathbf{v}$-constant bimoulds.

Eulerian multizetas and generators of $\mathrm{ARI}^{\mathrm{ent}/\mathbb{Z}_2}_{\underline{\mathrm{al/al}}}$.
For $r=1$ and d even we set :

$$\mathrm{bela}^{w_1}_{1,d} = u_1^d \;(resp.\;\; (2^{-d}-1)\,u_1^d)) \quad if \quad v_1 = 0 \;(resp.\;\; v_1 = 1/2) \tag{228}$$

and for $r \geq 1$ and d even we set :

$$\mathrm{bela}^{\bullet}_{r,d} := \mathrm{belam}^{\bullet}_{r,\mathrm{xaxi}} \quad with \quad \mathrm{xa}(t) := t^d \;;\; \mathrm{xi}(0) := 0 \;;\; \mathrm{xi}(1/2) := 1 \tag{229}$$

(i) *All bimoulds* $\mathrm{bela}^{\bullet}_{r,d}$ *(for* $r = 1,2,3\dots$ *and* $d = 2,4,6\dots$ *) are non-zero, bialternal,and self-correlated.*
(ii) *They freely generate a sub-algebra* $\underline{\mathrm{ARI}}^{\mathrm{ent}/\mathbb{Z}_2}_{\underline{\mathrm{al/al}}}$ *of* $\mathrm{ARI}^{\mathrm{ent}/\mathbb{Z}_2}_{\underline{\mathrm{al/al}}}$ *consisting of all bimoulds of type* $\underline{\mathrm{al/al}}$ *that possess an extension of type* $\underline{\mathrm{al/il}}$.

Eulerian multizetas and generators of $\mathrm{ARI}^{\mathrm{ent}/\mathbb{Z}_2}_{\underline{\mathrm{al/il}}}$.
(i) *Each bialternal* $\mathrm{bela}^{\bullet}_{r,d}$ *has a canonical counterpart or 'extension'* $\mathrm{bema}^{\bullet}_{r,d}$*, of alternal/alternil type, self-correlated, and with a first non-zero component (i.e. the one of length* r*) equal to the single non-zero component of* $\mathrm{bela}^{\bullet}_{r,d}$.
(ii) *These* $\mathrm{bema}^{\bullet}_{r,d}$ *freely generate the algebra* $\mathrm{ARI}^{\mathrm{ent}/\mathbb{Z}_2}_{\underline{\mathrm{al/il}}}$.
(iii) *The Eulerian irreducibles correspond one-to-one to the bialternals spanning* $\underline{\mathrm{ARI}}^{\mathrm{ent}/\mathbb{Z}_2}_{\underline{\mathrm{al/al}}}$*. More precisely, the number* $E_{s,r}$ *of independent irreducibles of weight* s *and length* r *coincides with the dimension of the cell of* $\mathrm{ARI}^{\mathrm{ent}/\mathbb{Z}_2}_{\underline{\mathrm{al/al}}}$ *consisting of bimoulds of length* r *and total degree* $d = s - r$.
(iv) *This establishes the Broadhurst-Kreimer conjecture (225) for the* formal *Eulerian multizetas* [78].

Plain multizetas and generators of $\mathrm{ARI}^{\mathrm{ent}/\mathbb{Z}_1}_{\underline{\mathrm{al/il}}}$.
(i) *For each even* d *there is a canonical pair of alternal/alternil swappees*

[78] Instead of Eulerian multizetas, Broadhurst and Kreimer speak of 'Euler sums'.

$\mathrm{ma}_d^\bullet/\mathrm{mi}_d^\bullet$ *with initial components* $\mathrm{ma}^{w_1} := u_1^d$, $\mathrm{ma}^{w_1} := v_1^d$ *and with a total of* d *non-zero components.*

(ii) *It is conjectured that the* $\mathrm{ma}_d^\bullet$ *freely generate* $\mathrm{ARI}^{\mathrm{ent}/\mathbb{Z}_1}_{\underline{\mathrm{al/il}}}$.

This conjecture (formulated in a different, less flexible framework) has been around for quite a while now, but it is rather crude and can be considerably sharpened, by reasoning on bialternals. Indeed:

Plain multizetas and generators of $\mathrm{ARI}^{\mathrm{ent}/\mathbb{Z}_1}_{\underline{\mathrm{al/al}}}$.

We require three integer sequences α,β,γ (with $\alpha(d)\equiv\beta(d)+\gamma(d-2)$):

$$\sum \alpha(d)\; x^d := x^6\;(1-x^2)^{-1}\;(1-x^4)^{-1} \tag{230}$$

$$\sum \beta(d)\; x^d := x^6\;(1-x^2)^{-1}\;(1-x^6)^{-1} \tag{231}$$

$$\sum \gamma(d)\; x^d := x^8\;(1-x^4)^{-1}\;(1-x^6)^{-1} \tag{232}$$

and three series of bialternals:

$$\begin{array}{rl} \mathrm{ekma}_d^\bullet/\mathrm{ekmi}_d^\bullet & d\ \textit{even} \geq 2 \\ \mathrm{doma}_{d,b}^\bullet/\mathrm{domi}_{d,b}^\bullet & d\ \textit{even} \geq 10,\ 1\leq b\leq \beta(d) \\ \mathrm{carma}_{d,c}^\bullet/\mathrm{carmi}_{d,c}^\bullet & d\ \textit{even} \geq 8,\ 1\leq c\leq \gamma(d) \end{array}$$

of total degree d and with a single non-zero component of length respectively 1,2,4 . The definition of the first two pairs is straightforward:

$$\mathrm{ekma}_d^{w_1} := u_1^d\ ;\ \mathrm{ekmi}_d^{w_1} := v_1^d \tag{233}$$

$$\mathrm{doma}_{d,b}^{w_1,w_2} \;:=\; \mathrm{fa}(u_1,u_2)\;(\mathrm{ga}(u_1,u_2))^{b-1}\;(\mathrm{ha}(u_1,u_2))^{d/2-3b} \tag{234}$$

$$\mathrm{domi}_{d,b}^{w_1,w_2} \;:=\; \mathrm{fi}(v_1,v_2)\;(\mathrm{gi}(v_1,v_2))^{b-1}\;(\mathrm{hi}(v_1,v_2))^{d/2-3b} \tag{235}$$

with

$$\begin{aligned} \mathrm{fa}(u_1,u_2) &:= u_1u_2(u_1-u_2)(u_1+u_2)(2u_1+u_2)(2u_2+u_1) \\ \mathrm{ga}(u_1,u_2) &:= (u_1+u_2)^2u_1^2u_2^2 \quad ; \quad \mathrm{ha}(u_1,u_2) := u_1^2+u_1u_2+u_2^2 \\ \mathrm{fi}(v_1,v_2) &:= v_1v_2(v_1-v_2)(v_1+v_2)(2v_1-v_2)(2v_2-v_1) \\ \mathrm{gi}(v_1,v_2) &:= (v_1-v_2)^2v_1^2v_2^2 \quad ; \quad \mathrm{hi}(v_1,v_2) := v_1^2-v_1v_2+v_2^2 \end{aligned}$$

The definition of the last pair, *carma*$^\bullet$/*carmi*$^\bullet$, is more roundabout. Observe first that the *ekma*$_d^\bullet$ are not *free* in ARI, but *bound* (for each degree d) by exactly $\gamma(d)$ independent relations of the form:

$$\sum_{d_1+d_2=d+2} R_c^{d_1,d_2}\;[\mathrm{ekma}_{d_1}^\bullet,\mathrm{ekma}_{d_2}^\bullet] \;=\; 0^\bullet \quad (1\leq c\leq\gamma(d)\,,\ R_c^{d_1,d_2}\in\mathbb{Q}) \tag{236}$$

which result from the expansions:

$$[\mathrm{ekma}^\bullet_{d_1}, \mathrm{ekma}^\bullet_{d_2}] = \sum_{1\leq b\leq \beta(d_1+d_2)} K^b_{d_1,d_2}\ \mathrm{doma}^\bullet_{d_1+d_2,b} \quad (K^b_{d_1,d_2}\in\mathbb{Q}) \tag{237}$$

Next, consider the moulds:

$$\mathrm{vima}^\bullet_{d,c} := \sum_{d_1+d_2=d+2} R^{d_1,d_2}_c\ [\mathrm{ma}^\bullet_{d_1}, \mathrm{ma}^\bullet_{d_2}] \ \neq\ 0^\bullet \tag{238}$$

with R^{d_1,d_2}_c as in (100). By construction:
(a) $\mathrm{vima}^\bullet_{d,c}$ *is of alternal/alternil type*
(b) *its components of length 1,2,3 vanish*
(c) *its (non-vanishing) component of length 4 defines a bialternal mould, which is precisely the sought-after mould* $\mathrm{carma}^\bullet_{d,c}$

Now, the crude conjecture at the end of the last para can be replaced by the much sharper, but also more tractable statements:

(i) *The moulds* $\mathrm{ekma}^\bullet_d$ *are free under the ari-bracket upto the contraints (100). More precisely, the number* $P^*_{s,r}$ *of linearly independent bialternals of length* r *, degree* d *(and weight* $s := d+r$*) generated by the* $\mathrm{ekma}^\bullet_d$ *is given by the BK-like formula:*

$$\prod_{s\geq 3, r\geq 1} (1-x^s y^r)^{P^*_{s,r}} = 1 - \frac{x^3 y}{1-x^2} + \frac{x^{12}y^2}{(1-x^4)(1-x^6)} \tag{239}$$

(ii) *The moulds* $\mathrm{carma}^\bullet_{d,k}$ *are free under the ari-bracket. As a consequence, the number* $P^{**}_{s,r}$ *of linearly independent bialternals of length* r *, degree* d *(and weight* $s := d+r$*) generated by the* $\mathrm{carma}^\bullet_{d,k}$ *is given by the BK-like formula:*

$$\prod_{s\geq 3, r\geq 1} (1-x^s y^r)^{P^{**}_{s,r}} = 1 - \frac{x^{12}y^4}{(1-x^4)(1-x^6)} \tag{240}$$

(iii) *In combination, the* $\mathrm{ekma}^\bullet_d$ *and* $\mathrm{carma}^\bullet_{d,k}$ *generate the bialternal algebra* $\mathrm{ARI}^{\mathrm{ent}/\mathbb{Z}_1}_{\underline{\mathrm{al/al}}}$*, freely upto the sole constraints (100). As a consequence, the total number* $P_{s,r}$ *of linearly independent bialternals of length* r *, degree* d *(and weight* $s := d+r$ *) is given by the BK formula (224).*

Unlike in the Eulerian case, the three above statements have been established only upto length $r = 7$ $(\forall d)$ and remain *conjectural* beyond that. But the supporting evidence is overwhelming ([E16]) and and in any case they have the merit of completely removing the weirdness of the artificial-looking corrective term $x^{12}y^2(1-y^2)/((1-x^4)(1-x^6))$ in the BK-formula: the explanation is simply that to each 'missing' bialternal of length 2 (– there just aren't 'enough' $doma^\bullet/domi^\bullet$ around –) there answers, under the transparent mechanism (100)+(101)+(102), a 'stop-gap' bialternal of length 4 (– namely the 'unexpected' $carma^\bullet/carmi^\bullet$–).

