Some toy models of self-organized criticality in percolation

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Abstract

We consider the Bernoulli percolation model in a finite box and we introduce an automatic control of the percolation probability, which is a function of the percolation configuration. For a suitable choice of this automatic control, the model is self-critical, i.e., the percolation probability converges to the critical point p_c when the size of the box tends to infinity. We study here three simple examples of such models, involving the size of the largest cluster, the number of vertices connected to the boundary of the box, or the distribution of the cluster sizes. Along the way, we prove a general geometric inequality for subgraphs of $\mathbb{Z}^d$, which is of independent interest.

1 Introduction

Many interesting physical models present a phenomenon called phase transition: there is a critical point or a critical curve in the parameter space separating two distinct regions characterized by very different macroscopic behaviours. In such systems, the behaviour of the model at criticality is of particular interest and presents some general features (e.g. fractal geometry or power-law temporal and spatial correlations) which are universal across a wide range of systems and do not depend much on the microscopic details of the system. In their seminal paper [BTW88], the physicists Bak, Tang and Wiesenfeld pointed out that these "critical features" are very common in nature, which is rather surprising because it seems that the parameters need to be finely tuned for a system to be critical. To explain this paradox, they showed that some systems tend to be naturally attracted by critical points, without any fine tuning of the parameters. They call this phenomenon self-organized criticality.

To illustrate this idea, they defined a simple model inspired by the dynamics of a sandpile. This system is said to be self-critical because it is naturally attracted by a critical slope, which is the slope at which large-scale avalanches appear. But despite a very simple dynamics, their model turns out to be very difficult to analyze mathematically [BF09, JR08]. Some other simple models presenting this phenomenon of self-organized criticality were studied for example in [Ber12] or [dBDF$^+$94]. In [CG16], Cerf and Gorny constructed a self-critical model as a variant of the Curie-Weiss Ising model, by replacing the temperature with a function depending on the spin configuration. In this paper, we implement the same principle of a feedback from the configuration to the parameter, but within the framework of Bernoulli percolation. This technique to obtain self-organized criticality by "artificially" replacing the control parameter with a feedback function depending on the state of the model, which is explained in section 15.4.2 of [Sor06], was implemented by physicists to imagine self-critical variants of percolation in [Sor92, SWdA$^+$00]. However, the understanding of such models often relies on computer simulations and few models are amenable to rigorous mathematical analysis.

We construct here a simple model of self-organized criticality based on Bernoulli percolation in finite boxes. Let $\Lambda(n)$ be the box of side n centered at 0 in $\mathbb{Z}^d$ with $d \geqslant 2$, and let $\mathbb{E}_n$ be the set of

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edges between nearest neighbours of $\Lambda(n)$. Consider a sequence of functions $F_n : \{0,1\}^{\mathbb{E}_n} \to \mathbb{N}$ and a parameter $a > 0$ and set, for $\omega : \mathbb{E}_n \to \{0,1\}$ a percolation configuration on the edges of the box,

$$p_n(\omega) \;=\; \varphi_n(F_n(\omega)) \qquad \text{where} \qquad \varphi_n(x) \;=\; \exp\Big(-\frac{x}{n^a}\Big)\,.$$

This function p_n will be our automatic control of the percolation probability, and in this paper we will study three examples of such a control, involving different sequences F_n. The model we consider is given by the following probability distribution on the configurations, which is obtained by replacing the parameter p of Bernoulli percolation with our feedback function p_n, with the appropriate normalization. Let

$$\mu_n \;:\; \omega \in \{0,1\}^{\mathbb{E}_n} \;\longmapsto\; \frac{1}{Z_n}\mathbb{P}_{p_n(\omega)}(\omega)$$

where

$$Z_n \;=\; \sum_{\omega\in\{0,1\}^{\mathbb{E}_n}} \mathbb{P}_{p_n(\omega)}(\omega)$$

is the partition function, and

$$\mathbb{P}_p(\omega) \;=\; \prod_{e\in\mathbb{E}_n} p^{\omega(e)}(1-p)^{1-\omega(e)}\,.$$

For $x \in \Lambda(n)$ and $\omega : \mathbb{E}_n \to \{0,1\}$, we write

$$C(x,\,\omega) \;=\; \Big\{\, y \in \Lambda(n) \;:\; x \stackrel{\omega}{\longleftrightarrow} y \,\Big\}$$

for the cluster of x in the configuration ω. We show the following convergence result, valid in any dimension $d \geqslant 2$. The critical point of the Bernoulli percolation model is denoted by p_c.

Theorem 1. *If F_n is one of the following sequences of functions:*

- $F_n : \omega \longmapsto |C_{max}(\omega)| \;=\; \max_{x\in\Lambda(n)} |C(x,\,\omega)|$ *with* $0 < a < d\,;$
- $F_n : \omega \longmapsto |\mathcal{M}_n(\omega)| \;=\; \Big|\Big\{\, x \in \Lambda(n) \;:\; x \stackrel{\omega}{\longleftrightarrow} \partial\Lambda(n) \,\Big\}\Big|$ *with* $d-1 < a < d\,;$
- $F_n : \omega \longmapsto B_n^b(\omega) \;=\; \Big|\Big\{\, x \in \Lambda(n) \;:\; |C(x,\,\omega)| \geqslant n^b \,\Big\}\Big|$ *with* $\dfrac{5d}{6} < a < d$ *and* $0 < b < \dfrac{2a}{d} - \dfrac{5}{3}\,,$

then the law of p_n under μ_n converges to δ_{p_c} when $n \to \infty$, and we have the following control:

$$\forall \varepsilon > 0 \qquad \limsup_{n\to\infty}\; \frac{1}{n^v} \ln \mu_n\big(\; |p_n - p_c| \geqslant \varepsilon \;\big) \;<\; 0\,, \tag{1}$$

where exponent v is given by

$$\begin{cases} v = \min\big(a,\, d-1\big) & \text{if } F_n = |C_{max}|\;; \\ v = d-1 & \text{if } F_n = |\mathcal{M}_n|\;; \\ v = \dfrac{2d}{3} & \text{if } F_n = B_n^b\,. \end{cases}$$

