Topics in metric number theory

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CHAPTER 1

Elementary Diophantine approximation

1.1. Very well approximable numbers

Diophantine approximation is originally concerned with the approximation of real numbers by rational numbers or, more generally, the approximations of points in $\mathbb{R}^d$ by points with integer coordinates. The first result on this topic is due to Dirichlet and is a simple consequence of the pigeon-hole principle. In the statement, $|\cdot|_\infty$ stands for the supremum norm in $\mathbb{R}^d$.

THEOREM 1.1 (Dirichlet, 1842). *Let us consider a point $x \in \mathbb{R}^d$. Then, for any integer $Q > 1$, the system*

$$\begin{cases} 1 \leq q < Q^d \\ |qx - p|_\infty \leq 1/Q \end{cases}$$

admits a solution (p, q) in $\mathbb{Z}^d \times \mathbb{N}$.

PROOF. As mentioned above, this is an illustration of the pigeon-hole principle. Let us consider the points

$$0, 1, \{x\}, \{2x\}, \ldots, \{(Q^d - 1)x\},$$

where $\{\cdot\}$ denotes the coordinate-wise fractional part, and 1 is the point whose all coordinates are equal to one. These points all lie in the unit cube $[0, 1]^d$, which we may decompose as the disjoint union over $u_1, \ldots, u_d \in \{0, \ldots, Q - 1\}$ of the cubes

$$\prod_{i=1}^{d} \left[\frac{u_i}{Q}, \frac{u_i + 1}{Q} \right\rangle,$$

where $\rangle$ stands for the symbol $]$ if $u_i = Q - 1$, and for the symbol $)$ otherwise; in other words, the interval is closed if and only if $u_i = Q - 1$.

There are Q^d such subcubes, and $Q^d + 1$ points. Thus, the pigeon-hole principle ensures that there is at least one subcube that contains two of the points. As a result, there exist either two integers distinct integers r_1 and r_2 between zero and $Q^d - 1$ such that $\{r_1 x\}$ and $\{r_2 x\}$ are in the same subcube, or one integer r_2 between one and $Q^d - 1$ such that $\{r_2 x\}$ and 1 belong to the same subcube. In both cases, we deduce that there exist two integers r_1 and r_2 satisfying $0 \leq r_1 < r_2 < Q^d$, and two points with integers coordinates s_1 and s_2 in $\mathbb{Z}^d$ such that

$$|(r_1 x - s_1) - (r_2 x - s_2)|_\infty \leq \frac{1}{Q}.$$

The result now follows from letting $q = r_2 - r_1$ and $p = s_2 - s_1$. □

Theorem 1.1 means that the d real numbers $x_1, \ldots, x_d$ may simultaneously be approximated at a distance at most $1/Q$ by d rational numbers with common denominator an integer less than Q^d, namely, the rationals $p_1/q, \ldots, p_d/q$. In what follows, $\mathbb{P}_d$ is the set defined by

$$\mathbb{P}_d = \{(p, q) \in \mathbb{Z}^d \times \mathbb{N} \mid \gcd(p, q) = 1\},$$

where $\gcd(p,q)$ denotes the greatest common divisor of q and all the coordinates of the integer point p.

COROLLARY 1.1. *For any point $x \in \mathbb{R}^d \setminus \mathbb{Q}^d$, there exist infinitely many pairs $(p,q) \in \mathbb{P}_d$ such that*

$$\left| x - \frac{p}{q} \right|_\infty < \frac{1}{q^{1+1/d}}.$$

PROOF. For any point $x \in \mathbb{R}^d \setminus \mathbb{Q}^d$, let us consider the set

$$\mathcal{E}_x = \left\{ (p,q) \in \mathbb{P}_d \;\middle|\; \left| x - \frac{p}{q} \right|_\infty < \frac{1}{q^{1+1/d}} \right\}$$

and, for any integer $Q > 1$, the set

$$\mathcal{E}_x(Q) = \left\{ (p,q) \in \mathbb{Z}^d \times \mathbb{N} \;\middle|\; q < Q^d \text{ and } |qx - p|_\infty \leq \frac{1}{Q} \right\}.$$

Theorem 1.1 ensures that the sets $\mathcal{E}_x(Q)$ are all nonempty. Moreover, the mapping $(p,q) \mapsto (p,q)/\gcd(p,q)$ sends the sets $\mathcal{E}_x(Q)$ into $\mathcal{E}_x$, and reduces the value of $|qx - p|_\infty$. Thus,

$$\inf_{(p,q)\in\mathcal{E}_x} |qx - p|_\infty \leq \inf_{(p,q)\in\mathcal{E}_x(Q)} |qx - p|_\infty \leq \frac{1}{Q}.$$

Letting $Q \to \infty$, we deduce that the infimum of $|qx - p|_\infty$ over $(p,q) \in \mathcal{E}_x$ vanishes. Since x has no rational coordinates, this implies that $\mathcal{E}_x$ is necessarily infinite. $\square$

Corollary 1.1 ensures that for any point $x \in \mathbb{R}^d$, the Diophantine inequality $|x - p/q|_\infty < 1/q^{1+1/d}$ holds infinitely often. In other words, the set

$$J_{d,\tau} = \left\{ x \in \mathbb{R}^d \;\middle|\; \left| x - \frac{p}{q} \right|_\infty < \frac{1}{q^\tau} \quad \text{for i.m. } (p,q) \in \mathbb{Z}^d \times \mathbb{N} \right\} \tag{1}$$

is equal to the whole space $\mathbb{R}^d$ as soon as $\tau \leq 1 + 1/d$. In the above formula, i.m. stands for "infinitely many". Note that the mapping $\tau \mapsto J_{d,\tau}$ is nonincreasing; this enables us to introduce the following definition.

DEFINITION 1.1. Let us consider a point $x \in \mathbb{R}^d \setminus \mathbb{Q}^d$. The *irrationality exponent* of x is defined by

$$\tau(x) = \sup\{\tau \in \mathbb{R} \mid x \in J_{d,\tau}\} \geq 1 + \frac{1}{d}. \tag{2}$$

The point x is called *very well approximable* if its irrationality exponent satisfies

$$\tau(x) > 1 + \frac{1}{d}.$$

The set of very well approximable points is denoted by Well_d.

It is clear from the above definition that the irrationality exponent reflects the quality with which the points in $\mathbb{R}^d \setminus \mathbb{Q}^d$ are approximated by those with rational coordinates: the higher the exponent, the better the approximation. Besides, observe that the set of very well approximable points satisfies

$$\mathrm{Well}_d = (\mathbb{R}^d \setminus \mathbb{Q}^d) \cap \bigcup_{\tau > 1+1/d} J_{d,\tau}. \tag{3}$$

The main purpose of the metric theory of Diophantine approximation is then to describe the size properties of sets such as $J_{d,\tau}$, or generalizations thereof, in the case of course where they do not coincide with the whole space $\mathbb{R}^d$. To this purpose, the most basic tool, but also the less precise one, is the Lebesgue measure. As regards the specific case of the sets $J_{d,\tau}$, and their companion set Well_d, we plainly have the following result. The Lebesgue measure in $\mathbb{R}^d$ is denoted by $\mathcal{L}^d$ in what follows; we refer to Section 2.5 for its construction and its main properties.

PROPOSITION 1.1. *The set* Well_d *of very well approximable points has Lebesgue measure zero, that is,*

$$\mathcal{L}^d(\mathrm{Well}_d) = 0.$$

Equivalently, we also have

$$\forall \tau > 1 + \frac{1}{d} \qquad \mathcal{L}^d(J_{d,\tau}) = 0.$$

PROOF. The proof is elementary, and amounts to using an appropriate covering of the set $J_{d,\tau}$. To be specific, for any integer $Q \geq 1$, we have

$$J_{d,\tau} \cap [0,1]^d \subseteq \bigcup_{q \geq Q} \bigcup_{p \in \{0,\ldots,q\}^d} \mathrm{B}_\infty\left(\frac{p}{q}, \frac{1}{q^\tau}\right),$$

where $\mathrm{B}_\infty(x,r)$ denotes the open ball centered at x with radius r, in the sense of the supremum norm. As a result,

$$\mathcal{L}^d(J_{d,\tau} \cap [0,1]^d) \leq \sum_{q \geq Q} (q+1)^d \left(\frac{2}{q^\tau}\right)^d$$

The above series clearly converges when $\tau > 1 + 1/d$. Letting $Q \to \infty$, we deduce that the Lebesgue measure of $J_{d,\tau} \cap [0,1]^d$ vanishes. The set $J_{d,\tau}$ being invariant under the action of $\mathbb{Z}^d$, its Lebesgue measure thus vanishes in the whole space.

To establish that the set Well_d has Lebesgue measure zero as well, it suffices to observe that the union in (3) may be indexed by a countable dense subset of values of τ, because of the monotonicity of the sets $J_{d,\tau}$ with respect to τ. More precisely, letting for instance $\tau_n = (1 + 1/d) + 1/n$, we may write that

$$\mathcal{L}^d(\mathrm{Well}_d) \leq \mathcal{L}^d\left(\bigcup_{n=1}^{\infty} J_{d,\tau_n}\right) \leq \sum_{n=1}^{\infty} \mathcal{L}^d(J_{d,\tau_n}) = 0.$$

Finally, knowing that Well_d has Lebesgue measure zero, we can easily recover the fact that the sets $J_{d,\tau}$, for $\tau > 1 + 1/d$, all have Lebesgue measure zero as well. It suffices to make use of (3) again, and to recall that the set $\mathbb{Q}^d$ of points with rational coordinates is countable and therefore Lebesgue null. $\square$

It readily follows from Proposition 1.1 that, in the sense of Lebesgue measure, the irrationality exponent is minimal almost everywhere, that is,

$$\text{for } \mathcal{L}^d\text{-a.e. } x \in \mathbb{R}^d \setminus \mathbb{Q}^d \qquad \tau(x) = 1 + \frac{1}{d}, \tag{4}$$

where a.e. means "almost every". Moreover, as shown by Proposition 1.1, describing the size of the sets $J_{d,\tau}$ in terms of Lebesgue measure only is not very precise, as we just have the following dichotomy:

$$\begin{cases} \tau \leq 1 + 1/d \implies \mathcal{L}^d(\mathbb{R}^d \setminus J_{d,\tau}) = 0 \\ \tau > 1 + 1/d \implies \mathcal{L}^d(J_{d,\tau}) = 0. \end{cases}$$

A standard way of giving a more precise description is then to compute the Hausdorff dimension of the set $J_{d,\tau}$; this will be performed in Section 3.1 below.

1.2. Continued fractions

Throughout this section, we consider the one-dimensional case, thus assuming that $d = 1$. In that situation, we know from Corollary 1.1 that an arbitrary irrational number x may be approximated with precision at most $1/q^2$ by a sequence of rationals p/q; the *optimal* rational approximates p/q of x may then be computed through the continued fraction algorithm that we now discuss. The material developed in this section is very classical; our main references are [**24**, Chapter 3] and [**55**, Chapter 1].

1.2.1. Continued fraction expansions.

1.2.1.1. *Synthesis: from partial quotients to continued fractions.* Let a_0 be a nonnegative integer and, for any $n \in \mathbb{N}$, let a_n be a positive integer. The continued fraction associated with the sequence $(a_n)_{n\geq 0}$ is defined by

$$[a_0; a_1, a_2, a_3, \ldots] = a_0 + \cfrac{1}{a_1 + \cfrac{1}{a_2 + \cfrac{1}{a_3 + \ldots}}}. \tag{5}$$

At the moment, this definition is purely formal; we shall give it a rigorous sense later, see (12). In addition, we shall consider the finite fraction associated with the integers $a_0, \ldots, a_n$, namely,

$$[a_0; a_1, a_2, \ldots, a_n] = a_0 + \cfrac{1}{a_1 + \cfrac{1}{a_2 + \ldots + \cfrac{1}{a_{n-1} + \cfrac{1}{a_n}}}}. \tag{6}$$

In particular, using the above notation, we clearly have, for any choice of the integers $a_0, \ldots, a_n$,

$$[a_0; a_1, a_2, \ldots, a_n] = a_0 + \frac{1}{[a_1; a_2, \ldots, a_n]}.$$

The integers a_n are called the *partial quotients* of the continued fraction. Moreover, the irreducible rational numbers p_n/q_n defined by

$$\frac{p_n}{q_n} = [a_0; a_1, a_2, \ldots, a_n] \tag{7}$$

are called the *convergents* of the continued fraction. The next lemma gives an expression of the numerator and the denominator of the convergents in terms of the partial quotients.

LEMMA 1.1. *For any nonnegative integer a_0, and any sequence of positive integers $a_1, a_2, \ldots$, the irreducible rational numbers p_n/q_n defined by (7) satisfy*

$$\forall n \geq 0 \qquad \begin{pmatrix} p_n & p_{n-1} \\ q_n & q_{n-1} \end{pmatrix} = \begin{pmatrix} a_0 & 1 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} a_1 & 1 \\ 1 & 0 \end{pmatrix} \cdots \begin{pmatrix} a_n & 1 \\ 1 & 0 \end{pmatrix}. \tag{8}$$

with the convention that $p_{-1} = 1$, $q_{-1} = 0$, $p_0 = a_0$ and $q_0 = 1$.

PROOF. The lemma may be proven by induction. In view of the adopted conventions, the formula (8) is clearly true for $n = 0$. Moreover, let us assume that (8) holds up to $n = m$, regardless of the choice of the $m+1$ integers $a_0, \ldots, a_m$. Then, let us consider $m + 2$ integers denoted by $a_0, \ldots, a_{m+1}$; we need to prove that (8) holds for these integers, and for $n = m + 1$.

To this end, let us apply (8) to the $m+1$ integers $a_1, \ldots, a_{m+1}$. Thus,

$$\begin{pmatrix} p & p' \\ q & q' \end{pmatrix} = \begin{pmatrix} a_1 & 1 \\ 1 & 0 \end{pmatrix} \cdots \begin{pmatrix} a_{m+1} & 1 \\ 1 & 0 \end{pmatrix}.$$

where p/q and p'/q' respectively denote the irreducible rational numbers equal to $[a_1; a_2, \ldots, a_{m+1}]$ and $[a_1; a_2, \ldots, a_m]$. On the one hand, we deduce that

$$\begin{pmatrix} a_0 & 1 \\ 1 & 0 \end{pmatrix} \cdots \begin{pmatrix} a_{m+1} & 1 \\ 1 & 0 \end{pmatrix} = \begin{pmatrix} a_0 & 1 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} p & p' \\ q & q' \end{pmatrix} = \begin{pmatrix} a_0 p + q & a_0 p' + q' \\ p & p' \end{pmatrix}.$$

On the other hand, $a_0 p + q$ and p are coprime, and their quotient is equal to

$$\frac{a_0 p + q}{p} = a_0 + \frac{q}{p} = a_0 + \frac{1}{[a_1; a_2, \ldots, a_{m+1}]} = [a_0; a_1, \ldots, a_{m+1}].$$

Likewise, $a_0 p' + q'$ and p' are coprime and their quotient is equal to the fraction $[a_0; a_1, \ldots, a_m]$. This means that (8) holds for $n = m+1$, with the integers $a_0, \ldots, a_{m+1}$. □

It directly follows from (8) that for any integer $n \geq 0$,

$$\begin{pmatrix} p_{n+1} & p_n \\ q_{n+1} & q_n \end{pmatrix} = \begin{pmatrix} p_n & p_{n-1} \\ q_n & q_{n-1} \end{pmatrix} \begin{pmatrix} a_{n+1} & 1 \\ 1 & 0 \end{pmatrix},$$

from which we deduce the next recursive formulas for the convergents:

$$\begin{cases} p_{n+1} = a_{n+1} p_n + p_{n-1} \\ q_{n+1} = a_{n+1} q_n + q_{n-1}. \end{cases} \tag{9}$$

In particular, since $a_n \geq 1$ for all $n \geq 1$, it is easy to establish by induction that the numerators p_n and the denominators q_n of the convergents are at least $2^{(n-2)/2}$, for all integers $n \geq 1$. Furthermore, taking the determinant in (8), we readily obtain

$$p_n q_{n-1} - p_{n-1} q_n = (-1)^{n+1}, \tag{10}$$

so that

$$\frac{p_n}{q_n} - \frac{p_{n-1}}{q_{n-1}} = \frac{(-1)^{n+1}}{q_{n-1} q_n}. \tag{11}$$

As a result, the convergents p_n/q_n have a finite limit when $n \to \infty$, namely,

$$[a_0; a_1, a_2, \ldots, a_n] = \frac{p_n}{q_n} \xrightarrow[n \to \infty]{} a_0 + \sum_{n=0}^{\infty} \frac{(-1)^n}{q_n q_{n+1}}. \tag{12}$$

This means that the formula (5) is not merely formal, but defines a true real number that corresponds to

$$x = [a_0; a_1, a_2, a_3, \ldots] = a_0 + \sum_{n=0}^{\infty} \frac{(-1)^n}{q_n q_{n+1}}.$$

Then, $[a_0; a_1, a_2, a_3, \ldots]$ is called the *continued fraction expansion* of x.

Note that the above series converges because it satisfies the alternating series test. Indeed, it is clear that the sequence $(q_n q_{n+1})_{n \geq 0}$ monotonically diverges to infinity. (In fact, the series is also absolutely convergent, since $q_n q_{n+1} \geq 2^{n-3/2}$ for all $n \geq 1$.) Thus, the even terms p_{2m}/q_{2m} increase to x, while the odd terms p_{2m+1}/q_{2m+1} decrease to x, and moreover

$$\forall n \geq 0 \qquad \left| x - \frac{p_n}{q_n} \right| < \frac{1}{q_n q_{n+1}} \leq \frac{1}{q_n^2}, \tag{13}$$

where the latest inequality is due to the fact that the sequence $(q_n)_{n \geq 0}$ is nondecreasing. This means that the convergents of the continued fraction expansion of x yield a sequence of irreducible rational numbers p_n/q_n that approximate x with an error smaller than $1/q_n^2$. This is clearly in accordance with Theorem 1.1.

Note in passing that x is necessarily irrational. As a matter of fact, let us assume that x can be written as an irreducible fraction of the form p/q. Then,

$$\forall n \geq 0 \qquad |pq_n - p_n q| < \frac{q}{q_n}.$$

Thus, as $q_n \to \infty$, the integer $pq_n - p_n q$ necessarily vanishes for n large enough. In view of the coprimeness of p and q, and that of p_n and q_n, this implies that $p_n = p$ and $q_n = q$ for n large enough, which contradicts the fact that $q_n \to \infty$. We shall show in Section 1.2.1.2 below that, conversely, any irrational real number has a continued fraction expansion, and this expansion is unique.

1.2.1.2. *Analysis: continued fraction expansion of an irrational number.* Let us begin by establishing the uniqueness of the continued fraction expansion; this is the purpose of the next proposition.

PROPOSITION 1.2. *The following mapping is injective:*

$$\begin{array}{ccc} \mathbb{N}_0 \times \mathbb{N}^{\mathbb{N}} & \longrightarrow & (0,\infty) \\ (a_n)_{n\geq 0} & \longmapsto & [a_0; a_1, a_2, a_3, \ldots]. \end{array}$$

PROOF. Note that a continued fraction expansion is clearly always positive, and recall the inductive relation

$$[a_0; a_1, a_2, a_3, \ldots] = a_0 + \frac{1}{[a_1; a_2, a_3, \ldots]} = a_0 + \cfrac{1}{a_1 + \cfrac{1}{[a_2; a_3, \ldots]}}$$

Thus, letting x denote the left-hand side above, we have

$$a_0 < x < a_0 + \frac{1}{a_1} \leq a_0 + 1,$$

so that x uniquely determines a_0. Applying the above argument to

$$[a_1; a_2, a_3, \ldots] = \frac{1}{x - a_0},$$

we deduce that x also uniquely determines a_1. We can clearly iterate this procedure; this shows that x uniquely determines all the integers a_n. $\square$

The procedure employed in the above proof suggests a way of computing the continued fraction expansion of a given irrational number. Let us first consider the irrational numbers between zero and one. Specifically, let us define the set $X = [0,1) \setminus \mathbb{Q}$ and the mapping T from X onto itself given by

$$T(x) = \left\{ \frac{1}{x} \right\} \tag{14}$$

for all $x \in X$. The mapping T is called the *Gauss map*, or *continuous fraction map*. The Gauss map enables one to compute the continued fraction expansion of an irrational number in X. As a matter of fact, for any irrational number $x \in X$ and any integer $n \geq 1$, let us define

$$a_n(x) = \left\lfloor \frac{1}{T^{n-1}(x)} \right\rfloor, \tag{15}$$

where $\lfloor \cdot \rfloor$ denotes integer part. Moreover, for any sequence $(a_n)_{n\geq 1}$ of positive integers, let

$$[a_1, a_2, \ldots] = [0; a_1, a_2, \ldots]\,;$$

this is merely the continued fraction defined by (5) with partial quotient a_0 equal to zero, and thus belonging to $[0,1)$. We then have the following result.

PROPOSITION 1.3. *For any irrational number $x \in X$, we have the following continued fraction expansion*

$$x = [a_1(x), a_2(x), \ldots].$$

PROOF. Let us prove by induction on $n \geq 0$ that for any irrational $x \in X$,

$$[a_1(x), \ldots, a_{2n}(x)] < x < [a_1(x), \ldots, a_{2n+1}(x)]. \tag{16}$$

When $n = 0$, this amounts to proving that $0 < x < 1/a_1(x)$, which readily follows from the definition of $a_1(x)$. Let us suppose that the result holds for a given integer $n \geq 0$ and for all $x \in X$. Then, applying this result to $T(x)$ instead of x, we obtain in particular

$$T(x) < [a_1(T(x)), \ldots, a_{2n+1}(T(x))],$$

which gives

$$\frac{1}{x} - a_1(x) < [a_2(x), \ldots, a_{2(n+1)}(x)],$$

that is,

$$x > [a_1(x), \ldots, a_{2(n+1)}(x)]\,;$$

this is the lower bound in (16) with $n+1$ instead of n. Replacing x by $T(x)$ again in the above inequality, and repeating the procedure, we also get

$$x < [a_1(x), \ldots, a_{2(n+1)+1}(x)],$$

which is the upper bound in (16) with $n+1$ instead of n. Finally, (16) holds for all $n \geq 0$ and all $x \in X$. To conclude, it suffices to recall that the both bounds in (16) both converge to the continued fraction $[a_1(x), a_2(x), \ldots]$. □

We may now give the continued fraction expansion of an irrational number that does not necessarily belong to the interval $[0, 1)$. If x denotes a positive irrational number, its fractional part $\{x\}$ then belongs to X, and we may extend (15) by letting

$$a_n(x) = a_n(\{x\})$$

for any integer $n \geq 1$. In addition, let us define $a_0(x)$ as the integer part $\lfloor x \rfloor$. We now deduce that

$$x = \lfloor x \rfloor + \{x\} = a_0(x) + [a_1(x), a_2(x), \ldots] = [a_0(x); a_1(x), a_2(x), \ldots], \tag{17}$$

as an immediate consequence of Proposition 1.3.

1.2.2. Implications for Diophantine approximation.

1.2.2.1. *Better rational approximants.* Let x be an irrational number with continued fraction expansion $[a_0; a_1, a_2, \ldots]$ as above and let p_n/q_n denote the corresponding convergents, defined by (7). Due to (13) and in accordance with Theorem 1.1, these convergents yield a sequence of irreducible rational numbers p_n/q_n that approximate x with an error smaller than $1/q_n^2$. This property can be improved by the next two results.

PROPOSITION 1.4 (Vahlen, 1895). *Let x be an irrational number with continued fraction expansion $[a_0; a_1, a_2, \ldots]$, and let p_n/q_n denote the corresponding convergents. For any fixed integer $n \geq 0$, at least one among the two convergents p_n/q_n and p_{n+1}/q_{n+1} satisfies*

$$\left| x - \frac{p}{q} \right| < \frac{1}{2q^2}.$$

Proof. We begin by observing that

$$\left|\frac{p_{n+1}}{q_{n+1}} - x\right| + \left|x - \frac{p_n}{q_n}\right| = \left|\frac{p_{n+1}}{q_{n+1}} - \frac{p_n}{q_n}\right|. \tag{18}$$

In fact, as the convergents tend to the limit x in an alternating manner, the three terms above all have the same sign, so that we can remove the absolute values around them, thus ending with a trivial equality. Using (11) and the fact that $uv < (u^2+v^2)/2$ for any distinct real numbers u and v, we deduce that

$$\left|\frac{p_{n+1}}{q_{n+1}} - x\right| + \left|x - \frac{p_n}{q_n}\right| = \frac{1}{q_n q_{n+1}} < \frac{1}{2q_{n+1}^2} + \frac{1}{2q_n^2},$$

and the result follows. □

Before stating the second improvement on the approximation property (13), let us point out a useful relationship between a given continued fraction expansion $x = [a_0; a_1, a_2, \ldots]$ and its n-th tail defined by $x_n = [a_n; a_{n+1}, a_{n+2}, \ldots]$. For any $k \geq 0$, Lemma 1.1 ensures that

$$\begin{aligned}\begin{pmatrix} p_{n+k} \\ q_{n+k} \end{pmatrix} &= \begin{pmatrix} a_0 & 1 \\ 1 & 0 \end{pmatrix} \cdots \begin{pmatrix} a_n & 1 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} a_{n+1} & 1 \\ 1 & 0 \end{pmatrix} \cdots \begin{pmatrix} a_{n+k} & 1 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} 1 \\ 0 \end{pmatrix} \\ &= \begin{pmatrix} p_n & p_{n-1} \\ q_n & q_{n-1} \end{pmatrix} \begin{pmatrix} p_{k-1}(x_{n+1}) & p_{k-2}(x_{n+1}) \\ q_{k-1}(x_{n+1}) & q_{k-2}(x_{n+1}) \end{pmatrix} \begin{pmatrix} 1 \\ 0 \end{pmatrix},\end{aligned}$$

where $p_k(x_{n+1})/q_k(x_{n+1})$ denotes the k-th convergent to the $(n+1)$-th tail. It follows that

$$\frac{p_{n+k}}{q_{n+k}} = \frac{p_n p_{k-1}(x_{n+1}) + p_{n-1} q_{k-1}(x_{n+1})}{q_n p_{k-1}(x_{n+1}) + q_{n-1} q_{k-1}(x_{n+1})}.$$

Letting k go to infinity, we finally deduce that

$$x = \frac{p_n x_{n+1} + p_{n-1}}{q_n x_{n+1} + q_{n-1}}. \tag{19}$$

This formula will come into play in the proof of the following improvement on (13).

Proposition 1.5 (Borel, 1903). *Let x be an irrational number with continued fraction expansion $[a_0; a_1, a_2, \ldots]$, and let p_n/q_n denote the corresponding convergents. For any fixed integer $n \geq 0$, at least one among the three convergents p_n/q_n, p_{n+1}/q_{n+1} and p_{n+2}/q_{n+2} satisfies*

$$\left|x - \frac{p}{q}\right| < \frac{1}{\sqrt{5}q^2}.$$

Proof. Let x_{n+1} denote the $(n+1)$-th tail of the continued fraction expansion of x. Then, owing to (10) and (19), we have

$$q_n x - p_n = q_n \frac{p_n x_{n+1} + p_{n-1}}{q_n x_{n+1} + q_{n-1}} - p_n = \frac{(-1)^n}{q_n x_{n+1} + q_{n-1}}. \tag{20}$$

As a consequence, letting β_n denote the ratio q_{n-1}/q_n, we have

$$q_n |q_n x - p_n| = \frac{1}{x_{n+1} + \beta_n}.$$

The proof now reduces to establishing that at least one among the three real numbers $x_{n+1} + \beta_n$, $x_{n+2} + \beta_{n+1}$ and $x_{n+3} + \beta_{n+2}$ is larger than $\sqrt{5}$.

Let us assume that $x_{n+1} + \beta_n$ and $x_{n+2} + \beta_{n+1}$ are both bounded above by $\sqrt{5}$. Note that $x_{n+1} = a_{n+1} + 1/x_{n+2}$ and, in view of (9),

$$\frac{1}{\beta_{n+1}} = \frac{q_{n+1}}{q_n} = \frac{a_{n+1} q_n + q_{n-1}}{q_n} = a_{n+1} + \beta_n, \tag{21}$$

from which we deduce that $1/x_{n+2} + 1/\beta_{n+1} = x_{n+1} + \beta_n$. The supposed bounds on $x_{n+1} + \beta_n$ and $x_{n+2} + \beta_{n+1}$ then imply that

$$1 = x_{n+2} \cdot \frac{1}{x_{n+2}} \leq \left(\sqrt{5} - \beta_{n+1}\right)\left(\sqrt{5} - \frac{1}{\beta_{n+1}}\right),$$

which means that the polynomial $Z^2 - \sqrt{5}Z + 1$ takes a nonpositive value when evaluated at β_{n+1}. In particular, β_{n+1} is larger than or equal to the smallest root of this polynomial. However, β_{n+1} is rational, so the inequality is strict, specifically,

$$\beta_{n+1} > \frac{\sqrt{5} - 1}{2}.$$

Likewise, assuming that $x_{n+2} + \beta_{n+1}$ and $x_{n+3} + \beta_{n+2}$ are both bounded above by $\sqrt{5}$ leads to the same lower bound on β_{n+2}. Using (21) with $n+1$ instead of n, along with the above bounds, we then conclude that

$$1 \leq a_{n+2} = \frac{1}{\beta_{n+2}} - \beta_{n+1} < \frac{2}{\sqrt{5} - 1} - \frac{\sqrt{5} - 1}{2} = 1,$$

which is a contradiction. □

The next result shows that, conversely, an approximation result that beats (13) is necessarily realized by some convergent.

Proposition 1.6 (Legendre). *Let x be an irrational real number with continued fraction expansion $[a_0; a_1, a_2, \ldots]$, and let p_n/q_n denote the corresponding convergents. Then, for any pair of coprime integers $(p, q) \in \mathbb{P}_1$,*

$$\left| x - \frac{p}{q} \right| < \frac{1}{2q^2} \qquad \Longrightarrow \qquad \exists n \geq 0 \quad \frac{p}{q} = \frac{p_n}{q_n}.$$

Proof. Let (p, q) denote a pair in $\mathbb{P}_1$ such that $|x - p/q| < 1/(2q^2)$. Then, there exist $\varepsilon \in \{-1, 1\}$ and $\theta \in (0, 1/2)$ such that

$$x - \frac{p}{q} = \frac{\varepsilon\theta}{q^2}.$$

Moreover, it is easy to prove by induction on q that the rational number p/q has exactly two finite continued fraction expansions, specifically,

$$\frac{p}{q} = [c_0; c_1, \ldots, c_k] = [c_0; c_1, \ldots, c_{k-1}, c_k - 1, 1],$$

with $c_k \geq 2$ unless k is equal to zero, in which case p/q is an integer. Among these two representations, we may thus privilege that with odd length if $\varepsilon = 1$, and that with even length if $\varepsilon = -1$. This yields a decomposition of the form

$$\frac{p}{q} = [b_0; b_1, \ldots, b_n],$$

where $b_0 \in \mathbb{N}_0$, $a_1, \ldots, a_n \in \mathbb{N}$ and $n \geq 0$ is such that $(-1)^n = \varepsilon$. For $k \in \{0, \ldots, n\}$, let r_k/s_k denote the convergents corresponding to the above continued fraction expansion. In particular, $r_n/s_n = p/q$. As x is irrational, we may define

$$\omega = \frac{r_{n-1} - s_{n-1}x}{s_n x - r_n}.$$

Then, let us observe that, in view of (10),

$$\frac{\varepsilon\theta}{s_n^2} = \frac{\varepsilon\theta}{q^2} = x - \frac{p}{q} = \frac{r_n\omega + r_{n-1}}{s_n\omega + s_{n-1}} - \frac{r_n}{s_n} = \frac{(-1)^n}{s_n(s_n\omega + s_{n-1})}.$$

Solving for ω, we infer that

$$\omega = \frac{1}{\theta} - \frac{s_{n-1}}{s_n} > 2 - 1 = 1.$$

Furthermore, note that ω is irrational, so we may consider its continued fraction expansion, specifically,

$$\omega = [b_{n+1}; b_{n+2}, \ldots].$$

Since ω is larger than one, all its partial quotients are positive. This means that we may concatenate the continued fraction expansion of p/q with that of ω, thereby recovering x. As a matter of fact, owing to (19), we have

$$[b_0; b_1, \ldots, b_n, b_{n+1}, \ldots] = \frac{r_n\omega + r_{n-1}}{s_n\omega + s_{n-1}} = x.$$

As x is irrational, its continued fraction expansion is unique, see Proposition 1.2. In particular, $b_k = a_k$ for all $k \in \{0, \ldots, n\}$, so that $p/q = r_n/s_n = p_n/q_n$. □

1.2.2.2. *The golden ratio and Hurwitz's theorem.* The most simple example of continued fraction expansion is certainly that of the golden ratio

$$\phi = \frac{1+\sqrt{5}}{2}. \tag{22}$$

It is clear that $\phi - 1$ is equal to $1/\phi$ and belongs to the interval $(0, 1)$. Thus the partial quotients of the golden ratio are all equal to one, that is, its continued fraction expansion is given by

$$\phi = [1; 1, 1, \ldots] = 1 + \cfrac{1}{1 + \cfrac{1}{1 + \ldots}}.$$

Moreover, in view of (9) and the initial value of the convergents p_n/q_n, one easily checks that $p_n = f_{n+2}$ and $q_n = f_{n+1}$ for all $n \geq 0$, where $(f_n)_{n\geq 0}$ denotes the Fibonacci sequence, defined by the recursive relation $f_{n+2} = f_{n+1} + f_n$, along with the initial terms $f_0 = 0$ and $f_1 = 1$. It is then straightforward to establish Binet's formula, namely,

$$\forall n \geq 0 \qquad f_n = \frac{\phi^n - (-\phi)^{-n}}{\sqrt{5}}$$

Hence, the convergents p_n/q_n to the golden ratio ϕ satisfy

$$q_n(q_n\phi - p_n) = f_{n+1}(f_{n+1}\phi - f_{n+2}) = \frac{1}{\sqrt{5}}\left((-1)^n + \frac{1}{\phi^{2n}}\right).$$

As a consequence, we end up with

$$\phi - \frac{p_n}{q_n} \sim \frac{(-1)^n}{\sqrt{5}q_n^2} \tag{23}$$

as $n \to \infty$. The next result shows that the same property holds for any irrational number whose continued fraction expansion is ultimately constant equal to one.

Proposition 1.7. *Given $a_0 \in \mathbb{N}_0$ and $(a_1, \ldots, a_k) \in \mathbb{N}^k$, let x denote the irrational number with continued fraction expansion $[a_0; a_1, \ldots, a_k, 1, 1, \ldots]$, and let p_n/q_n denote the corresponding convergents. Then, as n goes to infinity,*

$$x - \frac{p_n}{q_n} \sim \frac{(-1)^n}{\sqrt{5}q_n^2}.$$

Proof. We adopt the same notations as in the proof of Proposition 1.5. In particular, recall that (20) yields

$$\frac{(-1)^n}{q_n(q_nx - p_n)} = x_{n+1} + \beta_n,$$

where x_{n+1} is $(n+1)$-th tail of the continued fraction expansion of x, and β_n is the ratio q_{n-1}/q_n. Note that x_{n+1} is equal to the golden ratio ϕ when $n \geq k$. Furthermore, β_n satisfies

$$\frac{1}{\beta_n} = \frac{q_n}{q_{n-1}} = [a_n; a_{n-1}, \ldots, a_{k+1}, a_k, \ldots, a_1, a_0] = [\underbrace{1; 1, \ldots, 1}_{n-k \text{ times}}, a_k, \ldots, a_1, a_0],$$

so that $1/\beta_n$ is between the two convergents of the form $[1; 1, \ldots, 1]$ whose lengths are $n-k-1$ and $n-k$. These convergents both tend to ϕ as $n \to \infty$. Finally,

$$x_{n+1} + \beta_n \xrightarrow[n\to\infty]{} \phi + \frac{1}{\phi} = \sqrt{5},$$

and the announced result follows. □

The above results lead to the following optimal refinement of the corollary to Dirichlet's theorem, namely, Corollary 1.1 in the one-dimensional case.

THEOREM 1.2 (Hurwitz, 1891). *For any irrational number x, there are infinitely many pairs $(p, q) \in \mathbb{P}_1$ such that*

$$\left| x - \frac{p}{q} \right| < \frac{1}{\sqrt{5} q^2}.$$

Moreover, this property does not hold when $\sqrt{5}$ is replaced by any larger constant.

PROOF. The first part of the theorem readily follows from applying Proposition 1.5 to the absolute value of x. In order to prove the optimality of the constant, let us assume that the inequality holds for all irrational number x and infinitely many pairs $(p, q) \in \mathbb{P}_1$, with $\sqrt{5}$ replaced by some larger constant A. In particular, applying this to the golden ratio yields an infinite number of coprime integers p and q such that $|\phi - p/q| < 1/(Aq^2)$. However, A is larger than two, so Proposition 1.6 ensures that p/q is a convergent to ϕ. Thus, there exists an increasing sequence $(n_k)_{k\geq 1}$ of nonnegative integers such that

$$\left| \phi - \frac{p_{n_k}}{q_{n_k}} \right| < \frac{1}{A q_{n_k}^2}$$

for all $k \geq 1$; this contradicts (23). □

For any real number x, let us define the exponent

$$\kappa(x) = \liminf_{q\to\infty} q \left\| qx \right\|, \tag{24}$$

where $\|y\|$ denotes the distance from a real y to the integers, that is, the infimum of $|y - p|$ over all $p \in \mathbb{Z}$. Note that $\kappa(x)$ clearly vanishes when x is rational; we shall see in Section 1.3 that this exponent also characterizes the *badly approximable* numbers. Moreover, Theorem 1.2 implies that $\kappa(x)$ is bounded above by $1/\sqrt{5}$, and its proof shows that the bound is attained by the golden ratio. Thus,

$$\sup_{x\in\mathbb{R}} \kappa(x) = \frac{1}{\sqrt{5}}.$$

In fact, Proposition 1.7 shows that the irrational numbers with continued fraction expansion ultimately equal to one also satisfy (23); this implies that they also attain the above bound. Furthermore, Hurwitz showed that the bound is attained by these numbers only; in fact, every irrational number x with infinitely many partial quotients strictly greater than one satisfies $\kappa(x) \leq 1/\sqrt{8}$, see [**55**, Theorem 6C].

1.2.2.3. *Optimality of the convergents.* The convergents p_n/q_n yield the optimal rational approximants to the irrational number x in the sense of Theorem 1.3 below. The proof of this result calls upon the following simple lemma.

LEMMA 1.2. *Let x be an irrational number with continued fraction expansion $[a_0; a_1, a_2, \ldots]$, and let p_n/q_n denote the corresponding convergents. Then, the sequence $(|q_n x - p_n|)_{n\geq 0}$ is decreasing.*

PROOF. In view of (11) and (13), we can deduce from (18) that

$$\frac{1}{q_n q_{n+1}} > \left| x - \frac{p_n}{q_n} \right| > \frac{1}{q_n q_{n+1}} - \frac{1}{q_{n+1} q_{n+2}} = \frac{q_{n+2} - q_n}{q_n q_{n+1} q_{n+2}} = \frac{a_{n+2}}{q_n q_{n+2}} ;$$

the latest equality follows from the recursive formula for q_n. As a_{n+2} is greater than or equal to one, this readily implies that

$$\frac{1}{q_{n+2}} < |q_n x - p_n| < \frac{1}{q_{n+1}}. \tag{25}$$

The result follows. □

We are now in position to show the optimality of the rational approximants supplied by the convergents.

THEOREM 1.3 (Lagrange, 1770). *Let x be an irrational number with continued fraction expansion $[a_0; a_1, a_2, \ldots]$, and let p_n/q_n denote the corresponding convergents. Then, for any integer $n \geq 1$ and any pair $(p, q) \in \mathbb{P}_1$ such that $0 < q \leq q_n$,*

$$\frac{p}{q} = \frac{p_n}{q_n} \qquad \text{or} \qquad |q_n x - p_n| < |qx - p|.$$

In the latter case, we also have

$$\left| x - \frac{p_n}{q_n} \right| < \left| x - \frac{p}{q} \right| .$$

PROOF. We begin by dealing with the elementary case where $q = q_n$. In that situation, if $p/q \neq p_n/q_n$, we deduce from (13) and the fact that $q_{n+1} \geq 2$ that

$$\left| x - \frac{p}{q} \right| \geq \left| \frac{p}{q} - \frac{p_n}{q_n} \right| - \left| x - \frac{p_n}{q_n} \right| \geq \frac{1}{q_n} - \frac{1}{q_n q_{n+1}} \geq \frac{1}{2q_n} \geq \frac{1}{q_n q_{n+1}} > \left| x - \frac{p_n}{q_n} \right| ,$$

which gives $|qx - p| > |q_n x - p_n|$.

Let us now assume that $q_{n-1} < q < q_n$. There are two integers a and b in $\mathbb{Z}$ such that

$$\begin{pmatrix} p_n & p_{n-1} \\ q_n & q_{n-1} \end{pmatrix} \begin{pmatrix} a \\ b \end{pmatrix} = \begin{pmatrix} p \\ q \end{pmatrix} .$$

Indeed, the above matrix has integer-valued entries and determinant ± 1, so its inverse exists and also has integer-valued entries. Note that the integers a and b are nonvanishing, as we would have $q \in \{q_{n-1}, q_n\}$ otherwise. Moreover, $q = aq_n + bq_{n-1} < q_n$, so that a and b must be of opposite signs. This is also the case of $q_n x - p_n$ and $q_{n-1} x - p_{n-1}$, because the convergents tend to x in an alternating manner. Thus, the products $a(q_n x - p_n)$ and $b(q_{n-1} x - p_{n-1})$ are of the same sign; their sum is equal to $qx - p$, and is also of the same sign. Therefore,

$$|qx - p| = |a(q_n x - p_n)| + |b(q_{n-1} x - p_{n-1})| > |q_n x - p_n|.$$

Thus, we have proven the result for any integer $n \geq 1$ and any integers p and q such that $\gcd(p, q) = 1$ and $q_{n-1} < q \leq q_n$.

Now, let us assume that $n \geq 1$ is fixed, and that $q_m < q \leq q_{m+1}$ for some integer $m \in \{0, \ldots, n-1\}$. Then, applying what precedes with $m+1$ instead of n, we deduce that for any integer p such that $\gcd(p, q) = 1$, we have either

$$\frac{p}{q} = \frac{p_{m+1}}{q_{m+1}} \qquad \text{or} \qquad |q_{m+1}x - p_{m+1}| < |qx - p|.$$

Given that $0 \leq m+1 < n$, we now deduce from Lemma 1.2 that

$$|q_n x - p_n| < |q_{m+1}x - p_{m+1}| \leq |qx - p|.$$

Finally, it remains to address the case where $q = q_0 = 1$. If $q_1 = 1$, then we may use the beginning of the present proof to infer that

$$p \neq p_1 \qquad \Longrightarrow \qquad |qx - p| > |q_1 x - p_1|,$$

so the result holds for $n = 1$. In particular, regardless of the value of p, the large inequality holds and Lemma 1.2 implies that for $n \geq 2$ and for any p,

$$|qx - p| \geq |q_1 x - p_1| > |q_n x - p_n|.$$

If $q_1 > 1$, then for any integer $p \neq p_0$, making use of (13), we have

$$|qx - p| \geq |p - p_0| - \left| x - \frac{p_0}{q_0} \right| > 1 - \frac{1}{q_1} \geq \frac{1}{2} \geq \frac{1}{q_1} > \left| x - \frac{p_0}{q_0} \right| = |q_0 x - p_0|,$$

so that regardless of the value of p, the left-hand side is greater than or equal to the right-hand side. The result follows from Lemma 1.2. $\square$

1.2.2.4. *Characterization of the irrationality exponent.* Recall that, according to Definition 1.1, the irrationality exponent of an irrational real number x is defined as the supremum of all reals τ such that the inequality $|x - p/q| < q^{-\tau}$ has infinitely many solutions $(p, q) \in \mathbb{Z} \times \mathbb{N}$. In addition, due to Corollary 1.1, the irrationality exponent of an irrational number is bounded below by two. The following result shows that the irrationality exponent directly depends on the growth rate of the denominators of the convergents.

Proposition 1.8. *Let x be an irrational number with convergents p_n/q_n. Then, the irrationality exponent of x satisfies*

$$\tau(x) = 1 + \limsup_{n \to \infty} \frac{\log q_{n+1}}{\log q_n}.$$

Proof. The right-hand side is clearly bounded below by two. Thus, in order to prove the upper bound on $\tau(x)$, we may assume that $\tau(x) > 2$. Then, for any real number τ strictly between two and $\tau(x)$, there are infinitely many pairs $(p, q) \in \mathbb{P}_1$ such that

$$\left| x - \frac{p}{q} \right| < \frac{1}{q^\tau} \leq \frac{1}{2q^2}.$$

Owing to Proposition 1.6, each of these rationals p/q actually corresponds to a convergent p_n/q_n. Now, it follows from (11), (13) and (18) that

$$\frac{1}{q_n^\tau} > \left| x - \frac{p_n}{q_n} \right| > \frac{1}{q_n q_{n+1}} - \frac{1}{q_{n+1} q_{n+2}} \geq \frac{1}{2 q_n q_{n+1}}.$$

For the last inequality, we used the fact that $q_{n+2} \geq 2q_n$, owing to (9). We straightforwardly infer that

$$\tau < 1 + \frac{\log 2 + \log q_{n+1}}{\log q_n}$$

for infinitely many integers $n \geq 1$, from which we deduce that $\tau(x) - 1$ is bounded above by the upper limit of $\log q_{n+1} / \log q_n$.

For the lower bound, let us consider a real number τ such that $\tau - 1$ is smaller than the aforementioned upper limit. Then, one easily checks that $q_{n+1} > q_n^{\tau - 1}$

for infinitely many integers $n \geq 1$. We finally make use of (13) to conclude that $\tau \leq \tau(x)$, and the result follows. □

Thanks to the recursive relation on the denominators of the convergents, we may give an alternate expression to that given above, specifically,

$$\tau(x) = 2 + \limsup_{n\to\infty} \frac{\log a_{n+1}}{\log q_n}.$$

This is actually a direct consequence of Proposition 1.8, together with the observation that q_{n+1} is between $a_{n+1}q_n$ and $2a_{n+1}q_n$, owing to (9).

1.3. Badly approximable points

1.3.1. Definition and first properties. This section is devoted to the study of a class of points that are very particular from the perspective of Diophantine approximation: the badly approximable points, which are defined as follows.

DEFINITION 1.2. A point $x \in \mathbb{R}^d$ is called *badly approximable* if the following condition is satisfied:

$$\exists\varepsilon > 0 \quad \forall(p, q) \in \mathbb{Z}^d \times \mathbb{N} \qquad \left| x - \frac{p}{q} \right|_\infty \geq \frac{\varepsilon}{q^{1+1/d}}.$$

The set of badly approximable points is denoted by Bad_d. In dimension $d = 1$, the badly approximable points are called *badly approximable numbers.*

As the name seems to indicate, the elements of Bad_d are badly approximated by the points with rational coordinates. Indeed, the irrationality exponent, introduced by Definition 1.1, satisfies

$$\forall x \in \mathrm{Bad}_d \qquad \tau(x) = 1 + \frac{1}{d}.$$

This means that the points in Bad_d attain the bound imposed by Dirichlet's theorem and its corollary, that is, Theorem 1.1 and Corollary 1.1. In other words,

$$\mathrm{Bad}_d \subseteq (\mathbb{R}^d \setminus \mathbb{Q}^d) \setminus \mathrm{Well}_d, \tag{26}$$

where Well_d denotes the set of points that are very well approximable, see Definition 1.1. Due to Proposition 1.1, the set in the right-hand side of (26) has full Lebesgue measure in $\mathbb{R}^d \setminus \mathbb{Q}^d$. The badly approximable points thus supply specific examples of points for which the typical property (4) holds.

Turning our attention to the left-hand side of (26), we now establish the following result. Its proof relies on the corollary to Dirichlet's theorem, along with general tools from measure theory that are presented in Chapter 4; we postpone it to Section 1.3.2 for the sake of clarity. The one-dimensional case may also be settled with the help of continued fractions, as detailed in Section 1.3.3.

PROPOSITION 1.9. *The set* Bad_d *of badly approximable points has Lebesgue measure zero, that is,*

$$\mathcal{L}^d(\mathrm{Bad}_d) = 0.$$

The above measure theoretic considerations directly imply that the inclusion in (26) is strict. As a matter of fact, Lebesgue-almost every point in the set $\mathbb{R}^d \setminus \mathbb{Q}^d$ is neither very well nor badly approximable. The next step in the description of the size properties of the set Bad_d would be to consider its Hausdorff dimension; this will be discussed in Sections 3.3 and 12.2.

1.3.2. Size properties. This section details the proof of Proposition 1.9. We begin by observing that the set of badly approximable points satisfies

$$\mathbb{R}^d \setminus \mathrm{Bad}_d \supseteq \bigcap_{\varepsilon>0} \widetilde{J}_{d,\varepsilon}, \tag{27}$$

where $\widetilde{J}_{d,\varepsilon}$ denotes the set obtained when replacing by $\varepsilon/q^{1+1/d}$ the approximation radii $1/q^\tau$ in the definition (1) of the set $J_{d,\tau}$. To be more specific,

$$\widetilde{J}_{d,\varepsilon} = \left\{ x \in \mathbb{R}^d \;\middle|\; \left| x - \frac{p}{q} \right|_\infty < \frac{\varepsilon}{q^{1+1/d}} \quad \text{for i.m. } (p,q) \in \mathbb{Z}^d \times \mathbb{N} \right\}.$$

It is clear that the mapping $\varepsilon \mapsto \widetilde{J}_{d,\varepsilon}$ is nondecreasing, so that the intersection in (27) may be taken on a sequence of positive values of ε that converge to zero, such as $\varepsilon_n = 1/n$ for instance. In order to show that Bad_d has Lebesgue measure zero, it thus suffices to prove that

$$\forall \varepsilon > 0 \qquad \mathcal{L}^d(\mathbb{R}^d \setminus \widetilde{J}_{d,\varepsilon}) = 0. \tag{28}$$

As a matter of fact, assuming that (28) holds, we would then be able to write that

$$\mathcal{L}^d(\mathrm{Bad}_d) \leq \mathcal{L}^d\left(\bigcup_{n=1}^{\infty} \mathbb{R}^d \setminus \widetilde{J}_{d,\varepsilon_n} \right) \leq \sum_{n=1}^{\infty} \mathcal{L}^d(\mathbb{R}^d \setminus \widetilde{J}_{d,\varepsilon_n}) = 0,$$

which would directly lead to Proposition 1.9. The proof now reduces to establishing (28). To proceed, we begin by remarking that this assertion holds for $\varepsilon = 1$. In fact, the corollary to Dirichlet's theorem, namely, Corollary 1.1 implies that

$$\widetilde{J}_{d,1} = J_{d,1+1/d} = \mathbb{R}^d. \tag{29}$$

In view of the monotonicity of the sets $\widetilde{J}_{d,\varepsilon}$ with respect to ε, the assertion also holds *a fortiori* for $\varepsilon > 1$.

The remaining case in which $\varepsilon \in (0,1)$ may be settled by means of general measure theoretic tools for sets of limsup type that are detailed in Chapter 4. Specifically, Proposition 4.4 therein directly leads to the following weaker statement. Recall that the limsup of a sequence $(E_n)_{n\geq 1}$ of subsets of $\mathbb{R}^d$ is defined by

$$\limsup_{n\to\infty} E_n = \bigcap_{m=1}^{\infty} \bigcup_{n=m}^{\infty} E_n,$$

and consists of the points that belong to infinitely many sets of the form E_n.

LEMMA 1.3. *Let us consider a sequence $(x_n)_{n\geq 1}$ in $\mathbb{R}^d$ and a sequence $(r_n)_{n\geq 1}$ in $(0,1]$ such that for every integer $m \geq 1$, only finitely many indices $n \geq 1$ satisfy both $|x_n|_\infty < m$ and $r_n > 1/m$. Then,*

$$\mathbb{R}^d = \limsup_{n\to\infty} \mathrm{B}_\infty(x_n, r_n) \quad \Longrightarrow \quad \forall c > 0 \quad \mathcal{L}^d\left(\mathbb{R}^d \setminus \limsup_{n\to\infty} \mathrm{B}_\infty(x_n, c\, r_n) \right) = 0.$$

It is clear that the sets $\widetilde{J}_{d,\varepsilon}$ fit nicely in the setting supplied by Lemma 1.3. In fact, letting $(p_n, q_n)_{n\geq 1}$ denote an enumeration of the countable set $\mathbb{Z}^d \times \mathbb{N}$, and then defining $x_n = p_n/q_n$ and $r_n = 1/q_n^{1+1/d}$, we easily see that for any $\varepsilon > 0$,

$$\widetilde{J}_{d,\varepsilon} = \limsup_{n\to\infty} \mathrm{B}_\infty(x_n, \varepsilon\, r_n).$$

Moreover, as a result of (29), the above limsup set coincides with the whole space $\mathbb{R}^d$ when $\varepsilon = 1$, so that the assumptions of the lemma are fulfilled by the sequences $(x_n)_{n\geq 1}$ and $(r_n)_{n\geq 1}$. We may conclude that all the sets $\widetilde{J}_{d,\varepsilon}$ have full Lebesgue measure in $\mathbb{R}^d$. This leads to (28), and thus to Proposition 1.9.

1.3.3. Link with continued fractions. We assume in this section that the dimension d of the ambient space is equal to one. For any real number x, recall that the exponent $\kappa(x)$ is defined by (24). This exponent characterizes the badly approximable numbers: Definition 1.2 directly ensures that

$$x \in \mathrm{Bad}_1 \qquad \Longleftrightarrow \qquad \kappa(x) > 0. \tag{30}$$

Moreover, we showed in Section 1.2.2.2 that $\kappa(x)$ is bounded above by $1/\sqrt{5}$, and the bound is attained by the irrational numbers whose continued fraction expansion is ultimately equal to one, so in particular by the golden ratio ϕ defined by (22). These numbers may therefore be seen as the "most badly" approximable one.

The emblematic example of the golden ratio hints at the following characterization of the badly approximable numbers in terms of the partial quotients of their continued fraction expansion.

Proposition 1.10. *Let x be a positive irrational real number with continued fraction expansion $[a_0; a_1, a_2, \ldots]$. Then,*

$$x \in \mathrm{Bad}_1 \qquad \Longleftrightarrow \qquad \sup_{n \geq 0} a_n < \infty.$$

Proof. Let us assume that x is badly approximable. Then, for some $\varepsilon > 0$ and all $n \geq 0$, the corresponding convergents p_n/q_n satisfy

$$\frac{\varepsilon}{q_n^2} \leq \left| x - \frac{p_n}{q_n} \right| < \frac{1}{q_n q_{n+1}} \leq \frac{1}{a_{n+1} q_n^2}$$

in view of (9) and (13). This implies that the partial quotients a_{n+1} are bounded by $1/\varepsilon$ for all $n \geq 0$.

Conversely, let us assume that the partial quotients are bounded by some real $M > 0$. Then, making use of (13) again, we see that $q_{n+1} \leq (M+1)q_n$ for all $n \geq 0$. Now, let us consider a pair $(p, q) \in \mathbb{P}_1$. By virtue of the optimality of the convergents, combined with (25), we have

$$\left| x - \frac{p}{q} \right| \geq \left| x - \frac{p_n}{q_n} \right| > \frac{1}{q_n q_{n+2}} \geq \frac{1}{(M+1)^4 q_{n-1}^2} > \frac{1}{(M+1)^4 q^2}$$

if n is chosen in such a way that $q_{n-1} < q \leq q_n$, see Theorem 1.3. Thus, the number x is badly approximable. □

The previous proposition yields a description of the size of the set of badly approximable numbers in terms of cardinality.

Corollary 1.2. *There exist continuum many badly approximable numbers, and there exist continuum many numbers that are not badly approximable. In other words, the sets Bad_1 and $\mathbb{R} \setminus \mathrm{Bad}_1$ have cardinality equal to that of $\mathbb{R}$.*

The results of Section 3.2 give the asymptotic behavior of the continued fraction expansion of typical irrational numbers. In particular, Proposition 3.3 ensures that for Lebesgue-almost every irrational number, the mean of the n first partial quotients tends to infinity as $n \to \infty$. The partial quotients thus grow typically somewhat fast to infinity and, from this perspective, the badly approximable numbers behave very peculiarly. This observation implies that the set Bad_1 has Lebesgue measure zero. We therefore recover Proposition 1.9 in the one-dimensional case.

1.4. Quadratic irrationals

Recall that a quadratic irrational is an irrational real number x such that there are integers a, b and c with $ax^2 + bx + c = 0$, or equivalently such that $\mathbb{Q}(x)$ is a field extension of degree two over $\mathbb{Q}$. The golden ratio defined by (22) provides a simple example of quadratic irrational, and also happens to supply the most simple example of continued fraction expansion that is ultimately periodic. This coincidence is actually emblematic of a result due to Euler and Lagrange that characterizes the quadratic irrationals in terms of the partial quotients of their continued fraction expansion. In order to state this result, let us begin by a definition.

DEFINITION 1.3. A continued fraction is *eventually periodic* if there are integers $m \geq 0$ and $k \geq 1$ such that $a_{n+k} = a_n$ for all integers $n \geq m$. Such a continued fraction is written

$$[a_0; a_1, \dots, a_{m-1}, \overline{a_m, \dots, a_{m+k-1}}].$$

The aforementioned characterization of the quadratic irrationals is then given by the following result.

THEOREM 1.4 (Euler, 1737; Lagrange, 1770). *Let x be an irrational positive real number. Then, the continued fraction expansion of x is ultimately periodic if and only if x is a quadratic irrational.*

PROOF. The first proof of the direct part is due to Euler. Let us assume that x has a strictly periodic continued fraction expansion, namely, $x = [\overline{a_0; a_1, \dots, a_k}]$. As a consequence, the $(k+1)$-th tail of the continued fraction expansion of x is equal to x itself, and (19) implies that

$$x = \frac{p_k x + p_{k-1}}{q_k x + q_{k-1}},$$

so that x is a root of the polynomial $q_k Z^2 + (q_{k-1} - p_k)Z - p_{k-1}$, and is therefore a quadratic irrational. Note in passing that the discriminant of this polynomial is equal to $(p_k + q_{k-1})^2 + 4(-1)^k$ owing to (10), and thus cannot be a perfect square; this is compatible with the fact that x is irrational.

Let us now consider the general case in which x has a continued fraction expansion that is periodic only ultimately. Then, the continued fraction expansion of x is of the form $[a_0; a_1, \dots, a_{m-1}, \overline{a_m, \dots, a_{m+k-1}}]$. In particular, its m-th tail x_m has a strictly periodic continued fraction expansion, thereby being a quadratic irrational. By virtue of (19) again, we have

$$x = \frac{p_{m-1} x_m + p_{m-2}}{q_{m-1} x_m + q_{m-2}},$$

which proves that the two field extensions $\mathbb{Q}(x)$ and $\mathbb{Q}(x_m)$ coincide. In particular, $\mathbb{Q}(x)$ is of degree two over $\mathbb{Q}$, so that x is a quadratic irrational.

The converse part is more difficult and was first established by Lagrange. Let us suppose that x is a quadratic irrational. Then, x is a root of a polynomial $R_0 = \alpha_0 Z^2 + \beta_0 Z + \gamma_0$ with coefficients α_0, β_0 and γ_0 in $\mathbb{Z}$, and with a discriminant $\delta = \beta_0^2 - 4\alpha_0\gamma_0$ that cannot be a perfect square. Moreover, letting x_n denote the n-th tail of the continued fraction expansion of x, we see again that the two field extensions $\mathbb{Q}(x)$ and $\mathbb{Q}(x_n)$ coincide, so that x_n is a root of a polynomial $R_n = \alpha_n Z^2 + \beta_n Z + \gamma_n$ of the above form.

It is possible to choose these polynomials in such a way that they satisfy a simple recurrence relation. Since $x_n = a_n + 1/x_{n+1}$, we see that

$$x_{n+1}^2 R_n \left(a_n + \frac{1}{x_{n+1}} \right) = (a_n^2 \alpha_n + a_n \beta_n + \gamma_n) x_{n+1}^2 + (2a_n\alpha_n + \beta_n)x)n + 1 + \alpha_n$$

vanishes, so that we may assume that the coefficients of the polynomial R_{n+1} are obtained from those of R_n thanks to the following relations:

$$\begin{cases} \alpha_{n+1} = a_n^2 \alpha_n + a_n \beta_n + \gamma_n \\ \beta_{n+1} = 2a_n \alpha_n + \beta_n \\ \gamma_{n+1} = \alpha_n. \end{cases}$$

In particular, these relations imply that the integer $\beta_n^2 - 4\alpha_n\gamma_n$ does not depend on n. As a result, all the polynomials R_n have discriminant δ, which cannot be a perfect square. Thus, $\alpha_n \neq 0$ for all integers $n \geq 0$.

Let us assume that there exists an integer $m \geq 0$ such that α_n is positive for any $n \geq m$. As a_n is also positive, it follows from the above recurrence relations that the sequence $(\beta_n)_{n\geq m}$ is increasing, and furthermore that the three integers α_n, β_n and γ_n are simultaneously positive for n large enough. This contradicts the fact that x_n is a positive root of R_n. We deduce that there is an infinite subset $\mathcal{N}$ of $\mathbb{N}$ such that $\alpha_{n-1}\alpha_n < 0$ for all $n \in \mathcal{N}$. In that case, we see that

$$0 \leq \beta_n^2 < \delta \qquad \text{and} \qquad 0 < -4\alpha_n\gamma_n \leq \delta.$$

This gives a bound on the coefficients of the polynomial R_n when $n \in \mathcal{N}$, namely,

$$|\beta_n| < \sqrt{\delta} \qquad \text{and} \qquad \max\{|\alpha_n|, |\gamma_n|\} \leq \frac{\delta}{4}.$$

This means that when the index n runs through the infinite set $\mathcal{N}$, there are only finitely many different polynomials R_n. As a consequence, there is at least a polynomial that is chosen infinitely often. In particular, there are three integers $n_1 < n_2 < n_3$ for which the polynomials R_{n_1}, R_{n_2} and R_{n_3} coincide. Thus, x_{n_1}, x_{n_2} and x_{n_3} are a root of the same polynomial. Since a quadratic polynomial has at most two zeros, we deduce that at least two among these three numbers coincide. This ensures that the continued fraction expansion of x is ultimately periodic. $\square$

Thanks to Proposition 1.10, one easily checks that Theorem 1.4 leads to the following corollary.

Corollary 1.3. *Any quadratic irrational is badly approximable.*

1.5. Inhomogeneous approximation

Inhomogeneous Diophantine approximation usually refers to the approximation of points in $\mathbb{R}^d$ by the system obtained by the points of the form $(p+\alpha)/q$, where as usual p is an integer point, and q is a positive integer, and where α is a point in $\mathbb{R}^d$ that is fixed in advance. When α is equal to zero, one obviously recovers the situation discussed in Section 1.1, which is referred to as the homogeneous one.

In this context, a point α being fixed arbitrarily in $\mathbb{R}^d$, the analog of the set $J_{d,\tau}$ defined by (1) is now the set

$$J_{d,\tau}^{\alpha} = \left\{ x \in \mathbb{R}^d \;\middle|\; \left| x - \frac{p+\alpha}{q} \right|_\infty < \frac{1}{q^\tau} \quad \text{for i.m. } (p,q) \in \mathbb{Z}^d \times \mathbb{N} \right\}. \tag{31}$$

Proposition 1.1 may straightforwardly be extended to the inhomogeneous setting. Specifically, one easily checks that the Lebesgue measure of the set $J_{d,\tau}^{\alpha}$ vanishes for any real number $\tau > 1 + 1/d$. Some more work is required to show that, just as in the homogeneous setting, the set $J_{d,\tau}^{\alpha}$ has full Lebesgue measure in the whole space $\mathbb{R}^d$ in the opposite case; this will actually appear in the statement of Corollary 7.1. A much more precise description of the size of the set $J_{d,\tau}^{\alpha}$ will in fact be given in this statement, and subsequently in that of Corollary 10.3 as well.

1.5.1. A theorem of Khintchine.

The main purpose of this section is to establish the following result due to Khintchine [**39**], which in some sense complements Dirichlet's theorem, namely, Theorem 1.1. Our proof sticks to Khintchine's method very closely, but we find it valuable to detail the arguments anyway, because Khintchine's original paper [**39**] is written in German.

THEOREM 1.5. *Let us consider a point $x \in \mathbb{R}^d$ and assume that there exists a real number $\gamma > 0$ such that for any integer $Q > 1$, the system*

$$\begin{cases} 1 \leq q < \gamma Q^d \\ |qx - p|_\infty \leq 1/Q \end{cases}$$

admits no solution (p, q) in $\mathbb{Z}^d \times \mathbb{N}$. Then, there exists a real number $\Gamma > 0$, which depends on γ and d only, such that for any point $\alpha \in \mathbb{R}^d$ and any integer $Q > 1$, the system

$$\begin{cases} 1 \leq q < \Gamma Q^d \\ |qx - p - \alpha|_\infty \leq 1/Q \end{cases}$$

admits a solution (p, q) in $\mathbb{Z}^d \times \mathbb{N}$.

The remainder of this section is devoted to establishing Theorem 1.5. Let us begin by introducing some notations. Let us consider a point x in $\mathbb{R}^d$ and an integer $Q > 1$. Theorem 1.1 ensures that the system

$$\begin{cases} 1 \leq q < Q^d \\ |qx - p|_\infty \leq 1/Q \end{cases}$$

admits a solution (p, q) in $\mathbb{Z}^d \times \mathbb{N}$; we assume that q is minimal. Combined with the assumption that bears on x in the statement of Theorem 1.5, this implies that

$$\gcd(p, q) = 1 \qquad \text{and} \qquad q \geq \gamma Q^d.$$

In particular, γ is necessarily smaller than one. Now, for any $i \in \{1, \dots, d\}$, let p_i denote the i-th coordinate of p, and let e_i, p'_i and q'_i be the integers defined by

$$\begin{cases} e_i = \gcd(p_i, q) \\ p_i = e_i p'_i \\ q = e_i q'_i. \end{cases}$$

In addition, since p'_i and q'_i are coprime, p'_i is invertible modulo q'_i, and we may find an integer b_i such that

$$\begin{cases} p'_i b_i = 1 \mod q'_i \\ \gcd(e_i, b_i) = 1. \end{cases}$$

As a matter of fact, the solutions of the first equation are of the form $b_i = b_i^* + z q'_i$, for $z \in \mathbb{Z}$, when b_i^* is already a solution. The fact that one of these solutions also satisfies the second condition is a plain consequence of the following fact.

LEMMA 1.4. *Let b and c be two integers in $\mathbb{Z}$ with $\gcd(b, c) = 1$. Then,*

$$\forall a \in \mathbb{Z} \quad \exists z \in \mathbb{Z} \qquad \gcd(a, b + zc) = 1.$$

PROOF. When a divides c, we have $\gcd(a, b) = 1$, and the result clearly holds. In the opposite case, let n denote the product of the prime numbers that divide a and do not divide c. Clearly, the integers c and n are coprime, so there exists an integer $z \in \mathbb{Z}$ such that

$$zc = 1 - b \mod n.$$

Let us consider a prime divisor ℓ of a, and let us observe that $\ell \nmid zc + b$. When $\ell \nmid c$, this comes from the fact that $\ell \mid n$. When $\ell \mid c$, this is because $\ell \nmid b$. Finally, no prime divisor of a divides $zc + b$, and the result follows. □

On top of that, let E denote the product of the integers e_i, that is,

$$E = e_1 \dots e_d.$$

We observe that $E \mid q^{d-1}$. Indeed, since $q^d = E q'_1 \dots q'_d$, it suffices to show that $q \mid q'_1 \dots q'_d$. Let us consider a prime number ℓ and an integer $s \geq 1$ such that $\ell^s \mid q$.

The components of p are mutually coprime with q, so we must have $\ell \nmid p_i$ for some i. As $e_i \mid p_i$, we necessarily have $\ell \nmid e_i$ as well. Since $q = e_i q_i'$, we deduce that ℓ^s divides q_i', thereby dividing $q_1' \dots q_d'$. It follows that $q \mid q_1' \dots q_d'$, and therefore that $E \mid q^{d-1}$. We may thus introduce the integer

$$c = \frac{q^{d-1}}{E}.$$

Following the lines of Khintchine's original proof, We now state and establish a series of lemmas.

LEMMA 1.5. *Let us consider d integers $n_1, \dots, n_d \in \mathbb{Z}$, with each n_i being less than e_i in absolute value, and let us further assume that*

$$n_1 b_1 q_1' = \dots = n_d b_d q_d' \mod q.$$

Then, the integers $n_1, \dots, n_d$ are all equal to zero.

PROOF. The assumption of the lemma directly yields

$$n_1 b_1 q_1' q^{d-1} = \dots = n_d b_d q_d' q^{d-1} \mod q^d.$$

We have $q^d = E q_1' \dots q_d'$, and $e_i = q/q_i'$ for each integer i, so that

$$n_1 b_1 \frac{E}{e_1} = \dots = n_d b_d \frac{E}{e_d} \mod E.$$

Let us consider an integer $i \in \{1, \dots, d\}$. We may obviously exclude the trivial case where e_i is equal to one. Thus, assuming that $e_i > 1$, we may consider a prime number ℓ and an integer $s \geq 1$ such that $\ell^s \mid e_i$. Since q and the coordinates of p are mutually coprime, there exists an integer $i' \neq i$ such that $\ell \nmid e_{i'}$. We have

$$n_i b_i e_{i'} = n_{i'} b_{i'} e_i \mod e_i e_{i'},$$

so that ℓ^s divides $n_i b_i e_{i'}$. Moreover, as b_i and e_i are coprime, the prime number ℓ cannot divide b_i. It does not divide $e_{i'}$ either, so we deduce that $\ell^s \mid n_i$. The previous analysis implies that the integer n_i is a multiple of e_i, and the result follows from the assumption that it is smaller than e_i in absolute value. $\square$

LEMMA 1.6. *There exist a real number $C_0 > 0$ and an integer $Q_0 \geq 1$ that depend on γ and d only such that if $Q > Q_0$, then there are $2d^2$ integers $x_i^{(k)}$ and $y_i^{(k)}$, for $i, k \in \{1, \dots, d\}$, such that the following conditions hold simultaneously:*

(1) *for any $k \in \{1, \dots, d\}$,*

$$x_1^{(k)} b_1 = \dots = x_d^{(k)} b_d \mod q\,;$$

(2) *for any $i, k \in \{1, \dots, d\}$,*

$$x_i^{(k)} = y_i^{(k)} \mod q_i'\,;$$

(3) *there exists an integer $a \geq 1$ such that*

$$\Delta = \begin{vmatrix} y_1^{(1)} & \cdots & y_d^{(1)} \\ \vdots & & \vdots \\ y_1^{(d)} & \cdots & y_d^{(d)} \end{vmatrix} = ac\,;$$

(4) *for any $i, k \in \{1, \dots, d\}$,*

$$|y_i^{(k)}| \leq C_0 \, \frac{q^{(d-1)/d}}{e_i}.$$

Specifically, one may choose C_0 as the $(d-1)$-th power of an arbitrary integer larger than $2(d-1)/\gamma^{1/d}$, and Q_0 as any integer larger than $C_0/\gamma^{1/d}$.

PROOF. Let C denote an arbitrary integer larger than $2(d-1)/\gamma^{1/d}$, and let $C_0 = C^{d-1}$. Moreover, let Q_0 denote an arbitrary integer larger than $C_0/\gamma^{1/d}$. We assume throughout the proof that the condition $Q > Q_0$ is verified.

Then, let us consider $2d$ integers z_i and j_i satisfying the conditions

$$0 \leq z_i \leq C_0 \, \frac{q^{(d-1)/d}}{e_i} \qquad \text{and} \qquad 0 \leq j_i < e_i,$$

for $i \in \{1, \ldots, d\}$, and let us then define

$$u_i = j_i q_i' + z_i.$$

We obviously have e_i possible values for the integer j_i, and $\lfloor C_0 q^{(d-1)/d}/e_i \rfloor + 1$ for z_i. Moreover, note that $Q > Q_0 > C_0/\gamma^{1/d}$ and $q \geq \gamma Q^d$, so that $q > C_0^d$. This implies that the maximal possible value for z_i is smaller than q_i', and thus that the set of all possible values of the d-tuple $(u_1, \ldots, u_d)$ has cardinality equal to

$$\prod_{i=1}^{d} \left(e_i \left(\left\lfloor \frac{C_0 q^{(d-1)/d}}{e_i} \right\rfloor + 1 \right) \right) > C_0^d q^{d-1}.$$

Let U denote this set, and let Φ be the mapping defined on U by

$$\Phi(u_1, \ldots, u_d) = (u_1 b_1 - u_2 b_2, \ldots, u_1 b_1 - u_d b_d) \mod q. \tag{32}$$

The mapping Φ sends the set U to a subset of $(\mathbb{Z}/q\mathbb{Z})^{d-1}$. Therefore, the preimage sets $\Phi^{-1}(\{f\})$, for $f \in (\mathbb{Z}/q\mathbb{Z})^{d-1}$, form a partition of U. As a result,

$$C_0^d q^{d-1} < \#U = \sum_{f \in (\mathbb{Z}/q\mathbb{Z})^{d-1}} \#\Phi^{-1}(\{f\}) \leq q^{d-1} \max_{f \in (\mathbb{Z}/q\mathbb{Z})^{d-1}} \#\Phi^{-1}(\{f\}).$$

Consequently, there necessarily exists an element in $(\mathbb{Z}/q\mathbb{Z})^{d-1}$ whose preimage has cardinality larger than C_0^d. Thus, we can find C_0^d+1 distinct d-tuples $(u_1^{(k)}, \ldots, u_d^{(k)})$, with $k \in \{0, \ldots, C_0^d\}$, whose images under the mapping Φ coincide. The corresponding values for the integers z_i and j_i are denoted by $z_i^{(k)}$ and $j_i^{(k)}$, respectively.

We consider in $\mathbb{Z}^d$ the vectors $y^{(k)} = (y_1^{(k)}, \ldots, y_d^{(k)})$ defined by $y_i^{(k)} = z_i^{(k)} - z_i^{(0)}$. Note that there are $C_0^d + 1$ such vectors, and that the null vector is obtained for k equal to zero. Let us assume that these vectors span a linear subspace of dimension at most $d-1$, so that they all lie in a hyperplane with normal vector denoted by $(a_1, \ldots, a_d)$. Without loss of generality, we may assume that $|a_i|/e_i$ is maximal when $i = 1$. We may thus write the equation of the hyperplane in the form

$$e_1 y_1 = \sum_{i=2}^{d} \nu_i e_i y_i,$$

where the real numbers ν_i are bounded above by one in absolute value.

Recalling that C is the positive integer for which $C_0 = C^{d-1}$, we may split each interval $[0, C_0 q^{(d-1)/d}/e_i]$ into C^d disjoint subintervals with common length $q^{(d-1)/d}/(Ce_i)$. Accordingly, the rectangle formed by the product of these intervals over all $i \in \{2, \ldots, d\}$ may be partitioned into C_0^d disjoint rectangles. The $C_0^d + 1$ points $(z_2^{(k)}, \ldots, z_d^{(k)})$ are all contained in the large rectangle. The pigeon-hole principle then ensures that at least two points lie in the same smaller rectangle. These points correspond to two distinct choices of the index k and, for simplicity, their components are denoted by z_i' and z_i'', respectively. We thus have

$$\forall i \in \{2, \ldots, d\} \qquad |z_i' - z_i''| \leq \frac{q^{(d-1)/d}}{Ce_i}.$$

It is clear that the corresponding components y'_i and y''_i satisfy the same inequalities because they are equal to $z'_i - z_i^{(0)}$ and $z''_i - z_i^{(0)}$, respectively. In addition, as the points y' and y'' both belong to the aforementioned hyperplane, we have

$$e_1|y'_1 - y''_1| \leq \sum_{i=2}^{d} |\nu_i|\, e_i\, |y'_i - y''_i| \leq \sum_{i=2}^{d} e_i \, \frac{q^{(d-1)/d}}{Ce_i} = \frac{d-1}{C} q^{(d-1)/d}.$$

We finally deduce that

$$\forall i \in \{1, \ldots, d\} \qquad |y'_i - y''_i| \leq \frac{d-1}{C} \cdot \frac{q^{(d-1)/d}}{e_i}. \tag{33}$$

Let us suppose that all the differences $y'_i - y''_i$ vanish, *i.e.* that the points z' and z'' coincide. As the corresponding d-tuples $(u'_1, \ldots, u'_d)$ and $(u''_1, \ldots, u''_d)$ have the same image under the mapping Φ, we have for any index $i \in \{2, \ldots, d\}$,

$$u'_1 b_1 - u'_i b_i = u''_1 b_1 - u''_i b_i \mod q, \tag{34}$$

that is,

$$(j'_1 q'_1 + z'_1)b_1 - (j'_i q'_i + z'_i)b_i = (j''_1 q'_1 + z''_1)b_1 - (j''_i q'_i + z''_i)b_i \mod q.$$

As a consequence, making use of the assumption that the points z' and z'' are the same, we deduce that

$$(j'_1 - j''_1)q'_1 b_1 = \ldots = (j'_d - j''_d)q'_d b_d \mod q.$$

However, every integer $j'_i - j''_i$ is smaller than e_i in absolute value, so we may apply Lemma 1.5 to conclude that it is equal to zero. This is a contradiction because the points u' and u'' were chosen to be distinct. Thus, all the differences $y'_i - y''_i$ cannot vanish simultaneously.

Moreover, we also deduce from (34) that there is an integer g such that

$$\begin{cases} (u'_1 - u''_1)b_1 = \ldots = (u'_d - u''_d)b_d = g \mod q \\ -q \leq 2g < q. \end{cases}$$

As a result, for each fixed i, since q'_i divides q and b_i is the inverse of p'_i modulo q'_i, we infer that

$$gp'_i = (u'_i - u''_i)b_i p'_i = (j'_i - j''_i)b_i p'_i q'_i + (z'_i - z''_i)b_i p'_i = y'_i - y''_i \mod q'_i.$$

This plainly means that gp'_i is equal to $y'_i - y''_i + n_i q'_i$ for some integer $n_i \in \mathbb{Z}$, which directly leads to

$$\left| g\frac{p_i}{q} - n_i \right| = \frac{e_i}{q} |gp'_i - n_i q'_i| = \frac{e_i}{q} |y'_i - y''_i| \leq \frac{d-1}{Cq^{1/d}},$$

thanks to (33). Meanwhile, we know that $|qx_i - p_i|$ is bounded above by $1/Q$. It then follows from the triangle inequality that

$$|gx_i - n_i| \leq |g| \left| x_i - \frac{p_i}{q} \right| + \left| g\frac{p_i}{q} - n_i \right| \leq \frac{|g|}{qQ} + \frac{d-1}{Cq^{1/d}} \leq \left(\frac{1}{2} + \frac{d-1}{C\gamma^{1/d}} \right) \frac{1}{Q},$$

where the latter inequality is due to the fact that $|g| \leq q/2$ and $q \geq \gamma Q^d$. We now recall that the integer C is larger than $2(d-1)/\gamma^{1/d}$; this implies that the upper bound above is at most $1/Q$, specifically,

$$|gx - n|_\infty \leq \frac{1}{Q}.$$

Along with the fact that $|g|$ is smaller than q, this contradicts the minimality of q, unless the integer g vanishes. Thus, the only possibility is that g is equal to zero, which means that

$$\forall i \in \{1, \ldots, d\} \qquad (u'_i - u''_i)b_i = 0 \mod q.$$

Given that q_i' divides q and is coprime with b_i, we deduce that the integers u_i' and u_i'' coincide modulo q_i'. The integers y_i' and y_i'' share the same property, specifically,

$$y_i' = z_i' - z_i^{(0)} = u_i' - u_i^{(0)} = u_i'' - u_i^{(0)} = z_i'' - z_i^{(0)} = y_i'' \mod q_i'.$$

On top of that, (33) implies that $|y_i' - y_i''|$ is smaller than q_i'. In fact, this holds because q is large enough, specifically,

$$q^{1/d} \geq \gamma^{1/d} Q > \gamma^{1/d} Q_0 > C_0 \geq \frac{d-1}{C}.$$

We deduce that the differences $y_i' - y_i''$ are all equal to zero, a contradiction with what precedes. This means that the vectors $y^{(k)}$ cannot belong to a common hyperplane, and thus that they span the whole space $\mathbb{R}^d$.

The upshot is that d of the vectors $y^{(k)}$ are linearly independent; up to reordering, we may assume that these vectors are those indexed by $k \in \{1, \ldots, d\}$ and that their determinant Δ is positive. These vectors satisfy the condition (4) appearing in the statement of the lemma, because of the bounds on the integers z_i. We now define $x_i^{(k)}$ as being equal to $u_i^{(k)} - u_i^{(0)}$ for any indices i and k in $\{1, \ldots, d\}$, so that

$$x_i^{(k)} = (j_i^{(k)} q_i' + z_i^{(k)}) - (j_i^{(0)} q_i' + z_i^{(0)}) = z_i^{(k)} - z_i^{(0)} = y_i^{(k)} \mod q_i',$$

i.e. the condition (2) is verified. Furthermore, the vectors $(u_1^{(k)}, \ldots, u_d^{(k)})$ were chosen in such a way that they have the same image under the mapping Φ. In particular, for any $i \in \{2, \ldots, d\}$ and any $k \in \{1, \ldots, d\}$,

$$u_1^{(k)} b_1 - u_i^{(k)} b_i = u_1^{(0)} b_1 - u_i^{(0)} b_i \mod q,$$

which directly leads to the condition (1).

The discriminant Δ of the integers $y_i^{(k)}$ is a positive integer but, in order to obtain the condition (3), it remains to prove that Δ is a multiple of the integer c. Let us consider the integers $t_i^{(k)} = e_i y_i^{(k)}$, and let Δ' denote their discriminant. Clearly, Δ' is equal to $E\Delta$, so it suffices to establish that $q^{d-1} \mid \Delta'$. This is the purpose of the remainder of the proof.

Given four indices $i, i', k, k' \in \{1, \ldots, d\}$, the condition (1) gives

$$\begin{cases} e_i e_{i'} x_i^{(k)} x_{i'}^{(k')} b_i = e_i e_{i'} x_{i'}^{(k)} x_{i'}^{(k')} b_{i'} \mod q e_i e_{i'} x_{i'}^{(k')} \\ e_i e_{i'} x_i^{(k')} x_{i'}^{(k)} b_i = e_i e_{i'} x_{i'}^{(k')} x_{i'}^{(k)} b_{i'} \mod q e_i e_{i'} x_{i'}^{(k)}. \end{cases} \tag{35}$$

Let us consider a prime number ℓ and an integer $s \geq 1$ such that $\ell^s \mid q$, and let r denote the maximal integer satisfying $\ell^r \mid t_i^{(k)}$ for all $i, k \in \{1, \ldots, d\}$. The condition (2), combined with the fact that q is equal to $e_i q_i'$, gives

$$t_i^{(k)} = e_i y_i^{(k)} = e_i x_i^{(k)} \mod q. \tag{36}$$

Case where $r \leq s$. We see that ℓ^r divides both $t_i^{(k)}$ and q, which itself divides $t_i^{(k)} - e_i x_i^{(k)}$. Thus, ℓ^r divides $e_i x_i^{(k)}$ for any choice of i and k. This means that $q e_i \ell^r$ divides both $q e_i e_{i'} x_{i'}^{(k')}$ and $q e_i e_{i'} x_{i'}^{(k)}$, and taking the two equations in (35) modulo $q e_i \ell^r$, we deduce that

$$e_i e_{i'} x_i^{(k)} x_{i'}^{(k')} b_i = e_i e_{i'} x_{i'}^{(k)} x_{i'}^{(k')} b_{i'} = e_i e_{i'} x_{i'}^{(k')} x_{i'}^{(k)} b_{i'} = e_i e_{i'} x_i^{(k')} x_{i'}^{(k)} b_i \mod q e_i \ell^r,$$

so that

$$e_i e_{i'} b_i (x_i^{(k)} x_{i'}^{(k')} - x_i^{(k')} x_{i'}^{(k)}) = 0 \mod q e_i \ell^r. \tag{37}$$

We now observe that b_i is coprime with $q e_i \ell^r$. As a matter of fact, assuming that this does not hold, let us consider a prime number n that divides both b_i and $q e_i \ell^r$. Since b_i and e_i are coprime, n does not divide e_i, and thus necessarily divides $q \ell^r$. Furthermore, if n is different from ℓ, it then divides $q = e_i q_i'$, thereby necessarily dividing q_i'. This is impossible because b_i and q_i' are coprime. Hence, the prime number n is equal to ℓ. As a consequence, ℓ^s divides q and is coprime with e_i, so it must divide q'_i. This means that n divides both b_i and q'_i, which is impossible because these two integers are coprime. Finally, b_i is invertible modulo $qe_i\ell^r$, a multiple of ℓ^{s+r}, and we may thus deduce from (37) that

$$e_i e_{i'}(x_i^{(k)}x_{i'}^{(k')} - x_i^{(k')}x_{i'}^{(k)}) = 0 \mod \ell^{s+r}. \tag{38}$$

Now, starting from (36) again, up to replacing i and k by i' and k', respectively, and recalling that the integer ℓ^r divides both $t_{i'}^{(k')}$ and $e_i x_i^{(k)}$, we also have

$$\begin{cases} t_i^{(k)}t_{i'}^{(k')} = e_i x_i^{(k)} t_{i'}^{(k')} \mod q\ell^r \\ t_{i'}^{(k')}e_i x_i^{(k)} = e_i e_{i'} x_i^{(k)} x_{i'}^{(k')} \mod q\ell^r, \end{cases}$$

from which we directly infer that

$$t_i^{(k)}t_{i'}^{(k')} = e_i e_{i'} x_i^{(k)} x_{i'}^{(k')} \mod \ell^{s+r}.$$

We may obviously exchange the rôle of k and k' and deduce a similar equality. Combined with (38), this leads to

$$t_i^{(k)}t_{i'}^{(k')} = t_i^{(k')}t_{i'}^{(k)} \mod \ell^{s+r}.$$

Since all the integers $t_i^{(k)}$ are divisible by ℓ^r, they may be written in the form $t_i^{(k)} = \ell^r v_i^{(k)}$ for some integer $v_i^{(k)}$. The previous equation thus gives

$$v_i^{(k)}v_{i'}^{(k')} - v_i^{(k')}v_{i'}^{(k)} = 0 \mod \ell^{s-r}. \tag{39}$$

The determinant of the integers $v_i^{(k)}$ is denoted by Δ'', and is thus equal to $\ell^{-dr}\Delta'$. The maximality of r ensures that there is a pair (ι, κ) of indices in $\{1, \dots, d\}$ such that the integer $v_\iota^{(\kappa)}$ is not divisible by ℓ. We now transform the discriminant Δ'' as follows: for each index $i \neq \iota$, we replace the i-th column by its product by $v_\iota^{(\kappa)}$, minus $v_i^{(\kappa)}$ times the ι-th column. Hence, if $i \neq \iota$, the coefficient $v_i^{(k)}$ is replaced by $v_i^{(k)}v_\iota^{(\kappa)} - v_i^{(\kappa)}v_\iota^{(k)}$ which, in view of (39), may be written in the form $\ell^{s-r}w_i^{(k)}$ for some $w_i^{(k)} \in \mathbb{Z}$. The newly obtained discriminant is thus equal to both $(v_\iota^{(\kappa)})^{d-1}\Delta''$ and $\ell^{(s-r)(d-1)}\Delta'''$, where Δ''' denotes the discriminant of the matrix formed by the integers $w_i^{(k)}$, for $i \neq \iota$, and the integers $v_\iota^{(k)}$. In particular, $(v_\iota^{(\kappa)})^{d-1}$ divides $\ell^{(s-r)(d-1)}\Delta'''$ and, since $\ell \nmid v_\iota^{(\kappa)}$, we deduce that $(v_\iota^{(\kappa)})^{d-1}$ divides Δ''', *i.e.* that Δ''' may be written in the form $(v_\iota^{(\kappa)})^{d-1}m$ for some $m \in \mathbb{Z}$. Finally,

$$\Delta' = \ell^{dr}\Delta'' = \ell^{dr}\frac{\ell^{(s-r)(d-1)}}{(v_\iota^{(\kappa)})^{d-1}}\Delta''' = \ell^{(d-1)s+r}m,$$

from which we conclude that $(\ell^s)^{d-1}$ divides Δ'.

Case where $r > s$. In that situation, all the integers $t_i^{(k)}$ are divisible by ℓ^s, so that their discriminant Δ' is clearly divisible by $(\ell^s)^{d-1}$.

The previous analysis shows that for any prime number ℓ and any integer $s \geq 1$ such that $\ell^s \mid q$, we have $(\ell^s)^{d-1} \mid \Delta'$. It follows that $q^{d-1} \mid \Delta'$ as required, and the condition (3) readily follows. □

Lemma 1.7. *Let us consider $2d^2$ integers $x_i^{(k)}$ and $y_i^{(k)}$, for $i, k \in \{1, \dots, d\}$, such the conditions (1) and (2) of the statement of Lemma 1.6 hold, such that the condition (3) holds with $a > 1$, and such that for any $i, k \in \{1, \dots, d\}$,*

$$|y_i^{(k)}| \leq \lambda_i \qquad \text{with} \qquad 0 < \lambda_i < \frac{q'_i}{d}.$$

Then, there are $2d^2$ integers $\bar{x}_i^{(k)}$ and $\bar{y}_i^{(k)}$, for $i, k \in \{1, \dots, d\}$, that satisfy the aforementioned conditions (1) and (2), and also the following conditions:

(3') *there exists an integer* $\bar{a} \in \{1, \ldots, a-1\}$ *such that*

$$\bar{\Delta} = \begin{vmatrix} \bar{y}_1^{(1)} & \cdots & \bar{y}_d^{(1)} \\ \vdots & & \vdots \\ \bar{y}_1^{(d)} & \cdots & \bar{y}_d^{(d)} \end{vmatrix} = \bar{a}c\,;$$

(4') *for any* $i, k \in \{1, \ldots, d\}$,

$$|\bar{y}_i^{(k)}| \leq d\lambda_i.$$

PROOF. The vectors $y^{(1)}, \ldots, y^{(d)}$ form a sublattice L_a of $\mathbb{Z}^d$ with dimension equal to d. Its fundamental domain, *i.e.* the half-open parallelepiped spanned by these vectors, is denoted by J_a. The Lebesgue measure of J_a is the fundamental volume of the lattice and is equal to $\Delta = ac$. The index of L_a in $\mathbb{Z}^d$ is the cardinality of the quotient $\mathbb{Z}^d/L_a$; it is equal to Δ and also gives the number of points in $\mathbb{Z}^d$ that belong to J_a, see *e.g.* [**58**, Lecture V] for details.

Let us then consider the set A formed by the $2d$-tuples $(j_1, \ldots, j_d, y_1, \ldots, y_d)$ with $(y_1, \ldots, y_d) \in J_a \cap \mathbb{Z}^d$ and $j_i \in \{0, \ldots, e_i - 1\}$ for each index i. The mapping Ψ defined on A by

$$\Psi(j_1, \ldots, j_d, y_1, \ldots, y_d) = (j_1 q_1' + y_1, \ldots, j_d q_d' + y_d) \tag{40}$$

is one-to-one. Indeed, let us assume that $j_i' q_i' + y_i' = j_i'' q_i' + y_i''$ for all i, for two distinct $2d$-tuples. There necessarily exists an index m for which $j_m' \neq j_m''$, as otherwise the two $2d$-tuples would coincide. Consequently,

$$|y_m' - y_m''| = |j_m' - j_m''| q_m' \geq q_m'.$$

Given that $(y_1', \ldots, y_d')$ and $(y_1'', \ldots, y_d'')$ both belong to the parallelepiped J_a, the distance between y_m' and y_m'' is bounded above by the diameter of the projection of J_a onto the m-th axis. Hence,

$$|y_m' - y_m''| \leq |y_m^{(1)}| + \ldots + |y_m^{(d)}| \leq d\lambda_m < q_m',$$

thereby giving a contradiction. The mapping Ψ being one-to-one, its image A' has cardinality equal to that of A, namely,

$$\#A' = \#A = e_1 \ldots e_d \cdot \#(J_a \cap \mathbb{Z}^d) = E\Delta = Eac = aq^{d-1}.$$

In particular, since the integer a is greater than one, the set A' has cardinality larger than that of $(\mathbb{Z}/q\mathbb{Z})^{d-1}$, namely, q^{d-1}. Thus, the mapping Φ defined on A' as in (32) cannot be one-to-one. This means that there exist two distinct d-tuples $(x_1', \ldots, x_d')$ and $(x_1'', \ldots, x_d'')$ in A' such that for any index $i \in \{2, \ldots, d\}$,

$$x_1' b_1 - x_i' b_i = x_1'' b_1 - x_i'' b_i \mod q.$$

Naturally, the corresponding $2d$-tuples in A are denoted by $(j_1', \ldots, j_d', y_1', \ldots, y_d')$ and $(j_1'', \ldots, j_d'', y_1'', \ldots, y_d'')$, respectively. For any i, we then define

$$\begin{cases} u_i = x_i' - x_i'' \\ \ell_i = j_i' - j_i'' \\ v_i = y_i' - y_i'' \end{cases}$$

and we point out that

$$\begin{cases} u_1 b_1 = \ldots = u_d b_b \mod q \\ |\ell_1| < e_1, \ldots, |\ell_d| < e_d. \end{cases} \tag{41}$$

Now fix an index $k \in \{1, \ldots, d\}$. Since the vector $y' = (y_1', \ldots, y_d')$ belongs to J_a, the parallelepiped spanned by the vectors $y^{(1)}, \ldots, y^{(k-1)}, y', y^{(k+1)}, \ldots, y^{(d)}$ is included in the parallelepiped J_a. By a volume comparison argument, we deduce that the determinant obtained when replacing the k-th line of Δ by $(y_1', \ldots, y_d')$

belongs to the interval $[0,\Delta)$. A similar argument holds for $y'' = (y_1'', \dots, y_d'')$. The difference of the two determinants obtained in this manner is thus less than Δ in absolute value; it is denoted by $\Delta^{(k)}$ and is equal to the Lebesgue measure of the parallelepiped spanned by the vectors $y^{(1)}, \dots, y^{(k-1)}, v, y^{(k+1)}, \dots, y^{(d)}$, where v stands for $(v_1, \dots, v_d)$. As in the proof of Lemma 1.6, we observe that $\Delta^{(k)}$ is a multiple of c, *i.e.* there exists an integer $a^{(k)}$ such that $\Delta^{(k)} = a^{(k)}c$. Up to exchanging the rôle of y' and y'', we may assume that $a^{(k)} \geq 0$. Furthermore, $\Delta^{(k)} = a^{(k)}c$ is smaller than $\Delta = ac$, so that $a^{(k)} < a$.

Let us now assume that the determinant $\Delta^{(k)}$ vanishes regardless of the value of k. Expanding $\Delta^{(k)}$ along the k-th line, we get

$$0 = \Delta^{(k)} = \sum_{i=1}^{d} v_i Y_i^{(k)},$$

where $Y_i^{(k)}$ denotes the (k,i)-cofactor in Δ, *i.e.* that in the same position as $y_i^{(k)}$. As a consequence of Cramer's rule, the determinant of the integers $Y_i^{(k)}$ is equal to Δ^{d-1}, and is therefore positive. It follows that all the integers v_i vanish. Thus, $u_i = \ell_i q_i'$ for all i, so that

$$\ell_1 q_1' b_1 = \ldots = \ell_d q_d' b_b \mod q.$$

Applying Lemma 1.5 with the help of (41), we deduce that all the integers ℓ_i vanish as well. Finally, the integers u_i are all equal to zero. This contradicts the distinctness of the d-tuples $(x_1', \dots, x_d')$ and $(x_1'', \dots, x_d'')$, and means that one of the determinants $\Delta^{(k)}$ is nonvanishing. Without loss of generality, we may thus assume that $\Delta^{(1)} > 0$. In particular, $a^{(1)} > 0$.

To conclude, we define as follows the $2d^2$ integers $\bar{x}_i^{(k)}$ and $\bar{y}_i^{(k)}$ announced in the statement of the lemma:

$$\begin{cases} \bar{x}_i^{(k)} = x_i^{(k)} & \text{if } k \geq 2 \\ \bar{x}_i^{(1)} = u_i & \\ \bar{y}_i^{(k)} = y_i^{(k)} & \text{if } k \geq 2 \\ \bar{y}_i^{(1)} = v_i & \end{cases}$$

The conditions (1) and (2) obviously hold for $k \geq 2$. When k is equal to one, the condition (1) follows from (41) above, and the condition (1) is due to the simple observation that u_i and v_i coincide modulo q_i' for any index i. On top of that, let us remark that the determinant $\bar{\Delta}$ of the integers $\bar{y}_i^{(k)}$ defined above is equal to $\Delta^{(1)}$; the condition (3') thus holds with $\bar{a} = a^{(1)}$. It remains to establish the condition (4'). The case where $k \geq 2$ is elementary since we then have

$$|\bar{y}_i^{(k)}| = |y_i^{(k)}| \leq \lambda_i \leq d\lambda_i$$

for all index i. To deal with the case where $k = 1$, we use of the fact that the vector v joins two points that belong to the parallelepiped J_a. Thus, its component satisfy

$$|\bar{y}_i^{(1)}| = |v_i| \leq |y_i^{(1)}| + \ldots + |y_i^{(d)}| \leq d\lambda_i \tag{42}$$

for all i, as announced. □

Lemma 1.8. *There exist a real number $C_1 > 0$ and an integer $Q_1 \geq 1$ that depend on γ and d only such that if $Q > Q_1$, then for any integers $m_2, \dots, m_d$, there are $2d$ integers x_i^* and y_i^*, for $i \in \{1, \dots, d\}$, such that the following conditions hold simultaneously:*

(1") *for any $i \in \{2, \dots, d\}$,*

$$x_1^* b_1 - x_i^* b_i = m_i \mod q\,;$$

(2") *for any* $i \in \{1, \dots, d\}$, *there exists an integer* $j_i \in \{0, \dots, e_i - 1\}$ *such that*

$$x_i^* = j_i q_i' + y_i^* \, ;$$

(4") *for any* $i \in \{1, \dots, d\}$,

$$|y_i^*| \le C_1 \, \frac{q^{(d-1)/d}}{e_i}.$$

Specifically, one may choose C_1 *as being equal to* $C_0 d^{C_0^d d^{d/2}}$, *and* Q_1 *as any integer larger than* $C_1/\gamma^{1/d}$.

PROOF. Let $C_1 = C_0 d^{C_0^d d^{d/2}}$, and let Q_1 denote an arbitrary integer larger than $C_1/\gamma^{1/d}$. We assume throughout the proof that $Q > Q_1$. In particular, Q is larger than Q_0, so we may start from the $2d^2$ integers $x_i^{(k)}$ and $y_i^{(k)}$ that are obtained with the help of Lemma 1.6. Each integer $e_i y_i^{(k)}$ is bounded above by $C_0 q^{(d-1)/d}$ in absolute value, so that the Euclidean norm of the vector $(e_1 y_1^{(k)}, \dots, e_d y_d^{(k)})$ satisfies

$$|(e_1 y_1^{(k)}, \dots, e_d y_d^{(k)})|_2 \le d^{1/2} \, |(e_1 y_1^{(k)}, \dots, e_d y_d^{(k)})|_\infty \le C_0 d^{1/2} q^{(d-1)/d}.$$

Moreover, the determinant of the integers $e_i y_i^{(k)}$ is equal to $E\Delta$, which is itself equal to aq^{d-1}. We then deduce from Hadamard's inequality that

$$aq^{d-1} = |E\Delta| \le \prod_{k=1}^{d} |(e_1 y_1^{(k)}, \dots, e_d y_d^{(k)})|_2 \le C_0^d d^{d/2} q^{d-1}.$$

It follows that the integer a is smaller than or equal to $C_0^d d^{d/2}$. Along with the assumption that $Q > Q_1$, this yields

$$q^{1/d} \ge \gamma^{1/d} Q > \gamma^{1/d} Q_1 > C_1 = C_0 d^{C_0^d d^{d/2}} \ge C_0 d^a \ge C_0 d. \tag{43}$$

We now consider for each index i the real number λ_i defined by

$$\lambda_i = C_0 \, \frac{q^{(d-1)/d}}{e_i} = q_i' \frac{C_0}{q^{1/d}}.$$

By virtue of (43), each λ_i is smaller than q_i'/d. If $a > 1$, we may therefore apply Lemma 1.7 to the integers $x_i^{(k)}$ and $y_i^{(k)}$, with the above values for the parameters λ_i. We end up with other integers $\bar{x}_i^{(k)}$ and $\bar{y}_i^{(k)}$ such that the condition (3) holds with a replaced by some other integer $\bar{a} \in \{1, \dots, a-1\}$. We may in fact apply Lemma 1.7 iteratively, thereby decreasing the value of a up to one, provided that the parameters λ_i remain sufficiently small. Specifically, we apply this lemma at most $a - 1$ times; this may be done if the initial values of the parameters satisfy $\lambda_i < q_i'/d^a$ for all i, a requirement that is guaranteed by (43). The upshot is that we may assume that $a = 1$ in what follows, up to multiplying by $d^{C_0^d d^{d/2} - 1}$ the upper bound appearing in the condition (4). Thus, we may finally consider $2d^2$ integers $x_i^{(k)}$ and $y_i^{(k)}$ that satisfy the conditions (1) and (2), the condition (3) with $a = 1$, and the condition (4) with C_0 replaced by $C_0 d^{C_0^d d^{d/2} - 1} = C_1/d$.