5.3.6 Canonical-explicit decomposition into irreducibles.

Let δ^n be the upper dilation operator on bimoulds and let τ^s denote the homogeneous projector that retains only the $\boldsymbol{u}$-homogeneous part of total $\boldsymbol{u}$-degree $s-r$:

$$\delta^n\, M^{\binom{u_1\,,\dots,\,u_r}{v_1\,,\dots,\,v_r}} \;:=\; n^{-r}\;\; M^{\binom{u_1/n\,,\dots,\,u_r/n}{v_1\,,\dots,\,v_r}} \tag{241}$$

$$\tau^s\, M^{\binom{u_1\,,\dots,\,u_r}{v_1\,,\dots,\,v_r}} \;:=\; M^{\binom{u_1\,,\dots,\,u_r}{v_1\,,\dots,\,v_r}}\|_{\boldsymbol{u}\text{-part of degree } s-r} \tag{242}$$

There exists an explicit bimould $loma^\bullet/lomi^\bullet$:
– of alternal/alternil type
– with components depending on the u_i's alone and regular at the origin
– carrying only rational coefficients, ie with components in $\mathbb{Q}[[u_1,\dots,u_r]]$
– with a length-1 component equal to $1/(1-u_1^2)$
– with a length-2 component that is a rational function of u_1, u_2
– with length-r components that are meromorphic functions of $u_1,\dots,u_r$ and possess a *'minimal'* and yet (for $r\geq 3$) *infinite* number of elementary multipoles over the multi-integers.

This bimould $loma^\bullet/lomi^\bullet$ possesses innumerable properties. A whole book, mostly devoted to it, is currently being written. It (the bimould!) holds the key to the arcanes of multizeta aritmetics, mainly because the "numerical" $Zag^\bullet$ and its factors can be reconstituted from it, by means of the following relations, which involve the anti-action $arit$ of ARI on $BIMU$ and the anti-action $garit$ of $GARI$ on $BIMU$:

$$\mathrm{garit}(\mathrm{Zag}^\bullet_{\mathrm{II}}) \;:=\; \sum_{r\ even} B_\star^{n_1,\dots,n_r}\;\, \mathrm{arit}(\delta^{n_r}\mathrm{loma}^\bullet)\dots\mathrm{arit}(\delta^{n_1}\mathrm{loma}^\bullet) \tag{243}$$

$$\mathrm{garit}(\mathrm{Zag}^\bullet_{\mathrm{III}}) \;:=\; \sum_{r\ odd} C_\star^{n_1,\dots,n_r}\;\, \mathrm{arit}(\delta^{n_r}\mathrm{loma}^\bullet)\dots\mathrm{arit}(\delta^{n_1}\mathrm{loma}^\bullet) \tag{244}$$

$$\mathrm{garit}(\mathrm{Zag}^\bullet_{\mathrm{II+III}}) \;:=\; \sum_{r\geq 1} D_\star^{n_1,\dots,n_r}\;\, \mathrm{arit}(\delta^{n_r}\mathrm{loma}^\bullet)\dots\mathrm{arit}(\delta^{n_1}\mathrm{loma}^\bullet) \tag{245}$$

$$\text{garit}(\tau^s \text{Zag}^\bullet_{\text{II}}) := \sum_{r\, even} B^{s_1,\ldots,s_r}\ \text{arit}(\tau^{s_r}\text{loma}^\bullet)\ldots\text{arit}(\tau^{s_1}\text{loma}^\bullet) \quad (246)$$

$$\text{garit}(\tau^s \text{Zag}^\bullet_{\text{III}}) := \sum_{r\, odd} C^{s_1,\ldots,s_r}\ \text{arit}(\tau^{s_r}\text{loma}^\bullet)\ldots\text{arit}(\tau^{s_1}\text{loma}^\bullet) \quad (247)$$

$$\text{garit}(\tau^s \text{Zag}^\bullet_{\text{II+III}}) := \sum_{r\geq 1} D^{s_1,\ldots,s_r}\ \text{arit}(\tau^{s_r}\text{loma}^\bullet)\ldots\text{arit}(\tau^{s_1}\text{loma}^\bullet) \quad (248)$$

In the first system of relations the summation extends to all integers $n_i \geq 1$ but the sums are absolutely convergent. All three moulds $B^\bullet_\star, C^\bullet_\star, D^\bullet_\star$ are *symmetral*, of *perinomal type*[79], and assume *rational* values.

In the second system of relations the summation extends to all *odd* integers $s_i \geq 3$ but the sums are finite since $s_1 + \cdots + s_r = s$. The three new moulds $B^\bullet, C^\bullet, D^\bullet$ are once again *symmetral* and of *perinomal type*, but *a priori* transcendental. The third mould $D^\bullet$ in particular constitutes (together with π^2 which comes from the factor I) a *complete and free*[80] *system of irreducibles* for the formal multizetas. This also leads to the construction of *abstract numerical derivations* – by all accounts an extremely promising development ([E16],[E17]).

6 Reminders and Complements.

6.1 Moulds/bimoulds/comoulds. Basic symmetries and operations.

6.1.1 Main operations on moulds and bimoulds.

Moulds are *functions of a variable number of variables*: they depend on sequences $\boldsymbol{\omega} := (\omega_1, \ldots, \omega_r)$ of arbitrary length $r = r(\boldsymbol{\omega})$. The sum $\|\boldsymbol{\omega}\|$ of a sequence is simply $\sum_1^r \omega_i$. Sequences are systematically written in boldface, with upper indexation when such is called for, and with the product denoting concatenation: e.g. $\boldsymbol{\omega} = \boldsymbol{\omega}^1.\boldsymbol{\omega}^2$. The elements ω_i which make up these sequences are written in normal print, with lower indexation. The sequences themselves are affixed to the moulds as upper indices $A^\bullet = \{A^{\boldsymbol{\omega}}\}$, since moulds are meant to be contracted

$$A^\bullet, B_\bullet \mapsto < A^\bullet, B_\bullet > := \sum A^{\boldsymbol{\omega}}\, B_{\boldsymbol{\omega}}$$

[79]This is a new class of functions, which are omnipresent in multizeta arithmetics. Their properties are reminiscent of *polynomial, periodic, and modular* functions all at once!

[80]free, of course, *up to the symmetrality relations*, but one can most easily derive from it a system of absolute irreducibles.

with dual objects (often differential operators or elements of an associative algebra), the so-called co-moulds $B_\bullet = \{B_{\boldsymbol{\omega}}\}$, which carry their own indices in lower position.

Basic mould operations :
Moulds may be *added, multiplied, composed.*
Mould addition is what you expect : components of equal length get added.
Mould multiplication (*mu* or $\times$) is associative, but non-commutative :

$$C^\bullet = A^\bullet \times B^\bullet \Longleftrightarrow C^{\boldsymbol{\omega}} = \sum_{\boldsymbol{\omega} = \boldsymbol{\omega}^1 . \boldsymbol{\omega}^2} A^{\boldsymbol{\omega}^1} B^{\boldsymbol{\omega}^2} \tag{249}$$

(This includes the trivial decompositions $\boldsymbol{\omega} = \boldsymbol{\omega}.\boldsymbol{\emptyset}$ and $\boldsymbol{\omega} = \boldsymbol{\emptyset}.\boldsymbol{\omega}$).

Mould composition ($\circ$) too is associative and non-commutative :

$$C^\bullet = (A^\bullet \circ B^\bullet) \quad \Longleftrightarrow \quad C^{\boldsymbol{\omega}} = \sum_{\boldsymbol{\omega} = \boldsymbol{\omega}^1 \ldots \boldsymbol{\omega}^s} A^{\|\boldsymbol{\omega}^1\|,\ldots,\|\boldsymbol{\omega}^s\|} B^{\boldsymbol{\omega}^1} \ldots B^{\boldsymbol{\omega}^s} \tag{250}$$

with a sum extending to all possible decompositions of $\boldsymbol{\omega}$ into $s \leq r(\boldsymbol{\omega})$ non-empty factor sequences $\boldsymbol{\omega}^{\boldsymbol{i}}$

The operations $(+, \times, \circ)$ on moulds interact in exactly the same way as their namesakes for power series. Thus $(A^\bullet \times B^\bullet) \circ C^\bullet \equiv (A^\bullet \circ C^\bullet) \times (B^\bullet \circ C^\bullet)$

Basic bimould operations :
Bimoulds are moulds with indices of the form $w_i = (\begin{smallmatrix} u_i \\ v_i \end{smallmatrix})$. Some twenty-odd operations are defined on them. All are defined via four types of contractions $\mathbf{w} \mapsto \lceil \mathbf{w}, \lfloor \mathbf{w}, \mathbf{w} \rceil, \mathbf{w} \rfloor$ (see §5.1) which *add* the u_i's and *subtract* the v_i's in a "symplectic" way, ie respecting both $\sum u_i v_i$ and $\sum du_i \wedge dv_i$. We always use the following short-hand for **u**-sums and **v**-differences :
$u_{12} = u_1 + u_2, \;\; u_{123} = u_1 + u_2 + u_3, \;\; \ldots, \;\; v_{1:2} := v_1 - v_2, \;\; etc$

The main operations on bimoulds are
– the ordinary mould multiplication *mu*, giving rise to the algebra BIMU
– the Lie bracket *ari*, giving rise to the Lie algebra ARI
– the associative, non-commutative law *gari*, giving rise to the group GARI
– the anti-action *arit* of ARI in BIMU [81]
– the anti-action *garit* of GARI in BIMU [82]

But there also exist no less than five sets of operations that run parallel to *ari*, *gari*, *arit*, *garit*. There is also an interesting and useful structure of Lie

[81] it should be carefully distinguished from the *adjoint action adari* of ARI on itself.
[82] it should be carefully distinguished from the *adjoint action adgari* of GARI on ARI.

super-algebra SUARI which mirrors that of ARI. These are neither artefacts nor idle constructions. They all play an important part in unravelling the arithmetics of multizetas, and more generally in investigating *dimorphy.*

6.1.2 Main symmetries for moulds and bimoulds.

Most useful moulds fall into a few basic *symmetry types.*
A mould $A^\bullet$ is said to be *symmetral* (resp. *alternal*) iff :

$$\sum_{\boldsymbol{\omega}\in \mathrm{sha}(\boldsymbol{\omega^1},\boldsymbol{\omega^2})} A^{\boldsymbol{\omega}} = A^{\boldsymbol{\omega^1}} A^{\boldsymbol{\omega^2}} \quad (resp.\ 0) \qquad \forall\, \boldsymbol{\omega^1} \neq \emptyset\,, \forall\, \boldsymbol{\omega^2} \neq \emptyset \tag{251}$$

A mould $A^\bullet$ is said to be *symmetrel* (resp. *alternel*) iff :

$$\sum_{\boldsymbol{\omega}\in \mathrm{she}(\boldsymbol{\omega^1},\boldsymbol{\omega^2})} A^{\boldsymbol{\omega}} = A^{\boldsymbol{\omega^1}} A^{\boldsymbol{\omega^2}} \quad (resp.\ 0) \qquad \forall, \boldsymbol{\omega^1} \neq \emptyset\,, \forall\, \boldsymbol{\omega^2} \neq \emptyset \tag{252}$$

Here $\mathrm{sha}(\boldsymbol{\omega^1},\boldsymbol{\omega^2})$ (resp. $\mathrm{she}(\boldsymbol{\omega^1},\boldsymbol{\omega^2}\ )$) denotes the set of all sequences $\boldsymbol{\omega}$ obtained from $\boldsymbol{\omega^1}$ and $\boldsymbol{\omega^2}$ under ordinary (resp. contracting) shuffling. In a contracting shuffle, two adjacent indices ω_i and ω_j stemming from $\boldsymbol{\omega^1}$ and $\boldsymbol{\omega^2}$ respectively may coalesce into $\omega_{ij} := \omega_i + \omega_j$.