Let us explain briefly the heuristics which lead to the choice of the sequence F_n. The function p_n introduces a negative feedback which assigns low values $p_n(\omega) \ll p_c$ to configurations which are "typical" of the supercritical phase $p > p_c$, and high values $p_n(\omega) \gg p_c$ to configurations which are "typical" of the subcritical phase $p < p_c$. For example, if $F_n = |C_{max}|$, a configuration ω with a largest cluster containing a number of vertices of order n^d will be assigned a very low value $p_n(\omega) \ll p_c$. Yet, for this value of the parameter p in Bernoulli percolation, it is very unlikely to have such a large cluster, which will give ω a very low weight in the measure μ_n. Indeed, we will show that under μ_n, configurations which are either "typically subcritical" or "typically supercritical" have a very low probability. Therefore, the mass of μ_n concentrates on configurations ω with $p_n(\omega)$ sufficiently close to p_c, hence the self-critical behaviour of our model. Note that our parameter a does not need to be finely tuned for our result to hold, showing the robustness of the construction.

Concerning the third model, an estimate on the convergence speed can be easily obtained, provided that we assume the existence of the critical exponents β and γ. The existence of these exponents was proven for dimension 2 in the case of the triangular lattice in [SW01], with $\beta = 5/36$ and $\gamma = 43/18$.

Theorem 2. *Take $F_n = B_n^b$. Assume that there exist real constants $\beta,\, \gamma > 0$ such that*

$$\limsup_{\substack{p\to p_c\\ p>p_c}} \frac{\ln\theta(p)}{\ln(p-p_c)} \;\leqslant\; \beta \qquad \text{and} \qquad \liminf_{\substack{p\to p_c\\ p<p_c}} \frac{\ln\chi(p)}{\ln(p_c-p)} \;\geqslant\; -\gamma\,.$$

Then, for any real parameters a, b and c, we have

$$\frac{5d}{6} < a < d, \quad 0 < b < \frac{2a}{d} - \frac{5}{3} \quad \text{and} \quad c \;<\; \min\left(\frac{b}{2\gamma},\, \frac{1-b}{\beta},\, \frac{d-a}{\beta}\right) \;\Rightarrow\; n^c(p_n - p_c) \;\xrightarrow{\;\mathcal{L}\;}\; 0\,.$$

Note that our list of three models is not comprehensive, since many variants could be defined using the same approach. For example, the case of the largest cluster can be extended to the largest cluster in the torus, which means we can set periodic boundary conditions on the box $\Lambda(n)$. In the model defined with B_n^b, one could consider the distribution of the cluster diameters instead of the cluster sizes, by setting

$$\widetilde{B}_n^b(\omega) \;=\; \left|\left\{x \in \Lambda(n),\; x \overset{\omega}{\longleftrightarrow} \left(x + \partial\Lambda\left(\lceil n^b \rceil\right)\right) \cap \Lambda(n)\right\}\right|\,,$$

which gives exactly the same convergence result, under the same conditions for a and b, and with the same estimate on the convergence speed.

Outline of the paper: This article is organized as follows. First, we give some standard definitions and notations in section 2. We then prove a general geometric inequality for finite subgraphs of $\mathbb{Z}^d$ in section 3. The last three sections are devoted to the proofs of the three cases of theorem 1, theorem 2 being proved at the end of section 6. Each of these sections is divided in two main steps, the first being an exponential decay result on the control functions F_n in the subcritical and supercritical phases, and the second being a lower bound on the partition function Z_n.

While for the last model, the lower bound on Z_n is based on a property of independence which is specific to the function B_n^b considered, our technique for the two first models (see sections 4.3 and 5.3) is more general. It relies on a monotone coupling between configurations and on a careful study of the behaviour of F_n as p decreases towards p_c. Our problem is therefore closely related to the study of finite-size scaling, i.e., the behaviour of the model when one takes $n \to \infty$ and $p \to p_c$ simultaneously (see [BCKS01, GPS18]). Yet, we are able to bypass the recourse to (unproven) scaling laws thanks to the geometric argument of section 3, which is more general and does not rely on the near-critical behaviour of F_n.

An important goal is to build a similar model of self-organized criticality associated with the Ising model. A natural strategy consists in adapting the results presented here to the FK percolation model. However, our efforts in this direction have not succeeded so far. A major complication arises with the FK model. Indeed, in a dynamical coupling of the FK processes, there is already a phenomenon of self-organized criticality in the way the edges become open when one approaches the critical point from below [DCGP14]. As a consequence, our proof for the lower bound on the partition function is not valid in this context.

2 Definitions and notations

2.1 The box

We fix an integer $d \geqslant 2$ for the whole paper. Let $\mathbb{E}^d$ be the set of edges between nearest neighbours of the lattice $\mathbb{Z}^d$,

$$\mathbb{E}^d \;=\; \Big\{ \{x,y\} \in \big(\mathbb{Z}^d\big)^2 \;:\quad \|x-y\|_1 = 1 \Big\} .$$

Let $n \geqslant 1$. Let us consider the box centered at 0 and containing n^d vertices,

$$\Lambda(n) \;=\; \Big[-\frac{n}{2}, \frac{n}{2} \Big[^d \cap \mathbb{Z}^d \;=\; \Big\{ -\Big\lfloor \frac{n}{2} \Big\rfloor, \dots, \Big\lfloor \frac{n-1}{2} \Big\rfloor \Big\}^d .$$

For $V \subset \mathbb{Z}^d$ a set of vertices, we write

$$\mathbb{E}\left[V\right] \;=\; \Big\{ \{x,y\} \in V^2 \;:\; \|x-y\|_1 = 1 \Big\} \;=\; \mathbb{E}^d \cap V^2$$

for the set of edges in $\mathbb{E}^d$ connecting two vertices of V. We write in particular

$$\mathbb{E}_n \;=\; \mathbb{E}\left[\Lambda(n)\right] .$$

The boundary of the box $\Lambda(n)$ will be denoted

$$\partial\Lambda(n) \;=\; \Big\{ x \in \Lambda(n) \;:\quad \exists\, y \in \mathbb{Z}^d \backslash \Lambda(n) \quad \|x-y\|_1 = 1 \Big\} .$$