We now proceed as in the proof of Lemma 1.7, except that $a = 1$ and the bounds λ_i on the integers $y_i^{(k)}$ satisfy

$$\lambda_i = \frac{C_1}{d} \cdot \frac{q^{(d-1)/d}}{e_i} < \frac{q_i'}{d}. \tag{44}$$

Specifically, we consider the parallelepiped J_1 spanned by the vectors $y^{(1)}, \dots, y^{(d)}$, and we also consider the corresponding set A'. Here, the integers $a^{(k)}$ satisfy

$$0 \le a^{(k)} < a = 1.$$

Thus, all the determinants $\Delta^{(k)}$ necessarily vanish. This implies that the mapping Φ defined on the set A' as in (32) is one-to-one. Also, again because $a = 1$, we have

$$\#A' = q^{d-1} = \#(\mathbb{Z}/q\mathbb{Z})^{d-1}.$$

We deduce that the mapping Φ is a bijection from A' onto $(\mathbb{Z}/q\mathbb{Z})^{d-1}$. As a consequence, for every integers $m_2, \ldots, m_d$ in $\mathbb{Z}$,

$$\exists!(x_1, \ldots, x_d) \in A' \qquad \Phi(x_1, \ldots, x_d) = (m_2, \ldots, m_d) \mod q. \tag{45}$$

Furthermore, as shown in the proof of Lemma 1.7, the mapping Ψ defined on the set A by (40) is a bijection onto A'. Hence,

$$\exists!(j_1, \ldots, j_d, y_1, \ldots, y_d) \in A \qquad \Psi(j_1, \ldots, j_d, y_1, \ldots, y_d) = (x_1, \ldots, x_d). \tag{46}$$

To conclude, it suffices to define $x_i^* = x_i$ and $y_i^* = y_i$ for all indices i. In fact, the conditions (1") and (2") follow straightforwardly from (45) and (46), respectively. The condition (4") is a direct consequence of the approach developed in the proof of Lemma 1.7, along with the values (44) of the bounds λ_i. More precisely, the point $(y_1^*, \ldots, y_d^*)$ belonging to the parallelepiped J_1, its i-th component is bounded by $d\lambda_i$ in absolute value, in a way similar to (42), and the condition (4") finally holds. □

LEMMA 1.9. *There exists a real number $C_2 > 0$ that depends on γ and d only such that*

$$\max_{1 \le i \le d} e_i \le C_2\, q^{(d-1)/d}.$$

Specifically, we may choose C_2 as any real number larger than $2\gamma^{(1-d)/d}$.

PROOF. Let us fix an arbitrary real number $C_2 > 2\gamma^{(1-d)/d}$ and let us suppose that the reverse inequality $e_m > C_2\, q^{(d-1)/d}$ holds for some index m. Thus,

$$Q^{d-1} \le \left(\frac{q}{\gamma}\right)^{(d-1)/d} < \frac{e_m}{C_2} \cdot \frac{C_2}{2} = \frac{e_m}{2}.$$

We now consider the point $(q'_m x_1, \ldots, q'_m x_{m-1}, q'_m x_{m+1}, \ldots, q'_m x_d)$ in $\mathbb{R}^{d-1}$ and apply Dirichlet's theorem, that is, Theorem 1.1. Accordingly, we infer the existence of an integer k and a $(d-1)$-tuple of integers $(n_1, \ldots, n_{m-1}, n_{m+1}, \ldots, n_d)$ such that

$$1 \le k < Q^{d-1} \qquad \text{and} \qquad \forall i \ne m \quad |kq'_m x_i - n_i| \le \frac{1}{Q}.$$

In addition, regarding the m-th component, we have

$$|kq'_m x_m - kp'_m| = \frac{k}{e_m}|qx_m - p_m| \le \frac{k}{e_m Q} < \frac{Q^{d-1}}{e_m Q} < \frac{1}{2Q}.$$

Therefore, letting n_m stand for the product kp'_m, and letting n denote as usual the d-tuple $(n_1, \ldots, n_d)$, we end up with

$$|kq'_m x - n|_\infty \le \frac{1}{Q}.$$

The minimality of the integer q implies that it is less than or equal to kq'_m, so that k is bounded below by e_m. This contradicts the fact that $k < Q^{d-1} < e_m/2$, and the result follows. □

We are now in position to finish the proof of Theorem 1.5. We thus consider a point $\alpha = (\alpha_1, \ldots, \alpha_d)$ in $\mathbb{R}^d$, and for any index i, we define

$$s_i = \lfloor q'_i \alpha_i \rfloor = \left\lfloor \frac{q}{e_i} \alpha_i \right\rfloor.$$

Given that p'_i and q'_i are coprime, there exists an integer r_i such that

$$p'_i r_i = s_i \mod q'_i.$$

We then define $m_i = r_i - r_1$ for any $i \in \{2, \dots, d\}$. We assume that $Q > Q_1$, so as to apply Lemma 1.8. Therefore, we obtain $2d$ integers x_i^* and y_i^*, for $i \in \{1, \dots, d\}$, such that the conditions (1"), (2") and (4") hold simultaneously. In particular, the condition (1") implies that for any $i \in \{2, \dots, d\}$,

$$x_1^* b_1 - x_i^* b_i = r_i - r_1 \mod q, \tag{47}$$

so that the value of $x_i^* b_i + r_i$ modulo q does not depend on the choice of i. This common value is denoted by k and taken in $\{0, \dots, q-1\}$. Therefore, using the fact that q_i' divides q, we get

$$p_i' k = p_i'(x_i^* b_i + r_i) = p_i' b_i x_i^* + p_i' r_i = x_i^* + s_i = y_i^* + s_i \mod q_i'.$$

Note that the last equality above follows directly from the condition (2"). In view of the condition (4"), this implies that there exists an integer y_i such that

$$|p_i' k - q_i' y_i - s_i| = |y_i^*| \leq C_1 \frac{q^{(d-1)/d}}{e_i}.$$

Consequently, due to the definition of the integers s_i, we get

$$|p_i' k - q_i' y_i - q_i' \alpha_i| \leq |p_i' k - q_i' y_i - s_i| + |s_i - q_i' \alpha_i| \leq 1 + C_1 \frac{q^{(d-1)/d}}{e_i}.$$

Multiplying by e_i and making use of Lemma 1.9, we obtain

$$|p_i k - q y_i - q\alpha_i| = e_i |p_i' k - q_i' y_i - q_i' \alpha_i| \leq e_i + C_1 \, q^{(d-1)/d} \leq (C_1 + C_2) q^{(d-1)/d}.$$

Using the approximation property satisfied by the rational number p_i/q with respect to the real number x_i, we deduce that

$$|k x_i - y_i - \alpha_i| \leq \left| k \frac{p_i}{q} - y_i - \alpha_i \right| + k \left| x_i - \frac{p_i}{q} \right| \leq \frac{C_1 + C_2}{q^{1/d}} + \frac{k}{qQ},$$

and consequently that

$$|kx - y - \alpha|_\infty < \frac{C_1 + C_2 + 1}{q^{1/d}}. \tag{48}$$

To conclude, we consider an integer $\mathcal{Q} > 1$, and we suppose that $\mathcal{Q}$ is sufficiently large to ensure that the above arguments may be applied with $Q = \lfloor C_3 \mathcal{Q} \rfloor + 1$, where C_3 stands for $(C_1 + C_2 + 1)/\gamma^{1/d}$. To be more specific, we assume that $Q > Q_1$ or, equivalently, that $\mathcal{Q} \geq \mathcal{Q}_1$, where $\mathcal{Q}_1 = \lceil Q_1 / C_3 \rceil$ and $\lceil \cdot \rceil$ denotes the ceiling function. Recalling that $\gamma Q^d \leq q < Q^d$ and that $k < q$, and defining y as the d-tuple of integers $(y_1, \dots, y_d)$, we may then write that

$$\begin{cases} 1 \leq k < Q^d \leq (2C_3)^d \mathcal{Q}^d \\ |kx - y - \alpha|_\infty \leq (C_1 + C_2 + 1) q^{-1/d} \leq C_3 Q^{-1} \leq \mathcal{Q}^{-1}. \end{cases}$$

In the opposite case where $\mathcal{Q} \leq \mathcal{Q}_1$, we apply what precedes with $Q = \lfloor C_3 \mathcal{Q}_1 \rfloor + 1$, and we therefore obtain

$$\begin{cases} 1 \leq k < Q_1^d \leq (2C_3 \mathcal{Q}_1)^d < (C_3 + Q_1)^d \mathcal{Q}^d \\ |kx - y - \alpha|_\infty \leq (C_1 + C_2 + 1) q^{-1/d} \leq C_3 Q_1^{-1} \leq \mathcal{Q}_1^{-1} \leq \mathcal{Q}^{-1}. \end{cases}$$

We thus finally see that the conclusion of Theorem 1.5 holds with the real number Γ being equal for instance to the maximum of $(2C_3)^d$ and $(C_3 + Q_1)^d$, a value that depends on γ and d only.

1.5.2. A companion result. Inspecting the proof of Theorem 1.5, we may easily establish the next complementary result, which will be called upon in the proof of Theorem 7.3. For any point $x \in \mathbb{R}^d$ and any integer $Q > 1$, let us define

$$q(x,Q) = \inf \left\{ q \in \mathbb{N} \;\middle|\; |qx - p|_\infty \leq \frac{1}{Q} \text{ for some } p \in \mathbb{Z}^d \right\}.$$

It follows from Dirichlet's theorem that $q(x,Q)$ is finite; in fact, $q(x,Q)$ is less than Q^d, see Theorem 1.1 above.

PROPOSITION 1.11. *For any real number $\gamma \in (0,1)$, there exist a real number $\Gamma_* > 1$ and an integer $Q_* \geq 1$, both depending on γ and d only, such that the following property holds: for any points x and α in $\mathbb{R}^d$ and for any integer $Q > Q_*$,*

$$q(x,Q) \geq \gamma Q^d \quad \Longrightarrow \quad \exists (p,q) \in \mathbb{Z}^d \times \mathbb{N} \quad \begin{cases} q(x,Q) \leq q < 2q(x,Q) \\ |qx - p - \alpha|_\infty \leq \Gamma_* / q(x,Q)^{1/d}. \end{cases}$$

PROOF. It suffices to recast the last part of the proof of Theorem 1.5. Indeed, assuming that $Q > Q_1$ and applying Lemma 1.8, we ended up therein with (47), and then with some crucial integer k, that will play the rôle of q in the statement of Proposition 1.11. Note that k is determined modulo $q(x,Q)$ so, instead of choosing this integer between zero and $q(x,Q) - 1$ as in the proof of Theorem 1.5, we may choose it between $q(x,Q)$ and $2q(x,Q) - 1$. The required approximation property is then a reformulation of (48). This means in particular that the real number Γ_* corresponds to the term $C_1 + C_2 + 1$ in the proof of Theorem 1.5, and that the integer Q_* may be chosen to be equal to Q_1. □

1.5.3. Converse to the theorem. Khintchine actually showed in [**39**] that Theorem 1.5 gives a characterization of the uniform inhomogeneous approximation. As a matter of fact, it is quite easy to establish the following converse result.

PROPOSITION 1.12. *Let us consider a point $x \in \mathbb{R}^d$ and let us assume that there exists a real number $\Gamma > 0$ such that for any point $\alpha \in \mathbb{R}^d$ and any integer $Q > 1$, the system*

$$\begin{cases} 1 \leq q < \Gamma Q^d \\ |qx - p - \alpha|_\infty \leq 1/Q \end{cases}$$

admits a solution (p,q) in $\mathbb{Z}^d \times \mathbb{N}$. Then, there exists another real number $\gamma > 0$ such that for any integer $Q > 1$, the system

$$\begin{cases} 1 \leq q < \gamma Q^d \\ |qx - p|_\infty \leq 1/Q \end{cases}$$

admits no solution (p,q) in $\mathbb{Z}^d \times \mathbb{N}$.

PROOF. We argue by contradiction. Thus, for any real number $\varepsilon > 0$, there exists an integer $Q > 1$, and a pair $(p,q) \in \mathbb{Z}^d \times \mathbb{N}$ satisfying

$$1 \leq q < \varepsilon^{d(d+2)} Q^d \qquad \text{and} \qquad |qx - p|_\infty \leq \frac{1}{Q}.$$

Now, letting $\overline{\mathrm{B}}_\infty(x,r)$ denote the closed ball centered at x with radius r, in the sense of the supremum norm, we define the set

$$M_\varepsilon = \bigcup_{k=1}^{q} \bigcup_{y \in \mathbb{Z}^d} \overline{\mathrm{B}}_\infty \left(k\frac{p}{q} + y, \frac{2\varepsilon}{q^{1/d}} \right).$$

We assume from now on that ε is smaller than $1/12$. For each integer k, there are at most 3^d points y in $\mathbb{Z}^d$ for which the above closed ball meets the cube $[0,1)^d$. This means that the volume occupied in the unit cube by the set M_ε satisfies

$$\mathcal{L}^d([0,1)^d \cap M_\varepsilon) \leq 3^d q \left(\frac{4\varepsilon}{q^{1/d}}\right)^d = (12\varepsilon)^d < 1.$$

We may therefore consider a point α in the unit cube $[0,1)^d$ that does not belong to the set M_ε. We also introduce the integer $\mathcal{Q} = \lceil q^{1/d}/\varepsilon \rceil$. The point α verifies

$$\left|\alpha - k\frac{p}{q} - y\right|_\infty > \frac{2\varepsilon}{q^{1/d}}$$

for all points $y \in \mathbb{Z}^d$ and all integers $k \in \{1, \dots, q\}$. In addition, if the integer k is smaller than $\mathcal{Q}^d/(2^d\varepsilon)$, then it is *a fortiori* smaller than q/ε^{d+1}, so that

$$\left|kx - k\frac{p}{q}\right|_\infty = \frac{k}{q}|qx - p|_\infty < \frac{k}{q} \cdot \frac{\varepsilon^{d+2}}{q^{1/d}} < \frac{\varepsilon}{q^{1/d}}.$$

We thus built the point α and the integer $\mathcal{Q}$ in such a way that for any point y in $\mathbb{Z}^d$ and any positive integer k smaller than $\mathcal{Q}^d/(2^d\varepsilon)$, we have

$$|kx - y - \alpha|_\infty > \frac{\varepsilon}{q^{1/d}} \geq \frac{1}{\mathcal{Q}}.$$

We deduce that Γ must be larger than $1/(2^d\varepsilon)$. However, the above arguments are valid for arbitrarily small values of ε. This leads to a contradiction. $\square$

Among Khintchine's works, Theorem 1.5 and its converse, namely, Proposition 1.12 may be regarded as an anticipation of his deep transference principle that relates homogeneous and inhomogeneous problems, see *e.g.* [**16**, Chapter V].

CHAPTER 2

Hausdorff measures and dimension

The material discussed in this section is standard; our main references are [**29**, Chapters 2 and 4] and [**46**, Chapter 4], as well as [**51**]. The notion of Hausdorff dimension relies on that of Hausdorff measure; the first definitions and properties of Hausdorff measures were established by Carathéodory (1914) and Hausdorff (1919). Throughout this section, we restrict our attention to the space $\mathbb{R}^d$, even if the discussed notions may be defined in more general metric spaces.

2.1. Outer measures and measurability

Before dealing with Hausdorff measures, we introduce general definitions and establish standard results from geometric measure theory. We shall not follow here the standard approach that originates in the work of Radon and consists in defining measures on prespecified σ-fields. Instead, our viewpoint is that initiated by Carathéodory: considering *outer measures* on all the subsets of the space $\mathbb{R}^d$, and then discussing further *measurability* properties of the subsets. The collection of all subsets of $\mathbb{R}^d$ is denoted by $\mathcal{P}(\mathbb{R}^d)$.

Definition 2.1. A function $\mu : \mathcal{P}(\mathbb{R}^d) \to [0,\infty]$ is called an *outer measure* if the following conditions are fulfilled:

(1) $\mu(\emptyset) = 0$;
(2) for any sets E_1 and E_2 in $\mathcal{P}(\mathbb{R}^d)$,
$$E_1 \subseteq E_2 \qquad \Longrightarrow \qquad \mu(E_1) \leq \mu(E_2)\,;$$
(3) for any sequence $(E_n)_{n\geq 1}$ in $\mathcal{P}(\mathbb{R}^d)$,
$$\mu\left(\bigcup_{n=1}^{\infty} E_n\right) \leq \sum_{n=1}^{\infty} \mu(E_n).$$

Hence, outer measures are defined on the whole collection $\mathcal{P}(\mathbb{R}^d)$. However, they will enjoy further properties when restricted to the subcollection formed by the sets that are *measurable*.

Definition 2.2. Let μ be an outer measure. Then, a set E in $\mathcal{P}(\mathbb{R}^d)$ is called *μ-measurable* if for all sets A and B in $\mathcal{P}(\mathbb{R}^d)$,
$$\begin{cases} A \subseteq E \\ B \subseteq \mathbb{R}^d \setminus E \end{cases} \qquad \Longrightarrow \qquad \mu(A \sqcup B) = \mu(A) + \mu(B).$$
The collection of all μ-measurable sets is denoted by $\mathcal{F}_\mu$.

The two sets A and B arising in the above definition are said to be *separated* by the set E. Thus, a set E is μ-measurable if the outer measure μ is additive on sets that are separated by E. Let us also mention that it suffices to consider the case in which the sets A and B have finite μ-mass, and to prove that $\mu(A \cup B)$ is bounded below by the sum of $\mu(A)$ and $\mu(B)$.

The connection with the standard approach of measures on σ-fields is then given by the following result. In its statement, we say that a set $N \in \mathcal{P}(\mathbb{R}^d)$ is *μ-negligible* if its measure vanishes, namely, $\mu(N) = 0$.

THEOREM 2.1. *Let μ be an outer measure, and let $\mathcal{F}_\mu$ denote the collection of all μ-measurable sets. Then, the following properties hold:*

(1) *the collection $\mathcal{F}_\mu$ is a σ-field of $\mathbb{R}^d$;*
(2) *every μ-negligible set in $\mathcal{P}(\mathbb{R}^d)$ belongs to $\mathcal{F}_\mu$;*
(3) *for any sequence $(E_n)_{n\geq 1}$ of* disjoint *sets in $\mathcal{F}_\mu$, we have*

$$\mu\left(\bigsqcup_{n=1}^{\infty} E_n\right) = \sum_{n=1}^{\infty} \mu(E_n).$$

PROOF. We begin by establishing (2). To proceed, let us consider a set N in $\mathcal{P}(\mathbb{R}^d)$ such that $\mu(N) = 0$. Then, for any sets $A \subseteq N$ and $B \subseteq \mathbb{R}^d \setminus N$, we have

$$\mu(B) \leq \mu(A \cup B) \leq \mu(A) + \mu(B) \leq \mu(N) + \mu(B) = \mu(B),$$

from which we deduce that $\mu(A \cup B)$ is equal to the sum of $\mu(A)$ and $\mu(B)$. This implies that N belongs to $\mathcal{F}_\mu$.

In particular, as the empty set is μ-negligible, it belongs to the collection $\mathcal{F}_\mu$. Furthermore, the definition of a μ-measurable set is symmetric, in such a way that if a set E belongs to $\mathcal{F}_\mu$, then its complement $\mathbb{R}^d \setminus E$ belongs to $\mathcal{F}_\mu$ as well.

Let us now consider two sets E_1 and E_2 in $\mathcal{F}_\mu$ and show that their union $E_1 \cup E_2$ belongs to $\mathcal{F}_\mu$. To this purpose, let A and B denote two sets with finite μ-mass that satisfy $A \subseteq E_1 \cup E_2$ and $B \subseteq \mathbb{R}^d \setminus (E_1 \cup E_2)$. Now, remark that the two sets $A \cap E_1$ and $(A \cup B) \cap (\mathbb{R}^d \setminus E_1)$ are separated by the measurable set E_1 and that their union reduces to $A \cup B$. Hence,

$$\mu(A \cup B) = \mu(A \cap E_1) + \mu((A \cup B) \cap (\mathbb{R}^d \setminus E_1)).$$

Moreover, the sets $A \cap (\mathbb{R}^d \setminus E_1)$ and B are separated by the measurable set E_2, and their union is equal to the set whose measure corresponds to the second term above. Therefore,

$$\mu(A \cup B) = \mu(A \cap E_1) + \mu(A \cap (\mathbb{R}^d \setminus E_1)) + \mu(B).$$

However, the sets arising in the first two terms are clearly separated by E_1 and their union is equal to A, so the sum of these two terms reduces to $\mu(A)$. This means that $E_1 \cup E_2$ is μ-measurable.

Now, let us consider a sequence $(E_n)_{n\geq 1}$ of disjoint sets in $\mathcal{F}_\mu$, and let us show that their union, denoted by E, belongs to the collection $\mathcal{F}_\mu$, and that the formula in (3) holds. To proceed, let A denote a subset of E and let B denote a subset of its complement $\mathbb{R}^d \setminus E$. Fixing an integer $m \geq 1$ and applying what precedes iteratively to the sets $E_1, \ldots, E_m$, we infer that the union of these sets is μ-measurable, so

$$\mu(A \cup B) \geq \mu\left(\left(A \cap \bigsqcup_{n=1}^{m} E_n\right) \cup B\right) = \mu\left(A \cap \bigsqcup_{n=1}^{m} E_n\right) + \mu(B),$$

because the aforementioned union separates its intersection with the set A from the set B. Furthermore, the set E_m is disjoint from the sets $E_1, \ldots, E_{m-1}$ and is μ-measurable, so the first term in the right-hand side is equal to

$$\mu\left(\left(A \cap \bigsqcup_{n=1}^{m-1} E_n\right) \sqcup (A \cap E_m)\right) = \mu\left(A \cap \bigsqcup_{n=1}^{m-1} E_n\right) + \mu(A \cap E_m).$$

Iterating this procedure, we infer that this term is equal to the sum of $\mu(A \cap E_n)$ over all $n \in \{1, \ldots, m\}$. Thus, letting m go to infinity, we end up with

$$\mu(A \cup B) \geq \sum_{n=1}^{\infty} \mu(A \cap E_n) + \mu(B) \geq \mu\left(A \cap \bigsqcup_{n=1}^{\infty} E_n\right) + \mu(B) = \mu(A) + \mu(B),$$

from which we derive that the set E is μ-measurable. Furthermore, letting B be the empty set, we readily deduce that

$$\mu(A) = \sum_{n=1}^{\infty} \mu(A \cap E_n)\,; \tag{49}$$

the formula in (3) now follows from choosing A to be equal to the whole set E.

In order to establish (1), it remains to show that when $(E_n)_{n \geq 1}$ is a sequence of sets in $\mathcal{F}_\mu$ that are not necessarily disjoint, the union of these sets also belongs to $\mathcal{F}_\mu$. Given an integer $m \geq 1$, what precedes ensures that the union $E_1 \cup \ldots \cup E_{m-1}$ is μ-measurable, as well as the set

$$E_m \cap \left(\mathbb{R}^d \setminus \bigcup_{n=1}^{m-1} E_n\right) = \mathbb{R}^d \setminus \left((\mathbb{R}^d \setminus E_m) \cup \bigcup_{n=1}^{m-1} E_n\right).$$

Here, we adopt the convention that the union is equal to the empty set if m is equal to one. When m varies, the latter sets form a sequence of disjoint measurable sets, and what precedes implies that their union, which coincides with the union of the original sets E_n, belongs to $\mathcal{F}_\mu$. □

Theorem 2.1 helps clarifying the connection between the standard viewpoint, and the approach of outer measures that we adopt here. To be specific, this result ensures that the restriction of an outer measure μ to the σ-field $\mathcal{F}_\mu$ is a measure in the usual sense of for instance [**61**, Chapter 1]. Conversely, let us consider a measure ν defined on some σ-field $\mathcal{F}$ of subsets of $\mathbb{R}^d$. We may extend ν to the whole collection $\mathcal{P}(\mathbb{R}^d)$ by letting

$$\nu^*(E) = \inf_{\substack{F \in \mathcal{F} \\ F \supseteq E}} \nu(F) \tag{50}$$

for any set $E \in \mathcal{P}(\mathbb{R}^d)$. This way, we obtain an outer measure, and the σ-field of all ν^*-measurable sets contains the original σ-field $\mathcal{F}$. This is indeed a particular case of a more general construction that we now present.

2.2. From premeasures to outer measures: the abstract viewpoint

Rather than just building an outer measure as the extension of a usual measure, we shall explain how to obtain an outer measure starting from a function defined on a class of subsets of $\mathbb{R}^d$.

Definition 2.3. A *premeasure* is a function of the form $\zeta : \mathcal{C} \to [0, \infty]$, where $\mathcal{C}$ is a collection of subsets of $\mathbb{R}^d$ containing the empty set, that satisfies $\zeta(\emptyset) = 0$.

The construction makes use of the standard notion of covering. Given a set E in $\mathcal{P}(\mathbb{R}^d)$ and a collection $\mathcal{C}$ of subsets of $\mathbb{R}^d$ containing the empty set, recall that a sequence of sets $(C_n)_{n \geq 1}$ in $\mathcal{C}$ is called a *covering* of E if

$$E \subseteq \bigcup_{n=1}^{\infty} C_n.$$

Note that this definition encompasses the case of coverings by finitely many sets, as we can choose the sets C_n to be empty when n is large enough. The next result gives a general method to build an outer measure starting from a premeasure.

THEOREM 2.2. *Let $\mathcal{C}$ be a collection of subsets of $\mathbb{R}^d$ containing the empty set, and let ζ be a premeasure defined on $\mathcal{C}$. Then, the function ζ^* defined on $\mathcal{P}(\mathbb{R}^d)$ by*

$$\zeta^*(E) = \inf_{\substack{E \subseteq \bigcup_n C_n \\ C_n \in \mathcal{C}}} \sum_{n=1}^{\infty} \zeta(C_n) \tag{51}$$

is an outer measure. Here, the infimum is taken over all coverings of the set E by sequences $(C_n)_{n\geq 1}$ of sets that belong to $\mathcal{C}$.

PROOF. It is clear from the definition that ζ^* is a function defined on $\mathcal{P}(\mathbb{R}^d)$ with values in $[0, \infty]$. Moreover, as the empty set belongs to the collection $\mathcal{C}$, it can be used to cover itself, so that $\zeta^*(\emptyset) \leq \zeta(\emptyset) = 0$. In addition, if two subsets E_1 and E_2 of $\mathbb{R}^d$ are such that $E_1 \subseteq E_2$, then any covering of E_2 is also a covering of E_1, so that $\zeta^*(E_1) \leq \zeta^*(E_2)$.

The only nontrivial property is thus the subadditivity of ζ^*. To prove this fact, let us consider a sequence $(E_n)_{n\geq 1}$ of subsets of $\mathbb{R}^d$, and let E denote their union. We may clearly assume that the sum of the ζ^*-masses of the sets E_n is finite. In particular, every set E_n has finite measure, so that if some real $\varepsilon > 0$ is fixed in advance, we have

$$\sum_{m=1}^{\infty} \zeta(C^n_m) \leq \zeta^*(E_n) + \varepsilon 2^{-n}$$

for some covering $(C^n_m)_{m\geq 1}$ of the set E_n by sets from the collection $\mathcal{C}$. Then, the doubly-indexed sequence $(C^n_m)_{m,n\geq 1}$ clearly forms a covering of the set E by sets from the collection $\mathcal{C}$. Hence,

$$\zeta^*(E) \leq \sum_{n=1}^{\infty} \sum_{m=1}^{\infty} \zeta(C^n_m) \leq \sum_{n=1}^{\infty} \left(\zeta^*(E_n) + \varepsilon 2^{-n}\right) \leq \left(\sum_{n=1}^{\infty} \zeta^*(E_n)\right) + \varepsilon,$$

and the result follows by letting ε go to zero. □

The next result is elementary and shows that the above procedure is "closed", in the sense that it leaves the outer measures unchanged.

PROPOSITION 2.1. *Let μ be an outer measure. Then, μ may be seen as a premeasure on $\mathcal{P}(\mathbb{R}^d)$ and the outer measure μ^* defined via (51) coincides with μ.*

PROOF. Let E denote a subset of $\mathbb{R}^d$. Covering the set E by itself and the empty set, we infer that $\mu^*(E) \leq \mu(E)$. Conversely, let us observe that for any covering $(C_n)_{n\geq 1}$ of the set E by subsets of $\mathbb{R}^d$,

$$\mu(E) \leq \mu\left(\bigcup_{n=1}^{\infty} C_n\right) \leq \sum_{n=1}^{\infty} \mu(C_n).$$

Taking the infimum over all the possible coverings in the right-hand side, we deduce that $\mu(E) \leq \mu^*(E)$. □

The next result now gives a rigorous justification to the remarks that we made around the formula (50) above. In particular, we shall show that if ν^* is defined through (51), then it actually takes the simpler form (50).

PROPOSITION 2.2. *Let ν be a measure defined on a σ-field $\mathcal{F}$ of subsets of $\mathbb{R}^d$. Then, ν may be seen as a premeasure on $\mathcal{F}$ and the outer measure ν^* defined via (51) satisfies the following properties:*

(1) *the σ-field $\mathcal{F}_{\nu^*}$ of all ν^*-measurable sets contains $\mathcal{F}$;*
(2) *the restriction of ν^* to the σ-field $\mathcal{F}$ coincides with ν;*

(3) *the formula (50) holds, namely,*

$$\nu^*(E) = \inf_{\substack{F \in \mathcal{F} \\ F \supseteq E}} \nu(F)$$

for any set $E \in \mathcal{P}(\mathbb{R}^d)$, and the infimum is attained.

PROOF. Let us consider a subset E of $\mathbb{R}^d$ and a covering $(F_n)_{n\geq 1}$ of the set E by sets belonging to the σ-field $\mathcal{F}$. Then, the union F of the sets F_n belongs to $\mathcal{F}$, as well as the sets $G_n = F_n \setminus (F_1 \cup \ldots \cup F_{n-1})$. The latter sets form a partition of F and each of them is included in the corresponding set F_n, so that

$$\sum_{n=1}^{\infty} \nu(F_n) \geq \sum_{n=1}^{\infty} \nu(G_n) = \nu\left(\bigcup_{n=1}^{\infty} G_n\right) = \nu(F).$$

Taking the infima above, we deduce that $\nu^*(E)$ is bounded below by the infimum of $\nu(F)$ over all sets $F \in \mathcal{F}$ such that $F \supseteq E$.

Conversely, let us consider a set $F \in \mathcal{F}$ satisfying $F \supseteq E$. Covering the set E by F and the empty set, we infer that $\nu^*(E) \leq \nu(F)$. We may now take the infimum over all sets F. Combined with what precedes, this ensures that (50) holds.

Moreover, (50) ensures that there exists a sequence $(F_n)_{n\geq 1}$ of supersets of E that belong to $\mathcal{F}$ and satisfy $\nu(F_n) \leq \nu^*(E) + 1/n$ for all $n \geq 1$. Now, the intersection F of these sets belongs to $\mathcal{F}$ and contains E, so that

$$\nu^*(E) \leq \nu(F) \leq \nu(F_n) \leq \nu^*(E) + \frac{1}{n}$$

for all $n \geq 1$. Letting n go to infinity, we deduce that $\nu^*(E) = \nu(F)$, so that the infimum in (50) is attained. Besides, note that (50) also ensures that $\nu^*(E)$ coincides with $\nu(E)$ when E belongs to $\mathcal{F}$.

It remains to establish that any set F in $\mathcal{F}$ is ν^*-measurable. To proceed, let us consider a subset A of F and a subset B of $\mathbb{R}^d \setminus F$. As the infimum is attained in (50), there exists a superset G of $A \cup B$ that belongs to the σ-field $\mathcal{F}$ and satisfies $\nu^*(A \cup B) = \nu(G)$. Then, the sets $F \cap G$ and $(\mathbb{R}^d \setminus F) \cap G$ are disjoint, belong to $\mathcal{F}$, and contain the sets A and B, respectively. Hence,

$$\nu^*(A) + \nu^*(B) \leq \nu(F \cap G) + \nu((\mathbb{R}^d \setminus F) \cap G) = \nu(G) = \nu^*(A \cup B),$$

from which we deduce that F is ν^*-measurable. $\square$

Note that the restriction of an outer measure μ to the σ-field $\mathcal{F}_\mu$ of its measurable sets is a measure in the classical sense. It is then natural to ask whether we can recover the outer measure μ by applying the above procedure to its restriction. This will not happen in general, except if the outer measure μ is *regular* in the following sense.

DEFINITION 2.4. An outer measure μ on $\mathbb{R}^d$ is said to be *regular* if for any set $E \in \mathcal{P}(\mathbb{R}^d)$, there exists a set $F \in \mathcal{F}_\mu$ such that

$$E \subseteq F \qquad \text{and} \qquad \mu(E) = \mu(F).$$

Using the terminology of this definition, we may deduce from Proposition 2.2 that for any measure ν defined on a σ-field $\mathcal{F}$, the outer measure ν^* defined via (51) is regular. Indeed, given a set $E \in \mathcal{P}(\mathbb{R}^d)$, Proposition 2.2(3) ensures that there exists a set $F \in \mathcal{F}$ such that $E \subseteq F$ and $\nu^*(E) = \nu(F)$, which coincides with $\nu^*(F)$ by virtue of Proposition 2.2(2). The above discussion can be summarized in the following statement.

PROPOSITION 2.3. *Let μ be an outer measure, and let ν denote the restriction of μ to the σ-field $\mathcal{F}_\mu$ of its measurable sets. Then, ν may be seen as a premeasure on $\mathcal{F}_\mu$ and the outer measure ν^* defined via (51) satisfies the following properties:*

(1) *the outer measure* ν^* *is regular;*
(2) *all* μ*-measurable sets are* ν^**-measurable, that is,* $\mathcal{F}_\mu \subseteq \mathcal{F}_{\nu^*}$*;*
(3) *all* ν^**-measurable sets of* finite ν^*-mass *are* μ*-measurable;*
(4) ν^* *coincides with* μ *if and only if* μ *is regular.*

Proof. To begin with, Theorem 2.1 ensures that ν is a measure on the σ-field $\mathcal{F}_\mu$. Then, as already mentioned above, Proposition 2.2 ensures that the outer measure ν^* is regular, coincides with ν on $\mathcal{F}_\mu$, and satisfies

$$\nu^*(E) = \inf_{\substack{F \in \mathcal{F}_\mu \\ F \supseteq E}} \nu(F)$$

for any set $E \in \mathcal{P}(\mathbb{R}^d)$, where the infimum is attained. Moreover, Proposition 2.2 also ensures that $\mathcal{F}_\mu \subseteq \mathcal{F}_{\nu^*}$.

Conversely, let us now consider a set E in $\mathcal{F}_{\nu^*}$ and assume that E has finite ν^*-mass. Then, as the infimum above is attained, there exists a set $F \in \mathcal{F}_\mu$ that contains E and satisfies $\nu(F) = \nu^*(E)$, the latter quantity being equal to $\nu^*(F)$ because ν^* and ν coincide on $\mathcal{F}_\mu$. We deduce that

$$\nu^*(E) = \nu^*(F) = \nu^*(F \setminus E) + \nu^*(E).$$

Given that $\nu^*(E)$ is finite, it follows that $F \setminus E$ is ν^*-negligible. Thus, using again the fact that the above infimum is attained, we infer that there exists a set $G \in \mathcal{F}_\mu$ that contains $F \setminus E$ and satisfies $\nu(G) = \nu^*(E \setminus F) = 0$. Since ν coincides with the outer measure μ on $\mathcal{F}_\mu$, we see that $\mu(E \setminus F) \leq \mu(G) = \nu(G) = 0$. Thus, the set $E \setminus F$ is μ-negligible, and is therefore μ-measurable, by virtue of Theorem 2.1. Recalling that F is μ-measurable, we conclude that E is μ-measurable as well.

Finally, if μ coincides with ν^*, then it is necessarily regular, because ν^* is so. Conversely, if μ is regular, then

$$\mu(E) = \inf_{\substack{F \in \mathcal{F}_\mu \\ F \supseteq E}} \mu(F) = \inf_{\substack{F \in \mathcal{F}_\mu \\ F \supseteq E}} \nu(F) = \nu^*(E)$$

for any set $E \in \mathcal{P}(\mathbb{R}^d)$, so that the outer measures μ and ν^* coincide. □

2.3. Further properties of measurable sets

Let us now mention some useful properties satisfied by the measurable sets.

Proposition 2.4. *Let* μ *denote an outer measure on* $\mathbb{R}^d$ *and let* $(F_n)_{n\geq 1}$ *be a nondecreasing sequence of subsets of* $\mathbb{R}^d$*. The following properties hold:*

(1) *if the sets* F_n *are* μ*-measurable and* E *is an arbitrary subset of* $\mathbb{R}^d$*, then*

$$\mu\left(E \cap \bigcup_{n=1}^{\infty} \uparrow F_n\right) = \lim_{n\to\infty} \uparrow \mu(E \cap F_n)\,;$$

(2) *if the outer measure* μ *is regular, then*

$$\mu\left(\bigcup_{n=1}^{\infty} \uparrow F_n\right) = \lim_{n\to\infty} \uparrow \mu(F_n).$$

Proof. Let us begin by assuming that the sets F_n are μ-measurable and that E is a set in $\mathcal{P}(\mathbb{R}^d)$. Given that the sequence $(F_n)_{n\geq 1}$ is nondecreasing, we obtain a sequence $(G_n)_{n\geq 1}$ of disjoint μ-measurable sets simply by letting $G_1 = F_1$ and $G_n = F_n \setminus F_{n-1}$ for any integer $n \geq 2$. Then, for any subset A of the union of the sets G_n, it follows from (49) that $\mu(A)$ is the sum of $\mu(A \cap G_n)$ over all $n \geq 1$.

Thus, on the one hand, choosing A to be the intersection of the set E with the union of all the sets F_n, we deduce that

$$\mu\left(E \cap \bigcup_{n=1}^{\infty} \uparrow F_n\right) = \sum_{n=1}^{\infty} \mu(E \cap G_n)$$

On the other hand, fixing an integer $m \geq 1$ and letting A be the intersection of E with the union of the sets $G_1, \ldots, G_m$, we get

$$\sum_{n=1}^{m} \mu(E \cap G_n) = \mu\left(E \cap \bigsqcup_{n=1}^{m} G_n\right) = \mu(E \cap F_m).$$

The first part of the result then follows from letting m go to infinity.

Let us now drop the measurability assumption on the sets F_n and suppose instead that the outer measure μ is regular. First, it is clear that the μ-mass of every single set F_n is bounded by that of the union of these sets. Thus,

$$\lim_{n\to\infty} \uparrow \mu(F_n) \leq \mu\left(\bigcup_{n=1}^{\infty} \uparrow F_n\right). \tag{52}$$

For the reverse inequality, let us observe that for any integer $n \geq 1$, the regularity of μ ensures the existence of a μ-measurable superset H_n of F_n that has the same μ-mass. Then, the monotonicity of the sequence $(F_n)_{n\geq 1}$ implies that $F_n \subseteq I_n \subseteq H_n$ for all n, where I_n is defined as the intersection over all $m \geq n$ of the sets H_m. Now, observe that $(I_n)_{n\geq 1}$ is a nondecreasing sequence of μ-measurable sets, each of them having the same μ-mass as its counterpart in the original sequence $(F_n)_{n\geq 1}$. As a consequence, the first part of the proof above ensures that

$$\mu\left(\bigcup_{n=1}^{\infty} \uparrow F_n\right) \leq \mu\left(\bigcup_{n=1}^{\infty} \uparrow I_n\right) = \lim_{n\to\infty} \uparrow \mu(I_n) = \lim_{n\to\infty} \uparrow \mu(F_n),$$

and the result follows. □

PROPOSITION 2.5. *Let μ denote an outer measure on $\mathbb{R}^d$ and let $(F_n)_{n\geq 1}$ be a nonincreasing sequence of μ-measurable sets. Then, for any subset E of $\mathbb{R}^d$ such that $\mu(E \cap F_n) < \infty$ for some integer $n \geq 1$,*

$$\mu\left(E \cap \bigcap_{n=1}^{\infty} \downarrow F_n\right) = \lim_{n\to\infty} \downarrow \mu(E \cap F_n).$$

PROOF. Let m denote an integer for which $\mu(E \cap F_m)$ is finite. Then, let us consider the sets $G_n = F_m \setminus F_{m+n}$, for $n \geq 1$; they form a nondecreasing sequence of μ-measurable sets to which we may apply Proposition 2.4(1), thereby getting

$$\mu\left(E \cap \bigcup_{n=1}^{\infty} \uparrow G_n\right) = \lim_{n\to\infty} \uparrow \mu(E \cap G_n)$$

Now, the subsequence $(F_n)_{n\geq m+1}$ is formed of μ-measurable sets, and the set $E\cap F_m$ has finite μ-mass, so that the left-hand side of this equality is equal to

$$\mu\left(E \cap F_m \setminus \bigcap_{n=m+1}^{\infty} \downarrow F_n\right) = \mu(E \cap F_m) - \mu\left(E \cap F_m \cap \bigcap_{n=m+1}^{\infty} \downarrow F_n\right).$$

Likewise, its right-hand side is the limit as n goes to infinity of

$$\mu(E \cap F_m \setminus F_n) = \mu(E \cap F_m) - \mu(E \cap F_m \cap F_n).$$

It is now plain that the above equalities lead to desired result. □

2.4. From premeasures to outer measures: the metric viewpoint

We explained in Section 2.2 how to build an outer measure starting from a premeasure defined on a class of subsets of $\mathbb{R}^d$. Let us now present another way of extending a premeasure into an outer measure, by taking additionally into account the metric structure of the ambiant space $\mathbb{R}^d$. Accordingly, the next result is the counterpart of Theorem 2.2. The diameter of an arbitrary set $E \in \mathcal{P}(\mathbb{R}^d)$ is denoted by $|E|$ in what follows.

THEOREM 2.3. *Let $\mathcal{C}$ be a collection of subsets of $\mathbb{R}^d$ containing the empty set, and let ζ be a premeasure defined on $\mathcal{C}$. Then, the function ζ_* defined on $\mathcal{P}(\mathbb{R}^d)$ by*

$$\zeta_*(E) = \lim_{\delta \downarrow 0} \uparrow \zeta_\delta(E) \qquad \textit{with} \qquad \zeta_\delta(E) = \inf_{\substack{E \subseteq \bigcup_n C_n \\ C_n \in \mathcal{C}, |C_n| \leq \delta}} \sum_{n=1}^{\infty} \zeta(C_n) \tag{53}$$

is an outer measure. Here, the infimum is taken over all coverings of the set E by sequences $(C_n)_{n \geq 1}$ of sets belonging to $\mathcal{C}$ with diameter at most δ.

PROOF. The result follows straightforwardly from Theorem 2.2, combined with a simple observation. As a matter of fact, for any fixed $\delta > 0$, Theorem 2.2 implies that ζ_δ is an outer measure, namely, that obtained from the restriction of the premeasure ζ to the collection of sets in $\mathcal{C}$ whose diameter is at most δ. It now suffices to observe that ζ_* may also be written as the supremum over all $\delta > 0$ of the outer measures ζ_δ, and make use of the obvious fact that the supremum of an arbitrary family of outer measures is also an outer measure. □

Let us mention that it is obvious from (51) and (53) that for any premeasure ζ and any subset E of $\mathbb{R}^d$, we have $\zeta^*(E) \leq \zeta_\delta(E)$ for all $\delta > 0$; thus, taking the limit as δ goes to zero, we deduce that

$$\forall E \subseteq \mathbb{R}^d \qquad \zeta^*(E) \leq \zeta_*(E). \tag{54}$$

The main advantage of the above construction over that given by Theorem 2.2 is that one does not need to check whether two given disjoint sets are measurable when intending to apply the additivity property of the outer measure ζ_* to their union. Thus, ζ_* falls into the category of *metric outer measures* that we now define.

DEFINITION 2.5. An outer measure μ on $\mathbb{R}^d$ is said to be *metric* if for all sets A and B in $\mathcal{P}(\mathbb{R}^d) \setminus \{\emptyset\}$,

$$\mathrm{d}(A, B) > 0 \qquad \Longrightarrow \qquad \mu(A \sqcup B) = \mu(A) + \mu(B).$$

In the previous definition, $\mathrm{d}(A, B)$ denotes the distance between the sets A and B, that is, the infimum of $|a - b|$ over all $a \in A$ and $b \in B$. When this distance is positive, the sets are said to be *positively separated*. The previous remark now takes the form of the following precise result.

PROPOSITION 2.6. *For any choice of the premeasure ζ, the outer measure ζ_* defined via (53) is metric.*

PROOF. Let us consider two nonempty subsets A and B of $\mathbb{R}^d$, and let us assume that $\mathrm{d}(A, B) > 0$. As ζ_* is an outer measure, it suffices to prove that the sum of the ζ_*-masses of these sets A and B is at most the ζ_*-mass of their union, which we may assume to be finite. For any $\varepsilon, \delta_1, \delta_2 > 0$, letting $\delta = \min\{\delta_1, \delta_2, \mathrm{d}(A, B)/2\}$, we deduce from (53) that there exists a sequence $(C_n)_{n \geq 1}$ of sets in $\mathcal{C}$ with diameter at most δ such that

$$A \sqcup B \subseteq \bigcup_{n=1}^{\infty} C_n \qquad \text{and} \qquad \sum_{n=1}^{\infty} \zeta(C_n) \leq \zeta_*(A \sqcup B) + \varepsilon.$$

Note that none of the sets C_n can intersect both A and B. Indeed, in that situation, there would exist two points $a \in A \cap C_n$ and $b \in B \cap C_n$, which would lead to

$$2|a-b| \leq 2|C_n| \leq 2\delta \leq \mathrm{d}(A,B) \leq |a-b|,$$

a contradiction with the disjointness of the sets A and B. As a consequence, letting $A_n = C_n$ if C_n intersects A and $A_n = \emptyset$ otherwise, and letting $B_n = C_n$ if C_n intersects B and $B_n = \emptyset$ otherwise, we have

$$\sum_{n=1}^{\infty} \zeta(C_n) \geq \sum_{n=1}^{\infty} \zeta(A_n) + \sum_{n=1}^{\infty} \zeta(B_n).$$

Moreover, one easily checks that $(A_n)_{n\geq 1}$ and $(B_n)_{n\geq 1}$ are two sequences of sets in $\mathcal{C}$ with diameter at most δ_1 and δ_2, respectively, that cover the sets A and B, respectively. Thus, we end up with

$$\zeta_*(A \sqcup B) + \varepsilon \geq \zeta_{\delta_1}(A) + \zeta_{\delta_2}(B).$$

We conclude by letting δ_1, δ_2 and ε go to zero. $\square$

On account of the fact that the outer measures of the form ζ_* are metric, we may now state an analogue of Proposition 2.4 where the measurability assumptions are replaced by positive separateness conditions.

PROPOSITION 2.7. *Let ζ_* be the outer measure defined in terms of a given premeasure ζ through (53), and let $(E_n)_{n\geq 1}$ denote a nondecreasing sequence of subsets of $\mathbb{R}^d$. If $\mathrm{d}(E_n, \mathbb{R}^d \setminus E_{n+1})$ is positive for any integer $n \geq 1$, then*

$$\zeta_*\left(\bigcup_{n=1}^{\infty} \uparrow E_n\right) = \lim_{n\to\infty} \uparrow \zeta_*(E_n).$$

PROOF. It is clear that (52) holds for the sets E_n. We thus need to prove the reverse inequality, and we may assume that the sequence $(\zeta_*(E_n))_{n\geq 1}$ is bounded. Now, let us consider the sets $F_1 = E_1$ and $F_n = E_n \setminus E_{n-1}$ for all $n \geq 2$. Then, for any integer $n \geq 1$, the set $F_1 \sqcup F_3 \sqcup \ldots \sqcup F_{2n-1}$ is included in E_{2n-1} and the set F_{2n+1} is included in $\mathbb{R}^d \setminus E_{2n}$, so that

$$\mathrm{d}(F_1 \sqcup F_3 \sqcup \ldots \sqcup F_{2n-1}, F_{2n+1}) \geq \mathrm{d}(E_{2n-1}, \mathbb{R}^d \setminus E_{2n}) > 0.$$

By virtue of Proposition 2.6, the outer measure ζ_* is metric, and therefore

$$\zeta_*(F_1 \sqcup F_3 \sqcup \ldots \sqcup F_{2n+1}) = \zeta_*(F_1 \sqcup F_3 \sqcup \ldots \sqcup F_{2n-1}) + \zeta_*(F_{2n+1}).$$

Iterating this procedure, we readily deduce that

$$\sum_{n=1}^{N} \zeta_*(F_{2n-1}) = \zeta_*\left(\bigsqcup_{n=1}^{N} F_{2n-1}\right) \leq \zeta_*(E_{2N-1}).$$

We may obviously apply the same ideas to the sets F_n, for the even values of n, thereby inferring that

$$\sum_{n=1}^{N} \zeta_*(F_{2n}) = \zeta_*\left(\bigsqcup_{n=1}^{N} F_{2n}\right) \leq \zeta_*(E_{2N}).$$

Recalling that the sequence $(\zeta_*(E_n))_{n\geq 1}$ is bounded, we deduce that the series $\sum_n \zeta_*(F_n)$ converges. Now, for all $N \geq 1$, we have

$$\zeta_*\left(\bigcup_{n=1}^{\infty} \uparrow E_n\right) = \zeta_*\left(E_N \sqcup \bigsqcup_{n=N+1}^{\infty} F_n\right) \leq \zeta_*(E_N) + \sum_{n=N+1}^{\infty} \zeta_*(F_n),$$

and the desired inequality follows from letting N go to infinity. $\square$

We are now in position to state the main result concerning the outer measures of the form ζ_*, namely, that the Borel sets are measurable. The Borel σ-field is denoted by $\mathcal{B}$ in what follows.

THEOREM 2.4. *Let ζ_* be the outer measure obtained from a given premeasure ζ through (53). Then, the Borel subsets of $\mathbb{R}^d$ are ζ_*-measurable, that is, $\mathcal{B} \subseteq \mathcal{F}_{\zeta_*}$.*

PROOF. We know from Theorem 2.1(1) that the ζ_*-measurable subsets of $\mathbb{R}^d$ form a σ-field denoted by $\mathcal{F}_{\zeta_*}$. In order to show that the Borel σ-field is included in $\mathcal{F}_{\zeta_*}$, it thus suffices to establish that every closed subset of $\mathbb{R}^d$ is ζ_*-measurable.

Given a closed subset F of $\mathbb{R}^d$, let us consider two sets A and B in $\mathcal{P}(\mathbb{R}^d)$ that are included in F and $\mathbb{R}^d \setminus F$, respectively. We may suppose that A and B are nonempty. Now, for any integer $n \geq 1$, let B_n denote the set of points $b \in B$ such that $\mathrm{d}(\{b\}, F) > 1/n$. The sets B_n clearly form a nondecreasing sequence of subsets of B. Moreover, if b denotes a point in B, then $\mathrm{d}(\{b\}, F)$ is positive, because the set F is closed and cannot contain b. Thus, the point b belongs to B_n for n large enough. It follows that

$$B = \bigcup_{n=1}^{\infty} \uparrow B_n.$$

For any integer $n \geq 1$, let us consider two points $b \in B_n$ and $c \in \mathbb{R}^d \setminus B_{n+1}$. Then, the distance between the point c and the set F is at most $1/(n+1)$, so that there exists a point $f \in F$ satisfying $|c - f| \leq 2/(2n+1)$. Hence,

$$|b - c| \geq |b - f| - |c - f| \geq \mathrm{d}(\{b\}, F) - |c - f| > \frac{1}{n} - \frac{2}{2n+1} = \frac{1}{n(2n+1)} > 0.$$

We may thus conclude that the distance between the sets B_n and $\mathbb{R}^d \setminus B_{n+1}$ is positive, regardless of the value of n. The sets $A \sqcup B_n$ satisfy the same property:

$$\mathrm{d}(A \sqcup B_n, \mathbb{R}^d \setminus (A \sqcup B_{n+1})) \geq \min\{\mathrm{d}(A, \mathbb{R}^d \setminus B_{n+1}), \mathrm{d}(B_n, \mathbb{R}^d \setminus B_{n+1})\} > 0,$$

where the distance between A and B_{n+1} is clearly positive in view of the definition of B_{n+1} and the fact that A is contained in F. This means that we may apply Proposition 2.7 to the sequence of sets $(A \sqcup B_n)_{n\geq 1}$, as well as to the mere sequence $(B_n)_{n\geq 1}$, thereby obtaining

$$\zeta_*(A \sqcup B) = \lim_{n\to\infty} \uparrow \zeta_*(A \sqcup B_n) = \zeta_*(A) + \lim_{n\to\infty} \uparrow \zeta_*(B_n) = \zeta_*(A) + \zeta_*(B).$$

Here, we also used the fact that the outer measure ζ_* is metric: this enabled us to write the ζ_*-mass of the union of the sets A and B_n as the sum of their ζ_*-masses, because the distance separating them is positive. We may thus conclude that the set F is ζ_*-measurable. □

2.5. Lebesgue measure

The general theory developed in Sections 2.2 and 2.4 may be applied to define the important example of Lebesgue measure and recover its main properties. The starting point is the premeasure υ defined on the open rectangles of $\mathbb{R}^d$ by

$$\upsilon\left(\prod_{i=1}^{d}(a_i, b_i)\right) = \prod_{i=1}^{d}(b_i - a_i) \tag{55}$$

for any choice of points $(a_1, \ldots, a_d)$ and $(b_1, \ldots, b_d)$ in the space $\mathbb{R}^d$ such that the condition $a_i \leq b_i$ holds for any $i \in \{1, \ldots, d\}$.

DEFINITION 2.6. Let υ be the premeasure defined by (55) on the open rectangles of $\mathbb{R}^d$. The *d-dimensional Lebesgue outer measure* $\mathcal{L}^d$ is the outer measure on $\mathcal{P}(\mathbb{R}^d)$ defined with the help of (51) from the premeasure υ, namely,

$$\mathcal{L}^d = \upsilon^*.$$

The *d-dimensional Lebesgue measure*, still denoted by $\mathcal{L}^d$, is then the restriction of this outer measure to the σ-field of its measurable sets.

A noteworthy property that readily follows from Definition 2.6 is that the Lebesgue outer measure is translation invariant and homogeneous of degree d under dilations. We also observe that, according to this definition, the Lebesgue outer measure is obtained through (51). It is therefore an outer measure, as its name suggests, as a consequence of Theorem 2.2. However, as shown by the next result, the Lebesgue outer measure may also been obtained recovered with the help of (53). It will thus satisfy the additional metric properties discussed in Section 2.4.

PROPOSITION 2.8. *The Lebesgue outer measure $\mathcal{L}^d$ coincides with the outer measure defined on $\mathcal{P}(\mathbb{R}^d)$ from the premeasure υ with the help of (53), that is,*

$$\mathcal{L}^d = \upsilon_*.$$

PROOF. In view of (54), we already know that $\mathcal{L}^d(E)$ is smaller than or equal to $\upsilon_*(E)$, for any subset E of $\mathbb{R}^d$. In order to prove the reverse inequality, we may clearly assume that $\mathcal{L}^d(E)$ is finite and, given a real number $\varepsilon > 0$, consider a sequence $(C_n)_{n\geq 1}$ of open rectangles such that

$$E \subseteq \bigcup_{n=1}^{\infty} C_n \qquad \text{and} \qquad \sum_{n=1}^{\infty} \upsilon(C_n) \leq \mathcal{L}^d(E) + \varepsilon.$$

A real number $\delta > 0$ being fixed, we now need to derive from the sequence $(C_n)_{n\geq 1}$ a covering of the set E with open rectangles with diameter at most δ.

To proceed, we shall make use of the following elementary observation. We consider an open rectangle R that is determined by two points $(a_1, \ldots, a_d)$ and $(b_1, \ldots, b_d)$ in $\mathbb{R}^d$. For any integer $q \geq 1$ and any real number $\eta > 0$, the set R is clearly contained in the union of the open rectangles

$$R_p = \prod_{i=1}^{d} \left(a_i + \frac{p_i - 1}{q}(b_i - a_i), a_i + \frac{p_i + \eta}{q}(b_i - a_i) \right),$$

where $p = (p_1, \ldots, p_d)$ ranges in the set $\{1, \ldots, q\}^d$. Letting c denote a positive real number such that $|x| \leq c\,|x|_\infty$ for all $x \in \mathbb{R}^d$, we see that the diameter of each set R_p satisfies

$$|R_p| \leq c\,\frac{1+\eta}{q}|b-a|_\infty \leq \delta,$$

where the last inequality holds for an appropriate choice of q and η. Furthermore, turning our attention to the premeasure υ, we have

$$\sum_{p \in \{1,\ldots,q\}^d} \upsilon(R_p) = q^d \prod_{i=1}^{d} \left(\frac{1+\eta}{q}(b_i - a_i) \right) = (1+\eta)^d \upsilon(R),$$

a value that may be arbitrarily close to $\upsilon(R)$ if η is sufficiently small.

The upshot is that every rectangle C_n may be covered by finitely many open rectangles $C_{n,1}, \ldots, C_{n,m_n}$ with diameter at most δ and such that

$$\sum_{m=1}^{m_n} \upsilon(C_{n,m}) \leq \upsilon(C_n) + \varepsilon\, 2^{-n}.$$

Collecting all the rectangles $C_{n,m}$, we obtain a covering of the set E with sets of diameter bounded above by δ, and therefore

$$\upsilon_\delta(E) \leq \sum_{n=1}^{\infty}\sum_{m=1}^{m_n} \upsilon(C_{n,m}) \leq \sum_{n=1}^{\infty}(\upsilon(C_n) + \varepsilon\, 2^{-n}) \leq \mathcal{L}^d(E) + 2\varepsilon,$$

where υ_δ is defined as in (53). We conclude by letting δ, and then ε, go to zero. □

It follows from Proposition 2.8 that the Lebesgue outer measure enjoys all the properties presented in Section 2.4. For instance, Theorem 2.4 ensures that the Borel subsets of $\mathbb{R}^d$ are measurable with respect to the Lebesgue outer measure. Equivalently, the Lebesgue measure is well defined on Borel sets.

We finish this discussion of Lebesgue measure by a simple expected result that however does not follow from the general theory presented in the previous sections.

Proposition 2.9. *For any open rectangle R of $\mathbb{R}^d$,*

$$\mathcal{L}^d(R) = \upsilon(R).$$

Proof. Clearly, Definition 2.6 ensures that $\mathcal{L}^d(R)$ is bounded above by $\upsilon(R)$ for any open rectangle R. For the reverse inequality, we consider a closed hyper-rectangle S delimited by two points $(a_1, \ldots, a_d)$ and $(b_1, \ldots, b_d)$ satisfying $a_i < b_i$ for all $i \in \{1, \ldots, d\}$, namely,

$$S = \prod_{i=1}^{d}[a_i, b_i].$$

We further consider a covering $(C_n)_{n\geq 1}$ of the rectangle S composed of open rectangles. The set S is compact and the sets C_n are open, so there exists a finite subset $\mathcal{N}$ of $\mathbb{N}$ such that the rectangles C_n, for $n \in \mathcal{N}$, cover and intersect the set S. Defining $R = \operatorname{int} S$, the interior of S, we then observe that for each $n \in \mathcal{N}$, the intersection set $R \cap C_n$ is a nonempty open rectangle; its endpoints are denoted by $(a_{n,1}, \ldots, a_{n,d})$ and $(b_{n,1}, \ldots, b_{n,d})$. For each i, we introduce a nondecreasing rearrangement of the real numbers $a_{n,i}$ and $b_{n,i}$, specifically,

$$a_i = c_{1,i} \leq \ldots \leq c_{2q,i} = b_i,$$

where q denotes the cardinality of the index set $\mathcal{N}$.

Then, for each integer point $p = (p_1, \ldots, p_d)$ in the set $\{1, \ldots, 2q-1\}^d$, let us examine the open rectangle

$$R_p = \prod_{i=1}^{d}(c_{p_i,i}, c_{p_i+1,i}).$$

When R_p is nonempty, its midpoint lies in R, therefore belonging to some open rectangle C_n, with $n \in \mathcal{N}$. However, the above rearrangement procedure guarantees that the whole rectangle R_p is actually contained in the intersection $R \cap C_n$. Thus, any R_p is fully contained in some $R \cap C_n$. Moreover, for the same reason, the value assigned by the premeasure υ to the set $R \cap C_n$ coincides with the sum of those assigned to the sets R_p that it contains:

$$\upsilon(R \cap C_n) = \sum_{\substack{p \in \{1,\ldots,2q-1\}^d \\ R_p \subseteq R \cap C_n}} \upsilon(R_p).$$

These observations enable us to deduce that

$$\sum_{n=1}^{\infty} \upsilon(C_n) \geq \sum_{n\in\mathcal{N}} \upsilon(R\cap C_n) = \sum_{n\in\mathcal{N}} \sum_{\substack{p\in\{1,\ldots,2q-1\}^d \\ R_p\subseteq R\cap C_n}} \upsilon(R_p)$$

$$\geq \sum_{p\in\{1,\ldots,2q-1\}^d} \prod_{i=1}^{d}(c_{p_i+1,i}-c_{p_i,i}) = \prod_{i=1}^{d}(c_{2q,i}-c_{1,i}) = \prod_{i=1}^{d}(b_i-a_i).$$

Taking the infimum over all coverings $(C_n)_{n\geq 1}$ in the left-hand side, we deduce that

$$\mathcal{L}^d(S) \geq \prod_{i=1}^{d}(b_i-a_i).$$

Finally, if R denotes a nonempty open rectangle determined by two points $(a_1,\ldots,a_d)$ and $(b_1,\ldots,b_d)$, it is clear that the closed rectangle S_η delimited by the points $(a_1+\eta,\ldots,a_d+\eta)$ and $(b_1-\eta,\ldots,b_d-\eta)$ is contained in R, with the proviso that the positive parameter η is sufficiently small. As $\mathcal{L}^d$ is an outer measure, we deduce from what precedes that

$$\mathcal{L}^d(R) \geq \mathcal{L}^d(S_\eta) \geq \prod_{i=1}^{d}(b_i-a_i-2\eta).$$

The right-hand side clearly tends to $\upsilon(R)$ as $\eta\to 0$, and the result follows. □

A simple consequence of Proposition 2.9 is that if R is the closed rectangle determined by the points $(a_1,\ldots,a_d)$ and $(b_1,\ldots,b_d)$, then we have

$$R = \prod_{i=1}^{d}[a_i,b_i] \qquad \text{and} \qquad \mathcal{L}^d(R) = \prod_{i=1}^{d}(b_i-a_i). \tag{56}$$

In fact, on the one hand, R obviously contains its interior, denoted by $\operatorname{int} R$, which is the open rectangle delimited by the same endpoints. For any $\eta>0$, on the other hand, R is also included in the open rectangle R_η that is delimited by the points $(a_1-\eta,\ldots,a_d-\eta)$ and $(b_1+\eta,\ldots,b_d+\eta)$. Consequently, in view of Proposition 2.9 and the fact that $\mathcal{L}^d$ is an outer measure, we get

$$\upsilon(\operatorname{int} R) = \mathcal{L}^d(\operatorname{int} R) \leq \mathcal{L}^d(R) \leq \mathcal{L}^d(R_\eta) = \upsilon(R_\eta),$$

from which we straightforwardly deduce that

$$\prod_{i=1}^{d}(b_i-a_i) \leq \mathcal{L}^d(R) \leq \prod_{i=1}^{d}(b_i-a_i+2\eta),$$

and the right-hand side coincides with the left-hand side when we take the limit as η goes to zero. Note that the same result also holds if R is, for instance, a half-open rectangle of $\mathbb{R}^d$.

2.6. Hausdorff measures

2.6.1. Definition and main properties. As shown by Proposition 2.14 below, the Lebesgue measure discussed in Section 2.5 falls into the category of Hausdorff measures that we now present. To begin with, the Hausdorff measures are obtained by applying Theorem 2.3 to the premeasures that are defined in terms of the class of *gauge functions*.

DEFINITION 2.7. A *gauge function* is a function g defined on $[0,\infty]$ which is nondecreasing in a neighborhood of zero and satisfies the conditions

$$\lim_{r\to 0} g(r) = g(0) = 0 \qquad \text{and} \qquad g(\infty)=\infty.$$

The convention that gauge functions take an infinite value at infinity has very little importance and is only aimed at lightening some of the statements below. Note in addition that we do not exclude *a priori* the possibility that a gauge function assigns an infinite value to some positive real numbers.

DEFINITION 2.8. Let $g\circ|\cdot|$ be a shorthand for the premeasure defined on $\mathcal{P}(\mathbb{R}^d)$ by $E \mapsto g(|E|)$. For any gauge function g, the *Hausdorff g-measure* $\mathcal{H}^g$ is the outer measure on $\mathcal{P}(\mathbb{R}^d)$ defined with the help of (53) from the premeasure $g\circ|\cdot|$, namely,

$$\mathcal{H}^g = (g \circ |\cdot|)_*.$$

In view of this definition, the properties obtained in Section 2.4 are satisfied by the Hausdorff measures. In particular, it readily follows from Theorem 2.4 that the Borel subsets of $\mathbb{R}^d$ are measurable with respect to the Hausdorff measures. It is also important and useful to remark that the Hausdorff measures are translation invariant. Besides, for any real number $\delta > 0$, we shall also use the outer measures

$$\mathcal{H}^g_\delta = (g \circ |\cdot|)_\delta$$

defined by (53) in terms of the premeasure $g\circ|\cdot|$. Note that they are indeed outer measures as a result of Theorem 2.2.

2.6.2. Normalized gauge functions. We shall hardly be interested in the precise value of the Hausdorff g-measure of a set, but only in its finiteness or its positiveness. Thus, it will be useful to compare the Hausdorff g-measures with simpler objects obtained for instance by making further assumptions on the gauge function g or the form of the coverings. This is the purpose of the next two results. The first statement calls upon the following notion of normalized gauge functions.

DEFINITION 2.9. For any gauge function g, we consider the function g_d defined for all real numbers $r > 0$ by

$$g_d(r) = r^d \inf_{0<\rho\le r} \frac{g(\rho)}{\rho^d}, \tag{57}$$

along with $g_d(0) = 0$ and $g_d(\infty) = \infty$; the function g_d is then called the *d-normalization of g*. Moreover, we say that a gauge function is *d-normalized* if it coincides with its d-normalization in a neighborhood of zero.

The next result shows that the Hausdorff measure associated with some gauge function is comparable with the measure associated with its *d-normalization*.

PROPOSITION 2.10. *For any gauge function g, the function g_d defined above is a gauge function for which the mapping $r \mapsto g_d(r)/r^d$ is nonincreasing on $(0,\infty)$. Moreover, there exists a real number $\kappa \ge 1$ such that for any gauge function g and any subset E of $\mathbb{R}^d$,*

$$\mathcal{H}^{g_d}(E) \le \mathcal{H}^g(E) \le \kappa\, \mathcal{H}^{g_d}(E).$$

PROOF. First, it is obvious from (57) that the mapping $r \mapsto g_d(r)/r^d$ is nonincreasing on $(0,\infty)$, and that

$$\forall r > 0 \qquad 0 \le g_d(r) \le g(r), \tag{58}$$

which ensures the right-continuity at zero of g_d. Let us show that g_d is nondecreasing in a neighborhood of the origin. Recall that g is nondecreasing on the interval $[0,\varepsilon]$ for some $\varepsilon > 0$. Now, if $0 \le r < r' \le \varepsilon$, then we have $g_d(r) \le g_d(r')$, because

$$g_d(r) \le r'^d \inf_{0<\rho\le r} \frac{g(\rho)}{\rho^d} \quad \text{and} \quad g_d(r) \le g(r) \le \inf_{r<\rho\le r'} g(\rho) \le r'^d \inf_{r<\rho\le r'} \frac{g(\rho)}{\rho^d}.$$

To show that the Hausdorff measures $\mathcal{H}^g$ and $\mathcal{H}^{g_d}$ are comparable, let us consider a real $c \geq 1$ such that $|x|_\infty/c < |x| < c|x|_\infty$ for all $x \in \mathbb{R}^d \setminus \{0\}$, and a subset E of $\mathbb{R}^d$. We shall show that

$$\mathcal{H}^{g_d}(E) \leq \mathcal{H}^g(E) \leq (4c^2)^d \mathcal{H}^{g_d}(E).$$

The leftmost inequality clearly follows from the definition of the Hausdorff measures, along with (58). In order to show the rightmost inequality, let us consider a sequence $(C_n)_{n\geq 1}$ of sets in $\mathcal{P}(\mathbb{R}^d)$ with diameter at most some $\delta \in (0, \varepsilon]$ and such that $E \subseteq \bigcup_n C_n$. If the set C_n has positive diameter, then there exists a real $\rho_n \in (0, |C_n|]$ such that

$$|C_n|^d \frac{g(\rho_n)}{{\rho_n}^d} \leq g_d(|C_n|) + \delta 2^{-n},$$

and there exists a point $x_n \in C_n$, so that $C_n \subseteq \mathrm{B}_\infty(x_n, c|C_n|)$. Furthermore, the latter ball is covered by $m_n = \lceil 2c^2|C_n|/\rho_n \rceil^d$ closed cubes with sidelength ρ_n/c, denoted by $K_{n,1}, \ldots, K_{n,m_n}$. Hence,

$$\begin{aligned}
\delta + \sum_{n=1}^{\infty} g_d(|C_n|) &\geq \sum_{\substack{n\geq 1\\ |C_n|>0}} |C_n|^d \frac{g(\rho_n)}{{\rho_n}^d} \geq \frac{1}{(4c^2)^d} \sum_{\substack{n\geq 1\\ |C_n|>0}} m_n g(\rho_n) \\
&\geq \frac{1}{(4c^2)^d} \left(\sum_{\substack{n\geq 1\\ |C_n|=0}} g(|C_n|) + \sum_{\substack{n\geq 1\\ |C_n|>0}} \sum_{m=1}^{m_n} g(|K_{n,m}|) \right) \geq \frac{\mathcal{H}^g_\delta(F)}{(4c^2)^d},
\end{aligned}$$

and the desired inequality follows from taking the infimum over all the sequences $(C_n)_{n\geq 1}$, and finally letting δ go to zero. □

2.6.3. Net measures. The second statement shows that we may restrict our attention to coverings with dyadic cubes when estimating Hausdorff measures of sets. The main advantage of working with coverings by dyadic cubes is that they may easily be reduced to coverings by disjoint cubes; this is due to the fact that two dyadic cubes are either disjoint or contained in one another. Recall that a dyadic cube is a set of the form

$$\lambda = 2^{-j}(k + [0, 1)^d),$$

with $j \in \mathbb{Z}$ and $k \in \mathbb{Z}^d$. We also adopt the convention that the empty set is a dyadic cube. The collection of all dyadic cubes, including the empty set, is denoted by Λ. Given a gauge function g, let us consider the premeasure that maps each set λ in Λ to $g(|\lambda|)$, and which is denoted by $g \circ |\cdot|_\Lambda$ for brevity. Then, Theorem 2.3 enables us to introduce the outer measure

$$\mathcal{M}^g = (g \circ |\cdot|_\Lambda)_*, \tag{59}$$

and the results of Section 2.4 show in particular that the Borel sets are measurable with respect to $\mathcal{M}^g$; this outer measure is usually termed as a *net measures*. Furthermore, for any real $\delta > 0$, let $\mathcal{M}^g_\delta$ stand for the outer measure $(g \circ |\cdot|_\Lambda)_\delta$ that is defined as in (53).

Proposition 2.11. *There exists a real $\kappa' \geq 1$ such that for any gauge function g and any subset E of $\mathbb{R}^d$,*

$$\mathcal{H}^g(E) \leq \mathcal{M}^g(E) \leq \kappa' \, \mathcal{H}^g(E).$$

Proof. The leftmost inequality is clear, because a cover by dyadic cubes is a particular case of a cover by arbitrary sets. To prove the rightmost inequality, let us consider a real $c \geq 1$ such that $|x|_\infty/c < |x| < c|x|_\infty$ for all $x \in \mathbb{R}^d \setminus \{0\}$, and a real $\varepsilon > 0$ such that g is nondecreasing on $[0, \varepsilon]$, just as in the proof of Proposition 2.10.

Then, let E denote a subset of $\mathbb{R}^d$ and let $(C_n)_{n\geq 1}$ be a sequence of sets in $\mathcal{P}(\mathbb{R}^d)$ with diameter at most some $\delta \in (0, \varepsilon]$ and such that $E \subseteq \bigcup_n C_n$.

If the set C_n has positive diameter, then it contains a point x_n, so that C_n is contained in the ball $\mathrm{B}_\infty(x_n, c|C_n|)$. Furthermore, the latter ball is covered by $\lfloor 4c^2 \rfloor^d$ dyadic cubes with sidelength 2^{-j_n}, where j_n is the only integer satisfying $2^{-j_n} \leq |C_n|/c < 2^{-j_n+1}$; the cubes are denoted by $\lambda_{n,1}, \ldots, \lambda_{n,\lfloor 4c^2 \rfloor^d}$. Furthermore, if the diameter of C_n vanishes, then this set is either empty or reduced to a singleton $\{x_n\}$. In the first case, we let $\lambda_n = \emptyset$. In the second case, we let λ_n be an arbitrary dyadic cube with sidelength at most ε_n that contains x_n, where ε_n is chosen small enough to ensure that $c\varepsilon_n \leq \varepsilon$ and $g(c\varepsilon_n) \leq \delta 2^{-n}$. As a consequence,

$$\begin{aligned}
\mathcal{M}^g_\delta(E) &\leq \sum_{\substack{n\geq 1 \\ |C_n|>0}} \sum_{m=1}^{\lfloor 4c^2 \rfloor^d} g(|\lambda_{n,m}|) + \sum_{\substack{n\geq 1 \\ |C_n|=0}} g(|\lambda_n|) \\
&\leq (4c^2)^d \sum_{\substack{n\geq 1 \\ |C_n|>0}} g(|C_n|) + \sum_{\substack{n\geq 1 \\ \#C_n=1}} \delta 2^{-n} \leq (4c^2)^d \sum_{n=1}^{\infty} g(|C_n|) + \delta,
\end{aligned}$$

and the result follows from taking the infimum over all the sequences $(C_n)_{n\geq 1}$, and letting δ tend to zero. $\square$

Note that Proposition 2.11 may be straightforwardly extended to coverings by m-adic cubes. Such a generalization will be used in Section 3.4.

2.6.4. Further properties. In the same vein, we may derive from the relative behavior at zero of two given gauge functions g and h a comparison between the corresponding Hausdorff measures. This is the purpose of the next result.

PROPOSITION 2.12. *For any gauge functions g and h and for any set $E \subseteq \mathbb{R}^d$,*

$$\left(\liminf_{r\to 0} \frac{g(r)}{h(r)}\right) \mathcal{H}^h(E) \leq \mathcal{H}^g(E) \leq \left(\limsup_{r\to 0} \frac{g(r)}{h(r)}\right) \mathcal{H}^h(E),$$

except if the lower or upper bound is of the indeterminate form $0 \cdot \infty$, in which case the corresponding inequality has no meaning.

PROOF. Let us consider a sequence $(C_n)_{n\geq 1}$ of subsets of $\mathbb{R}^d$ with diameter at most some $\delta > 0$, and let us assume that $E \subseteq \bigcup_n C_n$. Then, it is clear that

$$\left(\inf_{0<r\leq\delta} \frac{g(r)}{h(r)}\right) \sum_{n=1}^{\infty} h(|C_n|) \leq \sum_{n=1}^{\infty} g(|C_n|) \leq \left(\sup_{0<r\leq\delta} \frac{g(r)}{h(r)}\right) \sum_{n=1}^{\infty} h(|C_n|),$$

and we conclude by taking the infima over $(C_n)_{n\geq 1}$ and letting δ tend to zero. $\square$

Let us now explain how the Hausdorff measures behave when taking the image of the set of interest under a mapping that satisfies a form of Lipschitz condition.

PROPOSITION 2.13. *Let V be a nonempty open subset of $\mathbb{R}^d$ and let f be a mapping defined on V with values in $\mathbb{R}^{d'}$. Let us assume that there exists a continuous increasing function φ defined on the interval $[0, \infty)$ such that $\varphi(0) = 0$ and*

$$\forall x, y \in V \qquad |f(x) - f(y)| \leq \varphi(|x - y|).$$

Then, for any gauge function g, the function $g \circ \varphi^{-1}$ may be extended to a gauge function, and for any subset E of V,

$$\mathcal{H}^{g\circ\varphi^{-1}}(f(E)) \leq \mathcal{H}^g(E).$$

PROOF. First, note that $g \circ \varphi^{-1}$ is nondecreasing on an interval of the form $[0, \varepsilon]$. As usual, let us consider a sequence $(C_n)_{n \geq 1}$ of subsets of $\mathbb{R}^d$ with diameter at most a given $\delta > 0$ for which $\varphi(\delta) \leq \varepsilon$, and such that $E \subseteq \bigcup_n C_n$. Thus, the image set $f(E)$ is covered by the sets $f(C_n \cap V)$. In addition,

$$|f(C_n \cap V)| = \sup_{x,y \in C_n \cap V} |f(x) - f(y)| \leq \sup_{x,y \in C_n \cap V} \varphi(|x-y|) \leq \varphi(|C_n|)$$

for every integer $n \geq 1$, from which it follows that

$$\mathcal{H}^{g \circ \varphi^{-1}}_{\varphi(\delta)}(f(E)) \leq \sum_{n=1}^{\infty} g \circ \varphi^{-1}(|f(C_n \cap V)|) \leq \sum_{n=1}^{\infty} g(|C_n|),$$

and we conclude again by taking the infimum on $(C_n)_{n \geq 1}$ and the limit as the parameter δ tends to zero. □

Proposition 2.13 is typically applied to mappings f that are Lipischitz, or even uniform Hölder, on an open set V; the function φ is therefore of the form $r \mapsto Cr^{\alpha}$.

2.6.5. Connection with Lebesgue measure. Finally, it is important to observe that the Lebesgue measure $\mathcal{L}^d$, already discussed in Section 2.5, is a particular example of Hausdorff measure.

PROPOSITION 2.14. *There exists a real number $\kappa'' > 0$ such that for any set B in the Borel σ-field $\mathcal{B}$,*

$$\mathcal{H}^{r \mapsto r^d}(B) = \kappa'' \mathcal{L}^d(B). \tag{60}$$

PROOF. Letting c denote a positive real such that $|x|_\infty / c \leq |x| \leq c|x|_\infty$ for all $x \in \mathbb{R}^d$, one easily checks that $\mathcal{M}^{r \mapsto r^d}([0,1)^d) \leq c^d$. Using Proposition 2.11, we infer that $\mathcal{H}^{r \mapsto r^d}([0,1)^d) \leq c^d$. Conversely, let us consider a sequence $(\lambda_n)_{n \geq 1}$ of dyadic cubes with diameter at most a given $\delta > 0$ such that $[0,1)^d \subseteq \bigcup_n \lambda_n$. Therefore, as (56) holds for half-open rectangles, we have

$$1 = \mathcal{L}^d([0,1)^d) \leq \sum_{n=1}^{\infty} \mathcal{L}^d(\lambda_n) \leq c^d \sum_{n=1}^{\infty} |\lambda_n|^d\,;$$

taking the infimum over all sequences $(\lambda_n)_{n \geq 1}$ and the limit as δ goes to zero, we thus deduce that $\mathcal{M}^{r \mapsto r^d}([0,1)^d) \geq c^{-d}$. Using Proposition 2.11 and the notations therein, we now infer that $\mathcal{H}^{r \mapsto r^d}([0,1)^d) \geq c^{-d}/\kappa'$. It follows that

$$\kappa'' = \mathcal{H}^{r \mapsto r^d}([0,1)^d) \in (0, \infty).$$

Given that the Lebesgue measure of the unit cube is equal to one, we deduce that (60) holds when the Borel set B is equal to the unit cube $[0,1)^d$.

Let us now consider an integer $j \geq 0$. The unit cube is the disjoint union of the dyadic cubes of the form $2^{-j}(k + [0,1)^d)$ with $k \in \{0, \ldots, 2^j - 1\}^d$. By virtue of Theorem 2.4, these dyadic cubes are measurable with respect to $\mathcal{H}^{r \mapsto r^d}$, so that

$$\mathcal{H}^{r \mapsto r^d}([0,1)^d) = \sum_{k \in \{0, \ldots, 2^j - 1\}^d} \mathcal{H}^{r \mapsto r^d}(2^{-j}(k + [0,1)^d)).$$

Due to the translation invariance of the Hausdorff measure $\mathcal{H}^{r \mapsto r^d}$, the value of the summand in the right-hand side does not depend on the value of k. We deduce that for any dyadic cube $\lambda \subseteq [0,1)^d$ with sidelength 2^{-j}, we have

$$\mathcal{H}^{r \mapsto r^d}(\lambda) = \kappa'' 2^{-dj} = \kappa'' \mathcal{L}^d(\lambda).$$

The latter equality is due to the obvious fact that the dyadic cube λ has Lebesgue measure equal to 2^{-dj}, see the discussion at the end of Section 2.5. The upshot is that (60) holds when the set B is an arbitrary dyadic subcube of $[0,1)^d$.

Finally, in view of Theorem 2.4, we obtain two finite measures on the unit cube $[0,1)^d$ by restricting the outer measures $\kappa''\mathcal{L}^d$ and $\mathcal{H}^{r\mapsto r^d}$ to the Borel sets therein. Moreover, the above discussion shows that these two measures coincide on the dyadic subcubes of $[0,1)^d$, which form a π-system that generate the Borel sets. We deduce from the uniqueness of extension lemma that the measures $\kappa''\mathcal{L}^d$ and $\mathcal{H}^{r\mapsto r^d}$ agree on the Borel subsets of $[0,1)^d$, see *e.g.* [**61**, Lemma 1.6(a)]. By translation invariance and countable additivity on measurable sets, we conclude that (60) holds on all the Borel subsets of $\mathbb{R}^d$. □

If the space $\mathbb{R}^d$ is endowed with the Euclidean norm, it can be shown that the constant κ'' arising in the statement of Proposition 2.14 is given by

$$\kappa'' = \left(\frac{4}{\pi}\right)^{d/2} \gamma_d \qquad \text{with} \qquad \gamma_d = \Gamma\left(\frac{d}{2}+1\right) = \begin{cases} \left(\dfrac{d}{2}\right)! & \text{if } d \text{ is even} \\ \dfrac{d!\sqrt{\pi}}{2^d\left(\frac{d-1}{2}\right)!} & \text{if } d \text{ is odd,} \end{cases}$$

where Γ denotes the gamma function, see [**51**, pp. 56–58] for a detailed proof.

Furthermore, the ideas developed in the proof of Proposition 2.14 also lead to the following noteworthy result for general Hausdorff measures.

PROPOSITION 2.15. *Let g denote a gauge function, and let ℓ_g be the parameter defined in $[0,\infty]$ by the formula*

$$\ell_g = \liminf_{r\to 0} \frac{g(r)}{r^d}. \tag{61}$$

Then, depending on the value of ℓ_g, one of the three following situations occurs:

(1) *if $\ell_g = \infty$, then for any Borel subset B of $\mathbb{R}^d$,*

$$\mathcal{L}^d(B) > 0 \qquad \Longrightarrow \qquad \mathcal{H}^g(B) = \infty\,;$$

(2) *if $\ell_g \in (0,\infty)$, then there exists a real number $\kappa_g > 0$ such that for any Borel subset B of $\mathbb{R}^d$,*

$$\mathcal{H}^g(B) = \kappa_g\, \mathcal{L}^d(B)\,;$$

(3) *if $\ell_g = 0$, then the outer measure $\mathcal{H}^g$ is equal to zero.*

PROOF. Let g_d denote the d-normalization, defined by (57), of the gauge function g. Thanks to Proposition 2.10, we know that g_d is a gauge function for which the mapping $r \mapsto g_d(r)/r^d$ is nonincreasing on the interval $(0,\infty)$, and that there exists a real number $\kappa \geq 1$ independent on g such that for any Borel set $B \in \mathcal{B}$,

$$\mathcal{H}^{g_d}(B) \leq \mathcal{H}^g(B) \leq \kappa\, \mathcal{H}^{g_d}(B). \tag{62}$$

On top of that, let us observe that $g_d(r)/r^d$ tends to ℓ_g when r goes to zero. Hence, Proposition 2.12 implies that we also have

$$\mathcal{H}^{g_d}(B) = \ell_g \mathcal{H}^{r\mapsto r^d}(B),$$

except if the right-hand side is of the indeterminate form $0\cdot\infty$. Letting κ'' denote the positive real number appearing in (60), we deduce from Proposition 2.14 that, except in the aforementioned indeterminate case, we further have

$$\mathcal{H}^{g_d}(B) = \kappa''\ell_g \mathcal{L}^d(B). \tag{63}$$

This directly yields (1). As a matter of fact, if the parameter ℓ_g is infinite and B denotes a set in the Borel σ-field $\mathcal{B}$, we then have

$$\mathcal{L}^d(B) > 0 \qquad \Longrightarrow \qquad \mathcal{H}^g(B) \geq \mathcal{H}^{g_d}(B) = \infty.$$

In order to prove (2) and (3), let us assume that the parameter ℓ_g is finite. Then, the $\mathcal{H}^g$-mass of the unit cube $[0,1)^d$, denoted by κ_g, is finite as well. Indeed, applying (62) and (63) to the unit cube, we get

$$\kappa_g = \mathcal{H}^g([0,1)^d) \leq \kappa\, \mathcal{H}^{g_d}([0,1)^d) = \kappa\kappa''\ell_g\mathcal{L}^d([0,1)^d) = \kappa\kappa''\ell_g < \infty.$$

Moreover, if ℓ_g vanishes, then κ_g vanishes as well. The countable subadditivity and the translation invariance of the outer measure $\mathcal{H}^g$ imply that the whole space $\mathbb{R}^d$ has zero $\mathcal{H}^g$-mass. This means that (3) holds. To establish (2), let us suppose that, in addition to being finite, the parameter ℓ_g is positive. Applying (62) and (63) to the unit cube, we also get

$$\kappa_g = \mathcal{H}^g([0,1)^d) \geq \mathcal{H}^{g_d}([0,1)^d) = \kappa''\ell_g\mathcal{L}^d([0,1)^d) = \kappa''\ell_g > 0.$$

Hence, κ_g is both positive and finite. We now proceed as in the proof of Proposition 2.14. The measurability of the dyadic cubes with respect to $\mathcal{H}^g$ and the translation invariance of that outer measure imply that for any dyadic cube $\lambda \subseteq [0,1)^d$,

$$\mathcal{H}^g(\lambda) = \kappa_g\mathcal{L}^d(\lambda).$$

Using the uniqueness of extension lemma just as in the proof of Proposition 2.14, we may conclude that the measures $\kappa_g\mathcal{L}^d$ and $\mathcal{H}^g$ agree on the Borel subsets of $[0,1)^d$, and finally that (2) holds. □

Note that, in the first case addressed by Proposition 2.15, the statement may be applied to nonempty open sets. As a consequence, when ℓ_g is infinite, we have

$$\forall U \neq \emptyset \text{ open} \qquad \mathcal{H}^g(U) = \infty.$$

This follows from the obvious fact that nonempty open sets are Borel and have nonvanishing Lebesgue measure.

2.7. Hausdorff dimension

The Hausdorff measures associated with general gauge functions enable to give a precise description of the size of a subset of $\mathbb{R}^d$. However, it is arguably more intuitive, and often sufficient, to restrict to a specific class of gauge functions, namely, the power functions $r \mapsto r^s$, for $s > 0$. This approach gives rise to the notion of Hausdorff dimension.

For these particular gauge functions, we use the notation $\mathcal{H}^s$ instead of $\mathcal{H}^{r\mapsto r^s}$, for brevity, and we call this outer measure the s-dimensional Hausdorff measure. It is clear that the gauge function $r \mapsto r^s$ is normalized if and only if $s \leq d$; when s is larger than d, the corresponding d-normalization is the zero function and, on account of Proposition 2.10, the s-dimensional Hausdorff measure is constant equal to zero. Furthermore, it is convenient to define $\mathcal{H}^0$ as the outer measure obtained by applying Theorem 2.3 to the premeasure that maps a given subset of $\mathbb{R}^d$ to one if the set is nonempty and to zero otherwise; it is then easy to see that $\mathcal{H}^0$ coincides with the counting measure $\#$ on $\mathbb{R}^d$.

Specializing Proposition 2.12 to the power gauge functions, it is easy to observe that for any nonempty subset E of $\mathbb{R}^d$, there exists a critical value $s_0 \in [0,d]$ such that for all $s \geq 0$,

$$\begin{cases} s < s_0 & \Longrightarrow \qquad \mathcal{H}^s(E) = \infty \\ s > s_0 & \Longrightarrow \qquad \mathcal{H}^s(E) = 0. \end{cases}$$

Note however that one cannot conclude in general as regards the exact value of $\mathcal{H}^{s_0}(E):$ it may well be zero, infinite, or both positive and finite. In the latter case, E is called an *s_0-set*. We may now define the notion of Hausdorff dimension.

DEFINITION 2.10. The *Hausdorff dimension* of a nonempty subset E of $\mathbb{R}^d$ is defined by the formula

$$\dim_{\mathrm{H}} E = \sup\{s \in [0,d] \mid \mathcal{H}^s(E) = \infty\} = \inf\{s \in [0,d] \mid \mathcal{H}^s(E) = 0\},$$

with the convention that the supremum and the infimum are equal to zero and d, respectively, if the inner sets are empty. Moreover, we adopt the convention that the Hausdorff dimension of the empty set is equal to $-\infty$.

We may in fact specialize to the power gauge functions the results of Section 2.6, thereby obtaining the following proposition.

PROPOSITION 2.16. *Hausdorff dimension satisfies the following properties.*

(1) Monotonicity*: for any subsets E_1 and E_2 of $\mathbb{R}^d$,*

$$E_1 \subseteq E_2 \qquad \Longrightarrow \qquad \dim_{\mathrm{H}} E_1 \leq \dim_{\mathrm{H}} E_2.$$

(2) Countable stability*: for any sequence $(E_n)_{n\geq 1}$ of subsets of $\mathbb{R}^d$,*

$$\dim_{\mathrm{H}} \bigcup_{n=1}^{\infty} E_n = \sup_{n\geq 1} \dim_{\mathrm{H}} E_n.$$

(3) Countable sets*: if a subset E of $\mathbb{R}^d$ is both nonempty and countable, then $\dim_{\mathrm{H}} E = 0$.*

(4) Sets with positive Lebesgue measure*: if a subset E of $\mathbb{R}^d$ has positive Lebesgue measure, then $\dim_{\mathrm{H}} E = d$.*

(5) Action of uniform Hölder mappings*: let V be an open subset of $\mathbb{R}^d$ and let $f : V \to \mathbb{R}^{d'}$ be a mapping such that*

$$\exists c, \alpha > 0 \quad \forall x, y \in V \qquad |f(x) - f(y)| \leq c|x-y|^{\alpha}\,;$$

then, for any subset E of V,

$$\dim_{\mathrm{H}} f(E) \leq \frac{1}{\alpha} \dim_{\mathrm{H}} E.$$

(6) Invariance under bi-Lipschitz mappings*: let V be an open subset of $\mathbb{R}^d$ and let $f : V \to \mathbb{R}^{d'}$ be a bi-Lipschitz mapping with constant $c_f \geq 1$,* i.e. *a mapping such that*

$$\forall x, y \in V \qquad \frac{|x-y|}{c_f} \leq |f(x) - f(y)| \leq c_f |x-y|\,; \tag{64}$$

then, for any subset E of V,

$$\dim_{\mathrm{H}} f(E) = \dim_{\mathrm{H}} E.$$

(7) Differentiable manifolds*: if M is a C^1-submanifold of $\mathbb{R}^d$ with dimension m, then $\dim_{\mathrm{H}} M = m$.*

PROOF. All these properties basically follow from the definition of Hausdorff dimension, along with the properties of Hausdorff measures obtained in Section 2.6. Specifically, the monotonicity property (1) follows from the monotonicity property of the outer measures $\mathcal{H}^s$. The countable stability property (2) is due to the monotonicity and the countable additivity of the outer measures $\mathcal{H}^s$. Then, (3) results from the countable stability of Hausdorff dimension, along with the obvious fact that singletons have dimension zero. Now, Proposition 2.14 ensures that a subset of $\mathbb{R}^d$ with positive Lebesgue measure also has positive $\mathcal{H}^d$-mass; this leads to (4). Finally, (5) follows from specializing Proposition 2.13 to the power gauge functions, (6) is a plain consequence of (5), and (7) is a corollary of (4) and (6). □

Using $\mathcal{M}^s$ as a shorthand for the net measures $\mathcal{M}^{r\mapsto r^s}$ introduced in Section 2.6 and obtained when restricting to coverings by dyadic cubes, we directly deduce from Proposition 2.11 that the Hausdorff dimension of a nonempty subset E of $\mathbb{R}^d$ is also characterized by the formula

$$\dim_{\mathrm{H}} E = \sup\{s \in [0,d] \mid \mathcal{M}^s(E) = \infty\} = \inf\{s \in [0,d] \mid \mathcal{M}^s(E) = 0\}.$$

Finally, let us mention for completeness that $\mathcal{M}^0$ is defined just as $\mathcal{H}^0$, and coincides with the counting measure on $\mathbb{R}^d$.

2.8. Upper bounds on Hausdorff dimensions for limsup sets

Deriving upper bounds on Hausdorff dimensions or, more generally, obtaining an upper bound on the Hausdorff measure of a set is usually elementary: it suffices to make use a well chosen covering of the set. There is a situation that we shall often encounter where the choice of the covering is natural: when the set under study is a limsup of simpler sets, such as balls for instance.

LEMMA 2.1. *Let $(E_n)_{n\geq 1}$ be a sequence of subsets of $\mathbb{R}^d$, and let*

$$E = \limsup_{n\to\infty} E_n.$$

Then, for any gauge function g, the following implication holds:

$$\sum_{n=1}^{\infty} g(|E_n|) < \infty \qquad \Longrightarrow \qquad \mathcal{H}^g(E) = 0.$$

In particular, the Hausdorff dimension of E satisfies

$$\dim_{\mathrm{H}} E \leq \inf\left\{ s \in [0,d] \;\middle|\; \sum_{n=1}^{\infty} |E_n|^s < \infty \right\}.$$

PROOF. Let us consider a real $\delta > 0$ and a gauge function g such that the series $\sum_n g(|E_n|)$ converges. In particular, $g(|E_n|)$ tends to zero as $n \to \infty$; thus, unless g is the zero function in a neighborhood of the origin, in which case the result is trivial, we deduce that $|E_n| \leq \delta$ for all n larger than some integer $n_0 \geq 1$. We then choose an integer $m > n_0$ and cover E by the sets E_n, for $n \geq m$, thereby obtaining

$$\mathcal{H}^g_\delta(E) \leq \sum_{n=m}^{\infty} g(|E_n|).$$

The series being convergent, the right-hand side tends to zero as m goes to infinity, and the result follows from letting δ tend to zero. Finally, the upper bound on the Hausdorff dimension is a plain consequence of specializing the above result to the power gauge functions. □

A typical application of Lemma 2.1 is the derivation of an upper bound on the Hausdorff dimension of the set $J_{d,\tau}$, see Section 3.1. Recall that this set is defined by (1) and consists of the points that are approximable at rate at least τ by the points with rational coordinates.

Lemma 2.1 may also be used to compute an upper bound on the Hausdorff dimension of a very classical fractal set: the middle-third Cantor set, denoted by $\mathbb{K}$. There are several ways of defining this set; the most condensed one is certainly to write $\mathbb{K}$ as the image of the symbolic set $\{0,1\}^{\mathbb{N}}$ under the mapping

$$(u_j)_{j\geq 1} \mapsto \sum_{j=1}^{\infty} 2u_j 3^{-j},$$

which amounts to saying that a real number between zero and one belongs to $\mathbb{K}$ if and only if the digits in its 3-adic expansion are all equal to zero or two. Another way, which is probably more suitable for dimension estimates, is to write

$$\mathbb{K} = \bigcap_{j=0}^{\infty} \downarrow \bigsqcup_{u \in \{0,1\}^j} I_u. \tag{65}$$

Here, I_u denotes the closed interval with left endpoint $2u_1/3 + \ldots + 2u_j/3^j$ and length 3^{-j}, if u is the word $u_1 \ldots u_j$ in $\{0,1\}^j$. For consistency, we adopt the convention that the set $\{0,1\}^0$ contains only one element, the empty word $\varnothing$, and that the set $I_\varnothing$ is equal to the whole interval $[0,1]$.

The upper bound on the dimension of $\mathbb{K}$ that results from Lemma 2.1 is then given by the following statement.

Proposition 2.17. *The middle-third Cantor set satisfies*

$$\dim_{\mathrm{H}} \mathbb{K} \leq \frac{\log 2}{\log 3}.$$

Proof. Note that every point of the Cantor set $\mathbb{K}$ belongs to one of the intervals I_u with $u \in \{0,1\}^j$, for every integer $j \geq 0$. In particular, $\mathbb{K}$ may be seen as the limsup of the intervals I_u. Applying Lemma 2.1, we are reduced to inspecting the convergence of the series $\sum_j 2^j (3^{-j})^s$, and the result follows.

Note that this upper bound may be obtained more directly by covering the Cantor set $\mathbb{K}$ by the intervals I_u, for $u \in \{0,1\}^j$, and then by letting j tend to infinity. This method also yields an upper bound on the s-dimensional Hausdorff measure of $\mathbb{K}$ at the critical value $s = \log 2/\log 3$. To be precise, the aforementioned covering implies that for $\delta > 0$ and $j \geq 0$ such that $3^{-j} \leq \delta$,

$$\mathcal{H}^s_\delta(\mathbb{K}) \leq 2^j (3^{-j})^s = 1.$$

Taking the limit as $\delta \to 0$, we deduce that $\mathcal{H}^s(\mathbb{K}) \leq 1$. □

We shall exhibit below a lower bound on the Hausdorff dimension of $\mathbb{K}$ that matches the upper bound given by Proposition 2.17.

2.9. Lower bounds on Hausdorff dimensions

2.9.1. The mass distribution principle. Whereas deriving upper bounds on Hausdorff dimensions often amounts to finding appropriate coverings, a standard way of establishing lower bounds is to build a clever outer measure on the set under study. This remark is embodied by the next simple, but crucial, result.

Lemma 2.2 (mass distribution principle). *Let E be a subset of $\mathbb{R}^d$ and let μ be an outer measure on $\mathbb{R}^d$ such that $\mu(E) > 0$. Let us assume that there exist a gauge function g and two real numbers $c, \delta_0 > 0$ such that for any subset C of $\mathbb{R}^d$ with diameter at most δ_0,*

$$\mu(C) \leq c\, g(|C|).$$

Then, the set E has positive Hausdorff g-mass, specifically,

$$\mathcal{H}^g(E) \geq \frac{\mu(E)}{c} > 0.$$

In particular, if g is the power function $r \mapsto r^s$ for some $s \in (0,d]$, then the s-dimensional Hausdorff measure of E is positive, and $\dim_{\mathrm{H}} E \geq s$.

PROOF. Let us consider a real $\delta \in (0, \delta_0]$ and a sequence $(C_n)_{n\geq 1}$ of subsets of the space $\mathbb{R}^d$ with diameter at most δ that satisfies $E \subseteq \bigcup_n C_n$. Then,

$$\mu(E) \leq \mu\left(\bigcup_{n=1}^{\infty} C_n\right) \leq \sum_{n=1}^{\infty} \mu(C_n) \leq c \sum_{n=1}^{\infty} g(|C_n|),$$

and the result follows as usual from taking the infimum over all sequences $(C_n)_{n\geq 1}$ and letting δ go to zero. □

Let us apply Lemma 2.2 to derive a lower bound on the Hausdorff dimension of the Cantor set $\mathbb{K}$; this will complement Proposition 2.17 above in an optimal way.

PROPOSITION 2.18. *The middle-third Cantor set satisfies*

$$\dim_{\mathrm{H}} \mathbb{K} \geq \frac{\log 2}{\log 3}.$$

PROOF. Let $\mathcal{C}$ denote the collection formed by the empty set and all the intervals I_u, for $u \in \{0,1\}^j$ and $j \geq 0$. We define a premeasure ζ on $\mathcal{C}$ by letting $\zeta(\emptyset) = 0$, and $\zeta(I_u) = 2^{-j}$ if the word u has length j. Theorem 2.2 enables us to extend via the formula (51) the premeasure ζ to an outer measure ζ^* on all the subsets of $\mathbb{R}$. One then easily checks that the function μ that maps a subset E of $\mathbb{R}$ to the value $\zeta^*(E \cap \mathbb{K})$ is also an outer measure.