The definition of *symmetral/ alternal* carries over, unchanged, to the case of bimoulds. The definition of *symmetril/ alternil* for bimoulds resembles that of symmetrel/alternel for moulds except that the contractions $\omega_i + \omega_j$ get replaced by $w_i \otimes w_j$ with:

$$A^{\dots,w_i\otimes w_j,\dots} := v_{i:j}^{-1}\, A^{\dots,w_{i/j},\dots} + v_{j:i}^{-1}\, A^{\dots,w_{j/i},\dots}$$

$$\text{and} \qquad v_{i:j} := v_i - v_j \quad , \quad w_{i/j} := (\begin{smallmatrix} u_i + u_j \\ v_i \end{smallmatrix})$$

6.1.3 Arborification and co-arborification.

Straightforward mould expansions $\sum_{\boldsymbol{\omega}} A^{\boldsymbol{\omega}}\ B_{\boldsymbol{\omega}}$, which typically pair a symmetral or alternal mould $A^\bullet$ with a cosymmetral comould $B_\bullet$ (or a symmetrel or alternel mould $A^\bullet$ with a cosymmetrel comould $B_\bullet$) often fail to converge absolutely, ie $\sum_{\boldsymbol{\omega}} \|A^{\boldsymbol{\omega}}\ B_{\boldsymbol{\omega}}\| = +\infty$, although the underlying series may well be convergent. Fortunately, there is an extremely general method for restoring convergence. In essence, it replaces expansions indexed by *totally ordered* sequences $\boldsymbol{\omega}$ by others whose indices are *arborescent sequences* $\boldsymbol{\omega}^{\prec}$ or $\boldsymbol{\omega}^{\not\prec}$, like this:

$$\sum_{\boldsymbol{\omega}} A^{\boldsymbol{\omega}}\ B_{\boldsymbol{\omega}} \ \mapsto\ \sum_{\boldsymbol{\omega}^{\prec}} A^{\boldsymbol{\omega}^{\prec}}\ B_{\boldsymbol{\omega}^{\prec}} \qquad (ordinary\ \ arborification) \tag{253}$$

$$\sum_{\boldsymbol{\omega}} A^{\boldsymbol{\omega}}\ B_{\boldsymbol{\omega}} \ \mapsto\ \sum_{\boldsymbol{\omega}^{\not\prec}} A^{\boldsymbol{\omega}^{\not\prec}}\ B_{\boldsymbol{\omega}^{\not\prec}} \qquad (contracting\ \ arborification) \tag{254}$$

The dual arborification/coarborification transforms:

$$
\begin{array}{llcll}
\text{arborification} \;\Longrightarrow & & & & \\
\textit{ordinary}: & A^{\boldsymbol{\omega}^{\prec}} & := & \sum_{\boldsymbol{\omega}^{\prec}<\boldsymbol{\omega}} A^{\boldsymbol{\omega}} & (\textit{complete definition}) \\
\textit{contracting}: & A^{\boldsymbol{\omega}^{\preccurlyeq}} & := & \sum_{\boldsymbol{\omega}^{\preccurlyeq}\ll\boldsymbol{\omega}} A^{\boldsymbol{\omega}} & (\textit{complete definition})
\end{array}
$$

$$
\begin{array}{llcll}
\text{coarborification} \;\Longrightarrow & & & & \\
\textit{ordinary}: & B_{\boldsymbol{\omega}} & := & \sum_{\boldsymbol{\omega}^{\prec}<\boldsymbol{\omega}} B_{\boldsymbol{\omega}^{\prec}} & (\textit{mere constraint}) \\
\textit{contracting}: & B_{\boldsymbol{\omega}} & := & \sum_{\boldsymbol{\omega}^{\preccurlyeq}\ll\boldsymbol{\omega}} B_{\boldsymbol{\omega}^{\preccurlyeq}} & (\textit{mere constraint})
\end{array}
$$

are devised in such a way as to:

(1) leave the expansions formally unchanged: they amount to a simple redistribution of terms.

(2) drastically reduce the size of the comould part: it typically gets divided by a factor of order $r! := r(\boldsymbol{\omega})!$

(3) prevent a concomitant increase of the mould part: it typically retains the same order of magnitude, despite being changed into a sum of almost $r!$ similar terms!

But whereas the reduction (2) is automatic and universal, the non-increase (3) relies on specific identities, of an algebraic or combinatorial nature, which can never be taken for granted, and yet tend to take place, with providential regularity, whenever we require them!

For the *coaborification* rule, we refer to [E5] or [EV3].

The ordinary (resp contracting) *arborification* rule boils down to summing all the terms $A^{\boldsymbol{\omega}}$ with totally ordered sequences $\boldsymbol{\omega}$ whose order is compatible with the arborecent order of $\boldsymbol{\omega}^{\prec}$ (resp with that of $\boldsymbol{\omega}^{\preccurlyeq}$, but allowing contractions of consecutive elements ω_i). The following example should make this amply clear. Assume:

$$
\begin{array}{cccccc}
 & & & \omega_4 & & \\
 & & \nearrow & & & \\
\boldsymbol{\omega}^{\prec}\ (\textit{or}\ \boldsymbol{\omega}^{\preccurlyeq}\ ) \ := & \omega_1 & \rightarrow & \omega_2 & \rightarrow & \omega_3
\end{array}
$$

Then the arborification rules means:

$$
\begin{aligned}
A^{\boldsymbol{\omega}^{\prec}} &:= A^{\omega_1,\omega_2,\omega_3,\omega_4} + A^{\omega_1,\omega_2,\omega_4,\omega_3} + A^{\omega_1,\omega_4,\omega_2,\omega_3} \\
A^{\boldsymbol{\omega}^{\preccurlyeq}} &:= A^{\omega_1,\omega_2,\omega_3,\omega_4} + A^{\omega_1,\omega_2,\omega_4,\omega_3} + A^{\omega_1,\omega_4,\omega_2,\omega_3} + A^{\omega_1,\omega_2,\omega_3+\omega_4} + A^{\omega_1,\omega_2+\omega_4,\omega_3} \\
&:= A^{\boldsymbol{\omega}^{\prec}} + A^{\omega_1,\omega_2,\omega_3+\omega_4} + A^{\omega_1,\omega_2+\omega_4,\omega_3}
\end{aligned}
$$

6.1.4 Some elementary yet useful moulds.

Constant-type moulds.

mould	value	symmetry	associated series
$\mathbf{1}^\bullet$	$1\ if\ r=0$ $(0\ otherwise)$	*symmetral*	1
$\mathrm{I}^\bullet$	$1\ if\ r=1$ $(0\ otherwise)$	*alternal*	x
$\log^\bullet$	$\frac{(-1)^{r-1}}{r}$	*alternel*	$\log(1+x)$
$\exp_a^\bullet$	$\frac{a^r}{r!}$	*symmetral*	$e^{a\,x}$
$\mathrm{tu}_a^\bullet$	$\frac{(-1)^r}{r!}\frac{\Gamma(r-a)}{\Gamma(-a)}$	*symmetrel*	$(1+x)^a$

Difference-type flat moulds.

$$
\begin{aligned}
\mathrm{sad}^{\emptyset} &:= 1\\
\mathrm{sad}^{t_1,\ldots,t_r} &:= 1\ \ if\ \ t_1<t_2<\cdots<t_r\\
\mathrm{sad}^{t_1,\ldots,t_r} &:= 0\ \ otherwise
\end{aligned}
$$

$$
\begin{aligned}
\mathrm{lad}^{\emptyset} &:= 0\\
\mathrm{lad}^{t_1,\ldots,t_r} &:= (-1)^q\,\frac{p!\,q!}{(p+q+1)!} \;=\; (-1)^q\,\frac{p!\,q!}{r!}\\
with\quad & p:=\sum_{t_i<t_{i+1}} 1 \quad and \quad q:=\sum_{t_i>t_{i+1}} 1
\end{aligned}
$$

Difference-type polar moulds.
Relevant to the construction of w.-b. averages and w.-b. monomials.

$$
\begin{aligned}
\mathrm{tas}_{a,b}^{\emptyset} &:= 1\\
\mathrm{tas}_{a,b}^{t_1} &:= \frac{a-b}{(a-t_1)(t_1-b)}\\
\mathrm{tas}_{a,b}^{t_1,\ldots,t_r} &:= \frac{a-b}{(a-t_1)(t_1-t_2)\ldots(t_{r-1}-t_r)(t_r-b)}
\end{aligned}
$$

$$
\begin{aligned}
\mathrm{tas}_{\star}^{\emptyset} &:= 0\\
\mathrm{tas}_{\star}^{t_1} &:= \frac{1}{(-t_1)(t_1)}\\
\mathrm{tas}_{\star}^{t_1,\ldots,t_r} &:= \frac{1}{(-t_1)(t_1-t_2)\ldots(t_{r-1}-t_r)(t_r)}
\end{aligned}
$$

Identities:

$$\begin{aligned}\mathrm{tas}^\bullet_{a,b}\times\mathrm{tas}^\bullet_{b,c} &= \mathrm{tas}^\bullet_{a,c}\\ \mathrm{tas}^\bullet_{a,b}\times\mathrm{tas}^\bullet_{b,a} &= \mathbf{1}^\bullet\end{aligned}$$

Sum-type flat moulds.
We use the auxiliary notations:

$$\begin{aligned}\boldsymbol{x} &:= (x_1,\dots,x_r)\\ \check{x}_i &:= x_1+\cdots+x_i\\ \hat{x}_i &:= x_i+\cdots+x_r\\ \|\boldsymbol{x}\| &:= x_1+\cdots+x_r=\hat{x}_1=\check{x}_r\\ \sigma(x) &:= +1\ \ if\ x>0\ \ (resp\ -1\ if\ \ x<0)\\ \delta &:= dirac\end{aligned}$$

$$\begin{aligned}\mathrm{sofo}^{\boldsymbol{x}}_\pm &:= (-1)^r\ \sigma_\pm(\check{x}_1)\dots\sigma_\pm(\check{x}_r)\\ \mathrm{antisofo}^{\boldsymbol{x}}_a &:= (-1)^r\ \sigma_\pm(\check{x}_1)\dots\sigma_\pm(\check{x}_r)\\ \mathrm{sefo}^{\boldsymbol{x}}_\pm &:= (-1)^{r-1}\sigma_\pm(\check{x}_1)\dots\sigma_\pm(\check{x}_{r-1})\ \sigma_\mp(\check{x}_r)\\ \mathrm{antisefo}^{\boldsymbol{x}}_\pm &:= (-1)^{r-1}\sigma_\mp(\check{x}_1)\ \sigma_\pm(\check{x}_{r-1})\dots\sigma_\pm(\check{x}_r)\\ \mathrm{lefo}^{\boldsymbol{x}}_\pm &:= (-1)^r\sigma_\pm(\check{x}_1)\dots\sigma_\pm(\check{x}_{r-1})\ \delta(\check{x}_r)\\ \mathrm{antilefo}^{\boldsymbol{x}}_\pm &:= (-1)^r\delta(\check{x}_1)\ \sigma_\pm(\check{x}_{r-1})\dots\sigma_\pm(\check{x}_r)\end{aligned}$$