2.2 Bernoulli percolation

For $0 \leqslant p \leqslant 1$, on the space $\{0,1\}^{\mathbb{E}^d}$ equipped with the σ-field generated by events depending on finitely many edges, let $\mathbb{P}_p$ be the product measure such that the state of each edge follows a Bernoulli law of parameter p. An element $\omega : \mathbb{E}^d \to \{0,1\}$ is called a percolation configuration. Edges $e \in \mathbb{E}^d$ such that $\omega(e) = 1$ will be said open in ω, and the other edges will be said closed in ω. Under the law $\mathbb{P}_p$, each edge is open with probability p and the states of different edges are independent of each other. For any configuration $\omega : \mathbb{E}^d \to \{0,1\}$ and any edge $e \in \mathbb{E}^d$, we will write

$$\omega^e \;:\; f \in \mathbb{E}^d \;\longmapsto\; \begin{cases} 1 & \text{if } f = e\,, \\ \omega(f) & \text{otherwise} \end{cases} \qquad \text{and} \qquad \omega_e \;:\; f \in \mathbb{E}^d \;\longmapsto\; \begin{cases} 0 & \text{if } f = e\,, \\ \omega(f) & \text{otherwise} \end{cases}$$

for the configurations obtained from ω by opening or closing the edge e. Similarly, for any configuration $\omega : \mathbb{E}^d \to \{0,1\}$ and any set of edges $H \subset \mathbb{E}^d$, we will write

$$\omega^H \;:\; f \in \mathbb{E}^d \;\longmapsto\; \begin{cases} 1 & \text{if } f \in H\,, \\ \omega(f) & \text{otherwise} \end{cases} \qquad \text{and} \qquad \omega_H \;:\; f \in \mathbb{E}^d \;\longmapsto\; \begin{cases} 0 & \text{if } f \in H\,, \\ \omega(f) & \text{otherwise} \end{cases}$$

for the configurations obtained from ω by opening or closing all the edges of H. All these notations are naturally extended to configurations $\omega : \mathbb{E}_n \to \{0,1\}$ on the edges of the box $\Lambda(n)$. We will also write $\mathbb{P}_p$ for the induced probability distribution on these configurations. Therefore, for any $\omega : \mathbb{E}_n \to \{0,1\}$, we have

$$\mathbb{P}_p(\omega) \;=\; \prod_{e \in \mathbb{E}_n} p^{\omega(e)} (1-p)^{1-\omega(e)} \;=\; p^{o(\omega)} (1-p)^{|\mathbb{E}_n| - o(\omega)} ,$$

where $o(\omega)$ is the number of edges in $\mathbb{E}_n$ which are open in the configuration ω.

2.3 Clusters

Let $\omega : \mathbb{E}^d \to \{0,1\}$ be a percolation configuration on $\mathbb{Z}^d$. For $x, y \in \mathbb{Z}^d$, we write

$$x \overset{\omega}{\longleftrightarrow} y$$

if there exists a path of open edges in the configuration ω joining x and y. For $x \in \mathbb{Z}^d$, we will write

$$C(x) \;=\; C(x,\,\omega) \;=\; \Big\{ y \in \mathbb{Z}^d \;:\; x \overset{\omega}{\longleftrightarrow} y \Big\}$$

for the connected component of x, which is called the cluster of x, in the configuration ω. If $x \in \mathbb{Z}^d$ and $Y \subset \mathbb{Z}^d$, we write

$$x \overset{\omega}{\longleftrightarrow} Y \quad\iff\quad \exists\, y \in Y \quad x \overset{\omega}{\longleftrightarrow} y\,.$$

All these notations naturally extend to percolation configurations on the edges of the box $\Lambda(n)$. Thus, for $\omega : \mathbb{E}_n \to \{0,1\}$ and $x \in \Lambda(n)$, we will write $C(x,\omega)$ (or $C(x)$) for the set of vertices in $\Lambda(n)$ which are connected to x in $\Lambda(n)$ by an open path in the configuration ω. When it is not clear whether we consider paths which stay in the box or not, for example if ω is defined on $\mathbb{E}^d$, we will specify $C_{\Lambda(n)}(x)$ to denote the set of vertices which are connected to x by an open path with all its intermediate vertices belonging to $\Lambda(n)$, i.e., the cluster of x in the configuration restricted to $\mathbb{E}_n$.

For a percolation configuration $\omega : \mathbb{E}_n \to \{0,1\}$ in the box $\Lambda(n)$, we will denote by $C_{max}(\omega)$ (or sometimes $C_{max}(\Lambda(n))$) the largest cluster in ω, speaking in terms of number of vertices. In case of equality between several maximal clusters, we choose one of them with an arbitrary order on subsets of $\Lambda(n)$. For $\omega : \mathbb{E}_n \to \{0,1\}$, we define $\mathcal{M}_n(\omega)$ as the set of vertices of the box $\Lambda(n)$ which are connected by an open path in ω to the boundary of the box,

$$\mathcal{M}_n(\omega) \;=\; \Big\{ x \in \Lambda(n) \;:\; x \overset{\omega}{\longleftrightarrow} \partial\Lambda(n) \Big\}\,,$$

and, for a real parameter $b > 0$, we set

$$B_n^b(\omega) \;=\; \Big| \Big\{ x \in \Lambda(n) \;:\; \big|C_{\Lambda(n)}(x,\,\omega)\big| \geqslant n^b \Big\} \Big|\,.$$

Given $p \in [0,1]$, let

$$\theta(p) \;=\; \mathbb{P}_p\big(\,|C(0)| = \infty\big)$$

be the probability that the origin lies in an infinite open cluster in a percolation configuration drawn according to $\mathbb{P}_p$. We will write p_c for the critical point of Bernoulli percolation in dimension d, defined by

$$p_c \;=\; \inf\Big\{ p \in [0,1],\; \theta(p) > 0 \Big\}\,,$$

and which is such that $0 < p_c < 1$ (see for this theorem 1.10 of [Gri99]).

3 Geometrical interlude

3.1 Main result

The purpose of this section is to show the following geometric inequality, which one could sum up as "separating a cluster of a given size in a graph (V, E) requires $O(|V|^{(d-1)/d})$ edges".