Given a subset C of $\mathbb{R}$ with diameter at most one, we now derive an appropriate upper bound on $\mu(C)$. We may clearly assume that $C \cap \mathbb{K}$ is nonempty, as otherwise $\mu(C)$ vanishes. Moreover, if C has positive diameter, there is a unique integer $j \geq 0$ such that $3^{-(j+1)} \leq |C| < 3^{-j}$. The intervals I_u, for $u \in \{0,1\}^j$, are separated by a distance at least 3^{-j}. Hence, the set C intersects only one of these intervals, which is denoted by $I(C)$. Therefore, $C \cap \mathbb{K}$ is included in $I(C)$, so that

$$\mu(C) = \zeta^*(C \cap \mathbb{K}) \leq \zeta(I(C)) = 2^{-j} = (3^{-j})^s \leq 3^s |C|^s = 2|C|^s,$$

where s is equal to $\log 2/\log 3$. The same bound holds when C has diameter zero. Actually, in that case, C is reduced to a single point in $\mathbb{K}$. For each integer $j \geq 0$, there is a unique $u \in \{0,1\}^j$ such that this point belongs to I_u, so that

$$\mu(C) = \zeta^*(C \cap \mathbb{K}) \leq \zeta(I_u) = 2^{-j} \xrightarrow[j\to\infty]{} 0.$$

To conclude, it suffices to observe that $\mu(\mathbb{K})$ is at least one. Indeed, thanks to Lemma 2.2, this implies that $\mathcal{H}^s(\mathbb{K}) \geq 1/2$, which eventually leads to the result.

For completeness, let us briefly explain why $\mu(\mathbb{K})$ is at least one. Let us consider a sequence $(C_n)_{n\geq 1}$ in $\mathcal{C}$ such that $\mathbb{K} \subseteq \bigcup_n C_n$. Since the intervals I_u are either disjoint or included in one another, there exists a subset $\mathcal{N}$ of $\mathbb{N}$ such that the sets C_n, for $n \in \mathcal{N}$, are disjoint intervals that still cover the set $\mathbb{K}$. Moreover, if C_n has length 3^{-j}, let C'_n denote the open interval formed by the points at a distance less than $3^{-(j+1)}$ from C_n. One easily checks that the open intervals C'_n are also disjoint and cover $\mathbb{K}$. By compactness of the latter set, we can extract from $\mathcal{N}$ a finite subset $\mathcal{N}'$ such that the intervals C'_n, for $n \in \mathcal{N}'$, cover $\mathbb{K}$. However, for these values of n, we have $\mathbb{K} \cap C'_n = \mathbb{K} \cap C_n$, by disjointness of the sets C'_n. It follows that $\mathbb{K}$ is covered by the finitely many intervals C_n, for $n \in \mathcal{N}'$. Among these intervals, let us pick one that has minimal diameter and that is denoted by C_{n_1}. Then, there necessarily exists an index $n_2 \in \mathcal{N}'$ such that C_{n_2} is the "neighbor" of C_{n_1} in the Cantor set construction: C_{n_1} and C_{n_2} have same length, 3^{-j} say, and are separated by a distance equal to 3^{-j}. Thus, $C_{n_1} \sqcup C_{n_2}$ is included in a set $D \in \mathcal{C}$ with length equal to $3^{-(j-1)}$. Along with the set D, the sets C_n, for $n \in \mathcal{N}' \setminus \{n_1, n_2\}$, cover

$\mathbb{K}$. Moreover, $\zeta(C_{n_1}) + \zeta(C_{n_2})$ and $\zeta(D)$ are both equal to $2^{-(j-1)}$, so that

$$\sum_{n \in \mathcal{N}'} \zeta(C_n) = \zeta(D) + \sum_{n \in \mathcal{N}' \setminus \{n_1, n_2\}} \zeta(C_n).$$

We can repeat this procedure until ending with the trivial covering of the Cantor set $\mathbb{K}$ by the whole interval $[0,1]$, thereby deducing that

$$\sum_{n=1}^{\infty} \zeta(C_n) \geq \sum_{n \in \mathcal{N}'} \zeta(C_n) = \zeta([0,1]) = 1.$$

Taking the infimum in the left-hand side, we conclude that $\zeta^*(\mathbb{K}) \geq 1$. Besides, it is clear that $\zeta^*(\mathbb{K}) \leq \zeta([0,1]) = 1$. Therefore, the μ-mass of the Cantor set $\mathbb{K}$ is in fact equal to one. □

Propositions 2.17 and 2.18 together imply that the Hausdorff dimension of the middle-third Cantor set $\mathbb{K}$ is equal to $s = \log 2 / \log 3$. Inspecting the proofs also shows that $1/2 \leq \mathcal{H}^s(\mathbb{K}) \leq 1$. One can actually prove that the exact value matches the upper bound, *i.e.* is equal to one.

2.9.2. The general Cantor construction. The above approach may be extended to a natural generalization of the middle-third Cantor set. It is convenient to assume that the construction is indexed by a *tree*, that is, a subset T of the set

$$\mathbb{U} = \bigcup_{j=0}^{\infty} \mathbb{N}^j$$

such that the three following properties hold:

- The empty word $\varnothing$ belongs to T.
- If the word $u = u_1 \dots u_j$ is not empty and belongs to T, then the word $\pi(u) = u_1 \dots u_{j-1}$ also belongs to T; this word is the *parent* of u.
- For every word u in T, there exists an integer $k_u(T) \geq 0$ such that the word uk belongs to T if and only if $1 \leq k \leq k_u(T)$; the number of *children* of u in T is then equal to $k_u(T)$.

Let us recall here that, in accordance with a convention adopted previously, the set $\mathbb{N}^0$ arising in the definition of $\mathbb{U}$ is reduced to the singleton $\{\varnothing\}$; the empty word $\varnothing$ clearly corresponds to the *root* of the tree.

To each element u of the tree T, we may then associate a compact subset I_u of $\mathbb{R}^d$, and a possibly infinite nonnegative value $\zeta(I_u)$. Defining in addition $\zeta(\emptyset) = 0$, we thus obtain a premeasure ζ on the collection $\mathcal{C}$ formed by the empty set together with all the sets I_u. We assume these objects are compatible with the tree structure, in the sense that for every $u \in T$,

$$I_u \supseteq \bigsqcup_{k=1}^{k_u(T)} I_{uk} \qquad \text{and} \qquad \zeta(I_u) \leq \sum_{k=1}^{k_u(T)} \zeta(I_{uk}). \tag{66}$$

In particular, nodes $u \in T$ such that $k_u(T)$ vanishes, *i.e.* childless nodes, are not excluded *a priori* but the corresponding sets necessarily satisfy $\zeta(I_u) = 0$. More generally, $\zeta(I_u)$ surely vanishes when the subtree of T formed by the descendants of u is finite; this is easily seen by induction on the height of this subtree.

Thanks to Theorem 2.2, we may then extend the premeasure ζ to an outer measure ζ^* on all the subsets of $\mathbb{R}^d$ through the formula (51). This finally enables us to consider the limiting set

$$K = \bigcap_{j=0}^{\infty} \downarrow \bigsqcup_{u \in T \cap \mathbb{N}^j} I_u, \tag{67}$$

together with the outer measure μ that maps a set $E \subseteq \mathbb{R}^d$ to the value $\zeta^*(E \cap K)$. If the tree T is finite, it is clear that K is empty and μ is the zero measure, and so the construction is pointless. The next result discusses the basic properties of K and μ in the opposite situation.

LEMMA 2.3. *Let us assume that the tree T is infinite. Then, K is a nonempty compact subset of $I_\varnothing$. Moreover, the outer measure μ has total mass $\mu(K) = \zeta(I_\varnothing)$.*

PROOF. Let us assume that the tree T is infinite. Then, K is the intersection of a nonincreasing nested sequence of nonempty compact sets, and is therefore itself nonempty by virtue of Cantor's intersection theorem.

Moreover, the set K is clearly included in the initial compact set $I_\varnothing$. It follows that the total mass of μ satisfies

$$\mu(\mathbb{R}^d) = \mu(K) = \zeta^*(K) \leq \zeta(I_\varnothing).$$

In order to establish the reverse inequality, let us consider a sequence $(C_n)_{n\geq 1}$ in $\mathcal{C}$ such that $K \subseteq \bigcup_n C_n$. We may now follow essentially the proof of Proposition 2.18. Indeed, as the compact sets I_u are either disjoint or included in one another, there exists a subset $\mathcal{N}$ of $\mathbb{N}$ such that the sets C_n indexed by $n \in \mathcal{N}$ are disjoint, have a nonempty intersection with K and still cover this set. Moreover, if C_n is a compact indexed by a node in $T \cap \mathbb{N}^j$ with $j \geq 1$, let us define C'_n as the open set formed by the points at a distance less than $\min\{\varepsilon_1, \ldots, \varepsilon_j\}/3$ from C_n, where

$$\varepsilon_j = \min_{\substack{u,v \in T \cap \mathbb{N}^j \\ u \neq v}} \mathrm{d}(I_u, I_v) > 0. \tag{68}$$

In the trivial case where C_n is merely equal to $I_\varnothing$, we choose C'_n to be an arbitrary open superset of C_n. We do the same thing if $\varepsilon_1 = \ldots = \varepsilon_j = \infty$, which means that C_n is a compact set indexed by the word $1 \ldots 1$ with length j, and that the tree begins by a single spine connecting the root $\varnothing$ to the node encoded by the above word. Now that the open sets C'_n, for $n \in \mathcal{N}$, are properly defined, one easily checks that they are disjoint and cover K. The latter set being compact, we may extract from $\mathcal{N}$ a finite subset $\mathcal{N}'$ such that the sets C'_n, for $n \in \mathcal{N}'$, still cover K. However, for these values of n, we have $K \cap C'_n = K \cap C_n$. It follows that K is covered by the finitely many compacts C_n, for $n \in \mathcal{N}'$.

Among these sets, we choose one that is indexed by a node with maximal generation in the tree T; this node is denoted by u^*. Then, the siblings of u^* in the tree T are of the form $\pi(u^*)k$ with $1 \leq k \leq k_{u^*}(T)$. If a set of the form $I_{\pi(u^*)k}$ intersects K, then it must intersect a unique set C_{n_0} with $n_0 \in \mathcal{N}'$. The generation of C_{n_0} cannot be larger than that of u^*, *i.e.* that of $\pi(u^*)k$, so that C_{n_0} contains $I_{\pi(u^*)k}$. Moreover, the latter inclusion cannot be strict, as otherwise C_{n_0} would also contain I_{u^*}, which would contradict the disjointness of the sets C_n, for $n \in \mathcal{N}'$. It follows that the sets $I_{\pi(u^*)k}$ that exhibit a nonempty intersection with K may be written in the form $C_{n_1}, \ldots, C_{n_i}$ with $n_1, \ldots, n_i \in \mathcal{N}'$. In the opposite case where $I_{\pi(u^*)k} \cap K = \emptyset$, then the subtree of T formed by the descendants of $\pi(u^*)k$ is necessarily finite and, as a result of a remark made right after (66), this demands that $\zeta(I_{\pi(u^*)k}) = 0$. Therefore, using (66), we end up with

$$\begin{aligned}\sum_{n\in\mathcal{N}'} \zeta(C_n) &= \sum_{k=1}^{k_{\pi(u^*)}(T)} \zeta(I_{\pi(u^*)k}) + \sum_{n\in\mathcal{N}'\setminus\{n_1,\ldots,n_i\}} \zeta(C_n) \\ &\geq \zeta(I_{\pi(u^*)}) + \sum_{n\in\mathcal{N}'\setminus\{n_1,\ldots,n_i\}} \zeta(C_n),\end{aligned}$$

together with the fact that the sets C_n, for $n \in \mathcal{N}' \setminus \{n_1, \ldots, n_i\}$, combined with the set $I_{\pi(u^*)}$ cover K. We can finally replicate this procedure until obtaining the trivial covering of the set K by the initial compact $I_\varnothing$. This leads to

$$\sum_{n=1}^{\infty} \zeta(C_n) \geq \sum_{n\in\mathcal{N}'} \zeta(C_n) \geq \zeta(I_\varnothing).$$

Taking the infimum in the left-hand side, we conclude that $\zeta^*(K) \geq \zeta(I_\varnothing)$. $\square$

Let us remark that the second condition in (66) may easily be replaced by an equality if necessary. Indeed, it suffices to replace ζ by the premeasure ξ defined on $\mathcal{C}$ by $\xi(I_\varnothing) = \zeta(I_\varnothing)$ and the recurrence relation

$$\xi(I_{uk}) = \frac{\zeta(I_{uk})}{\sum\limits_{l=1}^{k_u(T)} \zeta(I_{ul})} \xi(I_u),$$

for $u \in T$ and $k \in \{1, \ldots, k_u(T)\}$. When the denominator vanishes, the numerator vanishes as well, and we adopt the convention that the quotient is zero. Note that the premeasure thus obtained bounds ζ from below.

Under further conditions on the compact sets I_u, we may use Lemma 2.2, *i.e.* the mass distribution principle, in order to derive a lower bound on the Hausdorff dimension of the limiting set K. This is the purpose of the next result. In its statement, $(\varepsilon_j)_{j\geq 1}$ is the sequence given by (68) and $(m_j)_{j\geq 1}$ is defined by

$$m_j = \min_{u\in T\cap\mathbb{N}^{j-1}} k_u(T), \tag{69}$$

thereby indicating the smallest number of children among the nodes of the tree at a given generation.

LEMMA 2.4. *Let us assume that the sequence $(\varepsilon_j)_{j\geq 1}$ is decreasing and that the sequence $(m_j)_{j\geq 1}$ is positive. Then,*

$$\dim_{\mathrm{H}} K \geq \liminf_{j\to\infty} \frac{\log(m_1 \ldots m_{j-1})}{-\log(m_j^{1/d} \varepsilon_j)}.$$

PROOF. We may assume that the right-hand side in the formula is positive. Indeed, the integers m_j being positive, the tree T is infinite, and Lemma 2.3 ensures that the set K is nonempty, thereby having dimension at least zero.

Moreover, note that the sequence $(\varepsilon_j)_{j\geq 1}$ necessarily converges to zero and thus, as the right-hand side in the formula is positive, that the sequence $(m_j)_{j\geq 1}$ has infinitely many terms larger than one. As a matter of fact, let us assume by contradiction the existence of a real $\delta > 0$ such that $\varepsilon_j \geq \delta$ for all $j \geq 1$. Since the previous sequence is decreasing, for each $j \geq 0$, there exists a node $u \in T \cap \mathbb{N}^j$ such that $k_u(T) \geq 2$ and the sets $I_{u1}, \ldots, I_{uk_u(T)}$ are separated by a distance at least ε_{j+1}. Hence, $I_u \setminus (I_{u1} \sqcup \ldots \sqcup I_{uk_u(T)})$ contains an open ball with diameter δ. We thus obtain infinitely many disjoint balls with diameter δ that are included in $I_\varnothing$, which contradicts the boundedness of this set.

Let us now consider the premeasure ζ defined recursively on the collection $\mathcal{C}$ by $\zeta(I_\varnothing) = 1$ and

$$\forall u \in T \setminus \{\varnothing\} \qquad \zeta(I_u) = \frac{\zeta(I_{\pi(u)})}{k_{\pi(u)}(T)}.$$

It is clear that ζ satisfies (66), and that in fact equality holds therein. We may thus consider the outer measure μ defined on K as above. By Lemma 2.3 again, its total mass is equal to one.

Now, let C denote a subset of $\mathbb{R}^d$ such that $C \cap K \neq \emptyset$ and $0 < |C| < \varepsilon_1/2$. Then, C is contained in a closed ball B with diameter twice that of C, namely, $|B| = 2|C| < \varepsilon_1$. Let j denote the unique integer such that $\varepsilon_j \leq |B| < \varepsilon_{j-1}$. There exists a node $u^* \in T \cap \mathbb{N}^{j-1}$ such that $B \cap I_{u^*} \neq \emptyset$, and this node is unique because the compact sets of the $(j-1)$-th generation are separated by a distance at least ε_{j-1}. Therefore, the set $B \cap K$ is covered by the sets I_{u^*k} that intersect B, so that

$$\mu(B) = \zeta^*(B \cap K) \leq \sum_{\substack{1 \leq k \leq k_{u^*}(T) \\ B \cap I_{u^*k} \neq \emptyset}} \zeta(I_{u^*k}) = \frac{\zeta(I_{u^*})}{k_{u^*}(T)} \# \chi_B,$$

where χ_B denotes the set of $k \in \{1, \ldots, k_{u^*}(T)\}$ such that B intersects I_{u^*k}. For k in this set, let x_k denote a point lying in B and I_{u^*k} simultaneously. Thus, the open balls with radius $\varepsilon_j/2$ centered at x_k, for $k \in \chi_B$, are disjoint and all included in the ball obtained by doubling B. This leads to

$$\sum_{k \in \chi_B} \mathcal{L}^d\left(\overline{\mathrm{B}}\left(x_k, \frac{\varepsilon_j}{2}\right)\right) \leq \mathcal{L}^d(\overline{\mathrm{B}}(x, |B|)),$$

where x denotes the center of B. We deduce that, in addition to being bounded above by $k_{u^*}(T)$, the cardinality of the set χ_B is also at most $(2|B|/\varepsilon_j)^d$. Hence, for any real number $s \in [0, d]$,

$$\mu(B) \leq \frac{\zeta(I_{u^*})}{k_{u^*}(T)} (k_{u^*}(T))^{1-s/d} \left(\left(\frac{2|B|}{\varepsilon_j}\right)^d\right)^{s/d} = 2^s |B|^s \frac{\zeta(I_{u^*})}{k_{u^*}(T)^{s/d} \varepsilon_j^s}.$$

In view of the relationship between the set C and the ball B, and the definition of the integers m_j, we infer that

$$\frac{\mu(C)}{|C|^s} \leq \frac{4^s}{m_1 \ldots m_{j-1} (m_j^{1/d} \varepsilon_j)^s}.$$

If s is smaller than the lower bound given in the statement of the lemma, then the right-hand side is bounded above by 4^s for j large enough. Thus, letting κ denote the supremum over $j \geq 1$ of this right hand-side, we have $\kappa < \infty$ and therefore

$$\mu(C) \leq \kappa |C|^s$$

for all subsets C of $\mathbb{R}^d$ such that $C \cap K \neq \emptyset$ and $0 < |C| < \varepsilon_1/2$. Now, if C does not intersect K, the latter bound still holds in an obvious manner since $\mu(C)$ must vanish. Finally, the bound also holds when C intersects K and has diameter zero, because $\mu(C)$ vanishes as well. Indeed, $C \cap K$ is then reduced to a singleton $\{x\}$, which is covered by a nested sequence of compact sets I_u, so that

$$\mu(C) = \zeta^*(\{x\}) \leq \sup_{u \in T \cap \mathbb{N}^j} \zeta(I_u) \leq \frac{1}{m_1 \ldots m_j},$$

which goes to zero as $j \to \infty$, because m_j must be at least two for infinitely many values of j. We conclude using the mass distribution principle, see Lemma 2.2. □

2.10. Iterated function systems

We now turn our attention to a class of fractal sets that satisfy a kind of selfsimilarity property, meaning that the sets locally look like the global object. We shall eventually derive upper and lower bounds on the Hausdorff dimension of these sets. Let F denote a closed subset of $\mathbb{R}^d$. A mapping $f : F \to F$ is called a *contraction* if

$$\exists c \in (0,1) \quad \forall x, y \in F \qquad |f(y) - f(x)| \leq c\,|y - x|. \tag{70}$$

From its very definition, a contraction is clearly continuous. Furthermore, we call an *iterated function system* any finite collection $\{f_1, \ldots, f_m\}$ of contractions with cardinality $m \geq 2$. As shown by the next statement, any such iterated function system determines a unique *attractor*, that is, a nonempty compact $K \subseteq F$ with

$$K = \bigcup_{k=1}^{m} f_k(K).$$

To establish this result, we endow the collection $\mathcal{C}(F)$ of all nonempty compact subsets of F with the *Hausdorff metric* defined by

$$\delta(A, B) = \inf\{\delta > 0 \mid A \subseteq B_\delta \text{ and } B \subseteq A_\delta\},$$

where A_δ denotes the δ-neighborhood of the set A, that consists of the points $x \in F$ such that $\mathrm{d}(x, A) \leq \delta$. Let us mention that $\mathcal{C}(F)$ is a complete metric space.

Proposition 2.19. *Let us consider an iterated function system $\{f_1, \ldots, f_m\}$ on a closed set $F \subseteq \mathbb{R}^d$. Then, the system has a unique attractor, denoted by K. More precisely, letting f be the mapping that sends a set $A \in \mathcal{C}(F)$ to*

$$f(A) = \bigcup_{k=1}^{m} f_k(A),$$

and choosing A to be stable under each contraction f_k, we have

$$K = \bigcap_{j=0}^{\infty} \downarrow f^j(A),$$

where f^j denotes the j-th iterate of the mapping f.

Proof. Note that f maps $\mathcal{C}(F)$ to itself, and that a set in $\mathcal{C}(F)$ is an attractor if and only if it is a fixed point of the mapping f. Then, if A and B are two nonempty compact subsets of F, we have

$$\delta(f(A), f(B)) \leq \max_{1 \leq k \leq m} \delta(f_k(A), f_k(B)) \leq \delta(A, B) \max_{1 \leq k \leq m} c_k, \tag{71}$$

where c_k comes from (70) for the contraction f_k. Thus, f is a contraction on the complete metric space $\mathcal{C}(F)$. The Banach fixed point theorem now ensures that f admits a unique fixed point, *i.e.* the iterated function system admits a unique attractor, denoted by K. Moreover, K may be obtained as the limit as $j \to \infty$ of the j-th iterate of an arbitrary set $A \in \mathcal{C}(F)$. In particular, if A is stable under each f_k, then it is stable under f, that is, $f(A) \subseteq A$. Hence, the sets $f^j(A)$ form a nonincreasing sequence of compacts, and one easily checks that their intersection coincides with K.

Note that we can always find a set $A \in \mathcal{C}(F)$ that is A is stable under each f_k. If F itself is compact, then we can obviously pick $A = F$. Otherwise, letting x_0 denote an arbitrary point in F, we can choose $A = F \cap \overline{\mathrm{B}}(x_0, r)$ for r sufficiently large. Indeed, if $x \in F \cap \overline{\mathrm{B}}(x_0, r)$, then

$$|f_k(x) - x_0| \leq |f_k(x) - f_k(x_0)| + |f_k(x_0) - x_0| \leq c_k r + |f_k(x_0) - x_0| \leq r$$

if r is large enough to ensure that the latter inequality holds for all k. Again, the sets $f^j(A)$ are nonincreasing, and their intersection is a fixed point of f. This gives a more constructive proof of the existence of the attractor. The uniqueness may then be recovered by means of (71). □

The simplest example of attractor is certainly the middle-third Cantor set $\mathbb{K}$, already dealt with in Sections 2.8 and 2.9. As a matter of fact, it is easy to deduce from (65) that $\mathbb{K}$ is the attractor of the iterated function system $\{f_1, f_2\}$ formed by the two contracting similarity transformations from $[0, 1]$ to itself defined by

$$f_1(x) = \frac{x}{3} \qquad \text{and} \qquad f_2(x) = \frac{x+2}{3}. \tag{72}$$

We refer for instance to [**29**, Chapter 9] for other classical examples of fractal sets obtained through iterated function systems, like the Sierpiński triangle or the Koch curve and its generalizations.

Our purpose is now to give some estimates on the Hausdorff dimension of an attractor K. The next result gives an upper bound and holds in a general setting.

PROPOSITION 2.20. *Let K denote the attractor of an iterated function system $\{f_1, \ldots, f_m\}$ defined on a closed set $F \subseteq \mathbb{R}^d$, and let s be a positive real such that*

$$\sum_{k=1}^{m} c_k^s = 1, \tag{73}$$

where c_k comes from (70) for the contraction f_k. Then, the Hausdorff s-dimensional measure of the set K is finite, and in particular $\dim_{\mathrm{H}} K \leq s$.

PROOF. As usual for upper bounds, the proof reduces to finding an appropriate covering of the attractor K. Thanks to Proposition 2.19, we know that K is covered by the sets $f^j(K)$ for all $j \geq 0$. Moreover, $f^j(K)$ is the union over all integers $k_1, \ldots, k_j$ between one and m of the sets $f_{k_1} \circ \ldots \circ f_{k_j}(K)$. These sets satisfy

$$|f_{k_1} \circ \ldots \circ f_{k_j}(K)| \leq c_{k_1} \ldots c_{k_j} |K|,$$

so that for all $\delta > 0$ and for all j large enough,

$$\mathcal{H}^s_\delta(K) \leq \sum_{1 \leq k_1, \ldots, k_j \leq m} |f_{k_1} \circ \ldots \circ f_{k_j}(K)|^s \leq |K|^s \sum_{1 \leq k_1, \ldots, k_j \leq m} (c_{k_1} \ldots c_{k_j})^s = |K|^s.$$

Letting δ go to zero, we deduce that $\mathcal{H}^s(K)$ is bounded above by $|K|^s$, which is finite because K is compact. Therefore, K has Hausdorff dimension at most s. □

Obtaining a lower bound on the Hausdorff dimension of the attractor is less straightforward and requires additional assumptions. The classical setting consists in assuming that the contractions f_k that form the iterated function system are similarity transformations, *i.e.* satisfy the condition

$$\exists c_k \in (0,1) \quad \forall x, y \in F \qquad |f_k(y) - f_k(x)| = c_k\, |y - x|$$

instead of the mere (70), and then supposing that the *open set condition* holds, namely, that there exists a nonempty bounded open subset V of F such that

$$V \supseteq \bigsqcup_{k=1}^{m} f_k(V).$$

It is known from Proposition 2.19 that the attractor K is the union of its images $f_k(K)$ under the contractions. The open set condition roughly means that these components $f_k(K)$ do not overlap too much, and that the union is nearly disjoint. Following this intuition and exploiting the fact that the contractions f_k are similarities, a nonrigorous heuristic approach then consists in writing that

$$\mathcal{H}^s(K) = \sum_{k=1}^{m} \mathcal{H}^s(f_k(K)) = \mathcal{H}^s(K) \sum_{k=1}^{m} c_k^s,$$

so that the only plausible value for the Hausdorff dimension is the solution of (73). It is actually possible to make this approach correct, and to prove that, under the above assumptions, the Hausdorff s-dimensional measure of K is both positive and finite, so that in particular $\dim_{\mathrm{H}} K = s$, where s solves (73). We refer for example to [**29**, Theorem 9.3] for a precise statement and a detailed proof.

In the number-theoretic applications that we shall discuss in Section 3.3 below, the contractions that form the iterated function system are not similarity transformations, and the aforementioned classical setting is therefore irrelevant. Instead, we shall call upon the following result that applies to quite general contractions, but relies on a stronger assumption than the open set condition.

PROPOSITION 2.21. *Let us consider an iterated function system* $\{f_1, \ldots, f_m\}$ *defined on a closed set* $F \subseteq \mathbb{R}^d$ *and satisfying*

$$\forall k \in \{1, \ldots, m\} \quad \exists b_k \in (0,1) \quad \forall x, y \in F \qquad |f_k(y) - f_k(x)| \geq b_k\,|y - x|,$$

and let s *be a positive real such that*

$$\sum_{k=1}^{m} b_k^s = 1.$$

Let us assume that the attractor, denoted by K*, of the iterated function system* $\{f_1, \ldots, f_m\}$ *verifies*

$$K = \bigsqcup_{k=1}^{m} f_k(K). \tag{74}$$

Then, the Hausdorff s*-dimensional measure of the set* K *is positive, and in particular* $\dim_{\mathrm{H}} K \geq s$.

PROOF. We are in the setting of the general Cantor construction introduced in Section 2.9.2. Here, the construction is indexed by the m-ary tree T_m formed by the words of finite length over the alphabet $\{1, \ldots, m\}$, the compact sets are

$$I_u = f_{u_1} \circ \ldots \circ f_{u_j}(K)$$

for any word $u = u_1 \ldots u_j$, and the associated premeasure ζ is defined by

$$\zeta(I_u) = (b_{u_1} \ldots b_{u_j})^s,$$

in addition to $\zeta(\emptyset) = 0$. In accordance with the standard conventions, we have in particular $I_\varnothing = K$ and $\zeta(I_\varnothing) = 1$, where $\varnothing$ denotes the empty word, which represents the root of the tree. The compatibility conditions (66) are plainly satisfied. Indeed, for any word $u = u_1 \ldots u_j$ and any integer k between one and m, we have

$$f_{u_1} \circ \ldots \circ f_{u_j}(K) = \bigsqcup_{k=1}^{m} f_{u_1} \circ \ldots \circ f_{u_j} \circ f_k(K)\,;$$

the union is disjoint due to (74) and the injectivity of the contractions. Thus, every compact set I_u is the disjoint union of the sets I_{uk} indexed by its children. Moreover, the choice of s ensures that

$$\zeta(I_u) = (b_{u_1} \ldots b_{u_j})^s = \sum_{k=1}^{m} (b_{u_1} \ldots b_{u_j} b_k)^s = \sum_{k=1}^{m} \zeta(I_{uk}).$$

Now, thanks to Proposition 2.19, the limiting compact set defined by (67) coincides with the attractor K. We then use Theorem 2.2 to extend via the formula (51) the premeasure ζ to an outer measure ζ^* on all the subsets of $\mathbb{R}^d$. The function μ that maps a subset E of $\mathbb{R}^d$ to the value $\zeta^*(E \cap K)$ is an outer measure as well, and Lemma 2.3 implies that μ has total mass equal to $\mu(K) = \zeta(I_\varnothing) = 1$.

With a view to applying the mass distribution principle, let us estimate the μ-mass of sets in terms of their diameter. We begin by considering the closed balls $\overline{\mathrm{B}}(x, r)$ with $x \in K$ and $r \in (0, \varepsilon)$, where

$$\varepsilon = \min_{1 \leq k < k' \leq m} \mathrm{d}(f_k(K), f_{k'}(K)) > 0\,;$$

note that two distinct compact sets $f_k(K)$ are positively separated because they are disjoint. According to (67), for every integer $j \geq 0$, there exists a unique word $u^{(j)}$ with length j such that x belongs to the set $I_{u^{(j)}}$. Necessarily, the parent of the node $u^{(j+1)}$ is the node $u^{(j)}$; in addition to the fact that $0 < b_k < 1$ for all k, this ensures that the sequence $(\rho_j)_{j\geq 0}$ defined by $\rho_j = \varepsilon b_{u_1^{(j)}} \dots b_{u_j^{(j)}}$ is decreasing and converges to zero (again, due to the standard conventions, $\rho_0 = \varepsilon$). In particular, there exists a unique integer $j \geq 1$ such that $\rho_j \leq r < \rho_{j-1}$.

For this choice of the integer j, let us write u as a shorthand for $u^{(j)}$, and let us consider another word v with length j. Let w denote the closest common ancestor of u and v, and let l denote the length of w. Let x and y belong to I_u and I_v, respectively. In particular, x belongs to $I_{wu_{l+1}}$, so there exists a unique $x' \in f_{u_{l+1}}(K)$ such that $x = f_{w_1} \circ \dots \circ f_{w_l}(x')$. Likewise, y is in $I_{wv_{l+1}}$, and there is a unique $y' \in f_{v_{l+1}}(K)$ such that $y = f_{w_1} \circ \dots \circ f_{w_l}(y')$. Thus,

$$|x - y| = |f_{w_1} \circ \dots \circ f_{w_l}(x') - f_{w_1} \circ \dots \circ f_{w_l}(y')| \geq b_{w_1} \dots b_{w_l} |x' - y'|.$$

As u_{l+1} and v_{l+1} are distinct, the distance between x' and y' is at least ε. Taking the infimum over x and y in the left-hand side, we finally deduce that

$$\mathrm{d}(I_u, I_v) \geq \varepsilon b_{w_1} \dots b_{w_l} \geq \varepsilon b_{u_1} \dots b_{u_{j-1}} = \rho_{j-1}.$$

The latter inequality holds because the word w is a prefix of $u_1 \dots u_{j-1} = u^{(j-1)}$, and the reals b_k are again strictly between zero and one. The upshot is that the set $\overline{\mathrm{B}}(x,r) \cap K$ is contained in no other component of K of the j-th generation than I_u. Indeed, should v be another word with length j such that $\overline{\mathrm{B}}(x,r) \cap I_v \neq \emptyset$, the distance between I_u and I_v would be at most r, while the above ensures that this distance is at least ρ_{j-1} ; this would eventually lead to $\rho_{j-1} \leq r$, in contradiction with the choice of j with respect to r. We infer that

$$\mu(\overline{\mathrm{B}}(x,r)) = \zeta^*(K \cap \overline{\mathrm{B}}(x,r)) \leq \zeta(I_u) = (b_{u_1} \dots b_{u_j})^s = \left(\frac{\rho_j}{\varepsilon}\right)^s \leq \frac{r^s}{\varepsilon^s}.$$

Now, let C be a subset of $\mathbb{R}^d$ with diameter less than ε. If C does not intersect the attractor K, then the μ-mass of C obviously vanishes. Otherwise, there exists a point $x \in C \cap K$, and the set C is plainly included in the closed ball centered at x with diameter $|C|$. Therefore,

$$\mu(C) \leq \mu(\overline{\mathrm{B}}(x, |C|)) \leq \frac{|C|^s}{\varepsilon^s}.$$

Lemma 2.2, namely, the mass distribution principle finally ensures that the attractor K has positive Hausdorff s-dimensional measure. In particular, its Hausdorff dimension is bounded below by s. $\square$

Let us mention that the middle-third Cantor set $\mathbb{K}$ clearly falls into the above setting. Indeed, as mentioned previously, $\mathbb{K}$ is the attractor of the system formed by the two contractions f_1 and f_2 defined by (72), and these contractions clearly meet the requirements of Propositions 2.20 and 2.21 with all the parameters b_k and c_k being equal to $1/3$. Moreover, (74) holds for the set $\mathbb{K}$ together with the two contractions f_1 and f_2. We deduce that the Hausdorff s-dimensional measure of $\mathbb{K}$ is both positive and finite if s is a solution of the equation $2(1/3)^s = 1$, *i.e.* if $s = \log 2/\log 3$. We conclude that this value of s is the Hausdorff dimension of $\mathbb{K}$, thus recovering Propositions 2.17 and 2.18.

2.11. Connection with local density expressions

We end this chapter with a remarkable link between the Hausdorff dimension of a set and the local density properties of the measures that it supports. To proceed, we need the following classical covering lemma due to Vitali. In the statement, if B denotes an open ball of $\mathbb{R}^d$, then $5B$ stands for the open ball concentric to B with radius five times that of B.

LEMMA 2.5 (Vitali's covering lemma). *Let $\mathcal{C}$ denote an arbitrary collection of open balls of $\mathbb{R}^d$ such that*

$$\delta_{\mathcal{C}} = \sup_{B\in\mathcal{C}} |B| < \infty.$$

Then, there exists a countable subcollection $\mathcal{C}'$ of disjoint balls in $\mathcal{C}$ such that

$$\bigcup_{B\in\mathcal{C}} B \subseteq \bigcup_{B\in\mathcal{C}'} 5B.$$

PROOF. The proof makes a thorough use of the Hausdorff maximal principle. For any integer $j \geq 0$, let $\mathcal{C}_j$ denote the subcollection of $\mathcal{C}$ formed by the balls B with diameter satisfying $\delta_{\mathcal{C}} 2^{-(j+1)} < |B| \leq \delta_{\mathcal{C}} 2^{-j}$. We now define recursively a sequence of subcollections $\mathcal{C}'_j$ of $\mathcal{C}_j$ in the following manner. To begin with, $\mathcal{C}'_0$ is any maximal collection of disjoint balls in $\mathcal{C}_0$. Then, for any $j \geq 0$, assuming that $\mathcal{C}'_0, \ldots, \mathcal{C}'_j$ have been defined, we decide that $\mathcal{C}'_{j+1}$ is any maximal disjoint collection among the balls $B \in \mathcal{C}_{j+1}$ such that $B \cap B' = \emptyset$ for every ball B' in $\mathcal{C}'_0 \cup \ldots \cup \mathcal{C}'_j$. The union, denoted by $\mathcal{C}'$, of the collections $\mathcal{C}'_j$ over $j \geq 0$ is therefore a countable collection of disjoint balls in $\mathcal{C}$.

It remains to prove the covering property. Let us consider a ball $B \in \mathcal{C}$. There is an index $j \geq 0$ such that $B \in \mathcal{C}_j$. The maximality of $\mathcal{C}'_j$ ensures that there exists a ball B' in $\mathcal{C}'_0 \cup \ldots \cup \mathcal{C}'_j$ that intersects B. The diameter of B' is larger than $\delta_{\mathcal{C}} 2^{-(j+1)}$, while that of B is bounded above by $\delta_{\mathcal{C}} 2^{-j}$; we deduce that $|B| < 2|B'|$. Thus, the ball B is clearly contained in $5B'$, and the result follows. □

Now, let us consider an outer measure μ for which the Borel subsets of $\mathbb{R}^d$ are measurable, *i.e.* such that $\mathcal{B} \subseteq \mathcal{F}_\mu$. For any real $s \geq 0$, we define the *upper s-density* of the outer measure μ at a given point $x \in \mathbb{R}^d$ by

$$\overline{\Theta}^s(\mu, x) = \limsup_{r\to 0} \frac{\mu(\mathrm{B}(x,r))}{r^s}.$$

It is useful to observe that the function $x \mapsto \overline{\Theta}^s(\mu, x)$ is Borel-measurable, see [**46**, Remark 2.10] for details. The connection with Hausdorff measures is given by the following result.

PROPOSITION 2.22. *Let μ be an outer measure on $\mathbb{R}^d$ for which the Borel sets are measurable, let F be a Borel subset of $\mathbb{R}^d$, and let c be a positive real.*
(1) *If $\overline{\Theta}^s(\mu, x) < c$ for all $x \in F$, then $\mathcal{H}^s(F) \geq \mu(F)/c$.*
(2) *If $\overline{\Theta}^s(\mu, x) > c$ for all $x \in F$, then $\mathcal{H}^s(F) \leq 10^s \mu(\mathbb{R}^d)/c$.*

PROOF. In order to prove (1), let us consider a real number $\delta > 0$ and the subset of F defined by

$$F_\delta = \{x \in F \mid \mu(\mathrm{B}(x,r)) < c\, r^s \text{ for all } r \in (0,\delta]\}.$$

In view of [**46**, Remark 2.10], this is a Borel subset of F. Now, let $(C_n)_{n\geq 1}$ denote a sequence of sets in $\mathcal{P}(\mathbb{R}^d)$ with diameter at most $\delta/2$ and such that $F \subseteq \bigcup_n C_n$. In particular, the sets C_n cover the set F_δ. If n is such that $F_\delta \cap C_n$ contains a point denoted by x, then it is clear that for any $\varepsilon \in (0, \delta/2]$, the open ball centered at x with radius $|C_n| + \varepsilon$ contains the set C_n. Thus, by definition of F_δ, we have

$$\mu(C_n) \leq \mu(\mathrm{B}(x, |C_n| + \varepsilon)) < c(|C_n| + \varepsilon)^s.$$

Letting ε go to zero, we deduce that $\mu(C_n)$ is merely less than $c\,|C_n|^s$. As a consequence, the μ-mass of the set F_δ satisfies

$$\mu(F_\delta) \leq \sum_{F_\delta\cap C_n\neq\emptyset} \mu(C_n) \leq c \sum_{n=1}^{\infty} |C_n|^s.$$

Taking the infimum over all sequences $(C_n)_{n\geq 1}$ in the right-hand side, we deduce that $\mu(F_\delta) \leq c\,\mathcal{H}^s_{\delta/2}(F)$. Since the outer measures $\mathcal{H}^s_{\delta/2}$ increase to $\mathcal{H}^s$ as δ goes to zero, we have $\mu(F_\delta) \leq c\,\mathcal{H}^s(F)$. To conclude, it suffices to make use of Proposition 2.4(1) and to observe that $(F_{1/m})_{m\geq 1}$ is a nondecreasing sequence of Borel sets whose union is equal to the whole set F.

We now establish (2). To this purpose, let us consider a real $\delta > 0$ and the collection $\mathcal{C}$ of open balls defined by

$$\mathcal{C} = \{\mathrm{B}(x,r),\ x \in F \text{ and } r \in (0,\delta] \text{ such that } \mu(\mathrm{B}(x,r)) > c\,r^s\}$$

Then, the set F is covered by the balls in $\mathcal{C}$. We may apply Lemma 2.5 to obtain a countable subcollection $\mathcal{C}'$ of disjoint balls in $\mathcal{C}$ such that the enlarged balls $5B$, for $B \in \mathcal{C}'$, still cover the set F. These balls have diameter at most 10δ, so

$$\mathcal{H}^s_{10\delta}(F) \leq \sum_{B\in\mathcal{C}'} |5B|^s = 5^s \sum_{B\in\mathcal{C}'} |B|^s < \frac{10^s}{c} \sum_{B\in\mathcal{C}'} \mu(B') \leq \frac{10^s}{c}\mu(\mathbb{R}^d),$$

where the last inequality follows from the disjointness of the balls B in $\mathcal{C}'$, and the fact that these balls are μ-measurable. $\square$

Although Proposition 2.22 has many various applications, we shall not actually use this result as is in what follows. More specifically, when studying frequencies of digits in base m expansions, we shall use a variant of Proposition 2.22 where open balls are replaced by m-adic intervals, see Section 3.4.

CHAPTER 3

First applications in metric number theory

3.1. The Jarník-Besicovitch theorem

We shall apply the methods introduced in Sections 2.8 and 2.9 to determine the Hausdorff dimension of the set $J_{d,\tau}$ defined by (1) and formed by the points that are approximable at rate at least τ by the points with rational coordinates. Recall that this set is equal to the whole space $\mathbb{R}^d$ when $\tau \leq 1 + 1/d$, so that we may suppose that we are in the opposite case. The dimension of $J_{d,\tau}$ was obtained by Jarník in 1929 and, independently, Besicovitch in 1934, see [**7, 36**].

Theorem 3.1 (Jarník, Besicovitch). *For any real number $\tau > 1 + 1/d$, the Hausdorff dimension of the set $J_{d,\tau}$ is given by*

$$\dim_{\mathrm{H}} J_{d,\tau} = \frac{d+1}{\tau}.$$

The remainder of this section is devoted to the proof of Theorem 3.1; we shall establish the upper and the lower bound separately. We refer to Section 4.3 for another proof of this theorem, and a refinement thereof, based on the general theory of homogeneous ubiquitous systems.

3.1.1. Upper bound on the dimension of $J_{d,\tau}$.

The upper bound may be obtained by using Lemma 2.1. Indeed, the set $J_{d,\tau}$ may be written in the form

$$J_{d,\tau} = \bigcup_{k\in\mathbb{Z}^d} (k + J'_{d,\tau}) \qquad \text{with} \qquad J'_{d,\tau} = \bigcap_{Q=1}^{\infty} \bigcup_{q=Q}^{\infty} \bigcup_{p\in\{0,\ldots,q\}^d} \mathrm{B}_\infty\left(\frac{p}{q}, \frac{1}{q^\tau}\right).$$

The set $J'_{d,\tau}$ may be seen as the limsup of the balls $\mathrm{B}_\infty(p/q, q^{-\tau})$, for $p \in \{0,\ldots,q\}^d$ and $q \geq 1$. In view of Lemma 2.1, for any gauge function g such that the series $\sum_q (q+1)^d g(2q^{-\tau})$ converges, the Hausdorff g-mass of $J'_{d,\tau}$ vanishes. The subadditivity of the outer measure $\mathcal{H}^g$ then ensures that the same property holds for the whole set $J_{d,\tau}$. Note that, owing to Proposition 2.10, we can assume that the gauge function g is normalized, in which case the criterion boils down to the convergence of the slightly simpler series $\sum_q q^d g(q^{-\tau})$. Specializing to the power gauge functions, we end up with examining the convergence of the series $\sum_q q^{d-\tau s}$, so that the upper bound holds.

3.1.2. Lower bound on the dimension of $J_{d,\tau}$.

It suffices to give, for any $\sigma > \tau$, a lower bound on the Hausdorff dimension of the set $J''_{d,\sigma}$ defined by

$$J''_{d,\sigma} = \limsup_{q\to\infty} \bigcup_{p\in\{1,\ldots,q-1\}^d} B^\sigma_{p,q} \qquad \text{with} \qquad B^\sigma_{p,q} = \overline{\mathrm{B}}_\infty\left(\frac{p}{q}, \frac{1}{q^\sigma}\right),$$

because $J''_{d,\sigma}$ is clearly a subset of $J_{d,\tau}$. Instead of the open balls $\mathrm{B}_\infty(p/q, q^{-\tau})$, we choose to work with the closed balls $B^\sigma_{p,q}$ because we want to use some of them as the compact sets arising in the Cantor construction detailed in Section 2.9.2. To develop this construction here, we will call upon the next lemma.

LEMMA 3.1. *Let C be a closed subcube of $[0,1]^d$ with sidelength l and let n be an integer such that $l^{d+1}n \geq 2^{15d}$. Let $\mathcal{Q}_n$ denote the set of all integers q satisfying $2^{-6d}n \leq q \leq 2^d n$. Then, there exists a set $\mathcal{S}_n^\sigma(C) \subseteq \mathbb{Z}^d \times \mathcal{Q}_n$ with cardinality at least $2^{-18d^2} l^d n^{d+1}$ such that the balls $B_{p,q}^\sigma$, for $(p,q) \in \mathcal{S}_n^\sigma(C)$, are included in C and separated by a distance larger than $n^{-1-1/d}$.*

PROOF. Throughout the proof, we endow the space $\mathbb{R}^d$ with the supremum norm. We shall work with two parameters $\alpha, \beta > 1$ whose precise values will be tuned up later. Let us consider the subset C' of C formed by the points that are at a distance at least $\alpha n^{-1-1/d}$ from $\mathbb{R}^d \setminus C$. It is easily seen that C' is a cube with sidelength $l - 2\alpha n^{-1-1/d}$, with the proviso that this value is nonnegative. Hence,

$$\mathcal{L}^d(C') = (l - 2\alpha n^{-1-1/d})_+^d,$$

where $(\,\cdot\,)_+$ denotes the positive part function.

Furthermore, for each point $x \in C$, let $q(x)$ denote the minimal value of $q \in \mathbb{N}$ such that $|qx - p|_\infty \leq n^{-1/d}$ for some $p \in \mathbb{Z}^d$. Theorem 1.1, namely, Dirichlet's theorem ensures that $q(x)$ is less than $\lceil n^{1/d} \rceil^d$, which is clearly bounded above by $2^d n$. Let us now consider the set C'' formed by the points $x \in C$ such that $q(x) < n/\beta$. Then, C'' is covered by the closed balls with curvature $qn^{1/d}$ centered at the rational points p/q within distance $1/q$ of the cube C and with denominator $q < n/\beta$. For any fixed choice of q, there are at most $(ql+3)^d$ such points. Hence,

$$\begin{aligned}\mathcal{L}^d(C'') &\leq \sum_{q<n/\beta} (ql+3)^d \left(\frac{2}{qn^{1/d}}\right)^d = \frac{2^d}{n}\left(\sum_{q<1/l}\left(l+\frac{3}{q}\right)^d + \sum_{1/l \leq q < n/\beta}\left(l+\frac{3}{q}\right)^d\right)\\ &\leq \frac{2^d}{n}\left(\frac{4^d}{l} + (4l)^d\frac{n}{\beta}\right) = 8^d l^d \left(\frac{1}{\beta} + \frac{1}{l^{d+1}n}\right).\end{aligned}$$

We now define $\mathcal{Q}_n$ as the set of all integers q satisfying $n/\beta \leq q \leq 2^d n$, and subsequently $\mathcal{S}_n^\sigma(C)$ as any set of pairs $(p,q) \in \mathbb{Z}^d \times \mathcal{Q}_n$ indexing a maximal collection of rational points p/q with denominator in $\mathcal{Q}_n$ that are at a distance at least $(\beta/n)^{1+1/d}$ from the complement of C and are separated from each other by a distance at least $3(\beta/n)^{1+1/d}$. We readily see that for any pair $(p,q) \in \mathcal{S}_n^\sigma(C)$, the ball $B_{p,q}^\sigma$ is contained in C because its radius $q^{-\sigma}$ is at most $(\beta/n)^{1+1/d}$, which is a lower bound on the distance between its center and $\mathbb{R}^d \setminus C$. Moreover, for another pair $(p',q') \in \mathcal{S}_n^\sigma(C)$, the balls $B_{p,q}^\sigma$ and $B_{p',q'}^\sigma$ are clearly separated by a distance at least $(\beta/n)^{1+1/d}$, because their radius are at most $(\beta/n)^{1+1/d}$ and their center are at a distance at least $3(\beta/n)^{1+1/d}$. Given that $\beta > 1$, the balls are therefore separated by a distance larger than $n^{-1-1/d}$.

It remains us to derive the required lower bound on the cardinality of $\mathcal{S}_n^\sigma(C)$, and to adjust the values of the parameters α and β accordingly. For any point $x \in C' \setminus C''$, we have $q(x) \in \mathcal{Q}_n$, so that there exists a rational point p/q with denominator in $\mathcal{Q}_n$ for which

$$\left|x - \frac{p}{q}\right|_\infty \leq \frac{1}{qn^{1/d}} \leq \beta n^{-1-1/d}.$$

In particular, since x is at a distance at least $\alpha n^{-1-1/d}$ from the complement of C, the rational point p/q is surely at a distance at least $(\alpha-\beta)n^{-1-1/d}$ from $\mathbb{R}^d \setminus C$. If we assume in addition that $\alpha - \beta \geq \beta^{1+1/d}$, then p/q must be within distance $3(\beta/n)^{1+1/d}$ from a point p'/q' of the above collection, in view of the maximality property. Hence, by virtue of the triangle inequality,

$$\left|x - \frac{p'}{q'}\right|_\infty \leq \left|x - \frac{p}{q}\right|_\infty + \left|\frac{p}{q} - \frac{p'}{q'}\right|_\infty \leq \beta n^{-1-1/d} + 3\left(\frac{\beta}{n}\right)^{1+1/d} \leq 3\alpha n^{-1-1/d}.$$

Thus, the set $C' \setminus C''$ is covered by the closed balls with radius $3\alpha n^{-1-1/d}$ centered at the rational points indexed by $\mathcal{S}^\sigma_n(C)$. In particular,

$$\mathcal{L}^d(C' \setminus C'') \leq \frac{(6\alpha)^d}{n^{d+1}} \# \mathcal{S}^\sigma_n(C).$$

In the meantime, the Lebesgue measure of $C' \setminus C''$ is bounded below by

$$\mathcal{L}^d(C') - \mathcal{L}^d(C'') \geq (l - 2\alpha n^{-1-1/d})^d_+ - 8^d l^d \left(\frac{1}{\beta} + \frac{1}{l^{d+1} n} \right),$$

from which we deduce a lower bound on the cardinality of $\mathcal{S}^\sigma_n(C)$. It remains to adjust the parameters α and β in such a way that this bound is of the order of $l^d n^{d+1}$. It actually suffices to choose any real $\beta \geq 2^{4d+2}$, and then any real $\alpha \geq \beta(1+\beta^{1/d})$, and finally to impose that $l^{d+1} n \geq 4\alpha$ to obtain that

$$8^d \left(\frac{1}{\beta} + \frac{1}{l^{d+1} n} \right) \leq \frac{1}{2^{d+1}} \qquad \text{and} \qquad 1 - \frac{2\alpha}{l n^{1+1/d}} \geq 1 - \frac{2\alpha}{l^{d+1} n} \geq \frac{1}{2},$$

and then that the cardinality of $\mathcal{S}^\sigma_n(C)$ is bounded below by $l^d n^{d+1}/((6\alpha)^d 2^{d+1})$. We get the bounds of the statement of the lemma by choosing specifically $\alpha = 2^{13d}$ and $\beta = 2^{6d}$, imposing that $l^{d+1} n \geq 2^{15d}$, and noting that $(6\alpha)^d 2^{d+1} \leq 2^{18d^2}$. $\square$

We may now proceed with the general Cantor construction leading to the lower bound on the Hausdorff dimension of $J''_{d,\sigma}$. Lemma 3.1 will play a pivotal rôle in the construction. We introduce several constants whose specific value, though unimportant, will guarantee that this lemma may be applied throughout the proof. First, let us define

$$\kappa = 2^{(\sigma(d+1)+14)d-1} \qquad \text{and} \qquad \kappa' = 2^{d-(18+\sigma)d^2}.$$

The choice of the constants κ and κ' ensures that for any positive integers m and n and for any integer $q \in \mathcal{Q}_n$,

$$\begin{cases} m \geq \kappa\, n^{\sigma(d+1)} & \Longrightarrow \quad \left(\dfrac{2}{q^\sigma}\right)^{d+1} m \geq 2^{15d} \\[2ex] m^{d+1} > \dfrac{n^{\sigma d}}{\kappa'} & \Longrightarrow \quad 2^{-18d^2} \left(\dfrac{2}{q^\sigma}\right)^{d} m^{d+1} > 1. \end{cases} \tag{75}$$

Here, $\mathcal{Q}_n$ is the set of all integers q satisfying $2^{-6d} n \leq q \leq 2^d n$, in accordance with the statement of Lemma 3.1. We then fix an integer n_1 such that

$$n_1 > \max\{2^{15d}, 2^{18d^2/(d+1)}, 2^{(6d\sigma+1)d/(d\sigma-d-1)}\}. \tag{76}$$

The choice of n_1 ensures in particular that for all integers $n \geq n_1$ and $q \in \mathcal{Q}_n$,

$$\frac{2}{q^\sigma} < n^{-1-1/d}. \tag{77}$$

To begin with the construction, the unit cube $[0,1]^d$ is chosen to be the compact set $I_\varnothing$ indexed by the root of the underlying tree. Thanks to (76), we may apply Lemma 3.1 to this cube and the integer n_1, thus getting a set $\mathcal{S}^\sigma_{n_1}(I_\varnothing)$ contained in $\mathbb{Z}^d \times \mathcal{Q}_{n_1}$ with cardinality at least c_1 such that the balls $B^\sigma_{p,q}$, for $(p,q) \in \mathcal{S}^\sigma_{n_1}(I_\varnothing)$, are included in $I_\varnothing$ and separated by a distance larger than d_1, where

$$c_1 = 2^{-18d^2} n_1^{d+1} > 1 \qquad \text{and} \qquad d_1 = n_1^{-1-1/d}.$$

Accordingly, we choose the balls $B^\sigma_{p,q}$, for $(p,q) \in \mathcal{S}^\sigma_{n_1}(I_\varnothing)$, to be the compact sets I_k indexed by the children of the root. In particular, $k_\varnothing(T)$ is equal to $\#\mathcal{S}^\sigma_{n_1}(I_\varnothing)$.

Note that each set I_k is in fact a closed subcube of $[0,1]^d$ with sidelength equal to $2/q^\sigma$ for some $q \in \mathcal{Q}_{n_1}$. In view of (75), we may then apply Lemma 3.1 to each of these cubes and an arbitrary integer

$$n_2 > \max\left\{\kappa\, n_1^{\sigma(d+1)}, \left(\frac{n^{\sigma d}}{\kappa'}\right)^{1/(d+1)}\right\}$$

This yields subsets $\mathcal{S}^\sigma_{n_2}(I_1), \ldots, \mathcal{S}^\sigma_{n_2}(I_{k_\varnothing(T)})$ of $\mathbb{Z}^d \times \mathcal{Q}_{n_2}$ with cardinality at least c_2 such that for each k, the balls $B^\sigma_{p,q}$, for $(p,q) \in \mathcal{S}^\sigma_{n_2}(I_k)$, are included in I_k and separated by a distance larger than d_2, where

$$c_2 = \kappa'\, n_2^{d+1} n_1^{-d\sigma} > 1 \qquad \text{and} \qquad d_2 = n_2^{-1-1/d}.$$

It is then natural to choose the balls $B^\sigma_{p,q}$, for $(p,q) \in \mathcal{S}^\sigma_{n_2}(I_{u_1})$, to be the compact sets $I_{u_1 u_2}$ indexed by the children of a node $u_1 \in \{1, \ldots, k_\varnothing(T)\}$.

We may obviously repeat this procedure *ad infinitum*. We thus obtain a sequence $(n_j)_{j\geq 1}$ of integers and a family of closed cubes $(I_u)_{u\in T}$ indexed by a tree T such that the following properties hold for any integer $j \geq 1$:

- we have $n_{j+1} > \kappa\, n_j^{\sigma(d+1)}$;
- for each node $u \in T \cap \mathbb{N}^j$, the cube I_u is a closed ball of the form $B^\sigma_{p,q}$ with $(p,q) \in \mathcal{S}^\sigma_{n_j}(I_{\pi(u)})$;
- there are at least $c_j = \kappa' n_j^{d+1} n_{j-1}^{-d\sigma} > 1$ siblings at the j-th generation;
- the distance between the cubes indexed by two distinct nodes of the j-th generation is larger than $d_j = n_j^{-1-1/d}$.

Note that we adopt here the convention that $n_0 = 2^{1/\sigma - d}$ for the sake of consistency. Moreover, we recall for completeness that the initial cube is merely $I_\varnothing = [0,1]^d$.

It is clear that each point of the limiting compact set K belongs to infinitely many balls $B^\sigma_{p,q}$, and therefore K is included in $J''_{d,\sigma}$. Moreover, we are in the setting of Lemma 2.4 with

$$m_j = \min_{u \in T\cap\mathbb{N}^{j-1}} k_u(T) \geq c_j > 1 \qquad \text{and} \qquad \varepsilon_j = \min_{\substack{u,v\in T\cap\mathbb{N}^j \\ u\neq v}} \mathrm{d}(I_u, I_v) > d_j.$$

In particular, the sequence $(\varepsilon_j)_{j\geq 1}$ is decreasing, as a consequence of (77). Applying Lemma 2.4, we end up with

$$\dim_{\mathrm{H}} K \geq \liminf_{j\to\infty} \frac{\log(m_1 \ldots m_{j-1})}{-\log(m_j^{1/d}\varepsilon_j)} \geq \liminf_{j\to\infty} \frac{\log(c_1 \ldots c_{j-1})}{-\log(c_j^{1/d} d_j)}.$$

It remains to elucidate the lower limit appearing in the right-hand side. At each step of the above construction, the integer n_j may be chosen arbitrarily large: in particular, we may assume that $n_{j+1} \geq n_j^j$ for all $j \geq 0$. The numerator in the previous formula, namely,

$$(d+1)\sum_{k=1}^{j-1} \log n_k - d\sigma \sum_{k=0}^{j-2} \log n_k + (j-1)\log\kappa'$$

is therefore equivalent to $(d+1)\log n_{j-1}$ as j goes to infinity. Furthermore, the denominator is equal to

$$-\frac{1}{d}\log\kappa' + \sigma \log n_{j-1}.$$

We conclude that the lower limit is equal to $(d+1)/\sigma$, and the lower bound on the dimension of $J_{d,\tau}$ follows from letting σ tend to τ.

As shown above, the lower bound relies heavily on Lemma 3.1, which enables one to perform the general Cantor construction. In dimension $d = 1$, it is possible to use a variant form of this lemma that is slightly weaker but also much easier to establish. This method is used in Falconer's book [**29**] and we reproduce it here for the sake of completeness. In the next statement, Π_n denotes the set of primes numbers between $n+1$ and $2n$.

LEMMA 3.2. *Let I be a closed subinterval of $[0,1]$ with length l and let n be a positive integer. Then, there exists a set $\mathcal{S}_n^\sigma(I) \subseteq \mathbb{Z} \times \Pi_n$ with cardinality at least $(ln-3)\#\Pi_n$ such that the intervals $B_{p,q}^\sigma$, for $(p,q) \in \mathcal{S}_n^\sigma(I)$, are included in I and separated by a distance larger than $(2n)^{-2} - 2n^{-\sigma}$.*

PROOF. If the interval I has length $l \in (0,1]$, then it may be written in the form $x + [0,l]$ for some point $x \in [0, 1-l]$. A pair $(p,q) \in \mathbb{Z} \times \Pi_n$ is such that $B_{p,q}^\sigma \subseteq I$ as soon as p is between $qx+1$ and $q(x+l)-1$, a condition that is verified by at least $lq-3$ integers p. Thus, the total number of pairs $(p,q) \in \mathbb{Z} \times \Pi_n$ such that $B_{p,q}^\sigma \subseteq I$ is at least $(ln-3)\#\Pi_n$. To conclude, it suffices to observe that if (p,q) and (p',q') are two distinct pairs in $\mathbb{Z} \times \Pi_n$, then

$$\left|\frac{p}{q} - \frac{p'}{q'}\right| = \frac{|pq' - p'q|}{qq'} \geq \frac{1}{qq'} \geq \frac{1}{4n^2},$$

which gives the required lower bound on the distance between $B_{p,q}^\sigma$ and $B_{p',q'}^\sigma$. □

We may then use the previous lemma instead of Lemma 3.1 to develop the general Cantor construction in the one-dimensional case. The appropriate estimates on the minimal distance d_j between the intervals of the construction follow from the obvious fact that $(2n)^{-2} - 2n^{-\sigma}$ is larger than $(3n)^{-2}$ for n large enough, because $\sigma > 2$. The estimates on the minimal number of siblings c_j at the j-th generation call upon the prime number theorem, according to which $\#\Pi_n$ is larger than $n/(2\log n)$ for all n sufficiently large. Despite additional logarithmic terms, this yields the same lower bound on the Hausdorff dimension of $J_{1,\tau}$, namely, $2/\tau$.

3.2. Typical behavior of continued fraction expansions

3.2.1. The Gauss measure. We adopt the notations of Section 1.2.1.2 for the set X of all irrational numbers between zero and one, and for the Gauss map T thereon. The *Gauss measure* is then the probability measure μ on X defined by

$$\mu(A) = \frac{1}{\log 2} \int_A \frac{\mathrm{d}x}{1+x}$$

for any Borel subset A of X. The relationship between the Gauss measure and the Gauss map is stated in the following lemma.

LEMMA 3.3 (Gauss, 1845). *The Gauss map preserves the Gauss measure.*

PROOF. The sets $[0,s] \cap X$, for $s \in (0,1)$, form a π-system that generates the Borel subsets of X. By the uniqueness of extension lemma, it suffices to show that the measure μ and its pushforward under the mapping T, namely, $\mu \circ T^{-1}$ agree on that π-system, see *e.g.* [**61**, Lemma 1.6(a)]. Hence, let us show that for any $s \in (0,1)$, the sets $T^{-1}([0,s] \cap X)$ and $[0,s] \cap X$ have the same measure. We have

$$T^{-1}([0,s] \cap X) = \{x \in X \mid 0 < T(x) \leq s\} = \bigsqcup_{n=1}^{\infty} \left(\left[\frac{1}{s+n}, \frac{1}{n}\right) \cap X \right),$$

and the union in the right-hand side is disjoint. Therefore, the countable additivity of the measure μ implies that

$$\begin{aligned}\mu(T^{-1}([0,s]\cap X)) &= \frac{1}{\log 2}\sum_{n=1}^{\infty}\int_{\frac{1}{s+n}}^{\frac{1}{n}}\frac{\mathrm{d}x}{1+x}\\ &= \frac{1}{\log 2}\sum_{n=1}^{\infty}\left(\log\left(1+\frac{1}{n}\right)-\log\left(1+\frac{1}{s+n}\right)\right)\\ &= \frac{1}{\log 2}\sum_{n=1}^{\infty}\left(\log\left(1+\frac{s}{n}\right)-\log\left(1+\frac{s}{n+1}\right)\right)\\ &= \frac{\log(1+s)}{\log 2} = \mu([0,s]\cap X),\end{aligned}$$

and the result follows. □

3.2.2. Ergodicity of the Gauss map. With the help of the results of Section 1.2.1.2, observe that the following diagram commutes:

$$\begin{tikzcd} \mathbb{N}^{\mathbb{N}} \arrow[r, "\sigma"] \arrow[d] & \mathbb{N}^{\mathbb{N}} \arrow[d] \\ X \arrow[r, "T"] & X. \end{tikzcd}$$

The Gauss map may thus be represented as the shift σ on the symbolic space $\mathbb{N}^{\mathbb{N}}$. Moreover, for any vector $a = (a_1, \ldots, a_n) \in \mathbb{N}^n$, let us consider the subset $I(a)$ of X defined by

$$I(a) = \left\{[b_1, b_2, \ldots] \;\middle|\; b_1 = a_1, \ldots, b_n = a_n\right\}. \tag{78}$$

If n is equal to zero, we adopt the convention that $\mathbb{N}^n$ is reduced to the singleton $\{\varnothing\}$ formed by the empty word, and that $I(\varnothing)$ is equal to the whole set X. Each set $I(a)$ can be seen as either a cylinder in the symbolic space $\mathbb{N}^{\mathbb{N}}$ or the intersection of the set X with an interval. To be more precise, we have the following characterization of the sets $I(a)$.

Lemma 3.4. *For any integer $n \geq 0$, any vector $a = (a_1, \ldots, a_n) \in \mathbb{N}^n$ and any irrational real $x \in X$,*

$$x \in I(a) \qquad \Longleftrightarrow \qquad x = \frac{p_n + p_{n-1}T^n(x)}{q_n + q_{n-1}T^n(x)},$$

where p_{n-1}/q_{n-1} and p_n/q_n are defined by (7) with $a_0 = 0$. Moreover, we adopt the same conventions as in the statement of Lemma 1.1 when $n = 0$.

Proof. Let $p_n(x)/q_n(x)$ denote the convergents of the continued fraction expansion of x. Using (19) and noting that the $(n+1)$-th tail of the continued fraction expansion of x coincides with $1/T^n(x)$, we have

$$x = \frac{p_n(x) + p_{n-1}(x)T^n(x)}{q_n(x) + q_{n-1}(x)T^n(x)}.$$

If the irrational number x belongs to the set $I(a)$, we therefore have

$$x = \frac{p_n + p_{n-1}T^n(x)}{q_n + q_{n-1}T^n(x)}.$$

Note that the right-hand side is a monotonic function of $T^n(x)$. Thus, if conversely the latter equality holds, then x is between the rationals

$$\frac{p_n}{q_n} = [a_1, \ldots, a_n] \qquad \text{and} \qquad \frac{p_n + p_{n-1}}{q_n + q_{n-1}} = [a_1, \ldots, a_{n-1}, a_n + 1].$$

Let us show by induction on n that this implies that $x \in I(a)$. First, note that the result is a tautology if $n = 0$. Besides, if these bounds on x hold, then $1/x - a_1$ is between $[a_2, \ldots, a_n]$ and $[a_2, \ldots, a_{n-1}, a_n + 1]$. This means that $a_1(x) = a_1$ and that $T(x) = 1/x - a_1$. Applying the induction hypothesis to $T(x)$, we deduce that $T(x) \in I(a_2, \ldots, a_n)$, so that $a_{k+1}(x) = a_k(T(x)) = a_{k+1}$ for all $k \in \{2, \ldots, n\}$. As a result, x belongs to $I(a)$. $\square$

The above lemma will be called upon in the proof of the main result of this section, namely, the ergodicity of the Gauss map.

THEOREM 3.2. *The Gauss map T is ergodic on X with respect to the Gauss measure μ, that is, for any Borel subset A of X,*

$$T^{-1}(A) = A \qquad \Longrightarrow \qquad \mu(A) \in \{0, 1\}.$$

PROOF. The main part of the proof consists in establishing that for any integer $n \geq 0$, any vector $a = (a_1, \ldots, a_n) \in \mathbb{N}^n$ and any Borel subset A of X,

$$\frac{1}{4}\mu(A)\mu(I(a)) \log 2 \leq \mu(T^{-n}(A) \cap I(a)) \leq 8\mu(A)\mu(I(a)) \log 2, \tag{79}$$

where $I(a)$ is the subset of X defined by (78). Note that the Borel sets A for which (79) holds clearly form a monotone class; the monotone class theorem then ensures that it suffices to prove (79) for $A = [\alpha, \beta] \cap X$ with $0 < \alpha < \beta < 1$, see for instance [**24**, Appendix A].

Applying Lemma 3.4 and observing that $y \mapsto (p_n + p_{n-1}y)/(q_n + q_{n-1}y)$ is a continuous and monotonic mapping on the interval $(0, 1)$, we infer that the set $T^{-n}([\alpha, \beta] \cap X) \cap I(a)$ is an interval with endpoints

$$\frac{p_n + p_{n-1}\alpha}{q_n + q_{n-1}\alpha} \qquad \text{and} \qquad \frac{p_n + p_{n-1}\beta}{q_n + q_{n-1}\beta}.$$

As a consequence, its Lebesgue measure satisfies

$$\begin{aligned} \mathcal{L}^1(T^{-n}([\alpha, \beta] \cap X) \cap I(a)) &= \left| \frac{p_n + p_{n-1}\alpha}{q_n + q_{n-1}\alpha} - \frac{p_n + p_{n-1}\beta}{q_n + q_{n-1}\beta} \right| \\ &= \frac{\beta - \alpha}{(q_n + q_{n-1}\alpha)(q_n + q_{n-1}\beta)}. \end{aligned}$$

Furthermore, the Lebesgue measure of the set $I(a)$ is obtained by choosing above α and β to be equal to zero and one, respectively. Also, note that the ratio between the Lebesgue measure of a subset of X and its Gauss measure is between $\log 2$ and $2 \log 2$. Therefore,

$$\frac{\log 2}{2} \leq \frac{\mu(T^{-n}([\alpha, \beta] \cap X) \cap I(a))}{\mu([\alpha, \beta] \cap X)\mu(I(a))} \cdot \frac{(q_n + q_{n-1}\alpha)(q_n + q_{n-1}\beta)}{q_n(q_n + q_{n-1})} \leq 4 \log 2.$$

However, given that $0 < \alpha < \beta < 1$ and $q_n \geq q_{n-1}$, it is easily seen that

$$\frac{1}{2} \leq \frac{(q_n + q_{n-1}\alpha)(q_n + q_{n-1}\beta)}{q_n(q_n + q_{n-1})} \leq 2.$$

We finally deduce that (79) holds for $A = [\alpha, \beta] \cap X$, and the monotone class argument ensures that (79) still holds for an arbitrary Borel subset A of X.

Let us now suppose that A is invariant under the action of the Gauss map, that is, $T^{-1}(A) = A$. Then, (79) reduces to

$$\frac{1}{4}\mu(A)\mu(I(a)) \log 2 \leq \mu(A \cap I(a)) \leq 8\mu(A)\mu(I(a)) \log 2, \tag{80}$$

for any vector $a = (a_1, \ldots, a_n)$ of positive integers. Note that the sets $I(a)$, for $a \in \mathbb{N}^n$, form a partition of the set X and their diameter satisfies

$$|I(a)| = \mathcal{L}^1(I(a)) = \frac{1}{q_n(q_n + q_{n-1})} \leq 2^{2-n},$$

because q_n is at least $2^{(n-2)/2}$; these sets thus generate the Borel σ-field on X. The monotone class theorem then ensures that (80) still holds when $I(a)$ is replaced by an arbitrary Borel subset B of X. In particular, choosing B to be the set $X \setminus A$, we readily deduce that either $\mu(A)$ or $\mu(X \setminus A)$ vanishes. The ergodicity of the Gauss map with respect to the Gauss measure follows. □

3.2.3. Almost sure results. The ergodicity of the Gauss map, combined with Birkhoff's pointwise ergodic theorem, enables one to deduce well known properties on the distribution of the digits arising in the continued fraction expansion of almost every irrational number. Let us begin by recalling the statement of the ergodic theorem; we refer for instance to [**24**, Chapter 2] for details and a proof.

THEOREM 3.3 (Birkhoff). *Let $(X, \mathcal{F}, \mu, T)$ be a measure-preserving dynamical system, and assume that T is ergodic. Then, for any function $f \in L^1(\mu)$,*

$$\frac{1}{n}\sum_{j=0}^{n-1} f(T^j(x)) \xrightarrow[n\to\infty]{} \int_X f \,\mathrm{d}\mu\,;$$

convergence holds μ-almost everywhere and in $L^1(\mu)$.

Let us begin by a result on the frequencies of the partial quotients of a typical irrational number.

PROPOSITION 3.1. *For Lebesgue-almost every $x = [a_1, a_2, \ldots]$ in X, a given digit $b \geq 1$ appears with a frequency satisfying*

$$\lim_{n\to\infty} \frac{1}{n}\#\{j \leq n \mid a_j = b\} = \frac{2\log(b+1) - \log b - \log(b+2)}{\log 2}.$$

PROOF. For every irrational number $x = [a_1, a_2, \ldots]$ in X, the digit b appears in the first n digits with frequency equal to

$$\frac{1}{n}\#\{j \leq n \mid a_j = b\} = \frac{1}{n}\sum_{j=0}^{n-1} \mathbf{1}_{\left[\frac{1}{b+1}, \frac{1}{b}\right]}(T^j(x)).$$

Owing to Theorem 3.3, this converges μ-almost everywhere to

$$\mu\left(\left[\frac{1}{b+1}, \frac{1}{b}\right]\right) = \frac{1}{\log 2}\int_{\frac{1}{b+1}}^{\frac{1}{b}} \frac{\mathrm{d}y}{1+y} = \frac{2\log(b+1) - \log b - \log(b+2)}{\log 2}.$$

The result follows from the fact that the Gauss measure and the Lebesgue measure are absolutely continuous with respect to one another. □

We now study the asymptotic behavior of the product of the partial quotients of a typical irrational number.

PROPOSITION 3.2. *For Lebesgue-almost every $x = [a_1, a_2, \ldots]$ in X, we have*

$$\lim_{n\to\infty} (a_1 a_2 \ldots a_n)^{1/n} = \prod_{b=1}^{\infty} \left(\frac{(b+1)^2}{b(b+2)}\right)^{\frac{\log b}{\log 2}}.$$

PROOF. We begin by observing that $\log a_j = f(T^{j-1}(x))$ for any integer $j \geq 1$, where f is the function defined on X by

$$f = \sum_{b=1}^{\infty} \mathbf{1}_{\left[\frac{1}{b+1}, \frac{1}{b}\right]} \log b. \tag{81}$$

One easily checks that f is in $L^1(\mu)$; as a matter of fact,

$$\int_X f \,\mathrm{d}\mu = \sum_{b=1}^{\infty} \mu\left(\left[\frac{1}{b+1}, \frac{1}{b}\right]\right) \log b = \sum_{b=1}^{\infty} \frac{\log b}{\log 2} \log\left(1 + \frac{1}{b(b+2)}\right) < \infty.$$

Theorem 3.3 then ensures that for μ-almost every $x \in X$,

$$\frac{1}{n}\sum_{j=1}^{n} \log a_j = \frac{1}{n}\sum_{j=0}^{n-1} f(T^j(x)) \xrightarrow[n\to\infty]{} \int_X f \,\mathrm{d}\mu.$$

The result follows from composing with the exponential function in the above limit, using the previous computation for the integral of f with respect to μ, and observing that the Gauss measure and the Lebesgue measure have the same null sets. $\square$

The limiting value arising in the statement of Proposition 3.2 is called *Khintchine's constant*, and is approximately equal to 2.685452001.

Let us now turn our attention to the asymptotic behavior of the sums of the typical partial quotients. In the proof, it is tempting to apply Theorem 3.3 to the exponential of the function f defined by (81). However, this function fails to be integrable, and the above approach has to be refined.

PROPOSITION 3.3. *For Lebesgue-almost every $x = [a_1, a_2, \ldots]$ in X, we have*

$$\lim_{n\to\infty} \frac{1}{n}(a_1 + a_2 + \ldots + a_n) = \infty.$$

PROOF. Let g denote the function $\exp \circ f$, where f denotes the function defined by (81). Note that

$$\frac{1}{n}\sum_{j=1}^{n} a_j = \frac{1}{n}\sum_{j=0}^{n-1} g(T^j(x))\,;$$

however, the function g is not integrable, so that we cannot apply Theorem 3.3 directly. We first need to truncate the function g, namely, to fix an integer $N \geq 1$ and to consider the function $g_N = \min\{g, N\}$. The function g_N clearly belongs to $L^1(\mu)$, so Theorem 3.3 implies that for μ-almost every $x \in X$,

$$\liminf_{n\to\infty} \frac{1}{n}\sum_{j=0}^{n-1} g(T^j(x)) \geq \lim_{n\to\infty} \frac{1}{n}\sum_{j=0}^{n-1} g_N(T^j(x)) = \int_X g_N \mathrm{d}\mu = \sum_{b=1}^{N} \frac{b}{\log 2} \log \frac{(b+1)^2}{b(b+2)}.$$

The result follows from the fact that the right-hand side tends to infinity as $N \to \infty$, and again that the Gauss and Lebesgue measures share the same null sets. $\square$

We now study the typical behavior of the denominators of the convergents. This is somewhat more difficult than the previous results that were straightforward applications of Birkhoff's ergodic theorem.

PROPOSITION 3.4. *For Lebesgue-almost every x in X, the denominator of the convergents satisfy*

$$\lim_{n\to\infty} \frac{1}{n} \log q_n(x) = \frac{\pi^2}{12 \log 2}.$$

PROOF. First, note that the convergents $p_n(x)/q_n(x)$ of the continued fraction expansion $[a_1, a_2, \ldots]$ of x satisfy

$$\frac{p_n(x)}{q_n(x)} = \frac{1}{a_1 + [a_2, \ldots, a_n]} = \frac{1}{a_1 + \dfrac{p_{n-1}(T(x))}{q_{n-1}(T(x))}} = \frac{q_{n-1}(T(x))}{p_{n-1}(T(x)) + a_1 q_{n-1}(T(x))};$$

since the numerator and the denominator of the convergents are coprime, the left-hand side and the right-hand side are in their irreducible form, so that in particular $p_n(x) = q_{n-1}(T(x))$. As a consequence, applying this with $n-j$ instead of n and $T^j(x)$ instead of x, we have

$$\begin{aligned}\sum_{j=0}^{n-1} \log \frac{p_{n-j}(T^j(x))}{q_{n-j}(T^j(x))} &= \sum_{j=0}^{n-1} \log q_{n-(j+1)}(T^{j+1}(x)) - \log q_{n-j}(T^j(x)) \\ &= \log q_0(T^n(x)) - \log q_n(x) = -\log q_n(x).\end{aligned}$$

Thus, we may write $-\log q_n(x) = S_n(x) - R_n(x)$, where

$$S_n(x) = \sum_{j=0}^{n-1} \log T^j(x) \quad \text{and} \quad R_n(x) = \sum_{j=0}^{n-1} \left(\log T^j(x) - \log \frac{p_{n-j}(T^j(x))}{q_{n-j}(T^j(x))} \right).$$

Since the logarithm is integrable with respect to the Gauss measure, Theorem 3.3 ensures that for μ-almost every x in X,

$$\frac{S_n(x)}{n} \xrightarrow[n\to\infty]{} \int_X \log x \, \mu(\mathrm{d}x) = \frac{1}{\log 2} \int_0^1 \frac{\log x}{1+x} \, \mathrm{d}x = -\frac{\pi^2}{12 \log 2}.$$

As the Gauss and Lebesgue measures share the same null sets, the above convergence result also holds Lebesgue-almost everywhere. For completeness, let us recall that the above integral may be computed as follows:

$$-\int_0^1 \frac{\log x}{1+x} \, \mathrm{d}x = \int_0^1 \frac{\log(1+x)}{x} \, \mathrm{d}x = \int_0^1 \sum_{n=0}^{\infty} \frac{(-1)^n}{n+1} x^n \, \mathrm{d}x = \sum_{n=1}^{\infty} \frac{(-1)^{n-1}}{n^2} = \frac{\pi^2}{12}.$$

To conclude, we shall show that $(R_n(x))_{n\geq 1}$ is a bounded sequence for every $x \in X$. To this purpose, observe that the convergents satisfy

$$\left| \frac{x}{p_k(x)/q_k(x)} - 1 \right| = \frac{q_k(x)}{p_k(x)} \left| x - \frac{p_k(x)}{q_k(x)} \right| \leq \frac{1}{p_k(x) q_{k+1}(x)}.$$

Recall that the numerator and the denominator of the n-th convergent are both at least $2^{(n-2)/2}$ for all $n \geq 1$. Thus, $p_k(x) q_{k+1}(x) \geq 2^{k-3/2}$ for all $k \geq 1$. However, this bound can easily be improved when k is equal to one or two: specifically, $p_1(x) q_2(x) \geq 2$ and $p_2(x) q_3(x) \geq 3$. As a consequence, the right-hand side above cannot be larger than $1/2$. Given that the positive function $u \mapsto \log u/(u-1)$ is bounded above by $2 \log 2$ on the interval $[2, \infty)$, we deduce that

$$\left| \log x - \log \frac{p_k(x)}{q_k(x)} \right| \leq 2 \log 2 \left| \frac{x}{p_k(x)/q_k(x)} - 1 \right| \leq 2^{5/2-k} \log 2$$

for all $x \in X$ and all $k \geq 1$. This readily implies that for every x in X,

$$|R_n(x)| \leq \sum_{j=0}^{n-1} \left| \log T^j(x) - \log \frac{p_{n-j}(T^j(x))}{q_{n-j}(T^j(x))} \right| \leq \sum_{j=0}^{n-1} 2^{5/2-(n-j)} \log 2 \leq 2^{5/2} \log 2,$$

so that $(R_n(x))_{n\geq 1}$ is a bounded sequence, as announced previously. □

The exponential of the limiting value obtained in Proposition 3.4 is called *Lévy's constant*, and is approximately equal to 3.2758229187. It is therefore the almost sure limit of $q_n(x)^{1/n}$ as n goes to infinity.

The last result gives the asymptotic behavior of the error made when replacing a typical irrational number by the convergents of its continued fraction expansion.

COROLLARY 3.1. *For Lebesgue-almost every x in X, the convergents satisfy*

$$\lim_{n\to\infty} \frac{1}{n} \log \left| x - \frac{p_n(x)}{q_n(x)} \right| = -\frac{\pi^2}{6\log 2}.$$

PROOF. This directly follows from Proposition 3.4, along with the fact that

$$\log q_n + \log q_{n+1} < -\log \left| x - \frac{p_n(x)}{q_n(x)} \right| < \log q_n + \log q_{n+2},$$

as a consequence of (25). □

3.3. Prescribed continued fraction expansions

3.3.1. An emblematic example. The theory of iterated function systems introduced in Section 2.10 allows us to study the Hausdorff dimension of certain sets of positive real numbers that are defined through conditions on the continued fraction expansions. Rather than developing a systematic theory, we content ourselves with discussing the following emblematic example.

Given an integer $m \geq 2$ and using the notation (17) for the continued fraction expansion of a positive irrational real number, we may consider the set

$$K_m = \left\{ x \in [0,\infty) \setminus \mathbb{Q} \;\middle|\; a_n(x) \in \{1,\ldots,m\} \text{ for all } n \geq 0 \right\}.$$

Equivalently, the set K_m is formed by the positive irrational real numbers with all partial quotients between one and m. The following result makes the connection with the iterated function systems, which enables us to give a nontrivial lower bound on the Hausdorff dimension of the set K_m.

PROPOSITION 3.5. *The set K_m is the attractor of the iterated function system $\{f_1,\ldots,f_m\}$ formed by the contractions defined by $f_a(x) = a + 1/x$, for x in the closed interval $F_m = [\alpha_m, m\alpha_m]$, where*

$$\alpha_m = \frac{1}{2} + \sqrt{\frac{1}{4} + \frac{1}{m}}.$$

Moreover, the Hausdorff dimension of the attractor K_m satisfies

$$\frac{\log m}{2\log(m\alpha_m)} \leq \dim_{\mathrm{H}} K_m \leq 1.$$

PROOF. For every number $x \in K_m$, the partial quotient $a_0(x)$ coincides with the integer part $\lfloor x \rfloor$ and is between one and m. This means that the set K_m is contained in the interval $[1, m+1]$. We may actually be slightly more precise by observing that the continued fraction $[a_0; a_1, a_2, a_3, \ldots]$ defined by (5) is a nondecreasing function of the partial quotients a_{2n} and a nonincreasing function of the partial quotients a_{2n+1}. Thus, the infimum and the supremum of the set K_m are respectively attained by the continued fractions

$$[1; m, 1, m, \ldots] = \alpha_m \qquad \text{and} \qquad [m; 1, m, 1, \ldots] = m\alpha_m.$$

As a consequence, the set K_m is included in the closed interval $F_m = [\alpha_m, m\alpha_m]$, which is clearly a proper subinterval of $(1, m+1)$.

Moreover, it is clear that the mappings f_a are differentiable on F_m and share the same derivative at every point x, namely, $f'_a(x) = -1/x^2$. Consequently, we have

$$\forall x \in F_m \qquad \frac{1}{m^2\alpha_m^2} \leq |f'_a(x)| \leq \frac{1}{\alpha_m^2}.$$

The mean value theorem then ensures that the mappings f_a fall into the setting of Propositions 2.20 and 2.21 with

$$b_a = \frac{1}{m^2 \alpha_m^2} \qquad \text{and} \qquad c_a = \frac{1}{\alpha_m^2}. \tag{82}$$

In particular, these mappings are contractive. Moreover, one easily checks that the interval F_m contains the disjoint union of its images under the mappings f_a. As shown by Proposition 2.19, there is a unique attractor to the iterated function system $\{f_1, \ldots, f_m\}$. Recall that the attractor is a compact subset of F_m that coincides with the union of its images under the mappings f_a ; in view of the previous remark, the union must be disjoint, and the attractor thus satisfies (74). This means that we may apply Propositions 2.20 and 2.21 in order to derive upper and lower bounds on the Hausdorff dimension of the attractor.

The point is that the attractor of the iterated function system $\{f_1, \ldots, f_m\}$ is precisely the set K_m defined above, as we now explain. In view of Proposition 2.19, the attractor is the intersection over all integers $j \geq 0$ of the sets $f^j(F_m)$, where f is the mapping that sends a nonempty compact subset of F_m to the union of its images under the contractions f_a. Moreover, for every integer $j \geq 1$ and every point $x \in f^j(F_m)$, there exists a point $x' \in F_m$ and a j-tuple $(a_0, \ldots, a_{j-1})$ of integers between one and m such that

$$x = f_{a_0} \circ \ldots \circ f_{a_{j-1}}(x') = [a_0; a_1, a_2, \ldots, a_{j-1}, x'],$$

using a notation that naturally extends (6) to the case where the last partial quotient is replaced by a real number larger than one. We may now follow the lines of the proof of Proposition 1.2 to deduce that $a_n(x) = a_n \in \{1, \ldots, m\}$ for all n between zero and $j-1$. Hence, every point in the attractor belongs to the set K_m. Conversely, if an irrational number x belongs to K_m, then its partial quotients a_n are all between one and m, so that for any integer $j \geq 0$,

$$x = [a_0; a_1, a_2, \ldots] = f_{a_0} \circ \ldots \circ f_{a_{j-1}}([a_j; a_{j+1}, a_{j+2}, \ldots]),$$

from which we deduce that x belongs to $f^j(F_m)$, and thus to the attractor of the iterated function system formed by the contractions f_a.

Now, applying Propositions 2.20 and 2.21, we infer that the Hausdorff dimension of the attractor K_m is bounded by the positive real numbers β_m and γ_m that satisfy the equations $b_1^{\beta_m} + \ldots + b_m^{\beta_m} = 1$ and $c_1^{\gamma_m} + \ldots + c_m^{\gamma_m} = 1$, respectively, where the coefficients b_a and c_a are given by (82). Straightforward computations then yield

$$\beta_m = \frac{\log m}{2 \log(m\alpha_m)} \qquad \text{and} \qquad \gamma_m = \frac{\log m}{2 \log \alpha_m}.$$

The lower bound given by β_m may not be accurate, but is at least nontrivial. Unfortunately, the upper bound supplied by γ_m is useless: as easily seen, γ_m is larger than one for any integer $m \geq 2$. We can therefore just conclude with the bounds given in the statement of the proposition. $\square$

The bounds on the Hausdorff dimension of K_m supplied by Proposition 3.5 are not very accurate, but there is a simple trick to improve them: it suffices to remark that K_m is also the attractor of the iterated function system formed by the m^2 contractions $f_a \circ f_{a'}$, for a and a' between one and m. Using the mean value theorem again, it is possible to prove that these contractions fall into the setting of Propositions 2.20 and 2.21 with (82) replaced by the appropriate values of b_a and c_a. It is even possible to use higher order iterates of the contractions f_a so as to refine the bounds on the Hausdorff dimension of the attractor K_m, see [**29**, Example 9.8] for details. This way, it is possible to show that the Hausdorff dimension of K_2 is approximately equal to 0.531280506, see [**29**, Example 10.2]. Finally, we also refer to [**31**, Section 9.1] for possible generalizations of the above problem.

3.3.2. Link with badly approximable numbers. Using Proposition 3.5, one can easily obtain a lower bound on the Hausdorff dimension of the set Bad_1 of badly approximable numbers introduced in Section 1.3. Indeed, recall from Proposition 1.10 that a positive irrational real number is badly approximable if and only if the sequence of its partial quotients is bounded. This means that

$$\bigcup_{m=1}^{\infty} \uparrow K_m \subseteq \mathrm{Bad}_1.$$

The lower bound on the dimension of K_m that is supplied by Proposition 3.5 clearly tends to one half as m goes to infinity. This directly leads to the following result.

Corollary 3.2. *The Hausdorff dimension of the set of badly approximable numbers satisfies*

$$\frac{1}{2} \leq \dim_{\mathrm{H}} \mathrm{Bad}_1 \leq 1.$$

We shall dramatically improve this result in Section 12.2 and show that the Hausdorff dimension of the set Bad_1 of badly approximable numbers is actually equal to one, see Corollary 12.1 for a precise statement. Let us recall in passing that, as shown by Proposition 1.9 and Corollary 1.2, the set Bad_1 has cardinality equal to that of $\mathbb{R}$ but has Lebesgue measure zero.

3.4. Frequencies of digits

Let us consider an integer $m \geq 2$ and a real number $x \in [0,1)$. It is well known that if x is not a m-adic number, *i.e.* a rational number with denominator of the form m^j for some integer $j \geq 0$, then x may be written in a unique manner as

$$x = \sum_{j=1}^{\infty} x_j m^{-j}, \tag{83}$$

where $(x_j)_{j\geq 1}$ is a sequence of digits between zero and $m-1$. The m-adic numbers have two representations: one that we choose to privilege, where the digits eventually vanish, and another one where they are eventually equal to $m-1$.

The frequency with which a given digit b appears among the first j digits of x is then given by

$$f_j(b,x) = \frac{1}{j}\,\#\{i \in \{1,\ldots,j\} \mid x_i = b\}.$$

A classical result due to Borel asserts that Lebesgue-almost every real number is normal to the base m, that is, the asymptotic frequencies of the digits are all the same. More rigorously, this means that the set

$$F_p = \left\{ x \in [0,1) \;\middle|\; \lim_{j\to\infty} f_j(b,x) = p_b \text{ for all } b \in \{0,\ldots,m-1\} \right\}$$

has full Lebesgue measure in the interval $[0,1)$ when the components of the vector $p = (p_0,\ldots,p_{m-1})$ are all equal to $1/m$. This is a plain consequence of Borel's strong law of large numbers, but we will also recover this result from the analysis below. Moreover, it follows that Lebesgue-almost every real number is normal to all bases, *i.e.* is normal to the base m for all $m \geq 2$.

We shall determine the size of the set F_p in terms of Hausdorff dimension for every choice of the probability vector p. Recall that a probability vector is one for which all the components are between zero and one, and have a sum equal to one. Moreover, the Shannon entropy (based on natural logarithms) is defined by

$$\mathrm{H}(p) = -\sum_{b=0}^{m-1} p_b \log p_b, \tag{84}$$

with the convention that $0 \log 0$ vanishes. The next result shows that the Hausdorff dimension of the set defined above is a simple function of the Shannon entropy.

PROPOSITION 3.6. *For every integer $m \geq 2$ and every probability vector p with m components,*

$$\dim_{\mathrm{H}} F_p = \frac{\mathrm{H}(p)}{\log m}.$$

The rest of this section is devoted to the proof of Proposition 3.6. Though a standard and natural approach relies on probabilistic methods, see *e.g.* [**29**, Proposition 10.1], we provide here a proof that is based solely on analytic and measure theoretic tools, thus being more consistent with the viewpoint of these notes.

We begin by letting B_p denote the set of all digits b in $\{0, \ldots, m-1\}$ such that $p_b > 0$. We suppose that the set B_p is not reduced to a singleton. The opposite case is elementary and will be discussed briefly at the very end of the proof.

Now, on the one hand, let us consider the subintervals of $[0, 1)$ that may be written in the form

$$I_u = u_1 m^{-1} + \ldots + u_j m^{-j} + [0, m^{-j}),$$

where $u = u_1 \ldots u_j$ is a word of finite length over the alphabet $\{0, \ldots, m-1\}$. We endow the collection of all m-adic intervals, along with the empty set, with the premeasure ζ_p defined by

$$\zeta_p(I_u) = p_{u_1} p_{u_2} \ldots p_{u_j}.$$

In particular, recalling that $\varnothing$ denotes the empty word, $I_\varnothing$ is the whole interval $[0, 1)$ and its ζ_p-mass is equal to one. Note that $\zeta_p(I_u)$ clearly vanishes as soon as the word u has at least a letter that does not belong to the set B_p. With the help of Theorem 2.2, we may extend the premeasure ζ_p to an outer measure ζ_p^* on all the subsets of $\mathbb{R}$ through the formula (51). We may then consider the outer measure μ_p that maps a subset E of $\mathbb{R}$ to the value $\zeta_p^*(E \cap [0, 1))$.

On the other hand, for any $b \in B_p$, let us consider the mapping $\chi_{p,b}$ defined on the interval $[0, 1)$ by

$$\chi_{p,b}(t) = p_0 + \ldots + p_{b-1} + p_b t.$$

It is clear that the ranges of the mappings $\chi_{p,b}$ form a partition of the whole interval $[0, 1)$ by consecutive subintervals. Thus, any point ξ in $[0, 1)$ belongs to a unique interval of the form $\chi_{p,\xi_{p,1}}([0, 1))$, where $\xi_{p,1}$ is an integer in B_p. Iterating this procedure, we end up with a sequence $(\xi_{p,j})_{j\geq 1}$ of integers in B_p such that

$$\xi \in \chi_{p,\xi_{p,1}} \circ \ldots \circ \chi_{p,\xi_{p,j}}([0, 1)) \tag{85}$$

for all $j \geq 1$, and this sequence is unique. It will be useful to remark that the mapping $\xi \mapsto (\xi_{p,j})_{j\geq 1}$ is nondecreasing when the sequence space is endowed with the lexicographic order. Moreover, note that the intervals that appear in (85) have length $p_{\xi_{p,1}} \ldots p_{\xi_{p,j}}$. Given that the set B_p is not reduced to a singleton, all the reals p_b are less than one, so the previous length tends to zero as j goes to infinity. Thus, for any given sequence $(\xi_{p,j})_{j\geq 1}$, there is at most one possible value of ξ satisfying (85). In other words, the mapping $\xi \mapsto (\xi_{p,j})_{j\geq 1}$ is injective. Finally, we may define in terms of the sequence $(\xi_{p,j})_{j\geq 1}$ the real number

$$h_p(\xi) = \sum_{j=1}^{\infty} \xi_{p,j} m^{-j}. \tag{86}$$

We thus obtain a mapping h_p from $[0,1)$ to $[0,1]$. The next lemma gives a connection between the outer measure μ_p, the mapping h_p and the Lebesgue measure $\mathcal{L}^1$.

LEMMA 3.5. *For any m-adic interval I_u,*

$$\mu_p(I_u) = \zeta_p(I_u) = \mathcal{L}^1(h_p^{-1}(I_u)).$$

PROOF. Let us consider a real $\xi \in [0,1)$ such that $h_p(\xi)$ is an m-adic number. The integers $\xi_{p,j}$ are eventually equal to zero or eventually equal to $m-1$. There is therefore only a countable number of possible values for the sequence $(\xi_{p,j})_{j\geq 1}$, and any such sequence corresponds to at most one value of ξ, because the mapping $\xi \mapsto (\xi_{p,j})_{j\geq 1}$ is injective. We deduce that there are at most countably many reals ξ in $[0,1)$ such that $h_p(\xi)$ is an m-adic number.

When computing the Lebesgue measure of the set of all reals $\xi \in [0,1)$ such that $h_p(\xi) \in I_u$, we may therefore assume that $h_p(\xi)$ is not an m-adic number. This means that (86) is the base m expansion of $h_p(\xi)$. As a result, in view of (85),

$$h_p(\xi) \in I_u \quad\Longleftrightarrow\quad u = \xi_{p,1} \ldots \xi_{p,j} \quad\Longleftrightarrow\quad \xi \in \chi_{p,u_1} \circ \ldots \circ \chi_{p,u_j}([0,1)).$$

This readily implies that

$$\mathcal{L}^1(h_p^{-1}(I_u)) = \mathcal{L}^1(\chi_{p,u_1} \circ \ldots \circ \chi_{p,u_j}([0,1))) = p_{u_1} p_{u_2} \ldots p_{u_j} = \zeta_p(I_u).$$

This value is obviously an upper bound on $\mu_p(I_u)$. To show that equality holds, let us consider a sequence $(C_n)_{n\geq 1}$ of m-adic intervals such that $I_u \subseteq \bigcup_n C_n$. Applying what precedes to these intervals, we have

$$\sum_{n=1}^{\infty} \zeta_p(C_n) = \sum_{n=1}^{\infty} \mathcal{L}^1(h_p^{-1}(C_n)) \geq \mathcal{L}^1\left(h_p^{-1}\left(\bigcup_{n=1}^{\infty} C_n\right)\right) \geq \mathcal{L}^1(h_p^{-1}(I_u)).$$

Taking the infimum over all sequences $(C_n)_{n\geq 1}$ in the left-hand side, we deduce that $\mu_p(I_u)$ is at least $\mathcal{L}^1(h_p^{-1}(I_u))$, and the result follows. □

The next crucial lemma indicates that the range of the mapping h_p essentially charges the set F_p under study.

LEMMA 3.6. *The set $h_p^{-1}(F_p)$ has full Lebesgue measure in $[0,1)$.*

PROOF. For any probability vector $q = (q_0, \ldots, q_{m-1})$, let us now consider the mapping $g_{p,q}$ defined on the interval $[0,1)$ by

$$g_{p,q}(\xi) = \lim_{j\to\infty} \uparrow \chi_{q,\xi_{p,1}} \circ \ldots \chi_{q,\xi_{p,j}}(0),$$

where $(\xi_{p,j})_{j\geq 1}$ is the sequence that is defined above in terms of the real number ξ. Note that the limit always exists because the involved sequence is nondecreasing and bounded; this is due to the obvious fact that every mapping $\chi_{q,b}$ is increasing. Furthermore, note that the mapping $g_{p,q}$ is nondecreasing. It is therefore differentiable at Lebesgue almost every point of $[0,1)$, see *e.g.* [**32**, p. 358]. As a result, there exists a subset $\Xi_{p,q}$ of $[0,1)$ with full Lebesgue measure on which the mapping $g_{p,q}$ is differentiable. Let us consider a point ξ in $\Xi_{p,q}$. Then, the derivative of $g_{p,q}$ at ξ exists and is equal to the limiting rate of change of $g_{p,q}$ on any sequence of intervals that shrink to ξ, see [**32**, p. 345]. Now, for any integer $j \geq 1$, the point ξ is between $\chi_{p,\xi_{p,1}} \circ \ldots \chi_{p,\xi_{p,j}}(0)$ and $\chi_{p,\xi_{p,1}} \circ \ldots \chi_{p,\xi_{p,j}}(1)$, and the value of the function

$g_{p,q}$ at these two points is equal to $\chi_{q,\xi_{p,1}} \circ \dots \chi_{q,\xi_{p,j}}(0)$ and $\chi_{q,\xi_{p,1}} \circ \dots \chi_{q,\xi_{p,j}}(1)$, respectively. The corresponding rate of change is therefore equal to

$$\frac{|\chi_{q,\xi_{p,1}} \circ \dots \chi_{q,\xi_{p,j}}([0,1))|}{|\chi_{p,\xi_{p,1}} \circ \dots \chi_{p,\xi_{p,j}}([0,1))|} = \frac{q_{\xi_{p,1}} \cdots q_{\xi_{p,j}}}{p_{\xi_{p,1}} \cdots p_{\xi_{p,j}}} = \prod_{b \in B_p} \left(\frac{q_b}{p_b}\right)^{j\mathfrak{f}_{p,j}(b,\xi)},$$

and tends to $g'_{p,q}(\xi)$ as j goes to infinity. Here, $\mathfrak{f}_{p,j}(b,\xi)$ is the frequency with which b appears among the first j terms of the sequence $(\xi_{p,j})_{j\geq 1}$, that is,

$$\mathfrak{f}_{p,j}(b,\xi) = \frac{1}{j} \#\{i \in \{1,\dots,j\} \mid \xi_{p,j} = b\}.$$

Finally, taking logarithms and dividing by j, we deduce that

$$\forall \xi \in \Xi_{p,q} \qquad \limsup_{j\to\infty} \sum_{b \in B_p} \mathfrak{f}_{p,j}(b,\xi) \log \frac{q_b}{p_b} \leq 0.$$

We now fix an integer $b_0 \in B_p$ and a positive real λ. Recall that all the reals p_b are less than one, so up to choosing λ close enough to one, we obtain a probability vector q by letting $q_{b_0} = 1 - \lambda(1 - p_{b_0})$, along with $q_b = \lambda p_b$ if $b \neq b_0$. Using the notation $\Xi_p^{b_0,\lambda}$ for the set $\Xi_{p,q}$, we then have

$$\forall \xi \in \Xi_p^{b_0,\lambda} \qquad \limsup_{j\to\infty} \mathfrak{f}_{p,j}(b_0,\xi) \log\left(1 + \frac{1-\lambda}{\lambda p_{b_0}}\right) \leq -\log \lambda.$$

Remark that the logarithm in the left-hand side is positive when λ is less than one, and is negative when λ is larger than one. Moreover, the ratio of the two logarithms tends to p_{b_0} when λ tends to one. Considering two sequences $(\underline{\lambda}_k)_{k\geq 1}$ and $(\overline{\lambda}_k)_{k\geq 1}$ that increase and decrease to one, respectively, and letting $\Xi_p^{b_0}$ denote the intersection of all the corresponding sets $\Xi_p^{b_0,\underline{\lambda}_k}$ and $\Xi_p^{b_0,\overline{\lambda}_k}$, we deduce that

$$\forall \xi \in \Xi_p^{b_0} \qquad \lim_{j\to\infty} \mathfrak{f}_{p,j}(b_0,\xi) = p_{b_0}.$$

To conclude, let Ξ_p denote the intersection over $b_0 \in B_p$ of the sets $\Xi_p^{b_0}$; this set has full Lebesgue measure in $[0,1)$. Given $\xi \in \Xi_p$, the reals $\xi_{p,j}$ cannot be eventually equal to zero or eventually equal to $m-1$; indeed, otherwise, the set B_p would be reduced to the singleton $\{0\}$ or the singleton $\{m-1\}$. Thus, (86) is the base m expansion of $h_p(\xi)$, so that in particular $f_j(b, h_p(\xi)) = \mathfrak{f}_{p,j}(b,\xi)$ for all $j \geq 1$ and all $b \in \{0,\dots,m-1\}$. Consequently, $h_p(\xi)$ belongs to F_p, and we finally have $\Xi_p \subseteq h_p^{-1}(F_p)$. $\square$

For any real $x \in [0,1)$ and any integer $j \geq 0$, let $I_j(x)$ denote the unique m-adic interval with length m^{-j} that contains x. The next result gives an estimate of the scaling behavior of the outer measure μ_p on the set F_p.