Sum-type polar moulds.
Relevant to the construction of the organic averages and alien derivations. As usual $\check{\omega}_i:=\omega_1+\cdots+\omega_i$ and $\hat{\omega}_i:=\omega_i+\cdots+\omega_r$.

$$\begin{aligned}\mathrm{sa}^{\boldsymbol{\omega}} &:= \textstyle\prod_{i=1}^{i=r}\frac{\omega_i}{\check{\omega}_i} & \mathrm{musa}^{\boldsymbol{\omega}} &:= (-1)^r\textstyle\prod_{i=1}^{i=r}\frac{\omega_i}{\hat{\omega}_i}\\ \mathrm{romo}^{\boldsymbol{\omega}}_a &:= \textstyle\prod_{i=1}^{i=r}(a\,\frac{\omega_i}{\check{\omega}_i}-1) & \mathrm{antiromo}^{\boldsymbol{\omega}}_a &:= \textstyle\prod_{i=1}^{i=r}(a\,\frac{\omega_i}{\hat{\omega}_i}-1)\\ \mathrm{remo}^{\boldsymbol{\omega}}_a &:= a\,\frac{\omega_r}{\check{\omega}_r}\textstyle\prod_{i=1}^{i=r-1}(a\,\frac{\omega_i}{\check{\omega}_i}-1) & \mathrm{antiremo}^{\boldsymbol{\omega}}_a &:= a\,\frac{\omega_1}{\hat{\omega}_1}\textstyle\prod_{i=2}^{i=r}(a\,\frac{\omega_i}{\hat{\omega}_i}-1)\end{aligned}$$

$$\mathrm{redom}^{\boldsymbol{\omega}} := \frac{(-1)^r}{2}\,\frac{\omega_1+\omega_r}{\omega_1+\cdots+\omega_r}$$

$$\begin{aligned}\mathrm{redo}^\bullet_a &:= -\,\Box^{-1}\,(\ \mathrm{romo}^\bullet_{1-a}\times\Box\,\mathrm{I}^\bullet\times\mathrm{antiromo}^\bullet_a\ )\\ &= -\,\Box^{-1}\,(\ \mathrm{romo}^\bullet_{1-a}\times\Box\,\mathrm{I}^\bullet\times(\mathbf{1}^\bullet+\mathrm{I}^\bullet)^{-1}\times(\mathrm{romo}^\bullet_{1-a})^{-1}\ )\end{aligned}$$

$$\mathrm{somo}^\bullet_{a,b} := \mathrm{remo}^\bullet_a \times \mathrm{antiromo}^\bullet_{1-b} \tag{255}$$

$$:= \mathrm{romo}^\bullet_a \times \mathrm{antiremo}^\bullet_{1-b} \tag{256}$$

$$= \mathrm{romo}^\bullet_{a/b} \times \mathrm{remo}^\bullet_b \tag{257}$$

$$\mathrm{somo}^\bullet_{\left[\begin{smallmatrix} a & b \\ c & d\end{smallmatrix}\right]} := \mathrm{somo}^\bullet_{\frac{c-b}{d-b}\,,\,\frac{a-b}{d-b}} \tag{258}$$

$$\mathrm{somo}^\bullet_{\left[\begin{smallmatrix} b & 0 \\ a & 1\end{smallmatrix}\right]} := \mathrm{somo}^\bullet_{a,b} \tag{259}$$

Identities :

$$\mathrm{remo}^\bullet_a \times \mathrm{antiromo}^\bullet_{1-a} = \mathbf{1}^\bullet \tag{260}$$

$$\mathrm{romo}^\bullet_a \times \mathrm{antiremo}^\bullet_{1-a} = \mathbf{1}^\bullet \tag{261}$$

multplicative inverse :	$\mathrm{somo}^\bullet_{\left[\begin{smallmatrix} a & b \\ c & d\end{smallmatrix}\right]}$	$\leftrightarrow$	$\mathrm{somo}^\bullet_{\left[\begin{smallmatrix} c & b \\ a & d\end{smallmatrix}\right]}$	(a , c *exchanged*)
composition inverse :	$\mathrm{somo}^\bullet_{\left[\begin{smallmatrix} a & b \\ c & d\end{smallmatrix}\right]}$	$\leftrightarrow$	$\mathrm{somo}^\bullet_{\left[\begin{smallmatrix} b & a \\ d & c\end{smallmatrix}\right]}$	(*columns exchanged*)
sequence reversal :	$\mathrm{somo}^\bullet_{\left[\begin{smallmatrix} a & b \\ c & d\end{smallmatrix}\right]}$	$\overset{\textbf{anti}}{\leftrightarrow}$	$\mathrm{somo}^\bullet_{\left[\begin{smallmatrix} c & d \\ a & b\end{smallmatrix}\right]}$	(*rows exchanged*)

multplication : $\mathrm{somo}^\bullet_{a_1,a_2} \times \mathrm{somo}^\bullet_{a_2,a_3} = \mathrm{somo}^\bullet_{a_1,a_3}$

composition : $\mathrm{somo}^\bullet_{a_1,b_1} \circ \mathrm{somo}^\bullet_{a_2,b_2} = \mathrm{somo}^\bullet_{(a_2-b_2)a_1+b_2\,,\,(a_2-b_2)b_1+b_2}$

multplication : $\mathrm{somo}^\bullet_{\left[\begin{smallmatrix} a_0 & b_1 \\ a_2 & b_2\end{smallmatrix}\right]} \times \mathrm{somo}^\bullet_{\left[\begin{smallmatrix} a_1 & b_1 \\ a_0 & b_2\end{smallmatrix}\right]} = \mathrm{somo}^\bullet_{\left[\begin{smallmatrix} a_1 & b_1 \\ a_2 & b_2\end{smallmatrix}\right]}$

composition : $\mathrm{somo}^\bullet_{\left[\begin{smallmatrix} a_1 & b_1 \\ a_2 & b_2\end{smallmatrix}\right]} \circ \mathrm{somo}^\bullet_{\left[\begin{smallmatrix} b_1 & c_1 \\ b_2 & c_2\end{smallmatrix}\right]} = \mathrm{somo}^\bullet_{\left[\begin{smallmatrix} a_1 & c_1 \\ a_2 & c_2\end{smallmatrix}\right]}$

Symmetry types.
All the above moulds fall into one or the other of the main symmetry types.

Alternal : $\mathrm{lad}^\bullet$, $\mathrm{tas}^\bullet_\star$

Symmetral : $\mathrm{exp}^\bullet_a$, $\mathrm{sad}^\bullet$, $\mathrm{tas}^\bullet_{a,b}$, $\mathrm{sa}^\bullet$, $\mathrm{musa}^\bullet$

Alternel : $\mathrm{log}^\bullet$, $\mathrm{lefo}^\bullet_\pm$, $\mathrm{redo}^\bullet_\pm$, $\mathrm{redom}^\bullet$

Symmetrel : $\mathrm{tu}^\bullet_a$, $\mathrm{sofo}^\bullet_\pm$, $\mathrm{sefo}^\bullet_\pm$, $\mathrm{romo}^\bullet_a$, $\mathrm{remo}^\bullet_a$

Moulds *something*$^\bullet$ and *antisomething*$^\bullet$ have the same symmetry type.

Smooth arborification.

All the above moulds possess the property of smooth arborification (meaning that their arborified variants admit essentially the same type of bounds) the only exception being the moulds $log^\bullet$ and $tu_a^\bullet$ for $a \notin \mathbb{Z}$ and in particular for $a = 1/2$. This is in relation to the fact that the *standard* alien derivations (which admit $log^\bullet$ as their left-lateral mould) and the *standard* or *median* convolution average (which admits $tu_{1/2}^\bullet$ as its right- *and* left-lateral mould) are not *well-behaved.*
Of course, for alternal or symmetral (resp alternel or symmetrel) moulds, one should take the ordinary (resp contracting) form of arborification.

Form-preserving arborification.
All the sum-type moulds listed above, ie all those moulds whose definition involves forward sums $\hat{x}_i$ or $\hat{\omega}_i$ (resp backward sums $\check{x}_i$ or $\check{\omega}_i$) have the stronger and very useful property of form-preserving arborification. This means that they retain their *outward analytical expression*, except that the sums $\hat{x}_i$ or $\hat{\omega}_i$ (resp $\check{x}_i$ or $\check{\omega}_i$) are now relative to the arborescent (resp antiarborescent) order. The same holds for the difference-type moulds $tas_{a,\infty}^\bullet$ and $tas_{\infty,b}^\bullet$.

6.2 Resurgent functions.

Minors/majors :
Real-majors and natural-majors

$$\hat{\varphi}(\zeta) = -\frac{1}{2\pi i}\,\big(\check{\varphi}_{\text{real}}(e^{\pi i}\zeta) - \check{\varphi}_{\text{real}}(e^{-\pi i}\zeta)\big) \tag{262}$$

$$\hat{\varphi}(\zeta) = \check{\varphi}_{\text{nat}}(\zeta) - \check{\varphi}_{\text{nat}}(e^{-2\pi i}\zeta) \tag{263}$$

$$\check{\varphi}_{\text{real}}(\zeta) \equiv 2\pi i\,\check{\varphi}_{\text{nat}}(e^{-\pi i}\zeta) \tag{264}$$

The formulae below use real-majors.

Standard Borel transform : $\quad \varphi(z) \to \overset{\diamond}{\varphi}(\zeta) = \big(\hat{\varphi}(\zeta), \check{\varphi}(\zeta)\big)$

$$\hat{\varphi}(\zeta) = \frac{1}{2\pi i}\int_{c-i\infty}^{c+i\infty} \exp(z\zeta)\,\varphi(z)\,dz \qquad (1 \ll c\,;\ \arg\zeta = 0) \tag{265}$$

$$\check{\varphi}(\zeta) = \int_{+u}^{+\infty} \exp(-z\zeta)\,\varphi(z)\,dz \qquad (1 \ll u\,;\ |\arg\zeta| \le \pi) \tag{266}$$

Standard Laplace transform : $\quad \overset{\diamond}{\varphi}(\zeta) = \big(\hat{\varphi}(\zeta), \check{\varphi}(\zeta)\big) \to \varphi(z)$

$$\varphi(z) = \int_{+0}^{+\infty} \exp(-z\zeta)\,\hat{\varphi}(\zeta)\,d\zeta \qquad (\mathit{for}\ \overset{\diamond}{\varphi}\ \mathit{integrable\ at}\ 0.) \tag{267}$$

$$\varphi(z) = \frac{1}{2\pi i}\int_{e^{-\pi i}\infty}^{e^{\pi i}\infty} \exp(z\zeta)\,\check{\varphi}(\zeta)\,d\zeta \qquad (\mathit{for\ any}\ \overset{\diamond}{\varphi}\,;\ \arg z = 0) \tag{268}$$

Elementary (standard) Borel/Laplace transforms :

$$\begin{aligned}
\varphi(z) &= z^{-\sigma} \qquad (for\ \sigma \in \mathbb{C} - \mathbb{N}^\star)\\
\hat{\varphi}(\zeta) &= \zeta^{\sigma-1}/\Gamma(\sigma)\\
\check{\varphi}(\zeta) &= \zeta^{\sigma-1}\Gamma(1-\sigma)
\end{aligned} \tag{269}$$

$$\begin{aligned}
\varphi(z) &= z^{-n} \qquad (for\ n \in \mathbb{N}^\star)\\
\hat{\varphi}(\zeta) &= \zeta^{n-1}/\Gamma(n)\\
\check{\varphi}(\zeta) &= (-1)^n\, \zeta^{n-1} \log\zeta\, /\Gamma(n)
\end{aligned} \tag{270}$$

6.3 Alien derivations.

$$\Delta_\omega\, \hat{\varphi}(\zeta) := \sum_{\epsilon_i = \pm} \mathbf{d}^{\binom{\epsilon_1\ ,\dots,\ \epsilon_r}{\omega_1\ ,\dots,\ \omega_r}}\ \hat{\varphi}^{\binom{\epsilon_1\ ,\dots,\ \epsilon_r}{\omega_1\ ,\dots,\ \omega_r}}(\zeta+\omega) \tag{271}$$

Standard alien derivations (weights **d=dun**):

$$\mathbf{d}^{\binom{\epsilon_1\ ,\dots,\ \epsilon_r}{\omega_1\ ,\dots,\ \omega_r}} := \epsilon_r\, \frac{p!\,q!}{(p+q+1)!} \qquad \begin{array}{l} p := \#\{1 \le i < r\,;\ \epsilon_i = +\}\\ q := \#\{1 \le i < r\,;\ \epsilon_i = -\}\end{array} \tag{272}$$

Lateral moulds: $redun^\bullet = -ledun^\bullet = log^\bullet$. See §6.1.4.