Lemma 1. *There exists a constant $K = K(d)$ such that, for any finite connected subgraph $G = (V, E)$ of $(\mathbb{Z}^d, \mathbb{E}^d)$, for any vertex $x \in V$ and for any integer m such that $1 \leqslant m \leqslant |V|$, there exists a subset $E_0 \subset E$ of edges of G with cardinality*

$$|E_0| \;\leqslant\; K\,|V|^{\frac{d-1}{d}}$$

such that the connected component of x in the graph $(V,\, E\backslash E_0)$ contains exactly m vertices.

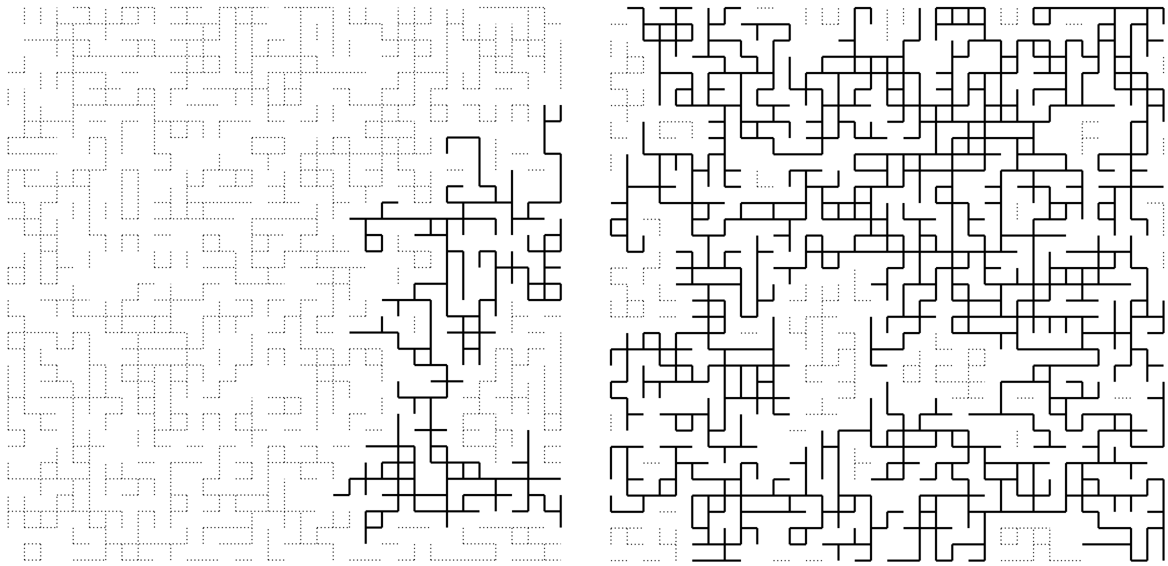

Figure 1: Percolation in the box $\Lambda(n)$ with, left, $p = 0.48$ and right, $p = 0.52$. Open edges belonging to the largest cluster are drawn in solid lines, while other open edges are in dotted lines.

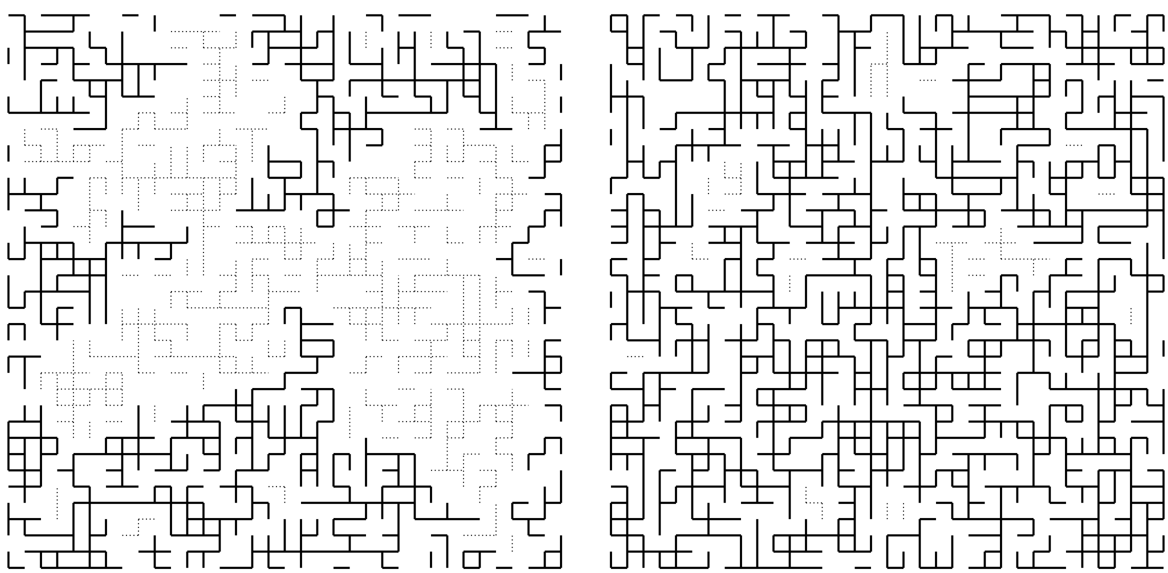

Figure 2: Percolation in the box $\Lambda(n)$ with, left, $p = 0.48$ and right, $p = 0.52$. Open edges connected to the boundary of the box by an open path are drawn in solid lines, while other open edges are in dotted lines.

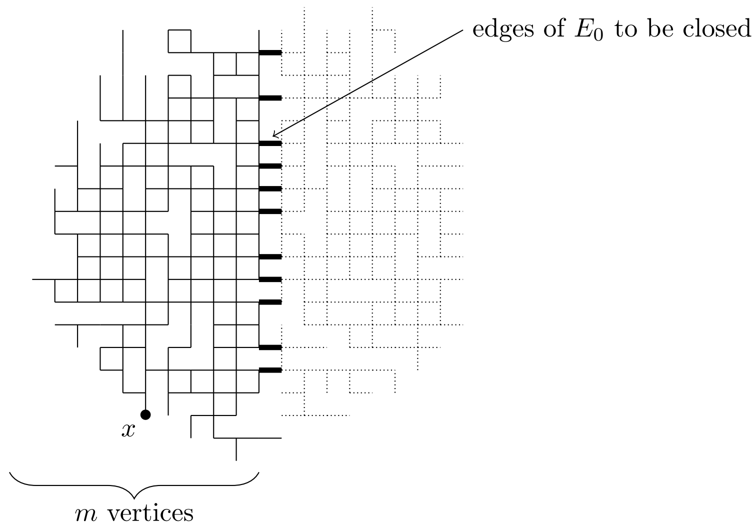

Figure 3: Closing the edges of E_0 (drawn in thick lines) cuts the graph in several connected components, such that x lies in a component (drawn in normal lines) containing the required number of vertices. Lemma 1 states that, in dimension 2, the subset E_0 can be chosen containing $O\big(\sqrt{|V|}\big)$ edges.