LEMMA 3.7. *For any real $x \in F_p$,*

$$\lim_{j\to\infty} \frac{\log \mu_p(I_j(x))}{\log |I_j(x)|} = \frac{\mathrm{H}(p)}{\log m}.$$

PROOF. As B_p is not reduced to a singleton, x is surely not an m-adic number. Hence, the interval $I_j(x)$ is clearly equal to $I_{x_1 \dots x_j}$, where $(x_j)_{j\geq 1}$ is the sequence of m-ary digits of x that is defined by (83). Lemma 3.5 now gives

$$\frac{\log \mu_p(I_j(x))}{\log |I_j(x)|} = -\frac{1}{j \log m} \sum_{i=1}^{j} \log p_{x_i} = -\frac{1}{\log m} \sum_{b \in B_p} f_j(b,x) \log p_b,$$

and the result readily follows. $\square$

We may now finish the proof with the help of Proposition 2.22 and a variant thereof. To be specific, let us consider a point $x \in [0,1)$, an integer $j \geq 1$ and a positive real s. The open interval centered at x with radius m^{-j} contains the m-adic interval $I_j(x)$, so that

$$\frac{\mu_p((x-m^{-j},x+m^{-j}))}{m^{-sj}} \geq \frac{\mu_p(I_j(x))}{|I_j(x)|^s}.$$

By virtue of Lemma 3.7, this ratio tends to infinity as j goes to infinity when s is larger than $\mathrm{H}(p)/\log m$ and x belongs to F_p. We infer that the upper s-density of the outer measure μ_p at any point $x \in F_p$ satisfies $\overline{\Theta}^s(\mu,x) = \infty$. In view of Proposition 2.22(2), we get $\mathcal{H}^s(F_p) \leq 10^s \mu_p(\mathbb{R})/c$ for all $c > 0$. We finally deduce that the Hausdorff dimension of F_p is bounded above by $\mathrm{H}(p)/\log m$.

For the lower bound on the dimension, we use Lemma 3.7 again to show that

$$\lim_{j\to\infty} \frac{\mu_p(I_j(x))}{|I_j(x)|^s} = 0$$

when s is less than $\mathrm{H}(p)/\log m$ and x belongs to F_p. Moreover, recall that Proposition 2.11 may easily be extended to coverings by m-adic cubes, specifically, the Hausdorff s-dimensional measures are comparable with those obtained by means of such coverings. Thus, using a variant of Proposition 2.22(1) where coverings by arbitrary sets are replaced by coverings by m-adic intervals, we may show that $\mathcal{H}^s(F_p) \geq \mu_p(F_p)/c$ for all $c > 0$. Meanwhile, it follows from Lemmas 3.5 and 3.6 that $\mu_p(F_p) \geq \mathcal{L}^1(h_p^{-1}(F_p)) = 1$. We deduce that the set F_p has Hausdorff dimension at least $\mathrm{H}(p)/\log m$.

It remains to deal with the degenerate situation where the set B_p is reduced to a singleton $\{b\}$, where b is an integer between zero and $m-1$. In that case, we assume that the ζ_p-mass of every m-adic interval is equal to one. It is then clear that the outer measure μ_p verifies the same property, and that Lemma 3.7 still holds. Proceeding as above, we deduce that the Hausdorff dimension of F_p is at most zero. Equality obviously holds because the set F_p is nonempty; indeed, it contains for instance the real number $\sum_{j=2}^{\infty} bm^{-j} = b/(m(m-1))$.

CHAPTER 4

Homogeneous ubiquity and dimensional results

The purpose of this chapter is to present an abstract setting into which the Jarník-Besicovitch theorem, that is, Theorem 3.1 fits naturally. The first step is to identify an appropriate notion of approximation system to generalize the combination of the approximating points p/q with the approximating radii $1/q^2$, or more generally $1/q^\tau$, that come into play in the homegeneous approximation problem. The second step is to introduce natural generalizations of the sets $J_{d,\tau}$ defined by (1. The third step is finally to provide optimal upper and lower bounds on the Hausdorff dimension of these generalized sets. As explained hereunder, through the remarkable notion of ubiquity, an *a priori* lower bound on the Hausdorff dimension can be derived from the sole knowledge that one of the sets has full Lebesgue measure. Thanks to ubiquity, the difficult lower bound in the Jarník-Besicovitch theorem will in fact quite amazingly be a straightforward consequence of a simple result, namely, Dirichlet's theorem.

Let us mention here that we do not need to specify the norm $|\cdot|$ the space $\mathbb{R}^d$ is endowed with. In fact, Proposition 4.4 below implies that the notions considered in this chapter do not depend on the chosen norm; let us recall in passing that this is also the case of Hausdorff dimension.

Definition 4.1. Let $\mathcal{I}$ be a countably infinite index set. We say that a family $(x_i, r_i)_{i\in\mathcal{I}}$ of elements of $\mathbb{R}^d \times (0,\infty)$ is an *approximation system* if

$$\sup_{i\in\mathcal{I}} r_i < \infty \qquad \text{and} \qquad \forall m \in \mathbb{N} \quad \#\left\{ i\in\mathcal{I} \;\middle|\; |x_i| < m \text{ and } r_i > \frac{1}{m} \right\} < \infty.$$

The emblematic example of approximation system to have in mind, and which indeed makes the connection with the Jarník-Besicovitch theorem, consists of the family formed by the pairs $(p/q, 1/q^2)$, for $p \in \mathbb{Z}^d$ and $q \in \mathbb{N}$. We shall discuss many other examples in Chapters 6 and 7. Replacing the system supplied by the rational points by an arbitrary approximation system $(x_i, r_i)_{i\in\mathcal{I}}$, the set $J_{d,\tau}$ defined by (1) may thus be generalized into

$$F_t = \left\{ x \in \mathbb{R}^d \;\middle|\; |x - x_i| < r_i^t \quad \text{for i.m. } i \in \mathcal{I} \right\}, \tag{87}$$

where $t \geq 1$. Moreover, extending the Jarník-Besicovitch theorem will then correspond to determining the Hausdorff dimension of the set F_t under appropriate assumptions on $(x_i, r_i)_{i\in\mathcal{I}}$.

Note that if x belongs to the set F_t, then there exists an injective sequence $(i_n)_{n\geq 1}$ of indices in $\mathcal{I}$ such that $|x - x_{i_n}| < r_{i_n}^t$ for all integers $n \geq 1$. Let us assume in addition that the family $(x_i, r_i)_{i\in\mathcal{I}}$ is an approximation system. Then, for any real number $\varepsilon > 0$ and any integer $n \geq 1$ such that $r_{i_n} > \varepsilon$, we have

$$|x_{i_n}| \leq |x| + |x - x_{i_n}| < |x| + \sup_{i\in\mathcal{I}} r_i^t.$$

Thus, letting m denote an integer larger than both $1/\varepsilon$ and the right-hand side above, we deduce that $|x_{i_n}| < m$ and $r_{i_n} > 1/m$, which means that there are only finitely many possible values of the integer n when ε is given. We readily deduce that, as $n \to \infty$, r_{i_n} tends to zero and x_{i_n} tends to x. The point x is thus approximated by the sequence $(x_{i_n})_{n\geq 1}$ at a rate given by the sequence $(r_{i_n}^t)_{n\geq 1}$; this justifies the terminology of the previous definition. Moreover, it is obvious and useful to remark that, up to extracting, we may suppose that the latter sequence is decreasing without losing the approximation property.

Our purpose is now to give an upper and a lower bound on the Hausdorff dimension of the set F_t defined by (87) when $(x_i, r_i)_{i\in\mathcal{I}}$ is a given approximation system. We shall subsequently extend the upcoming results in the direction of large intersection properties and Hausdorff measures associated with general gauge functions, see Chapters 5, 8 and 9.

4.1. Upper bound on the Hausdorff dimension

As suggested by the preceding discussion, the set F_t defined by (87) may essentially be seen as a limsup set, thereby falling in the setting deal with in Section 2.8. More precisely, for any bounded open subset U of $\mathbb{R}^d$, let

$$\mathcal{I}_U = \{i \in \mathcal{I} \mid x_i \in U\}. \tag{88}$$

If a given point x belongs to $F_t \cap U$, the above remark ensures that there exists a sequence $(i_n(x))_{n\geq 1}$ of indices in $\mathcal{I}$ such that $x_{i_n(x)}$ tends to x as $n \to \infty$. As the set U is open, the indices $i_n(x)$ thus belong to $\mathcal{I}_U$ for n sufficiently large. On top of that, for any real number $\varepsilon > 0$, we have

$$\#\{i \in \mathcal{I}_U \mid r_i > \varepsilon\} \leq \# \left\{ i \in \mathcal{I} \;\middle|\; |x_i| < m \text{ and } r_i > \frac{1}{m} \right\} < \infty$$

for m large enough. We may thus find an enumeration $(i_n)_{n\geq 1}$ of the set $\mathcal{I}_U$ such that the sequence $(r_{i_n})_{n\geq 1}$ is nonincreasing and tends to zero at infinity. We finally end up with an approximate local expression of the set F_t as a limsup set, namely,

$$F_t \cap U \subseteq \limsup_{n\to\infty} \mathrm{B}(x_{i_n}, r_{i_n}^t) \subseteq F_t \cap \overline{U}, \tag{89}$$

where $\overline{U}$ stands for the closure of the open set U.

In view of Section 2.8, it is thus natural to examine the convergence of the series $\sum_n |\mathrm{B}(x_{i_n}, r_{i_n}^t)|^s$, where s is a real parameter in the interval $[0, d]$. To be more specific, making a convenient change of variable, this amounts to considering the infimum of all s such that the series $\sum_{i\in\mathcal{I}_U} r_i^s$ is convergent. Note that this infimum is clearly a nondecreasing function of U. In order to cover the case where U is unbounded, and maybe also obtain a better value in the bounded case, we finally introduce the exponent

$$s_U = \inf_{U=\bigcup_\ell U_\ell} \sup_{\ell\geq 1} \inf \left\{ s > 0 \;\middle|\; \sum_{i\in\mathcal{I}_{U_\ell}} r_i^s < \infty \right\}, \tag{90}$$

where the infimum is taken over all sequences $(U_\ell)_{\ell\geq 1}$ of bounded open sets whose union is equal to U. Our approach thus leads to the following statement.

Proposition 4.1. *For any approximation system $(x_i, r_i)_{i\in\mathcal{I}}$, any open subset U of $\mathbb{R}^d$ and any real number $t \geq 1$,*

$$\dim_{\mathrm{H}}(F_t \cap U) \leq \frac{s_U}{t}.$$

Proof. Let $(U_\ell)_{\ell\geq 1}$ denote a sequence of bounded open sets whose union is equal to U. For any integer $\ell \geq 1$, the open set U_ℓ is bounded, so the inclusions (89) are valid. As a consequence, if s denotes a positive real number such that the sum $\sum_{i\in\mathcal{I}_{U_\ell}} r_i^s$ is finite, we may apply Lemma 2.1 with the gauge function $r \mapsto r^{s/t}$, thereby deducing that the set $F_t \cap U_\ell$ has dimension at most s/t. We conclude thanks to the countable stability of Hausdorff dimension, namely, Proposition 2.16(2). $\square$

In most situations, the naïve bound supplied by Proposition 4.1 gives the exact value of the Hausdorff dimension, and moreover the parameter s_U does not depend on the choice of the open set U. This happens for instance when the approximation system are derived from eutaxic sequences or optimal regular systems; these two notions are discussed in Chapters 6 and 7, respectively.

4.2. Lower bound on the Hausdorff dimension

Our goal is now to establish a lower bound on the Hausdorff dimension of the set F_t defined by (87) under the following simple assumption on the underlying approximation system $(x_i, r_i)_{i\in\mathcal{I}}$.

DEFINITION 4.2. Let $\mathcal{I}$ be a countably infinite index set, let $(x_i, r_i)_{i\in\mathcal{I}}$ be an approximation system in $\mathbb{R}^d \times (0,\infty)$ and let U be a nonempty open subset of $\mathbb{R}^d$. We call $(x_i, r_i)_{i\in\mathcal{I}}$ a *homogeneous ubiquitous system* in U if the set F_1 has full Lebesgue measure in U, *i.e.*

$$\text{for } \mathcal{L}^d\text{-a.e. } x \in U \quad \exists \text{ i.m. } i \in \mathcal{I} \qquad |x - x_i| < r_i.$$

Note that we do not impose that all the points x_i belong to the open set U. Actually, the approximation system is usually fixed at the beginning, and the open set is then allowed to change so that one can examine local approximation properties. Moreover, the fact that a given approximation system $(x_i, r_i)_{i\in\mathcal{I}}$ is homogeneously ubiquitous ensures that the approximating points x_i are well spread, in accordance with the corresponding approximation radii r_i. The following remarkable result, due to Jaffard [**34**], shows that this assumption suffices to establish an *a priori* lower bound on the Hausdorff dimension of the sets F_t.

THEOREM 4.1. *Let $(x_i, r_i)_{i\in\mathcal{I}}$ be a homogeneous ubiquitous system in some nonempty open subset U of $\mathbb{R}^d$. Then, for any real number $t > 1$,*

$$\dim_{\mathrm{H}}(F_t \cap U) \geq \frac{d}{t}.$$

More precisely, the set $F_t \cap U$ has positive Hausdorff measure with respect to the gauge function $r \mapsto r^{d/t}|\log r|$.

Combining Theorem 4.1 with Proposition 4.1 above, we remark that if $(x_i, r_i)_{i\in\mathcal{I}}$ is a homogeneous ubiquitous system in U, then the parameter s_U defined by (90) is necessarily bounded below by d. We also readily deduce the following result.

COROLLARY 4.1. *Let $(x_i, r_i)_{i\in\mathcal{I}}$ be a homogeneous ubiquitous system in some nonempty open subset U of $\mathbb{R}^d$. Let us assume that $s_U \leq d$. Then, for any $t > 1$,*

$$\dim_{\mathrm{H}}(F_t \cap U) = \frac{d}{t}.$$

Again, an emblematic situation where this holds is when the approximation system are issued from eutaxic sequences or optimal regular systems, see Chapters 6 and 7. The remainder of this section is devoted to the proof of Theorem 4.1. We thus fix a homogeneous ubiquitous system $(x_i, r_i)_{i\in\mathcal{I}}$ and a nonempty open subset U of $\mathbb{R}^d$. We may obviously assume that U has diameter at most one. Consequently, the index set $\mathcal{I}_U$ defined by (88) admits an enumeration $(i_n)_{n\geq 1}$ such that the sequence $(r_{i_n})_{n\geq 1}$ is nonincreasing and tends to zero at infinity.

4.2.1. A covering lemma. The proof of Theorem 4.1 calls upon a simple result in the spirit of Vitali's covering lemma, that is, Lemma 2.5 but with an additional measure theoretic flavor.

LEMMA 4.1. *For any nonempty open subset V of U and any real number $\rho > 0$, there exists a finite subset $\mathcal{I}(V,\rho)$ of $\mathcal{I}_U$ such that $r_i \leq \rho$ for all $i \in \mathcal{I}(V,\rho)$, and*

$$\bigsqcup_{i\in\mathcal{I}(V,\rho)} \overline{\mathrm{B}}(x_i,r_i) \subseteq V \qquad \text{and} \qquad \sum_{i\in\mathcal{I}(V,\rho)} \mathcal{L}^d(\overline{\mathrm{B}}(x_i,r_i)) \geq \frac{\mathcal{L}^d(V)}{2\cdot 3^d}.$$

PROOF. Let us consider a real number $\rho > 0$. Then, there exists an integer $n_\rho \geq 1$ such that $r_{i_n} \leq \rho$ for all integers $n \geq n_\rho$. We observe that $(x_{i_n}, r_{i_n})_{n\geq n_\rho}$ is a homogeneous ubiquitous system in U. As a consequence, every nonempty open set $V \subseteq U$ necessarily contains a closed ball of the form $\overline{\mathrm{B}}(x_{i_n}, r_{i_n})$, for $n \geq n_\rho$. Indeed, any such open set V contains an open ball of the form $\mathrm{B}(x_0, r_0)$, and the smaller ball $\mathrm{B}(x_0, r_0/2)$ contains a point x that belongs to infinitely many open balls of the form $\mathrm{B}(x_{i_n}, r_{i_n})$ with $n \geq n_\rho$; choosing n so large that r_{i_n} is smaller than $r_0/4$, we may use the point x to ensure that

$$\overline{\mathrm{B}}(x_{i_n}, r_{i_n}) \subseteq \mathrm{B}(x_0, r_0) \subseteq V.$$

Therefore, if V denotes a nonempty open subset of U, we can define

$$n_1 = \min\left\{n \geq n_\rho \;\middle|\; \overline{\mathrm{B}}(x_{i_n}, r_{i_n}) \subseteq V\right\}.$$

For any integer $K \geq 1$, the same argument allows us to define in a recursive manner

$$n_{K+1} = \min\left\{ n > n_K \;\middle|\; \overline{\mathrm{B}}(x_{i_n}, r_{i_n}) \subseteq V \setminus \bigcup_{k=1}^{K} \overline{\mathrm{B}}(x_{i_{n_k}}, r_{i_{n_k}}) \right\}.$$

We thus obtain a increasing sequence of positive integers $(n_K)_{K\geq 1}$. Then, recalling that the radii r_{i_n} monotonically tend to zero as $n \to \infty$, we infer that

$$V \cap \limsup_{n\to\infty} \mathrm{B}(x_{i_n}, r_{i_n}) \subseteq \bigcup_{k=1}^{\infty} \overline{\mathrm{B}}(x_{i_{n_k}}, 3r_{i_{n_k}}). \tag{91}$$

Indeed, if x belongs to the set in the left-hand side of (91), we necessarily have $x \in \overline{\mathrm{B}}(x_{i_n}, r_{i_n}) \subseteq V$ for some sufficiently large integer $n \geq n_1$. Letting K denote the unique integer such that $n_K \leq n < n_{K+1}$, we deduce from the mere definition of n_{K+1} that the ball $\overline{\mathrm{B}}(x_{i_n}, r_{i_n})$ meets at least one of the balls $\overline{\mathrm{B}}(x_{i_{n_k}}, r_{i_{n_k}})$, for $k \in \{1, \ldots, K\}$, at some point denoted by y. Hence,

$$|x - x_{i_{n_k}}| \leq |x - x_{i_n}| + |x_{i_n} - y| + |y - x_{i_{n_k}}| \leq r_{i_n} + r_{i_n} + r_{i_{n_k}} \leq 3r_{i_{n_k}},$$

where the latter bound results from the fact that $n \geq n_K \geq n_k$ and that the radii are nonincreasing. We deduce that x belongs to the right-hand side of (91)

Finally, since $(x_{i_n}, r_{i_n})_{n\geq 1}$ is a homogeneous ubiquitous system in U, the left-hand side of (91) has Lebesgue measure equal to $\mathcal{L}^d(V)$. Consequently, along with (91), the subadditivity and dilation behavior of Lebesgue measure imply that

$$\mathcal{L}^d(V) \leq \mathcal{L}^d\left(\bigcup_{k=1}^{\infty} \overline{\mathrm{B}}(x_{i_{n_k}}, 3r_{i_{n_k}})\right) \leq 3^d \sum_{k=1}^{\infty} \mathcal{L}^d(\overline{\mathrm{B}}(x_{i_{n_k}}, r_{i_{n_k}})).$$

For K large enough, the K-th partial sum of the series appearing in the right-hand side thus exceeds $\mathcal{L}^d(V)/(2\cdot 3^d)$. To conclude, it remains to define $\mathcal{I}(V,\rho)$ as the set of all indices i_{n_k}, for $k \in \{1, \ldots, K\}$. $\square$

4.2.2. The ubiquity construction. After fixing a real number $t > 1$, the proof of Theorem 4.1 now consists in applying Lemma 4.1 repeatedly in order to build a generalized Cantor set that is embedded in the set $F_t \cap U$, together with an appropriate outer measure thereon. We shall ultimately apply the mass distribution principle, namely, Lemma 2.2 to this outer measure. To this end, we shall need an estimate on the mass of balls, *i.e.* on the scaling properties of the outer measure.

The construction is modeled on that presented in Section 2.9.2; recall that it is indexed by a tree T and consists of a collection of compact sets $(I_u)_{u\in T}$ and a companion premeasure ζ such that the compatibility conditions (66) hold. However, we need to be more precise in the present construction, and we actually require the following more specific conditions:

(0) every node in the indexing tree T has at least one child, that is,

$$\min_{u\in T} k_u(T) \geq 1\,;$$

(1) the compact set $I_\varnothing$ indexed by the root of the tree is a closed ball contained in U with diameter in $(0,1)$ and

$$\zeta(I_\varnothing) = |I_\varnothing|^{d/t} \log \frac{1}{|I_\varnothing|}\,; \tag{92}$$

(2) for every node $u \in T \setminus \{\varnothing\}$, there exists an index $i_u \in \mathcal{I}_U$ such that

$$I_u = B_u^t \subset B_u \subseteq I_{\pi(u)}\,;$$

(3) for every node $u \in T \setminus \{\varnothing\}$, we have simultaneously

$$|B_u| \leq 2\exp\left(-\frac{2\cdot 6^d}{t}|I_{\pi(u)}|^{d(1/t-1)-1}\right),$$

in addition to both

$$\bigsqcup_{v\in S_u} B_v \subseteq I_{\pi(u)} \qquad \text{and} \qquad \sum_{v\in S_u} \mathcal{L}^d(B_v) \geq \frac{\mathcal{L}^d(I_{\pi(u)})}{2\cdot 3^d}\,;$$

(4) for every node $u \in T \setminus \{\varnothing\}$, the premeasure ζ satisfies

$$\zeta(I_u) = \frac{\mathcal{L}^d(B_u)}{\sum\limits_{v\in S_u} \mathcal{L}^d(B_v)}\,\zeta(I_{\pi(u)}).$$

In the above conditions, S_u denotes the set formed by a given node u and its siblings, namely, the nodes $v \in T$ such that $\pi(v) = \pi(u)$. Moreover, the sets B_u and B_u^t are the closed balls defined by

$$B_u = \overline{\mathrm{B}}(x_{i_u}, r_{i_u}) \qquad \text{and} \qquad B_u^t = \overline{\mathrm{B}}\left(x_{i_u}, \frac{r_{i_u}^t}{2}\right). \tag{93}$$

In addition, let us recall that $\pi(u)$ denotes the parent of a given node u, and $k_u(T)$ is the size of its progeny. Also, note that the compatibility conditions (66) easily result from (0–4) above; we even have equality in the compatibility condition that concerns the premeasure ζ. Lastly, it is useful to remark that the ball B_u involved in the construction all have diameter at most one, since they are included in U.

The construction is performed inductively on the generation of the indexing tree. In order to guarantee (1), we begin the construction by considering an arbitrary closed ball with diameter in $(0,1)$ that is contained in the nonempty open set U; this ball is the compact set $I_\varnothing$ indexed by the root of the tree. We also define $\zeta(I_\varnothing)$ by (92), in addition to the compulsory condition $\zeta(\emptyset) = 0$.

Furthermore, let us assume that the tree, the compact sets and the companion premeasure have been defined up to a given generation j in such a way that the conditions (0–4) above hold; we now build the tree, the compacts and the premeasure at the next generation $j+1$ in the following manner. For each node u of the j-th generation, we apply Lemma 4.1 to the interior of I_u and the real number

$$\rho_u = \exp\left(-\frac{2\cdot 6^d}{t}|I_u|^{d(1/t-1)-1}\right) ;$$

the resulting finite subset of $\mathcal{I}_U$ is denoted by $\mathcal{I}(\operatorname{int} I_u, \rho_u)$. We then decide that the progeny of the node u in the tree T has cardinality $k_u(T)$ equal to that of $\mathcal{I}(\operatorname{int} I_u, \rho_u)$. Furthermore, we let i_{uk}, for $k \in \{1,\dots,k_u(T)\}$, denote the elements of $\mathcal{I}(\operatorname{int} I_u, \rho_u)$. Making use of the notation (93), we therefore have

$$\bigsqcup_{k=1}^{k_u(T)} B_{uk} \subseteq \operatorname{int} I_u \subseteq I_u \qquad \text{and} \qquad \sum_{k=1}^{k_u(T)} \mathcal{L}^d(B_{uk}) \geq \frac{\mathcal{L}^d(I_u)}{2\cdot 3^d}.$$

On top of that, the radii of the balls B_{uk} are bounded above by ρ_u. Using the notation (93) again, we also define the compact sets I_{uk} as being equal to the closed balls B^t_{uk}, for $k \in \{1,\dots,k_u(T)\}$. This way, the condition (0) is satisfied by the nodes of the j-th generation, and the conditions (2–3) hold for those of the $(j+1)$-th generation. Finally, for $k \in \{1,\dots,k_u(T)\}$, we define

$$\zeta(I_{uk}) = \frac{\mathcal{L}^d(B_{uk})}{\sum\limits_{l=1}^{k_u(T)} \mathcal{L}^d(B_{ul})}\,\zeta(I_u),$$

so that (4) holds for the nodes of the $(j+1)$-th generation. Finally, the above procedure clearly implies that every node of the tree has at least one child, *i.e.* the condition (0) holds.

4.2.3. Scaling properties of the premeasure. The next result gives an upper bound on the premeasure ζ in terms of the diameters of sets.

LEMMA 4.2. *For any node $u \in T$,*

$$\zeta(I_u) \leq |I_u|^{d/t} \log\frac{1}{|I_u|}. \tag{94}$$

PROOF. Let us prove (94) by induction on the length of the word $u \in T$. First, equality holds when u is the empty word, due to the mere value of $\zeta(I_\varnothing)$ determined by (92). Moreover, if we consider a node $u \in T \setminus \{\varnothing\}$ and if we assume that (94) holds for its parent node $\pi(u)$, then the conditions (2–4) yield

$$\begin{aligned}\zeta(I_u) &\leq 2\cdot 3^d \mathcal{L}^d(B_u)\frac{\zeta(I_{\pi(u)})}{\mathcal{L}^d(I_{\pi(u)})} = 2\cdot 6^d |I_u|^{d/t}\frac{\zeta(I_{\pi(u)})}{|I_{\pi(u)}|^d}\\ &\leq 2\cdot 6^d |I_u|^{d/t}|I_{\pi(u)}|^{d(1/t-1)}\log\frac{1}{|I_{\pi(u)}|}.\end{aligned}$$

Finally, in view of the restriction on the diameter of the ball B_u imposed by the condition (3) and the obvious fact that $\log(1/r) \leq 1/r$ for all $r > 0$, we have

$$|I_{\pi(u)}|^{d(1/t-1)}\log\frac{1}{|I_{\pi(u)}|} \leq |I_{\pi(u)}|^{d(1/t-1)-1} \leq \frac{t}{2\cdot 6^d}\log\frac{2}{|B_u|} = \frac{1}{2\cdot 6^d}\log\frac{1}{|I_u|},$$

which leads to (94) for the node u itself. $\square$

4.2.4. The limiting outer measure and its scaling properties. With the help of Theorem 2.2, we may extend as usual the premeasure ζ to an outer measure ζ^* on all the subsets of $\mathbb{R}^d$ through the formula (51). We may also consider the limiting compact set K defined by (67), in addition to the outer measure μ that maps a set $E \subseteq \mathbb{R}^d$ to the value $\zeta^*(E \cap K)$. The tree T considered here is infinite, so Lemma 2.3 shows that K is a nonempty compact subset of $I_\varnothing$. Moreover, the outer measure μ has total mass $\mu(K) = \zeta(I_\varnothing)$. The next result shows that K is included in $F_t \cap U$ as required.

PROPOSITION 4.2. *The compact set K is contained in the intersection $F_t \cap U$. As a consequence,*

$$\mu(F_t \cap U) = \mu(K) = \zeta(I_\varnothing) = |I_\varnothing|^{d/t} \log \frac{1}{|I_\varnothing|}.$$

PROOF. On the one hand, we already mentioned that $K \subseteq I_\varnothing \subseteq U$. On the other hand, if a point x belongs to K, then there exists a sequence $(\xi_j)_{j\geq 1}$ of positive integers such that $x \in I_{\xi_1\ldots\xi_j}$ for all $j \geq 1$. Hence, the point x belong to the infinitely many nested balls $B^t_{\xi_1\ldots\xi_j} \subseteq \mathrm{B}(x_{i_{\xi_1\ldots\xi_j}}, r^t_{i_{\xi_1\ldots\xi_j}})$, and so ultimately belongs to the set F_t. □

Thanks to Lemma 4.2, we may now derive an upper bound on the μ-mass of sufficiently small closed balls $\mathbb{R}^d$.

PROPOSITION 4.3. *For any closed ball B of $\mathbb{R}^d$ with diameter less than $\mathrm{e}^{-d/t}$,*

$$\mu(B) \leq 2 \cdot 12^d |B|^{d/t} \log \frac{1}{|B|}.$$

PROOF. We may obviously assume that the ball B intersects the compact set K, as otherwise $\mu(B)$ clearly vanishes. Besides, if the ball B intersects only one compact set I_u at each generation, then there exists a sequence $(\xi_j)_{j\geq 1}$ of positive integers such that $B \cap K \subseteq I_{\xi_1\ldots\xi_j}$ for all $j \geq 1$, so that

$$\mu(B) = \zeta^*(B \cap K) \leq \zeta(I_{\xi_1\ldots\xi_j}) \leq |I_{\xi_1\ldots\xi_j}|^{d/t} \log \frac{1}{|I_{\xi_1\ldots\xi_j}|} \xrightarrow[j\to\infty]{} 0,$$

thanks to Lemma 4.2. The upshot is that we may suppose in what follows that there exists a node $u \in T$ such that the ball B intersects the compact set I_u, and at least two compacts indexed by the children of u. We further assume that u has minimal length, which in fact ensures its uniqueness.

The easy case is when the diameter of the ball B exceeds that of the compact set I_u ; indeed, as the intersection set $B \cap K$ is covered by the sole I_u, we may then deduce from Lemma 4.2 that

$$\mu(B) = \zeta^*(B \cap K) \leq \zeta(I_u) \leq |I_u|^{d/t} \log \frac{1}{|I_u|} \leq |B|^{d/t} \log \frac{1}{|B|}.$$

Note that the latter inequality holds because $|B|$ is small enough to ensure that the considered function of the diameter is nondecreasing.

Let us now deal with the opposite case in which $|B|$ is smaller than $|I_u|$. Let $\mathcal{K}$ denote the set of all integers k between one and $k_u(T)$ such that the compact set I_{uk} intersects the ball B. The proof calls upon the next simple volume estimate.

LEMMA 4.3. *For any integer $k \in \mathcal{K}$,*

$$\mathcal{L}^d(B \cap B_{uk}) \geq \frac{\mathcal{L}^d(B_{uk})}{4^d}.$$

PROOF. For any distinct k and k' in $\mathcal{K}$, the balls B_{uk} and $B_{uk'}$ are disjoint, so the distance between their center is larger than the sum of their radii; indeed, otherwise, we would have

$$\frac{r_{i_{uk'}} x_{i_{uk}} + r_{i_{uk}} x_{i_{uk'}}}{r_{i_{uk}} + r_{i_{uk'}}} \in B_{uk} \cap B_{uk'}.$$

Furthermore, let y_k denote a point that belongs to both I_{uk} and B. The previous fact and the triangle inequality yield

$$\begin{aligned} r_{i_{uk}} + r_{i_{uk'}} &< |x_{i_{uk'}} - x_{i_{uk}}| \\ &\leq |x_{i_{uk'}} - y_{k'}| + |x_{i_{uk}} - y_k| + |y_{k'} - y_k| \leq \frac{r_{i_{uk}}^t + r_{i_{uk'}}^t}{2} + |y_{k'} - y_k|, \end{aligned}$$

from which we deduce a lower bound on the distance between y_k and $y_{k'}$, and in fact a lower bound on the diameter of the ball B, namely,

$$|B| \geq |y_{k'} - y_k| \geq r_{i_{uk}} + r_{i_{uk'}} - \frac{r_{i_{uk}}^t + r_{i_{uk'}}^t}{2} \geq r_{i_{uk}} - \frac{r_{i_{uk}}^t}{2}.$$

Letting x_0 and r_0 denote the center and the radius of the ball B, respectively, and letting s_k denote half the right-hand side above, we deduce that $r_0 \geq s_k$.

Let us assume that the distance between $x_{i_{uk}}$ and x_0 is smaller than $r_0 - s_k$. Thus, the closed ball $\overline{\mathrm{B}}(x_{i_{uk}}, s_k)$ is included in both B and B_{uk}, so that

$$\mathcal{L}^d(B \cap B_{uk}) \geq \mathcal{L}^d(\overline{\mathrm{B}}(x_{i_{uk}}, s_k)) = \left(\frac{s_k}{r_{i_{uk}}}\right)^d \mathcal{L}^d(B_{uk}),$$

in view of the dilation behavior of Lebesgue measure. In the opposite case, thanks to the triangle inequality, we have

$$r_0 - s_k \leq |x_{i_{uk}} - x_0| \leq |x_{i_{uk}} - y_k| + |y_k - x_0| \leq \frac{r_{i_{uk}}^t}{2} + r_0 = r_0 + r_{i_{uk}} - 2s_k.$$

We may thus consider the barycenter defined by

$$m_k = \lambda_k x_{i_{uk}} + (1 - \lambda_k) x_0 \qquad \text{with} \qquad \lambda_k = \frac{r_0 - s_k}{|x_{i_{uk}} - x_0|} \in [0, 1].$$

It is clear that the distance between m_k and x_0 is equal to $r_0 - s_k$. Likewise, the distance between m_k and $x_{i_{uk}}$ satisfies

$$|m_k - x_{i_{uk}}| = (1 - \lambda_k)|x_{i_{uk}} - x_0| = |x_{i_{uk}} - x_0| - r_0 + s_k \leq r_{i_{uk}} - s_k.$$

We deduce that the closed ball $\overline{\mathrm{B}}(m_k, s_k)$ is contained in both B and B_{uk}, which gives as above

$$\mathcal{L}^d(B \cap B_{uk}) \geq \mathcal{L}^d(\overline{\mathrm{B}}(m_k, s_k)) = \left(\frac{s_k}{r_{i_{uk}}}\right)^d \mathcal{L}^d(B_{uk}).$$

The result follows from the fact that the radius of the ball B_{uk} is at most one. □

The previous lemma enables us to estimate the μ-mass of the ball B. Indeed, the ball intersects the compact set K inside the compact sets I_{uk}, for $k \in \mathcal{K}$, so the conditions (3) and (4) yield

$$\begin{aligned} \mu(B) &= \zeta^*(B \cap K) \\ &\leq \sum_{k \in \mathcal{K}} \zeta(I_{uk}) = \sum_{k \in \mathcal{K}} \frac{\mathcal{L}^d(B_{uk})}{\sum_{l=1}^{k_u(T)} \mathcal{L}^d(B_{ul})} \zeta(I_u) \leq 2 \cdot 3^d \frac{\zeta(I_u)}{\mathcal{L}^d(I_u)} \sum_{k \in \mathcal{K}} \mathcal{L}^d(B_{uk}). \end{aligned}$$

Now, applying Lemma 4.3 and making use of the disjointness of the balls B_{uk}, we infer that

$$\mu(B) \leq 2 \cdot 12^d \frac{\zeta(I_u)}{\mathcal{L}^d(I_u)} \sum_{k \in \mathcal{K}} \mathcal{L}^d(B \cap B_{uk}) \leq 2 \cdot 12^d \frac{\zeta(I_u)}{\mathcal{L}^d(I_u)} \mathcal{L}^d(B).$$

Combining the condition (2), the definition (93) of the balls B_u^t and the bound on the ζ-mass of I_u given by Lemma 4.2, we deduce that

$$\mu(B) \leq 2 \cdot 12^d |B|^d |I_u|^{d(1/t-1)} \log \frac{1}{|I_u|} \leq 2 \cdot 12^d |B|^{d/t} \log \frac{1}{|B|}.$$

For the latter bound, we have the fact that $t > 1$ and $|I_u| > |B|$. We conclude by combining this bound with the one obtained in the previous easier case. □

To finish the proof of Theorem 4.1, it remains to apply the mass distribution principle, namely, Lemma 2.2. In fact, any bounded subset C of $\mathbb{R}^d$ may be embedded in a closed ball B with radius equal to $|C|$. If we assume in addition that $|C| < \mathrm{e}^{-d/t}/2$, the ball B has diameter less than $\mathrm{e}^{-d/t}$, and Proposition 4.3 gives

$$\mu(C) \leq \mu(B) \leq 2 \cdot 12^d |B|^{d/t} \log \frac{1}{|B|} \leq 2 \cdot 12^d 2^{d/t} |C|^{d/t} \log \frac{1}{|C|}.$$

Letting g denote the gauge function $r \mapsto r^{d/t} |\log r|$, the mass distribution principle and Proposition 4.2 finally ensure that

$$\mathcal{H}^g(F_t \cap U) \geq \frac{\mu(F_t \cap U)}{2 \cdot 12^d 2^{d/t}} = \frac{g(|I_\varnothing|)}{2 \cdot 12^d 2^{d/t}} > 0,$$

from which we deduce that the set $F_t \cap U$ has Hausdorff dimension at least d/t.

4.3. Application to the Jarník-Besicovitch theorem

We already studied the Hausdorff dimension of the set $J_{d,\tau}$ formed by the points that are approximable at rate at least τ by the points with rational coordinates, see (1) for the exact definition of this set. Specifically, the Jarník-Besicovitch theorem discussed in Section 3.1 asserts that for any real $\tau > 1 + 1/d$,

$$\dim_{\mathrm{H}} J_{d,\tau} = \frac{d+1}{\tau},$$

see Theorem 3.1 for the precise statement. Also, let us recall that the set $J_{d,\tau}$ coincides with the whole space $\mathbb{R}^d$ when $\tau \leq 1 + 1/d$, as a consequence of Dirichlet's theorem, see Corollary 1.1.

The general theory discussed above enables us to give an alternative proof of the Jarník-Besicovitch theorem. Indeed, the set $J_{d,1+1/d}$ coincides with the whole $\mathbb{R}^d$, so it obviously has full measure therein, namely, for Lebesgue-almost every $x \in \mathbb{R}^d$, there are infinitely many pairs $(p,q) \in \mathbb{Z}^d \times \mathbb{N}$ such that $|x - p/q|_\infty < q^{-1-1/d}$. This means that the family $(p/q, q^{-1-1/d})_{(p,q) \in \mathbb{Z}^d \times \mathbb{N}}$ is a homogeneous ubiquitous system in $\mathbb{R}^d$. Besides, for any integer $M \geq 1$ and any real number $s > 0$, note that

$$\sum_{\substack{(p,q) \in \mathbb{Z}^d \times \mathbb{N} \\ p/q \in \mathrm{B}_\infty(0,M)}} (q^{-1-1/d})^s = \sum_{q=1}^{\infty} q^{-(1+1/d)s} \#(\mathbb{Z}^d \cap \mathrm{B}_\infty(0, qM)).$$

The cardinality appearing in the sum is of the order of $(qM)^d$, up to numerical constants. Hence, the critical value s for the convergence of the series is that for which $(1+1/d)s - d$ is equal to one. We deduce that for any open subset U of $\mathbb{R}^d$, the parameter s_U defined by (90) is bounded above by d. We are now in position to apply Corollary 4.1. After fixing a real number $\tau > 1 + 1/d$ and observing that the approximation radii $q^{-\tau}$ in the definition of $J_{d,\tau}$ may be written in the form

$(q^{-1-1/d})^t$ with $t = \tau d/(d+1) > 1$, we deduce from the aforementioned result that for any nonempty open subset U of $\mathbb{R}^d$,

$$\dim_{\mathrm{H}}(J_{d,\tau} \cap U) = \frac{d}{t} = \frac{d+1}{\tau}, \tag{95}$$

thereby obtaining a local version of the Jarník-Besicovitch theorem.

We can relate this result with the notion of irrationality exponent, supplied by Definition 1.1. In fact, for any real number $\tau \geq 1 + 1/d$,

$$J_{d,\tau} \setminus \mathbb{Q}^d \subseteq \{x \in \mathbb{R}^d \setminus \mathbb{Q}^d \mid \tau(x) \geq \tau\} = \bigcap_{\tau' < \tau} \downarrow J_{d,\tau'} \setminus \mathbb{Q}^d.$$

Due to (95) and the fact that the set $\mathbb{Q}^d$ has Hausdorff dimension zero, we deduce that for any nonempty open subset U of $\mathbb{R}^d$,

$$\dim_{\mathrm{H}}\{x \in U \setminus \mathbb{Q}^d \mid \tau(x) \geq \tau\} = \frac{d+1}{\tau}.$$

Theorem 4.1 gives actually a slightly more precise result, specifically, letting g_τ denote the gauge function $r \mapsto r^{(d+1)/\tau}|\log r|$, we have

$$\mathcal{H}^{g_\tau}(\{x \in U \setminus \mathbb{Q}^d \mid \tau(x) \geq \tau\}) \geq \mathcal{H}^{g_\tau}(J_{d,\tau} \cap U) > 0.$$

This allows us to determine the Hausdorff dimension of the set of points with irrationality exponent exactly equal to τ. As a matter of fact, let us observe that

$$\{x \in \mathbb{R}^d \setminus \mathbb{Q}^d \mid \tau(x) = \tau\} = \{x \in \mathbb{R}^d \setminus \mathbb{Q}^d \mid \tau(x) \geq \tau\} \setminus \bigcup_{\tau' > \tau} \uparrow J_{d,\tau'}. \tag{96}$$

Moreover, thanks to Proposition 2.12, we have for $\tau' > \tau$ and $\varepsilon > 0$ small enough to ensure that $(d+1)/\tau - \varepsilon$ is larger than $(d+1)/\tau'$,

$$\mathcal{H}^{g_\tau}(J_{d,\tau'}) \leq \left(\limsup_{r \to 0} \frac{g_\tau(r)}{r^{(d+1)/\tau - \varepsilon}}\right) \mathcal{H}^{(d+1)/\tau - \varepsilon}(J_{d,\tau'}) = 0$$

The mapping $\tau' \mapsto J_{d,\tau'}$ is nonincreasing, so the union in (96) may be written as a countable one, and Proposition 2.4(1) implies that its Hausdorff g_τ-measure vanishes. We deduce that

$$\dim_{\mathrm{H}}\{x \in U \setminus \mathbb{Q}^d \mid \tau(x) = \tau\} = \frac{d+1}{\tau}.$$

Indeed, the set in the left-hand side of (96) has positive g_τ-measure in U.

4.4. Behavior under uniform dilations

The next useful result shows that multiplying all the approximation radii by a common positive factor does not alter the property of being a homogeneous ubiquitous system. In particular, this implies that this property is independent on the choice of the norm the space $\mathbb{R}^d$ is endowed with.

PROPOSITION 4.4. *Let $(x_i, r_i)_{i \in \mathcal{I}}$ be a homogeneous ubiquitous system in some nonempty open subset U of $\mathbb{R}^d$. Then, for any real number $c > 0$, the family $(x_i, c\, r_i)_{i \in \mathcal{I}}$ is also a homogeneous ubiquitous system in U.*

PROOF. The family $(x_i, c\, r_i)_{i \in \mathcal{I}}$ is clearly an approximation system, so it remains to show that the set R_c of all points $x \in \mathbb{R}^d$ such that $|x - x_i| < c\, r_i$ for infinitely many indices $i \in \mathcal{I}$ has full Lebesgue measure in U. This is obvious if $c \geq 1$, because R_c contains R_1, which has full Lebesgue measure in U. We may thus restrict our attention to the case in which $c < 1$.

Let V be a nonempty bounded open subset of U and let j be a positive integer. By Lemma 4.1, there is a finite subset $\mathcal{I}_j = \mathcal{I}(V, 2^{-j})$ of $\mathcal{I}$ such that the balls

$\overline{\mathrm{B}}(x_i, r_i)$ are disjoint, contained in V, with radius at most 2^{-j}, and a total Lebesgue measure at least $\mathcal{L}^d(V)/(2\cdot 3^d)$. In particular,

$$R_c \cap V \supseteq \limsup_{j\to\infty} \bigsqcup_{i\in\mathcal{I}_j} \mathrm{B}(x_i, c\,r_i) = \bigcap_{j=1}^{\infty} \downarrow \bigcup_{j'=j}^{\infty} \bigsqcup_{i\in\mathcal{I}_{j'}} \mathrm{B}(x_i, c\,r_i).$$

The open set V is bounded, thereby having finite Lebesgue measure. Hence, Proposition 2.5 ensures that

$$\begin{aligned}\mathcal{L}^d(R_c \cap V) &\geq \lim_{j\to\infty} \downarrow \mathcal{L}^d\left(\bigcup_{j'=j}^{\infty} \bigsqcup_{i\in\mathcal{I}_{j'}} \mathrm{B}(x_i, c\,r_i)\right)\\ &\geq \limsup_{j\to\infty} \sum_{i\in\mathcal{I}_j} \mathcal{L}^d(\mathrm{B}(x_i, c\,r_i)) \geq \frac{c^d\,\mathcal{L}^d(V)}{2\cdot 3^d}.\end{aligned}$$

Let us assume that $\mathcal{L}^d(U\setminus R_c)$ is positive. Then $\mathcal{L}^d(U_m\setminus R_c)$ is positive for m large enough, where U_m denotes the set $U\cap(-m,m)^d$. Furthermore, there exists a compact subset K of $R_c\cap U_m$ such that

$$\mathcal{L}^d((R_c\cap U_m)\setminus K) < \frac{c^d\,\mathcal{L}^d(U_m\setminus R_c)}{2\cdot 3^d},$$

see for instance [**46**, Theorem 1.10]. Applying what precedes to the bounded open set $V = U_m\setminus K$, we obtain

$$\mathcal{L}^d(R_c\cap(U_m\setminus K)) \geq \frac{c^d\,\mathcal{L}^d(U_m\setminus K)}{2\cdot 3^d} \geq \frac{c^d\,\mathcal{L}^d(U_m\setminus R_c)}{2\cdot 3^d},$$

and we end up with a contradiction. Hence, R_c has full Lebesgue measure in U. $\square$

CHAPTER 5

Large intersection properties

5.1. The large intersection classes

The classes of sets with large intersection were introduced by Falconer [**26, 28**]. They are composed of subsets of $\mathbb{R}^d$ with Hausdorff dimension at least a given s satisfying the remarkable counterintuitive property that countable intersections of the sets also have Hausdorff dimension at least s. This is in stark contrast with, for instance, the case of two affine subspaces with dimension s_1 and s_2, respectively, where the intersection is generically expected to have dimension $s_1 + s_2 - d$. The aforementioned classes are formally defined as follows. Recall that a G_δ-set is one that may be written as the intersection of a countable sequence of open sets.

DEFINITION 5.1. For any real number $s \in (0, d]$, the class $\mathcal{G}^s(\mathbb{R}^d)$ of *sets with large intersection in* $\mathbb{R}^d$ *with dimension at least* s is the collection of all G_δ-subsets F of $\mathbb{R}^d$ such that

$$\dim_{\mathrm{H}} \bigcap_{n=1}^{\infty} \varsigma_n(F) \geq s$$

for any sequence $(\varsigma_n)_{n\geq 1}$ of similarity transformations of $\mathbb{R}^d$.

As shown later in these notes, numerous examples of sets with large intersection arise in metric number theory. Let us point out that the middle-third Cantor set $\mathbb{K}$ gives a typical example of set that is *not* with large intersection. Indeed, letting ς denote the mapping that sends a real number x to $(x+1)/3$, we readily observe that $\mathbb{K} \cap \varsigma(\mathbb{K})$ is reduced to the points $1/3$ and $2/3$, thereby having Hausdorff dimension zero, whereas the Cantor set $\mathbb{K}$ itself has dimension equal to $\log 2/\log 3$, see Propositions 2.17 and 2.18. More generally, the attractors of iterated function systems that are discussed in Section 2.10 do not satisfy the large intersection property.

As mentioned above, the main property of the large intersection classes $\mathcal{G}^s(\mathbb{R}^d)$ are their stability under countable intersections; remarkably, they are also stable under bi-Lipischitz transformations, *i.e.* mappings satisfying (64). This is the purpose of the next statement.

THEOREM 5.1. *For any real number* $s \in (0, d]$*, the class* $\mathcal{G}^s(\mathbb{R}^d)$ *is closed under countable intersections and bi-Lipschitz transformations of* $\mathbb{R}^d$.

The proof of Theorem 5.1 being quite long, we postpone it to Section 5.3 so as not to interrupt the flow of the presentation. Combined with the definition of the classes $\mathcal{G}^s(\mathbb{R}^d)$ given above, Theorem 5.1 directly yields the following maximality property with respect to countable intersections and similarities.

COROLLARY 5.1. *For any real number* $s \in (0, d]$*, the class* $\mathcal{G}^s(\mathbb{R}^d)$ *is the maximal class of* G_δ*-subsets of* $\mathbb{R}^d$ *with Hausdorff dimension at least* s *that is closed under countable intersections and similarity transformations.*

We now give several characterizations of the classes $\mathcal{G}^s(\mathbb{R}^d)$. Some of them are expressed in terms of outer net measures that are obtained by restricting to coverings by dyadic cubes. More precisely, let us recall from Section 2.6.3 that a dyadic cube is either the empty set or a set of the form $\lambda = 2^{-j}(k + [0,1)^d)$, with $j \in \mathbb{Z}$ and $k \in \mathbb{Z}^d$, and that the collection of all dyadic cubes is denoted by Λ. For any real number $s \in (0, d]$, let us consider the premeasure, denoted by $|\cdot|_\Lambda^s$, that maps a given $\lambda \in \Lambda$ to $|\lambda|^s$. Then, as in Section 2.6.3, Theorem 2.3 allows us to consider the net measure

$$\mathcal{M}^s = (|\cdot|_\Lambda^s)_*$$

defined by (53). In view of Proposition 2.11, this outer measure is comparable with the s-dimensional Hausdorff measure, in the sense that

$$\mathcal{H}^s(E) \le \mathcal{M}^s(E) \le \kappa\, \mathcal{H}^s(E)$$

for any subset E of $\mathbb{R}^d$ and for some real number $\kappa \ge 1$. In addition, Theorem 2.2 enables us to introduce the outer measure

$$\mathcal{M}^s_\infty = (|\cdot|_\Lambda^s)^* \tag{97}$$

that is defined by (51), and thus corresponds to coverings by dyadic cubes of arbitrary diameter. It is clear that the outer measures $\mathcal{M}^s_\infty$ bound the net measures $\mathcal{M}^s$ from below. Hence, for any subset E of $\mathbb{R}^d$,

$$\mathcal{M}^s_\infty(E) > 0 \qquad \Longrightarrow \qquad \dim_{\mathrm{H}} E \ge s. \tag{98}$$

Moreover, it is useful to observe that the $\mathcal{M}^s_\infty$-mass of the dyadic cubes may easily be expressed in terms of their diameters. This is the purpose of the next lemma.

LEMMA 5.1. *For any real number $s \in (0, d]$ and any dyadic cube $\lambda \in \Lambda$,*

$$\mathcal{M}^s_\infty(\lambda) = \mathcal{M}^s_\infty(\operatorname{int} \lambda) = |\lambda|^s.$$

PROOF. Given that $\mathcal{M}^s_\infty$ is an outer measure and that the considered dyadic cube λ may obviously be covered by itself, we directly infer that

$$\mathcal{M}^s_\infty(\operatorname{int} \lambda) \le \mathcal{M}^s_\infty(\lambda) \le |\lambda|^s.$$

In order to show that equality holds, let us consider a dyadic covering $(\lambda_n)_{n\ge1}$ of the interior of λ. If λ is contained in some cube λ_{n_0}, then we clearly have

$$|\lambda|^s \le |\lambda_{n_0}|^s \le \sum_{n=1}^{\infty} |\lambda_n|^s.$$

Otherwise, all the cubes λ_n are either disjoint from, or included inside, the cube λ. Thus, we may consider the subset $\mathcal{N}$ of $\mathbb{N}$ formed by the integers $n \ge 1$ for which λ_n is contained in λ. The cubes λ_n, for $n \in \mathcal{N}$, still cover the interior of λ and have a smaller diameter, so that

$$\sum_{n=1}^{\infty} |\lambda_n|^s \ge \sum_{n\in\mathcal{N}} |\lambda_n|^{s-d} |\lambda_n|^d \ge |\lambda|^{s-d} \sum_{n\in\mathcal{N}} \kappa'^d \mathcal{L}^d(\lambda_n) \ge |\lambda|^{s-d} \kappa'^d \mathcal{L}^d(\operatorname{int} \lambda) = |\lambda|^s,$$

where κ' is the diameter of the unit cube of $\mathbb{R}^d$, and only depends on the norm the space $\mathbb{R}^d$ is endowed with. We deduce the required inequality by finally taking the infimum over all coverings $(\lambda_n)_{n\ge1}$. □

We can now enumerate the properties that characterize the large intersection classes; note that the formulations given by Falconer [**28**] are slightly erroneous and one has to consider the corrected versions below, where s denotes a real number in the interval $(0, d]$ and F is a subset of $\mathbb{R}^d$:

(1) for any nonempty open subset U of $\mathbb{R}^d$ and any sequence $(f_n)_{n\geq 1}$ of bi-Lipschitz transformations from U to $\mathbb{R}^d$, we have

$$\dim_{\mathrm{H}} \bigcap_{n=1}^{\infty} f_n^{-1}(F) \geq s\,;$$

(2) for any sequence $(\varsigma_n)_{n\geq 1}$ of similarity transformations of $\mathbb{R}^d$, we have

$$\dim_{\mathrm{H}} \bigcap_{n=1}^{\infty} \varsigma_n(F) \geq s\,;$$

(3) for any positive real number $t < s$ and any dyadic cube $\lambda \in \Lambda$,

$$\mathcal{M}^t_\infty(F \cap \lambda) = \mathcal{M}^t_\infty(\lambda)\,;$$

(4) for any positive real number $t < s$ and any open subset V of $\mathbb{R}^d$,

$$\mathcal{M}^t_\infty(F \cap V) = \mathcal{M}^t_\infty(V)\,;$$

(5) for any positive real number $t < s$, there exists a real number $c \in (0, 1]$ such that for any dyadic cube $\lambda \in \Lambda$,

$$\mathcal{M}^t_\infty(F \cap \lambda) \geq c\,\mathcal{M}^t_\infty(\lambda)\,;$$

(6) for any positive real number $t < s$, there exists a real number $c \in (0, 1]$ such that any open subset V of $\mathbb{R}^d$,

$$\mathcal{M}^t_\infty(F \cap V) \geq c\,\mathcal{M}^t_\infty(V).$$

Note that the property (2) coincides with the definition of the large intersection class $\mathcal{G}^s(\mathbb{R}^d)$ under the assumption that F is a G_δ-set. The next result details the logical relationships between the previous properties, and in fact implies that they give equivalent characterizations of the large intersection classes.

Theorem 5.2. *Let us consider a real number $s \in (0, d]$ and a subset F of $\mathbb{R}^d$.*

- *The following implications hold:*

$$(1) \implies (2) \implies (3) \iff (4) \implies (5) \iff (6).$$

- *If F is a G_δ-set, then the properties (1–6) are all equivalent, and characterize the class $\mathcal{G}^s(\mathbb{R}^d)$.*

Just as that of Theorem 5.1, the proof of Theorem 5.2 is quite long and thus postponed to Section 5.3 for the sake of clarity. Note that the characterizations (5) and (6) still hold when changing the norm on $\mathbb{R}^d$; the large intersection classes are thus independent on the choice of the norm the space $\mathbb{R}^d$ is endowed with. Hereunder are several other noteworthy properties of these classes.

Proposition 5.1. *The large intersection classes $\mathcal{G}^s(\mathbb{R}^d)$, for $s \in (0, d]$, satisfy all the following properties.*

(1) *Any G_δ-subset of $\mathbb{R}^d$ that contains a set in the class $\mathcal{G}^s(\mathbb{R}^d)$ also belongs to the class $\mathcal{G}^s(\mathbb{R}^d)$.*
(2) *The mapping $s \mapsto \mathcal{G}^s(\mathbb{R}^d)$ is nonincreasing.*
(3) *The class $\mathcal{G}^s(\mathbb{R}^d)$ is the intersection over $t < s$ of the classes $\mathcal{G}^t(\mathbb{R}^d)$.*
(4) *For any sets $F \in \mathcal{G}^s(\mathbb{R}^d)$ and $F' \in \mathcal{G}^{s'}(\mathbb{R}^{d'})$, the product set $F \times F'$ belongs to the class $\mathcal{G}^{s+s'}(\mathbb{R}^{d+d'})$.*

Proof. We only need to detail the proof of the last property, because the others readily follow from Definition 5.1. To proceed, let us consider two sets $F \in \mathcal{G}^s(\mathbb{R}^d)$ and $F' \in \mathcal{G}^{s'}(\mathbb{R}^{d'})$, a real number $\varepsilon > 0$ and a dyadic cube of $\mathbb{R}^{d+d'}$ that is written in the form $\lambda \times \lambda'$, where λ is a dyadic cube of $\mathbb{R}^d$ and λ' is a dyadic cube of $\mathbb{R}^{d'}$. It is clear that $(F \times F') \cap (\lambda \times \lambda')$ is equal to $(F \cap \lambda) \times (F' \cap \lambda')$. Moreover, it follows from [**27**, Theorem 5.8] that

$$\mathcal{M}_\infty^{s+s'-\varepsilon}((F \cap \lambda) \times (F' \cap \lambda')) \geq c\,\mathcal{M}_\infty^{s-\varepsilon/2}(F \cap \lambda)\,\mathcal{M}_\infty^{s'-\varepsilon/2}(F' \cap \lambda'),$$

for some real constant $c > 0$. Using the property (3) of Theorem 5.2, together with Lemma 5.1, we deduce that

$$\begin{aligned}\mathcal{M}_\infty^{s+s'-\varepsilon}((F \times F') \cap (\lambda \times \lambda')) &\geq c\,|\lambda|^{s-\varepsilon/2}\,|\lambda'|^{s'-\varepsilon/2} \\ &= c\,c'\,|\lambda \times \lambda'|^{s+s'-\varepsilon} = \mathcal{M}_\infty^{s+s'-\varepsilon}(\lambda \times \lambda'),\end{aligned}$$

where c' is a positive real number that depends on the norms the spaces $\mathbb{R}^d$, $\mathbb{R}^{d'}$ and $\mathbb{R}^{d+d'}$ are endowed with. We conclude that $F \times F'$ belongs to $\mathcal{G}^{s+s'}(\mathbb{R}^{d+d'})$. □

Finally, note that a set with large intersection is necessarily dense in the whole space $\mathbb{R}^d$. This is easily seen for instance by considering the characterization (3) of the large intersection classes given by Theorem 5.2, and by making use of Lemma 5.1. However, in some applications, the considered sets are thought of satisfying a large intersection property in some nonempty open subset U of $\mathbb{R}^d$, but fail to be dense in the whole space $\mathbb{R}^d$ itself. We therefore need to introduce localized versions of the large intersection classes. In that situation, the use of similarity transformations is not suitable anymore; a convenient way of proceeding is thus to adjust the characterization (4) of the large intersection classes given by Theorem 5.2 in the following manner.

DEFINITION 5.2. For any real number $s \in (0, d]$ and any nonempty open subset U of $\mathbb{R}^d$, the class $\mathcal{G}^s(U)$ of *sets with large intersection in U with dimension at least s* is the collection of all G_δ-subsets F of $\mathbb{R}^d$ such that

$$\mathcal{M}_\infty^t(F \cap V) = \mathcal{M}_\infty^t(V)$$

for any positive real number $t < s$ and any open subset V of U.

Obviously, thanks to Theorem 5.2, the class $\mathcal{G}^s(U)$ defined above coincides with the initial class $\mathcal{G}^s(\mathbb{R}^d)$ introduced in Definition 5.1 when the open set U is equal to the whole space $\mathbb{R}^d$. We also directly obtain the following result; the second statement therein follows from (98), whereas the first one is proven in Section 5.3.

THEOREM 5.3. *Let $s \in (0, d]$ and let U be a nonempty open subset of $\mathbb{R}^d$. Then:*

(1) *the class $\mathcal{G}^s(U)$ is closed under countable intersections;*
(2) *for any set $F \in \mathcal{G}^s(U)$ and any nonempty open set $V \subseteq U$,*

$$\dim_{\mathrm{H}}(F \cap V) \geq s.$$

In view of the previous result, the large intersection property is actually a combination of a density property with a measure theoretic aspect. In that spirit, Theorem 5.1 may be thought of as a Hausdorff dimensional analog of the Baire category theorem.

5.2. Other notions of dimension

The sets with large intersection also display a remarkable behavior with respect to packing dimension. Let us explain how this notion of dimension, due to Tricot [**60**], is defined. First, given a gauge function g, we define on the collection of all subsets F of $\mathbb{R}^d$ the *packing g-premeasure* by

$$P^g(F) = \lim_{\delta \downarrow 0} \downarrow P^g_\delta(F) \qquad \text{with} \qquad P^g_\delta(F) = \sup \sum_{n=1}^{\infty} g(|B_n|),$$

where the supremum is taken over all sequences $(B_n)_{n\geq 1}$ of disjoint closed balls of $\mathbb{R}^d$ centered in the set F and with diameter less than δ. The premeasures P^g are only finitely subadditive; it is thus more convenient to work with the corresponding *packing g-measure*, defined by

$$\mathcal{P}^g = (P^g)^*$$

as in the formula (51), which is an outer measure on $\mathbb{R}^d$, as a consequence of Theorem 2.2. It is actually possible to show that the Borel subsets of $\mathbb{R}^d$ are $\mathcal{P}^g$-measurable, see [**46**, Chapter 5] for details.

The definition of packing dimension is then very similar to that of Hausdorff dimension, namely, Definition 2.10. Specifically, when the gauge function g is of the form $r \mapsto r^s$ with $s > 0$, it is customary to use $\mathcal{P}^s$ as a shorthand for $\mathcal{P}^g$, and the packing dimension of a nonempty set $F \subseteq \mathbb{R}^d$ is defined by

$$\dim_{\mathrm{P}} F = \sup\{s \in (0,d) \mid \mathcal{P}^s(F) = \infty\} = \inf\{s \in (0,d) \mid \mathcal{P}^s(F) = 0\}, \tag{99}$$

with the convention that $\sup\emptyset = 0$ and $\inf\emptyset = d$. When the set F is empty, we adopt the convention that the packing dimension is equal to $-\infty$. Moreover, one recovers the upper box-counting dimension $\overline{\dim}_{\mathrm{B}} E$ by considering the premeasures P^s instead of $\mathcal{P}^s$ in the latter formula.

The packing dimension of sets with large intersection is discussed in the next statement, which may be seen as an analog of Theorem 5.3(2), which deals with Hausdorff dimension.

PROPOSITION 5.2. *Let $s \in (0,d]$ and let U be a nonempty open subset of $\mathbb{R}^d$. Then, for any set $F \in \mathcal{G}^s(U)$ and for any nonempty open set $V \subseteq U$,*

$$\dim_{\mathrm{P}}(F \cap V) = d.$$

In other words, a set with large intersection has maximal packing dimension in any nonempty open set; the same property obviously holds for box-counting dimensions as well, because sets with large intersection are dense. Again, for the sake of clarity, the proof of Proposition 5.2 is postponed to Section 5.3.

5.3. Proof of the main results

5.3.1. Ancillary lemmas. The proofs make use of several technical lemmas concerning the outer measures $\mathcal{M}^s_\infty$ that we now state and establish.

LEMMA 5.2. *Let us consider two real numbers $s \in (0,d]$ and $c \in (0,1]$, a subset F of $\mathbb{R}^d$, and an open subset V of $\mathbb{R}^d$. Suppose that there is a $\delta > 0$ such that*

$$\mathcal{M}^s_\infty(F \cap \lambda) \geq c\,\mathcal{M}^s_\infty(\lambda)$$

for all dyadic cubes $\lambda \in \Lambda$ with diameter at most δ that are contained in V. Then,

$$\mathcal{M}^s_\infty(F \cap V) \geq c\,\mathcal{M}^s_\infty(V).$$

PROOF. Let $\Lambda_\delta(V)$ denote the collection of all dyadic cubes with diameter at most δ that are contained in V, and that are maximal for this property. Clearly, these cubes are disjoint and their union is equal to the whole open set V. Let us now consider a dyadic covering $(\lambda_n)_{n\geq 1}$ of the set $F \cap V$. Two dyadic cubes are either disjoint or included in one another, so there exists a subset $\mathcal{N}$ of $\mathbb{N}$ such that the cubes λ_n, for $n \in \mathcal{N}$, are disjoint and still cover $F \cap V$.

Moreover, for any cube $\lambda \in \Lambda_\delta(V)$, let $\mathcal{N}(\lambda)$ denote the set of all $n \in \mathcal{N}$ such that $\lambda_n \subseteq \lambda$. If $\mathcal{N}(\lambda) \neq \emptyset$, then the cubes λ_n, for $n \in \mathcal{N}(\lambda)$, cover $F \cap \lambda$, so that

$$\sum_{n\in\mathcal{N}(\lambda)} |\lambda_n|^s \geq \mathcal{M}^s_\infty(F \cap \lambda) \geq c\,\mathcal{M}^s_\infty(\lambda) = c\,|\lambda|^s,$$

where the last equality follows from Lemma 5.1. In addition, the sets $\mathcal{N}(\lambda)$ are disjoint. Hence, letting $\mathcal{N}'$ denote the complement of their union in $\mathcal{N}$, we have

$$\sum_{n=1}^{\infty} |\lambda_n|^s \geq \sum_{n\in\mathcal{N}'} |\lambda_n|^s + \sum_{\lambda\in\Lambda_\delta(V)} \sum_{n\in\mathcal{N}(\lambda)} |\lambda_n|^s \geq \sum_{n\in\mathcal{N}'} |\lambda_n|^s + \sum_{\substack{\lambda\in\Lambda_\delta(V)\\ \mathcal{N}(\lambda)\neq\emptyset}} c\,|\lambda|^s. \quad (100)$$

On top of that, let λ denote a cube in $\Lambda_\delta(V)$ for which the index set $\mathcal{N}(\lambda)$ is empty. The intersection $F\cap\lambda$ cannot be empty and is covered by the sets λ_n, for $n\in\mathcal{N}$. Thus, there is an integer $n_0\in\mathcal{N}$ such that the cubes λ and λ_{n_0} intersect. Necessarily, λ is a proper subcube of λ_{n_0}, and the index n_0 belongs to $\mathcal{N}'$. This means that the cubes λ_n, for $n\in\mathcal{N}'$, together with the cubes $\lambda\in\Lambda_\delta(V)$ such that $\mathcal{N}(\lambda)\neq\emptyset$ form a covering of the open set V. Hence, the right-hand side of (100) is bounded below by $c\,\mathcal{M}^s_\infty(V)$, and the result follows. □

LEMMA 5.3. *Let us consider two real numbers $s\in(0,d]$ and $c\in(0,1]$, a subset F of $\mathbb{R}^d$, and an open subset V of $\mathbb{R}^d$. Let us suppose that*

$$\mathcal{M}^s_\infty(F\cap\lambda)\geq c\,\mathcal{M}^s_\infty(\lambda)$$

for all dyadic cubes $\lambda\in\Lambda$ that are contained in V. Then,

$$\mathcal{M}^t_\infty(F\cap\lambda)=\mathcal{M}^t_\infty(\lambda)$$

for all dyadic cubes $\lambda\in\Lambda$ that are contained in V and all real numbers $t\in(0,s)$.

PROOF. Let us consider a dyadic cube λ contained in V with sidelength 2^{-j}, and a dyadic covering $(\lambda_n)_{n\geq 1}$ of the set $F\cap\lambda$. Again, two dyadic cubes are either disjoint or included in one another, so there exists a subset $\mathcal{N}$ of $\mathbb{N}$ such that the cubes λ_n, for $n\in\mathcal{N}$, are disjoint, included in λ, and still cover $F\cap\lambda$. Moreover, let j' denote an integer such that $2^{-(s-t)j'}\leq c\,2^{-(s-t)j}$.

Note that $j'\geq j$, so the cube λ may be written as the union of $2^{j'-j}$ disjoint subcubes with sidelength $2^{-j'}$. Let M denote the collection of these subcubes. As in the proof of Lemma 5.2, for any cube $\mu\in\mathrm{M}$, let $\mathcal{N}(\mu)$ denote the set of all indices $n\in\mathcal{N}$ such that $\lambda_n\subseteq\mu$. In that situation,

$$|\lambda_n|^t\geq|\mu|^{t-s}|\lambda_n|^s=(\kappa'\,2^{-j'})^{t-s}|\lambda_n|^s\geq\frac{1}{c}(\kappa'\,2^{-j})^{t-s}|\lambda_n|^s=\frac{1}{c}|\lambda|^{t-s}|\lambda_n|^s$$

where κ' is the diameter of the unit cube of $\mathbb{R}^d$, as in the proof of Lemma 5.1. Moreover, if $\mathcal{N}(\mu)\neq\emptyset$, then the cubes λ_n, for $n\in\mathcal{N}(\mu)$, cover $F\cap\mu$, so that

$$\begin{aligned}\sum_{n\in\mathcal{N}(\mu)}|\lambda_n|^t &\geq \frac{1}{c}|\lambda|^{t-s}\sum_{n\in\mathcal{N}(\mu)}|\lambda_n|^s\\ &\geq\frac{1}{c}|\lambda|^{t-s}\mathcal{M}^s_\infty(F\cap\mu)\geq|\lambda|^{t-s}\mathcal{M}^s_\infty(\mu)=|\lambda|^{t-s}|\mu|^s,\end{aligned}$$

where the last equality follows again from Lemma 5.1. Furthermore, let $\mathcal{N}'$ denote the complement of the union of the sets $\mathcal{N}(\mu)$ in $\mathcal{N}$. If n belongs to $\mathcal{N}'$, then

$$|\lambda_n|^t\geq|\lambda|^{t-s}|\lambda_n|^s,$$

and λ_n admits a proper subcube $\mu\in\mathrm{M}$. In fact, otherwise, all the cubes in M would be disjoint from λ_n ; this is impossible because λ_n is inside λ, which is covered by the cubes in M.

This means in particular that the cubes λ_n, for $n\in\mathcal{N}'$, along with the cubes $\mu\in\mathrm{M}$ for which $\mathcal{N}(\mu)\neq\emptyset$ form a covering of the cube λ. Hence, using also the disjointness of the index sets $\mathcal{N}(\mu)$, we infer that

$$\sum_{n=1}^{\infty}|\lambda_n|^t \geq \sum_{n\in\mathcal{N}'}|\lambda_n|^t + \sum_{\mu\in\mathrm{M}}\sum_{n\in\mathcal{N}(\mu)}|\lambda_n|^t$$
$$\geq |\lambda|^{t-s}\left(\sum_{n\in\mathcal{N}'}|\lambda_n|^s + \sum_{\substack{\mu\in\mathrm{M}\\ \mathcal{N}(\mu)\neq\emptyset}}|\mu|^s\right) \geq |\lambda|^{t-s}\mathcal{M}^s_\infty(\lambda) = |\lambda|^t.$$

Here again, the last equality follows from Lemma 5.1. We conclude by taking the infimum over all coverings in the left-hand side above. □

LEMMA 5.4. *Let U be a nonempty open subset of $\mathbb{R}^d$ and let f be a bi-Lipschitz mapping from U to $\mathbb{R}^d$ with constant $c_f \geq 1$, see (64). Let us consider two real numbers $s \in (0,d]$ and $c \in (0,1]$ and a subset F of $\mathbb{R}^d$, and suppose that*

$$\mathcal{M}^s_\infty(F\cap V) \geq c\,\mathcal{M}^s_\infty(V)$$

for any open subset V of $\mathbb{R}^d$. Then, for any open subset V of U,

$$\mathcal{M}^s_\infty(f^{-1}(F)\cap V) \geq \frac{c}{(3c_f)^{2d}}\mathcal{M}^s_\infty(V).$$

PROOF. The statement is clearly invariant under a change of norm, so we may assume throughout the proof that the space $\mathbb{R}^d$ is endowed with the supremum norm $|\cdot|_\infty$. Let us begin by observing that a Lipschitz mapping $g : U \to \mathbb{R}^d$ with constant $k \geq 1$ satisfies

$$\mathcal{M}^s_\infty(g(A)) \leq (3k)^d\mathcal{M}^s_\infty(A) \tag{101}$$

for any subset A of U. Indeed, if $(\lambda_n)_{n\geq 1}$ denotes a covering of the set A, then $g(A)$ is covered by the image sets $g(\lambda_n)$, and each of these sets is itself covered by $(\lceil k\rceil + 1)^d$ dyadic cubes with diameter equal to that of the initial cube λ_n.

Consequently, if V denotes an open subset of U, the set $f(V)$ is an open subset of $\mathbb{R}^d$, and we have

$$\mathcal{M}^s_\infty(V) \leq (3c_f)^d\mathcal{M}^s_\infty(f(V))$$
$$\leq \frac{(3c_f)^d}{c}\mathcal{M}^s_\infty(F\cap f(V)) \leq \frac{(3c_f)^{2d}}{c}\mathcal{M}^s_\infty(f^{-1}(F)\cap V),$$

which gives the required estimate. □

LEMMA 5.5. *Let U be a nonempty subset of $\mathbb{R}^d$ and let $s \in (0,d]$. Let us consider a sequence $(F_k)_{k\geq 1}$ of G_δ-subsets of $\mathbb{R}^d$ such that*

$$\mathcal{M}^s_\infty(F_k\cap V) = \mathcal{M}^s_\infty(V)$$

for any $k \geq 1$ and any open subset V of U. Then, for any open subset V of U,

$$\mathcal{M}^s_\infty\left(\bigcap_{k=1}^{\infty}F_k\cap V\right) \geq 3^{-d}\mathcal{M}^s_\infty(V).$$

PROOF. Throughout the proof, when V is an open set and δ is a positive real number, V_δ denotes the *inner δ-parallel body* of V, namely, the open set

$$V_\delta = \{x\in V \mid \mathrm{d}(x,\mathbb{R}^d\setminus V) > \delta\} \tag{102}$$

formed by the points in V at a distance larger than δ from its complement.

Let us first assume that the sets F_k are open and form a nonincreasing sequence. Let V be a bounded open subset of U and let $\varepsilon > 0$. We then define inductively a sequence $(V_k)_{k\geq 0}$ of open subsets of V and a sequence $(\delta_k)_{k\geq 1}$ of positive real numbers by letting $V_0 = V$ and

$$\forall k\geq 1 \qquad V_k = (F_k\cap V_{k-1})_{\delta_k},$$

where the real numbers δ_k are chosen in such a way that

$$\forall k \geq 1 \qquad \mathcal{M}^s_\infty(V_k) > \mathcal{M}^s_\infty(V) - \varepsilon.$$

The existence of δ_k is a consequence of the fact that Proposition 2.4(2) holds for the outer measure $\mathcal{M}^s_\infty$ even if it need not be regular, see [**51**, Theorem 52]. Indeed, the sets $(F_k \cap V_{k-1})_\delta$ are nonincreasing with respect to δ and their union is equal to the whole set $F_k \cap V_{k-1}$, so the previous remark ensures that

$$\lim_{\delta \downarrow 0} \uparrow \mathcal{M}^s_\infty((F_k \cap V_{k-1})_\delta) = \mathcal{M}^s_\infty(F_k \cap V_{k-1}) = \mathcal{M}^s_\infty(V_{k-1}).$$

The last equality follows from the hypothesis on the set F_k. As a consequence, it is possible to choose δ_k appropriately if the set V_{k-1} has been chosen so. Remark that $(\overline{V_k})_{k\geq 1}$ is a nonincreasing sequence of compact subsets of V, and that each compact set $\overline{V_k}$ is contained in the corresponding set F_k.

Let $(\lambda_n)_{n\geq 1}$ denote a covering of the intersection of the compact sets $\overline{V_k}$ by dyadic cubes. We have

$$\bigcap_{k=1}^{\infty} \downarrow \overline{V_k} \subseteq \bigcup_{n=1}^{\infty} \lambda_n \subseteq \bigcup_{n=1}^{\infty} \operatorname{int}(3\lambda_n),$$

where $3\lambda_n$ denotes the union formed by λ_n and the adjacent dyadic cubes. By compactness, there exists an integer $k \geq 1$ such that the set $\overline{V_k}$ is contained in the right-hand side above. Hence, $\overline{V_k}$ is covered by the dyadic cubes that belong to $3\lambda_n$, for $n \geq 1$. We deduce that

$$\mathcal{M}^s_\infty(V) - \varepsilon < \mathcal{M}^s_\infty(\overline{V_k}) \leq \sum_{n=1}^{\infty} 3^d |\lambda_n|^s.$$

Taking the infimum over all dyadic coverings in the right-hand side, we end up with

$$\mathcal{M}^s_\infty(V) - \varepsilon \leq 3^d \mathcal{M}^s_\infty\left(\bigcap_{k=1}^{\infty} \downarrow \overline{V_k}\right) \leq 3^d \mathcal{M}^s_\infty\left(\bigcap_{k=1}^{\infty} F_k \cap V\right).$$

By letting the parameter ε go to zero, we thus settle the case where the sets F_k are open and nonincreasing, and the open set V is bounded.

In order to drop the boundedness assumption on V, one may use the analog of Proposition 2.4(2) for the outer measure $\mathcal{M}^s_\infty$. To get rid of the assumption on the sets F_k, it suffices to observe the intersection of any sequence of G_δ-sets may be written as the intersection of a nonincreasing sequence of open sets. □

5.3.2. Proof of Theorem 5.2. We may now establish the various relationships between the properties (1–6) involved in the statement of Theorem 5.2.

5.3.2.1. *Proof that (1) implies (2).* This follows from the observation that the inverse of a similarity transformation of $\mathbb{R}^d$ is a bi-Lipschitz mapping.

5.3.2.2. *Proof that (2) implies (3).* Arguing by contradiction, we assume that there are two reals $t \in (0, s)$ and $c \in [0, 1)$ and a dyadic cube $\lambda \in \Lambda$ such that

$$\mathcal{M}^t_\infty(F \cap \lambda) < c\, \mathcal{M}^t_\infty(\lambda) = c\, |\lambda|^t.$$

Here again, the last equality is due to Lemma 5.1. As a result, there exists a dyadic covering $(\lambda_n)_{n\geq 1}$ of the intersection set $F \cap \lambda$ for which the total sum of $|\lambda_n|^t$ is smaller than $c\, |\lambda|^t$. Furthermore, there is a subset $\mathcal{N}$ of $\mathbb{N}$ such that the cubes λ_n, for $n \in \mathcal{N}$, are disjoint and included in λ, and still cover $F \cap \lambda$. For any integer $n \in \mathcal{N}$, let ς_n denote the natural affine mapping that sends λ onto λ_n. This is a similarity transformation of $\mathbb{R}^d$ and it is easy to check that for any set $A \subseteq \lambda$,

$$\mathcal{M}^t_\infty(\varsigma_n(A)) \leq \left(\frac{|\lambda_n|}{|\lambda|}\right)^t \mathcal{M}^t_\infty(A). \tag{103}$$

Furthermore, let us consider a point $x \in \lambda$ that belongs to all the image sets $\varsigma_{n_1} \circ \ldots \circ \varsigma_{n_k}$, for any choice $n_1, \ldots, n_k$ of integers in $\mathcal{N}$, and any integer $k \geq 0$. In particular, for $k = 0$, this means that the point x is in $F \cap \lambda$, thereby belonging to some dyadic cube λ_{n_1}, with $n_1 \in \mathcal{N}$. We can then write x in the form $\varsigma_{n_1}(x_1)$ for some $x_1 \in \lambda$. Applying the above hypothesis with $k = 1$, we observe that x_1 necessarily belongs to F as well. Thus, x_1 belongs to some dyadic cube λ_{n_2}, with $n_2 \in \mathcal{N}$. Iterating these arguments, we deduce that there exists a sequence $(n_k)_{k\geq 1}$ of integers in $\mathcal{N}$ and a sequence $(x_k)_{k\geq 0}$ of points in $F \cap \lambda$ such that $x_0 = x$ and $x_{k-1} = \varsigma_{n_k}(x_k)$ for all $k \geq 1$. As a consequence,

$$\bigcap_{k=1}^{\infty} \bigcap_{n_1,\ldots,n_k \in \mathcal{N}} \varsigma_{n_1} \circ \ldots \circ \varsigma_{n_k}(F) \cap \lambda \subseteq \bigcup_{(n_k)_{k\geq 1} \in \mathcal{N}^{\mathbb{N}}} \bigcap_{k=1}^{\infty} \varsigma_{n_1} \circ \ldots \circ \varsigma_{n_k}(F \cap \lambda).$$

On top of that, using the countable subadditivity of the outer measure $\mathcal{M}^t_\infty$ and applying (103) multiple times, we infer that for any integer $k \geq 1$,

$$\begin{aligned}
\mathcal{M}^t_\infty\left(\bigcup_{n_1,\ldots,n_k \in \mathcal{N}} \varsigma_{n_1} \circ \ldots \circ \varsigma_{n_k}(F \cap \lambda)\right) &\leq \sum_{n_1,\ldots,n_k \in \mathcal{N}} \mathcal{M}^t_\infty(\varsigma_{n_1} \circ \ldots \circ \varsigma_{n_k}(F \cap \lambda)) \\
&\leq \sum_{n_1,\ldots,n_k \in \mathcal{N}} \frac{|\lambda_{n_1}|^t \ldots |\lambda_{n_k}|^t}{|\lambda|^{kt}} \mathcal{M}^t_\infty(F \cap \lambda) \\
&= \left(\frac{1}{|\lambda|^t} \sum_{n \in \mathcal{N}} |\lambda_n|^t\right)^k \mathcal{M}^t_\infty(F \cap \lambda) \\
&\leq c^k \mathcal{M}^t_\infty(F \cap \lambda),
\end{aligned}$$

from which we readily deduce that

$$\mathcal{M}^t_\infty\left(\bigcap_{k=1}^{\infty} \bigcap_{n_1,\ldots,n_k \in \mathcal{N}} \varsigma_{n_1} \circ \ldots \circ \varsigma_{n_k}(F) \cap \lambda\right) \leq \inf_{k\geq 1} c^k \mathcal{M}^t_\infty(F \cap \lambda) = 0.$$

Finally, Theorem 2.2 enables us to consider the outer measure $\mathcal{H}^t_\infty = (|\cdot|^t)^*$ defined by (51) and corresponding to coverings by sets of arbitrary diameter. However, it is clear that this outer measure bounds $\mathcal{M}^t_\infty$ from below, so that

$$\mathcal{H}^t_\infty\left(\bigcap_{k=1}^{\infty} \bigcap_{n_1,\ldots,n_k \in \mathcal{N}} \varsigma_{n_1} \circ \ldots \circ \varsigma_{n_k}(F) \cap \lambda\right) = 0.$$

Now, let $(\tau_p)_{p\geq 1}$ denote a sequence of translations for which the image sets $\tau_p(\lambda)$, for $p \geq 1$, form a partition of the whole space $\mathbb{R}^d$. We have for each $p \geq 1$,

$$\mathcal{H}^t_\infty\left(\bigcap_{k=1}^{\infty} \bigcap_{n_1,\ldots,n_k \in \mathcal{N}} \tau_p \circ \varsigma_{n_1} \circ \ldots \circ \varsigma_{n_k}(F) \cap \tau_p(\lambda)\right) = 0,$$

which directly gives

$$\mathcal{H}^t_\infty\left(\bigcap_{p=1}^{\infty} \bigcap_{k=1}^{\infty} \bigcap_{n_1,\ldots,n_k \in \mathcal{N}} \tau_p \circ \varsigma_{n_1} \circ \ldots \circ \varsigma_{n_k}(F)\right) = 0. \tag{104}$$

Note that we can replace the outer measure $\mathcal{H}^t_\infty$ by the Hausdorff t-dimensional measure $\mathcal{H}^t$ in (104). As a matter of fact, any subset A of $\mathbb{R}^d$ may clearly be written as a countable union over $p \geq 1$ of sets A_p with diameter at most a given $\delta > 0$. Let us assume in addition that $\mathcal{H}^t_\infty(A)$ vanishes. Then, for each integer $p \geq 1$, it is clear that $\mathcal{H}^t_\infty(A_p)$ vanishes as well, so that there exists a covering $(C_{p,n})_{n\geq 1}$ of the set A_p with

$$\sum_{n=1}^{\infty} |C_{p,n}|^t \leq \varepsilon\, 2^{-p},$$

where ε is a positive real number fixed in advance. Up to replacing the sets $C_{p,n}$ by their intersection with A_p, we may assume that their diameter is at most δ. Thus, considering the sets $C_{p,n}$ altogether, we obtain a covering of A with sets with diameter at most δ, thereby inferring that

$$\mathcal{H}^t_\delta(A) \leq \sum_{p=1}^{\infty}\sum_{n=1}^{\infty} |C_{p,n}|^t \leq \sum_{p=1}^{\infty} \varepsilon\, 2^{-p} = \varepsilon.$$

Letting δ, and then ε, go to zero, we deduce that the Hausdorff t-dimensional measure of the set A is equal to zero.

We conclude that the set under study in (104) has Hausdorff dimension bounded above by t, and therefore smaller than s. As the mappings $\tau_p \circ \varsigma_{n_1} \circ \ldots \circ \varsigma_{n_k}$ form a countable sequence of similarity transformations, this contradicts (2).

5.3.2.3. *Proof that (3) is equivalent to (4), which implies (5), which itself is equivalent to (6).* This follows straightforwardly from Lemma 5.1, together with the observation that the interior of a dyadic cube λ is an open set with the same $\mathcal{M}^t_\infty$-mass than λ itself, by virtue of Lemma 5.2.

5.3.2.4. *Proof that (6) implies (1) for G_δ-sets.* Let us assume that F is a G_δ-set satisfying (6), and let $(f_n)_{n\geq 1}$ denote a sequence of bi-Lipschitz transformations defined on a nonempty open set U. For each $n \geq 1$, let c_n denote a constant such that f_n satisfies (64). Let t denote a positive real number smaller than s. Lemma 5.4 ensures that for any $t' \in (t, s)$, there is a real number $c \in (0, 1]$ such that for any open subset V of U,

$$\mathcal{M}^{t'}_\infty(f_n^{-1}(F) \cap V) \geq \frac{c}{(3c_n)^{2d}} \mathcal{M}^{t'}_\infty(V).$$

Applying this estimate to the interior of dyadic cubes and making use of Lemma 5.1, we get for every dyadic cube λ contained in U,

$$\begin{aligned}\mathcal{M}^{t'}_\infty(f_n^{-1}(F) \cap \lambda) &\geq \mathcal{M}^{t'}_\infty(f_n^{-1}(F) \cap \operatorname{int}\lambda)\\ &\geq \frac{c}{(3c_n)^{2d}} \mathcal{M}^{t'}_\infty(\operatorname{int}\lambda) = \frac{c}{(3c_n)^{2d}} \mathcal{M}^{t'}_\infty(\lambda).\end{aligned}$$

Then, it follows from Lemma 5.3 that for every dyadic cube λ contained in U,

$$\mathcal{M}^t_\infty(f_n^{-1}(F) \cap \lambda) = \mathcal{M}^t_\infty(\lambda),$$

and Lemma 5.2 now ensures that this also holds when λ is replaced by an arbitrary open subset of U in the above equality. Finally, Lemma 5.5 ensures that

$$\mathcal{M}^t_\infty\left(\bigcap_{n=1}^{\infty} f_n^{-1}(F) \cap U\right) \geq 3^{-d}\mathcal{M}^t_\infty(U) > 0.$$

To conclude, it remains to use (98) to deduce that the intersection of the sets $f_n^{-1}(F)$ has Hausdorff dimension at least t, and to let t tend to s.

5.3.3. Proof of Theorem 5.1. This is a direct consequence of Theorem 5.2. We deal with the stability under countable intersections and that under bi-Lipschitz mappings separately.

5.3.3.1. *Stability under countable intersections.* Let $(F_n)_{n\geq 1}$ denote a sequence of sets in the class $\mathcal{G}^s(\mathbb{R}^d)$. When t is a real number in $(0,s)$, the characterization (4) of this class ensures that all the sets F_n have maximal $\mathcal{M}^t_\infty$-mass in all the open subsets of $\mathbb{R}^d$. Lemma 5.5 implies that

$$\mathcal{M}^t_\infty\left(\bigcap_{n=1}^{\infty} F_n \cap V\right) \geq 3^{-d}\mathcal{M}^t_\infty(V)$$

for any open subset V of $\mathbb{R}^d$, and the characterization (6) shows that the intersection of the sets F_n belongs to the class $\mathcal{G}^s(\mathbb{R}^d)$.

5.3.3.2. *Stability under bi-Lipschitz mappings.* Let F be a set in the class $\mathcal{G}^s(\mathbb{R}^d)$ and let f denote a bi-Lipschitz mapping defined on $\mathbb{R}^d$. Again, when $t\in(0,s)$, the characterization (4) of this class ensures that the set F has maximal $\mathcal{M}^t_\infty$-mass in all the open subsets of $\mathbb{R}^d$. Lemma 5.4 then shows that for any open subset V of $\mathbb{R}^d$,

$$\mathcal{M}^t_\infty(f^{-1}(F)\cap V) \geq \frac{\mathcal{M}^t_\infty(V)}{(3c_f)^{2d}},$$

where c_f is a constant associated with f as in (64). We conclude that $f^{-1}(F)$ is in $\mathcal{G}^s(\mathbb{R}^d)$ thanks to the characterization (6) of this class.

5.3.4. Proof of Theorem 5.3(1). The proof is parallel to that of the stability under countable intersections of the classes $\mathcal{G}^s(\mathbb{R}^d)$ given in Section 5.3.3.1. It suffices to replace the characterization (4) of the class $\mathcal{G}^s(\mathbb{R}^d)$ by the definition of the generalized classes $\mathcal{G}^s(U)$, namely, Definition 5.2. As above, we then apply Lemma 5.5. Finally, we obtain an analog of the characterization (6) of the large intersection classes by applying Lemma 5.3.

5.3.5. Proof of Proposition 5.2. When the open set U is equal to the whole space $\mathbb{R}^d$, the result was obtained by Falconer in [**28**], see Theorem D(b) therein. We thus refer to that paper for the proof in the case where $U=\mathbb{R}^d$, and we content ourselves here with extending Falconer's result to arbitrary nonempty open sets U.

Let us consider a set $F\in\mathcal{G}^s(U)$, a nonempty open set $V\subseteq U$, and an arbitrary nonempty dyadic cube λ_0 contained in V. We write λ_0 in the form $2^{-j_0}(k_0+[0,1)^d)$ with $j_0\in\mathbb{Z}$ and $k_0\in\mathbb{Z}^d$, and we define

$$\widetilde{F} = \bigsqcup_{k\in\mathbb{Z}^d} (k2^{-j_0} + (F\cap \operatorname{int}\lambda_0)).$$

The fact that F is a G_δ-subset of $\mathbb{R}^d$ implies that $\widetilde{F}$ is a G_δ-set as well. Furthermore, for any dyadic cube λ with diameter at most that of λ_0, there exists a unique integer point $k\in\mathbb{Z}^d$ such that λ is contained in $k2^{-j_0}+\lambda_0$, so that

$$\widetilde{F}\cap\lambda = (k2^{-j_0} + (F\cap \operatorname{int}\lambda_0))\cap\lambda.$$

With the help of (101), we deduce that for any $t\in(0,s)$,

$$\begin{aligned}\mathcal{M}^t_\infty(\widetilde{F}\cap\lambda) &\geq 3^{-d}\mathcal{M}^t_\infty(F\cap\operatorname{int}\lambda_0\cap(-k2^{-j_0}+\lambda))\\ &\geq 3^{-d}\mathcal{M}^t_\infty(F\cap\operatorname{int}(-k2^{-j_0}+\lambda))\\ &= 3^{-d}\mathcal{M}^t_\infty(\operatorname{int}(-k2^{-j_0}+\lambda)) = 3^{-d}\mathcal{M}^t_\infty(\lambda).\end{aligned}$$

The last equality is due to Lemma 5.1. The previous one holds because the interior of $-k2^{-j_0}+\lambda$ is an open subset of U, and the set F is in $\mathcal{G}^s(U)$. Finally, Lemmas 5.2 and 5.3 enable us to deduce that $\widetilde{F}\in\mathcal{G}^s(\mathbb{R}^d)$, from which it follows that

$$\dim_{\mathrm{P}}(F\cap V)\geq\dim_{\mathrm{P}}(F\cap\lambda_0)\geq\dim_{\mathrm{P}}(F\cap\operatorname{int}\lambda_0)=\dim_{\mathrm{P}}(\widetilde{F}\cap\operatorname{int}\lambda_0)=d.$$

This results from applying [**28**, Theorem D(b)] to the set with large intersection $\widetilde{F}$ and the open set $\operatorname{int}\lambda_0$, and from the packing counterpart of the monotonicity property satisfied by Hausdorff dimension, see Proposition 2.16(1).

5.4. Connection with ubiquitous systems and application to the Jarník-Besicovitch theorem

We showed in Chapter 4 that if $(x_i, r_i)_{i\in\mathcal{I}}$ denotes a homogeneous ubiquitous system in some nonempty open subset U of $\mathbb{R}^d$, then for any real number $t > 1$, the set F_t defined by (87) has Hausdorff dimension at least d/t in the set U, that is,

$$\dim_{\mathrm{H}}(F_t \cap U) \geq \frac{d}{t},$$

see Theorem 4.1. The purpose of this section is to show that the set F_t belongs to the large intersection classes given by Definition 5.2.

THEOREM 5.4. *Let $(x_i, r_i)_{i\in\mathcal{I}}$ be a homogeneous ubiquitous system in some nonempty open subset U of $\mathbb{R}^d$. Then, for any real number $t > 1$,*

$$F_t \in \mathcal{G}^{d/t}(U).$$

PROOF. As mentioned in Sections 4.4 and 5.1, neither the notion of homogeneous ubiquitous system nor the large intersection classes depend on the choice of the norm. For convenience, we assume throughout the proof that the space $\mathbb{R}^d$ is endowed with the supremum norm; the diameter of a set E is denoted by $|E|_\infty$.

Let us consider two real numbers $\alpha \in (0, 1)$ and $s \in (0, d/t)$, and a nonempty dyadic cube $\lambda \subseteq U$ with diameter at most one. Dilating the closure of λ around its center, we obtain a closed ball B with diameter $\alpha|\lambda|_\infty$ that is contained in the interior of λ. We can reproduce the proof of Theorem 4.1 with U being the interior of λ and $I_\varnothing$ being the ball B. We thus obtain an outer measure μ supported in $F_t \cap \operatorname{int}\lambda$ with total mass given by (92) and such that Proposition 4.3 holds.

Moreover, let $(\lambda_n)_{n\geq 1}$ denote a covering of the set $F_t \cap \operatorname{int}\lambda$ by dyadic cubes. As already observed multiple times, there exists a subset $\mathcal{N}$ of $\mathbb{N}$ such that the cubes λ_n, for $n \in \mathcal{N}$, are disjoint and contained in λ, and still cover $\operatorname{int}\lambda$. If we assume in addition that the latter set has diameter less than $\mathrm{e}^{-d/t}/2$, we see that every cube λ_n with $n \in \mathcal{N}$ is included in a closed ball B_n with radius equal to $|\lambda_n|_\infty$, and thus diameter smaller than $\mathrm{e}^{-d/t}$. Applying Proposition 4.3, we get

$$\mu(\lambda_n) \leq \mu(B_n) \leq 2 \cdot 12^d |B_n|_\infty^{d/t} \log\frac{1}{|B_n|_\infty} \leq 2 \cdot 12^d 2^{d/t} |\lambda_n|_\infty^{d/t} \log\frac{1}{|\lambda_n|_\infty}.$$

Arguing as in the proof of the mass distribution principle, *i.e.* Lemma 2.2, we get

$$\begin{aligned}(\alpha|\lambda|_\infty)^{d/t}\log\frac{1}{\alpha|\lambda|_\infty} &= |I_\varnothing|_\infty^{d/t}\log\frac{1}{|I_\varnothing|_\infty} = \mu(F_t \cap \operatorname{int}\lambda)\\ &\leq 2 \cdot 12^d 2^{d/t} \sum_{n=1}^{\infty} |\lambda_n|_\infty^{d/t} \log\frac{1}{|\lambda_n|_\infty}.\end{aligned}$$

We then use the fact that the function $r \mapsto r^{d/t-s}\log(1/r)$ is nondecreasing near zero. Specifically, if the diameter of λ is less than $\mathrm{e}^{-t/(d-st)}$, we have

$$|\lambda_n|_\infty^{d/t}\log\frac{1}{|\lambda_n|_\infty} = |\lambda_n|_\infty^{s}|\lambda_n|_\infty^{d/t-s}\log\frac{1}{|\lambda_n|_\infty} \leq |\lambda_n|_\infty^{s}|\lambda|_\infty^{d/t-s}\log\frac{1}{|\lambda|_\infty}$$

for all $n \geq 1$. Combining this observation with the previous bound, and then taking the infimum over all dyadic coverings, we obtain

$$\mathcal{M}_\infty^s(F_t \cap \lambda) \geq \mathcal{M}_\infty^s(F_t \cap \operatorname{int}\lambda) \geq \frac{\alpha^{d/t}\log(\alpha|\lambda|_\infty)}{2 \cdot 12^d 2^{d/t}\log|\lambda|_\infty}|\lambda|_\infty^s,$$

with the proviso that the diameter of λ is smaller than $\delta_{s,t}$, defined as the minimum of $\mathrm{e}^{-d/t}/2$ and $\mathrm{e}^{-t/(d-st)}$. Now, thanks to Lemma 5.1, we may replace $|\lambda|_\infty^s$ by $\mathcal{M}_\infty^s(\lambda)$. Hence, letting α tend to one, we end up with

$$\mathcal{M}_\infty^s(F_t \cap \lambda) \geq \frac{\mathcal{M}_\infty^s(\lambda)}{2 \cdot 12^d 2^{d/t}}$$

for any dyadic cube $\lambda \subseteq U$ with diameter smaller than $\delta_{s,t}$. The restriction on the diameter may easily be removed. Indeed, if λ is an arbitrary dyadic cube contained in U, applying Lemma 5.2 to its interior, and then Lemma 5.1 again, we get

$$\mathcal{M}_\infty^s(F_t \cap \lambda) \geq \mathcal{M}_\infty^s(F_t \cap \operatorname{int} \lambda) \geq \frac{\mathcal{M}_\infty^s(\operatorname{int} \lambda)}{2 \cdot 12^d 2^{d/t}} = \frac{\mathcal{M}_\infty^s(\lambda)}{2 \cdot 12^d 2^{d/t}}$$

for all real numbers $s \in (0, d/t)$ and all dyadic cubes $\lambda \subseteq U$. Finally, Lemma 5.3 implies that for all such s and λ, we have in fact

$$\mathcal{M}_\infty^s(F_t \cap \lambda) = \mathcal{M}_\infty^s(\lambda).$$

The result follows from another utilization of Lemma 5.2. □

As an immediate application, let us show that the set $J_{d,\tau}$ defined by (1) is a set with large intersection. Recall that $J_{d,\tau}$ is formed by the points that are approximable at rate at least τ by those with rational coordinates. Moreover, a plain consequence of Dirichlet's theorem implies that this set coincides with the whole space $\mathbb{R}^d$ when $\tau \leq 1 + 1/d$, see Corollary 1.1. We also already established that $J_{d,\tau}$ has Hausdorff dimension $(d+1)/\tau$ in the opposite case where $\tau > 1+1/d$; this follows from the Jarník-Besicovitch theorem discussed in Section 3.1.