Organic alien derivations (weights **d=dom**):

$$\begin{aligned}
\mathbf{dun}^{\binom{\epsilon_1\ \dots\ \epsilon_r}{\omega_1\ \dots\ \omega_r}} &:= \tfrac{\epsilon_r}{2\pi i}\,\tfrac{p!\,q!}{r!} && with\ \ p := \#\{\epsilon_i = +\,,\ i \le r-1\}\\
& && and\ \ \ q := \#\{\epsilon_i = -\,,\ i \le r-1\}\\
\mathbf{dom}^{\binom{\epsilon_1\ \dots\ \epsilon_r}{\omega_1\ \dots\ \omega_r}} &:= \tfrac{\epsilon_r}{2}\,\tfrac{\omega_{p+1}}{\omega_1+\dots+\omega_r} && if\quad (\epsilon_1,\dots,\epsilon_r) = ((+)^p, (-)^q, \epsilon_r)\\
&:= \tfrac{\epsilon_r}{2}\,\tfrac{\omega_{q+1}}{\omega_1+\dots+\omega_r} && if\quad (\epsilon_1,\dots,\epsilon_r) = ((-)^q, (+)^p, \epsilon_r)\\
&:= 0 && \qquad otherwise
\end{aligned} \tag{273}$$

Lateral moulds: $redom^\bullet = -ledom^\bullet$. See §6.1.4.

Co-organic alien derivations (weights **d=don**):
Is capable of various definitions. Here, we define it via its:
Lateral moulds: $redon^\bullet = -ledon^\bullet = redo^\bullet_{1/2}$. See §6.1.4.

6.4 Convolution averages.

$$(\mathbf{m}.\hat{\varphi})(\zeta) := \sum_{\epsilon_i=\pm} \mathbf{m}^{\binom{\epsilon_1\ ,\dots,\ \epsilon_r}{\omega_1\ ,\dots,\ \omega_r}}\ \hat{\varphi}^{\binom{\epsilon_1\ ,\dots,\ \epsilon_r}{\omega_1\ ,\dots,\ \omega_r}}(\zeta) \tag{274}$$

Standard convolution average (weights **m=mun**):

$$\mathbf{m}^{\binom{\epsilon_1\ ,\dots,\ \epsilon_r}{\omega_1\ ,\dots,\ \omega_r}} := \frac{(2p)!\,(2q)!}{4^{p+q}\,(p+q)!\,p!\,q!} \qquad\qquad (p := nb\ of\ +\,;\ q := nb\ of\ -) \tag{275}$$

Lateral moulds: $remun^\bullet = lemun^\bullet = tu^\bullet_{1/2}$. See §6.1.4.

Organic convolution average (weights **m=mon**):

$$\mathbf{m}^{\binom{\epsilon_1\ ,\dots,\ \epsilon_r}{\omega_1\ ,\dots,\ \omega_r}} \ := \ \mathbf{m}^{\binom{\epsilon_1\ ,\dots,\ \epsilon_{r-1}}{\omega_1\ ,\dots,\ \omega_{r-1}}}.\Big(1-\frac{1}{2}\frac{\omega_r}{\omega_1+\dots+\omega_r}\Big) \qquad if\ \epsilon_{r-1}=\epsilon_r \tag{276}$$

$$\mathbf{m}^{\binom{\epsilon_1\ ,\dots,\ \epsilon_r}{\omega_1\ ,\dots,\ \omega_r}} \ := \ \mathbf{m}^{\binom{\epsilon_1\ ,\dots,\ \epsilon_{r-1}}{\omega_1\ ,\dots,\ \omega_{r-1}}}.\frac{1}{2}\,\frac{\omega_r}{\omega_1+\dots+\omega_r} \qquad if\ \epsilon_{r-1}\neq\epsilon_r \tag{277}$$

Lateral moulds: $remon^\bullet = lemon^\bullet = remo^\bullet_{1/2}$. See §6.1.4.

6.5 Comparative tables.

We take advantage of the relations:

$$\mathbf{m}^{\bar{\epsilon}_1,\dots,\bar{\epsilon}_r} \equiv +\,\mathbf{m}^{\epsilon_1,\dots,\epsilon_r} \qquad ; \qquad \mathbf{d}^{\bar{\epsilon}_1,\dots,\bar{\epsilon}_r} \equiv -\,\mathbf{d}^{\epsilon_1,\dots,\epsilon_r} \tag{278}$$

to register only the weights whose sequence $\boldsymbol{\epsilon}$ ends with $\epsilon_r = +$. Also, to further simplify, we assume all gaps ω_i to be of equal length.

As the following tables show, the weights of well-behaved operators (be they alien derivations or convolution averages) tend to be *very small* when the sign sequences are *strongly alternating*, ie when they correspond to *oft-crossing paths.*

object	*average*	*average*	*derivation*	*derivation*	*derivation*
species	*uniform*	*organic*	*uniform*	*organic*	*organic*
nature	*“bad”*	*“good”*	*“bad”*	*“good”*	*“good”*
name	**mun**	**mun**	**dun**	**don**	**dom**
$(+)$	1/2	1/2	1	1	1
sums	**1/2**	**1/2**	**1**	**1**	**1**
$(+,+)$	3/8	3/8	1/2	1/2	1/2
$(-,+)\ *$	1/8	1/8	1/2	1/2	1/2
sums	**1/2**	**1/2**	**1**	**1**	**1**
$(+,+,+)$	5/16	5/16	1/3	1/3	1/3
$(-,+,+)$	1/16	5/48	1/6	1/6	1/6
$(+,-,+)\ *$	1/16	1/48	1/6	1/6	1/6
$(-,-,+)$	1/16	1/16	1/3	1/3	1/3
sums	**1/2**	**1/2**	**1**	**1**	**1**
$(+,+,+,+)$	35/128	35/128	1/4	1/4	1/4
$(-,+,+,+)$	5/128	35/384	1/12	5/48	1/8
$(+,-,+,+)$	5/128	7/384	1/12	1/24	0
$(-,-,+,+)$	3/128	7/128	1/12	5/48	1/8
$(+,+,-,+)$	5/128	1/128	1/12	5/48	1/8
$(-,+,-,+)\ *$	3/128	1/384	1/12	1/24	0
$(+,-,-,+)$	3/128	5/384	1/12	5/48	1/8
$(-,-,-,+)$	5/128	5/128	1/4	1/4	1/4
sums	**1/2**	**1/2**	**1**	**1**	**1**
$(+,+,+,+,+)$	63/256	63/256	1/5	1/5	1/5
$(-,+,+,+,+)$	7/256	21/256	1/20	37/480	1/10
$(+,-,+,+,+)$	7/256	21/1280	1/20	11/480	0
$(-,-,+,+,+)$	3/256	63/1280	1/30	1/16	1/10
$(+,+,-,+,+)$	7/256	9/1280	1/20	11/480	0
$(-,+,-,+,+)$	3/256	3/1280	1/30	1/120	0
$(+,-,-,+,+)$	3/256	3/256	1/30	7/240	0
$(-,-,-,+,+)$	3/256	9/256	1/20	37/480	1/10
$(+,+,+,-,+)$	7/256	1/256	1/20	37/480	1/10
$(-,+,+,-,+)$	3/256	1/768	1/30	7/240	0
$(+,-,+,-,+)\ *$	3/256	1/3840	1/30	1/120	0
$(-,-,+,-,+)$	3/256	1/1280	1/20	11/480	0
$(+,+,-,-,+)$	3/256	7/1280	1/30	1/16	1/10
$(-,+,-,-,+)$	3/256	7/3840	1/20	11/480	0
$(+,-,-,-,+)$	3/256	7/768	1/20	37/480	1/10
$(-,-,-,-,+)$	7/256	7/256	1/5	1/5	1/5
sums	**1/2**	**1/2**	**1**	**1**	**1**

name	**mun**	**mun**	**dun**	**don**	**dom**
$(+,+,+,+,+,+)$	231/1024	231/1024	1/6	1/6	1/6
$(-,+,+,+,+,+)$	21/1024	77/1024	1/30	71/1152	1/12
$(+,-,+,+,+,+)$	21/1024	77/5120	1/30	23/1440	0
$(-,-,+,+,+,+)$	7/1024	231/5120	1/60	29/640	1/12
$(+,+,-,+,+,+)$	21/1024	33/5120	1/30	11/960	0
$(-,+,-,+,+,+)$	7/1024	11/5120	1/60	23/5760	0
$(+,-,-,+,+,+)$	7/1024	11/1024	1/60	47/2880	0
$(-,-,-,+,+,+)$	5/1024	33/1024	1/60	29/640	1/12
$(+,+,+,-,+,+)$	21/1024	11/3072	1/30	23/1440	0
$(-,+,+,-,+,+)$	7/1024	11/9216	1/60	11/1920	0
$(+,-,+,-,+,+)$	7/1024	11/46080	1/60	1/720	0
$(-,-,+,-,+,+)$	5/1024	11/15360	1/60	23/5760	0
$(+,+,-,-,+,+)$	7/1024	77/15360	1/60	47/2880	0
$(-,+,-,-,+,+)$	5/1024	77/46080	1/60	11/1920	0
$(+,-,-,-,+,+)$	5/1024	77/9216	1/60	13/576	0
$(-,-,-,-,+,+)$	7/1024	77/3072	1/30	71/1152	1/12
$(+,+,+,+,-,+)$	21/1024	7/3072	1/30	71/1152	1/12
$(-,+,+,+,-,+)$	7/1024	7/9216	1/60	13/576	0
$(+,-,+,+,-,+)$	7/1024	7/46080	1/60	11/1920	0
$(-,-,+,+,-,+)$	5/1024	7/15360	1/60	47/2880	0
$(+,+,-,+,-,+)$	7/1024	1/15360	1/60	23/5760	0
$(-,+,-,+,-,+)$ $*$	5/1024	1/46080	1/60	1/720	0
$(+,-,-,+,-,+)$	5/1024	1/9216	1/60	11/1920	0
$(-,-,-,+,-,+)$	7/1024	1/3072	1/30	23/1440	0
$(+,+,+,-,-,+)$	7/1024	3/1024	1/60	29/640	1/12
$(-,+,+,-,-,+)$	5/1024	1/1024	1/60	47/2880	0
$(+,-,+,-,-,+)$	5/1024	1/5120	1/60	23/5760	0
$(-,-,+,-,-,+)$	7/1024	3/5120	1/30	11/960	0
$(+,+,-,-,-,+)$	5/1024	21/5120	1/60	29/640	1/12
$(-,+,-,-,-,+)$	7/1024	7/5120	1/30	23/1440	0
$(+,-,-,-,-,+)$	7/1024	7/1024	1/30	71/1152	1/12
$(-,-,-,-,-,+)$	21/1024	21/1024	1/6	1/6	1/6
sums	**1/2**	**1/2**	**1**	**1**	**1**