The proof of this lemma turned out to be surprisingly difficult. We decompose it in two steps. In section 3.2, we prove "the butcher's lemma", which allows to cut a graph into small components, which may be too small, in particular the component of x might have a cardinality strictly smaller than the goal m. In section 3.3, we prove "the surgeon's lemma", which involves an adequate algorithm to reopen some of the edges closed by the butcher's lemma in order to reach the goal size m for the cluster of x.

3.2 "The butcher's lemma"

We start with an upper bound on the number of edges that one needs to remove from a connected graph to divide in into pieces which are smaller than half of the initial graph. For $x \in \mathbb{Z}^d$, we will write its coordinates $x = (x_1, \ldots, x_d)$. For any finite subset $V \subset \mathbb{Z}^d$ and any $i \in \{1, \ldots, d\}$, we define

$$\operatorname{diam}_i V \;=\; \max_{x\in V} x_i - \min_{x\in V} x_i \qquad \text{and} \qquad \operatorname{diam} V \;=\; \max_{1\leqslant i\leqslant d} \operatorname{diam}_i V\,.$$

If $i \in \{1, \ldots, d\}$ and $m \in \mathbb{Z}$, then

$$T_{i,m} \;=\; \Big\{ e = \{x, y\} \in \mathbb{E}^d, \quad x_i = m \text{ and } y_i = m+1 \Big\}$$

will denote the slice of edges cutting $\mathbb{Z}^d$ in two parts in the direction i between abscissa m and $m+1$. We first prove an auxiliary lemma.

Lemma 2. *For every $k \in \mathbb{N}$ and for any real number $A \geqslant 4$, given a subgraph $G = (V, E)$ of $(\mathbb{Z}^d, \mathbb{E}^d)$ such that $|V| \leqslant A^d$ and*

$$\operatorname{diam} V \;\leqslant\; \left(\frac{3}{2}\right)^k (A-1)\,,$$

there exists a subset $E_0 \subset E$ of edges of G with cardinality

$$|E_0| \;\leqslant\; 2A^{d-1} + 36d^2 \left(1 - \left(\frac{2}{3}\right)^k\right) A^{d-1}$$

such that any connected component of the graph $(V,\, E\backslash E_0)$ contains at most $\lceil A^d/2 \rceil$ vertices.

Remark 1. *In the sequel, this lemma will only be used with $A = |V|^{1/d}$ but it will be helpful for the proof to keep this parameter A fixed rather than have it depending on the graph.*

Proof. Fix $A \geqslant 4$. We will proceed by induction on k, and therefore we start with the case $k = 0$. Let $G = (V,\, E)$ be a subgraph of $(\mathbb{Z}^d,\, \mathbb{E}^d)$ such that $|V| \leqslant A^d$ and $\operatorname{diam} V \leqslant A - 1$. Without loss of generality, we can assume that $V \subset \Lambda(\operatorname{diam} V + 1)$. Let us choose

$$E_0 \;=\; E \cap (T_{0,-1} \cup T_{0,0})\;,$$

whose cardinality is

$$|E_0| \;\leqslant\; 2\big(\operatorname{diam} V + 1\big)^{d-1} \;\leqslant\; 2A^{d-1}\,.$$

If $C \subset V$ is a connected component of $(V,\, E\backslash E_0)$, then we have

$$|C| \;\leqslant\; \max\left(\left\lfloor \frac{\operatorname{diam} V}{2} \right\rfloor,\, \left\lfloor \frac{\operatorname{diam} V + 1}{2} \right\rfloor\right) \big(\operatorname{diam} V + 1\big)^{d-1} \;\leqslant\; \frac{\big(\operatorname{diam} V + 1\big)^{d}}{2} \;\leqslant\; \frac{A^d}{2}\,.$$

We now perform the induction step. Take $k \geqslant 1$ such that the result holds for $k - 1$. Let $G = (V, E)$ be a subgraph of $(\mathbb{Z}^d,\, \mathbb{E}^d)$ such that $|V| \leqslant A^d$ and

$$\operatorname{diam} V \;\leqslant\; \left(\frac{3}{2}\right)^k (A-1)\,.$$

We are going to trim the graph G to decrease its diameter by a factor $2/3$. To this end, we will remove slices of edges in directions i in which the diameter is "too big". Consider

$$\mathcal{I} \;=\; \left\{ i \in \{1,\, \dots,\, d\}\,,\quad \operatorname{diam}_i V > \left(\frac{3}{2}\right)^{k-1} (A-1) \right\}\,,$$

and take $i \in \mathcal{I}$. Without loss of generality, one can assume that $\min\limits_{x \in V} x_i = 0$. By the pigeonhole principle, there exists an integer k_i satisfying

$$\left\lfloor \frac{\operatorname{diam}_i V}{3} \right\rfloor \;<\; k_i \;\leqslant\; 2\left\lfloor \frac{\operatorname{diam}_i V}{3} \right\rfloor \qquad \text{and} \qquad |E \cap T_{i,k_i}| \;\leqslant\; \frac{|E|}{\left\lfloor \frac{\operatorname{diam}_i V}{3} \right\rfloor}\,.$$