We even refined this theorem in Section 4.3 above, starting from the following two observations: the family $(p/q, q^{-1-1/d})_{(p,q)\in\mathbb{Z}^d\times\mathbb{N}}$ is a homogeneous ubiquitous system in the whole space $\mathbb{R}^d$; for this system, the sets F_t defined by (87) coincide with the sets $J_{d,\tau}$, with the proviso that the parameters are such that $t = \tau d/(d+1)$. Thanks to Theorem 5.4, the same observations lead to the following statement.

Corollary 5.2. *For any real number $\tau > 1 + 1/d$, the set $J_{d,\tau}$ belongs to the class $\mathcal{G}^{(d+1)/\tau}(\mathbb{R}^d)$,* i.e. *is a set with large intersection in the whole space $\mathbb{R}^d$ with dimension at least $(d + 1)/\tau$.*

This result was already obtained by Falconer [**28**]. Combined with Proposition 5.2, this shows in particular that the set $J_{d,\tau}$ has packing dimension equal to d in every nonempty open subset of $\mathbb{R}^d$. For the sake of completeness, let us point out that in the opposite case where $\tau \leq 1 + 1/d$, the set $J_{d,\tau}$ clearly belongs to the class $\mathcal{G}^d(\mathbb{R}^d)$ because it coincides with the whole space $\mathbb{R}^d$ itself.

CHAPTER 6

Eutaxic sequences

The notion of eutaxic sequence was introduced by Lesca [**43**] and subsequently studied by Reversat [**49**]. It provides a nice setting to the study of Diophantine approximation properties, and we shall indeed use it in this chapter to analyze the approximation by fractional parts of sequences and by random sequences of points. With this notion, the emphasis is put on the sequence $(x_n)_{n\geq 1}$ of approximating points in $\mathbb{R}^d$, and one is ultimately interested in its uniform approximation behavior with respect to *all* possible sequences $(r_n)_{n\geq 1}$ of approximation radii.

Let us assume that the series $\sum_n r_n^d$ converges. It is clear that the set of all points $x \in \mathbb{R}^d$ for which

$$\exists \text{ i.m. } n \geq 1 \qquad |x - x_n| < r_n \tag{105}$$

has Lebesgue measure zero; this may indeed be deduced from applying Lemma 2.1 with the gauge function $r \mapsto r^d$, which essentially yields the Lebesgue measure, in view of Proposition 2.14. Note that in that situation, we may rearrange the points in such a way that the sequence $(r_n)_{n\geq 1}$ is nonincreasing and converges to zero. Now, eutaxy comes into play when one assumes that the series $\sum_n r_n^d$ is divergent, or equivalently that $(r_n)_{n\geq 1}$ belongs to the collection P_d of real sequences that is defined by the following condition:

$$(r_n)_{n\geq 1} \in \mathrm{P}_d \qquad \Longleftrightarrow \qquad \begin{cases} \forall n \geq 1 \quad r_{n+1} \leq r_n \\ \lim\limits_{n\to\infty} r_n = 0 \\ \sum\limits_{n=1}^{\infty} r_n^d = \infty. \end{cases} \tag{106}$$

As detailed hereunder, eutaxy will occur when (105) is satisfied by Lebesgue-almost every point of some open set of interest.

6.1. Definition and link with approximation

6.1.1. Sequencewise eutaxy. The simplest notion of eutaxy is obtained when specifying a sequence $(r_n)_{n\geq 1}$ in P_d and deciding on whether or not Lebesgue-almost every point may be approximated within distance r_n by some sequence of points x_n under consideration.

Definition 6.1. Let U be a nonempty open subset of $\mathbb{R}^d$, and let $(r_n)_{n\geq 1}$ be a sequence in P_d. A sequence $(x_n)_{n\geq 1}$ of points in $\mathbb{R}^d$ is called *eutaxic in U with respect to* $(r_n)_{n\geq 1}$ if the following condition holds:

$$\text{for } \mathcal{L}^d\text{-a.e. } x \in U \quad \exists \text{ i.m. } n \geq 1 \qquad |x - x_n| < r_n.$$

The notion of eutaxic sequence is naturally connected with those of approximation system and homogeneous ubiquitous system introduced by Definitions 4.1 and 4.2, respectively. However, the idea here is to restrict to families indexed by the positive integers, and to put a stress on the points x_n rather, to ultimately obtain uniform properties with respect to the sequence of radii r_n, see Section 6.1.2. The connection between the various notions is formalized by the next statement. We omit its proof because the result readily follows from the definitions of the various involved notions, namely, Definitions 4.1, 4.2 and 6.1.

PROPOSITION 6.1. *Let U be a nonempty open subset of $\mathbb{R}^d$, let $(r_n)_{n\geq 1}$ be a sequence in P_d, and let $(x_n)_{n\geq 1}$ be a sequence of points in $\mathbb{R}^d$. Then,*

(1) *the family $(x_n, r_n)_{n\geq 1}$ is an approximation system;*
(2) *the family $(x_n, r_n)_{n\geq 1}$ is a homogeneous ubiquitous system in U if and only if the sequence $(x_n)_{n\geq 1}$ is eutaxic in U with respect to $(r_n)_{n\geq 1}$.*

Combining Proposition 6.1 with Proposition 4.4, we easily observe that a sequence $(x_n)_{n\geq 1}$ is eutaxic with respect to $(r_n)_{n\geq 1}$ if and only if it is eutaxic with respect to $(c\,r_n)_{n\geq 1}$, for any fixed real number $c > 0$. Thus, the fact that a sequence is eutaxic does not depend on the choice of the norm on the space $\mathbb{R}^d$.

Besides, Proposition 6.1 invites us to consider the problem of the approximation within distances r_n by the points x_n. Accordingly, the sets F_t defined by (87) in the general setting are now given by

$$F_t = \left\{ x \in \mathbb{R}^d \;\middle|\; |x - x_n| < r_n^t \quad \text{for i.m. } n \geq 1 \right\}, \tag{107}$$

and their size and large intersection properties may be studied by specializing the results of Chapters 4 and 5. This results in the next statement.

THEOREM 6.1. *Let $(x_n)_{n\geq 1}$ be a sequence of points in $\mathbb{R}^d$ that is eutaxic in some nonempty open subset U of $\mathbb{R}^d$, with respect to some sequence $(r_n)_{n\geq 1}$ in P_d. We assume further that the series $\sum_n r_n^s$ is convergent for all $s > d$. Then, for any real number $t \geq 1$,*

$$\dim_{\mathrm{H}}(F_t \cap U) = \frac{d}{t} \qquad \textit{and} \qquad F_t \in \mathcal{G}^{d/t}(U).$$

PROOF. The convergence assumption on the series $\sum_n r_n^s$ implies that the parameter s_U defined by (90) is bounded above by d regardless of the choice of the open set U. Moreover, the family $(x_n, r_n)_{n\geq 1}$ is a homogeneous ubiquitous system in U, by virtue of Proposition 6.1. Therefore, we may apply Corollary 4.1, and deduce that the set $F_t \cap U$ has Hausdorff dimension equal to d/t for any real number $t > 1$. For the same reason, due to Theorem 5.4, the set F_t belongs to the large intersection class $\mathcal{G}^{d/t}(U)$. Finally, the result clearly holds for $t = 1$, because the set F_1 has full Lebesgue measure in U. □

6.1.2. Uniform eutaxy. Rather than the sequencewise, the notion of uniform eutaxy is the one that was introduced by Lesca [**43**] and subsequently studied by Reversat [**49**]. Uniform eutaxy is obtained when sequencewise eutaxy holds regardless of the choice of the sequence $(r_n)_{n\geq 1}$ in the collection P_d.

DEFINITION 6.2. Let U be a nonempty open subset of $\mathbb{R}^d$. A sequence $(x_n)_{n\geq 1}$ of points in $\mathbb{R}^d$ is called *uniformly eutaxic in U* if the following condition holds:

$$\forall (r_n)_{n\geq 1} \in \mathrm{P}_d \quad \text{for } \mathcal{L}^d\text{-a.e. } x \in U \quad \exists \text{ i.m. } n \geq 1 \qquad |x - x_n| < r_n.$$

As regards the aforementioned approximation problem, we may improve Theorem 6.1 when the eutaxy property of the underlying sequence $(x_n)_{n\geq 1}$ is uniform. Specifically, as shown by the next result, we may slightly relax the condition on the sequence $(r_n)_{n\geq 1}$ that comes into play in the definition (107) of the sets F_t, and obtain the same size and large intersection properties.

THEOREM 6.2. *Let $(x_n)_{n\geq 1}$ be a sequence of points in $\mathbb{R}^d$ that is uniformly eutaxic in some nonempty open subset U of $\mathbb{R}^d$, and let $(r_n)_{n\geq 1}$ be a nonincreasing sequence of positive real numbers such that*

$$\begin{cases} s < d \implies \sum_n r_n^s = \infty \\ s > d \implies \sum_n r_n^s < \infty. \end{cases} \tag{108}$$

Then, for any real number $t \geq 1$,

$$\dim_{\mathrm{H}}(F_t \cap U) = \frac{d}{t} \qquad \textit{and} \qquad F_t \in \mathcal{G}^{d/t}(U).$$

PROOF. The proof is an adaptation of that of Theorem 6.1. Again, due to the convergence assumption on the series, the parameter s_U defined by (90) is bounded above by d. The upper bound on the Hausdorff dimension then follows directly from Proposition 4.1. Furthermore, for any $s \in (0, d)$, the sequence $(r_n^{s/d})_{n\geq 1}$ belongs to P_d, so Proposition 6.1 implies that $(x_n, r_n^{s/d})_{n\geq 1}$ is a homogeneous ubiquitous system in U. Therefore, for any $t \geq 1$, we may apply Theorems 4.1 and 5.4 with the approximation radii raised to the power $dt/s > 1$ instead of t, thereby obtaining

$$\dim_{\mathrm{H}}(F_t \cap U) \geq \frac{s}{t} \qquad \text{and} \qquad F_t \in \mathcal{G}^{s/t}(U).$$

The required lower bound on the Hausdorff dimension clearly follows from letting s tend to d. The large intersection property follows the fact that the class $\mathcal{G}^{d/t}(U)$ is the intersection over $s \in (0, d)$ of the classes $\mathcal{G}^{s/t}(U)$, see Definition 5.2. □

It is clear that Theorem 6.2 may be extended to a wider range of sequences of approximating radii than those satisfying (108). More precisely, let us consider a nonincreasing sequence $\mathrm{r} = (r_n)_{n\geq 1}$ of positive real numbers such that

$$\begin{cases} s < s_{\mathrm{r}} \implies \sum_n r_n^s = \infty \\ s > s_{\mathrm{r}} \implies \sum_n r_n^s < \infty. \end{cases} \tag{109}$$

for some positive real number s_{r}. We may thus apply Theorem 6.2 with the sequence $(r_n^{s_{\mathrm{r}}/d})_{n\geq 1}$, because it satisfies (108). Performing the appropriate change of variable, we deduce a description of the size and large intersection properties of the sets F_t corresponding to the original sequence $\mathrm{r} = (r_n)_{n\geq 1}$; specifically,

$$\dim_{\mathrm{H}}(F_t \cap U) = \frac{s_{\mathrm{r}}}{t} \qquad \text{and} \qquad F_t \in \mathcal{G}^{s_{\mathrm{r}}/t}(U)$$

for any real $t \geq s_{\mathrm{r}}/d$. Besides, note that all the sets F_t, for $t < s_{\mathrm{r}}/d$, have Hausdorff dimension d and belong to the class $\mathcal{G}^d(U)$, because they contain $F_{s_{\mathrm{r}}/d}$.

6.2. Criteria for uniform eutaxy

6.2.1. A sufficient condition for uniform eutaxy. We now establish a criterion implying the uniform eutaxy of a sequence of points. This criterion is expressed in terms of the dyadic cubes of $\mathbb{R}^d$. Let us recall from Section 2.6.3 that a dyadic cube is either the empty set or a set of the form

$$\lambda = 2^{-j}(k + [0, 1)^d),$$

with $j \in \mathbb{Z}$ and $k \in \mathbb{Z}^d$, and that the collection of all dyadic cubes is denoted by Λ. Moreover, the generation of such a dyadic cube λ, *i.e.* the integer j, is denoted by $\langle\lambda\rangle$. Finally, for any point $x \in \mathbb{R}^d$ and any integer $j \in \mathbb{Z}$, there exists a unique dyadic cube with sidelength 2^{-j} that contains x ; this cube is denoted by $\lambda_j(x)$.

Let us now fix a sequence $(x_n)_{n\geq 1}$ of points in $\mathbb{R}^d$. For any nonempty dyadic cube $\lambda \in \Lambda$ and any integer $j \geq 0$, let us define a collection $\mathrm{M}((x_n)_{n\geq 1}; \lambda, j)$ of dyadic cubes by the following condition:

$$\lambda' \in \mathrm{M}((x_n)_{n\geq 1}; \lambda, j) \qquad \Longleftrightarrow \qquad \begin{cases} \lambda' \subseteq \lambda \\ \langle\lambda'\rangle = \langle\lambda\rangle + j \\ x_n \in \lambda' \text{ for some } n \leq 2^{d\langle\lambda'\rangle}. \end{cases}$$

It will be clear from the context what underlying sequence $(x_n)_{n\geq 1}$ is considered, and there should be no confusion if we decide to write $\mathrm{M}(\lambda, j)$ as a shorthand for

$\mathrm{M}((x_n)_{n\geq 1};\lambda,j)$. It is obvious that the cardinality of the set $\mathrm{M}(\lambda,j)$ is bounded above by 2^{dj}. When it is bounded below by a fraction of 2^{dj}, the sequence $(x_n)_{n\geq 1}$ is uniformly eutaxic, as shown by the following criterion.

THEOREM 6.3. *Let U be a nonempty open subset of $\mathbb{R}^d$ and let $(x_n)_{n\geq 1}$ be a sequence of points in $\mathbb{R}^d$. Let us assume that*

$$\text{for } \mathcal{L}^d\text{-a.e. } x\in U \qquad \liminf_{j_0,j\to\infty} 2^{-dj}\#\mathrm{M}((x_n)_{n\geq 1};\lambda_{j_0}(x),j)>0. \tag{110}$$

Then, the sequence $(x_n)_{n\geq 1}$ is uniformly eutaxic in U.

The remainder of this section is devoted to the proof of Theorem 6.3. It relies on the next useful measure-theoretic lemma that is excerpted from Sprindžuk's book [**59**] and that we establish first.

LEMMA 6.1. *Let μ be an outer measure on $\mathbb{R}^d$ such that $\mu(\mathbb{R}^d)$ is finite, and let $(E_n)_{n\geq 1}$ be a sequence of μ-measurable sets such that*

$$\sum_{n=1}^{\infty}\mu(E_n)=\infty. \tag{111}$$

Then, the set of points that belong to infinitely many sets E_n satisfies

$$\mu\left(\limsup_{n\to\infty} E_n\right)\geq \limsup_{N\to\infty}\frac{\left(\sum\limits_{n=1}^{N}\mu(E_n)\right)^2}{\sum\limits_{m=1}^{N}\sum\limits_{n=1}^{N}\mu(E_m\cap E_n)}.$$

PROOF. We begin by writing the limsup set under examination in the form

$$\limsup_{n\to\infty}E_n=\bigcap_{M=1}^{\infty}\downarrow\bigcup_{n=M}^{\infty}E_n.$$

Letting F_M^N denote the union of the sets E_n over all integers $n\in\{M,\ldots,N\}$, and using Proposition 2.5, we deduce that

$$\mu\left(\limsup_{n\to\infty}E_n\right)\geq \lim_{M\to\infty}\downarrow\lim_{N\to\infty}\uparrow\mu(F_M^N).$$

The μ-mass of the union set F_M^N may be estimated thanks to the second-moment method. To be specific, the Cauchy-Schwarz inequality gives

$$\left(\int_{\mathbb{R}^d}\mathbf{1}_{F_M^N}(y)\sum_{n=M}^{N}\mathbf{1}_{E_n}(y)\,\mu(\mathrm{d}y)\right)^2\leq\mu(F_M^N)\int_{\mathbb{R}^d}\left(\sum_{n=M}^{N}\mathbf{1}_{E_n}(y)\right)^2\mu(\mathrm{d}y).$$

The left-hand side above is clearly equal to the square of the sum over all integers $n\in\{M,\ldots,N\}$ of the μ-masses of the sets E_n, and is therefore equivalent to

$$\left(\sum_{n=1}^{N}\mu(E_n)\right)^2$$

as N goes to infinity and M remains fixed, due to (111). Likewise, the integral in the right-hand side coincides with the sum over all integers $m,n\in\{M,\ldots,N\}$ of the μ-masses of the sets $E_m\cap E_n$, which is equal to

$$\sum_{m=1}^{N}\sum_{n=1}^{N}\mu(E_m\cap E_n)+\mathrm{O}\left(\sum_{n=1}^{N}\mu(E_n)\right).$$

The result follows from combining all the previous estimates, and using (111) again in order to get rid of the remainder term above. □

We are now in position to detail the proof of Theorem 6.3. The fact that a sequence is uniformly eutaxic clearly does not depend on the choice of the norm; we thus assume throughout the proof that $\mathbb{R}^d$ is equipped with the supremum norm. Let us consider a nonempty open subset U of $\mathbb{R}^d$ and a sequence $(x_n)_{n\geq 1}$ of points in $\mathbb{R}^d$ such that (110) holds for Lebesgue-almost every $x \in U$. Our goal is to establish that for any sequence $(r_n)_{n\geq 1}$ chosen in advance in P_d, the set F_1, *i.e.* the set F_t obtained by choosing $t = 1$ in (107), has full Lebesgue measure in U. To proceed, let U_* denote the set of all points x in U such that (110) holds and none of the coordinates of x is a dyadic number. Then, U_* has full Lebesgue measure in U. Furthermore, for any $x \in U_*$, there exist a real number $\alpha(x) > 0$ and an integer $\underline{j}(x) \geq 0$ such that

$$\forall j_0, j \geq \underline{j}(x) \qquad \#\mathrm{M}(\lambda_{j_0}(x), j) \geq \alpha(x)\, 2^{dj}.$$

The proof now reduces to showing that there is a real number $\kappa > 0$ such that

$$\forall j_0 \geq \underline{j}(x) \qquad \mathcal{L}^d(F_1 \cap \lambda_{j_0}(x)) \geq \kappa\, \alpha(x)^2 \mathcal{L}^d(\lambda_{j_0}(x)). \tag{112}$$

Indeed, (112) implies that the density of the set F_1 at the point x is positive. Therefore, if this holds for any x in U_*, then the Lebesgue density theorem shows that Lebesgue-almost every point of U_* belongs to F_1, see [**46**, Corollary 2.14]. As a result, F_1 has full Lebesgue measure in U.

It now remains to show that any point x in U_* satisfies (112). For any fixed integer $j_0 \geq \underline{j}(x)$, we begin by observing that for any integer $j \geq \underline{j}(x)$, there exists a set $S_j(x, j_0) \subseteq \{1, \ldots, 2^{d(j_0+j)}\}$ with:

- $\#S_j(x, j_0) \geq \alpha(x)\, 2^{d(j-1)}$;
- $x_n \in \lambda_{j_0}(x)$ for any $n \in S_j(x, j_0)$;
- $|x_n - x_{n'}|_\infty \geq 2^{-(j_0+j)}$ for any distinct $n, n' \in S_j(x, j_0)$.

Indeed, for each $\beta \in \{0,1\}^d$, let us consider the cubes in $\mathrm{M}(\lambda_{j_0}(x), j)$ of the form $2^{-(j_0+j)}(k + [0,1)^d)$, where the coordinates of k are equal to those of β modulo two. For a suitable β, there are at least $2^{-d}\, \#\mathrm{M}(\lambda_{j_0}(x), j)$ such cubes. The result then follows from the observation that these cubes are at a distance at least $2^{-(j_0+j)}$ of each other and that each cube contains at least a point x_n with $n \leq 2^{d(j_0+j)}$.

Then, let us define $\widetilde{r}_n = \min\{r_n, 1/(2n^{1/d})\}$ for each $n \geq 1$. We thereby obtain another sequence $(\widetilde{r}_n)_{n\geq 1}$ in P_d. Indeed, otherwise, the sequence $(\widetilde{r}_n^d)_{n\geq 1}$ would be nonincreasing and have a finite sum, so that $n\widetilde{r}_n^d$ would tend to zero as n goes to infinity. Thus, $\widetilde{r}_n$ would be equal to r_n for n large enough and the series $\sum_n r_n^d$ would converge, contradicting the assumption that $(r_n)_{n\geq 1}$ belongs to P_d. Now, for any integer $j \geq \underline{j}(x)$, let us consider the set

$$V_j(x, j_0) = \bigcup_{n \in S_j(x, j_0)} \mathrm{B}_\infty(x_n, \rho_{j_0+j}),$$

where ρ_j is a shorthand for $\widetilde{r}_{2^{dj}}$. Since the sequence $(\widetilde{r}_n)_{n\geq 1}$ is nonincreasing and converges to zero, all the points in the limsup of these sets, except maybe those forming the sequence $(x_n)_{n\geq 1}$, belong to both the closure of $\lambda_{j_0}(x)$ and the set $\widetilde{F}_1$ obtained by replacing r_n by $\widetilde{r}_n$ in the definition of F_1. Therefore,

$$\mathcal{L}^d\left(\limsup_{j\to\infty} V_j(x, j_0)\right) \leq \mathcal{L}^d\left(\widetilde{F}_1 \cap \overline{\lambda_{j_0}(x)}\right) \leq \mathcal{L}^d(F_1 \cap \lambda_{j_0}(x)).$$

Hence, to obtain (112), it suffices to provide an appropriate lower bound on the Lebesgue measure of the limsup of the sets $V_j(x, j_0)$. This may be done with the help of Lemma 6.1. In fact, the sets $V_j(x, j_0)$ are all contained in the closure of the cube $\lambda_{j_0}(x)$, so that we may apply this lemma with the restriction of the Lebesgue measure to this closed cube. The resulting lower bound yields

$$\mathcal{L}^d(F_1 \cap \lambda_{j_0}(x)) \geq \limsup_{J\to\infty} \frac{\left(\sum\limits_{j=\underline{j}(x)}^{J} \mathcal{L}^d(V_j(x,j_0))\right)^2}{\sum\limits_{j=\underline{j}(x)}^{J}\sum\limits_{j'=\underline{j}(x)}^{J} \mathcal{L}^d(V_j(x,j_0)\cap V_{j'}(x,j_0))}. \tag{113}$$

However, we need to make sure that Lemma 6.1 may be applied, *i.e.* we need to check the divergence condition

$$\sum_{j=\underline{j}(x)}^{\infty} \mathcal{L}^d(V_j(x,j_0)) = \infty. \tag{114}$$

To this end, we observe that for any $j \geq \underline{j}(x)$ and any distinct n and n' in $S_j(x,j_0)$, the two open balls with common radius ρ_{j_0+j} and center x_n and $x_{n'}$, respectively, are disjoint. Otherwise, any point y in their intersection would satisfy

$$|x_n - x_{n'}|_\infty \leq |y-x_n|_\infty + |y-x_{n'}|_\infty < 2\rho_{j_0+j} \leq 2^{-(j_0+j)},$$

which would contradict the third property of the set $S_j(x,j_0)$ given above. As a result, the balls forming the set $V_j(x,j_0)$ are disjoint, so that

$$\mathcal{L}^d\left(V_j(x,j_0)\right) = (2\rho_{j_0+j})^d\,\#S_j(x,j_0) \geq \alpha(x)\,2^{dj}\,\rho_{j_0+j}^d. \tag{115}$$

In order to derive (114), we finally use the fact that the sequence $(\widetilde{r}_n)_{n\geq 1}$ is nonincreasing, as this enables us to write that

$$2^{dj_0}(2^d-1)\sum_{j=\underline{j}(x)}^{\infty} 2^{dj}\,\rho_{j_0+j}^d \geq \sum_{j=j_0+\underline{j}(x)}^{\infty}\ \sum_{n=2^{dj}}^{2^{d(j+1)}-1} \widetilde{r}_n^{\,d} = \infty. \tag{116}$$

To obtain (112), and thus complete the proof, it suffices to combine the lower bound (113) with the following inequality that holds for any integer J sufficiently large and that we now establish:

$$\sum_{j=\underline{j}(x)}^{J}\sum_{j'=\underline{j}(x)}^{J} \mathcal{L}^d\left(V_j(x,j_0)\cap V_{j'}(x,j_0)\right) \leq \frac{2^{d(j_0+4)}}{\alpha(x)^2}\left(\sum_{j=\underline{j}(x)}^{J}\mathcal{L}^d\left(V_j(x,j_0)\right)\right)^2. \tag{117}$$

Let us consider two integers j and j' such that $\underline{j}(x) \leq j < j'$. With a view to giving an upper bound on the Lebesgue measure of the intersection of the sets $V_j(x,j_0)$ and $V_{j'}(x,j_0)$, let us observe that for any integer $n \in S_j(x,j_0)$,

$$\mathrm{B}_\infty(x_n,\rho_{j_0+j})\cap V_{j'}(x,j_0) = \bigcup_{n'\in S_{j'}(x,j_0)} \left(\mathrm{B}_\infty(x_n,\rho_{j_0+j})\cap \mathrm{B}_\infty(x_{n'},\rho_{j_0+j'})\right).$$

The points $x_{n'}$, with $n' \in S_{j'}(x,j_0)$ such that this last intersection is nonempty, all lie in the open ball with center x_n and radius $2\rho_{j_0+j}$. Moreover, there are at most $(2^{j_0+j'+2}\rho_{j_0+j}+2)^d$ cubes with generation j_0+j' that intersect this ball and each of them contains at most one of the points $x_{n'}$. Thus,

$$\begin{aligned}\mathcal{L}^d(\mathrm{B}_\infty(x_n,\rho_{j_0+j})\cap V_{j'}(x,j_0)) &\leq (2^{j_0+j'+2}\rho_{j_0+j}+2)^d(2\rho_{j_0+j'})^d\\ &\leq 2^{3d-1}\rho_{j_0+j'}^d(1+2^{d(j_0+j'+1)}\rho_{j_0+j}^d).\end{aligned}$$

Along with the fact that there are at most 2^{dj} integers in $S_j(x,j_0)$, this yields

$$\mathcal{L}^d\left(V_j(x,j_0)\cap V_{j'}(x,j_0)\right) \leq 2^{d(j+3)-1}\rho_{j_0+j'}^d(1+2^{d(j_0+j'+1)}\rho_{j_0+j}^d).$$

As a consequence, for any integer $J \geq \underline{j}(x)$, the left-hand side of (117) is at most

$$2^d \sum_{j=\underline{j}(x)}^{J} 2^{dj} \rho_{j_0+j}^d + 2^{3d} \sum_{j,j'} 2^{dj} \rho_{j_0+j'}^d + 2^{4d} \sum_{j,j'} 2^{d(j_0+j+j')} \rho_{j_0+j}^d \rho_{j_0+j'}^d,$$

where the second and third sums are both over the integers j and j' that satisfy $\underline{j}(x) \leq j < j' \leq J$. Note that the second sum is equal to

$$\sum_{j'=\underline{j}(x)+1}^{J} 2^{dj'} \rho_{j_0+j'}^d \sum_{j=\underline{j}(x)}^{j'-1} 2^{d(j-j')} \leq \frac{1}{2^d - 1} \sum_{j'=\underline{j}(x)+1}^{J} 2^{dj'} \rho_{j_0+j'}^d,$$

and the third sum is obviously smaller than half the sum bearing on all the integers j and j' between $\underline{j}(x)$ and J. Thus, the left-hand side of (117) is at most

$$\left(2^d + \frac{2^{3d}}{2^d - 1}\right) 2^{-dj_0} \sum_{j=\underline{j}(x)}^{J} 2^{d(j_0+j)} \rho_{j_0+j}^d + 2^{4d-1} 2^{-dj_0} \left(\sum_{j=\underline{j}(x)}^{J} 2^{d(j_0+j)} \rho_{j_0+j}^d\right)^2.$$

In view of (116), the first sum tends to infinity as $J \to \infty$, thereby being larger than one, and thus smaller than its square, for J large enough. The left-hand side of (117) is therefore bounded above by

$$2^{-d(j_0-4)} \left(\sum_{j=\underline{j}(x)}^{J} 2^{d(j_0+j)} \rho_{j_0+j}^d\right)^2,$$

for any integer J sufficiently large, and this bound leads to the right-hand side of (117) with the help of (115). The proof of Theorem 6.3 is complete.

6.2.2. A necessary condition for uniform eutaxy. It is not known whether Theorem 6.3 also yields a necessary condition for uniform eutaxy. However, note that the sufficient condition (110) clearly holds if

$$\inf_{\substack{\lambda \in \Lambda \setminus \{\emptyset\} \\ \lambda \subseteq U}} \liminf_{j \to \infty} 2^{-dj} \#\mathrm{M}((x_n)_{n \geq 1}; \lambda, j) > 0. \tag{118}$$

Moreover, it is plain that this stronger assumption fails when the liminf vanishes for some nonempty dyadic cube λ. The next result shows that, in this situation, the sequence under consideration cannot be uniformly eutaxic.

THEOREM 6.4. *Let U be a nonempty open subset of $\mathbb{R}^d$ and let $(x_n)_{n \geq 1}$ be a sequence of points in $\mathbb{R}^d$. Let us assume that*

$$\exists \lambda \in \Lambda \setminus \{\emptyset\} \qquad \begin{cases} \lambda \subseteq U \\ \liminf_{j \to \infty} 2^{-dj} \#\mathrm{M}((x_n)_{n \geq 1}; \lambda, j) = 0. \end{cases}$$

Then, the sequence $(x_n)_{n \geq 1}$ is not uniformly eutaxic in U.

PROOF. As in the proof of Theorem 6.3, we endow $\mathbb{R}^d$ with the supremum norm. Let us consider an integer $j \geq 0$ and, on the one hand, let us define the set

$$U_j = \bigcup_{\substack{n \leq 2^{d(\langle\lambda\rangle+j)} \\ x_n \in \lambda}} \mathrm{B}_\infty(x_n, 2^{-(\langle\lambda\rangle+j)}). \tag{119}$$

If λ' is a nonempty dyadic subcube of λ, let $\widetilde{\lambda}'$ stand for the open cube concentric with λ' with triple sidelength. If moreover λ' has generation $\langle\lambda\rangle + j$ and contains some point x_n, then $\widetilde{\lambda}'$ contains the ball in (119) that is centered at this x_n. Hence,

$$U_j = \bigcup_{\substack{\lambda' \subseteq \lambda \\ \langle \lambda' \rangle = \langle \lambda \rangle + j}} \bigcup_{\substack{n \leq 2^{d\langle \lambda' \rangle} \\ x_n \in \lambda'}} \mathrm{B}_\infty(x_n, 2^{-\langle \lambda' \rangle}) \subseteq \bigcup_{\lambda' \in \mathrm{M}(\lambda, j)} \widetilde{\lambda}',$$

from which it directly follows that

$$\mathcal{L}^d(U_j) \leq 3^d 2^{-d(\langle \lambda \rangle + j)} \# \mathrm{M}(\lambda, j).$$

On the other hand, let us consider the set U'_j obtained by replacing in (119) the condition $x_n \in \lambda$ by the conjunction of the fact that $x_n \notin \lambda$ and that the open ball with center x_n and radius $2^{-(\langle \lambda \rangle + j)}$ meets the cube λ. In that case, the ball actually meets the boundary of the cube λ. This means that each point of U'_j is within distance $2^{1-(\langle \lambda \rangle + j)}$ from this boundary, and thus

$$\begin{aligned} \mathcal{L}^d(U'_j) &\leq (2^{-\langle \lambda \rangle} + 2^{2-(\langle \lambda \rangle + j)})^d - (2^{-\langle \lambda \rangle} - 2^{2-(\langle \lambda \rangle + j)})^d \\ &\leq 2^{3-d\langle \lambda \rangle - j} \sum_{\ell=0}^{d-1} (1 + 2^{2-j})^{d-1-\ell} (1 - 2^{2-j})^\ell \leq 5^d 2^{3-d\langle \lambda \rangle - j}, \end{aligned}$$

with the proviso that $j \geq 2$. As a consequence, summing the two above upper bounds and letting j go to infinity, we deduce that

$$\liminf_{j \to \infty} \mathcal{L}^d \left(\lambda \cap \bigcup_{n=1}^{2^{d(\langle \lambda \rangle + j)}} \mathrm{B}_\infty(x_n, 2^{-(\langle \lambda \rangle + j)}) \right) \leq 3^d 2^{-d\langle \lambda \rangle} \liminf_{j \to \infty} 2^{-dj} \# \mathrm{M}(\lambda, j),$$

because the set in the left-hand side is contained in the union of U_j and U'_j.

We now make use of the assumption bearing on the cube λ, namely, that the lower limit in the right-hand side vanishes. Thus, we may find an increasing sequence $(j_m)_{m \geq 1}$ of nonnegative integers such that $j_1 = 0$ and for all $m \geq 1$,

$$\mathcal{L}^d \left(\lambda \cap \bigcup_{n=2^{d(\langle \lambda \rangle + j_m)}+1}^{2^{d(\langle \lambda \rangle + j_{m+1})}} \mathrm{B}_\infty(x_n, 2^{-(\langle \lambda \rangle + j_{m+1})}) \right) \leq 2^{-m}.$$

For simplicity, we define $n_m = 2^{d(\langle \lambda \rangle + j_m)}$ for all $m \geq 1$, and also $n_0 = 0$. We then consider the unique sequence $(r_n)_{n \geq 1}$ such that

$$\forall m \geq 0 \quad \forall n \in \{n_m + 1, \ldots, n_{m+1}\} \qquad r_n = n_{m+1}^{-1/d}.$$

Clearly, this sequence is nonincreasing and converges to zero. Moreover, for any integer $m \geq 0$,

$$\sum_{n=n_m+1}^{n_{m+1}} r_n^d = 1 - \frac{n_m}{n_{m+1}} \geq 1 - 2^{-d},$$

so that the series $\sum_n r_n^d$ is divergent. We may therefore conclude that the sequence $(r_n)_{n \geq 1}$ belongs to the collection P_d.

On top of that, for any integer $\underline{m} \geq 1$, we have

$$\mathcal{L}^d \left(\lambda \cap \bigcup_{n=n_{\underline{m}}+1}^{\infty} \mathrm{B}_\infty(x_n, r_n) \right) \leq \sum_{m=\underline{m}}^{\infty} \mathcal{L}^d \left(\lambda \cap \bigcup_{n=n_m+1}^{n_{m+1}} \mathrm{B}_\infty(x_n, n_{m+1}^{-1/d}) \right).$$

By definition of the integers n_m, the summand in the right-hand side is bounded above by 2^{-m}, so that the whole sum is bounded by $2^{-\underline{m}+1}$. The left-hand side thus converges to zero when $\underline{m}$ tends to infinity. We deduce that

$$\mathcal{L}^d \left(\lambda \cap \limsup_{n \to \infty} \mathrm{B}_\infty(x_n, r_n) \right) \leq \inf_{m \geq 1} \mathcal{L}^d \left(\lambda \cap \bigcup_{n=m}^{\infty} \mathrm{B}_\infty(x_n, r_n) \right) = 0,$$

which implies that the sequence $(x_n)_{n\geq 1}$ cannot be uniformly eutaxic in U. $\square$

6.3. Fractional parts of linear sequences

We shall show in this section that the fractional parts of linear sequences yield emblematic examples of eutaxic sequences. Recall that $\{x\}$ stands for coordinate-wise fractional part of the point $x \in \mathbb{R}^d$, and belongs to the unit cube $[0,1)^d$. The sequences that we consider throughout are of the form $(\{nx\})_{n\geq 1}$ with x in $\mathbb{R}^d$.

6.3.1. Uniform distribution modulo one. We shall invoke below a well known property satisfied by the sequences $(\{nx\})_{n\geq 1}$, specifically, they derive from sequences $(nx)_{n\geq 1}$ that are uniformly distributed in the sense of the next definition.

DEFINITION 6.3. A sequence $(x_n)_{n\geq 1}$ of points in $\mathbb{R}^d$ is *uniformly distributed modulo one* if for any points $(a_1,\dots,a_d)$ and $(b_1,\dots,b_d)$ in $[0,1)^d$ such that $a_i \leq b_i$ for all $i \in \{1,\dots,d\}$, we have

$$\lim_{N\to\infty} \frac{1}{N} \# \left\{ n \in \{1,\dots,N\} \;\middle|\; \{x_n\} \in \prod_{i=1}^{d} [a_i, b_i) \right\} = \prod_{i=1}^{d} (b_i - a_i).$$

It is easy to remark that the notion is unchanged if the point $(a_1,\dots,a_d)$ is chosen to be equal to zero in the above definition. When trying to prove that a sequence is uniformly distributed modulo one, we may call upon the following criterion due to Weyl, see *e.g.* Theorems 1.4 and 1.19 in [**17**].

THEOREM 6.5 (Weyl's criterion). *For any sequence $(x_n)_{n\geq 1}$ of points in $\mathbb{R}^d$, the following assertions are equivalent:*

(1) *the sequence $(x_n)_{n\geq 1}$ is uniformly distributed modulo one;*
(2) *for any nonnegative $\mathbb{Z}^d$-periodic Riemann-integrable function f defined on $\mathbb{R}^d$, the following limit holds:*

$$\lim_{N\to\infty} \frac{1}{N} \sum_{n=1}^{N} f(x_n) = \int_{[0,1)^d} f(x)\,\mathrm{d}x\,; \tag{120}$$

(3) *for any complex-valued $\mathbb{Z}^d$-periodic continuous function f defined on $\mathbb{R}^d$, the limit (120) holds;*
(4) *for every vector $q \in \mathbb{Z}^d \setminus \{0\}$,*

$$\lim_{N\to\infty} \frac{1}{N} \sum_{n=1}^{N} \mathrm{e}^{2\imath\pi q\cdot x_n} = 0.$$

PROOF. We begin by proving that (1) entails (2), and that (2) itself implies (3). By linearity, it follows directly from (1) that (120) holds for $f(x) = \widetilde{f}(\{x\})$, where $\widetilde{f}$ is step function defined on $[0,1)^d$, *i.e.* a conical combination of indicator functions of half-open rectangles contained in $[0,1)^d$. Let us now suppose that $f(x) = \widetilde{f}(\{x\})$, where $\widetilde{f}$ is a nonnegative Riemann-integrable function defined on $[0,1)^d$. Then, for all $\varepsilon > 0$, there are two step functions $\widetilde{f}_1$ and $\widetilde{f}_2$ such that $\widetilde{f}_1 \leq \widetilde{f} \leq \widetilde{f}_2$ and

$$\int_{[0,1)^d} (\widetilde{f}_2(x) - \widetilde{f}_1(x))\,\mathrm{d}x < \varepsilon.$$

Observing that (120) holds for $\widetilde{f}_1(\{x\})$, we infer that

$$\int_{[0,1)^d} f(x)\,\mathrm{d}x - \varepsilon \leq \int_{[0,1)^d} \widetilde{f}_1(x)\,\mathrm{d}x = \lim_{N\to\infty} \frac{1}{N} \sum_{n=1}^{N} \widetilde{f}_1(\{x_n\}) \leq \liminf_{N\to\infty} \frac{1}{N} \sum_{n=1}^{N} f(x_n).$$

Similarly, since (120) holds for $\widetilde{f}_2(\{x\})$ as well, we also get

$$\limsup_{N\to\infty} \frac{1}{N}\sum_{n=1}^{N} f(x_n) \leq \int_{[0,1)^d} f(x)\,\mathrm{d}x + \varepsilon.$$

It is now clear that the function f satisfies (120) too, *i.e.* that (2) is valid. Furthermore, this result may straightforwardly be extended to complex-valued $\mathbb{Z}^d$-periodic functions, and (3) subsequently follows from the simple observation that continuous functions are Riemann-integrable.

Conversely, we observe that the indicator function of a subrectangle of $[0,1)^d$ may be sandwiched between two continuous functions whose integrals are arbitrarily close. Thus, the above approach may be adapted to establish that (3) implies (1).

Finally, specializing (3) to complex exponential functions, we readily obtain (4). Conversely, (4) implies by linearity that (120) holds for all trigonometric polynomials, and the Stone-Weierstrass theorem allows us to extend this property to general complex-valued $\mathbb{Z}^d$-periodic functions, thereby obtaining (3). □

Applying Theorem 6.5 to the sequences $(nx)_{n\geq 1}$ leads to the following statement. The proof is elementary and left to the reader.

THEOREM 6.6. *Let us consider a point $x = (x_1, \ldots, x_d)$ in $\mathbb{R}^d$. Then, the sequence $(nx)_{n\geq 1}$ is uniformly distributed modulo one if and only if the real numbers $1, x_1, \ldots, x_d$ are linearly independent over $\mathbb{Q}$.*

It is clear from Definition 6.3 that if a sequence $(x_n)_{n\geq 1}$ of points in $\mathbb{R}^d$ is uniformly distributed modulo one, then the reduced sequence $(\{x_n\})_{n\geq 1}$ is dense in $[0,1)^d$. Therefore, the above theorem enables us to recover a classical result due to Kronecker concerning the density of the sequence $(\{nx\})_{n\geq 1}$. One thus may regard Theorem 6.6 as a measure theoretic analog of Kronecker's result.

THEOREM 6.7 (Kronecker). *Let us consider a point $x = (x_1, \ldots, x_d)$ in $\mathbb{R}^d$. Then, the sequence $(\{nx\})_{n\geq 1}$ is dense in the unit cube $[0,1)^d$ if and only if the real numbers $1, x_1, \ldots, x_d$ are linearly independent over $\mathbb{Q}$.*

PROOF. If the real numbers $1, x_1, \ldots, x_d$ are linearly independent over $\mathbb{Q}$, the result is due to Theorem 6.6 and the observation that follows its statement. In the opposite case, there exist mutually coprime integers $r, s_1, \ldots, s_d$ with

$$s_1x_1 + \ldots + s_dx_d = r.$$

Hence, for any integer $n \geq 1$, the coordinates of the point $\{nx\}$ satisfy

$$s_1\{nx_1\} + \ldots + s_d\{nx_d\} = nr - s_1\lfloor nx_1\rfloor - \ldots - s_d\lfloor nx_d\rfloor \in \mathbb{Z}.$$

This means in particular that the point $\{nx\}$ lies in some hyperplane with normal vector $s = (s_1, \ldots, s_d)$ whose distance to the origin is an integer multiple of the inverse of the Euclidean norm of s. Only finitely many such hyperplanes intersect the cube $[0,1)^d$, so the sequence $(\{nx\})_{n\geq 1}$ is clearly not dense in $[0,1)^d$. □

The badly approximable points will play a particularly important rôle when studying uniform eutaxy properties in Section 6.3.3 below. Hence, it is worth pointing out now a simple connection with linear independence over the rationals. In accordance with Section 1.3 where it is defined, the set of badly approximable points is denoted by Bad_d in what follows.

LEMMA 6.2. *Let us consider a point $x = (x_1, \ldots, x_d)$ in Bad_d. Then, the real numbers $1, x_1, \ldots, x_d$ are linearly independent over $\mathbb{Q}$.*

Combining this result with Theorems 6.6 and 6.7, we directly deduce that when x is a badly approximable point, the sequence $(nx)_{n\geq 1}$ is uniformly distributed modulo one, and the reduced sequence $(\{nx\})_{n\geq 1}$ is dense in the unit cube $[0,1)^d$. We shall establish hereafter that the latter sequence is in fact uniformly eutaxic in the open cube $(0,1)^d$: this is Kurzweil's theorem, see Theorem 6.9.

The proof of Lemma 6.2 makes use of several notations that we now introduce. The distance to the nearest integer point is defined by

$$\|z\| = \inf_{p\in\mathbb{Z}^d} |z-p|_\infty \tag{121}$$

for every point z in $\mathbb{R}^d$. This enables us to extend the definition (24) of the exponent κ to the higher-dimensional case. Specifically, if x is in $\mathbb{R}^d$, we define

$$\kappa(x) = \liminf_{q\to\infty} q^{1/d}\,\|qx\|\,. \tag{122}$$

If the point x has rational coordinates, then $\kappa(x)$ clearly vanishes. Otherwise, we may use the corollary to Dirichlet's theorem, that is, Corollary 1.1 to prove that $\kappa(x)$ is bounded above by one. Finally, similarly to (30), the exponent κ characterizes the badly approximable points, namely,

$$x \in \mathrm{Bad}_d \qquad\Longleftrightarrow\qquad \kappa(x) > 0. \tag{123}$$

Now that these notations are set, we may detail the proof of the lemma.

Proof of Lemma 6.2. We argue by contradiction. Let us assume the existence of integers $r, s_1, \ldots, s_d$ that do not vanish simultaneously and satisfy

$$s_1x_1 + \ldots + s_dx_d = r.$$

Up to rearranging the coordinates of x and multiplying the above equation by minus one, we may assume that $s_d \geq 1$. Now, given q in $\mathbb{N}$ and $p = (p_1, \ldots, p_{d-1})$ in $\mathbb{Z}^{d-1}$, we define $q' = s_dq$, as well as $p'_i = s_dp_i$ for $i \in \{1, \ldots, d-1\}$ and

$$p'_d = rq - s_1p_1 - \ldots - s_{d-1}p_{d-1}.$$

If the index i is different from d, it is clear that $q'x_i - p'_i$ is equal to $s_d(qx_i - p_i)$. Moreover, concerning the d-th coordinate, we have

$$q'x_d - p'_d = s_1(p_1 - qx_1) + \ldots + s_{d-1}(p_{d-1} - qx_{d-1}).$$

Letting $|\cdot|_1$ stand as usual for the taxicab norm and letting s denote the d-tuple $(s_1, \ldots, s_d)$, we infer that

$$\max_{i\in\{1,\ldots,d\}} |q'x_i - p'_i|_\infty \leq |s|_1 \max_{i\in\{1,\ldots,d-1\}} |qx_i - p_i|_\infty.$$

Taking the infimum over all $(d-1)$-tuples p, we deduce that $\|s_dqx\|$ is bounded above by $|s|_1$ times $\|q(x_1, \ldots, x_{d-1})\|$, from which it follows that

$$(s_dq)^{1/d}\,\|s_dqx\| \leq \frac{|s|_1 s_d^{1/d}}{q^{1/(d(d-1))}} \left(q^{1/(d-1)}\,\|q(x_1, \ldots, x_{d-1})\|\right).$$

Since $\kappa(x_1, \ldots, x_{d-1})$ is bounded above by one, there is an infinite set of integers q on which the term in parentheses in the above right-hand side is bounded. As this term is then divided by $q^{1/(d(d-1))}$, the latter upper bound implies that $\kappa(x)$ vanishes, thereby contradicting the fact that x is badly approximable. □

6.3.2. Sequencewise eutaxy. We now turn our attention to the eutaxy of the fractional parts of linear sequences, and its consequences in terms of Diophantine approximation. We start with the sequencewise version of that notion. The main result is then the following.

THEOREM 6.8. *Let $(r_n)_{n\geq 1}$ be a sequence in* P_d. *Then, for $\mathcal{L}^d$-almost every point $x \in \mathbb{R}^d$, the sequence $(\{nx\})_{n\geq 1}$ is eutaxic in $(0,1)^d$ with respect to $(r_n)_{n\geq 1}$.*

PROOF. As mentioned above, changing the norm does not alter the notion of eutaxy, so we assume for convenience that the space $\mathbb{R}^d$ is endowed with the supremum norm. For any integer $n \geq 1$ and any point $p \in \mathbb{Z}^d$, we consider the set

$$U_{n,p} = \left\{(x,y) \in \mathbb{R}^d \times \mathbb{R}^d \;\middle|\; |y - nx - p|_\infty < r_n\right\}.$$

Such an integer n being fixed, the union over all points $p \in \mathbb{Z}^d$ of the sets $U_{n,p}$ is then denoted by V_n. We also consider the two sets defined by

$$S = [0,1)^d \times [0,1)^d \qquad \text{and} \qquad L = [0,1)^d \times \mathbb{R}^d.$$

Now, it is elementary to observe that (x,y) belongs to $U_{n,p}$ if and only if $(x, y-p)$ belongs to $U_{n,0}$. Moreover, the sequence $(r_n)_{n\geq 1}$ converges to zero, so we may assume that $r_n \leq 1/2$, up to choosing n sufficiently large. This guarantees the disjointness of the sets $U_{n,p}$, for p ranging in $\mathbb{Z}^d$, and enables us to write that

$$\mathcal{L}^{2d}(S \cap V_n) = \sum_{p \in \mathbb{Z}^d} \mathcal{L}^{2d}(S \cap U_{n,p}) = \sum_{p \in \mathbb{Z}^d} \mathcal{L}^{2d}(S_p \cap U_{n,0}) = \mathcal{L}^{2d}(L \cap U_{n,0}),$$

where S_p stands for the product of the cubes $[0,1)^d$ and $-p + [0,1)^d$. The last equality is due to the observation that the set L is the disjoint union of the sets S_p. Likewise, we have

$$\begin{aligned}\mathcal{L}^{2d}(S \cap V_m \cap V_n) &= \sum_{p \in \mathbb{Z}^d} \mathcal{L}^{2d}(S \cap V_m \cap U_{n,p}) \\ &= \sum_{p \in \mathbb{Z}^d} \mathcal{L}^{2d}(S_p \cap V_m \cap U_{n,0}) = \mathcal{L}^{2d}(L \cap V_m \cap U_{n,0}).\end{aligned}$$

Here, we used the additional observation that the set V_m is invariant under the translations of the form $(x,y) \mapsto (x, y-p)$, where p is in the set $\mathbb{Z}^d$.

In order to compute the Lebesgue measure of the set $L \cap U_{n,0}$, we consider two points $x = (x_1, \ldots, x_d)$ and $y = (y_1, \ldots, y_d)$ in $\mathbb{R}^d$, and we remark that

$$(x,y) \in L \cap U_{n,0} \qquad \Longleftrightarrow \qquad \forall i \in \{1, \ldots, d\} \quad \begin{cases} 0 \leq x_i < 1 \\ |y_i - nx_i| < r_n. \end{cases}$$

For each index i, the pairs (x_i, y_i) for which the latter condition holds form a set with Lebesgue measure clearly equal to $2r_n$. Therefore,

$$\mathcal{L}^{2d}(L \cap U_{n,0}) = (2r_n)^d.$$

In a similar fashion, the Lebesgue measure of the set $L \cap V_m \cap U_{n,0}$ may be determined by observing that

$$(x,y) \in L \cap V_m \cap U_{n,0} \quad \Longleftrightarrow \quad \forall i \in \{1, \ldots, d\}\ \exists p_i \in \mathbb{Z} \begin{cases} 0 \leq x_i < 1 \\ |y_i - mx_i - p_i| < r_m \\ |y_i - nx_i| < r_n. \end{cases}$$

We assume again that m is large enough to ensure that $r_m \leq 1/2$, and we also assume that $n > m$. Then, for every index i, the set of pairs (x_i, y_i) for which the latter condition holds is the disjoint union of $n - m + 1$ sets, each corresponding to a specific value of p_i in $\{0,\dots,n-m\}$. If $0<p_i<n-m$, the corresponding sets are parallelograms that are defined by the vectors

$$\frac{2r_n}{n-m}(1,m) \qquad \text{and} \qquad \frac{2r_m}{n-m}(1,n),$$

and that may be deduced from one another with the help of the translation by vector $(1,n)/(n-m)$. The area of each of these parallelograms is thus given by the determinant of the above vectors, namely, $4r_m r_n/(n-m)$. Besides, when p_i is equal to zero and to $n-m$, we obtain the two halves of a parallelogram of the previous form. Finally, the total Lebesgue measure of the $n-m+1$ disjoint sets is equal to $4r_m r_n$. We deduce that

$$n>m \qquad \Longrightarrow \qquad \mathcal{L}^{2d}(L\cap V_m\cap U_{n,0})=(4r_m r_n)^d.$$

The upshot is that for all integers m and n sufficiently large to ensure that r_m and r_n are both bounded above by $1/2$, we have

$$\mathcal{L}^{2d}(S\cap V_n)=(2r_n)^d \qquad \text{and} \qquad \mathcal{L}^{2d}(S\cap V_m\cap V_n)=(4r_m r_n)^d.$$

Moreover, in the opposite case where $r_n>1/2$, it is clear that the set V_n coincides with the whole space $\mathbb{R}^d\times\mathbb{R}^d$. Therefore, we may drop the assumption on the integers m and n, up to replacing r_n by $\widetilde{r}_n=\min\{r_n,1/2\}$ in the above formula and replacing r_m by a similar value $\widetilde{r}_m$. In particular, we have

$$\mathcal{L}^{2d}(S\cap V_m\cap V_n)=\mathcal{L}^{2d}(S\cap V_m)\cdot\mathcal{L}^{2d}(S\cap V_n)$$

for all integers $m,n\geq 1$. Moreover, given that the sequence $(r_n)_{n\geq 1}$ belongs to the collection P_d, we also have

$$\sum_{n=1}^{\infty}\mathcal{L}^{2d}(S\cap V_n)=\sum_{n=1}^{\infty}(2\widetilde{r}_n)^d=\infty.$$

The hypotheses of Lemma 6.1 are thus satisfied by the restriction of the Lebesgue measure to the set S, along with the sequence of sets $(V_n)_{n\geq 1}$. Applying this lemma, we conclude that

$$\mathcal{L}^{2d}\left(S\cap\limsup_{n\to\infty}V_n\right)\geq\limsup_{N\to\infty}\frac{\left(\sum\limits_{n=1}^{N}\mathcal{L}^{2d}(S\cap V_n)\right)^2}{\sum\limits_{m=1}^{N}\sum\limits_{n=1}^{N}\mathcal{L}^{2d}(S\cap V_m\cap V_n)}=1=\mathcal{L}^{2d}(S).$$

As the sets V_n are invariant under the translations of the form $(x,y)\mapsto(x+p,y+q)$, where p and q are in the set $\mathbb{Z}^d$, we deduce that

$$\mathcal{L}^{2d}\left(\mathbb{R}^{2d}\setminus\limsup_{n\to\infty}V_n\right)=0.$$

This means in particular that for Lebesgue-almost every point $x\in\mathbb{R}^d$, the set

$$Y_x=\{y\in(0,1)^d\mid(x,y)\in V_n \quad \text{for i.m. } n\geq 1\}$$

has full Lebesgue measure in $(0,1)^d$. Now, given a real number $\varepsilon\in(0,1/2)$, let us consider a point y belonging to both Y_x and $(\varepsilon,1-\varepsilon)^d$. Then, for infinitely many integers $n\geq 1$, there exists a point $p_n\in\mathbb{Z}^d$ such that $(x,y)\in U_{n,p_n}$, that is,

$$|y-nx-p_n|_\infty<r_n.$$

Letting $\lfloor\cdot\rfloor$ stand for the coordinate-wise floor function and letting h denote the point in $\mathbb{R}^d$ with all coordinates equal to $1/2$, we have

$$\begin{aligned}|\lfloor nx\rfloor+p_n|_\infty &\leq |y-nx-p_n|_\infty+|\{nx\}-h|_\infty+|y-h|_\infty\\ &< r_n+\frac{1}{2}+\left(\frac{1}{2}-\varepsilon\right)=1-\varepsilon+r_n<1.\end{aligned}$$

The last inequality holds for n large enough, because the sequence $(r_n)_{n\geq 1}$ converges to zero. In that situation, the point p_n is necessarily equal to $-\lfloor nx \rfloor$. Hence,

$$Y_x \cap (\varepsilon, 1-\varepsilon)^d \subseteq \left\{ y \in (0,1)^d \;\middle|\; |y - \{nx\}|_\infty < r_n \quad \text{for i.m. } n \geq 1 \right\}.$$

The set in the left-hand side has Lebesgue measure equal to $(1-2\varepsilon)^d$. We may then let ε tend to zero, thereby concluding that the set in the right-hand side has Lebesgue measure equal to one. □

We may now apply Theorem 6.1 to the example supplied by Theorem 6.8. Here, the formula (107) for the sets F_t gives rise to the sets

$$F_t(x) = \left\{ y \in \mathbb{R}^d \;\middle|\; |y - \{nx\}| < r_n^t \quad \text{for i.m. } n \geq 1 \right\},$$

where x is chosen according to the Lebesgue measure. Due to the aforementioned results, we then know that for any sequence $(r_n)_{n\geq 1}$ in P_d such that $\sum_n r_n^s$ converges for all $s > d$, and for Lebesgue-almost every point $x \in \mathbb{R}^d$, we have both

$$\dim_{\mathrm{H}}(F_t(x) \cap U) = \frac{d}{t} \qquad \text{and} \qquad F_t(x) \in \mathcal{G}^{d/t}(U) \tag{124}$$

for any real number $t \geq 1$ and for any nonempty open subset U of $(0,1)^d$. In the context of metric Diophantine approximation, it is customary to recast such a result with the help of the distance to the nearest integer point defined by (121). We may now easily deduce the next result from (124).

Corollary 6.1. *Let $(r_n)_{n\geq 1}$ be a sequence in P_d such that $\sum_n r_n^s$ converges for all $s > d$. For any real number $t \geq 1$, let us define the set*

$$F'_t(x) = \left\{ y \in \mathbb{R}^d \;\middle|\; \|y - nx\| < r_n^t \quad \text{for i.m. } n \geq 1 \right\}.$$

Then, for Lebesgue-almost every point $x \in \mathbb{R}^d$,

$$\forall t \geq 1 \qquad \dim_{\mathrm{H}} F'_t(x) = \frac{d}{t}.$$

Proof. One easily checks that for all $x \in \mathbb{R}^d$ and $t > 1$, the set $F'_t(x)$ contains the set $F_t(x) \cap (0,1)^d$. The lower bound on the dimension then readily follows from (124). For the upper bound, we begin by observing that the sets $F'_t(x)$ are invariant under the translations by vectors in $\mathbb{Z}^d$. It thus suffices to consider their intersection with the unit cube $[0,1)^d$. However, we clearly have

$$F'_t(x) \cap [0,1)^d \subseteq \limsup_{n\to\infty} \bigcup_{p \in \{-1,0,1\}^d} \mathrm{B}_\infty(\{nx\} + p, r_n^t),$$

and we conclude with the help of Lemma 2.1. □

An emblematic particular case is obtained by letting the sequence of approximating radii be given by $r_n = n^{-1/d}$. This sequence clearly satisfies the assumptions of Corollary 6.1 and, up to a simple change of parameter, we deduce that for Lebesgue-almost every point $x \in \mathbb{R}^d$ and for every real number $\sigma \geq 1/d$,

$$\dim_{\mathrm{H}} \left\{ y \in \mathbb{R}^d \;\middle|\; \|y - nx\| < \frac{1}{n^\sigma} \quad \text{for i.m. } n \geq 1 \right\} = \frac{1}{\sigma}. \tag{125}$$

In the one-dimensional setting, this result is well known, and even holds when x is an arbitrary irrational real number, see [**11**].

6.3.3. Uniform eutaxy: Kurzweil's theorem. Regarding the uniform eutaxy of the sequences $(\{nx\})_{n\geq 1}$, the main result is Theorem 6.9 below, which was first obtained by Kurzweil [**42**] and subsequently recovered by Lesca [**43**]. For the sake of completeness, let us mention in addition that Kurzweil also obtained in [**42**] an extension of Theorem 6.9 that deals with linear forms.

THEOREM 6.9 (Kurzweil). *For any point x in $\mathbb{R}^d$, the sequence $(\{nx\})_{n\geq 1}$ is uniformly eutaxic in $(0,1)^d$ if and only if x is badly approximable.*

In order to let the reader compare this result with Theorem 6.8, it is worth mentioning some metric properties of the set Bad_d of badly approximable points defined in Section 1.3. Specifically, Proposition 1.9 therein shows that Bad_d has Lebesgue measure zero. Moreover, Corollary 12.1 ensures that this set has Hausdorff dimension d in any nonempty open subset of $\mathbb{R}^d$.

The proof of Theorem 6.9 is postponed to the end of this section, and will make use of Propositions 6.2 and 6.3 below. These two propositions are more general than Theorem 6.9 in the sense that they concern fractional parts of the form $\{a_n x\}$, where a_n is the general term of an increasing sequence of positive integers. Such a sequence $(a_n)_{n\geq 1}$ being given, we define its lower asymptotic density by

$$\underline{\delta}((a_n)_{n\geq 1}) = \liminf_{N\to\infty} \frac{1}{N}\#\{n\geq 1 \mid a_n\leq N\}. \tag{126}$$

Moreover, we shall also use the exponent κ defined by (122), and we shall accordingly endow the space $\mathbb{R}^d$ with the supremum norm, which has no influence on the notion of eutaxy, as already observed above. Finally, let us recall that the exponent κ characterizes the badly approximable points, see (123). We then have the following result, established by Reversat [**49**].

PROPOSITION 6.2. *Let us consider an increasing sequence $(a_n)_{n\geq 1}$ of positive integers with positive lower asymptotic density, and a point $x=(x_1,\ldots,x_d)$ in $\mathbb{R}^d$ such that the real numbers $1, x_1, \ldots, x_d$ are linearly independent over $\mathbb{Q}$. Then, for any nonempty dyadic subcube λ of $[0,1)^d$,*

$$\liminf_{j\to\infty} 2^{-dj}\#\mathrm{M}((\{a_n x\})_{n\geq 1};\lambda,j) \leq 480^d \left(\frac{\kappa(x)}{\underline{\delta}((a_n)_{n\geq 1})}\right)^{d/(d+1)}.$$

PROOF. If δ denotes a positive real number smaller than $\underline{\delta}((a_n)_{n\geq 1})$, then we have $a_n\leq n/\delta$ for any sufficiently large integer n. Moreover, given $\kappa>\kappa(x)$, we know that there exists an infinite set $\mathcal{Q}\subseteq\mathbb{N}$ such that $\|qx\|\leq\kappa/q^{1/d}$ for all $q\in\mathcal{Q}$. We now fix a nonempty dyadic cube λ contained in $[0,1)^d$, an integer $q\in\mathcal{Q}$ and an integer $j\geq 0$ satisfying

$$c^{d/(d+1)}\,2^{d(\langle\lambda\rangle+j)} \leq q \leq c^{d/(d+1)}\,2^{d(\langle\lambda\rangle+j+1)}, \tag{127}$$

where c is a positive parameter that will be tuned up at the end of the proof.

Let us consider an integer $m\leq 2^{d(\langle\lambda\rangle+j)}$ such that $\{a_m x\}\in\lambda$. We decompose the integer a_m in the form $hq+r$ with $h\in\mathbb{N}_0$ and $r\in\{1,\ldots,q\}$. If the integer q is sufficiently large, the integer j is large as well and we may assume that

$$hq\leq a_m\leq a_{2^{d(\langle\lambda\rangle+j)}}\leq\frac{2^{d(\langle\lambda\rangle+j)}}{\delta}\qquad\text{and}\qquad 2^{j-1}\geq\frac{\kappa}{\delta c}.$$

As a consequence,

$$\|rx-a_m x\| = \|hqx\| \leq h\,\|qx\| \leq \kappa\frac{hq}{q^{1+1/d}} \leq \frac{\kappa}{\delta c}2^{-(\langle\lambda\rangle+j)} \leq 2^{-(\langle\lambda\rangle+1)}.$$

Letting y_λ denote the center of the cube λ, we deduce that for some point p in $\mathbb{Z}^d$,

$$|\{rx\}-p-y_\lambda|_\infty \leq |\{rx\}-\{a_m x\}-p|_\infty + |\{a_m x\}-y_\lambda|_\infty \leq 2^{-\langle\lambda\rangle}.$$

We conclude that $\{rx\}$ belongs to $U(\lambda)$, the set of points y in $[0,1)^d$ that are within distance $2^{-\langle\lambda\rangle}$ from $y_\lambda+\mathbb{Z}^d$. Therefore, the integer r is positive, bounded above by $c^{d/(d+1)}\,2^{d(\langle\lambda\rangle+j+1)}$, and verifies $\{rx\}\in U(\lambda)$; we define $R(\lambda,j)$ as the set of all integers that satisfy these three properties.

Furthermore, let λ' be the dyadic subcube of λ with generation $\langle\lambda\rangle+j$ that contains the point $\{a_m x\}$. We consider another integer $m'\leq 2^{d(\langle\lambda\rangle+j)}$ such that $a_{m'}$ may be written in the form $h'q+r$ for some nonnegative integer h'. We have

$$\|a_m x - a_{m'}x\| = \|(h-h')qx\| \leq |h-h'|\,\|qx\| \leq \kappa\frac{\max\{hq,h'q\}}{q^{1+1/d}} \leq \frac{\kappa}{\delta c}2^{-(\langle\lambda\rangle+j)}.$$

Thus, letting $y_{\lambda'}$ denote the center of the subcube λ', we observe that there exists a point p in $\mathbb{Z}^d$ such that

$$\begin{aligned}|\{a_{m'}x\}-p-y_{\lambda'}|_\infty &\leq |\{a_{m'}x\}-\{a_m x\}-p|_\infty + |\{a_m x\}-y_{\lambda'}|_\infty\\ &\leq \left(\frac{\kappa}{\delta c}+\frac{1}{2}\right)2^{-(\langle\lambda\rangle+j)}.\end{aligned}$$

This means that $\{a_{m'}x\}$ belongs to a closed ball centered at $p+y_{\lambda'}$ with radius the right-hand side above, that is denoted by ρ. Note that the number of dyadic cubes with generation $\langle\lambda\rangle+j$ that are required to cover this ball is bounded above by $((2\rho)2^{\langle\lambda\rangle+j}+2)^d$. In addition, it is easily seen that there are at most 5^d possible values for p, because the points $\{a_{m'}x\}$ and $y_{\lambda'}$ both belong to the unit cube. We conclude that the number of dyadic subcubes of λ with generation $\langle\lambda\rangle+j$ that may contain $\{a_{m'}x\}$ is bounded above by

$$5^d((2\rho)2^{\langle\lambda\rangle+j}+2)^d = 10^d\left(\frac{3}{2}+\frac{\kappa}{\delta c}\right)^d.$$

The upshot is that for every choice of r, the above value gives an upper bound on the number of dyadic subcubes of λ with generation $\langle\lambda\rangle+j$ that contain at least one point of the form $\{a_m x\}$, where $m\leq 2^{d(\langle\lambda\rangle+j)}$ and $a_m=hq+r$ for some nonnegative integer h. Recalling that r necessarily belongs to the set $R(\lambda,j)$ when such an integer a_m exists, we deduce that

$$\#\mathrm{M}(\lambda,j)\leq 10^d\left(\frac{3}{2}+\frac{\kappa}{\delta c}\right)^d\#R(\lambda,j).$$

This inequality is valid for infinitely many values of j, namely, for every integer j satisfying (127) for some $q\in\mathcal{Q}$. It follows that

$$\liminf_{j\to\infty}2^{-dj}\#\mathrm{M}(\lambda,j)\leq 10^d\left(\frac{3}{2}+\frac{\kappa}{\delta c}\right)^d\limsup_{j\to\infty}2^{-dj}\#R(\lambda,j). \tag{128}$$

Given that the real numbers $1,x_1,\ldots,x_d$ are linearly independent over $\mathbb{Q}$, we may conclude with the help of Theorem 6.6. Accordingly, the sequence $(rx)_{r\geq 1}$ is uniformly distributed modulo one, so that

$$\#R(\lambda,j)\sim\lfloor c^{d/(d+1)}\,2^{d(\langle\lambda\rangle+j+1)}\rfloor\mathcal{L}^d(U(\lambda))\qquad\text{as}\qquad j\to\infty.$$

One easily check that the set $U(\lambda)$ has Lebesgue measure at most $6^d2^{-d\langle\lambda\rangle}$. Hence, the limsup in (128) is bounded above by $12^dc^{d/(d+1)}$. We deduce that

$$\liminf_{j\to\infty}2^{-dj}\#\mathrm{M}(\lambda,j)\leq 120^dc^{d/(d+1)}\left(\frac{3}{2}+\frac{\kappa}{\delta c}\right)^d.$$

We conclude by choosing $c=2\kappa/\delta$, and then by letting δ and κ go to $\underline{\delta}((a_n)_{n\geq 1})$ and $\kappa(x)$, respectively. $\square$

The next result is a converse to Proposition 6.2 above. While the statement of Proposition 6.2 involves the exponent κ defined by (122), we rather consider here the exponent κ_* defined by

$$\kappa_*(x) = \inf_{q\in\mathbb{N}} q^{1/d} \|qx\|$$

for all x in $\mathbb{R}^d$. Clearly, $\kappa_*(x)$ is bounded above by $\kappa(x)$. Moreover, $\kappa(x)$ and $\kappa_*(x)$ are positive on the same set of values of x, namely, the set of badly approximable points. This means that κ_* satisfies a property similar to (123), specifically, this exponent also characterizes the badly approximable points:

$$x \in \mathrm{Bad}_d \qquad \Longleftrightarrow \qquad \kappa_*(x) > 0. \tag{129}$$

In connection with distributions modulo one, the statement below also calls upon the limiting ratios defined by

$$\underline{\rho}((x_n)_{n\geq 1}; \lambda) = \liminf_{N\to\infty} \frac{1}{N} \#\{n \in \{1, \ldots, N\} \mid \{x_n\} \in \lambda\} \tag{130}$$

when $(x_n)_{n\geq 1}$ denotes a sequence of points in $\mathbb{R}^d$ and λ is a nonempty dyadic subcube of $[0,1)^d$. As a direct consequence of Definition 6.3, each of these limiting ratios is equal to $\mathcal{L}^d(\lambda)$ if the sequence $(x_n)_{n\geq 1}$ is uniformly distributed modulo one. Again, the following result is due to Reversat [**49**].

PROPOSITION 6.3. *Let $(a_n)_{n\geq 1}$ be an increasing sequence of positive integers and let x be a point in $\mathbb{R}^d$. Then, for any nonempty dyadic subcube λ of $[0,1)^d$,*

$$\liminf_{j\to\infty} 2^{-dj} \#\mathrm{M}((\{a_n x\})_{n\geq 1}; \lambda, j) \geq \frac{\kappa_*(x)^d \underline{\delta}((a_n)_{n\geq 1})}{2^d \mathcal{L}^d(\lambda)} \, \underline{\rho}((a_n x)_{n\geq 1}; \lambda).$$

PROOF. We may obviously assume that $\kappa_*(x)$ and $\underline{\delta}((a_n)_{n\geq 1})$ are both positive. If κ is a positive real number smaller than $\kappa_*(x)$, it is clear that $\|qx\| > \kappa/q^{1/d}$ for all integers $q \geq 1$. Furthermore, if δ denotes a positive real number smaller than $\underline{\delta}((a_n)_{n\geq 1})$, we know that the inequality $a_n \leq n/\delta$ holds for n large enough. We now consider a nonempty dyadic subcube λ of $[0,1)^d$, an integer $j \geq 0$, and a dyadic cube λ' in the collection $\mathrm{M}(\lambda, j)$. In particular, the cube λ' contains a point of the form $\{a_m x\}$ for some integer $m \leq 2^{d(\langle\lambda\rangle+j)}$. If m' denotes another integer bounded above by $2^{d(\langle\lambda\rangle+j)}$ and for which $\{a_{m'} x\}$ belongs to λ' as well, then

$$|\{a_m x\} - \{a_{m'} x\}|_\infty \geq \|(a_m - a_{m'})x\| > \frac{\kappa}{|a_m - a_{m'}|^{1/d}} \geq \frac{\kappa\, \delta^{1/d}}{2^{\langle\lambda\rangle+j}}.$$

The last bound holds for j sufficiently large, because the positive integers a_m and $a_{m'}$ are then both bounded above by $2^{d(\langle\lambda\rangle+j)}/\delta$. We may naturally decompose the cube λ' as the disjoint union of $\lceil 1/(\kappa\, \delta^{1/d})\rceil^d$ half-open subcubes with sidelength equal to $2^{-(\langle\lambda\rangle+j)}/\lceil 1/(\kappa\, \delta^{1/d})\rceil$. Moreover, if we consider any of these subcubes, the above inequalities imply that at most one integer $m \leq 2^{d(\langle\lambda\rangle+j)}$ can be such that the point $\{a_m x\}$ lies in the cube. So, there can be no more than $\lceil 1/(\kappa\, \delta^{1/d})\rceil^d$ integers $m \leq 2^{d(\langle\lambda\rangle+j)}$ for which $\{a_m x\}$ is in λ'. As a consequence,

$$\#\{m \leq 2^{d(\langle\lambda\rangle+j)} \mid \{a_m x\} \in \lambda\} \leq \left\lceil \frac{1}{\kappa\, \delta^{1/d}} \right\rceil^d \#\mathrm{M}(\lambda, j),$$

from which we readily deduce that

$$2^{-dj} \#\mathrm{M}(\lambda, j) \geq \frac{\kappa^d \delta}{2^d \mathcal{L}^d(\lambda)} \, 2^{-d(\langle\lambda\rangle+j)} \#\{m \leq 2^{d(\langle\lambda\rangle+j)} \mid \{a_m x\} \in \lambda\}.$$

The result follows in a straightforward manner by letting j tend to infinity, and then by letting κ and δ go to $\kappa_*(x)$ and $\underline{\delta}((a_n)_{n\geq 1})$, respectively. $\square$

We are now in position to explain how to deduce Theorem 6.9 from the two propositions above, together with the necessary and sufficient conditions for eutaxy expressed by Theorems 6.3 and 6.4.

Proof of Theorem 6.9. The idea is to apply Propositions 6.2 and 6.3 to the sequence $(n)_{n\geq 1}$, which is increasing and has lower asymptotic density equal to one. Let us first assume that the point x is not badly approximable, and let $x_1, \ldots, x_d$ denote its coordinates. If the real numbers $1, x_1, \ldots, x_d$ are linearly dependent over the rationals, it follows from Kronecker's theorem, namely, Theorem 6.7 that the sequence $(\{nx\})_{n\geq 1}$ is not dense in $[0,1)^d$. This sequence is thus clearly not eutaxic in $(0,1)^d$. Now, if the above real numbers are linearly independent over $\mathbb{Q}$, we may apply Proposition 6.2, thereby inferring that for any point x in $\mathbb{R}^d$ and for any nonempty dyadic subcube λ of $[0,1)^d$,

$$\liminf_{j\to\infty} 2^{-dj} \#\mathrm{M}((\{nx\})_{n\geq 1}; \lambda, j) \leq 480^d \kappa(x)^{d/(d+1)}.$$

Since x is not badly approximable, the exponent $\kappa(x)$ vanishes by virtue of (123). The left-hand side above thus vanishes as well, and Theorem 6.4 ensures that the sequence $(\{nx\})_{n\geq 1}$ is not uniformly eutaxic in $(0,1)^d$.

Conversely, let us assume that x is badly approximable. Lemma 6.2 ensures that the real numbers $1, x_1, \ldots, x_d$ are linearly independent over $\mathbb{Q}$. We then deduce from Theorem 6.6 that the sequence $(nx)_{n\geq 1}$ is uniformly distributed modulo one, so that for any nonempty dyadic subcube λ of $[0,1)^d$, the limiting ratio $\underline{\rho}((nx)_{n\geq 1}; \lambda)$ defined by (130) is equal to $\mathcal{L}^d(\lambda)$. Applying Proposition 6.3, we thus infer that

$$\liminf_{j\to\infty} 2^{-dj} \#\mathrm{M}((\{nx\})_{n\geq 1}; \lambda, j) \geq 2^{-d} \kappa_*(x)^d.$$

Finally, in view of (129), the exponent $\kappa_*(x)$ is positive, and we conclude with the help of Theorem 6.3 that the sequence $(\{nx\})_{n\geq 1}$ is uniformly eutaxic in $(0,1)^d$. $\square$

In the vein of Corollary 6.1 and the discussion that precedes its statement, an interesting application is the study of the Diophantine approximation properties of the sequence $(\{nx\})_{n\geq 1}$ when x is a badly approximable point. That sequence being uniformly eutaxic, we end up with a much stronger result than Corollary 6.1, and actually a full and complete description of the size and large intersection properties of the sets $F_t(x)$ and $F'_t(x)$ considered at the end of Section 6.3.2. We refer to Section 10.1.1 for precise statements.

6.4. Fractional parts of other sequences

6.4.1. Sequencewise eutaxy. Theorem 6.8 may be extended to the case in which the underlying sequence is driven by a nonconstant polynomial with integer coefficients. In fact, Schmidt [**52**] established the following result.

Theorem 6.10. *Let P be a nonconstant polynomial with coefficients in $\mathbb{Z}$ and let $(r_n)_{n\geq 1}$ be a sequence in P_d. Then, for Lebesgue-almost every point $x \in \mathbb{R}^d$, the sequence $(\{P(n)x\})_{n\geq 1}$ is eutaxic in $(0,1)^d$ with respect to $(r_n)_{n\geq 1}$.*

Subsequently, Philipp [**48**] showed that, in dimension one, the above property still holds when the polynomial is replaced by the exponential function to a given integer base $b \geq 2$; this is related with the base b expansion of real numbers.

Theorem 6.11. *Let us consider an integer $b \geq 2$ and a sequence $(r_n)_{n\geq 1}$ in P_d. Then, for Lebesgue-almost every point $x \in \mathbb{R}$, the sequence $(\{b^n x\})_{n\geq 1}$ is eutaxic in $(0,1)$ with respect to $(r_n)_{n\geq 1}$.*

Philipp showed that this property also holds for x in a Lebesgue-full subset of the interval $[0,1)$ when the multiplication by b^n is replaced by the n-th iterate of either of the following mappings: the Gauss map for continued fractions defined by (14); the θ-adic expansion map $x \mapsto \{\theta x\}$, where $\theta > 1$. We refer to [**48**] for precise statements. In all those cases, we may reproduce the approach developed in Section 6.3.2 so as to obtain dimensional results analogous to Corollary 6.1.

6.4.2. Uniform eutaxy. The uniform analogs of Theorems 6.10 and 6.11 need not be valid, because the Lebesgue-null set of points x on which each of these results may fail depends on the choice of the sequence $(r_n)_{n\geq 1}$, and there are of course uncountably many sequences in P_d. In that direction, we have however the following one-dimensional statement, obtained by Reversat [**49**].

THEOREM 6.12. *Let $(a_n)_{n\geq 1}$ be a sequence of positive real numbers such that the series $\sum_n a_n/a_{n+1}$ converges. Then, for Lebesgue-almost every x in $\mathbb{R}$, the sequence $(\{a_n x\})_{n\geq 1}$ is uniformly eutaxic in $(0,1)$.*

With a view to establishing Theorem 6.12, we begin by deriving a simple estimate on integrals of products of fractional parts. To be specific, for any r-tuple $a = (a_1, \ldots, a_r)$ of positive real numbers and any r-tuple $I = (I_1, \ldots, I_r)$ of intervals contained in the unit interval $[0,1)$, we define

$$P_{a,I}(x) = \prod_{s=1}^{r} \mathbf{1}_{I_s}(\{a_s x\}). \tag{131}$$

We then integrate the function $P_{a,I}$ over bounded intervals of the real line. The next lemma gives an upper bound on the resulting integrals, under the assumption that the lengths of the r intervals forming I are bounded away from zero.

LEMMA 6.3. *Let $a = (a_1, \ldots, a_r)$ denote an r-tuple of positive real numbers, and let $I = (I_1, \ldots, I_r)$ denote an r-tuple of subintervals of $[0,1)$ satisfying*

$$\exists \delta > 0 \quad \forall s \in \{1, \ldots, r\} \qquad |I_s| \geq \delta.$$

Then, for any bounded subinterval I_0 of $\mathbb{R}$, we have

$$\int_{I_0} P_{a,I}(x)\,\mathrm{d}x \leq \left(|I_0| + \frac{2}{a_1}\right) \cdot \left(\prod_{s=1}^{r} |I_s|\right) \cdot \left(\prod_{s=1}^{r-1} \left(1 + \frac{2a_s}{\delta a_{s+1}}\right)\right).$$

PROOF. Without loss of generality, we may assume that the interval I_0 is of the form $[u,v]$, with $u < v$. Then, a simple change of variable implies that

$$\int_{I_0} P_{a,I}(x)\,\mathrm{d}x = \frac{1}{a_1} \int_{a_1 u}^{a_1 v} P_{a/a_1,I}(x)\,\mathrm{d}x.$$

The interval onto which the integral in right-hand side is computed is obviously covered by the intervals of the form $[p, p+1)$, where p is an integer between $\lceil a_1 u \rceil - 1$ and $\lfloor a_1 v \rfloor$. If x belongs to such an interval, we have

$$P_{a/a_1,I}(x) = \mathbf{1}_{I_1}(\{x\}) \prod_{s=2}^{r} \mathbf{1}_{I_s}\left(\left\{\frac{a_s}{a_1} x\right\}\right) = \mathbf{1}_{p+I_1}(x) P_{a',I'}(x),$$

where $p + I_1$ denotes the interval obtained by adding p to the elements of I_1, and where a' and I' stand for the $(r-1)$-tuples $(a_2/a_1, \ldots, a_r/a_1)$ and $(I_2, \ldots, I_r)$, respectively. As a consequence,

$$\int_p^{p+1} P_{a/a_1,I}(x)\,\mathrm{d}x \leq \int_{p+I_1} P_{a',I'}(x)\,\mathrm{d}x \leq \sup_{\substack{I'_1 \subseteq \mathbb{R} \\ |I'_1| = |I_1|}} \int_{I'_1} P_{a',I'}(x)\,\mathrm{d}x,$$

where the supremum is taken over all subintervals I_1' of $\mathbb{R}$ whose length is equal to that of I_1. Summing the above estimate over all integers p between $\lceil a_1 u\rceil - 1$ and $\lfloor a_1 v\rfloor$, we straightforwardly deduce that

$$\int_{I_0} P_{a,I}(x)\,\mathrm{d}x \leq \left(|I_0| + \frac{2}{a_1}\right) \sup_{\substack{I_1'\subseteq\mathbb{R}\\ |I_1'|=|I_1|}} \int_{I_1'} P_{a',I'}(x)\,\mathrm{d}x. \tag{132}$$

We may now conclude by induction on the integer r. Indeed, if the result holds for all appropriate $(r-1)$-tuples, then the integral in the right-hand side satisfies

$$\begin{aligned}\int_{I_1'} P_{a',I'}(x)\,\mathrm{d}x &\leq \left(|I_1'| + \frac{2}{a_2/a_1}\right)\cdot\left(\prod_{s=2}^{r}|I_s|\right)\cdot\left(\prod_{s=2}^{r-1}\left(1+\frac{2a_s/a_1}{\delta a_{s+1}/a_1}\right)\right)\\ &= \left(1+\frac{2a_1}{|I_1|a_2}\right)\cdot\left(\prod_{s=1}^{r}|I_s|\right)\cdot\left(\prod_{s=2}^{r-1}\left(1+\frac{2a_s}{\delta a_{s+1}}\right)\right),\end{aligned}$$

which yields the required upper bound because $|I_1|$ is bounded below by δ. It finally remains to observe that when r is equal to one, (132) reduces to

$$\int_{I_0} \mathbf{1}_{I_1}(\{a_1 x\})\,\mathrm{d}x \leq \left(|I_0| + \frac{2}{a_1}\right)|I_1|,$$

so that the required upper bound also holds in that case. □

The above ancillary lemma being proven, we are now in position to detail the proof of Theorem 6.12.

Proof of Theorem 6.12. Given that the series $\sum_n a_n/a_{n+1}$ is convergent, for any integer $j\geq 0$, we may find an integer $n_j\geq 0$ satisfying

$$S_{n_j} = \sum_{n=n_j+1}^{\infty}\frac{a_n}{a_{n+1}} \leq 2^{-j-2}. \tag{133}$$

Now, let us consider a dyadic interval $\lambda\subseteq[0,1)$, a real number $\alpha\in(0,1)$ and an integer $j\geq 0$. Let us assume that a real number x satisfies

$$\#\mathrm{M}((\{a_n x\})_{n\geq 1};\lambda,j) \leq \alpha\, 2^j. \tag{134}$$

This means that the first $2^{\langle\lambda\rangle+j}$ points $\{a_n x\}$ all belong to either the complement in $[0,1)$ of the interval λ, or some union of $\lfloor\alpha\, 2^j\rfloor$ dyadic subintervals of λ with generation equal to $\langle\lambda\rangle+j$. Letting $\lambda_1,\ldots,\lambda_{2^{\langle\lambda\rangle}-1}$ denote the dyadic intervals with the same generation as λ, excluding λ itself, and letting $\lambda_1',\ldots,\lambda'_{\lfloor\alpha\,2^j\rfloor}$ denote such subintervals of λ, we have in particular

$$\{a_{n_{\langle\lambda\rangle}+1}x\},\ldots,\{a_{2^{\langle\lambda\rangle+j}}x\} \in \lambda_1\sqcup\ldots\sqcup\lambda_{2^{\langle\lambda\rangle}-1}\sqcup\lambda_1'\sqcup\ldots\sqcup\lambda'_{\lfloor\alpha\,2^j\rfloor},$$

where the index $n_{\langle\lambda\rangle}$ is defined by (133). This means that, from now on, we forget the first $n_{\langle\lambda\rangle}$ points of the sequence and we assume that j is large enough to ensure that $2^{\langle\lambda\rangle+j}$ is greater than $n_{\langle\lambda\rangle}$. The intervals $\lambda_1,\ldots,\lambda_{2^{\langle\lambda\rangle}-1}$ and $\lambda_1',\ldots,\lambda'_{\lfloor\alpha\,2^j\rfloor}$ form a collection that is denoted by M. Moreover, these intervals are disjoint and their union is denoted by U. It will be useful to observe that, accordingly,

$$\mathbf{1}_U = \sum_{J\in\mathrm{M}}\mathbf{1}_J \qquad\text{and}\qquad \mathcal{L}^1(U) = \sum_{J\in\mathrm{M}}|J|.$$

Now, for any bounded interval I_0, adopting the notation (131) and letting a stand for the tuple formed by the real numbers $a_{n_{\langle\lambda\rangle}+1},\ldots,a_{2^{\langle\lambda\rangle+j}}$, we get

$$\int_{I_0}\prod_{n=n_{\langle\lambda\rangle}+1}^{2^{\langle\lambda\rangle+j}}\mathbf{1}_U(\{a_n x\})\,\mathrm{d}x = \sum_I\int_{I_0}P_{a,I}(x)\,\mathrm{d}x. \tag{135}$$

where the sum is over all choices of tuples of intervals $I_{n_{\langle\lambda\rangle}+1},\dots,I_{2^{\langle\lambda\rangle+j}}$ within the collection M. Observing that each of these intervals has length bounded below by $2^{-\langle\lambda\rangle-j}$, we may then use Lemma 6.3 to infer that the integral of the function $P_{a,I}$ over the interval I_0 is bounded above by

$$\left(|I_0|+\frac{2}{a_{n_{\langle\lambda\rangle}+1}}\right)\cdot\left(\prod_{n=n_{\langle\lambda\rangle}+1}^{2^{\langle\lambda\rangle+j}-1}\left(1+2^{\langle\lambda\rangle+j+1}\frac{a_n}{a_{n+1}}\right)\right)\cdot\left(\prod_{n=n_{\langle\lambda\rangle}+1}^{2^{\langle\lambda\rangle+j}}|I_n|\right).$$

Summing over all possible choices of I and then factorizing, we straightforwardly deduce that the expression in (135) is smaller than or equal to

$$\left(|I_0|+\frac{2}{a_{n_{\langle\lambda\rangle}+1}}\right)\cdot\left(\prod_{n=n_{\langle\lambda\rangle}+1}^{2^{\langle\lambda\rangle+j}-1}\left(1+2^{\langle\lambda\rangle+j+1}\frac{a_n}{a_{n+1}}\right)\right)\cdot(\mathcal{L}^1(U))^{2^{\langle\lambda\rangle+j}-n_{\langle\lambda\rangle}}.$$

This upper bound consists of three factors. The third one may easily be estimated after observing that

$$\begin{aligned}\mathcal{L}^1(U)&=|\lambda_1|+\ldots+|\lambda_{2^{\langle\lambda\rangle}-1}|+|\lambda'_1|+\ldots+|\lambda'_{\lfloor\alpha\,2^j\rfloor}|\\&=(2^{\langle\lambda\rangle}-1)2^{-\langle\lambda\rangle}+\lfloor\alpha\,2^j\rfloor 2^{-\langle\lambda\rangle-j}\le\exp(-(1-\alpha)2^{-\langle\lambda\rangle}).\end{aligned}$$

Here, we have used the obvious fact that $1+z\le \mathrm{e}^z$ for every real z. Combining this inequality with (133), we may also deal with the second factor, specifically,

$$\begin{aligned}\prod_{n=n_{\langle\lambda\rangle}+1}^{2^{\langle\lambda\rangle+j}-1}\left(1+2^{\langle\lambda\rangle+j+1}\frac{a_n}{a_{n+1}}\right)&\le\exp\sum_{n=n_{\langle\lambda\rangle}+1}^{2^{\langle\lambda\rangle+j}-1}2^{\langle\lambda\rangle+j+1}\frac{a_n}{a_{n+1}}\\&\le\exp(2^{\langle\lambda\rangle+j+1}S_{n_{\langle\lambda\rangle}})\le\exp(2^{j-1}).\end{aligned}$$

On top of that, note that the condition (134) introduced in the first place implies the choice of $\lfloor\alpha\,2^j\rfloor$ dyadic subintervals of λ with generation equal to $\langle\lambda\rangle+j$, among a total of 2^j possible intervals. We deduce that the set of all $x\in I_0$ for which the condition (134) holds has Lebesgue outer measure bounded above by

$$\binom{2^j}{\lfloor\alpha\,2^j\rfloor}\cdot\left(|I_0|+\frac{2}{a_{n_{\langle\lambda\rangle}+1}}\right)\cdot\exp(2^{j-1})\cdot\exp(-(1-\alpha)(2^j-n_{\langle\lambda\rangle}2^{-\langle\lambda\rangle})).$$

By virtue of Stirling's formula, the logarithm of the involved binomial coefficient is equivalent to $\mathrm{H}(\alpha)\,2^j$ as j goes to infinity, where $\mathrm{H}(\alpha)$ is a shorthand for the Shannon entropy of the probability vector $(\alpha,1-\alpha)$, as defined by (84). As a consequence, defining

$$m_j(\lambda,I_0,\alpha)=\mathcal{L}^1(\{x\in I_0\mid\#\mathrm{M}((\{a_nx\})_{n\ge1};\lambda,j)\le\alpha\,2^j\}),$$

we readily see that

$$\limsup_{j\to\infty}\frac{1}{2^j}\log m_j(\lambda,I_0,\alpha)\le\mathrm{H}(\alpha)+\alpha-\frac{1}{2}.$$

Clearly, the right-hand side vanishes for a unique value of $\alpha\in(0,1)$, denoted by α_0, and it is negative when $\alpha<\alpha_0$. In that case, we may conclude with the help of the Borel-Cantelli lemma. Indeed, for any j_0 sufficiently large, we have

$$\mathcal{L}^1\left(\limsup_{j\to\infty}\{x\in I_0\mid\#\mathrm{M}((\{a_nx\})_{n\ge1};\lambda,j)\le\alpha\,2^j\}\right)\le\sum_{j=j_0}^{\infty}m_j(\lambda,I_0,\alpha),$$

and the right-hand side tends to zero as j_0 goes to infinity, because the series is convergent when $\alpha<\alpha_0$. Making the interval I_0 increase to the whole real line, and the real number α increase to the critical value α_0 along countable sequences, we deduce that for Lebesgue-almost every $x \in \mathbb{R}$,

$$\liminf_{j\to\infty} 2^{-j} \#\mathrm{M}((\{a_n x\})_{n\geq 1}; \lambda, j) \geq \alpha_0.$$

As there are countably many dyadic intervals, it follows that for Lebesgue-almost every $x \in \mathbb{R}$, the sequence $(\{a_n x\})_{n\geq 1}$ satisfies (118) with $U = (0, 1)$. Hence, the weaker condition (110) is also verified and we conclude thanks to Theorem 6.3. □

Note that Theorem 6.12 does not apply to the case where $a_n = b^n$, which corresponds to the b-adic expansion of real numbers, simply because the corresponding series $\sum_n a_n/a_{n+1}$ does not converge. In fact, the hypothesis of Theorem 6.12 is satisfied if the sequence $(a_n)_{n\geq 1}$ grows superexponentially fast, such as for instance when $a_n = n^{(1+\varepsilon)n}$ for some $\varepsilon > 0$, or when $a_n = b^{n^2}$ for some $b > 1$.

Furthermore, we may combine Theorem 6.12 with the approach that we developed at the end of Section 6.3.2 above. This results in the following dimensional statement, in the vein of Corollary 6.1.

Corollary 6.2. *Let $(a_n)_{n\geq 1}$ be a sequence of positive real numbers such that the series $\sum_n a_n/a_{n+1}$ converges, and let $(r_n)_{n\geq 1}$ be a sequence in P_1 such that the series $\sum_n r_n^s$ converges for all $s > 1$. For any real number $t \geq 1$, let us define*

$$F'_t(x) = \left\{ y \in \mathbb{R} \;\middle|\; \|y - a_n x\| < r_n^t \quad \text{for i.m. } n \geq 1 \right\},$$

Then, for Lebesgue-almost every point $x \in \mathbb{R}$,

$$\forall t \geq 1 \qquad \dim_{\mathrm{H}} F'_t(x) = \frac{1}{t}.$$

In particular, if the approximating radii are given by $r_n = 1/n$, we end up with the following result: for Lebesgue-almost every $x \in \mathbb{R}$ and for every $\sigma \geq 1$,

$$\dim_{\mathrm{H}} \left\{ y \in \mathbb{R} \;\middle|\; \|y - a_n x\| < \frac{1}{n^\sigma} \quad \text{for i.m. } n \geq 1 \right\} = \frac{1}{\sigma}.$$

The tools introduced in the following chapters will enable us to substantially refine Corollary 6.2. In particular, Corollary 10.2 will give a precise and complete description of the size and large intersection properties of a family of sets that includes the above sets $F'_t(x)$. Let us also mention that a challenging problem is to understand how the Hausdorff dimension of sets of the form $F'_t(x)$ behaves when one considers their intersection with a given compact set. We do not address this problem here, and we refer to [**15**] for precise statements and motivations.

6.5. Random eutaxic sequences

The ideas pertaining in the proof of Theorem 6.12 above are in fact of a probabilistic nature. First, the proof calls upon the Borel-Cantelli lemma. Moreover, the ancillary lemma used therein, namely, Lemma 6.3 may actually be recast in terms of the correlations between the random variables $\{a_n X\}$, where X is uniformly distributed in the unit interval $[0, 1)$. This entices us to consider probabilistic models of eutaxic sequences. The simplest model consists of a sequence of points that are independently and uniformly distributed in some nonempty bounded open subset of $\mathbb{R}^d$. We shall also consider a model that is related with Poisson point processes.

6.5.1. Independent and uniform points. We consider a sequence $(X_n)_{n\geq 1}$ of points that are independently and uniformly distributed in a nonempty bounded open set $U \subseteq \mathbb{R}^d$. Hence, the random points X_n are stochastically independent and distributed according to the normalized Lebesgue measure $\mathcal{L}^d(\,\cdot\,\cap U)/\mathcal{L}^d(U)$. For any sequence $(r_n)_{n\geq 1}$ in P_d and any point x in U, we have

$$\mathbb{P}(x \in \mathrm{B}(X_n, r_n)) = \frac{\mathcal{L}^d(U \cap \mathrm{B}(x, r_n))}{\mathcal{L}^d(U)} = \frac{\mathcal{L}^d(\mathrm{B}(0,1))}{\mathcal{L}^d(U)}\, r_n^d$$

for n sufficiently large. Hence, the Borel-Cantelli lemma ensures that the inequality $|x - X_n| < r_n$ holds infinitely often with probability one. By virtue of Tonelli's theorem, this implies that the sequence $(X_n)_{n\geq 1}$ is almost surely eutaxic in U with respect to $(r_n)_{n\geq 1}$. Note that the almost sure event on which this property holds may depend on the sequence $(r_n)_{n\geq 1}$. In order to show that the sequence $(X_n)_{n\geq 1}$ is uniformly eutaxic in U, we need to develop the following additional arguments that are due to Reversat [**49**], and were already used in the proof of Theorem 6.12.

THEOREM 6.13. *Let $(X_n)_{n\geq 1}$ be a sequence of random points distributed independently and uniformly in a nonempty bounded open subset U of $\mathbb{R}^d$. Then, with probability one, the sequence $(X_n)_{n\geq 1}$ is uniformly eutaxic in U.*

PROOF. Let us consider a dyadic cube $\lambda \subseteq U$, a real number $\alpha \in (0,1)$ and an integer $j \geq 0$, and let us suppose that the condition

$$\#\mathrm{M}((X_n)_{n\geq 1}; \lambda, j) \leq \alpha\, 2^{dj} \tag{136}$$

holds. Then, the first $2^{d(\langle\lambda\rangle + j)}$ points X_n are contained in either the complement in $\mathbb{R}^d$ of the cube λ, or the union of $\lfloor \alpha\, 2^{dj} \rfloor$ subcubes of λ with generation $\langle\lambda\rangle + j$, denoted by $\lambda'_1, \ldots, \lambda'_{\lfloor \alpha\, 2^{dj} \rfloor}$. Each point X_n is uniformly distributed in U, so that

$$\mathbb{P}(X_n \in (\mathbb{R}^d \setminus \lambda) \sqcup \lambda'_1 \sqcup \ldots \sqcup \lambda'_{\lfloor \alpha\, 2^{dj} \rfloor}) = 1 - \frac{2^{-d\langle\lambda\rangle}}{\mathcal{L}^d(U)} + \lfloor \alpha\, 2^{dj} \rfloor \frac{2^{-d(\langle\lambda\rangle + j)}}{\mathcal{L}^d(U)}.$$

Moreover, combining the fact that the points X_n are independent with the obvious bound $1 + z \leq \mathrm{e}^z$, for z in $\mathbb{R}$, we deduce that

$$\mathbb{P}(X_1, \ldots, X_{2^{d(\langle\lambda\rangle + j)}} \in (\mathbb{R}^d \setminus \lambda) \sqcup \lambda'_1 \sqcup \ldots \sqcup \lambda'_{\lfloor \alpha\, 2^{dj} \rfloor}) \leq \exp\left(-\frac{1-\alpha}{\mathcal{L}^d(U)}\, 2^{dj}\right).$$

As a consequence, taking into account all the possible choices for the subcubes $\lambda'_1, \ldots, \lambda'_{\lfloor \alpha\, 2^{dj} \rfloor}$ that result from the assumption (136), we conclude that

$$\mathbb{P}(\#\mathrm{M}((X_n)_{n\geq 1}; \lambda, j) \leq \alpha\, 2^{dj}) \leq \binom{2^{dj}}{\lfloor \alpha\, 2^{dj} \rfloor} \exp\left(-\frac{1-\alpha}{\mathcal{L}^d(U)}\, 2^{dj}\right).$$

We now follow the lines of the proof of Theorem 6.12. The binomial coefficient above may again be estimated with the help of Stirling's formula: its logarithm is equivalent to $\mathrm{H}(\alpha)\, 2^{dj}$ as j goes to infinity, where $\mathrm{H}(\alpha)$ denotes the Shannon entropy of the probability vector $(\alpha, 1-\alpha)$, as defined by (84). Hence,

$$\limsup_{j\to\infty} \frac{1}{2^{dj}} \log \mathbb{P}(\#\mathrm{M}((X_n)_{n\geq 1}; \lambda, j) \leq \alpha\, 2^{dj}) \leq \mathrm{H}(\alpha) - \frac{1-\alpha}{\mathcal{L}^d(U)}.$$

The right-hand side vanishes for a unique value of $\alpha \in (0,1)$, that is denoted by α_0. Furthermore, the right-hand side is negative when $\alpha < \alpha_0$, and the Borel-Cantelli lemma ensures that almost surely, the condition (136) is satisfied for finitely many values of j only. Hence, for every dyadic cube $\lambda \subseteq U$ and every $\alpha \in (0, \alpha_0)$,

$$\text{a.s.} \qquad \liminf_{j\to\infty} 2^{-dj}\, \#\mathrm{M}((X_n)_{n\geq 1}; \lambda, j) \geq \alpha.$$

We may let α tend to α_0 along a countable sequence, and the limiting value α_0 does not depend on the choice of the dyadic cube λ. In addition, there are countably many dyadic cubes contained in U. The upshot is that the sequence $(X_n)_{n\geq 1}$ verifies (118) with probability one. Therefore, the weaker condition (110) is also satisfied almost surely, and we may conclude with the help of Theorem 6.3. $\square$

Blending Theorem 6.13 with Theorem 6.2, we get a first description of the size and large intersection properties of the random sets F_t defined for all $t \geq 1$ by

$$F_t = \left\{ x \in \mathbb{R}^d \;\middle|\; |x - X_n| < r_n^t \quad \text{for i.m. } n \geq 1 \right\}, \tag{137}$$

which is how (107) becomes in the present situation. More precisely, Theorems 6.2 and 6.13 directly lead to the following statement.