6.6 Difference between "good" and "bad".

We first set some notations:

$$\mathrm{cnb}(r_1, r_2) \;:=\; \sum_{k=0}^{k=r_2} \; (-1)^k \; \frac{r_2!}{k!\,(r_2-k)!} \; (r_2-k)^{r_1} \tag{279}$$

In particular:

$$\begin{aligned}
\mathrm{cnb}(r,1) &= 1 && && (280)\\
\mathrm{cnb}(r,r) &= r! && && (281)\\
\mathrm{cnb}(r,r_\star) &\geq 2 && \mathit{if} \quad 1 < r_\star < r && (282)\\
\mathrm{cnb}(r,r_\star) &= 0 && \mathit{if} \quad r < r_\star && (283)
\end{aligned}$$

Then we set:

$$\mathrm{f}(r_0, r_1) = \sum_{r_2=1}^{r_2=r_1} \; \mathrm{f}(r_0+r_2)\;\; \mathrm{cnb}(r_1, r_2) \tag{284}$$

Next, let $\boldsymbol{\omega}^{\prec}_{r_0,r_1}$ (resp. $\boldsymbol{\omega}^{\succ}_{r_0,r_1}$) be the arborescent (resp. antiarborescent) sequence obtained by *suffixing* (resp. *prefixing*) the *totally non-ordered* sequence $(\omega'_1, \ldots, \omega'_{r_1})$ to the *totally ordered* sequence $(\omega_1, \ldots, \omega_{r_0})$.

Assume now that $F^\bullet$ is some *constant-type mould* like $tu^\bullet_a$ (§6.1.4 *supra*), ie a mould whose values depend solely on the sequence length r, so that $F^{\omega_1,\ldots,\omega_r} \equiv f(r)$. In view of what precedes, it is clear that after a contracting arborification or antiarborification we get:

$$\mathrm{F}^{\boldsymbol{\omega}^{\prec}_{r_0,r_1}} \equiv \mathrm{F}^{\boldsymbol{\omega}^{\succ}_{r_0,r_1}} \equiv \mathrm{f}(r_0, r_1) \qquad\qquad \text{with} \quad \mathrm{f}(r_0,r_1) \text{ as in (284)} \tag{285}$$

If we take $F^\bullet := tu^\bullet_a$ with $a \in \mathbb{Z}$, then $tu^\bullet_a$ is well-behaved and indeed we can see (trivially for $a < 0$, less so for $a > 0$) that:

$$\limsup_{r_1 \to +\infty} \Big(\frac{log|\mathrm{f}(r_0,r_1)|}{r_0 + r_1} \Big) \;<\; +\infty \qquad\quad (\forall\, r_0 \;\text{ fixed and } \geq 1) \tag{286}$$

But if $a \notin \mathbb{Z}$, then $tu^\bullet_a$ is not well-behaved and we can show that:

$$\limsup_{r_1 = +\infty} \Big(\frac{log|\mathrm{f}(r_0,r_1)|}{r_0 + r_1} \Big) \;=\; +\infty \qquad\quad (\forall\, r_0 \;\text{ fixed and } \geq 1) \tag{287}$$

and in fact:

$$\limsup_{r_1 \to +\infty} \Big(\frac{log|\mathrm{f}(r_0,r_1)|}{(r_0+r_1)\,log(r_0+r_1)} \Big) \;>\; 0 \qquad (\forall\, r_0 \;\text{ fixed and } \geq 1) \tag{288}$$

6.7 Resurgence monomials.

6.7.1 Hyperlogarithms, perilogarithms, paralogarithms.

Perilogarithms have indices $\varpi_i = (\omega_i, \omega_i^\star) \in (\mathbb{C}^\star, \mathbb{C}^\star)$.
Hyperlogarithms have indices $\omega_i \in \mathbb{C}^\star$.
Perilogarithms have indices $\varpi_i = (\omega_i, \omega_i^\star) \in (\mathbb{C}^\star, \mathbb{C}^\star)$ with $\omega_i\, \omega_i^\star \in \mathbb{R}^+$.
Perilogarithms have indices $\varpi_i = (\omega_i, c^2 \bar{\omega}_i) \in (\mathbb{C}^\star, \mathbb{C}^\star)$.
Usually c is fixed, so that only ω_i is mentioned.

6.7.2 D- or Δ-friendly monomials and monics.

Monics depend only on the indices ω_i or ϖ_i.
Monomials depend on a variable z as well.
D-friendly monomials behave simply under ordinary z-differentiation, but less so under alien z-differentiation: their alien derivatives necessarily involve a number of so-called D-friendly monics.
Δ-friendly monomials behave simply under alien z-differentiation, but less so under ordinary z-differentiation: their ordinary derivatives necessarily involve a number of so-called Δ-friendly monics.
D-friendly (resp Δ-friendly) monomials all carry a calligraphic $\mathcal{V}$ (resp $\mathcal{U}$) in their names while the corresponding monics carry an upper-case V (resp. U), sometimes supplemented by a suitable string of pre- or suffixes.

6.7.3 Total closure.

The monomials, as functions of z, are acted upon by *one* ordinary derivation $D := \partial_z$ but by *infinitely many* independent alien derivations Δ_{ω_i} or their exponential-carrying variants $\mathbb{\Delta}_\omega := e^{-\omega z} \Delta_\omega$. The latter have the advantage of commuting with the ordinary derivation D, but at the cost of introducing an exponential factor and thus ceasing to act internally on the ring of formal power series of z^{-1}.
To highlight the $D \leftrightarrow \Delta$ duality, it is sometimes convenient to (formally) regroup all alien derivations under one single symbol:

$$\Delta := \sum \Delta_\omega \quad ; \quad \mathbb{\Delta} := \sum \mathbb{\Delta}_\omega$$

We also require the following mould derivations:

$$\begin{aligned}
\Box A^{\boldsymbol{\varpi}} &:= \|\omega\|\, A^{\boldsymbol{\varpi}} := (\sum \omega_i)\, A^{\boldsymbol{\varpi}}\\
\Box^\star A^{\boldsymbol{\varpi}} &:= \|\omega^\star\|\, A^{\boldsymbol{\varpi}} := (\sum \omega_i^\star)\, A^{\boldsymbol{\varpi}}\\
\Box_{\omega_i} &:= [\partial_{\omega_i}, \Box]\\
\Box^\star_{\omega_i} &:= [\partial_{\omega_i^\star}, \Box^\star]
\end{aligned}$$

6.7.4 Hyperlogarithmic monomials and monics.

Δ-*friendly monomials*	$\mathcal{U}^\bullet(z)\,,\, \mathcal{U}e^\bullet(z)$	*symmetral*
Δ-*friendly monics*	$U^\bullet$	*alternal*
Δ-*friendly monics*	$US^\bullet\,,\, SU^\bullet$	*symmetral*
D-*friendly monomials*	$\mathcal{V}^\bullet(z)\,,\, \mathcal{V}e^\bullet(z)$	*symmetral*
D-*friendly monics*	$V^\bullet$	*alternal*
D-*friendly monics*	$VS^\bullet\,,\, SV^\bullet$	*symmetral*

6.7.5 Basic relations.

$$\begin{aligned}
\mathcal{U}e^\bullet(z) &= \exp(z\,\Box)\,.\,\mathcal{U}^\bullet(z)\\
\mathcal{V}e^\bullet(z) &= \exp(z\,\Box)\,.\,\mathcal{V}^\bullet(z)\\
\mathcal{U}^\bullet(z) &= \mathcal{V}^\bullet(z)\circ U^\bullet\\
\mathcal{V}^\bullet(z) &= \mathcal{U}^\bullet(z)\circ V^\bullet\\
\mathbf{1}^\bullet &= U^\bullet\circ V^\bullet = V^\bullet\circ U^\bullet\\
\mathbf{1}^\bullet &= US^\bullet\times SU^\bullet = VS^\bullet\times SV^\bullet\\
U^\bullet &= US^\bullet\times I^\bullet\times SU^\bullet \qquad \textit{if all } \omega_i\in\mathbb{R}^+\\
V^\bullet &= VS^\bullet\times I^\bullet\times SV^\bullet \qquad \textit{if all } \omega_i\in\mathbb{R}^+
\end{aligned}$$

6.7.6 More relations.

$$\partial_{\omega_i}\mathcal{U}^\bullet(z) = -\mathcal{U}^\bullet(z)\times(\frac{\Box_i}{\Box}\,U^\bullet) - z\,\Box_i\,\mathcal{U}^\bullet(z) \tag{289}$$

$$z\,\partial_z\,\mathcal{U}^\bullet(z) = -z\,\Box\,\mathcal{U}^\bullet(z) - \mathcal{U}^\bullet(z)\times U^\bullet \tag{290}$$

$$\partial_{\omega_i}\mathcal{U}e^\bullet(z) = -\mathcal{U}e^\bullet(z)\times(\exp(z\,\Box).\frac{\Box_i}{\Box}.\,U^\bullet) \tag{291}$$

$$z\,\partial_z\,\mathcal{U}e^\bullet(z) = -\,\mathcal{U}^\bullet(z)\times(\exp(z\,\Box).U^\bullet) \tag{292}$$

$$\partial_{\omega_i} U^\bullet = +(\frac{\square_i}{\square} U^\bullet) \times U^\bullet - U^\bullet \times (\frac{\square_i}{\square} U^\bullet) \tag{293}$$

$$\partial_{\omega_i} US^\bullet = +(\frac{\square_i}{\square} U^\bullet) \times US^\bullet \tag{294}$$

$$\partial_{\omega_i} SU^\bullet = -U^\bullet \times (\frac{\square_i}{\square} SU^\bullet) \tag{295}$$

$\partial_{\boldsymbol{\omega}} := \sum \omega_i \partial_{\omega_i}$

$$\partial_{\boldsymbol{\omega}} U^\bullet = 0 \tag{296}$$

$$\partial_{\boldsymbol{\omega}} US^\bullet = +U^\bullet \times US^\bullet \tag{297}$$

$$\partial_{\boldsymbol{\omega}} SU^\bullet = -SU^\bullet \times U^\bullet \tag{298}$$

6.7.7 Ordinary and alien differentiation. The $D \leftrightarrow \Delta$ duality .

$$(z\,\partial_z + z\,\square)\,\mathcal{U}^\bullet(z) = -\mathcal{U}^\bullet(z) \times U^\bullet \tag{299}$$

$$(z\,\partial_z + z\,\square)\,\mathcal{V}^\bullet(z) = -\mathcal{V}^\bullet(z) \times I^\bullet \tag{300}$$

$$\Delta\,\mathcal{U}^\bullet(z) = I^\bullet \times \mathcal{U}^\bullet(z) \tag{301}$$

$$\Delta\,\mathcal{V}^\bullet(z) = V^\bullet \times \mathcal{V}^\bullet(z) \tag{302}$$

with $\square$ standing as usual for multiplication by $\|\bullet\| = \sum \omega_i$ and

$$I^{\omega_1} := 1 \qquad ; \qquad I^{\omega_1,\dots,\omega_r} := 0 \quad if \quad r \neq 1 \tag{303}$$