We choose such a k_i and we write, recalling that $A \geqslant 4$,

$$\begin{aligned}
\left\lfloor \frac{\operatorname{diam}_i V}{3} \right\rfloor \;&\geqslant\; \frac{\operatorname{diam}_i V}{3} - \frac{2}{3} \\
&\geqslant\; \frac{1}{3}\left(\frac{3}{2}\right)^{k-1} (A-1) - \frac{2}{3} \\
&=\; \frac{1}{9}\left(\frac{3}{2}\right)^{k-1} (A-1) \;+\; \frac{2}{9}\left(\left(\frac{3}{2}\right)^{k-1} (A-1) - 3\right) \\
&\geqslant\; \frac{1}{9}\left(\frac{3}{2}\right)^{k-1} (A-1) \\
&\geqslant\; \frac{1}{9}\left(\frac{3}{2}\right)^{k-1} \frac{3}{4}A \\
&=\; \frac{1}{12}\left(\frac{3}{2}\right)^{k-1} A\,.
\end{aligned}$$

Noting that $|E| \leqslant d\,|V| \leqslant dA^d$, we get

$$|E \cap T_{i,k_i}| \;\leqslant\; \left(\frac{2}{3}\right)^{k-1} \frac{12\,|E|}{A} \;\leqslant\; \left(\frac{2}{3}\right)^{k-1} \frac{12dA^d}{A} \;=\; 12d \left(\frac{2}{3}\right)^{k-1} A^{d-1} \,.$$

Consider now

$$E_1 \;=\; \bigcup_{i \in \mathcal{I}} \left(E \cap T_{i,k_i}\right),$$

whose cardinality is

$$|E_1| \;\leqslant\; 12d^2 \left(\frac{2}{3}\right)^{k-1} A^{d-1} \,.$$

Let $G' = (V', E')$ be a maximal connected component of the graph $(V,\, E \backslash E_1)$, in terms of number of vertices. By construction, we have that, for $i \in \mathcal{I}$,

$$\operatorname{diam}_i(V') \;\leqslant\; \max\Big(k_i,\; \operatorname{diam}_i V - (k_i + 1)\Big) \;\leqslant\; \frac{2}{3} \operatorname{diam}_i V \;\leqslant\; \left(\frac{3}{2}\right)^{k-1} (A-1)\,,$$

while for $i \notin \mathcal{I}$, the definition implies

$$\operatorname{diam}_i(V') \;\leqslant\; \operatorname{diam}_i V \;\leqslant\; \left(\frac{3}{2}\right)^{k-1} (A-1)\,.$$

Taking the maximum over i yields

$$\operatorname{diam}(V') \;\leqslant\; \left(\frac{3}{2}\right)^{k-1} (A-1)\,.$$

Besides, note that $|V'| \leqslant |V| \leqslant A^d$. Hence, by the induction hypothesis applied to the graph G', there exists $E_2 \subset E'$ such that

$$|E_2| \;\leqslant\; 2A^{d-1} + 36d^2 \left(1 - \left(\frac{2}{3}\right)^{k-1}\right) A^{d-1}\,,$$

and all connected components of the graph $(V',\, E' \backslash E_2)$ contain at most $\lceil A^d/2 \rceil$ vertices. Now take $E_0 = E_1 \cup E_2$. We have

$$\begin{aligned} |E_0| \;&=\; |E_1| + |E_2| \\ &\leqslant\; 12d^2 \left(\frac{2}{3}\right)^{k-1} A^{d-1} \;+\; 2A^{d-1} + 36d^2 \left(1 - \left(\frac{2}{3}\right)^{k-1}\right) A^{d-1} \\ &=\; 2A^{d-1} + 36d^2 \left(1 - \left(\frac{2}{3}\right)^{k}\right) A^{d-1}\,. \end{aligned}$$

If C is a connected component of the graph $(V,\, E \backslash E_0)$, then either $C \subset V \backslash V'$ which, by maximality of V', entails $|C| \leqslant |V|/2 \leqslant A^d/2$, or $C \subset V'$ in which case C turns out to be a connected component of the graph $(V',\, E' \backslash E_2)$, which implies $|C| \leqslant \lceil A^d/2 \rceil$. □

Let us now rephrase this result in a more convenient form, which can be summarized by "cutting a graph in two requires $O(|V|^{(d-1)/d})$ edges".

Lemma 3 (The butcher's lemma)**.** *For every finite subgraph $G = (V,\, E)$ of $(\mathbb{Z}^d,\, \mathbb{E}^d)$, there exists a subset $E_0 \subset E$ of edges of G with cardinality*

$$|E_0| \;\leqslant\; 4^{d+1} d^2\, |V|^{\frac{d-1}{d}}$$

such that any connected component of the graph $(V,\, E \backslash E_0)$ contains at most $\lceil |V|/2 \rceil$ vertices.

Proof. If $|V| \geqslant 4^d$, this is a straightforward consequence of lemma 2 with

$$A \;=\; |V|^{1/d} \qquad \text{and} \qquad k \;=\; \left\lceil \frac{d \ln A - \ln(A-1)}{\ln 3 - \ln 2} \right\rceil$$

because we then have

$$\operatorname{diam} V \;\leqslant\; |V| \;=\; \frac{A^d}{A-1}(A-1) \;\leqslant\; \left(\frac{3}{2}\right)^k (A-1)$$

and the lemma provides us with a subset $E_0 \subset E$ with cardinality

$$|E_0| \;\leqslant\; \left(2 + 36d^2\right) A^{d-1} \;\leqslant\; 4^{d+1} d^2 \, |V|^{\frac{d-1}{d}}$$

such that all connected components of $(V,\, E \backslash E_0)$ contain at most $\lceil A^d/2 \rceil = \lceil |V|/2 \rceil$ vertices. Otherwise, if $|V| < 4^d$, then $E_0 = E$ is solution of the problem. $\square$

3.3 "The surgeon's lemma"

The application of the butcher's lemma allows us to separate a graph into connected components which are at least twice smaller than the original graph. If the connected component of x in the remaining graph still contains more vertices than the goal size m, one can apply again the butcher's lemma to this component of x, to obtain a connected component which contains at most a fourth of the initial number of vertices. This operation can be repeated until the connected component of x contains strictly less than m edges, which means that we have closed too many edges. The surgeon's lemma will fix this problem, by reopening some of the edges closed by the butcher's lemma.