Corollary 6.3. *With probability one, for any nonincreasing sequence of positive real numbers $(r_n)_{n\geq 1}$ satisfying*

$$\begin{cases} s < d \implies \sum_n r_n^s = \infty \\ s > d \implies \sum_n r_n^s < \infty, \end{cases}$$

the following properties hold for all $t \geq 1$:

$$\dim_{\mathrm{H}}(F_t \cap U) = \frac{d}{t} \qquad \textit{and} \qquad F_t \in \mathcal{G}^{d/t}(U). \tag{138}$$

Note that the almost sure event on which the previous statement holds does not depend on the choice of the sequence $(r_n)_{n\geq 1}$. This is due to the fact that the almost sure eutaxy of the sequence $(X_n)_{n\geq 1}$ in the open set U is of uniform type. Furthermore, recall that we may easily extend Theorem 6.2 to sequences of approximating radii $\mathrm{r} = (r_n)_{n\geq 1}$ satisfying (109) for some positive real number s_{r}, instead of the mere (108). The same remark clearly applies to Corollary 6.3. Finally, restricting to power functions for the approximating radii, we have

$$\text{a.s.} \quad \forall c > 0 \quad \forall \sigma \geq \frac{1}{d} \qquad \dim_{\mathrm{H}} \left\{ x \in \mathbb{R}^d \;\middle|\; |x - X_n| < \frac{c}{n^\sigma} \quad \text{for i.m. } n \geq 1 \right\} = \frac{1}{\sigma}.$$

This follows from (138) with $r_n = (c^{1/\sigma}/n)^{1/d}$ and $t = \sigma d$. We thereby extend a result due to Fan and Wu [**30**], who addressed the case where $d = 1$ and $U = (0, 1)$.

The above study is related with the famous problem regarding random coverings of the circle raised in 1956 by Dvoretzky [**23**]. We now restrict our attention to the one-dimensional case. As mentioned above, the fact that a sequence $(r_n)_{n\geq 1}$ belongs to P_1 implies, through a simple application of the Borel-Cantelli lemma and Tonelli's theorem, that with probability one, *Lebesgue-almost every* point x of $(0, 1)$ is covered by the open interval centered at X_n with radius r_n, *i.e.* satisfies $|x - X_n| < r_n$, for infinitely many integers $n \geq 1$. Dvoretzky's question can then be recast as follows: find a necessary and sufficient condition on the sequence $(r_n)_{n\geq 1}$ to ensure that with probability one, *every* point of the open unit interval $(0, 1)$ satisfies the previous property. The problem raised the interest of many mathematicians such as Billard, Erdős, Kahane and Lévy, and was finally solved in 1972 by Shepp [**56**] who discovered that the condition is

$$\sum_{n=1}^{\infty} \frac{1}{n^2} \exp(2(r_1 + \ldots + r_n)) = \infty.$$

This criterion is very subtle in the sense that constants do matter: when r_n is of the specific form c/n with $c > 0$, the condition is satisfied if and only if $c \geq 1/2$. We refer to [**21**] and the references therein for more information on this topic.

We shall come back to the above random covering problem in Section 11.1 and give therein further results on the size and large intersection properties of the sets F_t, thus improving on Corollary 6.3.

6.5.2. Poisson point measures. Comparable results may be obtained when the approximating points and the approximation radii are distributed according to a Poisson point measure. We begin by briefly recalling some basic facts about Poisson measures; we refer to *e.g.* [**40, 47**] for additional details. The theory may be nicely developed for instance in locally compact topological spaces with a countable base. If S denotes such a topological space, we call a *point measure* on S any nonnegative measure ϖ on S that may be written as a sum of Dirac point masses, namely,

$$\varpi = \sum_{n\in\mathcal{N}} \delta_{s_n} \qquad \text{with} \qquad s_n \in S,$$

and that assigns a finite mass to each compact subset of S. Note that the above points s_n need not be distinct, but the index set $\mathcal{N}$ is necessarily countable. The set of all point measures may be endowed with the σ-field generated by the mappings $\varpi \mapsto \varpi(F)$, where F ranges over the Borel subsets of S. Naturally, a *random point measure* on S is then a measurable mapping Π defined on some abstract probability space and valued in the measurable space of point measures. One can show that the probability distribution of such a random point measure Π is characterized by the distributions of all the random vectors of the form $(\Pi(E_1), \ldots, \Pi(E_n))$, where the sets $E_1, \ldots, E_n$ range over any fixed class of relatively compact Borel subsets of S that is closed under finite intersections and generate the Borel σ-field on S. This enables us to now introduce our main definition.

DEFINITION 6.4. Let S be a locally compact topological space with a countable base, and let π be a positive Radon measure thereon. There exists a random point measure Π on S such that the following two properties hold:

- for every Borel subset E of S, the random variable $\Pi(E)$ is Poisson distributed with parameter $\pi(E)$;
- for all Borel subsets $E_1, \ldots, E_n$ of S that are pairwise disjoint, the random variables $\Pi(E_1), \ldots, \Pi(E_n)$ are independent.

The random point measure Π is called a *Poisson point measure* with intensity π, and its law is uniquely determined by the above two properties.

Note that we adopt the usual convention that a Poisson random variable with infinite parameter is almost surely equal to ∞. In addition to the aforementioned characterization, the distribution of a random point measure Π is also determined by its *Laplace functional*, namely, the mapping defined by the formula

$$\mathfrak{L}_\Pi(f) = \mathbb{E}\left[\exp\left(-\int_S f(s)\,\Pi(\mathrm{d}s)\right)\right],$$

where f is any nonnegative Borel measurable function defined on S. Thus, Π is a Poisson point measure with intensity π if and only if for any such f,

$$\mathfrak{L}_\Pi(f) = \exp\left(-\int_S (1-\mathrm{e}^{-f(s)})\,\pi(\mathrm{d}s)\right).$$

Throughout the remainder of this section, we shall restrict our attention to Poisson point measures on the interval $(0,1]$, the product space $(0,1]\times\mathbb{R}^d$, or subsets thereof. Let $\mathcal{R}$ be defined as the collection of all positive Radon measures ν on the interval $(0,1]$ such that ν has infinite total mass and

$$\forall\rho\in(0,1] \qquad \Phi_\nu(\rho) = \nu([\rho,1]) < \infty. \tag{139}$$

The function Φ_ν is then clearly nonincreasing on $(0,1]$. Moreover, at any given ρ, it is left-continuous with a finite right-limit, namely,

$$\Phi_\nu(\rho+) = \nu((\rho,1]).$$

Extending this notation to the case where ρ vanishes, we get that $\Phi_\nu(0+)$ is infinite because ν has infinite total mass. On top of that, given some nonempty open subset U of $\mathbb{R}^d$, we may consider on the product space

$$U_+ = (0,1] \times U$$

a Poisson point measure, denoted by Π, with intensity $\nu \otimes \mathcal{L}^d(\,\cdot\, \cap U)$. When the intensity measure has infinite total mass, the corresponding Poisson measure must almost surely have infinite total mass as well. As a result, there exists a sequence $(R_n, X_n)_{n\geq 1}$ of random pairs in U_+ such that with probability one,

$$\Pi = \sum_{n=1}^{\infty} \delta_{(R_n,X_n)}.$$

Our aim is now to study the approximation problem that results from distributing the approximating points and approximation radii according to the pairs (R_n, X_n). To be specific, in accordance with (107) again, we consider the random sets

$$F_t = \left\{ y \in \mathbb{R}^d \;\middle|\; |y - X_n| < R_n^t \quad \text{for i.m. } n \geq 1 \right\}, \tag{140}$$

for $t \geq 1$. Note that the Poisson point measure Π offers us an alternate way of defining the above sets. Indeed, F_t is also the set of points y in $\mathbb{R}^d$ such that

$$\int_{U_+} \mathbf{1}_{\{|y-x|<r^t\}}\, \Pi(\mathrm{d}r, \mathrm{d}x) = \sum_{n=1}^{\infty} \mathbf{1}_{\{|y-X_n|<R_n^t\}} = \infty.$$

The main result of this section is the following analog of Corollary 6.3 for the random sets F_t that are now under investigation.

THEOREM 6.14. *Let ν be a measure in $\mathcal{R}$, let U be a nonempty open subset of $\mathbb{R}^d$, and let Π be a Poisson point measure on U_+ with intensity $\nu \otimes \mathcal{L}^d(\,\cdot\, \cap U)$. For any real number $t \geq 1$, let us define*

$$F_t = \left\{ y \in \mathbb{R}^d \;\middle|\; \int_{U_+} \mathbf{1}_{\{|y-x|<r^t\}}\, \Pi(\mathrm{d}r, \mathrm{d}x) = \infty \right\}. \tag{141}$$

Let us assume that the measure ν satisfies the following integrability condition:

$$\begin{cases} s < d \implies \displaystyle\int_{(0,1]} r^s\, \nu(\mathrm{d}r) = \infty \\ s > d \implies \displaystyle\int_{(0,1]} r^s\, \nu(\mathrm{d}r) < \infty. \end{cases} \tag{142}$$

Then, with probability one, for all $t \geq 1$,

$$\dim_{\mathrm{H}}(F_t \cap U) = \frac{d}{t} \qquad \textit{and} \qquad F_t \in \mathcal{G}^{d/t}(U).$$

One may easily extend Theorem 6.14 to the more general case where d is replaced by some positive real number s_ν in the integrability condition (142). Indeed, for any real number $\alpha > 0$, the image Π_α of the Poisson point measure Π under the mapping $(r, x) \mapsto (r^\alpha, x)$ is a Poisson point measure on U_+ with intensity $\nu_\alpha \otimes \mathcal{L}^d(\,\cdot\, \cap U)$, where ν_α is the image of the measure ν under the mapping $r \mapsto r^\alpha$. Moreover, for each $t > 0$, the set $F_t^{(\alpha)}$ obtained when replacing Π by Π_α in (141) coincides with the original set $F_{\alpha t}$ corresponding to the Poisson point measure Π. Choosing $\alpha = s_\nu/d$, we easily check that the measure ν_α belongs to $\mathcal{R}$ and satisfies (142). We may then apply Theorem 6.14 to the corresponding Poisson point measure Π_α, thereby deducing that with probability one, for all $t \geq 1$,

$$\dim_{\mathrm{H}}(F_t^{(\alpha)} \cap U) = \frac{d}{t} \qquad \text{and} \qquad F_t^{(\alpha)} \in \mathcal{G}^{d/t}(U).$$

Performing a simple change of variable, we may transfer this statement to the original sets F_t, and thus conclude that with probability one, for all $t \geq s_\nu/d$,

$$\dim_{\mathrm{H}}(F_t \cap U) = \frac{s_\nu}{t} \qquad \text{and} \qquad F_t \in \mathcal{G}^{s_\nu/t}(U).$$

On top of that, with probability one, all the sets F_t, for $t < s_\nu/d$, have Hausdorff dimension d and belong to the class $\mathcal{G}^d(U)$; this easily follows from the observation that they all contain the set $F_{s_\nu/d}$.

The remainder of this section is devoted to establishing Theorem 6.14. We shall call upon a series of basic results that we now state and prove. The first lemma discusses how the sets F_t defined by (141) become distributed when one takes their intersection with an arbitrary nonempty bounded open subset of U.

LEMMA 6.4. *Let ν be a measure in $\mathcal{R}$, let U be a nonempty open subset of $\mathbb{R}^d$, and let Π be a Poisson point measure on U_+ with intensity $\nu \otimes \mathcal{L}^d(\,\cdot\, \cap U)$. Moreover, let V be a nonempty bounded open subset of U. For any real number $t \geq 1$, in addition to the set F_t given by (141), we define the set*

$$F_t^V = \left\{ y \in \mathbb{R}^d \;\middle|\; \int_{V_+} \mathbf{1}_{\{|y-x|<r^t\}}\, \Pi(\mathrm{d}r, \mathrm{d}x) = \infty \right\}. \tag{143}$$

Then, the following properties hold:

(1) *the restriction $\Pi(\,\cdot\, \cap V_+)$ is a Poisson point measure on V_+ with intensity*
$$\nu \otimes \mathcal{L}^d(\,\cdot\, \cap V)\,;$$
(2) *with probability one, for any real number $t \geq 1$,*
$$F_t \cap V \subseteq F_t^V \subseteq F_t \cap \overline{V}.$$

PROOF. The proof of (1) is easily obtained by computing the Laplace functional of the random point measure $\Pi(\,\cdot\, \cap V_+)$. In order to establish (2), we define V_1 as the set of points x in U such that $\mathrm{d}(x, V) < 1$, and we observe that for any $\rho \in (0, 1]$, the random variable $\Pi([\rho, 1] \times V_1)$ is Poisson distributed with parameter $\Phi_\nu(\rho)\mathcal{L}^d(V_1)$. This parameter is finite by virtue of (139) and the boundedness of V. Therefore, $\Pi([\rho, 1] \times V_1)$ is almost surely finite. However, this random variable is a monotonic function of ρ. We deduce that the probability that all the values $\Pi([\rho, 1] \times V_1)$, for $\rho \in (0, 1]$, are simultaneously finite is equal to one. From now on, we assume that the corresponding almost sure event holds.

Let us consider a point y in $F_t \cap V$. Given that the set V is open, it contains the open ball $\mathrm{B}(y, \delta)$ for some $\delta > 0$. Let us consider a pair (r, x) in U_+ satisfying $|y-x| < r^t$. Then, this pair actually belongs to V_+ when $r < \delta^{1/t}$, and to $[\delta^{1/t}, 1] \times V_1$ otherwise. As a consequence,

$$\infty = \int_{U_+} \mathbf{1}_{\{|y-x|<r^t\}}\, \Pi(\mathrm{d}r, \mathrm{d}x) \leq \int_{V_+} \mathbf{1}_{\{|y-x|<r^t\}}\, \Pi(\mathrm{d}r, \mathrm{d}x) + \Pi([\delta^{1/t}, 1] \times V_1).$$

On the almost sure event that we considered, the second term in the right-hand side of the above inequality is finite. It follows that the first term is infinite, *i.e.* the point y belongs to the set F_t^V.

Conversely, let us consider a point y in F_t^V. Given that V_+ is contained in U_+, the point y is then automatically in F_t. In order to show that y also belongs to the closure of V, it suffices to consider an arbitrary real number $\delta > 0$ and to prove that the ball $\mathrm{B}(y, \delta)$ meets V. If (r, x) denotes a pair V_+ with $|y - x| < r^t$, we remark that the point x belongs to the aforementioned ball if $r < \delta^{1/t}$, and simply to the set V_1 otherwise. Accordingly,

$$\infty = \int_{V_+} \mathbf{1}_{\{|y-x|<r^t\}}\, \Pi(\mathrm{d}r, \mathrm{d}x) \leq \Pi((0, 1] \times (\mathrm{B}(y, \delta) \cap V)) + \Pi([\delta^{1/t}, 1] \times V_1).$$

Again, the second term in the right-hand side is finite, so the first term is infinite, which means in particular that the sets $\mathrm{B}(y,\delta)$ and V intersect. □

Lemma 6.4 above will enable us to reduce the proof of Theorem 6.14 to the case of bounded open subsets of U. The advantage of working with bounded sets is that, with the help of the next lemma, we will be able to use a convenient representation of the Poisson point measure Π.

LEMMA 6.5. *Let ν be a measure in the collection $\mathcal{R}$, and let U be a nonempty bounded open subset of the space $\mathbb{R}^d$.*

(1) *Let N^U denote a Poisson point measure on the interval $(0,1]$ with intensity*
$$\nu^U = \mathcal{L}^d(U)\,\nu.$$
Then, there exists a nonincreasing sequence $(R_n)_{n\geq 1}$ of positive random variables that converges to zero such that with probability one,
$$\mathrm{N}^U = \sum_{n=1}^{\infty} \delta_{R_n}. \tag{144}$$

(2) *Let $(X_n)_{n\geq 1}$ be a sequence of random variables that are independently and uniformly distributed in U, and are also independent on N^U. Then,*
$$\mathrm{N}^U_+ = \sum_{n=1}^{\infty} \delta_{(R_n,X_n)} \tag{145}$$
is a Poisson point measure on U_+ with intensity $\nu\otimes\mathcal{L}^d(\,\cdot\,\cap U)$.

PROOF. In order to prove (1), we begin by observing that the Poisson point measure N^U must have infinite total mass with probability one, because its intensity ν^U has infinite total mass too. Thus, there is a sequence $(R_n)_{n\geq 1}$ of positive random variables such that (144) holds. However, the assumption (139) implies that
$$\forall \rho > 0 \qquad \mathbb{E}[\#\{n\geq 1 \mid R_n \geq \rho\}] = \mathbb{E}[\mathrm{N}^U([\rho,1])] = \Phi_{\nu^U}(\rho) < \infty.$$
Thus, $(R_n)_{n\geq 1}$ converges to zero with probability one. Now, up to rearranging the terms, we can assume that this sequence is nonincreasing and still verifies (144).

The property (2) may be established by computing the Laplace functional of the random point measure N^U_+. Let f denote a nonnegative Borel measurable function defined on U_+. Then, we have
$$\mathfrak{L}_{\mathrm{N}^U_+}(f) = \mathbb{E}\left[\exp\left(-\sum_{n=1}^{\infty} f(R_n,X_n)\right)\right] = \mathbb{E}\left[\prod_{n=1}^{\infty}\left(\int_U \mathrm{e}^{-f(R_n,x)}\frac{\mathrm{d}x}{\mathcal{L}^d(U)}\right)\right].$$
The right-hand side may be rewritten as the Laplace functional of the random point measure N^U evaluated at the nonnegative Borel measurable function
$$r \mapsto -\log\int_U \mathrm{e}^{-f(r,x)}\frac{\mathrm{d}x}{\mathcal{L}^d(U)}.$$
Since N^U is a Poisson point measure with intensity ν^U, we finally deduce that for every nonnegative Borel measurable function f defined on the set U_+, we have
$$\mathfrak{L}_{\mathrm{N}^U_+}(f) = \exp\left(-\int_{U_+}(1-\mathrm{e}^{-f(r,x)})\,\nu^U(\mathrm{d}r)\otimes\frac{\mathrm{d}x}{\mathcal{L}^d(U)}\right),$$
from which we may determine the law of the random point measure N^U_+. □

The representation supplied by Lemma 6.5 calls upon a sequence of independent uniform random points. In view of Theorem 6.13, it thus establishes a connection with eutaxy that we shall exploit in the upcoming proof of Theorem 6.14.

PROOF OF THEOREM 6.14. We begin by assuming that the open set U is bounded, thereby finding ourselves into the convenient setting of Lemma 6.5. The random point measures Π and N^U_+, appearing in the statement of Theorem 6.14 and that of Lemma 6.5, respectively, share the same distribution: both are Poisson point measures on U_+ with intensity $\nu\otimes\mathcal{L}^d(\,\cdot\cap U)$. We may therefore assume that Π is replaced by N^U_+ in the definition (141) of the random sets F_t under investigation. Equivalently, we may define the sets F_t through the formula (140), where the points X_n and the radii R_n are those given by Lemma 6.5.

Now, we infer from Theorem 6.13 that with probability one, the sequence $(X_n)_{n\geq 1}$ is almost surely uniformly eutaxic in U. On top of that, evaluating the Laplace functional of the Poisson point measure N^U at the functions $r\mapsto\theta\, r^s$, for all positive values of s and θ, we get

$$\mathbb{E}\left[\exp\left(-\theta\sum_{n=1}^{\infty}R_n^s\right)\right]=\exp\left(-\mathcal{L}^d(U)\int_{(0,1]}(1-\mathrm{e}^{-\theta\, r^s})\,\nu(\mathrm{d}r)\right).$$

Since ν is in the collection $\mathcal{R}$ and satisfies the integrability condition (142), the integral in the right-hand side is infinite if $s<d$. The expectation in the left-hand side is thus equal to zero, which means that the series $\sum_n R_n^s$ diverges almost surely. Furthermore, using twice the obvious fact that $1-\mathrm{e}^{-z}\leq z$ for all real numbers z, we deduce from the above equality that

$$\mathbb{E}\left[\frac{1}{\theta}\left(1-\exp\left(-\theta\sum_{n=1}^{\infty}R_n^s\right)\right)\right]\leq\mathcal{L}^d(U)\int_{(0,1]}r^s\,\nu(\mathrm{d}r),$$

where the right-hand side is finite if $s>d$. However, as θ goes to zero, the random variable in the expectation monotonically tends to the sum $\sum_n R_n^s$. We deduce from the monotone convergence theorem that this sum has finite expectation if $s>d$, thereby being finite almost surely. As a consequence, with probability one, $(R_n)_{n\geq1}$ is a nonincreasing sequence of positive real numbers satisfying (108), *i.e.* such that the series $\sum_n R_n^s$ is divergent for all $s<d$, and convergent for all $s>d$. Finally, it follows from Theorem 6.2 that with probability one, for any real number $t\geq 1$,

$$\dim_{\mathrm{H}}(F_t\cap U)=\frac{d}{t}\qquad\text{and}\qquad F_t\in\mathcal{G}^{d/t}(U).$$

The result is thus proven in the case where the open set U is bounded.

Let us drop the boundedness assumption on U. In order to recover the previous case, we consider a sequence $(U^{(\ell)})_{\ell\geq1}$ of bounded open subsets of U such that

$$U=\bigcup_{\ell=1}^{\infty}\uparrow U^{(\ell)}\qquad\text{with}\qquad\overline{U^{(\ell)}}\subseteq U^{(\ell+1)}.$$

For instance, we may define these sets through inner parallel bodies as in (102); specifically, the sets

$$U^{(\ell)}=\{x\in U\cap\mathrm{B}(0,\ell)\mid \mathrm{d}(x,\mathbb{R}^d\setminus(U\cap\mathrm{B}(0,\ell)))>1/\ell\}\tag{146}$$

are easily seen to verify the above properties. There is an integer $\ell_0\geq1$ such that the set $U^{(\ell_0)}$ is nonempty. Each subsequent set $U^{(\ell)}$ is therefore a nonempty bounded open set, and we may deduce from Lemma 6.4(1) that the restriction $\Pi(\,\cdot\cap U^{(\ell)}_+)$ is a Poisson point measure on $U^{(\ell)}_+$ with intensity $\nu\otimes\mathcal{L}^d(\,\cdot\cap U^{(\ell)})$. It follows from the bounded case that the corresponding approximation sets, defined as in (143), are such that with probability one, for any $t\geq1$ and any $\ell\geq\ell_0$,

$$\dim_{\mathrm{H}}(F_t^{U^{(\ell)}}\cap U^{(\ell)})=\frac{d}{t}\qquad\text{and}\qquad F_t^{U^{(\ell)}}\in\mathcal{G}^{d/t}(U^{(\ell)}).$$

On top of that, combining Lemma 6.4(2) with the properties of the sets $U^{(\ell)}$, we observe that for any real number $t \geq 1$,

$$\bigcup_{\ell=\ell_0}^{\infty} \uparrow (F_t \cap U^{(\ell)}) \subseteq \bigcup_{\ell=\ell_0}^{\infty} \uparrow F_t^{U^{(\ell)}} \subseteq \bigcup_{\ell=\ell_0}^{\infty} \uparrow (F_t \cap \overline{U^{(\ell)}}) \subseteq \bigcup_{\ell=\ell_0}^{\infty} \uparrow (F_t \cap U^{(\ell+1)}), \quad (147)$$

where the leftmost and the rightmost sets are both equal to $F_t \cap U$. In particular, due to Proposition 2.16(2), we deduce that

$$\dim_{\mathrm{H}}(F_t \cap U) = \dim_{\mathrm{H}} \bigcup_{\ell=\ell_0}^{\infty} \uparrow F_t^{U^{(\ell)}} = \sup_{\ell \geq \ell_0} \dim_{\mathrm{H}}(F_t^{U^{(\ell)}} \cap U^{(\ell)}) = \frac{d}{t}.$$

In order to prove that each set F_t belongs to the large intersection class $\mathcal{G}^{d/t}(U)$, Definition 5.2 requires us to show that it is a G_δ-subset of $\mathbb{R}^d$ and that for any positive real number $s < d/t$ and any open subset $\widetilde{U}$ of U,

$$\mathcal{M}^s_\infty(F_t \cap \widetilde{U}) = \mathcal{M}^s_\infty(\widetilde{U})$$

The first property follows straightforwardly from (140). Moreover, as regards the second property, Lemma 5.2 implies that it suffices to establish the above equality for all dyadic cubes contained in U rather than for all such open sets $\widetilde{U}$. Specifically, since there are countably many such cubes, it suffices to fix a nonempty dyadic cube $\lambda \subseteq U$ and to prove that with probability one, for all $t \geq 1$ and $s \in (0, d/t)$,

$$\mathcal{M}^s_\infty(F_t \cap \lambda) = \mathcal{M}^s_\infty(\lambda).$$

This property follows from the bounded case. Indeed, since the interior of the cube λ is a nonempty bounded open subset of U, what precedes ensures that almost surely, for every $t \geq 1$, the set $F_t^{\operatorname{int}\lambda}$ defined as in (143) belongs to the class $\mathcal{G}^{d/t}(\operatorname{int}\lambda)$. Hence, making also use of Lemmas 5.1 and 6.4, we deduce that for all $s \in (0, d/t)$,

$$\mathcal{M}^s_\infty(F_t \cap \lambda) \geq \mathcal{M}^s_\infty(F_t^{\operatorname{int}\lambda} \cap \operatorname{int}\lambda) = \mathcal{M}^s_\infty(\operatorname{int}\lambda) = \mathcal{M}^s_\infty(\lambda),$$

which gives the required result. □

Much more precise results, actually a full and complete description of the size and large intersection properties of Poisson random coverings, will be given in Section 11.2. Besides, in the spirit of Dvoretzky's covering problem briefly discussed in Section 6.5.1, one may ask for a necessarily and sufficient condition on the measure ν to ensure that with probability one, *all* the points of the open set U are covered by the Poisson distributed balls, *i.e.* that the set F_1 obtained by choosing $t = 1$ in (141) contains the whole open set U almost surely. This problem was posed by Mandelbrot [**45**] and solved by Shepp [**57**] in dimension $d = 1$ when the open set U is equal to the whole real line. We refer to [**8**] and the references therein for further results in that direction.

CHAPTER 7

Optimal regular systems

The notion of optimal regular system was introduced by Baker and Schmidt [**1**], and subsequently refined by Beresnevich [**3**]. These systems result from the combination of a countably infinite subset $\mathcal{A}$ of $\mathbb{R}^d$ with a *height* function $H : \mathcal{A} \to (0, \infty)$. As we shall explain below, they encompass many relevant examples arising in the metric theory of Diophantine approximation. On top of that, they naturally give rise to uniformly eutaxic sequences; we shall thus be able to apply Theorem 6.2 to determine the basic size and large intersection properties of the set F_t defined by (107) when the considered sequences result from an optimal regular system.

However, in the metric theory of Diophantine approximation, the notion of optimal regular system is usually employed without a detour to eutaxic sequences. In that spirit, considering such a system $(\mathcal{A}, H)$, we shall replace the set F_1 obtained by letting $t = 1$ in (107) by the set

$$F_\varphi = \left\{ x \in \mathbb{R}^d \;\middle|\; |x - a| < \varphi(H(a)) \quad \text{for i.m. } a \in \mathcal{A} \right\} \tag{148}$$

associated with some positive nonincreasing continuous function φ defined on the interval $[0, \infty)$, and more generally the sets F_t by the sets F_{φ^t} obtained by replacing the function φ by its t-th power in (148). The pair $(\mathcal{A}, H)$ has to be *admissible*, in the sense that the following condition holds:

$$\forall m \in \mathbb{N} \qquad \# \left\{ a \in \mathcal{A} \;\middle|\; |a| < m \text{ and } H(a) \leq m \right\} < \infty. \tag{149}$$

In order to justify this admissibility condition, we may point out that if φ also tends to zero at infinity, then (149) implies that the family $(a, \varphi(H(a)))_{a \in \mathcal{A}}$ of elements of $\mathbb{R}^d \times (0, \infty)$ is an approximation system in the sense of Definition 4.1.

The relationship with Diophantine approximation is discussed more thoroughly in Section 7.2. Examples of optimal regular systems include the points with rational coordinates and the real algebraic numbers of bounded degree associated with suitable height functions. They will be dealt with in Sections 7.3 and 7.4, along with their implications in the metric theory of Diophantine approximation.

7.1. Definition and connection with eutaxy

Our purpose now is to define the notion of optimal regular system, and to discuss the link with eutaxic sequences.

Definition 7.1. Let $\mathcal{A}$ be a countably infinite subset of $\mathbb{R}^d$, let $H : \mathcal{A} \to (0, \infty)$ be a *height* function, and let U be a nonempty open subset of $\mathbb{R}^d$.

(1) The pair $(\mathcal{A}, H)$ is called a *regular system in U* if it is admissible and if one may find a real number $\kappa > 0$ such that for any open ball $B \subseteq U$, there is a real number $h_B > 0$ such that for all $h > h_B$, there exists a subset $\mathcal{A}_{B,h}$ of $\mathcal{A} \cap B$ with

$$\begin{cases} \#\mathcal{A}_{B,h} \geq \kappa |B|^d h^d \\ \forall a \in \mathcal{A}_{B,h} \qquad H(a) \leq h \\ \forall a, a' \in \mathcal{A}_{B,h} \qquad a \neq a' \quad \Longrightarrow \quad |a - a'| \geq 1/h. \end{cases}$$

(2) The pair $(\mathcal{A}, H)$ is called an *optimal system in* U if it is admissible and if for any open ball B, there exist two real numbers $\kappa'_B > 0$ and $h'_B > 0$ such that for all $h > h'_B$,

$$\#\{a \in \mathcal{A} \cap U \cap B \mid H(a) \leq h\} \leq \kappa'_B \, h^d. \tag{150}$$

Throughout what follows, we shall freely employ the notations of Definition 7.1 without necessarily reintroducing them. It is elementary to remark that any regular system in U is also regular in every nonempty open subset of U ; the same observation holds for the optimality property. Moreover, when the set U is bounded, the next lemma shows that any regular system therein may be enumerated monotonically with respect to the height function. The resulting enumerations will play a key rôle in the connection between optimal regular systems and eutaxic sequences.

Lemma 7.1. *Let U be a nonempty bounded open subset of $\mathbb{R}^d$, and let $(\mathcal{A}, H)$ denote a regular system in U. Then, there exists an enumeration $(a_n)_{n\geq 1}$ of the set $\mathcal{A} \cap U$ such that $H(a_n)$ monotonically tends to infinity as $n \to \infty$.*

Proof. On the one hand, the regularity property of the system $(\mathcal{A}, H)$ ensures that the set $\mathcal{A} \cap U$ is countably infinite. On the other hand, as the set U is bounded, it is contained in the open ball $\mathrm{B}(0, m)$, for m sufficiently large, and the admissibility condition (149) implies that for any $h > 0$, only finitely many points in $\mathcal{A} \cap U$ have height bounded above by h. We deduce the existence of an increasing sequence $(h_j)_{j\geq 1}$ of nonnegative integers with initial term zero and such that all the sets

$$A_j = \{a \in \mathcal{A} \cap U \mid h_j < H(a) \leq h_{j+1}\}$$

are both nonempty and finite. For each integer $j \geq 1$, we write the elements of the set A_j in the form $a_1^{(j)}, \ldots, a_{\#A_j}^{(j)}$, in such a way that

$$H(a_1^{(j)}) \leq \ldots \leq H(a_{\#A_j}^{(j)}).$$

It is clear that for any integer $n \geq 1$, there is a unique pair of integers (j, k), with $j \geq 1$ and $k \in \{1, \ldots, \#A_j\}$, such that

$$n = \#A_1 + \ldots + \#A_{j-1} + k.$$

We then define a_n as being equal to $a_k^{(j)}$, and it is elementary to check that the sequence $(a_n)_{n\geq 1}$ fulfills the conditions of the lemma. □

Any sequence $(a_n)_{n\geq 1}$ resulting from Lemma 7.1 will be called a *monotonic enumeration* of the regular system $(\mathcal{A}, H)$ in the set U. We now present the first part of the connection between optimal regular systems and eutaxic sequences.

Proposition 7.1. *Let U be a nonempty bounded open subset of $\mathbb{R}^d$, let $(\mathcal{A}, H)$ be an optimal regular system in U, and let $(a_n)_{n\geq 1}$ denote a monotonic enumeration of $(\mathcal{A}, H)$ in U. Then, the sequence $(a_n)_{n\geq 1}$ is uniformly eutaxic in U. In fact,*

$$\inf_{\substack{\lambda \in \Lambda \setminus \{\emptyset\} \\ \lambda \subseteq U}} \liminf_{j\to\infty} 2^{-dj} \#\mathrm{M}((a_n)_{n\geq 1}; \lambda, j) > 0. \tag{151}$$

Proof. The set U being bounded, it is contained in some open ball B. We consider a real number $\gamma \in (0, 1)$ such that $\kappa'_B \gamma \leq |[0, 1)^d|^d$, and a nonempty dyadic cube λ contained in U. Observe that there exists an open ball $B' \subseteq \lambda$ satisfying $|B'| = |\lambda|$. Then, let j be a nonnegative integer so large that

$$h = \gamma^{1/d} \frac{2^j}{|\lambda|} > \max\{h'_B, h_{B'}\}.$$

The choice of h ensures that any dyadic subcube λ' of λ with generation equal to $\langle\lambda\rangle + j$ cannot contain more than one point of the set $\mathcal{A}_{B',h}$. Otherwise, we would have two distinct points in $\mathcal{A}_{B',h}$ at a distance bounded above by

$$|\lambda'| = 2^{-j}|\lambda| = \frac{\gamma^{1/d}}{h} < \frac{1}{h},$$

which would contradict the third property satisfied by $\mathcal{A}_{B',h}$. Moreover, every point contained in $\mathcal{A}_{B',h}$ has height bounded above by h and belongs to the set $\mathcal{A} \cap U$, thereby being of the form a_n for some $n \geq 1$. The monotonicity of the enumeration implies that n is actually bounded above by

$$\#\{a \in \mathcal{A} \cap U \cap B \mid H(a) \leq h\} \leq \kappa'_B\, h^d = \kappa'_B \left(\gamma^{1/d}\frac{2^j}{|\lambda|}\right)^d \leq \left(|[0,1)^d|\frac{2^j}{|\lambda|}\right)^d,$$

so that $n \leq 2^{d(\langle\lambda\rangle+j)}$. Lastly, all the points of $\mathcal{A}_{B',h}$ are contained in B', and thus belong to some dyadic subcube of λ with generation $\langle\lambda\rangle + j$. We deduce that

$$\#\mathrm{M}((a_n)_{n\geq 1};\lambda,j) \geq \#\mathcal{A}_{B',h} \geq \kappa|B'|^d h^d = \kappa\left(|\lambda|\gamma^{1/d}\frac{2^j}{|\lambda|}\right)^d = \kappa\gamma 2^{dj},$$

and we end up with (151) by letting j tend to infinity. Hence, the sequence $(a_n)_{n\geq 1}$ satisfies the condition (118), and so the weaker condition (110) holds as well. The uniform eutaxy of the sequence thus follows from Theorem 6.3. □

Further investigating the connection between optimal regular systems and eutaxic sequences, we now give a converse result to Proposition 7.1. We start from the property (151) that already appeared in the statement of this proposition and is in fact stronger than uniform eutaxy. This means that we assume that the sequence under consideration satisfies a condition of the form (118). As already observed, this condition implies the sufficient condition (110) that guarantees uniform eutaxy.

Proposition 7.2. *Let U be a nonempty open subset of $\mathbb{R}^d$, and let $(a_n)_{n\geq 1}$ denote a sequence of points contained in U. We assume that (151) holds, so that in particular $(a_n)_{n\geq 1}$ is uniformly eutaxic in U. Moreover, let $\mathcal{A}$ denote the collection of all values a_n, for $n \geq 1$. We endow $\mathcal{A}$ with the height function H defined by*

$$H(a) = \inf\{n \geq 1 \mid a = a_n\}^{1/d}.$$

Then, the pair $(\mathcal{A}, H)$ is an optimal regular system in the open set U.

Proof. For any open ball B and any real number $h > 0$, it is clear that a point $a \in \mathcal{A} \cap U \cap B$ satisfying $H(a) \leq h$ is among the points $a_1, \ldots, a_{\lfloor h^d \rfloor}$. This proves that the pair $(\mathcal{A}, H)$ is admissible, and is in fact an optimal system in U.

Let us now establish that $(\mathcal{A}, H)$ is a also a regular system in U. Throughout, c denotes a real number such that $|x|_\infty/c \leq |x| \leq c|x|_\infty$ for all x in $\mathbb{R}^d$. Let B be a nonempty open ball contained in U, and let λ_B denote a nonempty dyadic cube contained in B with minimal generation. One easily checks that $|B| \leq 6c\, 2^{-\langle\lambda_B\rangle}$. Moreover, there is an integer $\underline{j}(\lambda_B) \geq 0$ such that

$$\forall j \geq \underline{j}(\lambda_B) \qquad \#\mathrm{M}((a_n)_{n\geq 1};\lambda_B,j) \geq \alpha\, 2^{d(j-1)},$$

where α denotes the left-hand side of (151). Thus, just as in the proof of Theorem 6.3, detailed in Section 6.2.1, we infer that for any integer $j \geq \underline{j}(\lambda_B)$, there exists a set $S_j(\lambda_B) \subseteq \{1, \ldots, 2^{d(\langle\lambda_B\rangle+j)}\}$ satisfying the following properties:

- $\#S_j(\lambda_B) \geq \alpha\, 2^{d(j-2)}$;
- $a_n \in \lambda_B$ for any $n \in S_j(\lambda_B)$;
- $|a_n - a_{n'}|_\infty \geq 2^{-(\langle\lambda_B\rangle+j)}$ for any distinct $n, n' \in S_j(\lambda_B)$.

For any real number h larger than $c\,2^{\langle\lambda_B\rangle+\underline{j}(\lambda_B)}$, letting j be equal to the integer $\lfloor\log_2(h/c)\rfloor-\langle\lambda_B\rangle$, where $\log_2$ is the base two logarithm, we have $j\geq\underline{j}(\lambda_B)$. Hence, we may define $\mathcal{A}_{B,h}$ as the collection of all points a_n, for n in $S_j(\lambda_B)$. It is then straightforward to check that $\mathcal{A}_{B,h}$ is a subset of $\mathcal{A}\cap B$ such that

$$\begin{cases} \#\mathcal{A}_{B,h}=\#S_j(\lambda_B)\geq\alpha\,2^{d(j-2)}\geq\alpha|B|^d h^d/(48c^2)^d \\ \forall a\in\mathcal{A}_{B,h} \qquad H(a)\leq(2^{d(\langle\lambda_B\rangle+j)})^{1/d}\leq h/c\leq h \\ \forall a,a'\in\mathcal{A}_{B,h} \qquad a\neq a' \quad\Longrightarrow\quad |a-a'|\geq 2^{-(\langle\lambda_B\rangle+j)}/c\geq 1/h, \end{cases}$$

and we deduce that the pair $(\mathcal{A},H)$ is a regular system in the set U. □

Combining Propositions 7.1 and 7.2, we may finally deduce that, rather than being equivalent to uniform eutaxy, the notion of optimal regular system is essentially comparable with the stronger condition (151).

7.2. Approximation by optimal regular systems

Proposition 7.1 can be combined with Theorem 6.2 to determine the basic size and large intersection properties of the set F_t defined by (107) when the considered sequences result from an optimal regular system. However, as mentioned at the beginning of Section 7.1, we shall follow the common practice from metric Diophantine approximation and state our results without a detour to eutaxic sequences. Thus, given an optimal regular system $(\mathcal{A},H)$, we replace the set F_1 obtained by letting $t=1$ in (107) by the set F_φ defined by (148), and more generally the sets F_t by the sets F_{φ^t} obtained when replacing φ by its t-th power. The basic size and large intersection properties of the sets F_{φ^t} are given by the next result.

THEOREM 7.1. *Let φ denote a positive nonincreasing continuous function defined on the interval $[0,\infty)$, and let I_φ be the integral defined by*

$$I_\varphi=\int_0^\infty \eta^{d-1}\varphi(\eta)^d\,\mathrm{d}\eta. \tag{152}$$

Moreover, let U denote a nonempty open subset of $\mathbb{R}^d$, and let $(\mathcal{A},H)$ denote an optimal regular system in U.

(1) *The set F_φ has full, or zero, Lebesgue measure in the open set U according to whether the integral I_φ diverges, or converges, respectively.*
(2) *Let us assume that the function φ tends to zero at infinity and that the integral I_φ diverges. Then, the family $(a,\varphi(H(a)))_{a\in\mathcal{A}}$ is a homogeneous ubiquitous system in U.*
(3) *Let us assume that the positive powers of the function φ are such that*

$$\begin{cases} t<1 \quad\Longrightarrow\quad I_{\varphi^t}=\infty \\ t>1 \quad\Longrightarrow\quad I_{\varphi^t}<\infty \end{cases}$$

Then, for any real number $t\geq 1$,

$$\dim_{\mathrm{H}}(F_{\varphi^t}\cap U)=\frac{d}{t} \qquad \textit{and} \qquad F_{\varphi^t}\in\mathcal{G}^{d/t}(U).$$

PROOF. The open set U may clearly be written as a countable union of open balls B_n. For instance, we can consider the open balls contained in U, with center in $\mathbb{Q}^d$ and radius in $\mathbb{Q}\cap(0,\infty)$. We deduce that

$$\mathcal{L}^d(U\setminus F_\varphi)\leq\sum_{n=1}^\infty\mathcal{L}^d(B_n\setminus F_\varphi) \qquad\text{and}\qquad \mathcal{L}^d(F_\varphi\cap U)\leq\sum_{n=1}^\infty\mathcal{L}^d(F_\varphi\cap B_n).$$

As a consequence, the proof of (1) reduces to establishing the next property:

(1') For any open ball B contained in U, the set F_φ has full, or zero, Lebesgue measure in B according to whether I_φ diverges, or converges, respectively.

We begin by proving (1') in the divergence case. If B denotes a nonempty open ball contained in U, the pair $(\mathcal{A}, H)$ is also an optimal regular system in B, so Lemma 7.1 enables us to consider a monotonic enumeration of $(\mathcal{A}, H)$ in B, denoted by $(a_n)_{n\geq 1}$. Then, it is clear that F_φ contains the set F_φ^B defined by

$$F_\varphi^B = \left\{x \in \mathbb{R}^d \;\middle|\; |x - a_n| < r_n \quad \text{for i.m. } n \geq 1\right\}, \tag{153}$$

where $r_n = \varphi(H(a_n))$ for any $n \geq 1$. By virtue of Proposition 7.1, the sequence $(a_n)_{n\geq 1}$ is uniformly eutaxic in B. Moreover, the sequence $(r_n)_{n\geq 1}$ is in P_d when the integral I_φ diverges, see below. It follows that for Lebesgue-almost every x in B, there are infinitely many integers $n \geq 1$ such that $|x - a_n| < r_n$. Hence, the set F_φ^B has full Lebesgue measure in B, owing to (153). The same property thus holds for the set F_φ as well, and we deduce (1') in the divergence case.

The fact that $(r_n)_{n\geq 1}$ is in P_d when I_φ diverges may be proven as follows. First, we may clearly assume that the function φ converges to zero at infinity; the result is elementary otherwise. Let ζ be the premeasure defined on the intervals of the form (h, h'), with $0 < h \leq h' < \infty$, by the formula $\zeta((h, h')) = \varphi(h)^d - \varphi(h')^d$, and let ζ_* be the outer measure defined by (53). It follows from Theorem 2.4 that the Borel sets contained in $(0, \infty)$ are ζ_*-measurable. The resulting Borel measure is called the *Lebesgue-Stieltjes measure* associated with the monotonic function φ^d, and we may integrate locally bounded Borel-measurable functions with respect to that measure. Adapting the proof of Proposition 2.8, we remark that the above outer measure ζ_* is also equal to the outer measure ζ^* defined by (51). We may also adapt the proof of Proposition 2.9 in order to prove that ζ_* coincides with the premeasure ζ on the intervals where it is defined. Combining this observation with Proposition 2.4(1) and the fact that φ tends to zero at infinity, we deduce in particular that $\zeta_*([h, \infty)) = \varphi(h)^d$ for any real number $h > 0$. Accordingly, using Tonelli's theorem and the regularity of the system, we have

$$\begin{aligned}
\sum_{n=1}^{\infty} r_n^d &= \sum_{n=1}^{\infty} \int_0^\infty \mathbf{1}_{\{H(a_n)\leq h\}}\, \zeta_*(\mathrm{d}h) = \int_0^\infty \#\{n \geq 1 \mid H(a_n) \leq h\}\, \zeta_*(\mathrm{d}h) \\
&\geq \int_0^\infty \kappa |B|^d h^d\, \zeta_*(\mathrm{d}h) + \underbrace{\int_0^{h_B} \left(\#\{n \geq 1 \mid H(a_n) \leq h\} - \kappa |B|^d h^d\right) \zeta_*(\mathrm{d}h)}_{R} \\
&= \kappa |B|^d \int_0^\infty \int_0^h d\, \eta^{d-1}\, \mathrm{d}\eta\, \zeta_*(\mathrm{d}h) + R = \kappa d |B|^d \int_0^\infty \eta^{d-1} \zeta_*([\eta, \infty))\, \mathrm{d}\eta + R \\
&= \kappa d |B|^d I_\varphi + R,
\end{aligned}$$

which proves that $(r_n)_{n\geq 1}$ belongs to P_d when I_φ is divergent.

We now prove (1') in the convergence case, using the above notations in addition to those of Definition 7.1. Note that the intersection $F_\varphi \cap B$ is contained in the set F_φ^B defined by (153). Indeed, let x denote a point in this intersection. The ball B being open, it contains a ball B' of the form $\mathrm{B}(x, r)$ for a sufficiently small $r > 0$. Moreover, the function φ necessarily tends to zero at infinity, in view of the convergence of the integral I_φ. This means that $\varphi(h) \leq r$ for any real number h larger than some h_0. Now, there exists an infinite subset $\mathcal{A}_x$ of $\mathcal{A}$ formed by points a satisfying $|x - a| < \varphi(H(a))$. In particular, all these points belong to the open ball centered at x with radius $\varphi(0)$, so that

$$\{a \in \mathcal{A}_x \mid H(a) \leq h_0\} \subseteq \left\{a \in \mathcal{A} \;\middle|\; |a| < |x| + \varphi(0) \text{ and } H(a) \leq h_0\right\}.$$

The latter set is finite in view of the admissibility condition (149). It follows that infinitely many points a in the set $\mathcal{A}_x$ have height larger than h_0, thereby satisfying $\varphi(H(a)) \leq r$. All these points thus belong to the ball B', and must then be of the form a_n for some integer $n \geq 1$. We deduce that x belongs to the set F_φ^B.

Furthermore, due to (153), the set F_φ^B is covered by the open balls centered at a_n with radius r_n, for n starting from any fixed $n_0 \geq 1$. Adapting the proof of Proposition 1.1, we get

$$\mathcal{L}^d(F_\varphi \cap B) \leq \mathcal{L}^d(F_\varphi^B) \leq \sum_{n=n_0}^{\infty} \mathcal{L}^d(\mathrm{B}(a_n, r_n)) = \mathcal{L}^d(\mathrm{B}(0,1)) \sum_{n=n_0}^{\infty} r_n^d.$$

The convergence part of (1') now follows from letting n_0 go to infinity and observing that the series appearing in the above bound is convergent when the integral I_φ is convergent. As a matter of fact, reproducing the above reasoning and using the optimality of the system, we obtain

$$\begin{aligned}
\sum_{n=1}^{\infty} r_n^d &= \int_0^\infty \#\{n \geq 1 \mid H(a_n) \leq h\}\, \zeta_*(\mathrm{d}h) \\
&\leq \int_0^\infty \kappa'_B h^d \, \zeta_*(\mathrm{d}h) + \underbrace{\int_0^{h'_B} \left(\#\{n \geq 1 \mid H(a_n) \leq h\} - \kappa'_B h^d\right) \zeta_*(\mathrm{d}h)}_{R'} \\
&= \kappa'_B d I_\varphi + R'.
\end{aligned}$$

Owing to the admissibility condition (149), there are finitely many points a_n with height bounded above by h'_B, so that the integral R' is finite. Finally, the series $\sum_n r_n^d$ converges when the integral I_φ does.

Let us turn our attention to (2). As mentioned at the beginning of Section 7.1, if φ tends to zero at infinity, the admissibility condition (149) implies that the family $(a, \varphi(H(a)))_{a\in\mathcal{A}}$ is an approximation system in the sense of Definition 4.1. Now, if the integral I_φ diverges, it follows from (1) that the set F_φ has full Lebesgue measure in U. The definition (148) of this set, and that of a homogeneous ubiquitous system, *i.e.* Definition 4.2, then straightforwardly lead to (2).

In order to establish (3), let us assume that the integral I_{φ^t} diverges for $t < 1$, and converges for $t > 1$. We consider a nonempty open ball $B \subseteq U$ and we adopt the same notations as in the proof of (1'). The above arguments imply that

$$F_{\varphi^t} \cap B \subseteq F_{\varphi^t}^B \subseteq F_{\varphi^t}, \tag{154}$$

where $F_{\varphi^t}^B$ denotes the set obtained by raising r_n to the power t in (153). Moreover, in view of the hypotheses on the integrals I_{φ^t}, the sequence $(r_n)_{n\geq1}$ satisfies (108), *i.e.* the series $\sum_n r_n^s$ diverges when $s < d$, and converges when $s > d$. Recalling that $(a_n)_{n\geq1}$ is uniformly eutaxic in B, we deduce from Theorem 6.2 that

$$\dim_{\mathrm{H}}(F_{\varphi^t}^B \cap B) = \frac{d}{t} \qquad \text{and} \qquad F_{\varphi^t}^B \in \mathcal{G}^{d/t}(B).$$

To conclude, recall that U may be written as a countable union of open balls B_n. Combining Proposition 2.16(2) with (154), we get

$$\dim_{\mathrm{H}}(F_{\varphi^t} \cap U) = \sup_{n\geq1} \dim_{\mathrm{H}}(F_{\varphi^t} \cap B_n) = \sup_{n\geq1} \dim_{\mathrm{H}}(F_{\varphi^t}^{B_n} \cap B_n) = \frac{d}{t}.$$

Furthermore, according to Definition 5.2, proving that F_{φ^t} belongs to the large intersection class $\mathcal{G}^{d/t}(U)$ amounts to establishing that

$$\mathcal{M}_\infty^s(F_{\varphi^t} \cap V) = \mathcal{M}_\infty^s(V) \tag{155}$$

for any positive real number $s < d/t$ and any open subset V of U. To this purpose, let us consider a dyadic cube $\lambda \in \Lambda$ contained in V. Thanks to (154), we have

$$\mathcal{M}^s_\infty(F_{\varphi^t} \cap \lambda) \geq \mathcal{M}^s_\infty(F^B_{\varphi^t} \cap \operatorname{int}\lambda) = \mathcal{M}^s_\infty(\operatorname{int}\lambda) = \mathcal{M}^s_\infty(\lambda).$$

where B denotes an arbitrary open ball sandwiched between λ and V. Here, we have combined Lemma 5.1 together with the fact that $F^B_{\varphi^t}$ belongs to $\mathcal{G}^{d/t}(B)$. It finally suffices to apply Lemma 5.2 to obtain (155). □

7.3. Application to homogeneous and inhomogeneous approximation

A simple example of optimal regular system is supplied by the points with rational coordinates; this corresponds to the classical problem of homogeneous Diophantine approximation. We now detail this example, as well as its inhomogeneous counterpart. We shall then state the corresponding metric results obtained by further applying Theorem 7.1, namely, a famous theorem by Khintchine [**38**] and an inhomogeneous analog of Theorem 3.1, *i.e.* the Jarník-Besicovitch theorem.

7.3.1. Homogeneous approximation. In order to study the regularity and the optimality of the set $\mathbb{Q}^d$ of all points with rational coordinates, we first endow it with the appropriate height function, specifically,

$$H_d(a) = \inf\{q \in \mathbb{N} \mid qa \in \mathbb{Z}^d\}^{1+1/d}. \tag{156}$$

The regularity and optimality properties of the resulting pair are in fact reminiscent of the statement of Lemma 3.1, which was crucial when establishing the lower bound in the Jarník-Besicovitch theorem, see Section 3.1.2. Accordingly, an easy adaptation of the proof of that lemma leads to the next statement.

THEOREM 7.2. *The pair $(\mathbb{Q}^d, H_d)$ is an optimal regular system in $\mathbb{R}^d$.*

PROOF. When the open set U is equal to the whole space $\mathbb{R}^d$ in Definition 7.1, one easily checks that the notion of optimal regular system does not depend on the choice of the norm. We thus choose to work with the supremum norm.

Establishing the optimality of the system is rather elementary. Indeed, let B denote the open ball with center x and radius r, and let a be a point in $\mathbb{Q}^d \cap B$ with height at most h. We write a in the form p/q, with $p \in \mathbb{Z}^d$ and $q \in \mathbb{N}$ as small as possible. As a result, the height $H_d(a)$ is equal to $q^{1+1/d}$, which means that q is bounded above by $h^{d/(d+1)}$. Moreover, the number of possible values for the point p is not greater than $(2rq+1)^d$. This follows from a volume comparison argument, along with the observation that the open balls with radius $1/(2q)$ centered at the points $p'/q \in B$, with $p' \in \mathbb{Z}^d$, are disjoint and contained in the open ball with center x and radius $r + 1/(2q)$. Hence,

$$\begin{aligned}\#\{a \in \mathbb{Q}^d \cap B \mid H_d(a) \leq h\} &\leq \sum_{1 \leq q \leq h^{d/(d+1)}} (2rq+1)^d \\ &\leq h^{d/(d+1)}(2rh^{d/(d+1)}+1)^d \leq (4r)^d h^d,\end{aligned}$$

where the last bound holds for $h \geq (2r)^{-1-1/d}$.

The proof of the regularity of the system is parallel to that of Lemma 3.1, and is in fact less technical. For any point y in $\mathbb{R}^d$, let $q(y)$ denote the minimal value of the integer $q \geq 1$ for which

$$\exists p \in \mathbb{Z}^d \qquad |qy - p|_\infty \leq \frac{1}{\lfloor h^{1/(d+1)} \rfloor}.$$

Dirichlet's theorem, namely, Theorem 1.1, ensures that $q(y) \leq h^{d/(d+1)}$. Actually, this holds if h is large enough to guarantee that $\lfloor h^{1/(d+1)} \rfloor$ is larger than one, *i.e.* if

$$h \geq 2^{d+1}, \tag{157}$$

a condition that we assume from now on. Moreover, the minimality of $q(y)$ implies that the integer $q(y)$ and the coordinates of the corresponding integer point p are mutually coprime. In particular,

$$H_d\left(\frac{p}{q(y)}\right) = q(y)^{1+1/d} \leq h. \tag{158}$$

Now, some parameters γ and δ being fixed in $(0,1)$, let B' denote the open ball concentric with B, with radius δ times that of B, and let B'' be the subset of B' formed by the points y such that $q(y) < \gamma h^{d/(d+1)}$. The set B'' is covered by the closed balls with radius $2/(qh^{1/(d+1)})$ centered at the rational points p/q within distance $1/q$ of the ball B' and with denominator $q < \gamma h^{d/(d+1)}$. For any fixed choice of q, there are at most $(2q\delta r+3)^d$ such points. Hence, the Lebesgue measure of the set B'' is at most

$$\sum_{1\leq q<\gamma h^{d/(d+1)}} (2q\delta r+3)^d \left(\frac{4}{qh^{1/(d+1)}}\right)^d = \frac{4^d}{h^{d/(d+1)}} \sum_{1\leq q<\gamma h^{d/(d+1)}} \left(2\delta r + \frac{3}{q}\right)^d.$$

In order to derive an upper bound on the sum in the right-hand side, we first consider the case in which $q < 3/(2\delta r)$. In that situation, the summand is clearly bounded by 6^d. In the opposite case, the summand is bounded by $(4\delta r)^d$. Thus,

$$\mathcal{L}^d(B'') \leq \frac{3\cdot 24^d}{2\delta r h^{d/(d+1)}} + (16\delta r)^d\gamma.$$

We may now define $\mathcal{A}_{B,h}$ as any maximal collection of points in $\mathbb{Q}^d \cap B$ with height at most h and separated from each other by a distance at least $1/(\gamma h)$, so in particular at least $1/h$. It remains us to establish a lower bound on the cardinality of $\mathcal{A}_{B,h}$, and to tune up the parameters γ and δ appropriately. Any point $y \in B' \setminus B''$ is such that $q(y)$ is between $\gamma h^{d/(d+1)}$ and $h^{d/(d+1)}$, so there exists an integer point p in $\mathbb{Z}^d$ such that the rational point $p/q(y)$ satisfies

$$\left|y - \frac{p}{q(y)}\right|_\infty \leq \frac{1}{q(y)\lfloor h^{1/(d+1)}\rfloor} \leq \frac{1}{\gamma h^{d/(d+1)}\lfloor h^{1/(d+1)}\rfloor} \leq \frac{2}{\gamma h}.$$

In particular, since y is in the ball B', the rational point $p/q(y)$ belongs to the ball B if the following condition holds:

$$\frac{2}{\gamma h} + \delta r \leq r. \tag{159}$$

In that situation, the point $p/q(y)$ is in $\mathbb{Q}^d \cap B$ and has height at most h, in view of (158). Therefore, the collection $\mathcal{A}_{B,h}$ being maximal, it must contain a point p'/q' located at a distance less than $1/(\gamma h)$ from $p/q(y)$. Hence,

$$\left|y - \frac{p'}{q'}\right|_\infty \leq \left|y - \frac{p}{q(y)}\right|_\infty + \left|\frac{p}{q(y)} - \frac{p'}{q'}\right|_\infty < \frac{2}{\gamma h} + \frac{1}{\gamma h} \leq \frac{3}{\gamma h}.$$

It follows that the set $B' \setminus B''$ is covered by the open balls with radius $3/(\gamma h)$ centered at the points in $\mathcal{A}_{B,h}$. As a consequence,

$$(2\delta r)^d - \frac{3\cdot 24^d}{2\delta r h^{d/(d+1)}} - (16\delta r)^d\gamma \leq \mathcal{L}^d(B'\setminus B'') \leq \left(\frac{6}{\gamma h}\right)^d \#\mathcal{A}_{B,h},$$

from which we deduce that

$$\frac{\#\mathcal{A}_{B,h}}{|B|^d h^d} \geq \left(\frac{\gamma\delta}{6}\right)^d \left(1 - 8^d\gamma - \frac{3\cdot 12^d}{2(\delta r)^{d+1}h^{d/(d+1)}}\right). \tag{160}$$

To conclude, it remains to adjust the values of the parameters γ and δ appropriately, and to specify how large h must be chosen in order to ensure that all the conditions above hold, in particular that (160) holds with a constant in the right-hand side. In fact, we choose γ smaller than 8^{-d}, and δ arbitrarily, and we require that h is large enough to ensure that (157) and (159) both hold, and that (160) holds with a constant in the right-hand side. More specifically, we may define $\gamma = 2^{-3d-1}$ and $\delta = 1/2$, and then assume that

$$h \geq \max\left\{2^{d+1}, \frac{2^{3(d+1)}}{r}, \left(\frac{2^{3d+2}3^{d+1}}{r^{d+1}}\right)^{1+1/d}\right\}.$$

As required, this ensures that (157) and (159) are both satisfied, and that (160) holds with constant $2^{-3d(d+1)-2} \cdot 3^{-d}$ in the right-hand side. This finally proves the regularity of the system $(\mathbb{Q}^d, H_d)$ of points with rational coordinates. □

7.3.2. Inhomogeneous approximation. Theorem 7.2 may be extended to the inhomogeneous case presented in Section 1.5 and obtained by shifting the approximating rational points p/q with the help of a chosen value α in $\mathbb{R}^d$. To be specific, the approximation is realized by the points that belong to the collection

$$\mathbb{Q}^{d,\alpha} = \left\{\frac{p+\alpha}{q}, \ (p,q) \in \mathbb{Z}^d \times \mathbb{N}\right\}$$

Obviously, when α vanishes, we recover the set $\mathbb{Q}^d$ of points with rational coordinates. The collection $\mathbb{Q}^{d,\alpha}$ is endowed with the height function H_d^α defined by

$$H_d^\alpha(a) = \inf\{q \in \mathbb{N} \mid qa - \alpha \in \mathbb{Z}^d\}^{1+1/d}.$$

Again, when α is zero, we get the height function H_d introduced in the above homogeneous case. We then have the following generalization of Theorem 7.2. The proof is essentially due to Bugeaud [**12**] and relies on an inhomogeneous approximation result derived in Section 1.5 above, specifically, Proposition 1.11.

THEOREM 7.3. *For any point α in $\mathbb{R}^d$, the pair $(\mathbb{Q}^{d,\alpha}, H_d^\alpha)$ is an optimal regular system in $\mathbb{R}^d$.*

PROOF. The proof is, to a certain extent, a generalization of that detailed in the homogeneous case. In particular, the optimality of the system $(\mathbb{Q}^{d,\alpha}, H_d^\alpha)$ may straightforwardly be established by adapting the arguments developed in the proof of Theorem 7.2, so we shall only detail the proof of the regularity.

On a more technical note, it is convenient here again to endow $\mathbb{R}^d$ with the supremum norm. For any point y in $\mathbb{R}^d$, we slightly modify the definition of the integer $q(y)$ coming into play in the homogeneous case: this is now the minimal value of the integer $q \geq 1$ for which

$$\exists p \in \mathbb{Z}^d \qquad |qy - p|_\infty \leq \frac{1}{\lfloor 2^{-1/d} h^{1/(d+1)} \rfloor}.$$

Dirichlet's theorem then shows that $2q(y)$ is bounded above by $h^{d/(d+1)}$, with the proviso that the following condition holds:

$$h \geq 2^{(d+1)^2/d}. \tag{161}$$

We assume from now on that this condition is satisfied. We consider an open ball B in $\mathbb{R}^d$, two parameters γ and δ in $(0,1)$, and then another ball B', exactly as in the proof of Theorem 7.2. We shall however slightly modify the definition of the set B'' : this is now the set of points y in B' such that $2q(y) < \gamma h^{d/(d+1)}$. Adapting the arguments developed in the proof of Theorem 7.2, we observe that

$$\mathcal{L}^d(B'') \leq \frac{3 \cdot 24^d}{\delta r h^{d/(d+1)}} + (16\delta r)^d \gamma.$$

Finally, we define $\mathcal{A}_{B,h}$ as any maximal collection of points belonging to the set $\mathbb{Q}^{d,\alpha} \cap B$ with height at most h and separated from each other by a distance at least $(2/\gamma)^{1+1/d}/h$, thus in particular at least $1/h$.

We now search for an appropriate lower bound on the cardinality of $\mathcal{A}_{B,h}$. Note that each point y in the set $B' \setminus B''$ satisfies

$$q(y) \geq \frac{\gamma}{2} h^{d/(d+1)} \geq \gamma \lfloor 2^{-1/d} h^{1/(d+1)} \rfloor^d.$$

This suggests us to apply Proposition 1.11 to the integer $\lfloor 2^{-1/d} h^{1/(d+1)} \rfloor$, the point α, and each point y in the set $B' \setminus B''$. We thereby infer the existence of two real numbers Γ_* and H_*, both larger than one and depending on γ and d only, such that the condition

$$h > H_* \tag{162}$$

implies that for each point y in the set $B' \setminus B''$, there is a pair (p,q) in $\mathbb{Z}^d \times \mathbb{N}$ with

$$q(y) \leq q < 2q(y) \qquad \text{and} \qquad |qy - p - \alpha|_\infty \leq \frac{\Gamma_*}{q(y)^{1/d}}.$$

In that situation, we straightforwardly deduce that

$$\left| y - \frac{p+\alpha}{q} \right|_\infty \leq \frac{\Gamma_*}{q(y)^{1+1/d}} \leq \frac{\Gamma_*}{h} \left(\frac{2}{\gamma} \right)^{1+1/d}.$$

Given that the point y is in the ball B', this means in particular that the point $(p+\alpha)/q$ belongs to the set $\mathbb{Q}^{d,\alpha} \cap B$ if the following condition holds:

$$\frac{\Gamma_*}{h} \left(\frac{2}{\gamma} \right)^{1+1/d} + \delta r \leq r. \tag{163}$$

On top of that, we observed previously that $2q(y)$ is bounded above by $h^{d/(d+1)}$, so we deduce that this point satisfies

$$H_d^\alpha \left(\frac{p+\alpha}{q} \right) \leq q^{1+1/d} < (2q(y))^{1+1/d} \leq h.$$

Since the collection $\mathcal{A}_{B,h}$ is maximal, it contains a point $(p'+\alpha)/q'$ located at a distance smaller than $(2/\gamma)^{1+1/d}/h$ from $(p+\alpha)/q$, so that

$$\left| y - \frac{p'+\alpha}{q'} \right|_\infty \leq \left| y - \frac{p+\alpha}{q} \right|_\infty + \left| \frac{p+\alpha}{q} - \frac{p'+\alpha}{q'} \right|_\infty < \frac{\Gamma_* + 1}{h} \left(\frac{2}{\gamma} \right)^{1+1/d}.$$

Hence, the set $B' \setminus B''$ is covered by the open balls centered at the points in $\mathcal{A}_{B,h}$ with radius the right-hand side above. Adapting the arguments of the homogeneous case, and making use of the fact that Γ_* is larger than one, we obtain

$$(2\delta r)^d - \frac{3 \cdot 24^d}{\delta r h^{d/(d+1)}} - (16\delta r)^d \gamma \leq \mathcal{L}^d(B' \setminus B'') \leq \left(\frac{4\Gamma_*}{h} \right)^d \left(\frac{2}{\gamma} \right)^{d+1} \# \mathcal{A}_{B,h},$$

from which we deduce that

$$\frac{\# \mathcal{A}_{B,h}}{|B|^d h^d} \geq \left(\frac{\delta}{4\Gamma_*} \right)^d \left(\frac{\gamma}{2} \right)^{d+1} \left(1 - 8^d \gamma - \frac{3 \cdot 12^d}{(\delta r)^{d+1} h^{d/(d+1)}} \right). \tag{164}$$

To conclude, we choose γ smaller than 8^{-d}, and δ arbitrarily, and we require that h is large enough to ensure that (161), (162) and (163) all hold, and that (164) holds with a constant that depends on d in the right-hand side. $\square$

Combining Proposition 7.1 and Theorem 7.3, we directly get the following property: for any nonempty bounded open subset U of $\mathbb{R}^d$, any monotonic enumeration of the optimal regular system $(\mathbb{Q}^{d,\alpha}, H_d^\alpha)$ in the set U is uniformly eutaxic. In particular, the arguably most natural enumeration of the rational numbers that are strictly between zero and one, namely, the sequence

$$\frac{1}{2}, \frac{1}{3}, \frac{2}{3}, \frac{1}{4}, \frac{3}{4}, \frac{1}{5}, \frac{2}{5}, \frac{3}{5}, \frac{4}{5}, \frac{1}{6}, \frac{5}{6}, \frac{1}{7}, \frac{2}{7}, \frac{3}{7}, \frac{4}{7}, \frac{5}{7}, \frac{6}{7}, \dots$$

is uniformly eutaxic in the open interval $(0,1)$.

7.3.3. Metrical implications for general approximating functions. We may use Theorem 7.3 in conjunction with Theorem 7.1 in order to describe the basic size and large intersection properties of the set

$$\mathfrak{Q}^{\alpha}_{d,\psi} = \left\{ x \in \mathbb{R}^d \;\middle|\; \left| x - \frac{p+\alpha}{q} \right|_{\infty} < \psi(q) \quad \text{for i.m. } (p,q) \in \mathbb{Z}^d \times \mathbb{N} \right\}, \tag{165}$$

where ψ denotes a positive nonincreasing continuous function that is defined on the interval $[0,\infty)$. When $\psi(q)$ coincides with $q^{-\tau}$ for all $q \geq 1$ and some $\tau > 0$, we clearly recover the set $J^{\alpha}_{d,\tau}$ defined by (31). Moreover, in the homogeneous case, *i.e.* when the point α is equal to the origin, we end up with the emblematic set $J_{d,\tau}$ defined by (1) and whose Hausdorff dimension is given by Theorem 3.1, *i.e.* the Jarník-Besicovitch theorem. Among other results, we shall therefore extend this theorem to the more general set $\mathfrak{Q}^{\alpha}_{d,\psi}$. This is the purpose of the next statement.

THEOREM 7.4. *Let α be a point in $\mathbb{R}^d$ and let ψ denote a positive nonincreasing continuous function defined on the interval $[0,\infty)$.*

(1) *The set $\mathfrak{Q}^{\alpha}_{d,\psi}$ has full, or zero, Lebesgue measure in $\mathbb{R}^d$ according to whether the integral $\mathrm{I}_{d,\psi}$ diverges, or converges, respectively, where*

$$\mathrm{I}_{d,\psi} = \int_0^{\infty} q^d \psi(q)^d \,\mathrm{d}q.$$

(2) *Let us assume that the integral $\mathrm{I}_{d,\psi}$ is convergent. Then, the parameter*

$$\theta_{d,\psi} = \limsup_{q\to\infty} \frac{(d+1)\log q}{-\log \psi(q)}$$

is bounded above by d. Moreover, if the parameter $\theta_{d,\psi}$ is positive, then the set $\mathfrak{Q}^{\alpha}_{d,\psi}$ satisfies

$$\dim_{\mathrm{H}} \mathfrak{Q}^{\alpha}_{d,\psi} = \theta_{d,\psi} \qquad \textit{and} \qquad \mathfrak{Q}^{\alpha}_{d,\psi} \in \mathcal{G}^{\theta_{d,\psi}}(\mathbb{R}^d).$$

PROOF. To establish (1), we observe that the set $\mathfrak{Q}^{\alpha}_{d,\psi}$ coincides with the set F_{φ} defined by (148) when the function φ satisfies $\varphi(\eta) = \psi(\eta^{d/(d+1)})$ for all $\eta \geq 0$, and the underlying system $(\mathcal{A}, H)$ is equal to $(\mathbb{Q}^{d,\alpha}, H^{\alpha}_d)$, which is optimal and regular in the whole space $\mathbb{R}^d$ by virtue of Theorem 7.3. Applying Theorem 7.1(1) and making the obvious change of variable, we deduce that the set $\mathfrak{Q}^{\alpha}_{d,\psi}$ has full, or zero, Lebesgue measure in the whole space $\mathbb{R}^d$ according to whether the following integral diverges, or converges, respectively:

$$I_{\varphi} = \int_0^{\infty} \eta^{d-1} \varphi(\eta)^d \,\mathrm{d}\eta = \left(1 + \frac{1}{d}\right) \int_0^{\infty} q^d \psi(q)^d \,\mathrm{d}q = \left(1 + \frac{1}{d}\right) \mathrm{I}_{d,\psi}.$$

With a view to proving (2), we begin by using the monotonicity of the function ψ in order to remark that for all positive real numbers s and Q,

$$\int_0^{\infty} q^d \psi(q)^s \,\mathrm{d}q \geq \int_{Q/2}^{Q} q^d \psi(q)^s \,\mathrm{d}q \geq \psi(Q)^s \left(\frac{Q}{2}\right)^{d+1}.$$

When s is equal to d, the integral in the left-hand side is finite because it coincides with $\mathrm{I}_{d,\psi}$. This implies that the function ψ converges to zero at infinity, and in fact that the parameter $\theta_{d,\psi}$ is bounded above by d.

Let us suppose that $s < \theta_{d,\psi}$. One may find a real number $\varepsilon > 0$ and a real sequence $(Q_n)_{n\geq 1}$ going to infinity such that $\psi(Q_n)^{s+\varepsilon}$ is larger than $1/Q_n^{d+1}$ for all $n \geq 1$. The above inequalities then yield

$$\int_0^\infty q^d \psi(q)^s \, \mathrm{d}q \geq \psi(Q_n)^s \left(\frac{Q_n}{2}\right)^{d+1} > 2^{-(d+1)} Q_n^{(d+1)\varepsilon/(s+\varepsilon)}.$$

Letting $n \to \infty$, we deduce that the integral in the left-hand side diverges. This means that the integral $\mathrm{I}_{d,\psi^{s/d}}$ diverges, where $\psi^{s/d}$ denotes the function ψ raised to the power s/d. Conversely, if $s > \theta_{d,\psi}$, there is a real number $\varepsilon > 0$ such that $\psi(Q)^{s-\varepsilon}$ is smaller than $1/Q^{d+1}$ for all Q sufficiently large; this readily implies that the integral $\mathrm{I}_{d,\psi^{s/d}}$ is convergent. The upshot is that

$$\begin{cases} s < \theta_{d,\psi} & \Longrightarrow \quad \mathrm{I}_{d,\psi^{s/d}} = \infty \\ s > \theta_{d,\psi} & \Longrightarrow \quad \mathrm{I}_{d,\psi^{s/d}} < \infty. \end{cases}$$

It remains to perform a simple change of function to exactly recover the setting of Theorem 7.1(3). To be specific, assuming that $\theta_{d,\psi} > 0$, we raise ψ to the power $\theta_{d,\psi}/d$, and we let ψ_* denote the resulting function. As in the proof of (1), the set $\mathfrak{Q}^\alpha_{d,\psi_*}$ then coincides with the set F_{φ_*} obtained for $\varphi_*(\eta) = \psi_*(\eta^{d/(d+1)})$. Observing that the integrals I_{d,ψ_*^t} and $I_{\varphi_*^t}$ share the same convergence properties, we get

$$\begin{cases} t < 1 & \Longrightarrow \quad I_{\varphi_*^t} = \infty \\ t > 1 & \Longrightarrow \quad I_{\varphi_*^t} < \infty. \end{cases} \tag{166}$$

We may now apply Theorem 7.1(3), thereby deducing that for any $t \geq 1$, the set $F_{\varphi_*^t}$ has Hausdorff dimension d/t and belongs to the class $\mathcal{G}^{d/t}(\mathbb{R}^d)$. Finally, when t is equal to $d/\theta_{d,\psi}$, the set $F_{\varphi_*^t}$ is equal to the set $\mathfrak{Q}^\alpha_{d,\psi}$, and the result follows. $\square$

Theorem 7.4(1) is essentially due to Khintchine [**38**] in the homogeneous case, and to Schmidt [**52**] in the general case. Note that the original proofs, however, do not call upon the methods that we develop here. Moreover, Theorem 7.4(2) follows from more general results from Jarník [**37**] and Bugeaud [**12**] that address the homogeneous, and the inhomogeneous case, respectively. These more general results will be presented in Section 10.2 below.

7.3.4. An inhomogeneous Jarník-Besicovitch theorem. As an immediate consequence of Theorem 7.4, we deduce the basic size and large intersection properties of the set $J^\alpha_{d,\tau}$ defined by (31). This corresponds to the case where the approximation function ψ is of the form $q \mapsto q^{-\tau}$ on the interval $[1, \infty)$, for some positive real number τ. Observe that the integral $\mathrm{I}_{d,\psi}$ arising in the statement of Theorem 7.4 converges if and only if $\tau > 1 + 1/d$. Furthermore, the parameter $\theta_{d,\psi}$ is clearly equal to $(d+1)/\tau$. Specializing Theorem 7.4 to this situation, we therefore end up with the next result.

Corollary 7.1. *For any point α in $\mathbb{R}^d$ and any real parameter τ, the set $J^\alpha_{d,\tau}$ defined by (31) has full, or zero, Lebesgue measure in $\mathbb{R}^d$ according to whether $\tau \leq 1 + 1/d$, or not, respectively. Moreover, in the latter situation, we have*

$$\dim_{\mathrm{H}} J^\alpha_{d,\tau} = \frac{d+1}{\tau} \qquad \textit{and} \qquad J^\alpha_{d,\tau} \in \mathcal{G}^{(d+1)/\tau}(\mathbb{R}^d).$$

Obviously, the set $J^\alpha_{d,\tau}$ is also a set with large intersection when $\tau \leq 1 + 1/d$. To be specific, $J^\alpha_{d,\tau}$ belongs to the class $\mathcal{G}^d(\mathbb{R}^d)$, just as any Lebesgue-full G_δ-set. Furthermore, in the homogeneous case where α vanishes, we obviously recover the introductory set $J_{d,\tau}$ defined by (1). Recall that its Hausdorff dimension is equal to $(d+1)/\tau$, due to the Jarník-Besicovitch theorem, and that it even belongs to the large intersection class $\mathcal{G}^{(d+1)/\tau}(\mathbb{R}^d)$, see Theorem 3.1 and Corollary 5.2. We may thus see Corollary 7.1 as an extension of these results to the inhomogeneous case.

Remarkably, the large intersection property allows us to consider countably many values of the parameter α and to study the size of the intersection of the corresponding sets $J^{\alpha}_{d,\tau}$, for possibly different values of the parameter τ. Indeed, let $(\alpha_n)_{n\geq 1}$ be a sequence of points in $\mathbb{R}^d$, and let $(\tau_n)_{n\geq 1}$ be a sequence of real numbers. We begin by assuming that the supremum

$$\tau_* = \sup_{n\geq 1} \tau_n$$

is both finite and larger than $1 + 1/d$. Thanks to Proposition 5.1(2) and Corollary 7.1, we know that each $J^{\alpha_n}_{d,\tau_n}$ is a set with large intersection in $\mathbb{R}^d$ with dimension at least $\min\{(d+1)/\tau_n, d\}$, and thus belongs to the class $\mathcal{G}^{(d+1)/\tau_*}(\mathbb{R}^d)$. In view of Theorem 5.1, the latter class is closed under countable intersections, thereby containing the intersection of the sets $J^{\alpha_n}_{d,\tau_n}$. In particular, this intersection has dimension at least $(d+1)/\tau_*$. The matching upper bound being a straightforward consequence of Proposition 2.16(1), *i.e.* the monotonicity property of Hausdorff dimension, we deduce that

$$\dim_{\mathrm{H}} \bigcap_{n=1}^{\infty} J^{\alpha_n}_{d,\tau_n} = \frac{d+1}{\tau_*}.$$

When τ_* is bounded above by $1 + 1/d$, the above intersection has Hausdorff dimension equal to d. Indeed, Corollary 7.1 ensures that all the sets $J^{\alpha_n}_{d,\tau_n}$ have full Lebesgue measure in $\mathbb{R}^d$, and so has their intersection. In the remaining case where τ_* is infinite, one may show that the intersection has Hausdorff dimension equal to zero; this will follow from more precise results established in Section 10.2.2.

7.3.5. Connection with fractional parts of linear sequences. Finally, Theorem 7.4 also enables us to recover the fact that the fractional parts of almost all linear sequences are eutaxic in the unit cube $(0,1)^d$, see Theorem 6.8. Let us consider a sequence $(r_n)_{n\geq 1}$ in the collection P_d. The sequence $(r_n/n)_{n\geq 1}$ is both positive and nonincreasing, so we may find a positive nonincreasing continuous function ψ defined on the interval $[0,\infty)$ that coincides with this sequence on the positive integers. Hence, the integral $I_{d,\psi}$ on which relies Theorem 7.4(1) satisfies

$$I_{d,\psi} = \int_0^{\infty} q^d \psi(q)^d \,\mathrm{d}q \geq \sum_{n=1}^{\infty} (n-1)^d \psi(n)^d = 2^{-d} \sum_{n=2}^{\infty} r_n^d = \infty.$$

We deduce that the sets $\mathfrak{Q}^{y}_{d,\psi}$, defined as in (165) for all points y in $\mathbb{R}^d$, have full Lebesgue measure in $\mathbb{R}^d$. In particular, if y belongs to the unit cube $(0,1)^d$, then $\mathcal{L}^d$-almost every point x in $\mathbb{R}^d$ satisfies

$$|nx - (p_n + y)|_\infty < n\psi(n) = r_n$$

with some integer point p_n, for infinitely many integers $n \geq 1$. For convenience, we work here and below with the supremum norm; recall from Section 6.1.1 that this choice does not alter the notion of eutaxy. Letting $h = (1/2, \ldots, 1/2)$, we have

$$|\lfloor nx \rfloor - p_n|_\infty \leq |nx - (p_n + y)|_\infty + |\{nx\} - h|_\infty + |y - h|_\infty < r_n + \frac{1}{2} + |y - h|_\infty$$

The right-hand side is smaller than one for n sufficiently large, because the sequence $(r_n)_{n\geq 1}$ converges to zero. The point p_n is then necessarily equal to $\lfloor nx \rfloor$. We deduce that for all $y \in (0,1)^d$ and for $\mathcal{L}^d$-almost all $x \in \mathbb{R}^d$, the inequality

$$|y - \{nx\}|_\infty < r_n$$

holds for infinitely many integers $n \geq 1$. This holds *a fortiori* for $\mathcal{L}^d$-almost every point y. Tonelli's theorem finally allows us to exchange the order of y and x, thus concluding that for $\mathcal{L}^d$-almost every $x \in \mathbb{R}^d$, the sequence $(\{nx\})_{n\geq 1}$ is eutaxic in the cube $(0,1)^d$ with respect to the sequence $(r_n)_{n\geq 1}$. This is exactly Theorem 6.8.

7.4. Application to the approximation by algebraic numbers

We now turn our attention to the examples supplied by the real algebraic numbers and the real algebraic integers. Our treatment will be somewhat brief, as for instance we shall not detail all the proofs; for further details, we refer to the seminal paper by Baker and Schmidt [**1**], subsequent important works by Beresnevich [**2**] and Bugeaud [**9**], and the references therein. We shall show that the algebraic numbers and integers lead to optimal regular systems, and we shall state the metrical results obtained from subsequently applying Theorem 7.1.

The collection of all real algebraic numbers is denoted by $\mathbb{A}$. The naïve height of a number a in $\mathbb{A}$, denoted by $\mathrm{H}(a)$, is the maximum of the absolute values of the coefficients of its minimal defining polynomial over $\mathbb{Z}$. Moreover, the set of all real algebraic numbers with degree at most n is denoted by $\mathbb{A}_n$. Baker and Schmidt [**1**] proved that the set $\mathbb{A}_n$, endowed with the height function

$$a \mapsto \frac{\mathrm{H}(a)^{n+1}}{(\max\{1, \log \mathrm{H}(a)\})^{3n^2}},$$

forms a regular system. The trouble is that, due to the logarithmic term, this height function does not lead to the best possible metrical statements. However, Beresnevich proved that the height function

$$H_n(a) = \frac{\mathrm{H}(a)^{n+1}}{(1+|a|)^{n(n+1)}}, \tag{167}$$

where there is no logarithmic term, is actually convenient. We shall therefore privilege the following statement when deriving metrical results underneath.

THEOREM 7.5 (Beresnevich). *For any integer $n \geq 1$, the pair $(\mathbb{A}_n, H_n)$ is an optimal regular system in $\mathbb{R}$.*

It is elementary to check that $(\mathbb{A}_n, H_n)$ is an optimal system. Establishing the regularity is much more difficult and relies on a fine knowledge of the distribution of real algebraic numbers; we refer to [**2**] for a detailed proof. Note that $\mathbb{A}_1$ obviously coincides with the set $\mathbb{Q}$ of rational numbers. Moreover, writing an element a in $\mathbb{A}_1$ in the form p/q for two coprime integers p and q, the latter being positive, we have

$$H_1(a) = \frac{\mathrm{H}(a)^2}{(1+|a|)^2} = \frac{\max\{|p|, q\}^2}{(1+|a|)^2} = \left(\frac{\max\{1, |a|\}}{1+|a|}\right)^2 q^2,$$

so that $H_1(a)$ is between $q^2/4$ and q^2. Hence, the height of a, viewed as an algebraic number with degree one, is comparable with its height when regarded as a rational point of the real line, see (156).

We shall now combine Theorem 7.5 with Theorem 7.1, in order to describe the basic size and large intersection properties of sets that arise naturally when studying the approximation of real numbers by real algebraic numbers. For any positive nonincreasing continuous function ψ defined on $[0, \infty)$, let us define

$$\mathfrak{A}_{n,\psi} = \left\{ x \in \mathbb{R} \;\middle|\; |x - a| < \psi(\mathrm{H}(a)) \quad \text{for i.m. } a \in \mathbb{A}_n \right\}. \tag{168}$$

The elementary size and large intersection properties of the set $\mathfrak{A}_{n,\psi}$ are detailed in the next statement, which should be thought of as an analog of Theorem 7.4 to the present situation.

THEOREM 7.6. *Let n be a positive integer and let ψ denote a positive nonincreasing continuous function defined on the interval $[0,\infty)$.*

(1) *The set $\mathfrak{A}_{n,\psi}$ has full, or zero, Lebesgue measure in $\mathbb{R}$ according to whether the integral $\mathrm{I}_{n,\psi}$ diverges, or converges, respectively, where*
$$\mathrm{I}_{n,\psi}=\int_0^\infty h^n\psi(h)\,\mathrm{d}h.$$
(2) *Let us assume that the integral $\mathrm{I}_{n,\psi}$ is convergent. Then, the parameter*
$$\theta_{n,\psi}=\limsup_{h\to\infty}\frac{(n+1)\log h}{-\log\psi(h)}$$
is bounded above by one. Moreover, if the parameter $\theta_{n,\psi}$ is positive, then the set $\mathfrak{A}_{n,\psi}$ satisfies
$$\dim_{\mathrm{H}}\mathfrak{A}_{n,\psi}=\theta_{n,\psi}\qquad\textit{and}\qquad\mathfrak{A}_{n,\psi}\in\mathcal{G}^{\theta_{n,\psi}}(\mathbb{R}).$$

PROOF. In order to prove (1), we begin by observing that the set $\mathfrak{A}_{n,\psi}$ may be approximated with the help of the sets F_φ defined by (148) when the underlying system $(\mathcal{A},H)$ is equal to $(\mathbb{A}_n,H_n)$ and the function φ is chosen appropriately. Indeed, for any integer $k\geq 1$, let φ_k denote the function defined for all $\eta\geq 0$ by $\varphi_k(\eta)=\psi(k\,\eta^{1/(n+1)})$. Note that, the larger k, the smaller F_{φ_k}. We then have
$$\bigcap_{k=1}^{\infty}\downarrow F_{\varphi_k}\subseteq\mathfrak{A}_{n,\psi}\subseteq F_{\varphi_1}. \tag{169}$$
Indeed, let x denote a point in the left-hand side and let k be chosen as any integer larger than or equal to $(1+|x|+\psi(0))^n$. Since the point x belongs to the set F_{φ_k}, there are infinitely many points a in $\mathbb{A}_n$ such that
$$|x-a|<\varphi_k(H_n(a))=\psi(k\,H_n(a)^{1/(n+1)})$$
However, the function ψ is nonincreasing and the integer k is bounded below by $(1+|x|+\psi(0))^n$, and thus by $(1+|a|)^n$. Hence, we have
$$|x-a|<\psi((1+|a|)^n\,H_n(a)^{1/(n+1)})=\psi(\mathrm{H}(a))$$
for infinitely many points a in $\mathbb{A}_n$, so that x is in $\mathfrak{A}_{n,\psi}$. Furthermore, in that situation, since the inequality $|x-a|<\psi(\mathrm{H}(a))$ holds for infinitely many points a in $\mathbb{A}_n$, we deduce that
$$|x-a|<\psi(\mathrm{H}(a))=\psi((1+|a|)^n\,H_n(a)^{1/(n+1)})\leq\psi(H_n(a)^{1/(n+1)})=\varphi_1(H_n(a)),$$
again because the function ψ is nonincreasing, so that the point x belongs to the set F_{φ_1} in the right-hand side of (169).

We may now finish the proof of (1). Thanks to (169), it suffices to prove that the set F_{φ_1} has Lebesgue measure zero in $\mathbb{R}$ when the integral $\mathrm{I}_{n,\psi}$ converges, and that all the sets F_{φ_k}, for $k\geq 1$, have full Lebesgue measure in $\mathbb{R}$ when the integral diverges. However, a simple change of variable implies that
$$I_{\varphi_k}=\int_0^\infty\varphi_k(\eta)\,\mathrm{d}\eta=\frac{n+1}{k^{n+1}}\int_0^\infty h^n\psi(h)\,\mathrm{d}h=\frac{n+1}{k^{n+1}}\mathrm{I}_{n,\psi}, \tag{170}$$
so we conclude with the help of Theorem 7.1(1) and the fact that $(\mathbb{A}_n,H_n)$ is an optimal regular system in $\mathbb{R}$ by virtue of Theorem 7.5.

Let us now turn our attention to the proof of (2). We suppose that the integral $\mathrm{I}_{n,\psi}$ is convergent. Then, adapting the proof of Theorem 7.4(2), we easily establish that $\theta_{n,\psi}$ is bounded above by one, and that
$$\begin{cases} s<\theta_{n,\psi} & \Longrightarrow\quad \mathrm{I}_{n,\psi^s}=\infty\\ s>\theta_{n,\psi} & \Longrightarrow\quad \mathrm{I}_{n,\psi^s}<\infty.\end{cases}$$

As a consequence, if $\theta_{n,\psi}$ is positive, then (166) holds here as well for $\varphi_* = \varphi_k^{\theta_{n,\psi}}$, where k denotes an arbitrary positive integer. Applying Theorem 7.1(3), we infer that for any $t \geq 1$, the set $F_{\varphi_*^t}$ has Hausdorff dimension $1/t$ and belongs to the class $\mathcal{G}^{1/t}(\mathbb{R})$. Choosing $t = 1/\theta_{n,\psi}$, we deduce that all the sets F_{φ_k} have Hausdorff dimension $\theta_{n,\psi}$ and belongs to the class $\mathcal{G}^{\theta_{n,\psi}}(\mathbb{R})$. We conclude with the help of (169). Indeed, on the one hand, the set $\mathfrak{A}_{n,\psi}$ is contained in the set F_{φ_1}, thereby having Hausdorff dimension at most $\theta_{n,\psi}$. On the other hand, the set $\mathfrak{A}_{n,\psi}$ is a G_δ-set that contains the intersection over all $k \in \mathbb{N}$ of the sets F_{φ_k}, which all belong to the class $\mathcal{G}^{\theta_{n,\psi}}(\mathbb{R})$. Hence, Theorem 5.1 and Proposition 5.1(1) imply that the set $\mathfrak{A}_{n,\psi}$ also belongs to $\mathcal{G}^{\theta_{n,\psi}}(\mathbb{R})$. In particular, its dimension is at least $\theta_{n,\psi}$. $\square$

Theorem 7.6(1) is due to Beresnevich [**2**] and the dimensional result in Theorem 7.6(2) was obtained by Baker and Schmidt [**1**]. We shall give a more precise description of the size and large intersection properties of the set $\mathfrak{A}_{n,\psi}$ in Section 10.3 below. We shall also discuss therein the connection with Koksma's classification of real transcendental numbers.

Let us mention that Bugeaud [**9**] obtained an analog of Theorem 7.5 for the set of real algebraic integers, that is, the real algebraic numbers whose minimal defining polynomial over $\mathbb{Z}$ is monic. In what follows, $\mathbb{A}'$ denotes the subset of $\mathbb{A}$ formed by the real algebraic integers, and $\mathbb{A}'_n$ denotes the intersection $\mathbb{A}' \cap \mathbb{A}_n$, that is, the set of all real algebraic integers with degree at most n.

THEOREM 7.7 (Bugeaud). *For any integer $n \geq 2$, the pair $(\mathbb{A}'_n, H_{n-1})$ is an optimal regular system in $\mathbb{R}$.*

Combining Theorem 7.7 with the above methods, we may describe the elementary size and large intersection properties of the set $\mathfrak{A}'_{n,\psi}$ defined as that obtained when replacing $\mathbb{A}_n$ by $\mathbb{A}'_n$ in (168), namely,

$$\mathfrak{A}'_{n,\psi} = \left\{ x \in \mathbb{R} \;\middle|\; |x - a| < \psi(\mathrm{H}(a)) \quad \text{for i.m. } a \in \mathbb{A}'_n \right\}.$$

To be precise, adapting the proof of Theorem 7.6, one easily checks that for any integer $n \geq 2$ and any positive nonincreasing continuous function ψ defined on the interval $[0, \infty)$, the set $\mathfrak{A}'_{n,\psi}$ has full, or zero, Lebesgue measure in $\mathbb{R}$ according to whether the integral

$$\mathrm{I}_{n-1,\psi} = \int_0^\infty h^{n-1} \psi(h) \, \mathrm{d}h$$

diverges, or converges, respectively. Moreover, if the latter integral is convergent, then the set $\mathfrak{A}'_{n,\psi}$ has Hausdorff dimension equal to

$$\theta_{n-1,\psi} = \limsup_{h \to \infty} \frac{n \log h}{-\log \psi(h)},$$

provided that this parameter is positive, and moreover it belongs to the large intersection class $\mathcal{G}^{\theta_{n-1,\psi}}(\mathbb{R})$.

CHAPTER 8

Transference principles

8.1. Mass transference principle

We begin by recalling the main results of Chapter 4, and shedding new light thereon. Let $\mathcal{I}$ be a countably infinite index set, let $(x_i, r_i)_{i\in\mathcal{I}}$ be an approximation system in the sense of Definition 4.1, and let F_t be the sets defined by (87), namely,

$$F_t = \left\{x \in \mathbb{R}^d \;\middle|\; |x - x_i| < r_i^t \quad \text{for i.m. } i \in \mathcal{I}\right\}.$$

Moreover, let U denote a nonempty open subset of $\mathbb{R}^d$. According to Definition 4.2, the family is a homogeneous ubiquitous system in U if the set F_1 has full Lebesgue measure in U. In that situation, Theorem 4.1 shows that for any real number $t > 1$,

$$\dim_{\mathrm{H}}(F_t \cap U) \geq \frac{d}{t}.$$

In fact, the set $F_t \cap U$ has positive Hausdorff measure with respect to the gauge function $r \mapsto r^{d/t}|\log r|$. Thus, the mere fact that the set F_1 has full Lebesgue measure in U yields an *a priori* lower bound on the Hausdorff dimension of the sets F_t, which are smaller than F_1 when t is larger than one.

We adopt a new perspective on this result by considering from now on that the set defined by

$$\mathfrak{F}((x_i, r_i)_{i\in\mathcal{I}}) = \left\{x \in \mathbb{R}^d \;\middle|\; |x - x_i| < r_i \quad \text{for i.m. } i \in \mathcal{I}\right\} \tag{171}$$

is that on which we seek an estimate on the size. In the above notations, this set coincides with the set F_1. However, for any real number $t \geq 1$, this set also coincides with the set F_t associated with the underlying family $(x_i, r_i^{1/t})_{i\in\mathcal{I}}$, which is an approximation system as well. In that new situation, Theorem 4.1 ensures that if the family $(x_i, r_i^{1/t})_{i\in\mathcal{I}}$ is a homogeneous ubiquitous system in U, that is, if

$$\text{for } \mathcal{L}^d\text{-a.e. } x \in U \quad \exists \text{ i.m. } i \in \mathcal{I} \qquad |x - x_i| < r_i^{1/t}, \tag{172}$$

then the set $\mathfrak{F}((x_i, r_i)_{i\in\mathcal{I}})$ has positive Hausdorff measure in the open set U with respect to the gauge function $r \mapsto r^{d/t}|\log r|$, so in particular

$$\dim_{\mathrm{H}}(\mathfrak{F}((x_i, r_i)_{i\in\mathcal{I}}) \cap U) \geq \frac{d}{t}.$$

A further way to recast this result is to let g denote the gauge function $r \mapsto r^{d/t}$, to rewrite the assumption (172) in the form

$$\mathcal{L}^d(U \setminus \mathfrak{F}((x_i, g(r_i)^{1/d})_{i\in\mathcal{I}})) = 0, \tag{173}$$

where the involved set is defined as in (171), and to reinterpret the conclusion as the fact that the set $\mathfrak{F}((x_i, r_i)_{i\in\mathcal{I}})$ has positive Hausdorff measure in U with respect to the gauge function $r \mapsto g(r)|\log r|$. Note that the gauge function g is d-normalized in the sense of Definition 2.9, because g coincides on the interval $(0, \infty)$ with its d-normalization g_d, defined by (57). Thus, the condition (173) still holds when g is replaced by g_d. In that situation, the approximation system $(x_i, r_i)_{i\in\mathcal{I}}$ will be called *g-ubiquitous*, in accordance with the following definition.

DEFINITION 8.1. Let $\mathcal{I}$ be a countably infinite index set, let $(x_i, r_i)_{i\in\mathcal{I}}$ be an approximation system in $\mathbb{R}^d \times (0, \infty)$, let g be a gauge function and let U be a nonempty open subset of $\mathbb{R}^d$. We say that $(x_i, r_i)_{i\in\mathcal{I}}$ is a *homogeneous g-ubiquitous system* in U if the following condition holds:

$$\mathcal{L}^d(U \setminus \mathfrak{F}((x_i, g_d(r_i)^{1/d})_{i\in\mathcal{I}})) = 0.$$

The latter condition means that for Lebesgue-almost every point x in the open set U, the inequality $|x - x_i| < g_d(r_i)^{1/d}$ holds for infinitely many indices i in $\mathcal{I}$. Hence, the previous definition may be seen as an extension of that of a homogeneous ubiquitous system. In fact, according to Definitions 4.2 and 8.1, respectively, an approximation system is a homogeneous ubiquitous system in some nonempty open set U if and only if it is homogeneously ubiquitous in U with respect to any gauge function whose d-normalization is $r \mapsto r^d$.

Remarkably, Beresnevich and Velani [**5**] managed to extend the above approach to any gauge function g, and also improved the above conclusion. Specifically, they established the following *mass transference principle* for the sets defined by (171).

THEOREM 8.1 (Beresnevich and Velani). *Let $\mathcal{I}$ be a countably infinite index set, let $(x_i, r_i)_{i\in\mathcal{I}}$ be an approximation system in $\mathbb{R}^d \times (0, \infty)$, let g be a gauge function and let U be a nonempty open subset of $\mathbb{R}^d$. If $(x_i, r_i)_{i\in\mathcal{I}}$ is a homogeneous g-ubiquitous system in U, then for every nonempty open subset V of U,*

$$\mathcal{H}^g(\mathfrak{F}((x_i, r_i)_{i\in\mathcal{I}}) \cap V) = \mathcal{H}^g(V).$$

A FEW WORDS ON THE PROOF. Some of the ideas supporting Theorem 8.1 are similar to those developed in the proof of Theorem 4.1 above. However, Theorem 4.1 being essentially concerned with Hausdorff dimension only, its proof does not require as high much accuracy as in the proof of Theorem 8.1, where Hausdorff measures associated with arbitrary gauge functions are considered. The proof of Theorem 8.1 is therefore somewhat technically involved. Consequently, we omit it from these notes, and we refer the reader to Beresnevich and Velani's paper [**5**].

We just mention that Theorem 8.1 is a straightforward consequence of Theorem 2 in [**5**], except that Beresnevich and Velani only considered d-normalized functions. However, this assumption may easily be removed with the help of Propositions 2.10 and 2.15. Indeed, let us suppose that Theorem 8.1 holds for d-normalized gauge functions. Then, let g be an arbitrary gauge function such that the approximation system $(x_i, r_i)_{i\in\mathcal{I}}$ is homogeneously g-ubiquitous in U. It is clear from Definition 8.1 that the system is also g_d-ubiquitous, where g_d denotes the d-normalization of g. Applying Theorem 8.1 to the d-normalized gauge function g_d, we infer that for every nonempty open subset V of U,

$$\mathcal{H}^{g_d}(\mathfrak{F}((x_i, r_i)_{i\in\mathcal{I}}) \cap V) = \mathcal{H}^{g_d}(V).$$

Thanks to Proposition 2.10, we may then compare the Hausdorff measures $\mathcal{H}^{g_d}$ and $\mathcal{H}^g$, thereby deducing that

$$\mathcal{H}^g(\mathfrak{F}((x_i, r_i)_{i\in\mathcal{I}}) \cap V) \geq \frac{\mathcal{H}^g(V)}{\kappa},$$

where κ is given by Proposition 2.10. There are now essentially three different possible situations, depending on the value of the parameter ℓ_g defined by (61). The case where ℓ_g vanishes is trivial: Proposition 2.15(3) ensures that the Haudorff measure $\mathcal{H}^g$ vanishes, and the conclusion of Theorem 8.1 clearly holds. Now, if ℓ_g is infinite, then Proposition 2.15(1) ensures that $\mathcal{H}^g(V)$ is infinite, and so that

$$\mathcal{H}^g(\mathfrak{F}((x_i, r_i)_{i\in\mathcal{I}}) \cap V) = \mathcal{H}^g(V) = \infty.$$

In the remaining case where ℓ_g is both positive and finite, we have $g_d(r) \leq 2\ell_g r^d$ for all $r > 0$, so that the approximation system $(x_i, r_i)_{i\in\mathcal{I}}$ is homogeneously ubiquitous in U with respect to the gauge function $r \mapsto 2\ell_g r^d$. By virtue of Proposition 4.4, we may remove the constant $2\ell_g$ in that property, specifically, $(x_i, r_i)_{i\in\mathcal{I}}$ is homogeneously ubiquitous in U with respect to $r \mapsto r^d$. This means that the set $\mathfrak{F}((x_i, r_i)_{i\in\mathcal{I}})$ has full Lebesgue measure in U. We conclude with the help of Proposition 2.15(2), which ensures that $\mathcal{H}^g$ is a multiple of the Lebesgue measure. □

Theorem 8.1 is remarkable because of its universality. It can in fact be applied to many approximation systems arising in metric number theory and probability; we shall give several examples in Chapters 10 and 11. However, our approach relies on the notion of describability introduced in Chapter 9, and at heart on the large intersection transference principle discussed in Section 8.2. Hence, the mass transference will never be used *per se* in what follows. The general philosophy behind this result is that it enables one to automatically convert a property concerning the Lebesgue measure of a limsup of balls to a property concerning the Hausdorff measure of a similar set where the balls are dilated. This leads in particular to a full description of the size properties of limsup of balls for which the description of the Lebesgue measure is known.