6.7.8 Perilogarithmic monomials and monics.

Perilogarithms have indices $\boldsymbol{\varpi} := (\varpi_1, \dots, \varpi_r)$ with $\varpi_i = (\omega_i, \omega_i^\star) \in (\mathbb{C}^\star, \mathbb{C}^\star)$ but with products $\omega_i\,\omega_i^\star \in \mathbb{R}^+$.
Antipodal involution :

$$\# : \qquad {}^{\#}M^{\varpi_1,\dots,\varpi_r} := M^{\varpi_r^\star,\dots,\varpi_1^\star} \quad \text{with} \quad \varpi^\star = (\omega_i^\star, \omega_i) \quad \text{if} \quad \varpi = (\omega_i, \omega_i^\star)$$

Δ-friendly perilogarithms :

	primary	*secundary*	*type*
monomials	$\mathcal{U}a^\bullet$	$\mathcal{U}^\bullet, \mathcal{U}e^\bullet$	*symmetral*
monics	$U^\bullet$	$UR^\bullet, UL^\bullet$	*alternal*
monics	$USS^\bullet, SSU^\bullet$	$US^\bullet, SU^\bullet$	*symmetral*

D-frienly perilogarithms :

	primary	*secundary*	*type*
monomials	$\mathcal{V}a^\bullet$	$\mathcal{V}^\bullet, \mathcal{V}e^\bullet$	*symmetral*
monics	$V^\bullet$	$VR^\bullet, VL^\bullet$	*alternal*
monics	$VSS^\bullet, SSV^\bullet$	$VS^\bullet, SV^\bullet$	*symmetral*

6.7.9 Basic relations.

$$\begin{aligned}
\mathcal{U}e^\bullet(z) &= \exp(\, z\,\square + \, z^{-1}\,\square^\star).\,\mathcal{U}a^\bullet(z) && (304)\\
\mathcal{U}^\bullet(z) &= \exp(\, z^{-1}\,\square^\star).\,\mathcal{U}a^\bullet(z) && (305)\\
\mathbf{1}^\bullet &= USS^\bullet \times SSU^\bullet = US^\bullet \times SU^\bullet && (306)\\
UR^\bullet &= USS^\bullet \times (\square^\star\, SSU^\bullet) && (307)\\
UL^\bullet &= USS^\bullet \times (\square\, SSU^\bullet) && (308)\\
USS^\bullet &= US^\bullet \times {}^{\#}US^\bullet && (309)\\
SSU^\bullet &= {}^{\#}SU^\bullet \times SU^\bullet && (310)\\
SU^\bullet &= \mathcal{U}a^\bullet(1) && (311)
\end{aligned}$$

6.7.10 Integral formulae for the Δ-friendly monomials and monics.

The main ingredients of the integral formulae are the SPA or “Standard Path Averaging” (see §3.6.4) and the CCI or “Common Core Integrand”:

$$CCI := \frac{1}{(2\pi i)^r}\frac{\exp(-\sum \omega_i t_i - \sum \omega_i^\star/t_i)}{(t_r - t_{r-1})\dots(t_3-t_2)(t_2-t_1)} \qquad (312)$$

and

$$\begin{array}{llllll}
\textbf{Monomials}: & & & & & \\
\mathcal{U}a^{\varpi}(z) & = & SPA\int_0^\infty CCI & (t_1-z)^{-1} & dt_1\ldots dt_r \\
\mathcal{U}e^{\varpi}(z) & = & e^{\|\boldsymbol{\omega}\|\,z+\|\boldsymbol{\omega}^\star\|\,z^{-1}} & \mathcal{U}a^{\varpi}(z) & \\
\mathcal{U}^{\varpi}(z) & = & e^{\|\boldsymbol{\omega}^\star\|\,z^{-1}} & \mathcal{U}a^{\varpi}(z) & \\
\textbf{Monics}: & & & & & \\
U^{\varpi} & = & SPA\int_0^\infty CCI & (\sum\omega_i) & dt_1\ldots dt_r \\
 & = & SPA\int_0^\infty CCI & (\sum\omega_i^\star/t_i^2) & dt_1\ldots dt_r \\
UL^{\varpi} & = & SPA\int_0^\infty CCI & (\sum\omega_i/t_i) & dt_1\ldots dt_r \\
UR^{\varpi} & = & SPA\int_0^\infty CCI & (\sum\omega_i^\star/t_i) & dt_1\ldots dt_r \\
SU^{\varpi} & = & SPA\int_0^\infty CCI & (t_1-1)^{-1} & dt_1\ldots dt_r \\
US^{\varpi} & = & SPA\int_0^\infty CCI & (1-t_r)^{-1} & dt_1\ldots dt_r \\
SSU^{\varpi} & = & SPA\int_0^\infty CCI & (1/t_1) & dt_1\ldots dy_r \\
USS^{\varpi} & = & SPA\int_0^\infty CCI & (-1/t_r) & dt_1\ldots dt_r
\end{array}$$

6.7.11 More relations for the Δ-friendly perilogarithms.

Monomials:

$$\partial_{\omega_i}\mathcal{U}a^\bullet(z) = -\mathcal{U}a^\bullet(z)\times\frac{\Box_i}{\Box}U^\bullet - z\,\Box_i\mathcal{U}a^\bullet(z) \tag{313}$$

$$\partial_{\omega_i^\star}\mathcal{U}a^\bullet(z) = +z^{-1}\mathcal{U}a^\bullet(z)\times USS^\bullet\times\Box_i^\star SSU^\bullet - z^{-1}\,\Box_i^\star\,\mathcal{U}a^\bullet(z) \tag{314}$$

$$z\,\partial_z\mathcal{U}a^\bullet(z) = (-z\,\Box+z^{-1}\,\Box^\star)\mathcal{U}a^\bullet(z) - \mathcal{U}a^\bullet(z)\times(U^\bullet+z^{-1}\,UR^\bullet) \tag{315}$$

Monics:

$$\partial_{\omega_i}U^\bullet = +(\frac{\Box_i}{\Box}U^\bullet)\times U^\bullet - U^\bullet\times(\frac{\Box_i}{\Box}U^\bullet) - \Box_i\,UR^\bullet \tag{316}$$

$$\partial_{\omega_i^\star}U^\bullet = +\Box\,((\Box_i^\star\,USS^\bullet)\times SSU^\bullet) \tag{317}$$

$$\partial_{\omega_i}USS^\bullet = +(\frac{\Box_i}{\Box}U^\bullet)\times USS^\bullet \tag{318}$$

$$\partial_{\omega_i^\star}USS^\bullet = +USS^\bullet\times(\frac{\Box_i^\star}{\Box^\star}\,{}^{\#}U^\bullet) \tag{319}$$

$$\partial_{\omega_i}\,SSU^\bullet = -SSU^\bullet\times(\frac{\Box_i}{\Box}U^\bullet) \tag{320}$$

$$\partial_{\omega_i^\star}\,SSU^\bullet = -(\frac{\Box_i^\star}{\Box^\star}\,{}^{\#}U^\bullet)\times SSU^\bullet \tag{321}$$

$$\partial_{\omega_i}\, \mathcal{US}^\bullet = +(\frac{\square_i}{\square}\, \mathcal{U}^\bullet) \times \mathcal{US}^\bullet - \square_i\, \mathcal{US}^\bullet \tag{322}$$
$$\partial_{\omega_i^\star}\, \mathcal{US}^\bullet = +\mathcal{US}^\bullet \times (\square_i^\star\, {}^{\#}\mathcal{US}^\bullet) \times \mathcal{SU}^\bullet \tag{323}$$
$$\partial_{\omega_i}\, {}^{\#}\mathcal{US}^\bullet = +\mathcal{SU}^\bullet \times (\square_i\, \mathcal{US}^\bullet) \times {}^{\#}\mathcal{US}^\bullet \tag{324}$$
$$\partial_{\omega_i^\star}\, {}^{\#}\mathcal{US}^\bullet = +\, {}^{\#}\mathcal{US}^\bullet \times (\frac{\square_i^\star}{\square^\star}\, {}^{\#}\mathcal{U}^\bullet) - \square_i^\star\, {}^{\#}\mathcal{US}^\bullet \tag{325}$$

$$\partial_{\omega_i}\, \mathcal{SU}^\bullet = -\mathcal{SU}^\bullet \times (\frac{\square_i}{\square}\, \mathcal{U}^\bullet) - \square_i\, \mathcal{SU}^\bullet \tag{326}$$
$$\partial_{\omega_i^\star}\, \mathcal{SU}^\bullet = +\, {}^{\#}\mathcal{US}^\bullet \times (\square_i^\star\, {}^{\#}\mathcal{SU}^\bullet) \times \mathcal{SU}^\bullet \tag{327}$$
$$\partial_{\omega_i}\, {}^{\#}\mathcal{SU}^\bullet = +\, {}^{\#}\mathcal{SU}^\bullet \times (\square_i\, \mathcal{SU}^\bullet) \times \mathcal{US}^\bullet \tag{328}$$
$$\partial_{\omega_i^\star}\, {}^{\#}\mathcal{SU}^\bullet = -(\frac{\square_i^\star}{\square^\star}\, {}^{\#}\mathcal{U}^\bullet) \times {}^{\#}\mathcal{SU}^\bullet - \square_i^\star\, {}^{\#}\mathcal{SU}^\bullet \tag{329}$$

6.7.12 Yet more relations for the Δ-friendly perilogarithms.

The partial differentiation rules relative to

$$\partial_{\boldsymbol{\omega}} := \sum \omega_i \partial_{\omega_i} \quad and \quad \partial_{\boldsymbol{\omega}^\star} := \sum \omega_i^\star \partial_{\omega_i^\star}$$

though deducible from the above, are also worth mentioning.

$$\partial_{\boldsymbol{\omega}} \mathcal{U}a^\bullet(z) = -\mathcal{U}a^\bullet(z) \times \mathcal{U}^\bullet - z\, \square\, \mathcal{U}a^\bullet(z) \tag{330}$$
$$\partial_{\boldsymbol{\omega}^\star} \mathcal{U}a^\bullet(z) = +\, z^{-1}\, \mathcal{U}a^\bullet(z) \times \mathcal{UR}^\bullet -\, z^{-1}\, \square\, \mathcal{U}a^\bullet(z) \tag{331}$$

$$\begin{array}{ccccccc} \partial_{\boldsymbol{\omega}}\, \mathcal{U}^\bullet & = & \partial_{\boldsymbol{\omega}^\star}\, \mathcal{U}^\bullet & = & \square\, ((\square^\star\, \mathcal{USS}^\bullet) \times \mathcal{SSU}^\bullet & = & -\square \mathcal{UR}^\bullet \\ \partial_{\boldsymbol{\omega}}\, \mathcal{USS}^\bullet & = & \partial_{\boldsymbol{\omega}^\star}\, \mathcal{USS}^\bullet & = & +\mathcal{U}^\bullet \times \mathcal{USS}^\bullet & = & \mathcal{USS}^\bullet \times {}^{\#}\mathcal{U}^\bullet \\ \partial_{\boldsymbol{\omega}}\, \mathcal{SSU}^\bullet & = & \partial_{\boldsymbol{\omega}^\star}\, \mathcal{SSU}^\bullet & = & -\mathcal{SSU}^\bullet \times \mathcal{U}^\bullet & = & -{}^{\#}\mathcal{U}^\bullet \mathcal{SSU}^\bullet \end{array}$$

$$\partial_{\boldsymbol{\omega}}\, \mathcal{US}^\bullet = +\mathcal{U}^\bullet \times \mathcal{US}^\bullet - \square\, \mathcal{US}^\bullet \tag{332}$$
$$\partial_{\boldsymbol{\omega}^\star}\, \mathcal{US}^\bullet = +\mathcal{US}^\bullet \times (\square^\star\, {}^{\#}\mathcal{US}^\bullet) \times^{\#} \mathcal{SU}^\bullet \tag{333}$$

$$\partial_{\boldsymbol{\omega}}\, \mathcal{SU}^\bullet = -\mathcal{SU}^\bullet \times \mathcal{U}^\bullet - \square\, \mathcal{SU}^\bullet \tag{334}$$
$$\partial_{\boldsymbol{\omega}^\star}\, \mathcal{SU}^\bullet = +{}^{\#}\mathcal{US}^\bullet \times (\square^\star\, {}^{\#}\mathcal{SU}^\bullet) \times \mathcal{SU}^\bullet \tag{335}$$