Lemma 4 (The surgeon's lemma)**.** *Let $k \in \mathbb{N}$ and let $G = (V,\, E)$ be a connected subgraph of $(\mathbb{Z}^d,\, \mathbb{E}^d)$ with $|V| \leqslant 2^k$. Let $x \in V$ and let m be an integer such that $1 \leqslant m \leqslant |V|$. There exists a subset $E_0 \subset E$ of edges of G with cardinality*

$$|E_0| \;\leqslant\; \frac{1-a^k}{1-a} 4^{d+1} d^2 \, |V|^{\frac{d-1}{d}} \,, \qquad \text{where} \qquad a = \frac{1}{2^{\frac{d-1}{d}}} \,,$$

such that, in the graph $(V,\, E \backslash E_0)$, the connected component of x contains exactly m vertices.

Proof. We proceed by induction on k. The result is trivial if $k = 0$, so we perform next the induction step. Take $k \geqslant 1$ such that the result holds for $k-1$. Let $G = (V,\, E)$ be a connected subgraph of $(\mathbb{Z}^d,\, \mathbb{E}^d)$ with $2^{k-1} < |V| \leqslant 2^k$, let $x \in V$ and let m be an integer such that $1 \leqslant m \leqslant |V|$. According to lemma 3, we can choose a subset $E_0 \subset E$ of cardinality

$$|E_0| \;\leqslant\; 4^{d+1} d^2 \, |V|^{\frac{d-1}{d}}$$

such that any connected component of the graph $(V,\, E \backslash E_0)$ contains at most 2^{k-1} vertices. The idea is to reopen the edges of E_0 one by one starting from the cluster of x, in order to make this cluster grow until it reaches the size m. Then we will apply the induction hypothesis on the last piece added, which contains at most 2^{k-1} vertices.

We are going to order the edges of E_0 by exploring them one by one starting from the cluster of x. We start by writing V_0 for the connected component of x in the graph $(V,\, E \backslash E_0)$. We have that $|V_0| \leqslant 2^{k-1} < |V|$, hence $V_0 \subsetneq V$. Yet the graph $(V,\, E)$ is connected, therefore we can choose an edge $e_1 \in E_0$ incident to this cluster V_0. Assume now that we have defined $e_1, \ldots, e_s \in E_0$ for some $s \geqslant 1$. Let V_s be the connected component of x in the graph

$$\big(V,\, E \backslash \left(E_0 \backslash \{e_1, \ldots, e_s\}\right)\big)\,.$$

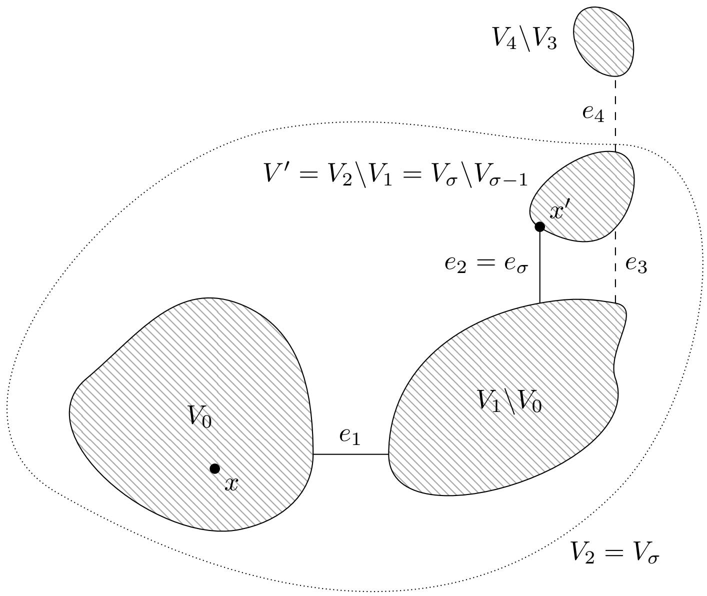

Figure 4: Illustration of the proof of lemma 4: closing the edges of $E_0 = \{e_1,\, e_2,\, e_3,\, e_4\}$ cuts the graph in pieces containing at most 2^{k-1} vertices. We reopen the edges e_i in this order until the number of vertices in the cluster of x reaches or exceeds m. In the case drawn here, $\sigma = 2$, and $V_3 = V_2$ because the edge e_3 connects two vertices which already belong to V_2.

If $s < |E_0|$, then we can choose an edge $e_{s+1} \in E_0$ incident to V_s. Such an edge exists because $(V,\, E)$ is connected. We proceed with this construction until all the edges of E_0 are ordered in a sequence $e_1,\, \ldots,\, e_r$ where $r = |E_0|$. We have then

$$x \in V_0 \ \subset\ V_1 \ \subset\ \ldots\ \subset\ V_r = V\,.$$

If we close all the edges of E_0 and then reopen these edges one by one in the order $e_1,\, \ldots,\, e_r$, then after having reopened s edges, the cluster of x is V_s. Therefore, we introduce

$$\sigma \ = \ \min\Big\{\, s \in \{0,\, \ldots,\, r\}\,, \quad |V_s| \geqslant m\,\Big\}$$

which is the number of reopened edges at which the size of the cluster of x reaches or exceeds the desired size m. This number σ is well-defined because $|V_r| = |V| \geqslant m$. Assume that $\sigma \geqslant 1$. By minimality of σ, we have $|V_{\sigma-1}| < m \leqslant |V_\sigma|$, hence $V_\sigma \neq V_{\sigma-1}$. In that case, the edge e_σ must connect a vertex of $V_{\sigma-1}$ to a vertex $x' \in V_\sigma \backslash V_{\sigma-1}$. Letting $m' = m - |V_{\sigma-1}|$, we have that

$$1 \ \leqslant\ m' \ \leqslant\ |V_\sigma| - |V_{\sigma-1}| \ = \ |V_\sigma \backslash V_{\sigma-1}|\ .$$

Otherwise, if $\sigma = 0$, we set $x' = x$ and $m' = m$, which entails $1 \leqslant m' \leqslant |V_0|$.