8.2. Large intersection transference principle

The purpose of this section is to give an analog of the mass transference principle for large intersection properties. In the spirit of Theorem 8.1, this result leads to a very precise description of the large intersection properties of a limsup of balls in terms of arbitrary gauge functions. Accordingly, we first need to introduce large intersection classes that are associated with arbitrary gauge functions, thereby generalizing the original classes introduced by Falconer and presented in Section 5.1. We adopt the same viewpoint as in the definition of the localized classes $\mathcal{G}^s(U)$, namely, Definition 5.2. In particular, the generalized classes are defined with the help of outer net measures; these are built in terms of general gauge functions and coverings by dyadic cubes.

8.2.1. Net measures revisited. We recall from Section 2.6.3 that a dyadic cube is either the empty set or a set of the form $\lambda = 2^{-j}(k+[0,1)^d)$, with $j \in \mathbb{Z}$ and $k \in \mathbb{Z}^d$, and that the collection of all dyadic cubes is denoted by Λ. We restrict ourselves to gauge functions that are d-normalized in the sense of Definition 2.9. Under this assumption, the resulting outer net measures satisfy additional properties that are in fact necessary to an appropriate definition of the generalized classes.

If g denotes a d-normalized gauge function, the set of all real numbers $\varepsilon > 0$ such that g is nondecreasing on $[0,\varepsilon]$ and $r \mapsto g(r)/r^d$ is nonincreasing on $(0,\varepsilon]$ is nonempty. We may thus define ε_g as the supremum of this set, and next Λ_g as the collection of all dyadic cubes with diameter less than ε_g. We then consider the premeasure $g \circ |\cdot|_{\Lambda_g}$ that sends each set λ in Λ_g to $g(|\lambda|)$, and Theorem 2.2 allows us to define similarly to (51) the outer measure

$$\mathcal{M}^g_\infty = (g \circ |\cdot|_{\Lambda_g})^*$$

resulting from coverings by dyadic cubes with diameter less than ε_g.

The outer measure $\mathcal{M}^g_\infty$ provides a lower bound on the corresponding net measure $\mathcal{M}^g$, which is defined by (59) and is comparable with the Hausdorff measure $\mathcal{H}^g$, see Proposition 2.11. As a consequence, there is a real number $\kappa \geq 1$ independent on g such that for any set $E \subseteq \mathbb{R}^d$,

$$\kappa\, \mathcal{H}^g(E) \geq \mathcal{M}^g_\infty(E). \tag{174}$$

Recall that the outer net measures $\mathcal{M}^s_\infty$, defined by (97) for $s \in (0,d]$, played a crucial rôle in the characterization of Falconer's classes and the definition of their localized counterparts $\mathcal{G}^s(U)$, see Theorem 5.2 and Definition 5.2, respectively. These outer measures are actually an instance of the above construction. Specifically, for any $s \in (0,d]$, the gauge function $r \mapsto r^s$ is clearly d-normalized and the parameter $\varepsilon_{r\mapsto r^s}$ is infinite. Hence, the collection $\Lambda_{r\mapsto r^s}$ coincides with the whole Λ, from which it follows that $\mathcal{M}^{r\mapsto r^s}_\infty$ is merely equal to $\mathcal{M}^s_\infty$. The outer measures $\mathcal{M}^g_\infty$ thus extend naturally those used in Chapter 5 ; this hints at why they will play a key rôle in the definition of the generalized large intersection classes.

Finally, it is useful to point out that the value in each dyadic cube of the $\mathcal{M}^g_\infty$-mass of Lebesgue-full sets has a very simple expression.

LEMMA 8.1. *For any d-normalized gauge function g, any dyadic cube λ in Λ_g, and any subset F of $\mathbb{R}^d$, the following implication holds:*

$$\mathcal{L}^d(\lambda\setminus F)=0 \qquad\Longrightarrow\qquad \mathcal{M}^g_\infty(F\cap\lambda)=g(|\lambda|).$$

PROOF. The proof borrows some ideas from that of Lemma 5.1. First, the intersection set $F\cap\lambda$ is obviously covered by the sole cube λ, so that

$$\mathcal{M}^g_\infty(F\cap\lambda)\le g(|\lambda|).$$

In order to prove the reverse inequality, let us consider a covering $(\lambda_n)_{n\ge1}$ of the intersection set $F\cap\lambda$ by dyadic cubes with diameter less than ε_g. If λ is contained in some cube λ_{n_0}, the fact that g is nondecreasing on $[0,\varepsilon_g)$ implies that

$$g(|\lambda|)\le g(|\lambda_{n_0}|)\le\sum_{n=1}^\infty g(|\lambda_n|).$$

Otherwise, we observe that the cubes $\lambda_n\subset\lambda$ suffice to cover the set $F\cap\lambda$. Along with the fact that the mapping $r\mapsto g(r)/r^d$ is nonincreasing on $(0,\varepsilon_g)$, this yields

$$\begin{aligned}\sum_{n=1}^\infty g(|\lambda_n|)&\ge\sum_{\substack{n\ge1\\ \lambda_n\subset\lambda}}\frac{g(|\lambda_n|)}{|\lambda_n|^d}|\lambda_n|^d\ge\frac{g(|\lambda|)}{|\lambda|^d}\sum_{\substack{n\ge1\\ \lambda_n\subset\lambda}}|\lambda_n|^d=\frac{g(|\lambda|)}{|\lambda|^d}\kappa'^d\sum_{\substack{n\ge1\\ \lambda_n\subset\lambda}}\mathcal{L}^d(\lambda_n)\\&\ge\frac{g(|\lambda|)}{|\lambda|^d}\kappa'^d\mathcal{L}^d(F\cap\lambda)=\frac{g(|\lambda|)}{|\lambda|^d}\kappa'^d\mathcal{L}^d(\lambda)=g(|\lambda|).\end{aligned}$$

Here, κ' stands for the diameter of the unit cube of $\mathbb{R}^d$, which depends on the choice of the norm. We conclude by taking the infimum over all coverings $(\lambda_n)_{n\ge1}$. $\square$

The previous result may be used to express the $\mathcal{M}^g_\infty$-mass of dyadic cubes in terms of their diameters. As a matter of fact, using the notations of Lemma 8.1, if the set F is chosen to be the cube λ itself, or its interior, we get

$$\mathcal{M}^g_\infty(\lambda)=\mathcal{M}^g_\infty(\operatorname{int}\lambda)=g(|\lambda|),\tag{175}$$

a formula which extends Lemma 5.1 to any d-normalized gauge function. Likewise, all the ancillary lemmas from Section 5.3.1 may be extended to such gauge functions; we refer to [**18**] for precise statements, see in particular Lemmas 10 and 12 therein.

8.2.2. Generalized large intersection classes. We are now in position to define the large intersection classes that are associated with general gauge functions. We defined those classes in [**18**], and we refer to that paper for all the proofs and details that are missing in the presentation below. As mentioned above, there is a lineage with the definition of the localized classes $\mathcal{G}^s(U)$, see Definition 5.2.

We write $h\prec g$ to indicate that two d-normalized gauge functions g and h are such that the quotient h/g monotonically tends to infinity at zero, that is,

$$h\prec g \qquad\Longleftrightarrow\qquad \lim_{r\downarrow0}\uparrow\frac{h(r)}{g(r)}=\infty.$$

This means essentially that h increases faster than g near the origin. Note that g may vanish in a neighborhood of zero; in that situation, we adopt the convention that $h \prec g$ for any choice of h, even if h also vanishes near zero.

DEFINITION 8.2. For any gauge function g and any nonempty open subset U of $\mathbb{R}^d$, the class $\mathcal{G}^g(U)$ of *sets with large intersection in U with respect to g* is the collection of all G_δ-subsets F of $\mathbb{R}^d$ such that

$$\mathcal{M}^h_\infty(F \cap V) = \mathcal{M}^h_\infty(V) \tag{176}$$

for any d-normalized gauge function h satisfying $h \prec g_d$, where g_d denotes the d-normalization of g defined by (57), and for any open subset V of U.

Note that the class $\mathcal{G}^g(U)$ associated with a given gauge function g coincides with that associated with its d-normalization, namely, the class $\mathcal{G}^{g_d}(U)$. One may therefore restrict oneself to d-normalized gauge functions when studying large intersection properties. Moreover, if two gauge functions are such that their respective d-normalizations match near the origin, the corresponding classes coincide.

With a view to detailing the connection with the localized classes $\mathcal{G}^s(U)$, we associate with any gauge function g the following dimensional parameter s_g.

DEFINITION 8.3. Let g be a gauge function with d-normalization denoted by g_d. The *dimension* of the gauge function g is the parameter defined by

$$s_g = \sup \{ s \in (0,d] \mid (r \mapsto r^s) \prec g_d \},$$

with the convention that the supremum is equal to zero if the inner set is empty.

Obviously, we have $s_g = \min\{s,d\}$ if the gauge function g is of the form $r \mapsto r^s$, with $s > 0$. The relationship between the generalized classes $\mathcal{G}^g(U)$ and the original classes $\mathcal{G}^s(U)$ is now detailed in the next statement.

PROPOSITION 8.1. *For any gauge function g with dimension satisfying $s_g > 0$ and for any nonempty open subset U of $\mathbb{R}^d$, the following inclusion holds:*

$$\mathcal{G}^g(U) \subseteq \mathcal{G}^{s_g}(U).$$

In particular, for any set F in $\mathcal{G}^g(U)$ and for any nonempty open set $V \subseteq U$,

$$\dim_{\mathrm{H}}(F \cap V) \geq s_g \qquad \text{and} \qquad \dim_{\mathrm{P}}(F \cap V) = d.$$

Moreover, the left-hand inequality above still holds if s_g vanishes.

PROOF. Let us assume that s_g is positive and let us consider a set F in the class $\mathcal{G}^g(U)$. First, F is a G_δ-subset of $\mathbb{R}^d$. Then, for any $s \in (0, s_g)$, we have $(r \mapsto r^s) \prec g_d$, and Definition 8.2 implies that

$$\mathcal{M}^{r\mapsto r^s}_\infty(F \cap V) = \mathcal{M}^{r\mapsto r^s}_\infty(V)$$

for any open subset V of U. Recalling that the outer measure $\mathcal{M}^{r\mapsto r^s}_\infty$ is identical to the outer measure $\mathcal{M}^s_\infty$ defined by (97), we deduce from Definition 5.2 that the set F belongs to the original localized class $\mathcal{G}^{s_g}(U)$.

Moreover, applying Theorem 5.3 and Proposition 5.2, we deduce that the set F has Hausdorff dimension at least s_g and packing dimension equal to d in every nonempty open subset V of U. Finally, in view of Definition 8.2, any set in the class $\mathcal{G}^g(U)$ has to be dense in U. Therefore, the Hausdorff dimension of $F \cap V$ is necessarily bounded below by zero, that is, by s_g when this value vanishes. $\square$

Choosing U equal to the whole space $\mathbb{R}^d$, we clearly deduce from Proposition 8.1 a statement bearing on Falconer's original classes $\mathcal{G}^s(\mathbb{R}^d)$. In addition, as easily seen for instance musing on the examples discussed in Chapters 10 and 11, the inclusion appearing in the statement of Proposition 8.1 is strict.

Let us now briefly discuss the case in which the gauge function g has a d-normalization g_d that vanishes in a neighborhood of zero. The d-normalized gauge function that is constant equal to zero is denoted by $\mathbf{0}$; let us mention in passing that its dimension clearly satisfies $s_{\mathbf{0}} = d$.

PROPOSITION 8.2. *For any nonempty open set $U \subseteq \mathbb{R}^d$, the large intersection class $\mathcal{G}^{\mathbf{0}}(U)$ is formed by the G_δ-subsets of $\mathbb{R}^d$ with full Lebesgue measure in U.*

PROOF. Let us consider a G_δ-subset F of $\mathbb{R}^d$ with full Lebesgue measure in U. Lemma 8.1, combined with (175), ensures that for any d-normalized gauge function g and any dyadic cube λ in Λ_g that is contained in U,

$$\mathcal{M}^g_\infty(F \cap \lambda) = g(|\lambda|) = \mathcal{M}^g_\infty(\lambda).$$

We finally conclude that F belongs to the class $\mathcal{G}^{\mathbf{0}}(U)$ thanks to the extension of Lemma 5.2 to arbitrary d-normalized gauge functions, see [**18**, Lemma 10].

Conversely, let us consider a set F in the class $\mathcal{G}^{\mathbf{0}}(U)$. First, F is necessarily a G_δ-set. Moreover, we know that (176) holds in particular for the d-normalized gauge function $r \mapsto r^d$ and for all open balls $\mathrm{B}(x, r)$ contained in U. Using (174) and (176), and letting κ'' be the constant appearing in Proposition 2.14, we get

$$\kappa\kappa''\mathcal{L}^d(F \cap \mathrm{B}(x, r)) = \kappa\mathcal{H}^d(F \cap \mathrm{B}(x, r)) \geq \mathcal{M}^d_\infty(F \cap \mathrm{B}(x, r)) = \mathcal{M}^d_\infty(\mathrm{B}(x, r)).$$

We consider a nonempty dyadic cube λ with minimal generation that is contained in $\mathrm{B}(x, r)$, and we know from the proof of Proposition 7.2 that $|\lambda| \geq c\,r$ for some $c > 0$ depending on the choice of the norm only. Lemma 5.1 then yields

$$\mathcal{M}^d_\infty(\mathrm{B}(x, r)) \geq \mathcal{M}^d_\infty(\lambda) = |\lambda|^d \geq c^d r^d = \frac{c^d}{\mathcal{L}^d(\mathrm{B}(0, 1))}\mathcal{L}^d(\mathrm{B}(x, r)),$$

where the last equality follows from fact that the Lebesgue measure is translation invariant and homogeneous with degree d with respect to dilations. Hence,

$$\frac{\mathcal{L}^d(F \cap \mathrm{B}(x, r))}{\mathcal{L}^d(\mathrm{B}(x, r))} \geq \frac{c^d}{\kappa\kappa''\mathcal{L}^d(\mathrm{B}(0, 1))} > 0$$

for any open ball $\mathrm{B}(x, r)$ contained in U. It follows from the Lebesgue density theorem that F has full Lebesgue measure in U, see [**46**, Corollary 2.14]. □

The various remarkable properties of the large intersection classes $\mathcal{G}^g(U)$ naturally extend those satisfied by Falconer's classes, see Section 5.1. We begin by stating the properties that follow immediately from the definition. The next result may be seen as a partial analog of Proposition 5.1 ; in its statement, $\mathfrak{G}$ stands for the collection of all gauge functions.

PROPOSITION 8.3. *Let g be a gauge function with d-normalization denoted by g_d, and let U be a nonempty open subset of $\mathbb{R}^d$.*

(1) *Any G_δ-subset of $\mathbb{R}^d$ that contains a set in $\mathcal{G}^g(U)$ also belongs to $\mathcal{G}^g(U)$.*
(2) *The following equalities hold:*

$$\mathcal{G}^g(U) = \bigcap_{\substack{V \text{ open} \\ \emptyset \neq V \subseteq U}} \mathcal{G}^g(V) \qquad \text{and} \qquad \mathcal{G}^g(U) = \bigcap_{\substack{h \in \mathfrak{G} \\ h_d \prec g_d}} \mathcal{G}^h(U).$$

A FEW WORDS ON THE PROOF. All the properties are essentially immediate from the definition of the generalized large intersection classes, and the proof is therefore omitted here. We just mention as a hint to the interested reader that if g and h denote two d-normalized gauge functions such that $h \prec g$, then $\sqrt{gh}$ is a d-normalized gauge function that satisfies $h \prec \sqrt{gh} \prec g$. □

The next result extends Theorem 5.1 to the large intersection classes $\mathcal{G}^g(U)$, thereby showing that they enjoy the same stability properties as Falconer's classes.

THEOREM 8.2. *Let g be a gauge function with d-normalization denoted by g_d and with dimension denoted by s_g, and let U be a nonempty open subset of $\mathbb{R}^d$. The following properties hold:*

(1) *the class $\mathcal{G}^g(U)$ is closed under countable intersections;*
(2) *for any bi-Lipschitz transformation $f : U \to \mathbb{R}^d$ and any set $F \subseteq \mathbb{R}^d$,*
$$F \in \mathcal{G}^g(f(U)) \qquad \Longrightarrow \qquad f^{-1}(F) \in \mathcal{G}^g(U)\,;$$
(3) *for any set F in the class $\mathcal{G}^g(U)$ and for every gauge function h,*
$$h_d \prec g_d \qquad \Longrightarrow \qquad \mathcal{H}^h(F \cap U) = \mathcal{H}^h(U).$$

A FEW WORDS ON THE PROOF. The result corresponds to Theorem 1 in [**18**], so we refer to that paper for the whole proof. Let us just mention that the statement in [**18**] only addresses the d-normalized gauge functions g for which the parameter ℓ_g defined by (61) is positive. In that situation, note that the Hausdorff h-measure of the set $F \cap U$ that appears in (3) is actually infinite, as a consequence of Propositions 2.12 and 2.15. Furthermore, the normalization assumption made in [**18**] may easily be dropped with the help of Proposition 2.10. In addition, Theorem 8.2 clearly holds for $\ell_g = 0$. Indeed, in that situation, the gauge function g_d vanishes near zero and Proposition 8.2 ensures that the class $\mathcal{G}^g(U)$ is formed by the G_δ-sets with full Lebesgue measure in U. All the properties are thus satisfied, even (3) which may be obtained with the help of Propositions 2.12 and 2.15. □

A plain consequence of Theorem 8.2 is that if $(F_n)_{n\geq 1}$ is a sequence of sets in $\mathcal{G}^g(U)$ and if h is a gauge function, then
$$h_d \prec g_d \qquad \Longrightarrow \qquad \mathcal{H}^h\left(\bigcap_{n=1}^{\infty} F_n \cap U\right) = \mathcal{H}^h(U). \tag{177}$$
Thanks to Proposition 2.15, the latter equality may be rewritten in various alternate forms depending on the value of the parameter ℓ_h defined as in (61). In addition, (177) implies that the intersection of all the sets F_n has Hausdorff dimension bounded below by s_g, and this bound is clearly attained if one of the sets has Hausdorff dimension at most s_g.

8.2.3. The transference principle. Now that the classes associated with arbitrary gauge functions have been defined, we may state the large intersection analog of Theorem 8.1, specifically, the mass transference principle dealt with in Section 8.1. While the latter result discusses the size properties of the set $\mathfrak{F}((x_i, r_i)_{i\in\mathcal{I}})$ defined by (171), the next statement concerns its large intersection properties.

THEOREM 8.3. *Let $\mathcal{I}$ be a countably infinite set, let $(x_i, r_i)_{i\in\mathcal{I}}$ be an approximation system in $\mathbb{R}^d \times (0,\infty)$, let g be a gauge function and let U be a nonempty open subset of $\mathbb{R}^d$. If $(x_i, r_i)_{i\in\mathcal{I}}$ is a homogeneous g-ubiquitous system in U, then*
$$\mathfrak{F}((x_i, r_i)_{i\in\mathcal{I}}) \in \mathcal{G}^g(U).$$

A FEW WORDS ON THE PROOF. The result is a straightforward consequence of Theorem 2 in [**18**]; we refer to that paper for a comprehensive proof. Similarly to the mass transference principle, some ideas supporting Theorem 8.3 are analogous to those developed in the proof of Theorem 4.1 above, and also that of Theorem 5.4 which is more specifically concerned with large intersection properties. □

Just as the mass transference principle extends Theorem 4.1 to arbitrary Hausdorff measures, the above *large intersection transference principle* may be seen as an extension of Theorem 5.4. As a matter of fact, let $(x_i, r_i)_{i\in\mathcal{I}}$ denote a homogeneous ubiquitous system in U in the sense of Definition 4.2. Thus, for any real number $t > 1$, the family $(x_i, r_i^t)_{i\in\mathcal{I}}$ is homogeneously ubiquitous in U with respect to the gauge function $r \mapsto r^{d/t}$. Theorem 8.3 then ensures that the set F_t defined by (87) is a set with large intersection in U with respect to the same gauge function. This gauge function clearly has dimension equal to d/t, so we deduce with the help of Corollary 8.1 that the set F_t belongs to Falconer's class $\mathcal{G}^{d/t}(U)$, which is exactly the conclusion of Theorem 5.4.

Furthermore, the large intersection transference principle nicely complements the mass transference principle: under similar hypotheses, it shows that the size properties of sets of the form (171) are in fact stable under countable intersections and bi-Lipschitz mappings. Also, due to Proposition 8.3(2) and Theorem 8.2(3), it implies that for any gauge function h and any nonempty open set $V \subseteq U$,

$$h_d \prec g_d \qquad \Longrightarrow \qquad \mathcal{H}^h(\mathfrak{F}((x_i, r_i)_{i\in\mathcal{I}}) \cap V) = \infty = \mathcal{H}^h(V).$$

Note that the last equality follows from Proposition 2.15(1), because $h(r)/r^d$ necessarily tends to infinity as r goes to zero. Unfortunately, we may not apply this with h being equal to g, thereby failing narrowly to recover the conclusion of the mass transference principle, specifically,

$$\mathcal{H}^g(\mathfrak{F}((x_i, r_i)_{i\in\mathcal{I}}) \cap V) = \mathcal{H}^g(V).$$

However, we may often in practice circumvent this problem and, through the notion of describability introduced in Chapter 9, the large intersection transference principle will be sufficient to describe both size *and* large intersection properties of limsup of balls for which the description of the Lebesgue measure is known. We shall apply this principle to the many examples studied in Chapters 10 and 11.

CHAPTER 9

Describable sets

Our purpose is to combine the mass and the large intersection principles discussed in Sections 8.1 and 8.2, respectively, and place them in a wider setting that we now define. This framework aims at describing in a complete and precise manner the size and large intersection properties of various subsets of $\mathbb{R}^d$ that are derived from eutaxic sequences and optimal regular systems, thereby being relevant to the applications already discussed in Chapters 6 and 7.

Note that the size and large intersection properties of Lebesgue-full sets are easily described as follows. Let E be a Borel subset of $\mathbb{R}^d$ and let U be a nonempty open subset of $\mathbb{R}^d$. If E has full Lebesgue measure in U, then Proposition 2.15 ensures that for any gauge function g and any nonempty open set $V \subseteq U$,

$$\mathcal{H}^g(E \cap V) = \mathcal{H}^g(V).$$

Furthermore, under the stronger assumption that E admits a G_δ-subset with full Lebesgue measure in U, Propositions 8.2 and 8.3(2) imply that for any gauge function g and any nonempty open set $V \subseteq U$,

$$\exists F \in \mathcal{G}^g(V) \qquad F \subseteq E.$$

The above description of the size and large intersection properties of Lebesgue-full sets being both precise and complete, we shall exclude such sets from our analysis.

Our framework will enable us to achieve a similar description for some Lebesgue-null sets. The collection of all Borel subsets of $\mathbb{R}^d$ that are Lebesgue-null in the open set U is denoted by $\mathcal{Z}(U)$, specifically,

$$\mathcal{Z}(U) = \{E \in \mathcal{B} \mid \mathcal{L}^d(E \cap U) = 0\},$$

where $\mathcal{B}$ is the Borel σ-field, in accordance with the notation initiated in Section 2.4. The starting point is the notion of majorizing and minorizing collections of gauge functions that we now introduce.

9.1. Majorizing and minorizing gauge functions

Let E be a set in $\mathcal{Z}(U)$. On the one hand, Proposition 2.15 ensures that for any gauge function g,

$$\ell_g < \infty \qquad \Longrightarrow \qquad \mathcal{H}^g(E \cap U) = 0,$$

where ℓ_g is defined by (61). Studying what happens for the other gauge functions, namely, those belonging to the set

$$\mathfrak{G}^\infty = \{g \in \mathfrak{G} \mid \ell_g = \infty\}$$

gives rise to the following notion of majorizing gauge function.

Definition 9.1. Let U be a nonempty open subset of $\mathbb{R}^d$ and let E be a set in $\mathcal{Z}(U)$. We say that a gauge function $g \in \mathfrak{G}^\infty$ is a *majorizing* for E in U if

$$\mathcal{H}^g(E \cap U) = 0.$$

Such gauge functions form the *majorizing collection* of E in U, denoted by $\mathfrak{M}(E, U)$.

It is plain from Proposition 2.10 that a gauge function $g \in \mathfrak{G}^\infty$ is majorizing for E in U if and only if its d-normalization g_d satisfies the same property. Also, as a simple example, let us point out that

$$E \cap U \text{ countable} \qquad \Longrightarrow \qquad \mathfrak{M}(E, U) = \mathfrak{G}^\infty, \tag{178}$$

because a countable set has Hausdorff g-measure zero for any gauge function g.

On the other hand, Proposition 8.2 shows that a G_δ-subset of $\mathbb{R}^d$ with Lebesgue measure zero in U cannot belong to the large intersection class $\mathcal{G}^{\mathbf{0}}(U)$, and therefore cannot belong to any of the classes $\mathcal{G}^g(U)$ for which $\ell_g = 0$. Similarly to the previous definition, looking at the other gauge functions, specifically, those in the set

$$\mathfrak{G}^* = \{g \in \mathfrak{G} \mid \ell_g \in (0, \infty]\}$$

results in the following notion of minorizing gauge function.

Definition 9.2. Let U be a nonempty open subset of $\mathbb{R}^d$ and let E be a set in $\mathcal{Z}(U)$. We say that a gauge function $g \in \mathfrak{G}^*$ is a *minorizing* for E in U if

$$\exists F \in \mathcal{G}^g(U) \qquad F \subseteq E.$$

Such gauge functions form the *minorizing collection* of E in U, denoted by $\mathfrak{m}(E, U)$.

Similarly to what happens for majorizing gauge functions, a gauge function $g \in \mathfrak{G}^*$ is minorizing for E in U if and only if g_d is; this follows from Definition 8.2. Moreover, if E is a G_δ-set for which g is minorizing in U, it follows from Proposition 8.3(1) that E belongs to the class $\mathcal{G}^g(U)$. Finally, we now have

$$E \cap U \text{ countable} \qquad \Longrightarrow \qquad \mathfrak{m}(E, U) = \emptyset, \tag{179}$$

because the existence of a minorizing gauge function requires that E is dense in U.

We now detail the basic properties of the majorizing and minorizing collections. As shown by the next result, their structure is reminiscent of that of two intervals of the real line whose intersection is at most a singleton.

Proposition 9.1. *Consider a nonempty open set $U \subseteq \mathbb{R}^d$, a set E in $\mathcal{Z}(U)$, and two gauge functions g and h with d-normalizations such that $g_d \prec h_d$. Then,*

$$\begin{cases} g \in \mathfrak{M}(E, U) & \Longrightarrow \qquad h \in \mathfrak{M}(E, U) \setminus \mathfrak{m}(E, U) \\ h \in \mathfrak{m}(E, U) & \Longrightarrow \qquad g \in \mathfrak{m}(E, U) \setminus \mathfrak{M}(E, U). \end{cases}$$

Proof. Let us suppose that g is majorizing for E in U. By virtue of Proposition 2.10, the same property holds for its d-normalization g_d. Proposition 2.12 then ensures that h_d is also majorizing. We conclude by Proposition 2.10 again that h is majorizing as well. Furthermore, if h were minorizing, h_d would be minorizing too, and Theorem 8.2(3) would finally contradict the fact that g_d is majorizing.

Assume now that h is minorizing for E in U. Proposition 8.3(2) shows that g is also minorizing. Finally, Theorem 8.2(3), combined with Proposition 2.15 and the fact that ℓ_g is infinite, implies that g cannot be majorizing. □

The next result enlightens the monotonicity properties of $\mathfrak{M}(E, U)$ and $\mathfrak{m}(E, U)$ when regarded as two functions defined on the set of pairs (E, U) such that U is a nonempty open subset of $\mathbb{R}^d$ and E is a set in $\mathcal{Z}(U)$.

Proposition 9.2. *The majorizing and minorizing collections satisfy the following monotonicity properties:*

1. *the mappings $E \mapsto \mathfrak{M}(E, U)$ and $U \mapsto \mathfrak{M}(E, U)$ are both nonincreasing;*
2. *the mappings $E \mapsto \mathfrak{m}(E, U)$ and $U \mapsto \mathfrak{m}(E, U)$ are nondecreasing and nonincreasing, respectively.*

PROOF. The properties on the majorizing collection hold because Hausdorff measures are outer measure. Moreover, $E \mapsto \mathfrak{m}(E,U)$ is nondecreasing because of Definition 9.2, and $U \mapsto \mathfrak{m}(E,U)$ is nonincreasing due to Proposition 8.3(2). $\square$

Let us now turn our attention to the behavior under countable unions and intersections of the majorizing and minorizing collections.

PROPOSITION 9.3. *Let us consider a nonempty open subset U of $\mathbb{R}^d$. Then, for any sequence $(E_n)_{n\geq 1}$ in the collection $\mathcal{Z}(U)$,*

$$\mathfrak{M}\left(\bigcup_{n=1}^{\infty} E_n, U\right) = \bigcap_{n=1}^{\infty} \mathfrak{M}(E_n,U) \quad \textit{and} \quad \mathfrak{m}\left(\bigcap_{n=1}^{\infty} E_n, U\right) = \bigcap_{n=1}^{\infty} \mathfrak{m}(E_n,U).$$

PROOF. The property satisfied by the majorizing collection results from the fact that Hausdorff measures are outer measure. The property concerning the minorizing collection is a consequence of the stability under countable intersections of the generalized large intersection classes, see Theorem 8.2(1). $\square$

9.2. Openness

With a view to pursuing our investigation of the majorizing and minorizing collections, we need to introduce a definition concerning subsets of gauge functions; the chosen terminology should not refer to any topological property but only comes from the aforementioned analogy with intervals of the real line.

We begin by remarking that for any d-normalized gauge function $g \in \mathfrak{G}^*$, we may build a d-normalized gauge function $\overline{g} \in \mathfrak{G}^*$ satisfying $\overline{g} \prec g$ by simply letting

$$\overline{g}(r) = \sqrt{g(r)}.$$

Studying whether this property holds for given subsets of $\mathfrak{G}^*$ yields the notion of left-openness. Here and below, $\mathfrak{G}_d$ is the collection of d-normalized gauge functions.

DEFINITION 9.3. Let $\mathfrak{H}$ denote a subset of $\mathfrak{G}^*$. We say that the collection $\mathfrak{H}$ is *left-open* if the following property holds:

$$\forall g \in \mathfrak{G}_d \cap \mathfrak{H} \quad \exists \overline{g} \in \mathfrak{G}_d \cap \mathfrak{H} \qquad \overline{g} \prec g.$$

The whole collection $\mathfrak{G}^*$ is thus left-open. With a view to defining the symmetrical notion of right-openness, we begin by observing that if a d-normalized gauge function $g \in \mathfrak{G}^*$ satisfies $\ell_g < \infty$, then no d-normalized gauge function $\underline{g} \in \mathfrak{G}^*$ can satisfy $g \prec \underline{g}$. To cope with this issue, we just exclude these gauge functions g, thus restricting ourselves to the set $\mathfrak{G}^\infty$. Indeed, if g is a d-normalized gauge function in $\mathfrak{G}^\infty$, we get a d-normalized gauge function $\underline{g} \in \mathfrak{G}^\infty$ with $g \prec \underline{g}$ by defining

$$\underline{g}(r) = r^{d/2}\sqrt{g(r)}.$$

Proceeding as above and considering a similar property for various subsets of $\mathfrak{G}^\infty$, we end up with the notion of right-openness.

DEFINITION 9.4. Let $\mathfrak{H}$ denote a subset of $\mathfrak{G}^\infty$. We say that the collection $\mathfrak{H}$ is *right-open* if the following property holds:

$$\forall g \in \mathfrak{G}_d \cap \mathfrak{H} \quad \exists \underline{g} \in \mathfrak{G}_d \cap \mathfrak{H} \qquad g \prec \underline{g}.$$

Clearly, the above constructions ensure that the collection $\mathfrak{G}^\infty$ is both left-open and right-open. The connexion with majorizing and minorizing collections comes from the following observation that may easily be established by combining Proposition 9.1 with the previous arguments: a majorizing collection is always right-open and a minorizing collection is always left-open. The next result shows that further properties arise when these collections are both left-open and right-open.

PROPOSITION 9.4. *Let us consider a nonempty open subset U of $\mathbb{R}^d$ and a set E belonging to the collection $\mathcal{Z}(U)$.*

(1) *If the collection $\mathfrak{M}(E,U)$ is left-open, then for any gauge function g in $\mathfrak{M}(E,U)$ and for any nonempty open subset V of U,*
$$\forall F \in \mathcal{G}^g(V) \qquad F \not\subseteq E,$$
and as a consequence,
$$\mathfrak{M}(E,U) \subseteq \mathfrak{G}^* \setminus \mathfrak{m}(E,U).$$

(2) *If the collection $\mathfrak{m}(E,U) \cap \mathfrak{G}^\infty$ is right-open, then for any gauge function g in $\mathfrak{m}(E,U) \cap \mathfrak{G}^\infty$ and for any nonempty open subset V of U,*
$$\mathcal{H}^g(E \cap V) = \infty,$$
and as a consequence,
$$\mathfrak{m}(E,U) \subseteq \mathfrak{G}^* \setminus \mathfrak{M}(E,U).$$

PROOF. To establish the first property, let us consider a majorizing gauge function g. Since g_d is also majorizing, the left-openness ensures that there is a majorizing gauge function $\overline{g} \in \mathfrak{G}_d$ such that $\overline{g} \prec g_d$. Now, given a nonempty open set $V \subseteq U$, let us assume that E contains a set $F \in \mathcal{G}^g(V)$. By Theorem 8.2(3) and Proposition 2.15, the set F has infinite Hausdorff $\overline{g}$-measure in V, which contradicts the fact that $\overline{g}$ is majorizing. Hence, E cannot contain any set in $\mathcal{G}^g(V)$. Choosing V equal to U, we deduce that g is not minorizing.

Similar arguments lead to the second property. Specifically, if g denotes a gauge function in $\mathfrak{m}(E,U) \cap \mathfrak{G}^\infty$, its d-normalization g_d belongs to the same collection and the right-openness yields a minorizing gauge function $\underline{g} \in \mathfrak{G}_d \cap \mathfrak{G}^\infty$ with $g_d \prec \underline{g}$. The class $\mathcal{G}^{\underline{g}}(U)$ thus contains a set $F \subseteq E$. Now, let V be a nonempty open subset of U. Proposition 8.3(2) shows that F is in the class $\mathcal{G}^{\underline{g}}(V)$. Theorem 8.2(3) and Proposition 2.15 then imply that F has infinite Hausdorff g_d-measure in V. Finally, the set E has infinite Hausdorff g-measure in V, owing to Proposition 2.10. Choosing $V = U$, we conclude that g is not majorizing. □

As a consequence of Proposition 9.4, if either of the collections $\mathfrak{M}(E,U)$ and $\mathfrak{m}(E,U) \cap \mathfrak{G}^\infty$ is simultaneously left-open and right-open, then
$$\mathfrak{M}(E,U) \cap \mathfrak{m}(E,U) = \emptyset,$$
meaning that no gauge function can be majorizing and minorizing at the same time. Under the stronger assumption that *both* collections are left-open and right-open simultaneously, Propositions 2.15 and 9.4 directly yield the next statement.

COROLLARY 9.1. *Consider a nonempty open set $U \subseteq \mathbb{R}^d$ and a set $E \in \mathcal{Z}(U)$, and assume that $\mathfrak{M}(E,U)$ and $\mathfrak{m}(E,U) \cap \mathfrak{G}^\infty$ are both left-open and right-open. Then, for any gauge function $g \in \mathfrak{G}^*$ and any nonempty open set $V \subseteq U$,*
$$\begin{cases} g \in \mathfrak{M}(E,U) \cup (\mathfrak{G}^* \setminus \mathfrak{G}^\infty) & \Longrightarrow \quad \mathcal{H}^g(E \cap V) = 0 \\ g \in \mathfrak{m}(E,U) \cap \mathfrak{G}^\infty & \Longrightarrow \quad \mathcal{H}^g(E \cap V) = \infty \end{cases}$$
and
$$\begin{cases} g \in \mathfrak{M}(E,U) & \Longrightarrow \quad \forall F \in \mathcal{G}^g(V) \quad F \not\subseteq E \\ g \in \mathfrak{m}(E,U) & \Longrightarrow \quad \exists F \in \mathcal{G}^g(V) \quad F \subseteq E. \end{cases}$$

We shall thus be able to describe precisely the size and large intersection properties of a given set E, once we know which gauge functions are majorizing and which are minorizing. Hence, an important question is to determine whether *all* gauge functions are either majorizing or minorizing for E.

9.3. Describability

In the ideal situation where we know that every gauge function is either majorizing or minorizing, the description of the size and large intersection properties of a set will be both precise and complete; we shall then say that the set if fully describable. A further question is to establish a criterion to determine whether a given gauge function is majorizing or minorizing; this will lead to the notions of $\mathfrak{n}$-describable and $\mathfrak{s}$-describable sets that are detailed afterward. As shown in Sections 9.4 and 9.5, these notions are naturally connected with those of eutaxic sequence and optimal regular system.

9.3.1. Fully describable set. To be more specific, we define the notion of fully describable set in the following manner.

DEFINITION 9.5. Let U be a nonempty open subset U of $\mathbb{R}^d$ and let E be a set in $\mathcal{Z}(U)$. We say that the set E is *fully describable in* U if

$$\mathfrak{G}^\infty \subseteq \mathfrak{M}(E,U) \cup \mathfrak{m}(E,U),$$

that is, if every gauge function g for which ℓ_g is infinite is either majorizing or minorizing in U for the set E.

Obviously, the notion of fully describable set is only relevant to the setting of sets with large intersection. For instance, the middle-third Cantor set $\mathbb{K}$ has positive Hausdorff measure in the dimension $s = \log 2/\log 3$, see the proof of Proposition 2.18. Thus, the gauge function $r \mapsto r^s$ cannot be majorizing for $\mathbb{K}$ in $(0,1)$. Furthermore, as already observed in Section 5.1, the set $\mathbb{K}$ cannot contain any set with large intersection. In particular, the previous gauge function cannot be minorizing either. Hence, the Cantor set $\mathbb{K}$ is not fully describable in $(0,1)$.

If U denotes again an arbitrary nonempty open subset of $\mathbb{R}^d$, we already discussed a trivial example of fully describable set in U, namely, the Borel subsets E of $\mathbb{R}^d$ for which the intersection $E \cap U$ is a countable set. We have indeed

$$\mathfrak{G}^\infty = \mathfrak{M}(E,U) \cup \mathfrak{m}(E,U),$$

as an immediate consequence of (178) and (179). Another situation where full describability arises is discussed in the next statement.

PROPOSITION 9.5. *Let U be a nonempty open subset of $\mathbb{R}^d$ and let E be a set in $\mathcal{Z}(U)$. Then, the following implication holds:*

$$\mathfrak{m}(E,U) \setminus \mathfrak{G}^\infty \neq \emptyset \qquad \Longrightarrow \qquad \begin{cases} \mathfrak{M}(E,U) = \emptyset \\ \mathfrak{m}(E,U) = \mathfrak{G}^*. \end{cases}$$

In particular, if there exists a minorizing gauge function g such that ℓ_g is finite, then the set E is fully describable in U.

PROOF. Let g denote a minorizing gauge function for which ℓ_g is finite. Since g is minorizing, ℓ_g is also necessarily positive, so the d-normalization g_d satisfies

$$g_d(r) \sim \ell_g\, r^d \qquad \text{as} \qquad r \to 0. \tag{180}$$

Let us now consider a gauge function h such that $h \prec (r \mapsto r^d)$. We already observed that the mapping $\underline{h}$ defined by

$$\underline{h}(r) = r^{d/2}\sqrt{h(r)}$$

is a gauge function satisfying the condition $h \prec \underline{h} \prec (r \mapsto r^d)$. The quotient $\underline{h}/g_d$ tends to infinity at zero. Adapting the proof of Proposition 2.10, it is straightforward to check that the function $\widetilde{h}$ defined by

$$\widetilde{h}(r) = g_d(r) \inf_{0<\rho\leq r} \frac{\underline{h}(\rho)}{g_d(\rho)}$$

for all $r > 0$, along with $\widetilde{h}(0) = 0$ and $\widetilde{h}(\infty) = \infty$, is a d-normalized gauge function that is bounded above by $\underline{h}$ and satisfies $\widetilde{h} \prec g_d$. On top of that, as g is minorizing, there exists a subset F of E in the class $\mathcal{G}^g(U)$. In view of Definition 8.2 and (175), this implies that for any dyadic cube λ in $\Lambda_{\widetilde{h}}$ that is contained in U,

$$\mathcal{M}^{\underline{h}}_\infty(F \cap \lambda) \geq \mathcal{M}^{\widetilde{h}}_\infty(F \cap \operatorname{int} \lambda) = \mathcal{M}^{\widetilde{h}}_\infty(\operatorname{int} \lambda) = \widetilde{h}(|\lambda|).$$

Furthermore, owing to (180), we know that $g_d(r)/r^d$ is between $\ell_g/2$ and $2\ell_g$ when r is small enough. In that situation, $\widetilde{h}(r)$ is clearly bounded below by $\underline{h}_d(r)/4$, where $\underline{h}_d$ denotes the d-normalization of $\underline{h}$. This coincides with $\underline{h}(r)/4$ again if r is sufficiently small, because the gauge function $\underline{h}$ is d-normalized. As a consequence, for any dyadic cube $\lambda \subseteq U$ whose diameter is small enough, we have

$$\mathcal{M}^{\underline{h}}_\infty(F \cap \lambda) \geq \frac{1}{4}\, \underline{h}(|\lambda|) = \frac{1}{4}\, \mathcal{M}^{\underline{h}}_\infty(\lambda),$$

where the last equality follows from (175). Thanks to the respective extensions of Lemmas 5.2 and 5.3 to arbitrary gauge functions, namely, Lemmas 10 and 12 in [**18**], we deduce that for any open set $V \subseteq U$,

$$\mathcal{M}^{h}_\infty(F \cap V) = \mathcal{M}^{h}_\infty(V).$$

This means that F is a set with large intersection in U with respect to the gauge function $r \mapsto r^d$, and more generally with respect to all gauge functions in $\mathfrak{G}^*$. Therefore, all these gauge functions are minorizing for E in U.

Finally, $\mathfrak{m}(E, U) \cap \mathfrak{G}^\infty$ coincides with the whole $\mathfrak{G}^\infty$, thereby being both left-open and right-open. Proposition 9.4 then ensures the disjointness of the majorizing and minorizing collections, which means that $\mathfrak{M}(E, U)$ must be empty. $\square$

9.3.2. $\mathfrak{n}$-describable sets. We now single out an important category of fully describable sets; they are characterized by the existence of a simple criterion to decide whether a given gauge function is majorizing or minorizing. This criterion is expressed in terms of integrability properties with respect to a given measure $\mathfrak{n}$ that belongs to the collection $\mathcal{R}$ defined in Section 6.5.2.

Let us recall that $\mathcal{R}$ is the collection of all positive Radon measures $\mathfrak{n}$ on the interval $(0, 1]$ such that $\mathfrak{n}$ has infinite total mass and (139) holds, namely, the proper subintervals of the form $[r, 1]$ all have finite mass. It is worth pointing out here that the d-normalization g_d of an arbitrary gauge function g is always Borel measurable and bounded on $(0, 1]$. Also, we shall use the notation

$$\langle \mathfrak{n}, g_d \rangle = \int_{(0,1]} g_d(r)\, \mathfrak{n}(\mathrm{d}r)$$

and we shall in fact restrict our attention to certain measures in $\mathcal{R}$ only, namely, those belonging to the subcollection

$$\mathcal{R}_d = \{\mathfrak{n} \in \mathcal{R} \mid \langle \mathfrak{n}, r \mapsto r^d \rangle < \infty\}. \tag{181}$$

For any $\mathfrak{n}$ in $\mathcal{R}$, the gauge functions $g \notin \mathfrak{G}^*$ clearly satisfy $\langle \mathfrak{n}, g_d \rangle < \infty$. If $\mathfrak{n}$ is in $\mathcal{R}_d$, this property actually holds for all gauge functions $g \notin \mathfrak{G}^\infty$. Indeed, the parameter ℓ_g is then finite, so that $g_d(r) \leq \ell_g\, r^d$ for all $r \in (0, 1]$. The finiteness of $\langle \mathfrak{n}, g_d \rangle$ therefore remains undecided only if g is in $\mathfrak{G}^\infty$; this motivates the introduction of the set

$$\mathfrak{G}(\mathfrak{n}) = \{g \in \mathfrak{G}^\infty \mid \langle \mathfrak{n}, g_d \rangle = \infty\},$$

along with its complement in $\mathfrak{G}^\infty$, which is denoted by $\mathfrak{G}(\mathfrak{n})^\complement$.

DEFINITION 9.6. Let U be a nonempty open subset of $\mathbb{R}^d$, let E be a set in $\mathcal{Z}(U)$, and let $\mathfrak{n}$ be a measure in $\mathcal{R}_d$. We say that the set E is $\mathfrak{n}$-*describable in* U if

$$\mathfrak{M}(E,U) = \mathfrak{G}(\mathfrak{n})^{\complement} \qquad \text{and} \qquad \mathfrak{m}(E,U) \cap \mathfrak{G}^{\infty} = \mathfrak{G}(\mathfrak{n}),$$

or equivalently if for any gauge function g in $\mathfrak{G}^{\infty}$,

$$\begin{cases} g \in \mathfrak{M}(E,U) & \Longleftrightarrow & \langle \mathfrak{n}, g_d \rangle < \infty \\ g \in \mathfrak{m}(E,U) & \Longleftrightarrow & \langle \mathfrak{n}, g_d \rangle = \infty. \end{cases}$$

It is clear from the definition that if E denotes $\mathfrak{n}$-describable set in U, then E is fully describable in U and the majorizing and minorizing collections are disjoint. We know that this situation occurs when either of the collections $\mathfrak{M}(E,U)$ and $\mathfrak{m}(E,U) \cap \mathfrak{G}^{\infty}$ is simultaneously left-open and right-open. The following lemma actually implies that both collections are left-open and right-open at the same time, which will enable us to subsequently apply Corollary 9.1. It also entails that $\mathfrak{m}(E,U) \cap \mathfrak{G}^{\infty}$ is nonempty, meaning that E contains a set with large intersection.

LEMMA 9.1. *For any measure $\mathfrak{n}$ in $\mathcal{R}_d$, the following properties hold:*

(1) *the set $\mathfrak{G}^* \setminus \mathfrak{G}(\mathfrak{n})$ is left-open;*
(2) *the set $\mathfrak{G}(\mathfrak{n})$ is right-open and nonempty.*

In particular, if a set E is $\mathfrak{n}$-describable in U, then both collections $\mathfrak{M}(E,U)$ and $\mathfrak{m}(E,U) \cap \mathfrak{G}^{\infty}$ are simultaneously left-open and right-open.

PROOF. In order to prove (1), let us consider a d-normalized gauge function $g \in \mathfrak{G}^*$ such that $\langle \mathfrak{n}, g_d \rangle < \infty$. We may build a decreasing sequence $(r_n)_{n\geq 1}$ of real numbers in $(0, \varepsilon_g)$, with ε_g being defined in Section 8.2.1, such that for all $n \geq 2$,

$$g(r_n) \leq g(r_{n-1})\,\mathrm{e}^{-1/n} \qquad \text{and} \qquad \int_{0<r\leq r_{n-1}} g(r)\,\mathfrak{n}(\mathrm{d}r) \leq \frac{1}{(n+1)^3}.$$

Note that the sequence $(r_n)_{n\geq 1}$ necessarily converges to zero. Indeed, $g(r_n)$ tends to zero as n goes to infinity, and the function g is nonvanishing and continuous on $(0, \varepsilon_g)$. Then, for any $n \geq 2$ and any $r \in (r_n, r_{n-1}]$, let us define

$$\xi(r) = n + \frac{\log g(r_{n-1}) - \log g(r)}{\log g(r_{n-1}) - \log g(r_n)}.$$

The function ξ is nonincreasing and continuous on $(0, r_1]$, goes to infinity at zero, and is such that $\xi(r) \in [n, n+1]$ for all $r \in (r_n, r_{n-1}]$ et $n \geq 2$. We now define

$$\overline{g}(r) = g(r)\xi(r)$$

for all $r \in (0, r_1]$. Then, for $n \geq 2$ and $r_n < r \leq r' \leq r_{n-1}$, the difference $\overline{g}(r') - \overline{g}(r)$ vanishes if $g(r') = g(r)$. Otherwise, it is equal to

$$\begin{aligned} g(r')\xi(r') - g(r)\xi(r) &= (\xi(r') - \xi(r))g(r) + \xi(r')(g(r') - g(r)) \\ &\geq (g(r') - g(r))\, n \left(1 - \frac{\log \frac{g(r')}{g(r)}}{\frac{g(r')}{g(r)} - 1} \cdot \frac{1}{n \log \frac{g(r_{n-1})}{g(r_n)}} \right) \geq 0. \end{aligned}$$

As a consequence, the function $\overline{g}$ is nondecreasing on each interval $(r_n, r_{n-1}]$. Since it is continuous on $(0, r_1]$, it is nondecreasing on that whole interval. Furthermore,

$$\begin{aligned} \int_{0<r\leq r_1} \overline{g}(r)\,\mathfrak{n}(\mathrm{d}r) &= \sum_{n=2}^{\infty} \int_{r_n<r\leq r_{n-1}} g(r)\xi(r)\,\mathfrak{n}(\mathrm{d}r) \\ &\leq \sum_{n=2}^{\infty} (n+1) \int_{0<r\leq r_{n-1}} g(r)\,\mathfrak{n}(\mathrm{d}r) \leq \sum_{n=2}^{\infty} \frac{1}{(n+1)^2} < \infty. \end{aligned}$$

In particular, $\overline{g}$ tends to zero at the origin. We thus extend it to a gauge function such that $\langle \mathfrak{n}, \overline{g}_d \rangle$ is finite by defining for instance $\overline{g}(r) = \overline{g}(r_1)$ for all $r > r_1$, as well as $\overline{g}(\infty) = \infty$. Finally, $\overline{g}/g = \xi$ monotonically tends to infinity at zero, so that $\overline{g}$ is d-normalized and satisfies $\overline{g} \prec g$.

To prove the right-openness in (2), let us suppose that $g \in \mathfrak{G}^\infty$ and $\langle \mathfrak{n}, g_d \rangle = \infty$. Let us define $r_1 = \varepsilon_g/2$, and also $\theta(r) = g(r)/r^d$ for all $r \in (0, r_1]$. The function θ is nonincreasing on $(0, r_1]$ and tends to infinity at zero. For any $n \geq 2$, there exists an $r_n \in (0, r_{n-1})$ with

$$\theta(r_n) \geq \theta(r_{n-1})\,\mathrm{e} \qquad \text{and} \qquad \int_{r_n < r \leq r_{n-1}} g(r)\,\mathfrak{n}(\mathrm{d}r) \geq 1.$$

The sequence $(r_n)_{n\geq 1}$ is decreasing and converges to zero, because $\theta(r_n)$ tends to infinity as n goes to infinity and because the function θ is continuous on $(0, r_1]$. For any $n \geq 2$ and any $r \in (r_n, r_{n-1}]$, let us then define

$$\xi(r) = n + \frac{\log\theta(r) - \log\theta(r_{n-1})}{\log\theta(r_n) - \log\theta(r_{n-1})}.$$

We thus obtain a function ξ which is nonincreasing, continuous and positive on $(0, r_1]$, tends to infinity at zero and satisfies $\xi(r) \leq n + 1$ for all $r \in (r_n, r_{n-1}]$. Let us define $\underline{g}(r) = g(r)/\xi(r)$ for all $r \in (0, r_1]$ and extend $\underline{g}$ to a gauge function by letting for instance $\underline{g}(r) = \underline{g}(r_1)$ for all $r > r_1$, as well as $\underline{g}(0) = 0$ and $\underline{g}(\infty) = \infty$. When $n \geq 2$ and $r_n < r \leq r' \leq r_{n-1}$, the difference $\underline{g}(r)/r^d - \underline{g}(r')/r'^d$ vanishes if $\theta(r') = \theta(r)$, and otherwise is equal to

$$\begin{aligned}\frac{\theta(r)}{\xi(r)} - \frac{\theta(r')}{\xi(r')} &= \frac{(\theta(r) - \theta(r'))\xi(r') + \theta(r')(\xi(r') - \xi(r))}{\xi(r)\xi(r')} \\ &\geq \frac{\theta(r) - \theta(r')}{\xi(r)\xi(r')}\, n \left(1 - \frac{\log\frac{\theta(r)}{\theta(r')}}{\frac{\theta(r)}{\theta(r')} - 1} \cdot \frac{1}{n \log\frac{\theta(r_n)}{\theta(r_{n-1})}} \right) \geq 0.\end{aligned}$$

Therefore, the mapping $r \mapsto \underline{g}(r)/r^d$ is continuous at r_n and nonincreasing on the interval $(r_n, r_{n-1}]$ for all $n \geq 2$, which implies that $\underline{g}$ is a d-normalized gauge function. Moreover, $g/\underline{g}$ coincides with ξ near zero, so that $g \prec \underline{g}$. Finally,

$$\begin{aligned}\int_{0<r\leq r_1} \underline{g}(r)\,\mathfrak{n}(\mathrm{d}r) &= \sum_{n=2}^{\infty} \int_{r_n<r\leq r_{n-1}} \frac{g(r)}{\xi(r)}\,\mathfrak{n}(\mathrm{d}r) \\ &\geq \sum_{n=2}^{\infty} \frac{1}{n+1} \int_{r_n<r\leq r_{n-1}} g(r)\,\mathfrak{n}(\mathrm{d}r) \geq \sum_{n=2}^{\infty} \frac{1}{n+1} = \infty,\end{aligned}$$

from which it follows that $\langle \mathfrak{n}, \underline{g}_d \rangle$ is infinite. To conclude, it remains to mention that $\underline{g}$ belongs to $\mathfrak{G}^\infty$; this easily follows from the observation that $\underline{g}(r_n)/r_n^d$ is equal to $\theta(r_n)/n$, which is bounded below by e^{n-1}/n for all $n \geq 1$.

The nonemptyness in (2) may be established by formally replacing the gauge function g above by $\mathbf{1}$, the indicator function of the interval $(0, 1]$. Indeed, although $\mathbf{1}$ is not a gauge function in the strict sense, it still verifies the next two properties that were crucial in the previous construction: the mapping $r \mapsto \mathbf{1}(r)/r^d$ monotonically tends to infinity at zero; the integral of $\mathbf{1}$ with respect to the measure $\mathfrak{n}$ is infinite. Note that the latter holds because $\mathfrak{n}$ belongs to the collection $\mathcal{R}$. We may therefore reproduce the above approach, and we end up with a gauge function $\underline{g}$ in $\mathfrak{G}^\infty$ such that $\langle \mathfrak{n}, \underline{g}_d \rangle$ is infinite. □

As mentioned above, Lemma 9.1 enables us to apply Corollary 9.1 to the $\mathfrak{n}$-describable sets. This boils down to the next statement, which gives a complete and precise description of the size and large intersection properties of those sets.

THEOREM 9.1. *Let U be a nonempty open subset of $\mathbb{R}^d$, let E be a set in $\mathcal{Z}(U)$, and let $\mathfrak{n}$ be a measure in $\mathcal{R}_d$. Let us assume that E is* $\mathfrak{n}$-describable in U. *For any nonempty open set $V \subseteq U$, the following properties hold:*

(1) *for any gauge function $g \in \mathfrak{G} \setminus \mathfrak{G}(\mathfrak{n})$,*

$$\begin{cases} \mathcal{H}^g(E \cap V) = 0 \\ \forall F \in \mathcal{G}^g(V) \quad F \not\subseteq E \end{cases}$$

(2) *for any gauge function $g \in \mathfrak{G}(\mathfrak{n})$,*

$$\begin{cases} \mathcal{H}^g(E \cap V) = \infty \\ \exists F \in \mathcal{G}^g(V) \quad F \subseteq E\,; \end{cases}$$

PROOF. The property (2) results directly from combining of Lemma 9.1 and Corollary 9.1. This is also the case of (1) when the gauge function g is in $\mathfrak{G}^\infty$. It remains us to prove (1) when g is not in $\mathfrak{G}^\infty$. Given that $E \in \mathcal{Z}(U)$, Proposition 2.15 leads to the first part of (1), and Proposition 8.2 implies the second part in the situation where ℓ_g vanishes. Finally, if g is in $\mathfrak{G}^*$, Lemma 9.1 ensures that there is a d-normalized gauge function $\overline{g} \prec g_d$ for which $\langle \mathfrak{n}, \overline{g} \rangle < \infty$. Necessarily, $\overline{g}$ is in $\mathfrak{G}^\infty$, thus verifying (1). Hence, $E \cap V$ has Hausdorff $\overline{g}$-measure zero, and we deduce from Theorem 8.2(3) the second part of (1) for the initial gauge function g. □

In the vein of (142), we may associate with every measure in the collection $\mathcal{R}_d$ a parameter that characterizes its integrability properties at the origin. Specifically, for every measure $\mathfrak{n}$ in $\mathcal{R}_d$, let us define the exponent

$$s_{\mathfrak{n}} = \sup\{s \in (0, d] \mid (r \mapsto r^s) \in \mathfrak{G}(\mathfrak{n})\} = \inf\{s \in (0, d] \mid (r \mapsto r^s) \notin \mathfrak{G}(\mathfrak{n})\}. \quad (182)$$

Note that the right-most set contains d, so that its infimum is well defined. The left-most set may however be empty and, in that situation, we adopt the convention that its supremum is equal to zero. By way of illustration, note that (142) implies that the above exponent $s_{\mathfrak{n}}$ is equal to d. Restricting Theorem 9.1 to the gauge functions $r \mapsto r^s$, we directly obtain the following dimensional statement.

COROLLARY 9.2. *Let U be a nonempty open subset of $\mathbb{R}^d$, let E be a set in $\mathcal{Z}(U)$, and let $\mathfrak{n}$ be a measure in $\mathcal{R}_d$. Let us assume that E is* $\mathfrak{n}$-describable in U. *Then, for any nonempty open set $V \subseteq U$,*

$$\dim_{\mathrm{H}}(E \cap V) = s_{\mathfrak{n}}.$$

Let us assume that $s_{\mathfrak{n}} > 0$. Then, for any nonempty open set $V \subseteq U$,

$$\dim_{\mathrm{P}}(E \cap V) = d.$$

Moreover, if E is a G_δ-set, it belongs to the large intersection class $\mathcal{G}^{s_{\mathfrak{n}}}(U)$.

PROOF. Let us assume that $s_{\mathfrak{n}} < d$. We deduce from Theorem 9.1(1) that $E \cap V$ has Hausdorff s-dimensional measure zero, for any $s \in (s_{\mathfrak{n}}, d]$. Hence, this set has Hausdorff dimension at most $s_{\mathfrak{n}}$. Obviously, this bound still holds if $s_{\mathfrak{n}} = d$.

If the parameter $s_{\mathfrak{n}}$ is positive, Theorem 9.1(2) implies that for any $s \in (0, s_{\mathfrak{n}})$, there exists a subset F_s of E that belongs to the generalized class $\mathcal{G}^{r \mapsto r^s}(V)$. Proposition 8.1 then ensures that each set F_s belongs to the original class $\mathcal{G}^s(V)$ and that its intersection with the open set V has Hausdorff dimension at least s and packing dimension equal to d. It follows that $E \cap V$ has Hausdorff dimension at least $s_{\mathfrak{n}}$ and packing dimension equal to d. Furthermore, if E is a G_δ-set itself, we choose $V = U$ above and deduce from Proposition 8.3(1) that the set E belongs to all the classes $\mathcal{G}^s(U)$, for $s \in (0, s_{\mathfrak{n}})$. In view of Definition 5.2, this implies that $E \in \mathcal{G}^{s_{\mathfrak{n}}}(U)$.

Finally, note that the lower bound on the Hausdorff dimension of $E \cap V$ still holds when $s_{\mathfrak{n}}$ vanishes. Indeed, by Lemma 9.1(2), there is a gauge function in

$\mathfrak{G}(\mathfrak{n})$. Applying Theorem 9.1(2) with such a gauge function, we infer that $E \cap V$ is nonempty, thus having nonnegative Hausdorff dimension. □

9.3.3. $\mathfrak{s}$-describable sets. This section is parallel to previous one. We consider another category of fully describable sets where we have at hand a criterion to decide whether a gauge function is majorizing or minorizing. This criterion is now expressed in terms of growth rates at the origin.

As a motivation, let us consider the measures $\mathfrak{n}_s$ defined for $s \in [0, d)$ by

$$\mathfrak{n}_s(\mathrm{d}r) = \frac{\mathrm{d}r}{r^{s+1}}. \tag{183}$$

It is elementary to check that each measure $\mathfrak{n}_s$ belongs to the collection $\mathcal{R}_d$, and that the associated exponent given by (182) is equal to s. In particular, in view of Corollary 9.2, every $\mathfrak{n}_s$-describable set has Hausdorff dimension equal to s. Moreover, note that the mapping $s \mapsto \mathfrak{G}(\mathfrak{n}_s)$ is nondecreasing.

The new category of fully describable sets that we introduce hereafter may be obtained by considering countable intersections of $\mathfrak{n}_s$-describable set. Let U be a nonempty open subset of $\mathbb{R}^d$, let $(E_n)_{n \geq 1}$ be a sequence of sets in $\mathcal{Z}(U)$, and let E denote the intersection of the sets E_n. Propositions 9.2 and 9.3 show that

$$\mathfrak{m}(E, U) = \bigcap_{n=1}^{\infty} \mathfrak{m}(E_n, U) \quad \text{and} \quad \mathfrak{M}(E, U) \supseteq \bigcup_{n=1}^{\infty} \mathfrak{M}(E_n, U). \tag{184}$$

Let us suppose the existence of a sequence $(s_n)_{n \geq 1}$ of real numbers in $[0, d)$ such that each set E_n is $\mathfrak{n}_{s_n}$-describable in U. Definition 9.6 implies that

$$\mathfrak{m}(E_n, U) \cap \mathfrak{G}^\infty = \mathfrak{G}(\mathfrak{n}_{s_n}) \qquad \text{and} \qquad \mathfrak{M}(E_n, U) = \mathfrak{G}(\mathfrak{n}_{s_n})^{\complement}. \tag{185}$$

It follows that the minorizing collection of E in U coincides on $\mathfrak{G}^\infty$ with the intersection of the collections $\mathfrak{G}(\mathfrak{n}_{s_n})$, and the majorizing collection of E in U contains the complement in $\mathfrak{G}^\infty$ of the latter intersection.

This entails in particular that the set E is fully describable. This also prompts the study of countable intersections of sets of the form $\mathfrak{G}(\mathfrak{n}_s)$. Those sets being monotonic with respect to the parameter s, we end up with an intersection set that is either of the previous form $\mathfrak{G}(\mathfrak{n}_s)$, or of a new form $\mathfrak{G}(\mathfrak{s})$, where $\mathfrak{s}$ is some real number in $[0, d)$. The latter sets are the subsets of $\mathfrak{G}^\infty$ defined by the condition

$$g \in \mathfrak{G}(\mathfrak{s}) \qquad \Longleftrightarrow \qquad \forall s > \mathfrak{s} \qquad g_d(r) \neq \mathrm{o}(r^s) \quad \text{as} \quad r \to 0,$$

and are linked with the former through the statement of Lemma 9.2 below. Note that $g_d(r) \neq \mathrm{o}(r^d)$ for any $g \in \mathfrak{G}^\infty$. So, in the previous condition, the only relevant values of s are those in $(\mathfrak{s}, d)$. Moreover, the mapping $\mathfrak{s} \mapsto \mathfrak{G}(\mathfrak{s})$ is clearly nondecreasing. Finally, the complement in $\mathfrak{G}^\infty$ of $\mathfrak{G}(\mathfrak{s})$ is denoted by $\mathfrak{G}(\mathfrak{s})^{\complement}$.

Lemma 9.2. *For any real number $\mathfrak{s} \in [0, d)$, we have*

$$\mathfrak{G}(\mathfrak{s}) = \bigcap_{s \in (\mathfrak{s}, d)} \downarrow \mathfrak{G}(\mathfrak{n}_s).$$

Proof. Let g be a gauge function in $\mathfrak{G}^\infty$. If g is not in $\mathfrak{G}(\mathfrak{s})$, then we have $g_d(r) \leq c\, r^{s_0}$ for all $r \in (0, 1]$, and some $s_0 \in (\mathfrak{s}, d)$ and some $c > 0$. Thus,

$$\langle \mathfrak{n}_s, g_d \rangle = \int_{(0,1]} g_d(r)\, \mathfrak{n}_s(\mathrm{d}r) \leq c \int_{(0,1]} r^{s_0 - s - 1}\, \mathrm{d}r$$

for every $s \in (\mathfrak{s}, d)$, and the latter integral is finite if $s < s_0$. It follows that g does not belong to any of the sets $\mathfrak{G}(\mathfrak{n}_s)$ with $s \in (\mathfrak{s}, s_0)$.

Conversely, let us assume that g is not in $\mathfrak{G}(\mathfrak{n}_s)$ for some $s \in (\mathfrak{s}, d)$. We deduce from the monotonicity properties satisfied by g_d that for any real number $r \in (0,1]$,

$$\langle \mathfrak{n}_s, g_d \rangle \geq \int_{r/2}^{r} \frac{g_d(\rho)}{\rho^d} \rho^{d-s-1} \, \mathrm{d}\rho \geq \frac{g_d(r)}{r^d} \int_{r/2}^{r} \rho^{d-s-1} \, \mathrm{d}\rho = \frac{1-2^{s-d}}{d-s} \cdot \frac{g_d(r)}{r^s}.$$

The finiteness of the left-hand side entails that $g_d(r) = \mathrm{O}(r^s)$ as r goes to zero. As a consequence, the gauge function g cannot belong to $\mathfrak{G}(\mathfrak{s})$. □

As we now explain, the sets $\mathfrak{G}(\mathfrak{s})$ play a pivotal rôle in the definition of the new category of fully describable sets.

Definition 9.7. Let U be a nonempty open subset of $\mathbb{R}^d$, let E be a set in $\mathcal{Z}(U)$, and let $\mathfrak{s}$ be in $[0, d)$. We say that the set E is *$\mathfrak{s}$-describable in U* if

$$\mathfrak{M}(E, U) = \mathfrak{G}(\mathfrak{s})^{\complement} \qquad \text{and} \qquad \mathfrak{m}(E, U) \cap \mathfrak{G}^{\infty} = \mathfrak{G}(\mathfrak{s}).$$

Similarly to what happens for $\mathfrak{n}$-describable sets, it is clear that $\mathfrak{s}$-describable sets are fully describable, with disjoint majorizing and minorizing collections. Moreover, we have the following analog of Lemma 9.1.

Lemma 9.3. *For any real number $\mathfrak{s} \in [0, d)$, the following properties hold:*

(1) *the set $\mathfrak{G}^* \setminus \mathfrak{G}(\mathfrak{s})$ is left-open;*
(2) *the set $\mathfrak{G}(\mathfrak{s})$ is right-open and nonempty.*

In particular, if a set E is $\mathfrak{s}$-describable in U, then both collections $\mathfrak{M}(E, U)$ and $\mathfrak{m}(E, U) \cap \mathfrak{G}^{\infty}$ are simultaneously left-open and right-open.

Proof. The left-openness of the set $\mathfrak{G}^* \setminus \mathfrak{G}(\mathfrak{s})$ is inherited from that of the sets $\mathfrak{G}^* \setminus \mathfrak{G}(\mathfrak{n})$, for $\mathfrak{n} \in \mathcal{R}_d$. Indeed, if g is d-normalized gauge function in $\mathfrak{G}^* \setminus \mathfrak{G}(\mathfrak{s})$, Lemma 9.2 ensures that $g \notin \mathfrak{G}(\mathfrak{n}_s)$ for some $s \in (\mathfrak{s}, d)$. By Lemma 9.1(1), there is a d-normalized gauge function $\overline{g}$ in $\mathfrak{G}^* \setminus \mathfrak{G}(\mathfrak{n}_s)$ such that $\overline{g} \prec g$. By Lemma 9.2 again, $\overline{g}$ does not belong to $\mathfrak{G}(\mathfrak{s})$, and we end up with (1).

Furthermore, let us recall that the mapping $s \mapsto \mathfrak{G}(\mathfrak{n}_s)$ is nondecreasing. Thanks to Lemma 9.2, we deduce that $\mathfrak{G}(\mathfrak{s})$ contains $\mathfrak{G}(\mathfrak{n}_{\mathfrak{s}})$. Lemma 9.1(2) shows that the latter set is nonempty, so the former is nonempty as well.

Finally, the right-openness property in (2) follows from the fact that, if g is a d-normalized gauge function in $\mathfrak{G}(\mathfrak{s})$, letting $\underline{g}(r) = g(r)/\log(g(r)/r^d)$ yields as required a d-normalized gauge function in $\mathfrak{G}(\mathfrak{s})$ such that $g \prec \underline{g}$. □

Owing to Lemma 9.3, if a set E is $\mathfrak{s}$-describable in U, then $\mathfrak{m}(E, U) \cap \mathfrak{G}^{\infty}$ is nonempty, so E necessarily contains a set with large intersection. Furthermore, both $\mathfrak{M}(E, U)$ and $\mathfrak{m}(E, U) \cap \mathfrak{G}^{\infty}$ are left-open and right-open at the same time. We may thus apply Corollary 9.1, and deduce the following complete and precise description of the size and large intersection properties of the set E.

Theorem 9.2. *Let U be a nonempty open subset of $\mathbb{R}^d$, let E be a set in $\mathcal{Z}(U)$, and let $\mathfrak{s}$ be in $[0, d)$. Let us assume that E is* $\mathfrak{s}$-describable in U. *For any nonempty open set $V \subseteq U$, the following properties hold:*

(1) *for any gauge function $g \in \mathfrak{G} \setminus \mathfrak{G}(\mathfrak{s})$,*

$$\begin{cases} \mathcal{H}^g(E \cap V) = 0 \\ \forall F \in \mathcal{G}^g(V) \quad F \not\subseteq E \end{cases}$$

(2) *for any gauge function $g \in \mathfrak{G}(\mathfrak{s})$,*

$$\begin{cases} \mathcal{H}^g(E \cap V) = \infty \\ \exists F \in \mathcal{G}^g(V) \quad F \subseteq E\,; \end{cases}$$

Theorem 9.2 above may be regarded as an analog of Theorem 9.1, and may be established by easily adapting the proof of the latter result. The proof is therefore omitted here. We just mention that one needs to use Lemma 9.3 instead of Lemma 9.1 whenever necessary, and that Corollary 9.1 is crucial in that proof too.

For all $s \in (0,d]$ and $\mathfrak{s} \in [0,d)$, one easily checks that the gauge function $r \mapsto r^s$ belongs to the set $\mathfrak{G}(\mathfrak{s})$ if and only if $s \leq \mathfrak{s}$. Therefore, restricting Theorem 9.2 to these specific gauge functions leads to the following dimensional statement which is parallel to Corollary 9.2. Again, the proof is very similar to that of the latter result; for that reason, it is left to the reader.

COROLLARY 9.3. *Let U be a nonempty open subset of $\mathbb{R}^d$, let E be a set in $\mathcal{Z}(U)$, and let $\mathfrak{s}$ be in $[0,d)$. Let us assume that E is* $\mathfrak{s}$-describable *in U. Then, for any nonempty open set $V \subseteq U$,*

$$\dim_{\mathrm{H}}(E \cap V) = \mathfrak{s} \qquad \textit{with} \qquad \mathcal{H}^{\mathfrak{s}}(E \cap V) = \infty.$$

Let us assume that $\mathfrak{s} > 0$. Then, for any nonempty open set $V \subseteq U$,

$$\dim_{\mathrm{P}}(E \cap V) = d.$$

Moreover, there exists a subset of E in the large intersection class $\mathcal{G}^{\mathfrak{s}}(U)$. In particular, if E is a G_δ-set itself, it belongs to the latter class.

We finish by going back to the motivational example supplied by the intersection of the $\mathfrak{n}_{s_n}$-describable sets E_n. As shown below, the set $\mathfrak{G}(\mathfrak{s})$ actually arises under the assumption that the infimum of the real numbers s_n is not attained.

PROPOSITION 9.6. *Let U be a nonempty open subset of $\mathbb{R}^d$ and, for each $n \geq 1$, let E_n be a set in $\mathcal{Z}(U)$ that is $\mathfrak{n}_{s_n}$-describable in U for some $s_n \in [0,d)$. Letting*

$$E = \bigcap_{n=1}^{\infty} E_n \qquad \textit{and} \qquad \mathfrak{s} = \inf_{n \geq 1} s_n,$$

we then have the following dichotomy:

- *if the infimum is attained at some n_0, then E is $\mathfrak{n}_{s_{n_0}}$-describable in U ;*
- *if the infimum is not attained, then E is $\mathfrak{s}$-describable in U.*

PROOF. To begin with, we learn from (184) and (185) that the minorizing and majorizing collections of E in U satisfy

$$\mathfrak{m}(E,U) \cap \mathfrak{G}^{\infty} = \bigcap_{n=1}^{\infty} \mathfrak{G}(\mathfrak{n}_{s_n}) \qquad \text{and} \qquad \mathfrak{M}(E,U) \supseteq \mathfrak{G}^{\infty} \setminus \bigcap_{n=1}^{\infty} \mathfrak{G}(\mathfrak{n}_{s_n}). \tag{186}$$

If the infimum is attained at a given integer n_0, the intersection over all $n \geq 1$ of the sets $\mathfrak{G}(\mathfrak{n}_{s_n})$ coincides with the sole $\mathfrak{G}(\mathfrak{n}_{s_{n_0}})$, so that

$$\mathfrak{m}(E,U) \cap \mathfrak{G}^{\infty} = \mathfrak{G}(\mathfrak{n}_{s_{n_0}}) \qquad \text{and} \qquad \mathfrak{M}(E,U) \supseteq \mathfrak{G}(\mathfrak{n}_{s_{n_0}})^{\complement}.$$

In particular, we deduce from Lemma 9.1(2) that the collection $\mathfrak{m}(E,U) \cap \mathfrak{G}^{\infty}$ is right-open. Proposition 9.4(2) then yields

$$\mathfrak{G}^{\infty} \setminus \mathfrak{M}(E,U) \supseteq \mathfrak{m}(E,U) \cap \mathfrak{G}^{\infty} = \mathfrak{G}(\mathfrak{n}_{s_{n_0}}).$$

It follows that the majorizing collection $\mathfrak{M}(E,U)$ is equal to the whole $\mathfrak{G}(\mathfrak{n}_{s_{n_0}})^{\complement}$. As a consequence, the set E is $\mathfrak{n}_{s_{n_0}}$-describable in U.

The proof is very similar in the opposite situation where the infimum is not attained. Indeed, using the monotonicity of the mapping $s \mapsto \mathfrak{G}(\mathfrak{n}_s)$ and combining Lemma 9.2 with (186), we now get

$$\mathfrak{m}(E,U) \cap \mathfrak{G}^{\infty} = \mathfrak{G}(\mathfrak{s}) \qquad \text{and} \qquad \mathfrak{M}(E,U) \supseteq \mathfrak{G}(\mathfrak{s})^{\complement}.$$

We deduce from Lemma 9.3(2) that $\mathfrak{m}(E,U)\cap\mathfrak{G}^\infty$ is right-open, and then from Proposition 9.4(2) that the majorizing collection $\mathfrak{M}(E,U)$ is equal to the whole $\mathfrak{G}(\mathfrak{s})^\complement$. Hence, the set E is $\mathfrak{s}$-describable in U. $\square$

Slightly modifying the above approach leads to another situation where $\mathfrak{s}$-describable sets arise naturally. Given a real number $\mathfrak{s}\in[0,d)$ and a nonempty open set U, we consider a sequence $(E_s)_{s\in(\mathfrak{s},d)}$ of sets in $\mathcal{Z}(U)$, and we assume that the mapping $s\mapsto E_s$ is increasing and that each set E_s is $\mathfrak{n}_s$-describable in U. We then choose in the interval $(\mathfrak{s},d)$ an arbitrary decreasing sequence $(s_n)_{n\geq 1}$ that converges to $\mathfrak{s}$. The monotonicity of the sets E_s with respect to s implies that their intersection is equal to that of the sets E_{s_n}. Moreover, the latter sets fall into the above setting because the infimum of the real numbers s_n is not attained. Hence, the intersection over all $s\in(\mathfrak{s},d)$ of the sets E_s is $\mathfrak{s}$-describable in U.

9.4. Link with eutaxic sequences

Eutaxic sequences were defined and thoroughly studied in Chapter 6. Our purpose is now to show that the limsup sets that are naturally associated with such sequences fall into the category of fully describable sets. The analysis below heavily relies on the large intersection transference principle presented in Section 8.2.

Let $(x_n)_{n\geq 1}$ be a sequence of points in $\mathbb{R}^d$ and let $(r_n)_{n\geq 1}$ be a nonincreasing sequence of positive real numbers that converges to zero. It is clear that the family $(x_n,r_n)_{n\geq 1}$ is an approximation system in the sense of Definition 4.1 ; this naturally prompts us to consider the associated limsup set defined as in (171), namely,

$$\mathfrak{F}((x_n,r_n)_{n\geq 1})=\left\{x\in\mathbb{R}^d \;\middle|\; |x-x_n|<r_n \quad \text{for i.m. } n\geq 1\right\}.$$

This set is unchanged if we remove a finite number of initial terms x_n and r_n, so there is no loss in generality in assuming that the real numbers r_n are in $(0,1]$.

Lemma 2.1 shows that for any gauge function g such that the series $\sum_n g_d(r_n)$ is convergent, the set $\mathfrak{F}((x_n,r_n)_{n\geq 1})$ has Hausdorff g_d-measure equal to zero. Here, g_d denotes as usual the d-normalization of the gauge function g. Proposition 2.10 allows us to transfer the previous property to the Hausdorff g-measure itself. As a consequence, for any gauge function g, the following implication holds:

$$\sum_{n=1}^{\infty} g_d(r_n)<\infty \qquad\Longrightarrow\qquad \mathcal{H}^g(\mathfrak{F}((x_n,r_n)_{n\geq 1}))=0. \tag{187}$$

Let us now recast this elementary result in terms of majorizing gauge functions. In what follows, r is a shorthand for $(r_n)_{n\geq 1}$, and $\mathfrak{n}_\mathrm{r}$ is the measure in $\mathcal{R}$ defined by

$$\mathfrak{n}_\mathrm{r}=\sum_{n=1}^{\infty}\delta_{r_n}. \tag{188}$$

We further assume that the series $\sum_n r_n^d$ is convergent, or equivalently that $\mathfrak{n}_\mathrm{r}$ belongs to $\mathcal{R}_d$, so as to ensure that the above limsup set has Lebesgue measure zero in $\mathbb{R}^d$. The previous result then yields the next statement.

Proposition 9.7. *Let $(x_n)_{n\geq 1}$ be a sequence in $\mathbb{R}^d$ and let $(r_n)_{n\geq 1}$ be a nonincreasing sequence of real numbers in $(0,1]$ such that $\sum_n r_n^d$ converges. Then,*

$$\mathfrak{F}((x_n,r_n)_{n\geq 1})\in\mathcal{Z}(\mathbb{R}^d) \qquad \text{and} \qquad \mathfrak{M}(\mathfrak{F}((x_n,r_n)_{n\geq 1}),\mathbb{R}^d)\supseteq\mathfrak{G}(\mathfrak{n}_\mathrm{r})^\complement.$$

Eutaxy will lead to a natural converse of that result. To be specific, when U denotes a nonempty open subset of $\mathbb{R}^d$, we recall from Definition 6.2 that a sequence $(x_n)_{n\geq 1}$ in $\mathbb{R}^d$ is uniformly eutaxic in U if for any sequence $(r_n)_{n\geq 1}$ in the set P_d defined by (106), the following condition holds:

$$\text{for } \mathcal{L}^d\text{-a.e. } x\in U \quad \exists \text{ i.m. } n\geq 1 \qquad |x-x_n|<r_n.$$

Transferring this property to the present setting, this means that the following implication holds:

$$\sum_{n=1}^{\infty} r_n^d = \infty \qquad \Longrightarrow \qquad \mathcal{L}^d(U \setminus \mathfrak{F}((x_n, r_n)_{n\geq 1})) = 0. \tag{189}$$

The trick is now to replace r_n by $g_d(r_n)^{1/d}$ in (189), if g denotes the gauge function under consideration. In fact, since the real numbers r_n are nonincreasing and tend to zero, the real numbers $g_d(r_n)^{1/d}$ tend to zero as well and, at least for n sufficiently large, are also nonincreasing. The limsup set $\mathfrak{F}((x_n, r_n)_{n\geq 1})$ being unchanged when removing initial terms, we end up with the implication

$$\sum_{n=1}^{\infty} g_d(r_n) = \infty \qquad \Longrightarrow \qquad \mathcal{L}^d(U \setminus \mathfrak{F}((x_n, g_d(r_n)^{1/d})_{n\geq 1})) = 0.$$

In other words, the divergence assumption bearing on g implies that the approximation system $(x_n, r_n)_{n\geq 1}$ is homogeneously g-ubiquitous in U in the sense of Definition 8.1. We are now in position to apply the large intersection transference principle, namely, Theorem 8.3. Accordingly, we deduce that

$$\sum_{n=1}^{\infty} g_d(r_n) = \infty \qquad \Longrightarrow \qquad \mathfrak{F}((x_n, r_n)_{n\geq 1}) \in \mathcal{G}^g(U).$$

Again, this result may be recast in terms of minorizing gauge functions. We have indeed the following statement.

Proposition 9.8. *Let U be a nonempty open subset of $\mathbb{R}^d$, let $(x_n)_{n\geq 1}$ be a sequence in $\mathbb{R}^d$ that is uniformly eutaxic in U, and let $(r_n)_{n\geq 1}$ be a nonincreasing sequence of real numbers in $(0,1]$ such that $\sum_n r_n^d$ converges. Then,*

$$\mathfrak{F}((x_n, r_n)_{n\geq 1}) \in \mathcal{Z}(U) \qquad \text{and} \qquad \mathfrak{m}(\mathfrak{F}((x_n, r_n)_{n\geq 1}), U) \supseteq \mathfrak{G}(\mathfrak{n}_\mathrm{r}).$$

Combining Propositions 9.7 and 9.7, we infer that the set $\mathfrak{F}((x_n, r_n)_{n\geq 1})$ is fully describable in U, under the assumptions that the sequence $(x_n)_{n\geq 1}$ is uniformly eutaxic in U and the series $\sum_n r_n^d$ converges. The next lemma will actually help us obtain a more precise statement.

Lemma 9.4. *Let U be a nonempty open subset of $\mathbb{R}^d$, let E be a set in $\mathcal{Z}(U)$, and let $\mathfrak{H}$ be a subset of $\mathfrak{G}^\infty$ with complement denoted by $\mathfrak{H}^\complement$. Let us assume that:*

- *a gauge function $g \in \mathfrak{G}^\infty$ is in $\mathfrak{H}$ if and only if its d-normalization g_d is;*
- *the collections $\mathfrak{m}(E, U)$ and $\mathfrak{M}(E, U)$ contain $\mathfrak{H}$ and $\mathfrak{H}^\complement$, respectively;*
- *the collection $\mathfrak{H}$ is right-open, or the collection $\mathfrak{H}^\complement$ is left-open.*

Then, the following equalities hold:

$$\mathfrak{M}(E, U) = \mathfrak{H}^\complement \qquad \text{and} \qquad \mathfrak{m}(E, U) \cap \mathfrak{G}^\infty = \mathfrak{H},$$

Proof. To begin with, let us assume for instance that the collection $\mathfrak{H}$ is right-open. Let us consider a gauge function g in $\mathfrak{M}(E, U)$, and assume by contradiction that g does not belong to $\mathfrak{H}^\complement$. Then, g_d belongs to $\mathfrak{H}$, and the right-openness assumption ensures the existence of a d-normalized gauge function $\underline{g} \in \mathfrak{H}$ with $g_d \prec \underline{g}$. Since $\mathfrak{H}$ is contained in $\mathfrak{m}(E, U)$, the gauge function $\underline{g}$ is minorizing for E in U, meaning that E admits a subset F in the class $\mathcal{G}^{\underline{g}}(U)$. Owing to Theorem 8.2(3) and Proposition 2.15, the set F has infinite Hausdorff g_d-measure in U. We deduce with the help of Proposition 2.10 that E has infinite Hausdorff g-measure in U, thereby contradicting the fact that g is majorizing for E in U.

The case where $\mathfrak{H}^\complement$ is left-open is treated similarly. To be specific, let us consider a gauge function g in $\mathfrak{m}(E, U) \cap \mathfrak{G}^\infty$, and suppose by contradiction that $g \notin \mathfrak{H}$. Thus, $g_d \in \mathfrak{H}^\complement$, and there is a d-normalized gauge function $\overline{g} \in \mathfrak{H}^\complement$ such that $\overline{g} \prec g_d$. The gauge function $\overline{g}$ is then in $\mathfrak{M}(E, U)$. By Theorem 8.2(3) and Proposition 2.15 again, this contradicts the fact that g_d is minorizing for E in U. $\square$

In view of Propositions 9.7 and 9.8, and under the assumptions that the sequence $(x_n)_{n\geq 1}$ is uniformly eutaxic in U and the series $\sum_n r_n^d$ converges, we may apply Lemma 9.4 to the set $\mathfrak{F}((x_n, r_n)_{n\geq 1})$, along with the collection $\mathfrak{G}(\mathfrak{n}_\mathrm{r})$. We end up with the next statement.

THEOREM 9.3. *Let U be a nonempty open subset of $\mathbb{R}^d$, let $(x_n)_{n\geq 1}$ be a sequence in $\mathbb{R}^d$ that is uniformly eutaxic in U, and let $(r_n)_{n\geq 1}$ be a nonincreasing sequence of real numbers in $(0, 1]$ such that the series $\sum_n r_n^d$ converges. Then, the set $\mathfrak{F}((x_n, r_n)_{n\geq 1})$ is $\mathfrak{n}_\mathrm{r}$-describable in U.*

This means that we may eventually apply Theorem 9.1 to the set $\mathfrak{F}((x_n, r_n)_{n\geq 1})$, thereby obtaining a complete and precise description of its size and large intersection properties. We may also apply Corollary 9.2 if only a dimensional result is needed. This will enable us to revisit in Chapters 10 and 11 the examples of eutaxic sequences already presented in Chapter 6 and to shed light on the size and large intersection properties of the associated limsup sets.

Let us finally recall from completeness that when the sequence $(x_n)_{n\geq 1}$ is uniformly eutaxic in U and the series $\sum_n r_n^d$ diverges, the set $\mathfrak{F}((x_n, r_n)_{n\geq 1})$ has full Lebesgue measure in U, see for instance (189). As explained at the beginning of this chapter, its size and large intersection properties are then trivial. This remark remains also valid when the sequence $(r_n)_{n\geq 1}$, while still being nonincreasing, does not converge to zero. In fact, the sequence $(r_n)_{n\geq 1}$ is not in P_d anymore, which prevents us from applying (189) directly. However, as already observed in Section 6.2.1, the sequence defined by $\widetilde{r}_n = \min\{r_n, 1/(2n^{1/d})\}$ for each $n \geq 1$ is necessarily in P_d. Applying (189) to this sequence, we deduce that the smaller set $\mathfrak{F}((x_n, \widetilde{r}_n)_{n\geq 1})$ has full Lebesgue measure in U, and thus $\mathfrak{F}((x_n, r_n)_{n\geq 1})$ as well.

9.5. Link with optimal regular systems

The notion of optimal regular system is the purpose of Chapter 7, and is closely related with the notion of eutaxic sequence discussed in Section 9.4 above. Hence, we may anticipate that optimal regular systems also share interesting connections with the material discussed in the present chapter.

We recall from Definition 7.1 that an optimal regular system results from combining a countably infinite set $\mathcal{A} \subseteq \mathbb{R}^d$ with a height function $H : \mathcal{A} \to (0, \infty)$. In the context of Diophantine approximation, the sets that are naturally associated with such a system are of those the form (148), namely,

$$F_\varphi = \left\{x \in \mathbb{R}^d \;\middle|\; |x - a| < \varphi(H(a)) \quad \text{for i.m. } a \in \mathcal{A}\right\},$$

where φ is a positive nonincreasing continuous function defined on $[0, \infty)$. A first description of the size and large intersection properties of those sets is given by Theorem 7.1. In particular, if U is a nonempty open subset of $\mathbb{R}^d$, and $(\mathcal{A}, H)$ is an optimal regular system in U, then the set F_φ has full Lebesgue measure in U if the integral defined by (152), namely,

$$I_\varphi = \int_0^\infty \eta^{d-1}\varphi(\eta)^d \,\mathrm{d}\eta$$

diverges. The size and large intersection properties of the set F_φ being trivial in that situation, we may rule out this situation in what follows.

Accordingly, we assume from now on that the integral I_φ is convergent. Since the function φ is nonincreasing, it necessarily tends to zero at infinity. Hence, the admissibility condition (149) entails that the family $(a, \varphi(H(a)))_{a\in\mathcal{A}}$ is an approximation system in the sense of Definition 4.1. Moreover, we necessarily have $\varphi(h) \leq 1$ for h large enough, and arguing as in the proof of the convergence case of Theorem 7.1(1), we see that F_φ is left unchanged when replacing φ by its minimum with the fonction that is constant equal to one on $[0,\infty)$. As a result, there is no loss in generality in assuming hereafter that the function φ is valued in $(0,1]$.

We then associate with the function φ the measure $\mathfrak{n}_\varphi$ characterized by the condition that for any nonnegative Borel measurable function f defined on $(0,1]$,

$$\int_{(0,1]} f(r)\,\mathfrak{n}_\varphi(\mathrm{d}r) = \int_0^\infty \eta^{d-1} f(\varphi(\eta))\,\mathrm{d}\eta. \tag{190}$$

It is clear that the measure $\mathfrak{n}_\varphi$ belongs to the collection $\mathcal{R}$. Moreover, the finiteness assumption on the integral I_φ is equivalent to the fact that $\mathfrak{n}_\varphi$ belongs to $\mathcal{R}_d$. Our purpose is to establish an analog of Theorem 9.3 for optimal regular systems; the measure $\mathfrak{n}_\varphi$ will in fact play the rôle of $\mathfrak{n}_\mathrm{r}$ in the present analysis.

We begin by studying the majorizing collection of F_φ in U. The next statement may be seen as a natural counterpart of Proposition 9.7 in the present setting.

Proposition 9.9. *Let U be a nonempty open subset of $\mathbb{R}^d$, let $(\mathcal{A}, H)$ be an optimal regular system in U, and let φ be a positive nonincreasing continuous function defined on $[0,\infty)$, valued in $(0,1]$ and such that I_φ converges. Then,*

$$F_\varphi \in \mathcal{Z}(U) \qquad \text{and} \qquad \mathfrak{M}(F_\varphi, U) \supseteq \mathfrak{G}(\mathfrak{n}_\varphi)^{\complement}.$$

Proof. To begin with, the set F_φ has Lebesgue measure zero in U owing to Theorem 7.1(1). Furthermore, learning from the proof of this theorem, let us disclose and exploit the limsup structure of the set F_φ. In fact, for any nonempty open ball $B \subseteq U$, the pair $(\mathcal{A}, H)$ is also an optimal regular system in B, and Lemma 7.1 yields a monotonic enumeration, denoted by $(a_n)_{n\geq 1}$, of $(\mathcal{A}, H)$ in B. Then, $F_\varphi \cap B$ is contained in the set F_φ^B defined by (153), namely,

$$F_\varphi \cap B \subseteq F_\varphi^B = \left\{x \in \mathbb{R}^d \;\middle|\; |x - a_n| < r_n \quad \text{for i.m. } n \geq 1\right\},$$

where $r_n = \varphi(H(a_n))$ for any $n \geq 1$. Combining Lemma 2.1 and Proposition 2.10, we deduce that for any gauge function g,

$$\sum_{n=1}^\infty g_d(r_n) < \infty \qquad \Longrightarrow \qquad \mathcal{H}^g(F_\varphi \cap B) = 0.$$

Now, the gauge function g_d is nondecreasing on the interval $[0, \varepsilon_{g_d})$, where ε_{g_d} is defined in Section 8.2.1, so we may consider a function $\widetilde{g}$ that is nondecreasing on $[0,\infty)$ and coincides with g_d on $[0, \varepsilon_{g_d})$. Still reasoning as in the proof of Theorem 7.1(1), we define a premeasure ζ by $\zeta((h,h')) = \widetilde{g}(\varphi(h)) - \widetilde{g}(\varphi(h'))$ when $0 < h \leq h' < \infty$, and then consider the outer measure ζ_* given by (53). We end up with a Borel measure on $(0,\infty)$ such that $\zeta_*([h,\infty)) = \widetilde{g}(\varphi(h))$ for any $h > 0$. Thanks to Tonelli's theorem and the optimality of the underlying system, we have

$$\begin{aligned}
\sum_{n=1}^\infty \widetilde{g}(r_n) &= \int_0^\infty \#\{n \geq 1 \mid H(a_n) \leq h\}\,\zeta_*(\mathrm{d}h) \\
&\leq \int_0^\infty \kappa'_B h^d\,\zeta_*(\mathrm{d}h) + \underbrace{\int_0^{h'_B} \left(\#\{n \geq 1 \mid H(a_n) \leq h\} - \kappa'_B h^d\right)\zeta_*(\mathrm{d}h)}_{R'} \\
&= \kappa'_B d \int_0^\infty \eta^{d-1}\widetilde{g}(\varphi(\eta))\,\mathrm{d}\eta + R' = \kappa'_B d \int_{(0,1]} \widetilde{g}(r)\,\mathfrak{n}_\varphi(\mathrm{d}r) + R',
\end{aligned}$$

where κ'_B and h'_B are given by Definition 7.1. Given that φ tends to zero at infnity, we may replace the function $\widetilde{g}$ by the gauge function g_d in the left-most and the right-most sides without altering the convergent or divergent nature of the involved series or integral. As a consequence,

$$\int_{(0,1]} g_d(r)\,\mathfrak{n}_\varphi(\mathrm{d}r) < \infty \qquad \Longrightarrow \qquad \mathcal{H}^g(F_\varphi \cap B) = 0.$$

We may finally replace the ball B above by the whole open set U, because the Hausdorff g-measure is an outer measure and every open set may be written as a countable union of inside open balls. □

Let us now naturally turn our attention to the minorizing collection of F_φ in U. The counterpart of Proposition 9.8 is the following result.

PROPOSITION 9.10. *Let U be a nonempty open subset of $\mathbb{R}^d$, let $(\mathcal{A}, H)$ be an optimal regular system in U, and let φ be a positive nonincreasing continuous function defined on $[0,\infty)$, valued in $(0,1]$ and such that I_φ converges. Then,*

$$F_\varphi \in \mathcal{Z}(U) \qquad \text{and} \qquad \mathfrak{m}(F_\varphi, U) \supseteq \mathfrak{G}(\mathfrak{n}_\varphi).$$

PROOF. Given $g \in \mathfrak{G}(\mathfrak{n}_\varphi)$, the idea is basically to apply Theorem 7.1(1) with the function $h \mapsto g_d(\varphi(h))^{1/d}$, denoted for short by $g_d^{1/d} \circ \varphi$, instead of φ. This new function might not be continuous and nonincreasing on the whole interval $[0,\infty)$, but surely satisfies these properties on the closed right-infinite interval formed by the real numbers $h \geq 0$ such that $\varphi(h) \leq \varepsilon_{g_d}/2$. Therefore, letting $\widetilde{\varphi}(h) = g_d(\min\{\varphi(h), \varepsilon_{g_d}/2\})^{1/d}$, we obtain a function that is continuous and nonincreasing on the whole $[0,\infty)$ and matches the function of interest near infinity.

Since the gauge function g is in $\mathfrak{G}(\mathfrak{n}_\varphi)$, the integral $I_{\widetilde{\varphi}}$ is divergent. We deduce from Theorem 7.1(1) that the set $F_{\widetilde{\varphi}}$ has full Lebesgue measure in U, and thus that the larger set $F_{g_d^{1/d}\circ\varphi}$ has full Lebesgue measure in U as well. As a consequence, the approximation system $(a, \varphi(H(a)))_{a\in\mathcal{A}}$ is homogeneously g-ubiquitous in U. We conclude that the set F_φ belongs to the class $\mathcal{G}^g(U)$ by means of the large intersection transference principle, namely, Theorem 8.3. □

Finally, if the assumptions of Propositions 9.7 and 9.8 are satisfied, these results ensure that we may apply Lemma 9.4 to the set F_φ and the collection $\mathfrak{G}(\mathfrak{n}_\varphi)$. This readily gives the next statement.

THEOREM 9.4. *Let U be a nonempty open subset of $\mathbb{R}^d$, let $(\mathcal{A}, H)$ be an optimal regular system in U, and let φ be a positive nonincreasing continuous function defined on $[0,\infty)$, valued in $(0,1]$ and such that the integral I_φ converges. Then, the set F_φ is $\mathfrak{n}_\varphi$-describable in U.*

Subsequently applying Theorem 9.1 to the set F_φ, we may obtain a complete and precise description of its size and large intersection properties. Also, if only a dimensional result is needed, it is possible and sufficient to use Corollary 9.2. We shall employ these ideas in Chapter 10 so as to revisit the examples from Diophantine approximation presented in Chapter 7.

CHAPTER 10

Applications to metric Diophantine approximation

Our aim is to review most of the examples from metric Diophantine approximation studied hitherto, and to complete the analysis of their size and large intersection properties in light of the theory of describable sets introduced in Chapter 9.

10.1. Approximation by fractional parts of sequences

10.1.1. Linear sequences. This section should be seen as a followup to Sections 6.1.2 and 6.3.3. Let us begin by recalling that Kurzweil's theorem, namely, Theorem 6.9 ensures that for any point $x \in \mathbb{R}^d$, the sequence $(\{nx\})_{n\geq 1}$ of fractional parts is uniformly eutaxic in the open cube $(0,1)^d$ if and only if x is a badly approximable point.

Let us place ourselves in that situation and let us consider a nonincreasing sequence $\mathrm{r} = (r_n)_{n\geq 1}$ of positive real numbers. Our aim is to detail the size and large intersection properties of the set

$$F_{\mathrm{r}}(x) = \left\{ y \in \mathbb{R}^d \;\middle|\; |y - \{nx\}| < r_n \quad \text{for i.m. } n \geq 1 \right\}.$$

We may rule out the case in which the series $\sum_n r_n^d$ diverges. Indeed, as observed at the end of Section 9.4, the eutaxy of the sequence $(\{nx\})_{n\geq 1}$ then implies that this set has full Lebesgue measure in $(0,1)^d$, so that its size and large intersection properties are trivially described.

As a consequence, we assume throughout that the series $\sum_n r_n^d$ converges. In particular, $(r_n)_{n\geq 1}$ converges to zero and, as the set $F_{\mathrm{r}}(x)$ is unchanged when removing a finite number of initial terms, there is no loss of generality in assuming that the real numbers r_n are in $(0,1]$. We may then define a real number s_{r} in the interval $[0,d]$ through the condition (109), namely,

$$\begin{cases} s < s_{\mathrm{r}} \implies \sum_n r_n^s = \infty \\ s > s_{\mathrm{r}} \implies \sum_n r_n^s < \infty. \end{cases}$$

If s_{r} is positive, the discussion that follows the statement of Theorem 6.2 implies that the set $F_{\mathrm{r}}(x)$ belongs to the class $\mathcal{G}^{s_{\mathrm{r}}}((0,1)^d)$ and actually has Hausdorff dimension equal to s_{r} in $(0,1)^d$. The ideas developed in Chapters 8 and 9 enable us to optimally refine this result without even requiring that s_{r} is positive. In particular, Section 9.4 suggests that we introduce the measure in $\mathcal{R}_d$ given by (188), that is,

$$\mathfrak{n}_{\mathrm{r}} = \sum_{n=1}^{\infty} \delta_{r_n},$$

and Theorem 9.3 therein leads straightforwardly to the next statement.

THEOREM 10.1. *For any point x in Bad_d and for any nonincreasing sequence $\mathrm{r} = (r_n)_{n\geq 1}$ in $(0,1]$ such that $\sum_n r_n^d$ is finite, $F_{\mathrm{r}}(x)$ is $\mathfrak{n}_{\mathrm{r}}$-describable in $(0,1)^d$.*

We may recast this result with the help of the distance to the nearest integer point defined by (121), thus considering instead of $F_{\mathrm{r}}(x)$ the companion set

$$F'_{\mathrm{r}}(x) = \left\{ y \in \mathbb{R}^d \;\middle|\; \|y - nx\| < r_n \quad \text{for i.m. } n \geq 1 \right\}.$$

The resulting statement bearing on this set is the following one. Note that the describability property is now valid on the whole space $\mathbb{R}^d$ instead of the mere open unit cube $(0,1)^d$; this is because the companion set $F'_{\mathrm{r}}(x)$ may basically be seen as the initial set $F_{\mathrm{r}}(x)$, along with its images under all translations by vectors in $\mathbb{Z}^d$.