6.7.13 A glimpse of the D-friendly perilogarithms.

Their inductive definition is :

$$(z\, \partial_z + z\, \square - z^{-1}\, \square^\star)\; \mathcal{V}a^\bullet(z) = -\mathcal{V}a^\bullet(z) \times Ja^\bullet(z) \tag{336}$$

with an elementary, one-component mould $Ja^\bullet$:

$$\begin{aligned} Ja^{\underline{\varpi}_1}(z) = Ja^{\binom{\varpi_1}{\tau_1}}(z) \;&:=\; 1 \quad if \quad \tau_1 = 0 \\ &:=\; cz^{-1} \quad if \quad \tau_1 = 1 \\ Ja^{\underline{\varpi}_1,\dots,\underline{\varpi}_r}(z) = Ja^{\binom{\varpi_1\ ,\dots,\ \varpi_r}{\tau_1\ ,\dots,\ \tau_r}}(z) \;&:=\; 0 \quad if \quad r \neq 1 \end{aligned}$$

6.7.14 From D- to Δ-friendly .

$$\mathcal{U}a^\bullet(z) = \mathcal{V}a^\bullet \circ U^\bullet \tag{337}$$

which is short-hand for

$$\mathcal{U}a^{\boldsymbol{\varpi}}(z) = \sum_{\left\{ {\boldsymbol{\varpi}^1 \dots \boldsymbol{\varpi}^s = \boldsymbol{\varpi} \atop s \geq 1\,,\, \tau_i \in \{0,1\}} \right\}} \mathcal{V}a^{\binom{\|\boldsymbol{\varpi}^1\|\ ,\dots,\ \|\boldsymbol{\varpi}^s\|}{\tau_1\ ,\dots,\ \tau_s}}(z)\;\; U^{\binom{\boldsymbol{\varpi}^1}{\tau_1}} \dots U^{\binom{\boldsymbol{\varpi}^s}{\tau_s}} \tag{338}$$

with

$$\begin{aligned} U^{\underline{\boldsymbol{\varpi}}} = U^{\binom{\boldsymbol{\varpi}}{\tau_0}} \;&:=\; U^{\boldsymbol{\varpi}} \quad if \quad \tau_0 = 0 \\ &:=\; UR^{\boldsymbol{\varpi}} \quad if \quad \tau_0 = 1 \end{aligned}$$

6.7.15 Resurgence equations.

$$\Delta\,\mathcal{U}^\bullet \;=\; I^\bullet \times \mathcal{U}^\bullet \qquad (\text{ with indices } \quad \varpi_i) \tag{339}$$

$$\Delta\,\mathcal{V}^\bullet \;=\; V^\bullet \times \mathcal{V}^\bullet \qquad (\text{ with indices } \quad \underline{\varpi}_i = (\begin{matrix}\varpi_i \\ \tau_i\end{matrix})) \tag{340}$$

The endearingly simple relation $I^\bullet = V^\bullet \circ U^\bullet$ connecting the Δ- and D-friendly hyperlogarithmic monics carries over to the perilogarithmic monics, but with doubled storeyed indices $\underline{\omega}_i = \binom{\omega_i}{\tau_i}$, with a double-storeyed $U^\bullet$ and a double-storeyed mould composition $\circ$ defined as above.

6.7.16 Paralogarithmic monomials and monics.

We now replace tha antipodal involution :

$$({}^{\#}M)^{\varpi_1,\dots,\varpi_r} := M^{c^2\bar{\varpi}_r,\dots,c^2\bar{\varpi}_1} \tag{341}$$

by the more convenient variant :

$$({}^{\natural}M)^{\varpi_1,\dots,\varpi_r} := M^{\bar{\varpi}_r,\dots,\bar{\varpi}_1} \tag{342}$$

and we get:

$$\begin{aligned}
{}^\sharp SSU^\bullet_c &= SSU^\bullet_c && (343)\\
{}^\sharp USS^\bullet_c &= USS^\bullet_c && (344)\\
{}^\sharp U^\bullet_c &= SSU^\bullet_c \times U^\bullet_c \times USS^\bullet_c && (345)\\
{}^\sharp UR^\bullet_c &= SSU^\bullet_c \times UL^\bullet_c \times USS^\bullet_c && (346)\\
{}^\sharp UL^\bullet_c &= SSU^\bullet_c \times UR^\bullet_c \times USS^\bullet_c && (347)
\end{aligned}$$

$$({}^\sharp\mathcal{U}a)^\bullet_c(z) \times (\mathcal{U}a)^\bullet_c(c^2/z) \equiv SSU^\bullet_c \qquad (348)$$

The integral formulae remain unchanged, except that the extreme factors $(y_1-1)^{-1}$ and $(1-y_r)^{-1}$ become $(y_1-c)^{-1}$ and $(c-y_r)^{-1}$. One should always integrate along the axes $Arg(\omega_i\, y_i) = Arg(\bar{\omega}_i/y_i) = 0$ and heed the "SPA" rules of mutual bypassing whenever several consecutive $Arg(\omega_i)$ coincide. The partial differentiation rules for the perilogarithms particularise to the paralogarithms.

PS. I wish to thank D. Sauzin for checking the formulae of §6.7.

6.8 Acceleration operators and acceleration kernels.

Acceleration/pseudoacceleration kernels:

Acceleration $z_1 \to z_2$ with $z_1 \ll z_2$ and $z_1 \equiv F(z_2)$:

$$\begin{aligned}
C_F(\zeta_2,\zeta_1) &:= \frac{1}{2\pi i}\int_{c-i\infty}^{c+\infty} e^{\,z_2\,\zeta_2 - z_1\,\zeta_1}\, dz_2\\
C^F(\zeta_2,\zeta_1) &:= \int_{+\infty}^{+u} e^{\,-z_2\,\zeta_2 + z_1\,\zeta_1}\, dz_2 \quad with \qquad u \gg 1
\end{aligned}$$

Pseudoacceleration $z_1 \to z_0$ with $z_0 = z_1 + F(z_1)$ and $1 \ll F(z) \ll z$:

$$\begin{aligned}
C_{I+F}(\zeta_1,\zeta_0) &:= C_F(\zeta_1 - \zeta 0, \zeta_0) \qquad (0 < \zeta_0 < \zeta_1)\\
C^{I+F}(\zeta_1,\zeta_0) &:= C^F(\zeta_1 - \zeta 0, \zeta_0) \qquad (0 < \zeta_0 < \zeta_1)
\end{aligned}$$

Acceleration integrals:

$$\begin{array}{rcl}
Acceleration\ of\ minors & : & infinite\ interval \\
\hat{\varphi}_2(\zeta_2) & = & \displaystyle\int_{+0}^{+\infty} C_F(\zeta_2,\zeta_1)\,\hat{\varphi}_1(\zeta_1)\,d\zeta_1 \\
Acceleration\ of\ majors & : & infinite\ loop \\
\check{\varphi}_2(\zeta_2) & = & \displaystyle\frac{1}{2\pi i}\int_{\infty_1}^{\infty_2} C^F(\zeta_2,\zeta_1)\,\check{\varphi}_1(\zeta_1)\,d\zeta_1 \\
Deceleration\ of\ minors & : & finite\ loop \\
\zeta_1\hat{\varphi}_1(\zeta_1) & = & \displaystyle\frac{1}{2\pi i}\int_{0_1}^{0_2} \zeta_2\,\hat{\varphi}_2(\zeta_2)\,C^F(\zeta_2,\zeta_1)\,d\zeta_2 \\
Deceleration\ of\ majors & : & finite\ interval \\
\zeta_1\check{\varphi}_1(\zeta_1) & = & \displaystyle\int_{+0}^{+v} \zeta_2\,\check{\varphi}_2(\zeta_2)\,C_F(\zeta_2,\zeta_1)\,d\zeta_2
\end{array}$$

Pseudoacceleration integrals:

$$\begin{array}{rcl}
Pseudodeceler.\ of\ minors & : & finite\ interval \\
\hat{\varphi}_1(\zeta_1) & = & \displaystyle\int_{0}^{\zeta_1} C_{I+F}(\zeta_1,\zeta_0)\hat{\varphi}_0(\zeta_0)\,d\zeta_0 \\
 & = & \displaystyle\int_{0}^{\zeta_1} C_F(\zeta_1-\zeta_0,\zeta_0)\hat{\varphi}_0(\zeta_0)\,d\zeta_0 \\
Pseudodeceler.\ of\ majors & : & finite\ loop \\
\check{\varphi}_1(\zeta_1) & = & \displaystyle\frac{1}{2\pi i}\int_{v_1}^{v_2} C^{I+F}(\zeta_1,\zeta_0)\check{\varphi}_0(\zeta_0)\,d\zeta_0 \\
 & = & \displaystyle\frac{1}{2\pi i}\int_{v_1}^{v_2} C^F(\zeta_1-\zeta_0,\zeta_0)\check{\varphi}_0(\zeta_0)\,d\zeta_0 \\
Pseudoacceler.\ of\ minors & : & finite\ loop \\
\zeta_0\hat{\varphi}_0(\zeta_0) & = & \displaystyle\frac{1}{2\pi i}\int_{0_1}^{0_2} \zeta_1\hat{\varphi}_1(\zeta_1)C^{I+F}(\zeta_1,\zeta_0)\,d\zeta_1 \\
 & = & \displaystyle\frac{1}{2\pi i}\int_{0_1}^{0_2} \zeta_1\hat{\varphi}_1(\zeta_1)C^F(\zeta_1-\zeta_0,\zeta_0)d\zeta_1 \\
Pseudoacceler.\ of\ majors & : & finite\ interval \\
\zeta_0\check{\varphi}_0(\zeta_0) & = & \displaystyle\int_{\zeta_0}^{v} \zeta_1\check{\varphi}_1(\zeta_1)C_{I+F}(\zeta_1,\zeta_0)\,d\zeta_1 \\
 & = & \displaystyle\int_{\zeta_0}^{v} \zeta_1\check{\varphi}_1(\zeta_1)C_F(\zeta_1-\zeta_0,\zeta_0)\,d\zeta_1
\end{array}$$

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