COROLLARY 10.1. *For any point x in* Bad_d *and for any nonincreasing sequence* $\mathrm{r} = (r_n)_{n\geq 1}$ *in* $(0,1]$ *such that* $\sum_n r_n^d$ *is finite,* $F'_{\mathrm{r}}(x)$ *is* $\mathfrak{n}_{\mathrm{r}}$*-describable in* $\mathbb{R}^d$.

PROOF. Let us consider a gauge function g in $\mathfrak{G}(\mathfrak{n}_{\mathrm{r}})$, a d-normalized gauge function h satisfying $h \prec g_d$, and a nonempty dyadic cube λ in the collection Λ_h introduced in Section 8.2.1. We also assume that λ has diameter at most that of the unit cube $[0,1)^d$, which is equal to one because we work with the supremum norm when considering the distance to the nearest integer point. Thus, λ is included in a dyadic cube of the form $k+[0,1)^d$ for some integer point $k \in \mathbb{Z}^d$. Now, it is clear that the companion set $F'_{\mathrm{r}}(x)$ contains the image of the initial set $F_{\mathrm{r}}(x)$ under the translation by vector k. Also, note that (101) remain valid for such translations, along with the net measures associated with general gauge functions. Hence,

$$\mathcal{M}^h_\infty(F'_{\mathrm{r}}(x)\cap\lambda) \geq \mathcal{M}^h_\infty(k+(F_{\mathrm{r}}(x)\cap(-k+\lambda))) \geq 3^{-d}\mathcal{M}^h_\infty(F_{\mathrm{r}}(x)\cap(-k+\lambda)).$$

In addition, the interior of $-k+\lambda$ is contained in the open unit cube $(0,1)^d$, and Theorem 10.1 implies that $F_{\mathrm{r}}(x)$ satisfies a large intersection property with respect to g in the latter open cube. Hence,

$$\mathcal{M}^h_\infty(F_{\mathrm{r}}(x)\cap\operatorname{int}(-k+\lambda)) = \mathcal{M}^h_\infty(\operatorname{int}(-k+\lambda)) = \mathcal{M}^h_\infty(\lambda),$$

where the last equality is due to (175). We deduce that the set $F'_{\mathrm{r}}(x)$ belongs to the class $\mathcal{G}^g(\mathbb{R}^d)$ by making make use of Lemmas 10 and 12 in [**18**], namely, the natural extension of Lemmas 5.2 and 5.3 to general gauge functions. Therefore,

$$\mathfrak{m}(F'_{\mathrm{r}}(x),\mathbb{R}^d) \supseteq \mathfrak{G}(\mathfrak{n}_{\mathrm{r}}).$$

Conversely, we recall from the proof of Corollary 6.1 that the set $F'_{\mathrm{r}}(x)$ is invariant under the translations by vectors in $\mathbb{Z}^d$, and that

$$F'_{\mathrm{r}}(x)\cap[0,1)^d \subseteq \limsup_{n\to\infty} \bigcup_{p\in\{-1,0,1\}^d} \mathrm{B}_\infty(\{nx\}+p, r_n).$$

Therefore, in the same vein as (187), we deduce from Lemma 2.1 and Proposition 2.10 that the set $F'_{\mathrm{r}}(x)$ has Hausdorff g-measure zero for any gauge function g for which the series $\sum_n g_d(r_n)$ converges. This means that

$$\mathfrak{M}(F'_{\mathrm{r}}(x),\mathbb{R}^d) \supseteq \mathfrak{G}(\mathfrak{n}_{\mathrm{r}})^{\complement}.$$

To conclude, it suffices to apply Lemma 9.4. □

A simple example is obtained by assuming that the sequence r is defined by $r_n = n^{-\sigma}$ for all $n \geq 1$, and for a fixed $\sigma > 1/d$. Indeed, one then easily checks that the set $\mathfrak{G}(\mathfrak{n}_{\mathrm{r}})$ coincides with the set $\mathfrak{G}(\mathfrak{n}_{1/\sigma})$, where the measure $\mathfrak{n}_{1/\sigma}$ is defined as in (183). If the point x is badly approximable, we thus deduce from Corollary 10.1 that the set of all points $y \in \mathbb{R}^d$ such that

$$\|y-nx\| < \frac{1}{n^\sigma} \qquad \text{for i.m. } n \geq 1$$

is $\mathfrak{n}_{1/\sigma}$-describable in $\mathbb{R}^d$, thereby ending up with a major improvement on (125).

As a typical application, we may describe the size and large intersection properties of the intersection of countably many sets of the form $F'_{\mathrm{r}}(x)$. Specifically, for each integer $m \geq 1$, let us consider a badly approximable point x_m and a nonincreasing sequence $\mathrm{r}_m = (r_{m,n})_{n\geq 1}$ in $(0,1]$ such that $\sum_n r^d_{m,n}$ is finite. This enables us to define the intersection, denoted by F' for simplicity, of all the sets

$F'_{\mathrm{r}_m}(x_m)$, for $m \geq 1$. Then, similarly to (186), we may combine Corollary 10.1 with Propositions 9.2 and 9.3 to infer that

$$\mathfrak{m}(F', \mathbb{R}^d) \cap \mathfrak{G}^\infty = \bigcap_{m=1}^{\infty} \mathfrak{G}(\mathfrak{n}_{\mathrm{r}_m}) \qquad \text{and} \qquad \mathfrak{M}(F', \mathbb{R}^d) \supseteq \mathfrak{G}^\infty \setminus \bigcap_{m=1}^{\infty} \mathfrak{G}(\mathfrak{n}_{\mathrm{r}_m}).$$

In particular, the set F' is fully describable in $\mathbb{R}^d$. Further assumptions on the sequences r_m can then enable us to make the intersection of the sets $\mathfrak{G}(\mathfrak{n}_{\mathrm{r}_m})$ more explicit, and then get more comprehensive results. For instance, if the measures $\mathfrak{n}_{\mathrm{r}_m}$ are all of the form (183), then Proposition 9.6 implies that F' is either $\mathfrak{n}_s$-describable for some $s \in [0, d)$, or $\mathfrak{s}$-describable for some $\mathfrak{s} \in [0, d)$. In the particular case where $r_{m,n} = n^{-\sigma_m}$ for all $n \geq 1$ and some $\sigma_m > 1/d$, we established above that each set $F'_{\mathrm{r}_m}(x_m)$ is $\mathfrak{n}_{1/\sigma_m}$-describable in $\mathbb{R}^d$. According to Proposition 9.6, we conclude that the set F' is either $\mathfrak{n}_{1/\sigma_*}$-describable or $(1/\sigma_*)$-describable, depending respectively on whether or not the supremum, denoted by σ_*, of all parameters σ_m is attained.

10.1.2. Sequences with very fast growth. This section is a sequel to Section 6.4.2. Since it is parallel to the previous one, some details will be omitted from the presentation below. We consider throughout a sequence $(a_n)_{n\geq 1}$ of positive real numbers such that

$$\sum_{n=1}^{\infty} \frac{a_n}{a_{n+1}} < \infty,$$

which is the case for instance when the sequence grows superexponentially fast. We recall from Theorem 6.12 that for Lebesgue-almost every x in $\mathbb{R}$, the sequence $(\{a_n x\})_{n\geq 1}$ is uniformly eutaxic in $(0, 1)$.

Given a nonincreasing sequence $\mathrm{r} = (r_n)_{n\geq 1}$ of positive real numbers, the set initially studied in Section 10.1.1 now becomes

$$F_{\mathrm{r}}(x) = \left\{ y \in \mathbb{R} \;\middle|\; |y - \{a_n x\}| < r_n \quad \text{for i.m. } n \geq 1 \right\}.$$

As above, our purpose is to describe the size and large intersection properties of this set, and we may again assume throughout that the series $\sum_n r_n$ converges and that the real numbers r_n are all in $(0, 1]$. We then introduce the measure in $\mathcal{R}_d$ given by (188), and readily deduce the next statement from Theorem 9.3.

Theorem 10.2. *For Lebesgue-almost every real number x and for any nonincreasing sequence $\mathrm{r} = (r_n)_{n\geq 1}$ in $(0, 1]$ such that $\sum_n r_n$ is finite, the set $F_{\mathrm{r}}(x)$ is $\mathfrak{n}_{\mathrm{r}}$-describable in $(0, 1)$.*

Note that this result is analogous to Theorem 10.1. We now rephrase it by means of the distance to the nearest integer point defined by (121), thereby dealing with the companion set

$$F'_{\mathrm{r}}(x) = \left\{ y \in \mathbb{R} \;\middle|\; \|y - a_n x\| < r_n \quad \text{for i.m. } n \geq 1 \right\}.$$

The statement bearing on this set is the following analog of Corollary 10.1.

Corollary 10.2. *For Lebesgue-almost every real number x and for any nonincreasing sequence $\mathrm{r} = (r_n)_{n\geq 1}$ in $(0, 1]$ such that $\sum_n r_n$ is finite, the set $F'_{\mathrm{r}}(x)$ is $\mathfrak{n}_{\mathrm{r}}$-describable in $\mathbb{R}$.*

The above corollary may be deduced from Theorem 10.2 by simply adapting the arguments employed to deduce Corollary 10.1 from Theorem 10.1. The proof is thus a simple modification of that of Corollary 10.1, and is left to the reader. Finally, note that Corollary 10.2 is a substantial improvement on Corollary 6.2, which only addressed dimensional properties. Moreover, in the particular case where $r_n = n^{-\sigma}$ for all $n \geq 1$ and some fixed $\sigma > 1$, we deduce from Corollary 10.1

that for Lebesgue-almost every real number x and for every $\sigma > 1$, the set of all points $y \in \mathbb{R}$ such that

$$\|y - a_n x\| < \frac{1}{n^\sigma} \qquad \text{for i.m. } n \geq 1$$

is $\mathfrak{n}_{1/\sigma}$-describable in $\mathbb{R}^d$. We may then consider countable intersections of such sets, in the same vein as at the end of Section 10.1.1.

10.2. Homogeneous and inhomogeneous approximation

10.2.1. General describability statement. This section is a continuation of Sections 7.3.1–7.3.3. Let us recall that the inhomogeneous Diophantine approximation problem consists in approximating the points in $\mathbb{R}^d$ by the points that belong to the collection

$$\mathbb{Q}^{d,\alpha} = \left\{ \frac{p+\alpha}{q},\ (p,q) \in \mathbb{Z}^d \times \mathbb{N} \right\},$$

where α is a point that is fixed in advance in $\mathbb{R}^d$. When α vanishes, $\mathbb{Q}^{d,\alpha}$ is obviously equal to the set $\mathbb{Q}^d$ of points with rational coordinates, and the problem reduces to the homogeneous one. The collection $\mathbb{Q}^{d,\alpha}$ is endowed with the height function

$$H^\alpha_d(a) = \inf\{q \in \mathbb{N} \mid qa - \alpha \in \mathbb{Z}^d\}^{1+1/d}.$$

Then, we know from Theorem 7.3 that the pair $(\mathbb{Q}^{d,\alpha}, H^\alpha_d)$ is an optimal regular system in $\mathbb{R}^d$. The material developed in Section 9.5 will then enable us to complete the description of the size and large intersection properties of the set $\mathfrak{Q}^\alpha_{d,\psi}$ that was initiated by Theorem 7.4. Let us recall that this set is defined by (165), namely,

$$\mathfrak{Q}^\alpha_{d,\psi} = \left\{ x \in \mathbb{R}^d \;\middle|\; \left| x - \frac{p+\alpha}{q} \right|_\infty < \psi(q) \quad \text{for i.m. } (p,q) \in \mathbb{Z}^d \times \mathbb{N} \right\},$$

where ψ is a positive nonincreasing continuous function defined on $[0,\infty)$. Moreover, Theorem 7.4(1) shows that $\mathfrak{Q}^\alpha_{d,\psi}$ has full Lebesgue measure in $\mathbb{R}^d$ when

$$\mathrm{I}_{d,\psi} = \int_0^\infty q^d \psi(q)^d \,\mathrm{d}q$$

is a divergent integral. As explained at the beginning of Chapter 9, the description of the size and large intersection properties of the set $\mathfrak{Q}^\alpha_{d,\psi}$ is then elementary. We therefore exclude this situation and assume throughout that $\mathrm{I}_{d,\psi}$ is convergent. As the function ψ is nonincreasing, it then necessarily tends to zero at infinity. The set $\mathfrak{Q}^\alpha_{d,\psi}$ is clearly left unchanged if we remove a finite number of possible values for q, so there is no loss in generality in assuming that ψ is valued in $(0,1]$.

Furthermore, we learn from the proof of Theorem 7.4 that the set $\mathfrak{Q}^\alpha_{d,\psi}$ coincides with the set defined by (148), namely,

$$F_\varphi = \left\{ x \in \mathbb{R}^d \;\middle|\; |x-a| < \varphi(H(a)) \quad \text{for i.m. } a \in \mathcal{A} \right\},$$

where φ is the function given by $\varphi(\eta) = \psi(\eta^{d/(d+1)})$ for all $\eta \geq 0$, and $(\mathcal{A}, H)$ is equal to the optimal regular system $(\mathbb{Q}^{d,\alpha}, H^\alpha_d)$. The convergence of $\mathrm{I}_{d,\psi}$ is then equivalent to that of the integral I_φ defined by (152) converges. The approach developed in Section 9.5 then invites us to consider the measure $\mathfrak{n}_\varphi$ defined in $\mathcal{R}_d$ by (190). However, as we want to express our results in terms of ψ rather than φ, we preferably introduce another measure $\mathfrak{n}_{d,\psi}$ in $\mathcal{R}_d$, defined through the condition

$$\int_{(0,1]} f(r)\,\mathfrak{n}_{d,\psi}(\mathrm{d}r) = \int_0^\infty q^d f(\psi(q))\,\mathrm{d}q$$

for any nonnegative Borel measurable function f defined on $(0,1]$. This yields an equivalent formulation because the sets $\mathfrak{G}(\mathfrak{n}_{d,\psi})$ and $\mathfrak{G}(\mathfrak{n}_\varphi)$ coincide. Applying Theorem 9.4, we end up with the next substantial improvement on Theorem 7.4.

THEOREM 10.3. *Let α be a point in $\mathbb{R}^d$ and let ψ denote a positive nonincreasing continuous function defined on $[0,\infty)$, valued in $(0,1]$ and such that the integral $\mathrm{I}_{d,\psi}$ converges. Then, the set $\mathfrak{Q}^\alpha_{d,\psi}$ is $\mathfrak{n}_{d,\psi}$-describable in $\mathbb{R}^d$.*

In the spirit of the end of Section 10.1.1, a possible application is then to consider a sequence $(\alpha_n)_{n\geq 1}$ of points in $\mathbb{R}^d$, and to use Theorem 10.3 in conjunction with the appropriate formulation of (186) in order to describe the size and large intersection properties of the intersection over all $n \geq 1$ of the sets $\mathfrak{Q}^{\alpha_n}_{d,\psi}$. The same ideas may be put into practice by considering a sequence $(\psi_n)_{n\geq 1}$ of approximating functions and analyzing the intersection over all $n \geq 1$ of the sets $\mathfrak{Q}^\alpha_{d,\psi_n}$. It is even possible to mix these two approaches by considering the intersection of a doubly indexed sequence of sets of the form $\mathfrak{Q}^{\alpha_m}_{d,\psi_n}$.

In view of Theorem 9.1, we may readily deduce from Theorem 10.3 a complete description of the size and large intersection properties of the set $\mathfrak{Q}^\alpha_{d,\psi}$. In particular, we infer that for any gauge function g and any nonempty open set $V \subseteq \mathbb{R}^d$,

$$\mathcal{H}^g(\mathfrak{Q}^\alpha_{d,\psi} \cap V) = \begin{cases} \infty & \text{if } \sum_q q^d g_d(\psi(q)) = \infty \\ 0 & \text{if } \sum_q q^d g_d(\psi(q)) < \infty. \end{cases}$$

Note that we also used the elementary fact that a gauge function g belongs to the set $\mathfrak{G}(\mathfrak{n}_{d,\psi})$ if and only if its d-normalization g_d is such that the above series diverges; this follows from the monotonicity of ψ and that of g_d near the origin. We thus recover the extension established by Bugeaud [**12**] of a classical statement due to Jarník [**37**]. Likewise, Theorems 9.1 and 10.3 allow us to recover the description of the large intersection properties of the set $\mathfrak{Q}^\alpha_{d,\psi}$ that was obtained in [**18**].

10.2.2. The inhomogeneous Jarník-Besicovitch theorem revisited. As in Section 7.3.4, let us focus on the particular case where the function ψ is of the form $q \mapsto q^{-\tau}$ on the interval $[1,\infty)$, for some positive real number τ. Then, $\mathfrak{Q}^\alpha_{d,\psi}$ reduces to the set defined by (31), namely,

$$J^\alpha_{d,\tau} = \left\{ x \in \mathbb{R}^d \;\middle|\; \left| x - \frac{p+\alpha}{q} \right|_\infty < \frac{1}{q^\tau} \quad \text{for i.m. } (p,q) \in \mathbb{Z}^d \times \mathbb{N} \right\}.$$

When α vanishes, the above set reduces to the introductory set $J_{d,\tau}$ defined by (1) and corresponding to the homogeneous setting. We complete the definition of the function ψ by assuming that it is constant equal to one on the interval $[0,1]$. One easily checks that

$$\begin{cases} \mathrm{I}_{d,\psi} < \infty \quad \Longleftrightarrow \quad \tau > 1 + 1/d \\ \mathfrak{G}(\mathfrak{n}_{d,\psi}) = \mathfrak{G}(\mathfrak{n}_{(d+1)/\tau}), \end{cases}$$

where the measure $\mathfrak{n}_{(d+1)/\tau}$ is defined as in (183). Theorem 10.3 then leads to the next major improvement on Corollary 7.1.

COROLLARY 10.3. *For any point $\alpha \in \mathbb{R}^d$ and any real parameter $\tau > 1 + 1/d$, the set $J^\alpha_{d,\tau}$ is $\mathfrak{n}_{(d+1)/\tau}$-describable in $\mathbb{R}^d$.*

Going back to the application mentioned at the end of Section 7.3.4, let us consider a sequence $(\alpha_n)_{n\geq 1}$ of points in $\mathbb{R}^d$, and a sequence $(\tau_n)_{n\geq 1}$ of real numbers with supremum denoted by τ_*. Under the assumption that τ_* is finite, we proved in Section 7.3.4 that

$$\dim_{\mathrm{H}} \bigcap_{n=1}^{\infty} J_{d,\tau_n}^{\alpha_n} = \min\left\{\frac{d+1}{\tau_*}, d\right\}.$$

This was a consequence of the large intersection property satisfied by the sets $J_{d,\tau_n}^{\alpha_n}$ that was expressed by Corollary 7.1. We now derive a full description of the size and large intersection properties of the intersection of the sets $J_{d,\tau_n}^{\alpha_n}$. Our analysis also covers the case in which τ_* is infinite that was left open at the end of Section 7.3.4. Note that we rule out, as trivial, the case where τ_* is bounded above by $1 + 1/d$, because the intersection of the sets $J_{d,\tau_n}^{\alpha_n}$ then has full Lebesgue measure in $\mathbb{R}^d$, as a consequence of Corollary 7.1.

Corollary 10.4. *Given a sequence $(\alpha_n)_{n\geq 1}$ of points in $\mathbb{R}^d$ and a sequence $(\tau_n)_{n\geq 1}$ of real numbers, let us consider*

$$J_{d,*} = \bigcap_{n=1}^{\infty} J_{d,\tau_n}^{\alpha_n} \qquad \text{and} \qquad \tau_* = \sup_{n\geq 1} \tau_n > 1 + 1/d.$$

Then, the set $J_{d,*}$ is either $\mathfrak{n}_{(d+1)/\tau_*}$-describable or $((d+1)/\tau_*)$-describable in $\mathbb{R}^d$ depending on whether or not the supremum τ_* is attained, respectively.

Proof. Let $\mathcal{N}$ be the set of all integers $n \geq 1$ such that $\tau_n > 1 + 1/d$. Our assumption on τ_* implies that $\mathcal{N}$ is nonempty. Moreover, τ_* is also the supremum of τ_n over $n \in \mathcal{N}$. Now, Proposition 9.2 yields on the one hand

$$\mathfrak{M}(J_{d,*}, \mathbb{R}^d) \supseteq \mathfrak{M}\left(\bigcap_{n\in\mathcal{N}} J_{d,\tau_n}^{\alpha_n}, \mathbb{R}^d\right).$$

On the other hand, let us consider a gauge function g that is minorizing in $\mathbb{R}^d$ for the intersection over $n \in \mathcal{N}$ of the sets $J_{d,\tau_n}^{\alpha_n}$. Due to Corollary 7.1, the intersection over $n \in \mathbb{N} \setminus \mathcal{N}$ of these sets has full Lebesgue measure in $\mathbb{R}^d$. By Propositions 8.2 and 8.3(2), any gauge function is minorizing in $\mathbb{R}^d$ for this set, and so is g in particular. This shows with Theorem 8.2(1) that g is minorizing for $J_{d,*}$. Hence,

$$\mathfrak{m}(J_{d,*}, \mathbb{R}^d) \supseteq \mathfrak{m}\left(\bigcap_{n\in\mathcal{N}} J_{d,\tau_n}^{\alpha_n}, \mathbb{R}^d\right).$$

Proposition 9.6 and Corollary 10.3 enable us to appropriately express the right-hand side of either of the two above inclusions in terms of either $\mathfrak{G}(\mathfrak{n}_{(d+1)/\tau_*})$ or $\mathfrak{G}((d+1)/\tau_*)$, depending on whether or not the supremum τ_* is attained, respectively. To conclude, it suffices to invoke Lemma 9.4, along with Lemma 9.1(2) in the first case, and Lemma 9.3(2) in the second. □

Where τ_* is infinite, we deduce from Corollary 10.4 that the intersection of the sets $J_{d,\tau_n}^{\alpha_n}$ is 0-describable in $\mathbb{R}^d$. By Corollary 9.3, its Hausdorff dimension is thus equal to zero, as announced without proof at the end of Section 7.3.4.

10.2.3. Inhomogeneous Liouville points. Note that the mapping $\tau \mapsto J_{d,\tau}^{\alpha}$ is decreasing. In the spirit of the end of Section 9.3.3, this prompts us to introduce

$$L_d^{\alpha} = \bigcap_{\tau > 1+1/d} \downarrow J_{d,\tau}^{\alpha}.$$

The monotonicity property satisfied by the sets $J_{d,\tau}^{\alpha}$ shows that L_d^{α} coincides for instance with the intersection over all $n \geq 1$ of the sets $J_{d,n}^{\alpha}$. Moreover, each of these sets is $\mathfrak{n}_{(d+1)/n}$-describable in $\mathbb{R}^d$, as a consequence of Corollary 10.3. We are in the setting of Proposition 9.6, with the infimum being equal to zero and not being attained. This yields the next statement.

COROLLARY 10.5. *For any point $\alpha \in \mathbb{R}^d$, the set L_d^α is 0-describable in $\mathbb{R}^d$.*

The complete description of the size and large intersection properties of the set L_d^α then follows from Theorem 9.2. Moreover, we deduce from Corollary 9.3 that this set has Hausdorff dimension equal to zero and packing dimension equal to d in every nonempty open subset of $\mathbb{R}^d$.

Let us now establish a connection between the set L_d^α and a natural extension to the inhomogeneous and multidimensional setting of the notion of Liouville number.

DEFINITION 10.1. Let α be a point in $\mathbb{R}^d$. A point x in $\mathbb{R}^d$ is called *α-Liouville* if x does not belong to $\mathbb{Q}^{d,\alpha}$ and if for any integer $n \geq 1$, there exists an integer $q \geq 1$ and a point $p \in \mathbb{Z}^d$ such that

$$\left| x - \frac{p+\alpha}{q} \right|_\infty < \frac{1}{q^n}.$$

For $\alpha = 0$ and $d = 1$, we obviously recover the condition that defines Liouville numbers. Excluding the points in $\mathbb{Q}^{d,\alpha}$ from this definition is analogous to excluding the irrationals from the classical definition of Liouville numbers. In fact, this ensures that for each integer $n \geq 1$, there are infinitely many pairs (p, q) such that the above inequality holds. As a consequence, the set of α-Liouville points in $\mathbb{R}^d$ is equal to the set $L_d^\alpha \setminus \mathbb{Q}^{d,\alpha}$. As shown by the next statement, removing the points in $\mathbb{Q}^{d,\alpha}$ does not alter the describability properties of the set L_d^α.

COROLLARY 10.6. *For any point α in $\mathbb{R}^d$, the set $L_d^\alpha \setminus \mathbb{Q}^{d,\alpha}$ of all α-Liouville points in $\mathbb{R}^d$ is 0-describable in $\mathbb{R}^d$.*

PROOF. The set $\mathbb{R}^d \setminus \mathbb{Q}^{d,\alpha}$ is clearly a Lebesgue-full G_δ-subset of $\mathbb{R}^d$. Owing to Propositions 8.2 and 8.3(2), it thus belongs to the class $\mathcal{G}^{\mathbf{0}}(\mathbb{R}^d)$, and in fact to all the classes $\mathcal{G}^g(\mathbb{R}^d)$, for g in $\mathfrak{G}$. In conjunction with Proposition 9.2 and Corollary 10.5, this implies that

$$\mathfrak{m}(L_d^\alpha \setminus \mathbb{Q}^{d,\alpha}, \mathbb{R}^d) \cap \mathfrak{G}^\infty = \mathfrak{m}(L_d^\alpha, \mathbb{R}^d) \cap \mathfrak{G}^\infty = \mathfrak{G}(0).$$

In addition, the same results straightforwardly show that

$$\mathfrak{M}(L_d^\alpha \setminus \mathbb{Q}^{d,\alpha}, \mathbb{R}^d) \supseteq \mathfrak{M}(L_d^\alpha, \mathbb{R}^d) = \mathfrak{G}(0)^{\complement}.$$

We conclude with the help of Lemmas 9.3(2) and 9.4. □

Let us mention a noteworthy consequence of Corollary 10.6. Let us consider an arbitrary gauge function g in $\mathfrak{G}(0)$. Then, Theorem 9.2 shows that the set of all α-Liouville points in $\mathbb{R}^d$, namely, $L_d^\alpha \setminus \mathbb{Q}^{d,\alpha}$ belongs to the large intersection class $\mathcal{G}^g(\mathbb{R}^d)$. Now, for any given point x in $\mathbb{R}^d$, the mapping $y \mapsto x - y$ is obviously bi-Lipschitz. We deduce from Theorem 8.2(1–2) that the set

$$(L_d^\alpha \setminus \mathbb{Q}^{d,\alpha}) \cap (x - (L_d^\alpha \setminus \mathbb{Q}^{d,\alpha}))$$

also belongs to the class $\mathcal{G}^g(\mathbb{R}^d)$. As a result, there are uncountably many ways of writing a give point x in $\mathbb{R}^d$ as the sum of two α-Liouville points. This substantially improves on a result by Erdős [**25**] according to which any real number may be written as a sum of two Liouville numbers. Of course, many variations are possible as one may freely replace $y \mapsto x - y$ above by any bi-Lipschitz mapping, or even a countable number thereof.

Finally, let us also point out that the set of Liouville numbers, *i.e.* the set L_d^0 in the above notations, also comes into play in the theory of dynamical systems, especially in the study of the homeomorphisms of the circle, see [**19**] for details.

10.3. Approximation by algebraic numbers

10.3.1. General describability statement. The purpose of this section is to continue the analysis initiated in Section 7.4. Let us recall that the collection of all real algebraic numbers is denoted by $\mathbb{A}$, the naïve height of a number a in $\mathbb{A}$ is denoted by $\mathrm{H}(a)$, and the set of all real algebraic numbers with degree at most n is denoted by $\mathbb{A}_n$. Moreover, a result due to Beresnevich shows that for any $n \geq 1$, the pair $(\mathbb{A}_n, H_n)$ is an optimal regular system in $\mathbb{R}$, where the appropriate height function is given by (167), that is,

$$H_n(a) = \frac{\mathrm{H}(a)^{n+1}}{(1+|a|)^{n(n+1)}},$$

see Theorem 7.5. This result already enabled us to describe the elementary size and large intersection properties of the set defined by (168), specifically,

$$\mathfrak{A}_{n,\psi} = \left\{ x \in \mathbb{R} \;\middle|\; |x-a| < \psi(\mathrm{H}(a)) \quad \text{for i.m. } a \in \mathbb{A}_n \right\},$$

where ψ is any positive nonincreasing continuous function defined on $[0,\infty)$, see Theorem 7.6 for a precise statement. In particular, we recovered a result due to Beresnevich [**2**], according to which the set $\mathfrak{A}_{n,\psi}$ has full Lebesgue measure in $\mathbb{R}$ when the integral

$$\mathrm{I}_{n,\psi} = \int_0^\infty h^n \psi(h)\,\mathrm{d}h$$

is divergent. The situation is now parallel to studied in Section 10.2. To be specific, the description of the size and large intersection properties of the set $\mathfrak{A}_{n,\psi}$ is trivial when $\mathrm{I}_{n,\psi}$ diverges, and so we rule out this situation. Assuming that $\mathrm{I}_{n,\psi}$ is convergent, we infer that ψ tends to zero at infinity. Finally, as the set $\mathfrak{A}_{n,\psi}$ is unchanged when assuming that the height of the approximating algebraic numbers exceeds a fixed threshold, we are further allowed to restrict our attention to the case in which ψ is valued in $(0,1]$.

Accordingly, we assume from now on that the integral $\mathrm{I}_{n,\psi}$ is convergent and that the function ψ is valued in the interval $(0,1]$. The proof of Theorem 7.6 informs us that $\mathfrak{A}_{n,\psi}$ is very close to sets of the form (148), namely,

$$F_\varphi = \left\{ x \in \mathbb{R}^d \;\middle|\; |x-a| < \varphi(H(a)) \quad \text{for i.m. } a \in \mathcal{A} \right\},$$

when the underlying optimal regular system $(\mathcal{A}, H)$ is equal to $(\mathbb{A}_n, H_n)$ and the function φ is well chosen. In fact, (169) shows that

$$\bigcap_{k=1}^{\infty} \downarrow F_{\varphi_k} \subseteq \mathfrak{A}_{n,\psi} \subseteq F_{\varphi_1},$$

where $\varphi_k(\eta) = \psi(k\,\eta^{1/(n+1)})$ for any real number $\eta \geq 0$ and any integer $k \geq 1$. We deduce from Propositions 9.2 and 9.3 that

$$\mathfrak{M}(\mathfrak{A}_{n,\psi}, \mathbb{R}^d) \supseteq \mathfrak{M}(F_{\varphi_1}, \mathbb{R}^d) \qquad \text{and} \qquad \mathfrak{m}(\mathfrak{A}_{n,\psi}, \mathbb{R}^d) \supseteq \bigcap_{k=1}^{\infty} \mathfrak{m}(F_{\varphi_k}, \mathbb{R}^d). \tag{191}$$

Our intent is now to apply Theorem 9.4 to all the sets F_{φ_k}, so as to obtain a simple expression for the majorizing and minorizing collections that come into play here. We first need to mention that the integrals I_{φ_k} defined as in (152), namely,

$$I_{\varphi_k} = \int_0^\infty \varphi_k(\eta)\,\mathrm{d}\eta$$

are all convergent; in view of (170), this property is indeed equivalent to the convergence of the integral $\mathrm{I}_{n,\psi}$. Moreover, we are enticed to introduce in $\mathcal{R}_1$ the measures $\mathfrak{n}_{\varphi_k}$ defined as in (190). However, similarly to Section 10.2, our results will be stated in terms of ψ, so we rather introduce the measure $\mathfrak{n}_{n,\psi}$ satisfying

$$\int_{(0,1]} f(r)\, \mathfrak{n}_{n,\psi}(\mathrm{d}r) = \int_0^\infty h^n f(\psi(h))\, \mathrm{d}h$$

for any nonnegative Borel measurable function f defined on $(0,1]$. A change of variable as in (170) shows that all the sets $\mathfrak{G}(\mathfrak{n}_{\varphi_k})$ coincide with $\mathfrak{G}(\mathfrak{n}_{n,\psi})$. Applying Theorem 9.4, we deduce that for any $k \geq 1$, the set F_{φ_k} is $\mathfrak{n}_{n,\psi}$-describable in $\mathbb{R}^d$. Then, making use of (191), we end up with

$$\mathfrak{M}(\mathfrak{A}_{n,\psi}, \mathbb{R}^d) \supseteq \mathfrak{G}(\mathfrak{n}_{n,\psi})^{\complement} \qquad \text{and} \qquad \mathfrak{m}(\mathfrak{A}_{n,\psi}, \mathbb{R}^d) \supseteq \mathfrak{G}(\mathfrak{n}_{n,\psi}),$$

and it suffices to invoke Lemmas 9.3(2) and 9.4 to secure the following major improvement on Theorem 7.6.

THEOREM 10.4. *Let n be a positive integer and let ψ denote a positive nonincreasing continuous function defined on $[0,\infty)$, valued in $(0,1]$ and such that the integral $\mathrm{I}_{n,\psi}$ converges. Then, the set $\mathfrak{A}_{n,\psi}$ is $\mathfrak{n}_{n,\psi}$-describable in $\mathbb{R}$.*

Combined with Theorem 9.1, the previous result yields a complete description of the size and large intersection properties of the set $\mathfrak{A}_{n,\psi}$. In particular, we recover the characterization of the Hausdorff g-measure of the set $\mathfrak{A}_{n,\psi}$, for any gauge function g, that was obtained independently by Beresnevich, Dickinson and Velani [**4**], and by Bugeaud [**10**]. To be specific, for any gauge function g and any nonempty open set $V \subseteq \mathbb{R}$,

$$\mathcal{H}^g(\mathfrak{A}_{n,\psi} \cap V) = \begin{cases} \infty & \text{if } \sum_h h^n g_1(\psi(h)) = \infty \\ 0 & \text{if } \sum_h h^n g_1(\psi(h)) < \infty. \end{cases}$$

We also used here the obvious fact that $g \in \mathfrak{G}(\mathfrak{n}_{n,\psi})$ if and only if the above series diverges, owing to the monotonicity of ψ and that of g_1 near the origin. Similarly, we recover in addition the complete description of the large intersection properties of the set $\mathfrak{A}_{n,\psi}$ that was obtained in [**18**].

10.3.2. Koksma's classification of real transcendental numbers. Let us now concentrate on the case in which the function ψ is of the form $h \mapsto h^{-\omega-1}$ on the interval $[1,\infty)$, for some real number $\omega > -1$. In order to stress on the rôle of ω and ensure some coherence with Koksma's notations, the set $\mathfrak{A}_{n,\psi}$ is denoted by $K^*_{n,\omega}$ in what follows, namely,

$$K^*_{n,\omega} = \left\{ x \in \mathbb{R} \;\middle|\; |x-a| < \mathrm{H}(a)^{-\omega-1} \quad \text{for i.m. } a \in \mathbb{A}_n \right\}.$$

Furthermore, to complete the definition of ψ, we suppose that it is constant equal to one on the interval $[0,1]$. We then clearly have

$$\begin{cases} \mathrm{I}_{n,\psi} < \infty \quad \Longleftrightarrow \quad \omega > n \\ \mathfrak{G}(\mathfrak{n}_{n,\psi}) = \mathfrak{G}(\mathfrak{n}_{(n+1)/(\omega+1)}), \end{cases}$$

where the measure $\mathfrak{n}_{(n+1)/(\omega+1)}$ is again defined as in (183). We then readily deduce the next statement from Theorems 7.6(1) and 10.4.

COROLLARY 10.7. *For any integer $n \geq 1$ and any real parameter $\omega > -1$, the following properties hold:*

(1) *if $\omega \leq n$, then the set $K^*_{n,\omega}$ has full Lebesgue measure in $\mathbb{R}$;*
(2) *if $\omega > n$, then the set $K^*_{n,\omega}$ is $\mathfrak{n}_{(n+1)/(\omega+1)}$-describable in $\mathbb{R}$.*

This result will be used to comment on a classification of real transcendental numbers that is due to Koksma [**41**] and that we now present. First, it is clear that the mapping $\omega \mapsto K^*_{n,\omega}$ is nonincreasing; for every real number x, we thus naturally introduce the exponent

$$\omega^*_n(x) = \sup\{\omega > -1 \mid x \in K^*_{n,\omega}\}.$$

Note that when $n = 1$ and x is irrational, one essentially recovers the irrationality exponent defined by (2). Indeed, as observed in Section 7.4, the set $\mathbb{A}_1$ coincides with $\mathbb{Q}$, and writing an element $a \in \mathbb{A}_1$ in the form p/q for two coprime integers p and q, the latter being positive, we have $\mathrm{H}(a) = \max\{|p|, q\}$. It is then easy to check that for all $\omega > 0$,

$$K^*_{1,\omega} \subseteq J_{1,\omega+1} \setminus \mathbb{Q} \subseteq \bigcap_{\varepsilon>0} \downarrow K^*_{1,\omega-\varepsilon},$$

and therefore that for any irrational number x,

$$\omega^*_1(x) = \tau(x) - 1.$$

Koksma introduced a classification of the real transcendental numbers x which is based on the way the exponents $\omega^*_n(x)$ evolve as n grows. This amounts to studying how the quality with which a real number x is approximated by algebraic numbers behaves when their degree is allowed to increase. Specifically, let us define

$$\omega^*(x) = \limsup_{n\to\infty} \frac{\omega^*_n(x)}{n}.$$

Koksma classifies the real transcendental numbers x according to whether or not $\omega^*(x)$ is finite, see [**13**, Section 3.3]. In the first situation, that is, if $\omega^*(x)$ is finite, he calls x an *S^*-number.* Besides, let us mention that a result due to Wirsing [**62**] shows that a real number x is transcendental if and only if $\omega^*(x)$ is positive, see [**13**].

As we now explain, Corollary 10.7 entails that Lebesgue-almost every real number x is an S^*-number satisfying $\omega^*_n(x) = n$ for every $n \geq 1$. In fact, for any real parameter $\omega > 0$, let $\widehat{K}^*_{n,\omega}$ denote the set of all real numbers x for which the exponent $\omega^*_n(x)$ is bounded below by $(n+1)\omega - 1$. Observing that

$$\widehat{K}^*_{n,\omega} = \bigcap_{\omega'<(n+1)\omega-1} \downarrow K^*_{n,\omega'},$$

we deduce from Corollary 10.7 that the set $\widehat{K}^*_{n,\omega}$ has full Lebesgue measure in $\mathbb{R}$ when $\omega \leq 1$, and Lebesgue measure zero otherwise.

Our aim is now to describe the size and large intersection properties of the set $\widehat{K}^*_{n,\omega}$. As usual, we may exclude the trivial case in which this set has full Lebesgue measure, and therefore suppose that $\omega > 1$. Due to the monotonicity of the mapping $\omega' \mapsto K^*_{n,\omega'}$, we may assume in the above intersection that ω' ranges over a sequence of real numbers strictly between n and $(n+1)\omega - 1$ that monotonically tends to the latter value. In view of Corollary 10.7, we fall into the setting of Proposition 9.6 in the case where the infimum is not attained. We end up with the next result.

COROLLARY 10.8. *For any integer $n \geq 1$ and any real parameter $\omega > 1$, the set $\widehat{K}^*_{n,\omega}$ is $(1/\omega)$-describable in $\mathbb{R}$.*

In order to make the connection with Koksma's classification, we need to consider all the integers n simultaneously. Accordingly, let us introduce the set

$$\widehat{K}^*_{\omega} = \bigcap_{n=1}^{\infty} \widehat{K}^*_{n,\omega}.$$

When $\omega \leq 1$, what precedes ensures that $\widehat{K}^*_\omega$ has full Lebesgue measure in $\mathbb{R}$, and its size and large intersection properties are trivially described. Let us assume oppositely that $\omega > 1$. Combining Corollary 10.8 with Propositions 9.2 and 9.3, we straightforwardly establish that

$$\mathfrak{M}(\widehat{K}^*_\omega, \mathbb{R}) \supseteq \mathfrak{G}(1/\omega)^{\complement} \qquad \text{and} \qquad \mathfrak{m}(\widehat{K}^*_\omega, \mathbb{R}) \supseteq \mathfrak{G}(1/\omega).$$

Applying Lemmas 9.3(2) and 9.4, we eventually get the following result.

COROLLARY 10.9. *For any $\omega > 1$, the set $\widehat{K}^*_\omega$ is $(1/\omega)$-describable in $\mathbb{R}$.*

Again, combining this result with Theorem 9.2, we obtain a complete and precise description of the size and large intersection properties of the set $\widehat{K}^*_\omega$, thereby recovering results previously established in [**14, 18**]. One may also use Corollary 9.3 if only dimensional results are desired. In particular, we observe that the set $\widehat{K}^*_\omega$ has Hausdorff dimension equal to $1/\omega$. We thus recover a seminal result established by Baker and Schmidt [**1**].

The connection with Koksma's classification now consists in making the obvious remark that for any real parameter $\omega > 0$, the set

$$\Omega^*_\omega = \{x \in \mathbb{R} \mid \omega^*(x) \geq \omega\}$$

contains $\widehat{K}^*_\omega$. In particular, we recover the fact that Ω^*_ω has full Lebesgue measure in $\mathbb{R}$ when $\omega \leq 1$. As regards size and large intersection properties, the opposite case is richer and is covered by the next result.

THEOREM 10.5. *For any real parameter $\omega > 1$, the set Ω^*_ω of all real numbers x such that $\omega^*(x) \geq \omega$ is $(1/\omega)$-describable in $\mathbb{R}$.*

PROOF. First, since the set Ω^*_ω contains $\widehat{K}^*_\omega$, we deduce from Proposition 9.2 and Corollary 10.9 that

$$\mathfrak{m}(\Omega^*_\omega, \mathbb{R}) \supseteq \mathfrak{m}(\widehat{K}^*_\omega, \mathbb{R}) \supseteq \mathfrak{G}(1/\omega).$$

Furthermore, let us consider a sequence $(\omega'_m)_{m\geq 1}$ of real numbers strictly between one and ω that monotonically tends to the latter value. We clearly have

$$\Omega^*_\omega \subseteq \bigcap_{m=1}^{\infty} \bigcup_{n=1}^{\infty} K^*_{n,(n+1)\omega'_m - 1}.$$

By virtue of Propositions 9.2 and 9.3, and also Corollary 10.7, this entails that

$$\mathfrak{M}(\Omega^*_\omega, \mathbb{R}) \supseteq \bigcup_{m=1}^{\infty} \bigcap_{n=1}^{\infty} \mathfrak{M}(K^*_{n,(n+1)\omega'_m - 1}, \mathbb{R}) = \bigcup_{m=1}^{\infty} \mathfrak{G}(\mathfrak{n}_{1/\omega'_m})^{\complement}.$$

Indeed, each set $K^*_{n,(n+1)\omega'_m - 1}$ is $\mathfrak{n}_{1/\omega'_m}$-describable in $\mathbb{R}$. We finally infer from Lemma 9.2 that the right-hand side is equal to $\mathfrak{G}(1/\omega)^{\complement}$, and we conclude thanks to Lemmas 9.3(2) and 9.4. □

It is possible to formally let ω tend to infinity in Theorem 10.5. This amounts to considering the intersection of the sets Ω^*_ω, in conjunction with the observation that the intersection of the sets $\mathfrak{G}(1/\omega)$ reduces to the set $\mathfrak{G}(0)$. Using the methods developed up to now, the reader should easily prove the next result.

COROLLARY 10.10. *The set Ω^*_∞ of all real numbers x such that $\omega^*(x) = \infty$ is 0-describable in $\mathbb{R}$.*

Note that, referring to Koksma's classification, the set Ω^*_∞ consists of the transcendental numbers x that are not S^*-numbers; they are call either T^*-*numbers* or U^*-*numbers*, depending respectively on whether $\omega^*_n(x)$ is finite for all $n \geq 1$, or infinite for from some n onwards. Let us finally mention that Koksma's classification is very close to that previously introduced by Mahler [**44**] and for which large intersection properties also come into play, see [**13, 18**] for details.

10.3.3. The case of algebraic integers. As explained at the end of Section 7.4, a result due to Bugeaud [**9**] shows that for any integer $n \geq 2$, the pair $(\mathbb{A}'_n, H_{n-1})$ is an optimal regular system in $\mathbb{R}$, where $\mathbb{A}'_n$ denotes the set of all real algebraic integers with degree at most n, and the height function H_{n-1} is defined as in (167), see Theorem 7.7. Combining this result with the above methods, we may obtain an analog of Theorem 10.4 for the set obtained when replacing $\mathbb{A}_n$ by $\mathbb{A}'_n$ in (168), namely,

$$\mathfrak{A}'_{n,\psi} = \left\{ x \in \mathbb{R} \;\middle|\; |x-a| < \psi(\mathrm{H}(a)) \quad \text{for i.m. } a \in \mathbb{A}'_n \right\}.$$

To be specific, we already observed in Section 7.4 that the set $\mathfrak{A}'_{n,\psi}$ has full Lebesgue measure in $\mathbb{R}$ when the integral $\mathrm{I}_{n-1,\psi}$ is divergent. This case being trivial as regards size and large intersection properties, we rather assume that $\mathrm{I}_{n-1,\psi}$ is convergent. Then, adapting the methods leading to Theorem 10.4, we end up with the fact that the set $\mathfrak{A}'_{n,\psi}$ is $\mathfrak{n}_{n-1,\psi}$-describable in $\mathbb{R}$.

CHAPTER 11

Applications to random coverings problems

Similarly to Chapter 10, the purpose of this chapter is to review some examples introduced before and, using the machinery of describable sets introduced in Chapter 9, to give a precise and complete description of the size and large intersection properties of the involved sets. We focus here on the examples from probability theory studied essentially in Section 6.5.

11.1. Uniform random coverings

This section is a sequel to Section 6.5.1. Throughout, U denotes a nonempty bounded open subset of $\mathbb{R}^d$ and $(X_n)_{n\geq 1}$ denotes a sequence of points that are independently and uniformly distributed in U. Let us recall from Theorem 6.13 that with probability one, the sequence $(X_n)_{n\geq 1}$ is uniformly eutaxic in U. Moreover, let us consider a nonincreasing sequence $\mathrm{r} = (r_n)_{n\geq 1}$ of positive real numbers. We wish to detail the size and large intersection properties of the random set

$$F_{\mathrm{r}} = \left\{ x \in \mathbb{R}^d \;\middle|\; |x - X_n| < r_n \quad \text{for i.m. } n \geq 1 \right\}.$$

Note that this set is equal to that obtained when choosing $t = 1$ in (137). As usual, we exclude the case in which the above set has full Lebesgue measure in U, because the size and large intersection properties are then trivial. As mentioned in Section 6.5.1, this case is obtained when the series $\sum_n r_n^d$ diverges, as a simple consequence of the Borel-Cantelli lemma and Tonelli's theorem.

We therefore suppose from now on that the series $\sum_n r_n^d$ converges. We see that the sequence $(r_n)_{n\geq 1}$ then converges to zero and that the set F_{r} is unaltered when removing a finite number of initial terms. Without loss of generality, we thus also assume from now on that the real numbers r_n all belong to $(0, 1]$. The material of Section 9.4 prompts us to consider the measure in $\mathcal{R}_d$ given by (188), namely,

$$\mathfrak{n}_{\mathrm{r}} = \sum_{n=1}^{\infty} \delta_{r_n}.$$

In the present situation, Theorem 9.3 turns into the following result.

THEOREM 11.1. *Almost surely, for any nonincreasing sequence* $\mathrm{r} = (r_n)_{n\geq 1}$ *in the interval* $(0, 1]$ *such that* $\sum_n r_n^d$ *converges, the set* F_{r} *is* $\mathfrak{n}_{\mathrm{r}}$*-describable in* U.

In combination with Theorem 9.1, the above result yields a precise and complete description of the size and large intersection properties of the random set F_{r} ; such a description was first obtained in [**21**]. Furthermore, as far as dimensional results are concerned, Corollary 9.2 is sufficient. By way of illustration, let us apply this result here. Note that the exponent associated with the measure $\mathfrak{n}_{\mathrm{r}}$ through (182) is nothing but the critical exponent s_{r} for the convergence of the series $\sum_n r_n^s$ that is defined in the interval $[0, d]$ through the condition (109), namely,

$$\begin{cases} s < s_{\mathrm{r}} & \Longrightarrow \quad \sum_n r_n^s = \infty \\ s > s_{\mathrm{r}} & \Longrightarrow \quad \sum_n r_n^s < \infty. \end{cases}$$

Corollary 9.2 now implies that almost surely, for any nonempty open set $V \subseteq U$,

$$\begin{cases} \dim_{\mathrm{H}}(F_{\mathrm{r}} \cap V) = s_{\mathrm{r}} \\ \dim_{\mathrm{P}}(F_{\mathrm{r}} \cap V) = d \\ F_{\mathrm{r}} \in \mathcal{G}^{s_{\mathrm{r}}}(V), \end{cases}$$

where the last two properties are valid under the additional assumption that s_{r} is positive. We thus recover a property briefly mentioned after Corollary 6.3.

11.2. Poisson random coverings

This section is a follow-up to Section 6.5.2. Given a measure $\nu \in \mathcal{R}$ and a nonempty open set $U \subseteq \mathbb{R}^d$, we consider on $U_+ = (0,1] \times U$ a Poisson point measure Π with intensity $\nu \otimes \mathcal{L}^d(\,\cdot \cap U)$, and furthermore the set

$$F_\nu = \left\{ y \in \mathbb{R}^d \;\middle|\; \int_{U_+} \mathbf{1}_{\{|y-x|<r\}}\, \Pi(\mathrm{d}r, \mathrm{d}x) = \infty \right\}.$$

Note that this set is equal to that obtained when letting $t = 1$ in (141). We use here the notation F_ν to stress the dependence on ν. Of course, there is no loss in generality in assuming that $t = 1$, because replacing r^t by r amounts to replacing ν by its pushforward under the mapping $r \mapsto r^t$, and our analysis will be valid for all measures ν. Our main result is the following full and precise description of the size and large intersection properties of the set F_ν. We recall from (181) that the measure ν belongs to $\mathcal{R}_d$ if and only if

$$\langle \nu, r \mapsto r^d \rangle = \int_{(0,1]} r^d \,\nu(\mathrm{d}r) < \infty.$$

THEOREM 11.2. *For any measure $\nu \in \mathcal{R}$ and a nonempty open set $U \subseteq \mathbb{R}^d$, the following properties hold:*

- *if $\nu \notin \mathcal{R}_d$, then the set F_ν almost surely has full Lebesgue measure in U;*
- *if $\nu \in \mathcal{R}_d$, then the set F_ν is almost surely ν-describable in U.*

Before establishing Theorem 11.2, let us make some comments. The description of the size and large intersection properties of the set F_ν follows indeed from the combination of that result with Theorem 9.1. As usual, if one is only interested in dimensional results, Corollary 9.2 is enough, and actually implies that with probability one, for any nonempty open set $V \subseteq U$,

$$\begin{cases} \dim_{\mathrm{H}}(F_\nu \cap V) = s_\nu \\ \dim_{\mathrm{P}}(F_\nu \cap V) = d \\ F_\nu \in \mathcal{G}^{s_\nu}(V), \end{cases}$$

where the last two properties hold if s_ν is positive. We thus recover a result shortly mentioned after the statement of Theorem 6.14. Here, the exponent s_ν is defined by the following integrability condition:

$$\begin{cases} s < s_\nu \quad \Longrightarrow \quad \displaystyle\int_{(0,1]} r^s \,\nu(\mathrm{d}r) = \infty \\ s > s_\nu \quad \Longrightarrow \quad \displaystyle\int_{(0,1]} r^s \,\nu(\mathrm{d}r) < \infty. \end{cases}$$

Let us mention in passing that, as already observed in Section 6.5.2 and as suggested by the last property, F_ν is almost surely a G_δ-subset of $\mathbb{R}^d$.

With that level of generality, Theorem 11.2 does not appear anywhere in the literature. However, in dimension $d = 1$, results of the same flavor have been obtained in [**20**] with a view to studying the singularity sets of Lévy processes. Similar results are also used in [**22**] to perform the multifractal analysis of multivariate extensions of Lévy processes; this also corresponds to the case where $d = 1$, but the approximating points are replaced by Poisson distributed hyperplanes.

The remainder of this section is devoted to the proof of Theorem 11.2. Since it is quite long, we split it into several parts.

11.2.1. Reduction to the bounded case. We begin by reducing the study to the case in which the ambient open set is bounded. To this end, we adopt a strategy similar to that employed in the proof of Theorem 6.14. Specifically, for any bounded open subset of U, let us introduce the set

$$F_\nu^V = \left\{ y \in \mathbb{R}^d \;\middle|\; \int_{V_+} \mathbf{1}_{\{|y-x|<r\}}\, \Pi(\mathrm{d}r, \mathrm{d}x) = \infty \right\}$$

defined by restriction from F_ν as in (143), and let us recall from Lemma 6.4(1) that $\Pi(\,\cdot \cap V_+)$ is a Poisson point measure on V_+ with intensity $\nu \otimes \mathcal{L}^d(\,\cdot \cap V)$. Moreover, for any integer $\ell \geq 1$, the sets $U^{(\ell)}$ defined by (146), namely,

$$U^{(\ell)} = \{x \in U \cap \mathrm{B}(0,\ell) \mid \mathrm{d}(x, \mathbb{R}^d \setminus (U \cap \mathrm{B}(0,\ell))) > 1/\ell\}$$

form a nondecreasing sequence of bounded open sets with union equal to U, and we get from (147) that

$$F_\nu \cap U = \bigcup_{\ell=1}^{\infty} \uparrow (F_\nu^{U^{(\ell)}} \cap U^{(\ell)}). \tag{192}$$

In addition, there exists an integer $\ell_0 \geq 1$ such that $U^{(\ell_0)}$ is nonempty, and in fact all the subsequent sets $U^{(\ell)}$ are nonempty as well.

Let us assume that Theorem 11.2 holds for bounded open sets, and let us begin by supposing that the measure ν is not in $\mathcal{R}_d$. Then, for any $\ell \geq \ell_0$, with probability one, the set $F_\nu^{U^{(\ell)}}$ is Lebesgue-full in $U^{(\ell)}$. We readily deduce from (192) and the basic properties of Lebesgue measure that F_ν is almost surely Lebesgue-full in U.

Let us now suppose that ν is in $\mathcal{R}_d$. Then, for any $\ell \geq \ell_0$, with probability one, any gauge function in $\mathfrak{G}(\nu)^\complement$ is majorizing for $F_\nu^{U^{(\ell)}}$ in $U^{(\ell)}$. Hence, with probability one, for any such gauge function g, we have

$$\mathcal{H}^g(F_\nu \cap U) \leq \sum_{\ell=\ell_0}^{\infty} \mathcal{H}^g(F_\nu^{U^{(\ell)}} \cap U^{(\ell)}) = 0,$$

because of (192) and the fact that the Hausdorff g-measure is an outer measure. In other words, we have established that

$$\text{a.s.} \qquad \mathfrak{M}(F_\nu, U) \supseteq \mathfrak{G}(\nu)^\complement.$$

Furthermore, we also know that for any $\ell \geq \ell_0$, with probability one, any gauge function in $\mathfrak{G}(\nu)$ is minorizing for $F_\nu^{U^{(\ell)}}$ in $U^{(\ell)}$. Thus, with probability one, for any such gauge function g, each set $F_\nu^{U^{(\ell)}}$ with $\ell \geq \ell_0$ belongs to the class $\mathcal{G}^g(U^{(\ell)})$. By Definition 8.2, this means that for any d-normalized gauge function $h \prec g_d$ and any open set $\widetilde{U} \subseteq U$, we have

$$\mathcal{M}^h_\infty(F_\nu^{U^{(\ell)}} \cap U^{(\ell)} \cap \widetilde{U}) = \mathcal{M}^h_\infty(U^{(\ell)} \cap \widetilde{U}),$$

because $U^{(\ell)} \cap \widetilde{U}$ is then an open subset of $U^{(\ell)}$. The sets in the right-hand side are nondecreasing with respect to ℓ and their union is equal to $\widetilde{U}$. Owing to (192), the sets in the left-hand side satisfy the same monotonicity property, with an union equal to $F_\nu \cap \widetilde{U}$. We now use the fact that Proposition 2.4(2) holds for the outer measure $\mathcal{M}^h_\infty$ even if it need not be regular, see [**51**, Theorem 52]. We end up with

$$\mathcal{M}^h_\infty(F_\nu \cap \widetilde{U}) = \mathcal{M}^h_\infty(\widetilde{U}).$$

We have thus proved that with probability one, for any gauge function g in $\mathfrak{G}(\nu)$, the set F_ν belongs to the large intersection class $\mathcal{G}^g(U)$. As a result,

$$\text{a.s.} \qquad \mathfrak{m}(F_\nu, U) \supseteq \mathfrak{G}(\nu).$$

To conclude that the set F_ν is almost surely ν-describable in U, it suffices to apply Lemmas 9.1(2) and 9.4. The upshot is that we may assume in what follows that the open set U that comes into play in the statement of Theorem 11.2 is bounded.

11.2.2. Integrability with respect to a Poisson measure. The proof will make a crucial use of the following result on the integrability of gauge functions with respect to Poisson random measures.

LEMMA 11.1. *Let ν be a measure in $\mathcal{R}_d$, let U be a nonempty bounded open subset of $\mathbb{R}^d$, and let N^U denote a Poisson point measure on the interval $(0,1]$ with intensity equal to $\mathcal{L}^d(U)\,\nu$. Then, with probability one,*

$$\mathrm{N}^U \in \mathcal{R}_d \qquad \textit{and} \qquad \mathfrak{G}(\mathrm{N}^U) = \mathfrak{G}(\nu).$$

In order to establish Lemma 11.1, let us begin by proving that N^U is in $\mathcal{R}_d$ with probability one. We observe that N^U must have finite total mass almost surely, because its intensity has infinite total mass, since ν is in $\mathcal{R}_d$. Moreover, evaluating the Laplace functional of N^U at the functions $r \mapsto \theta\, r^d$, for all $\theta > 0$, we get

$$\mathbb{E}\left[\exp\left(-\theta \int_{(0,1]} r^d\, \mathrm{N}^U(\mathrm{d}r)\right)\right] = \exp\left(-\mathcal{L}^d(U) \int_{(0,1]} (1 - \mathrm{e}^{-\theta\, r^d})\, \nu(\mathrm{d}r)\right).$$

We obviously have $1 - \mathrm{e}^{-z} \leq z$ for all $z \in \mathbb{R}$. Using this bound twice, we deduce from the above equality that

$$\mathbb{E}\left[\frac{1}{\theta}\left(1 - \exp\left(-\theta \int_{(0,1]} r^d\, \mathrm{N}^U(\mathrm{d}r)\right)\right)\right] \leq \mathcal{L}^d(U) \int_{(0,1]} r^d\, \nu(\mathrm{d}r).$$

Given that ν belongs to $\mathcal{R}_d$, the right-hand side is finite. In addition, as θ goes to zero, the random variable in the expectation monotonically tends to the integral of $r \mapsto r^d$ with respect to N^U. We deduce from the monotone convergence theorem that this integral has finite expectation. Hence, with probability one,

$$\int_{(0,1]} r^d\, \mathrm{N}^U(\mathrm{d}r) < \infty \qquad \text{and} \qquad \forall \rho \in (0,1] \quad \Phi_{\mathrm{N}^U}(\rho) = \mathrm{N}^U([\rho, 1]) < \infty. \tag{193}$$

As a consequence, the Poisson point measure N^U is almost surely in $\mathcal{R}_d$. It remains to prove that the two sets $\mathfrak{G}(\mathrm{N}^U)$ and $\mathfrak{G}(\nu)$ coincide with probability one. As we now show, this follows from the next property:

$$\text{a.s.} \qquad \Phi_{\mathrm{N}^U}(\rho) \sim \mathcal{L}^d(U)\, \Phi_\nu(\rho) \quad \text{as} \quad \rho \to 0, \tag{194}$$

where $\Phi_\nu(\rho)$ is equal to $\nu([\rho, 1])$, as defined by (139).

Let us suppose that (194) holds and let us consider a gauge function g in $\mathfrak{G}^\infty$ with d-normalization denoted by g_d as usual. The function g_d is nondecreasing and continuous near zero, but need not satisfy this property on the whole interval $[0,1]$. However, g_d clearly coincides near zero with some function denoted by $\widetilde{g}$ which is both nondecreasing and continuous on the whole $[0,1]$. Moreover, due to (139) and the observation that g_d is bounded on $(0,1]$, we have

$$g \in \mathfrak{G}(\nu) \qquad \Longleftrightarrow \qquad \int_{(0,1]} \widetilde{g}(r)\, \nu(\mathrm{d}r) = \infty.$$

Also, on the almost sure event on which (193) holds, the above characterization remains valid if ν is replaced by the Poisson point measure N^U.

Now, similarly to the proof of Theorem 7.1, let us introduce the Lebesgue-Stieltjes measure associated with the monotonic function $\widetilde{g}$. Specifically, let ζ be the premeasure satisfying $\zeta((r,r']) = \widetilde{g}(r') - \widetilde{g}(r)$ when $0 \le r \le r' \le 1$, and let ζ_* be the outer measure defined by (53). Theorem 2.4 shows that the Borel sets contained in $(0,1]$ are ζ_*-measurable, and we may thus integrate locally bounded Borel-measurable functions with respect to ζ_*. Adapting the proof of Proposition 2.9 and using Proposition 2.4(1), one may prove that ζ_* coincides with the premeasure ζ on the intervals where it is defined, and in particular that $\zeta_*((0,r]) = \widetilde{g}(r)$ for any real number $r \in (0,1]$. Using Tonelli's theorem, we deduce that

$$\int_{(0,1]} \widetilde{g}(r)\,\nu(\mathrm{d}r) = \int_{(0,1]} \Phi_\nu(\rho)\,\zeta_*(\mathrm{d}\rho),$$

and that the same property holds when ν is replaced by N^U. As a consequence, the sets $\mathfrak{G}(\mathrm{N}^U)$ and $\mathfrak{G}(\nu)$ coincide with probability one if (194) holds.

It remains us to establish (194). To proceed, let us introduce the countable set $\mathcal{D}_0$ of all real numbers $r \in (0,1]$ such that $\nu(\{r\}) \ge 1$, and also its complement in $(0,1]$, denoted by $\mathcal{D}_1$. Then, for all $\ell \in \{0,1\}$ and $\rho \in (0,1]$, let

$$\Phi_{\nu,\ell}(\rho) = \nu(\mathcal{D}_\ell \cap [\rho,1]) \qquad \text{and} \qquad \Phi_{\mathrm{N}^U,\ell}(\rho) = \mathrm{N}^U(\mathcal{D}_\ell \cap [\rho,1]).$$

Note that there necessarily exists an index ℓ such that $\nu(\mathcal{D}_\ell)$ is infinite. Moreover, if $\nu(\mathcal{D}_\ell)$ is finite, $\Phi_{\nu,\ell}(\rho)$ tends to a finite limit as ρ goes to zero, and $\Phi_{\mathrm{N}^U,\ell}(\rho)$ as well, with probability one. Hence, the proof reduces to showing that for any index $\ell \in \{0,1\}$ such that $\nu(\mathcal{D}_\ell)$ is infinite, we have

$$\text{a.s.} \qquad \Phi_{\mathrm{N}^U,\ell}(\rho) \sim \mathcal{L}^d(U)\,\Phi_{\nu,\ell}(\rho) \quad \text{as} \quad \rho \to 0. \tag{195}$$

For any $\xi > 0$ and any $\rho > 0$ small enough to ensure that $\Phi_{\nu,\ell}(\rho) > 0$, we assert that the following bound holds:

$$\mathbb{P}\left(\left|\frac{\Phi_{\mathrm{N}^U,\ell}(\rho)}{\mathcal{L}^d(U)\,\Phi_{\nu,\ell}(\rho)} - 1\right| \ge \xi\right) \le 2\exp\left(-\frac{3\xi^2}{2\xi+6}\mathcal{L}^d(U)\,\Phi_{\nu,\ell}(\rho)\right). \tag{196}$$

This is indeed a consequence of Bernstein's inequality for integrals with respect to compensated Poisson point measures. To be specific, if S is a locally compact topological space with a countable base, π is a positive Radon measure thereon, and Π is a Poisson point measure with intensity π, then for any real-valued Borel measurable function f defined on S such that

$$M = \sup_S |f| \qquad \text{and} \qquad V = \int_S f^2\,\mathrm{d}\pi$$

are both positive and finite, we have for all positive values of ξ,

$$\mathbb{P}\left(\left|\int_S f\,\mathrm{d}\Pi - \int_S f\,\mathrm{d}\pi\right| \ge \xi\right) \le 2\exp\left(-\frac{3\xi^2}{2M\xi+6V}\right).$$

The above bound may be obtained for instance with the help of [**33**, Corollary 5.1] or [**50**, Proposition 7].

Let us consider a decreasing enumeration $(a_n)_{n\ge1}$ of $\mathcal{D}_0$, and let us suppose that $\nu(\mathcal{D}_0)$ is infinite. The sequence $(a_n)_{n\ge1}$ then necessarily converges to zero. In addition, (196) implies that for all integers $m,n \ge 1$,

$$\mathbb{P}\left(\left|\frac{\Phi_{\mathrm{N}^U,0}(a_n)}{\mathcal{L}^d(U)\,\Phi_{\nu,0}(a_n)} - 1\right| \ge \frac{1}{m}\right) \le 2\exp\left(-\frac{3\mathcal{L}^d(U)\,n}{(6m+2)m}\right), \tag{197}$$

because $\Phi_{\nu,0}(a_n) = \nu(\{a_1,\dots,a_n\}) \ge n$. Summing these inequalities over n for each fixed value of m, we infer from the Borel-Cantelli lemma that for any integer

$m \geq 1$, with probability one, for all n large enough,

$$1 - \frac{1}{m} \leq \frac{\Phi_{\mathrm{N}^U,0}(a_n)}{\mathcal{L}^d(U)\,\Phi_{\nu,0}(a_n)} \leq 1 + \frac{1}{m}. \tag{198}$$

For any real number $\rho \in (0, 1]$, let $n(\rho)$ stand for the number of integers $n \geq 1$ such that $a_n \geq \rho$. Note that $\Phi_{\nu,0}(\rho)$ coincides with $\Phi_{\nu,0}(a_{n(\rho)})$, and the same property holds when ν is replaced by N^U. Hence, if ρ is sufficiently close to zero, we may replace a_n by ρ is the above inequalities. As a result,

$$\text{a.s.} \quad \forall m \geq 1 \qquad \limsup_{\rho \to 0} \left| \frac{\Phi_{\mathrm{N}^U,0}(\rho)}{\mathcal{L}^d(U)\,\Phi_{\nu,0}(\rho)} - 1 \right| \leq \frac{1}{m},$$

and we obtain (195) for $\ell = 0$ by letting $m \to \infty$.

Let us now assume that $\nu(\mathcal{D}_1)$ is infinite. For each integer $n \geq 1$, let us define

$$\rho_n = \sup\left\{\rho \in (0, 1] \;\middle|\; \Phi_{\nu,1}(\rho) \geq n\right\}.$$

We thus obtain a nonincreasing sequence in $(0, 1)$ that converges to zero. In addition, $\Phi_{\nu,1}(\rho_n) \geq n$ for all $n \geq 1$. Applying (196) again, we infer that the bounds (197) still hold when a_n is replaced by ρ_n, and $\Phi_{\mathrm{N}^U,0}$ and $\Phi_{\nu,0}$ are replaced by $\Phi_{\mathrm{N}^U,1}$ and $\Phi_{\nu,1}$, respectively. Using the Borel-Cantelli lemma, we deduce likewise that (198) holds when the same substitutions are performed. On top of that, by definition of $\mathcal{D}_1$ and ρ_n, we have

$$n \leq \Phi_{\nu,1}(\rho_n) = \nu(\mathcal{D}_1 \cap \{\rho_n\}) + \lim_{\rho \downarrow \rho_n} \uparrow \Phi_{\nu,1}(\rho) \leq 1 + n.$$

Making use of the monotonicity of $\Phi_{\mathrm{N}^U,1}$ and $\Phi_{\nu,1}$, we conclude that with probability one, for all n large enough and all $\rho \in [\rho_{n+1}, \rho_n]$,

$$\frac{n}{n+2}\left(1 - \frac{1}{m}\right) \leq \frac{\Phi_{\mathrm{N}^U,1}(\rho)}{\mathcal{L}^d(U)\,\Phi_{\nu,1}(\rho)} \leq \frac{n+2}{n}\left(1 + \frac{1}{m}\right),$$

and finally that (195) holds for $\ell = 1$. The proof of Lemma 11.1 is complete.

11.2.3. Proof in the bounded case. It remains us to establish Theorem 11.2 in the case where the open set U is bounded. Given a measure ν in $\mathcal{R}$, let N^U denote a Poisson point measure on $(0, 1]$ with intensity $\mathcal{L}^d(U)\,\nu$. Lemma 6.5(1) ensures the existence of a nonincreasing sequence $(R_n)_{n\geq1}$ of positive random variables that converges to zero such that (144) holds with probability one, namely,

$$\text{a.s.} \qquad \mathrm{N}^U = \sum_{n=1}^{\infty} \delta_{R_n}.$$

Moreover, let $(X_n)_{n\geq1}$ be a sequence of random variables that are independently and uniformly distributed in U, and are also independent on N^U. Lemma 6.5(2) now implies that the random point measure defined on U_+ by (145), specifically,

$$\mathrm{N}^U_+ = \sum_{n=1}^{\infty} \delta_{(R_n, X_n)}$$

is Poisson distributed with intensity $\nu \otimes \mathcal{L}^d(\,\cdot\, \cap U)$. Hence, the random point measures Π and N^U_+ share the same law. The upshot is that we may assume that Π is replaced by N^U_+ in the definition of the random set F_ν. This enables us to write this set in the alternate form

$$F_\nu = \left\{y \in \mathbb{R}^d \;\middle|\; |y - X_n| < R_n \quad \text{for i.m. } n \geq 1\right\}.$$

On top of that, Theorem 6.13 ensures that with probability one, the sequence $(X_n)_{n\geq1}$ is almost surely uniformly eutaxic in U.

Evaluating the Laplace functional of the Poisson point measure N^U at the function $r \mapsto r^d$, we obtain

$$\mathbb{E}\left[\exp\left(-\sum_{n=1}^{\infty} R_n^d\right)\right] = \exp\left(-\mathcal{L}^d(U)\int_{(0,1]} (1-\mathrm{e}^{-r^d})\,\nu(\mathrm{d}r)\right).$$

Therefore, if the measure ν is not in $\mathcal{R}_d$, the integral in the right-hand side is infinite, so that the expectation in the left-hand side vanishes. This means that the series $\sum_n R_n^d$ diverges almost surely, and thus that the sequence $(R_n)_{n\geq 1}$ belongs to the set P_d characterized by (106). By Definition 6.2, the set F_ν almost surely has full Lebesgue measure in U.

Lastly, let us deal with the case where ν belongs to $\mathcal{R}_d$. We infer from Lemma 11.1 that with probability one, the Poisson point measure N^U belongs to $\mathcal{R}_d$, so that the series $\sum_n R_n^d$ converges. Applying Theorem 9.3, we then deduce that with probability one, the set F_ν is N^U-describable in U. However, Lemma 11.1 ensures that the sets $\mathfrak{G}(\mathrm{N}^U)$ and $\mathfrak{G}(\nu)$ coincide almost surely. It follows that F_ν is almost surely ν-describable in U.

CHAPTER 12

Schmidt's game and badly approximable points

12.1. Schmidt's game

We shall study the following game introduced by Schmidt [**55**]. Let us consider two real numbers $\alpha, \beta \in (0,1)$ and a subset S of $\mathbb{R}^d$. Two players, called Alice and Bob, successively choose nested closed balls of $\mathbb{R}^d$, namely,

$$B_1 \supseteq A_1 \supseteq B_2 \supseteq A_2 \supseteq \ldots$$

with the condition that for any integer $i \geq 1$,

$$|A_i| = \alpha |B_i| \qquad \text{and} \qquad |B_{i+1}| = \beta |A_i|.$$

Alice picks the balls A_i and Bob chooses the balls B_i. Within this setting, Cantor's intersection theorem ensures that intersections $\bigcap_i A_i$ and $\bigcap_i B_i$ are both reduced to the same nonempty compact set with diameter zero, a singleton denoted by $\{\omega\}$. Alice wins the game if ω belongs to S, and Bob wins the game otherwise. The question now is to determine whether or not, depending on the choice of the initial set S, there exists a strategy that Alice can follow in order to be surely the winner, no matter how Bob plays.

More formally, for any closed ball D of $\mathbb{R}^d$ and any real number $\rho \in (0,1)$, let $\mathcal{D}_\rho(D)$ denote the collection of all closed balls $D' \subseteq D$ such that $|D'| = \rho |D|$. For any integer $i \geq 1$, let $\mathcal{F}_{\rho,i}$ be the set of all functions f defined on the i-tuples $(D_1, \ldots, D_i)$ of closed balls of $\mathbb{R}^d$ for which $f(D_1, \ldots, D_i) \in \mathcal{D}_\rho(D_i)$. The strategies that Alice can follow are defined in the next manner.

Definition 12.1. Let α and β be two real numbers in $(0,1)$ and let S be a subset of $\mathbb{R}^d$.

- We call an *α-strategy* any sequence of functions $(f_i)_{i\geq 1}$ such that $f_i \in \mathcal{F}_{\alpha,i}$ for any integer $i \geq 1$.
- An α-strategy $(f_i)_{i\geq 1}$ is called *$(\alpha, \beta; S)$-winning* if for all sequences $(A_i)_{i\geq 1}$ and $(B_i)_{i\geq 1}$ of closed balls of $\mathbb{R}^d$,

$$\left[\forall i \geq 1 \quad \begin{cases} A_i = f_i(B_1, \ldots, B_i) \\ B_{i+1} \in \mathcal{D}_\beta(A_i) \end{cases} \right] \qquad \Longrightarrow \qquad \bigcap_{i=1}^{\infty} A_i = \bigcap_{i=1}^{\infty} B_i \subseteq S.$$

- The set S is called *(α, β)-winning* if there exists an α-strategy that is $(\alpha, \beta; S)$-winning.
- The set S is called *α-winning* if it is (α, β)-winning for all $\beta \in (0,1)$.

Within this formalism, a game then corresponds to the choice oftwo sequences $(A_i)_{i\geq 1}$ and $(B_i)_{i\geq 1}$ of closed balls of $\mathbb{R}^d$ such that $A_i \in \mathcal{D}_\alpha(B_i)$ et $B_{i+1} \in \mathcal{D}_\beta(A_i)$ for all $i \geq 1$. An α-strategy represents the way with which Alice will choose the balls A_i given the balls $B_1, \ldots, B_i$ previously chosen by Bob. If S is an (α, β)-winning set, and if $(f_i)_{i\geq 1}$ denotes an α-strategy that is $(\alpha, \beta; S)$-winning, then Alice will always win if she systematically picks the balls A_i in the form $f_i(B_1, \ldots, B_i)$.

The following notion of chain, which keeps track of the balls chosen by Bob, will also play a useful rôle in the sequel.

DEFINITION 12.2. Let $(B_i)_{i\geq 1}$ denote a sequence of closed balls of $\mathbb{R}^d$ with positive diameter and let $(f_i)_{i\geq 1}$ be an α-strategy.

- For any integer $j \geq 1$, we say that $(B_1, \ldots, B_j)$ is an $(f_1, \ldots, f_j)$-*chain* if for all $i \in \{1, \ldots, j-1\}$,

$$B_{i+1} \in \mathcal{D}_\beta(f_i(B_1, \ldots, B_i)). \tag{199}$$

- We say that $(B_i)_{i\geq 1}$ is an $(f_i)_{i\geq 1}$-*chain* if (199) holds for any $j \geq 1$.

It is clear from the above definitions that if the α-strategy $(f_i)_{i\geq 1}$ is $(\alpha, \beta; S)$-winning, then the intersection of any $(f_i)_{i\geq 1}$-chain is a singleton contained in S. Moreover, if $(B_i)_{i\geq 1}$ denotes an $(f_i)_{i\geq 1}$-chain, then $|B_{i+1}| = \alpha\beta|B_i|$ for all $i \geq 1$.

In the case where $\alpha \leq 1/2$, there exists α-winning sets that do not coincide with the whole $\mathbb{R}^d$, see for instance the important example of badly approximable points discussed in Section 12.2 below. However, it is quite intuitive that α-winning sets have to be somewhat large. This intuition is confirmed by the following result.

THEOREM 12.1. *Let α be a real number in $(0, 1)$, and let S be an α-winning subset of $\mathbb{R}^d$. Then, for any nonempty open subset U of $\mathbb{R}^d$,*

$$\dim_{\mathrm{H}}(S \cap U) = d.$$

PROOF. Let $\beta \in (0, 1/2)$ and let m denote the maximal number of disjoint closed balls with radius 2β that may be included in the closed unit ball of $\mathbb{R}^d$. One easily checks that $\kappa \leq (2\beta)^d m \leq 1$ for some real $\kappa \in (0, 1)$ that depends on the norm the space $\mathbb{R}^d$ is endowed with. Moreover, the set S is (α, β)-winning, so there exists an α-strategy $(f_i)_{i\geq 1}$ that is $(\alpha, \beta; S)$-winning.

The proof makes use of the setting of the general Cantor construction introduced in Section 2.9.2. The construction is indexed by the m-ary tree T_m formed by the words of finite length over the alphabet $\{1, \ldots, m\}$. We define as follows a collection $(I_u)_{u\in T_m}$ of closed balls of $\mathbb{R}^d$ satisfying the following properties:

- for any u in T_m, the balls $I_{u1}, \ldots, I_{um}$ are disjoint and included in I_u ;
- for any integer $j \geq 1$ and for any distinct u and v in $\{1, \ldots, m\}^j$, the distance between I_u and I_v is at least $(\alpha\beta)^j |I_\varnothing|$;
- for any sequence $(\xi_i)_{i\geq 1}$ of integers between one and m, the sequence $(I_{\xi_1\ldots\xi_i})_{i\geq 1}$ is an $(f_i)_{i\geq 1}$-chain.

We proceed by induction on the height of the tree. First, the ball $I_\varnothing$ indexed by the root is an arbitrary closed ball with positive diameter that is contained in U. Second, the ball $I_\varnothing$ contains m disjoint closed balls with diameter $2\alpha\beta|I_\varnothing|$. The balls concentric to them with half their radius are denoted by $I_1, \ldots, I_m$; they have diameter $\alpha\beta|I_\varnothing|$ and are separated by a distance at least $\alpha\beta|I_\varnothing|$ as well. It is clear that each of these balls forms an (f_1)-chain; in fact, every closed ball of $\mathbb{R}^d$ is an (f_1)-chain. Then, let us consider an integer $j \geq 1$ and let us assume that the balls I_u, for $u \in T_m$ with length at most j, have been defined appropriately. In particular, the set $A_j = f_j(I_{u_1}, \ldots, I_u)$ is a closed ball of $\mathbb{R}^d$ with diameter $\alpha|I_u|$. Therefore, it contains m disjoint closed balls with diameter $2\alpha\beta|I_u|$, so that we can find m balls, denoted by $I_{u1}, \ldots, I_{um}$, in the collection $\mathcal{D}_\beta(A_j)$ that are separated by a distance at least $\alpha\beta|I_u|$. For each k, the $(j+1)$-tuple $(I_{u_1}, \ldots, I_u, I_{uk})$ is an $(f_1, \ldots, f_{j+1})$-chain. This implies in particular that $\alpha\beta|I_u| = (\alpha\beta)^j |I_{u_1}| = (\alpha\beta)^{j+1}|I_\varnothing|$. We thus have built appropriately the balls indexed by the words with length $j + 1$.

Now, given that the α-strategy $(f_i)_{i\geq 1}$ is $(\alpha, \beta; S)$-winning, the limiting compact set K defined by (67) is contained in S. Indeed, for any point x in K, there exists a sequence $(\xi_i)_{i\geq 1}$ in $\{1, \ldots, m\}$ such that x belongs to the ball $I_{\xi_1\ldots\xi_i}$ for any $i \geq 1$; since these balls form an $(f_i)_{i\geq 1}$-chain, their intersection is a singleton contained in S, and this singleton is necessarily equal to $\{x\}$.

Here, the sequence $(\varepsilon_j)_{j\geq 1}$ defined by (68) is given by $\varepsilon_j = (\alpha\beta)^j |I_\varnothing|$ for all $j \geq 1$, and the sequence $(m_j)_{j\geq 1}$ defined by (69) is constant equal to m. In particular, the sequence $(\varepsilon_j)_{j\geq 1}$ is decreasing and the sequence $(m_j)_{j\geq 1}$ is positive. We may therefore apply Lemma 2.4; this yields the lower bound

$$\dim_{\mathrm{H}} K \geq \liminf_{j\to\infty} \frac{\log(m_1 \dots m_{j-1})}{-\log(m_j^{1/d}\varepsilon_j)}.$$

Recalling that K is included in $S\cap U$ and replacing ε_j and m_j by the above values, we deduce that

$$\dim_{\mathrm{H}}(S\cap U) \geq \liminf_{j\to\infty} \frac{\log(m^{j-1})}{-\log(m^{1/d}(\alpha\beta)^j |I_\varnothing|)} = \frac{\log m}{|\log(\alpha\beta)|}.$$

We conclude by recalling that $m \geq \kappa(2\beta)^{-d}$, and finally by letting the parameter β go to zero. □

12.2. The set of badly approximable numbers

We consider in this section an emblematic example of winning set: the set, denoted by Bad_1, of badly approximable numbers that we defined in Section 1.3. The main result is the following, and is proven at the end of this section.

THEOREM 12.2. *The set* Bad_1 *is* (α,β)*-winning for any pair* (α,β) *of real numbers in* $(0,1)$ *satisfying* $2\alpha < 1+\alpha\beta$.

Combined with Theorem 12.1, the above result directly enables us to determine the value of the Hausdorff dimension of the set of badly approximable numbers, thereby obtaining a definitive improvement on Corollary 3.2.

COROLLARY 12.1. *For any nonempty open subset* U *of* $\mathbb{R}$*, the badly approximable numbers that belong to* U *form a set with Hausdorff dimension satisfying*

$$\dim_{\mathrm{H}}(\mathrm{Bad}_1 \cap U) = 1.$$

PROOF. If α denotes a real number in the interval $(0,1/2]$, then for any real β in $(0,1)$, we have $2\alpha \leq 1 < 1+\alpha\beta$, so that the set Bad_1 is (α,β)-winning, by virtue of Theorem 12.2. We deduce that the set Bad_1 is α-winning. Its Hausdorff dimension is therefore equal to one, as a consequence of Theorem 12.1. □

Corollary 12.1 may be extended to badly approximable points, that is, to the d-dimensional setting. In fact, a result of Schmidt [**54**] shows that the Hausdorff dimension of the set Bad_d is equal to d.

The remainder of this section is now devoted to the proof of Theorem 12.2. Let us consider two real numbers α and β in the interval $(0,1)$, and let us assume that $\gamma = 1+\alpha\beta-2\alpha$ is positive. For any real number $\ell > 0$, let us define

$$\delta(\ell) = \frac{\gamma}{4}\min\left\{\ell, (\alpha\beta)^2\frac{\gamma}{4}\right\}.$$

First reduction of the problem. The proof of Theorem 12.2 reduces to that of the following statement.

PROPOSITION 12.1. *The exists an* α*-strategy* $(f_i)_{i\geq 1}$ *such that for all sequences* $(A_i)_{i\geq 1}$ *and* $(B_i)_{i\geq 1}$ *of nonempty closed intervals of* $\mathbb{R}$ *satisfying*

$$|B_1| \leq \frac{\alpha\beta\gamma}{4} \qquad \text{and} \qquad \forall i\geq 1 \quad \begin{cases} A_i = f_i(B_1,\dots,B_i) \\ B_{i+1} \in \mathcal{D}_\beta(A_i), \end{cases}$$

the intersections $\bigcap_i A_i$ *and* $\bigcap_i B_i$ *are both reduced to the same singleton* $\{\omega\}$*, where* ω *is such that*

$$\forall (p,q) \in \mathbb{Z} \times \mathbb{N} \qquad \left| \omega - \frac{p}{q} \right| > \frac{\delta(|B_1|)}{q^2}. \tag{200}$$

In order to explain how Theorem 12.2 derives from Proposition 12.1, let us consider an α-strategy $(f_i)_{i\geq 1}$ satisfying the property given in the latter statement. We built another α-strategy $(f_i^*)_{i\geq 1}$ as follows. Let us fix an integer $i \geq 1$ and an i-tuple $(I_1, \ldots, I_i)$ of closed intervals of $\mathbb{R}$. In the situation where the condition

$$|I_i| = \alpha\beta |I_{i-1}| = \ldots = (\alpha\beta)^{i-1} |I_1| \leq \frac{\alpha\beta\gamma}{4} \tag{201}$$

holds, we let j denote the smallest positive integer such that $|I_j| \leq \alpha\beta\gamma/4$, so that j is necessarily less than or equal to i, and we define

$$f_i^*(I_1, \ldots, I_i) = f_{i-j+1}(I_j, \ldots, I_i).$$

Otherwise, we decide that $f_i^*(I_1, \ldots, I_i)$ is an arbitrary element of $\mathcal{D}_\alpha(I_i)$, *e.g.* the interval concentric to I_i with length α times that of I_i.

Let us show that the α-strategy $(f_i^*)_{i\geq 1}$ is $(\alpha, \beta; \mathrm{Bad}_1)$-winning. Let us consider two sequences $(A_i)_{i\geq 1}$ and $(B_i)_{i\geq 1}$ of closed intervals of $\mathbb{R}$ such that for all $i \geq 1$,

$$A_i = f_i^*(B_1, \ldots, B_i) \qquad \text{and} \qquad B_{i+1} \in \mathcal{D}_\beta(A_i). \tag{202}$$

We need to show that the intersection of the intervals A_i or, equivalently, that of the intervals B_i is contained in the set Bad_1 of badly approximable numbers. To proceed, we may obviously assume that the intervals A_i and B_i are nonempty; the aforementioned intersection is thus reduced to a singleton $\{\omega\}$. We now observe that $(B_i)_{i\geq 1}$ is an $(f_i)_{i\geq 1}$-chain. In particular, $|B_i| = (\alpha\beta)^{i-1}|B_1|$ for all $i \geq 1$. Letting j denote the smallest positive integer such that $|B_j| \leq \alpha\beta\gamma/4$, we deduce that (201) is satisfied by the intervals $B_1, \ldots, B_i$ as soon as $i \geq j$. As a consequence, in view of (202), the intervals $A_i^j = A_{j+i-1}$ and $B_i^j = B_{j+i-1}$ verify for all $i \geq 1$,

$$A_i^j = f_i(B_1^j, \ldots, B_i^j) \qquad \text{and} \qquad B_{i+1}^j \in \mathcal{D}_\beta(A_i^j),$$

in addition to $|B_1^j| \leq \alpha\beta\gamma/4$. Applying Proposition 12.1, we deduce that ω satisfies (200) with $\delta(|B_1^j|)$, that is, $\delta(|B_j|)$ instead of $\delta(|B_1|)$ in the bound. However, these two values coincide. Clearly, this is the case if $|B_1| \leq \alpha\beta\gamma/4$, because $j = 1$ then. Moreover, in the opposite situation, the minimality of j ensures that $|B_j| > (\alpha\beta)^2\gamma/4$, so that

$$\delta(|B_j|) = \frac{\gamma}{4} \min\left\{|B_j|, (\alpha\beta)^2 \frac{\gamma}{4}\right\} = \frac{\gamma}{4} \min\left\{|B_1|, (\alpha\beta)^2 \frac{\gamma}{4}\right\} = \delta(|B_1|).$$

As a consequence, ω satisfies (200). As a consequence, ω is badly approximable, *i.e.* belongs to the set Bad_1.

Since the α-strategy $(f_i^*)_{i\geq 1}$ is $(\alpha, \beta; \mathrm{Bad}_1)$-winning, the set Bad_1 is (α, β)-winning, and Theorem 12.2 holds. We are thus reduced to establishing Proposition 12.1.

Second reduction of the problem. To proceed with the proof of Proposition 12.1, let us consider the unique integer $t \geq 1$ such that

$$\frac{\alpha\beta\gamma}{2} \leq (\alpha\beta)^t < \frac{\gamma}{2},$$

along with the unique positive real number R such that $R^2(\alpha\beta)^t = 1$, and let us introduce the following definition.

DEFINITION 12.3. Let us consider an integer $k \geq 0$ and a real number ℓ in $(0, \alpha\beta\gamma/4]$. A nonempty closed interval I of $\mathbb{R}$ is called *(k, ℓ)-appropriate* if its length satisfies $|I| = R^{-2k}\ell$ and if the following holds:

$$\forall x \in I \quad \forall (p, q) \in \mathbb{P}_1 \qquad q < R^k \quad \Longrightarrow \quad \left| x - \frac{p}{q} \right| > \frac{\delta(\ell)}{q^2}.$$

It is elementary and useful to observe that any closed interval I with length in $(0, \alpha\beta\gamma/4]$ is $(0, |I|)$-appropriate. The proof of Proposition 12.1 now relies on the following result.

PROPOSITION 12.2. *For any integer $k \geq 0$ and any real $\ell \in (0, \alpha\beta\gamma/4]$, there are some functions $g^\ell_{kt+1}, \ldots, g^\ell_{(k+1)t}$ in $\mathcal{F}_{\alpha,1}, \ldots, \mathcal{F}_{\alpha,t}$, respectively, such that for any nonempty closed intervals $A_{kt+1}, \ldots, A_{(k+1)t}$ and $B_{kt+1}, \ldots, B_{(k+1)t+1}$ satisfying*

$$\forall i \in \{kt+1, \ldots, (k+1)t\} \qquad \begin{cases} A_i = g^\ell_i(B_{kt+1}, \ldots, B_i) \\ B_{i+1} \in \mathcal{D}_\beta(A_i), \end{cases} \tag{203}$$

the following implication holds:

$$B_{kt+1} \text{ is } (k, \ell)\text{-appropriate} \qquad \Longrightarrow \qquad B_{(k+1)t+1} \text{ is } (k+1, \ell)\text{-appropriate.}$$

As a matter of fact, Proposition 12.2 yields functions g^ℓ_i which enables us to define an α-strategy $(f_i)_{i\geq 1}$ by

$$f_i(I_1, \ldots, I_i) = g_i^{\varepsilon_k(|I_{kt+1}|)}(I_{kt+1}, \ldots, I_i)$$

for any integer $i \geq 1$ and any i-tuple $(I_1, \ldots, I_i)$ of closed intervals of $\mathbb{R}$. Here, k is the unique integer such that $i = kt + r$ for some $r \in \{1, \ldots, t\}$, and

$$\varepsilon_k(l) = \min\left\{ R^{2k} l, \frac{\alpha\beta\gamma}{4} \right\}.$$

Note that the function $g_i^{\varepsilon_k(|I_{kt+1}|)}$ belongs to $\mathcal{F}_{\alpha,r}$, so that f_i belongs to $\mathcal{F}_{\alpha,i}$ as required. Let us now consider two sequences $(A_i)_{i\geq 1}$ and $(B_i)_{i\geq 1}$ of nonempty closed intervals with

$$|B_1| \leq \frac{\alpha\beta\gamma}{4} \qquad \text{and} \qquad \forall i \geq 1 \quad \begin{cases} A_i = f_i(B_1, \ldots, B_i) \\ B_{i+1} \in \mathcal{D}_\beta(A_i), \end{cases}$$

In particular, for any integer $k \geq 0$, the interval B_{kt+1} has length $(\alpha\beta)^{kt}$ times that of the interval B_1. Thus, $\varepsilon_k(|B_{kt+1}|)$ is constant equal to $|B_1|$. We deduce that (203) holds for all $k \geq 0$, with ℓ equal to $|B_1|$. On top of that, the interval B_1 is $(0, |B_1|)$-appropriate. Applying Proposition 12.2 with $\ell = |B_1|$, we may thus prove by induction on the integer $k \geq 0$ that each interval B_{kt+1} is $(k, |B_1|)$-appropriate. As a consequence, the intersection $\{\omega\}$ of the intervals B_i satisfies the following property for every integer $k \geq 0$, every integer $q \in \{1, \ldots, R^k - 1\}$ and every integer $p \in \mathbb{Z}$,

$$\gcd(p, q) = 1 \qquad \Longrightarrow \qquad \left| \omega - \frac{p}{q} \right| > \frac{\delta(|B_1|)}{q^2}.$$

This readily implies (200), and we conclude that Proposition 12.1 holds. We are thus finally reduced to proving Proposition 12.2.

End of the proof. In order to establish Proposition 12.2, let us consider an integer $k \geq 0$, a real number $\ell \in (0, \alpha\beta\gamma/4]$, some functions $g^\ell_{kt+1}, \ldots, g^\ell_{(k+1)t}$ belonging to $\mathcal{F}_{\alpha,1}, \ldots, \mathcal{F}_{\alpha,t}$, respectively, and some nonempty closed intervals $A_{kt+1}, \ldots, A_{(k+1)t}$

and $B_{kt+1}, \dots, B_{(k+1)t+1}$ satisfying (203). Let us also assume that B_{kt+1} is (k,ℓ)-appropriate. Furthermore, let us introduce the set

$$U_k^\ell = \bigcup_{\substack{(p,q)\in\mathbb{P}_1 \\ R^k \le q < R^{k+1}}} \left[\frac{p}{q} - \frac{\delta(\ell)}{q^2}, \frac{p}{q} + \frac{\delta(\ell)}{q^2}\right], \tag{204}$$

and let us assume that the interval $B_{(k+1)t+1}$ does not meet the set U_k^ℓ. Thus, for every pair of coprime integers (p,q) such that $R^k \le q < R^{k+1}$, any point in $B_{(k+1)t+1}$ is at a distance larger than $\delta(\ell)/q^2$ from the rational number p/q. As the interval B_{kt+1} is (k,ℓ)-appropriate and contains $B_{(k+1)t+1}$, this actually holds for every positive integer q less than R^{k+1}. Moreover, the length of $B_{(k+1)t+1}$ is $(\alpha\beta)^t$ times that of B_{kt+1} ; we deduce that this interval is $(k+1,\ell)$-appropriate.

It thus suffices to find a strategy that forces the interval $B_{(k+1)t+1}$ to fit into $B_{kt+1} \setminus U_k^\ell$. To proceed, let us study the intersection set $B_{kt+1} \cap U_k^\ell$ more precisely. Recall that the set U_k^ℓ is a union of intervals that are indexed by pairs of integers; let us assume that there are two distinct pairs (p,q) and (p',q') such that the corresponding intervals meet the set B_{kt+1} at some point x and some point x', respectively. Then, on the one hand,

$$\left|\frac{p}{q} - \frac{p'}{q'}\right| = \frac{|pq'-p'q|}{qq'} \ge \frac{1}{qq'} > R^{-2(k+1)} = (\alpha\beta)^t R^{-2k},$$

because the aforementioned pairs are formed by coprime integers. On the other hand, the triangle inequality yields

$$\begin{aligned}\left|\frac{p}{q} - \frac{p'}{q'}\right| &\le \left|x - \frac{p}{q}\right| + \left|x' - \frac{p'}{q'}\right| + |x - x'| \le \frac{\delta(\ell)}{q^2} + \frac{\delta(\ell)}{q'^2} + |B_{kt+1}| \\ &\le 2\left(\frac{\alpha\beta\gamma}{4}\right)^2 R^{-2k} + (\alpha\beta)^{kt}\ell \le \frac{\alpha\beta\gamma}{4}\left(\frac{\alpha\beta\gamma}{2} + 1\right) R^{-2k} < \frac{\alpha\beta\gamma}{2} R^{-2k}.\end{aligned}$$

These bounds are due to the fact that $\delta(\ell) \le (\alpha\beta\gamma/4)^2$ and that $\gamma \le 2$. We directly deduce that $(\alpha\beta)^t < \alpha\beta\gamma/2$, which contradicts the choice of the integer t. This means that, among the intervals that compose the set U_k^ℓ, at most one can meet the set B_{kt+1}. As a consequence, the intersection set $B_{kt+1} \cap U_k^\ell$ is a (possibly empty) closed interval with diameter at most $2\delta(\ell)R^{-2k}$.

Let b_{kt+1} denote the center of the interval B_{kt+1}. If we assume furthermore that the interval $B_{kt+1} \cap U_k^\ell$ is nonempty and centered at the left of b_{kt+1}, then its right bound is at most

$$b_{kt+1} + \frac{|B_{kt+1} \cap U_k^\ell|}{2} \le b_{kt+1} + \delta(\ell)R^{-2k} = b_{kt+1} + \frac{\delta(\ell)}{\ell}|B_{kt+1}| \le b_{kt+1} + \frac{\gamma}{4}|B_{kt+1}|,$$

from which we directly deduce that

$$B_{kt+1} \cap U_k^\ell \subseteq B_{kt+1} \cap \left(-\infty, b_{kt+1} + \frac{\gamma}{4}|B_{kt+1}|\right].$$

Now, let h^+ be the function in $\mathcal{F}_{\alpha,1}$ which maps every interval of the form $[c-\rho, c+\rho]$, with $c \in \mathbb{R}$ and $\rho > 0$, to the interval $[c+(1-2\alpha)\rho, c+\rho]$. We suppose that $A_i = h^+(B_i)$ for all $i \in \{kt+1, \dots, (k+1)t\}$. The interval B_{kt+2} is contained in $h^+(B_{kt+1})$, so its left bound satisfies

$$b_{kt+2} - \frac{|B_{kt+2}|}{2} \ge b_{kt+1} + (1-2\alpha)\frac{|B_{kt+1}|}{2}.$$

Moreover, the length of B_{kt+2} is $\alpha\beta$ times that of B_{kt+1}. As a consequence,

$$b_{kt+2} \ge b_{kt+1} + (1-2\alpha)\frac{|B_{kt+1}|}{2} + \alpha\beta\frac{|B_{kt+1}|}{2} = b_{kt+1} + \frac{\gamma}{2}|B_{kt+1}|.$$

The choice of the function h^+ implies that the center of each interval B_{i+1} is necessarily on the right of that B_i. In particular, $b_{(k+1)t+1}$ is larger than or equal to b_{kt+2}, and the above lower bound on b_{kt+2} implies that the left bound of the interval $B_{(k+1)t+1}$ satisfies

$$b_{(k+1)t+1} - \frac{|B_{(k+1)t+1}|}{2} \geq b_{kt+1} + \frac{\gamma}{2}|B_{kt+1}| - \frac{(\alpha\beta)^t}{2}|B_{kt+1}| > b_{kt+1} + \frac{\gamma}{4}|B_{kt+1}|,$$

which finally yields

$$B_{(k+1)t+1} \subseteq B_{kt+1} \cap \left(b_{kt+1} + \frac{\gamma}{4}|B_{kt+1}|, \infty\right).$$

Thanks to the previous analysis, we may now explain how to establish Proposition 12.2. First, when I is a nonempty bounded interval, we let $c(I)$ denote its center. Concerning the empty set, we adopt the arbitrary convention that $c(\emptyset)$ is equal to ∞. The situation detailed above thus corresponds to the case where $c(B_{kt+1} \cap U_k^\ell) \leq c(B_{kt+1})$, and the relevant function is therefore h^+. A similar approach can be developed in the case where $c(B_{kt+1} \cap U_k^\ell) > c(B_{kt+1})$, *i.e.* if the interval $B_{kt+1} \cap U_k^\ell$ is either empty or centered on the right of b_{kt+1}. In that situation, the relevant function is the function h^- in $\mathcal{F}_{\alpha,1}$ which sends every interval of the form $[c-\rho, c+\rho]$, with $c \in \mathbb{R}$ and $\rho > 0$, to the interval $[c-\rho, c-(1-2\alpha)\rho]$. It is now natural to define the functions $g_{kt+1}^\ell, \ldots, g_{(k+1)t}^\ell$ as follows: for any integer $i \in \{1, \ldots, t\}$ and for any i-tuple of intervals $(I_1, \ldots, I_i)$,

$$g_{kt+i}^\ell(I_1, \ldots, I_i) = \mathbf{1}_{\{c(I_1 \cap U_k^\ell) \leq c(I_1)\}} h^+(I_i) + \mathbf{1}_{\{c(I_1 \cap U_k^\ell) > c(I_1)\}} h^-(I_i).$$

It is clear that each function g_{kt+i}^ℓ belongs to $\mathcal{F}_{\alpha,i}$. Moreover, for any nonempty closed intervals $A_{kt+1}, \ldots, A_{(k+1)t}$ and $B_{kt+1}, \ldots, B_{(k+1)t+1}$ such that (203) holds, it results from the previous analysis that the interval $B_{(k+1)t+1}$ cannot meet the set U_k^ℓ, thereby being $(k+1, \ell)$-appropriate. This finishes the proof of Proposition 12.2, and in fact of Theorem 12.2.

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