Let us consider the graph $G' = (V',\, E')$ of the connected component of x' in $(V,\, E \backslash E_0)$. The choice of E_0 ensures that $|V'| \leqslant 2^{k-1}$. What's more, we have that $V' = V_\sigma \backslash V_{\sigma-1}$ if $\sigma \geqslant 1$ and $V' = V_0$ otherwise, which in both cases leads to $1 \leqslant m' \leqslant |V'|$. The induction hypothesis applied to the graph $G' = (V',\, E')$ gives us a subset $E'_0 \subset E'$ satisfying

$$|E'_0| \ \leqslant\ \frac{1 - a^{k-1}}{1-a} 4^{d+1} d^2\, |V'|^{\frac{d-1}{d}} \ \leqslant\ \frac{1 - a^{k-1}}{1-a} 4^{d+1} d^2 a\, |V|^{\frac{d-1}{d}}$$

and such that the connected component of x' in $(V',\, E'\backslash E_0')$, which will be denoted $V'_{x'}$, contains exactly m' vertices. Now, we consider the set

$$E_0'' \;=\; \{e_{\sigma+1},\,\dots,\,e_r\}\cup E_0'\,,$$

which is such that

$$\begin{aligned}|E_0''| \;&=\; (r-\sigma)+|E_0'|\\ &\leqslant\; 4^{d+1}d^2\,|V|^{\frac{d-1}{d}}+\frac{a-a^k}{1-a}4^{d+1}d^2\,|V|^{\frac{d-1}{d}}\\ &=\; \frac{1-a^k}{1-a}4^{d+1}d^2\,|V|^{\frac{d-1}{d}}\;.\end{aligned}$$

If $\sigma\,=\,0$, then the connected component of x in the graph $(V,\, E\backslash E_0'')$ is $V'_{x'}$ and thus it contains exactly $m'=m$ vertices. Otherwise, if $\sigma\geqslant 1$, then this connected component is $V_{\sigma-1}\cup V'_{x'}$, which contains $|V_{\sigma-1}|+m'=m$ vertices. □

4 The largest cluster

This section is devoted to the proof of theorem 1 for the case of the first model, i.e., the one defined with the largest cluster in the box $\Lambda(n)$. The first step will be to show the exponential decay of the distribution of $|C_{max}|$ in the subcritical and supercritical phases, and the second step will be to prove a lower bound on the partition function.

4.1 Exponential decay in the subcritical phase

We need the following upper bound:

Lemma 5. *For any real parameter $a>0$, we have*

$$\forall p<p_c\quad \forall A>0\qquad \limsup_{n\to\infty}\,\frac{1}{n^a}\ln\mathbb{P}_p\Big(\big|C_{max}\big(\Lambda(n)\big)\big|>An^a\Big)\;<\;0\,.$$

Proof. Let $a>0$, $p<p_c$ and $A>0$. For all $n\geqslant 1$, we have that

$$\begin{aligned}\mathbb{P}_p\Big(\big|C_{max}\big(\Lambda(n)\big)\big|>An^a\Big)\;&=\;\mathbb{P}_p\left(\max_{v\in\Lambda(n)}\,\big|C_{\Lambda(n)}\,v\big|>An^a\right)\\ &\leqslant\;\mathbb{P}_p\left(\max_{v\in\Lambda(n)}\,|C(v)|>An^a\right)\\ &\leqslant\; n^d\mathbb{P}_p\Big(\,|C(0)|>An^a\Big)\,.\end{aligned}$$

According to theorem 6.75 in [Gri99], there exists a constant $\lambda(p)>0$ such that, for all $m\geqslant 1$,

$$\mathbb{P}_p\Big(\,|C(0)|>m\Big)\;\leqslant\; e^{-m\lambda(p)}\,.$$

It follows that, for all $n\geqslant 1$,

$$\mathbb{P}_p\Big(\big|C_{max}\big(\Lambda(n)\big)\big|>An^a\Big)\;\leqslant\; n^d\exp\big(-A\lambda(p)n^a\big)\,,$$

which implies the desired inequality. □

4.2 Exponential decay in the supercritical phase

We establish a similar result in the supercritical regime.

Lemma 6. *For all real $a < d$, we have*

$$\forall p > p_c \quad \forall A > 0 \qquad \limsup_{n\to\infty} \frac{1}{n^{d-1}} \ln \mathbb{P}_p\Big(\big|C_{max}(\Lambda(n))\big| < An^a \Big) \;<\; 0\,.$$

Proof. We will prove a stronger result, namely that

$$\forall p > p_c \quad \forall A > 0 \qquad \limsup_{n\to\infty} \frac{1}{n^{d-1}} \ln \mathbb{P}_p\left(\big|C_{max}(\Lambda(n))\big| \leqslant \frac{\theta(p)n^d}{2} \right) \;<\; 0\,.$$

Assume first that $d \geqslant 3$. From theorem 1.2 of [Pis96], it follows that, for $d \geqslant 3$, for all $p > \widehat{p_c}$ (where $\widehat{p_c}$ denotes the slab-percolation threshold),

$$\limsup_{n\to\infty} \frac{1}{n^{d-1}} \ln \mathbb{P}_p\left(|C_{max}(\Lambda(n))| \leqslant \frac{\theta(p)n^d}{2} \right) \;<\; 0\,.$$

In addition, Grimmett and Marstrand proved the identity $p_c = \widehat{p_c}$ for $d \geqslant 3$ in [GM90]. The claim for $d \geqslant 3$ thus follows immediately.

Consider now the case $d = 2$. Theorem 6.1 of [ACC90] implies that, for all $p > p_c$, if we consider a percolation configuration on $\mathbb{Z}^d$ and write $C_\infty \subset \mathbb{Z}^d$ for the unique infinite cluster of the configuration, then

$$\lim_{n\to\infty} \frac{1}{n} \ln \mathbb{P}_p\left(|C_\infty \cap \Lambda(n)| \leqslant \frac{\theta(p)n^2}{2} \right) \;<\; 0\,.$$

Thereby, there exists $L > 0$ such that, for all $n \geqslant 1$,

$$\mathbb{P}_p\left(|C_\infty \cap \Lambda(n)| \leqslant \frac{\theta(p)n^2}{2} \right) \;\leqslant\; e^{-Ln}\,. \tag{2}$$

Besides, if we set, for $m \geqslant k \geqslant 1$,

$$L_{k,\,m} \;=\; \Big\{ \text{The rectangle } \{0,\,\dots,\,k\} \times \{0,\,\dots,\,m\} \text{ is crossed by an open path in its long direction} \Big\}\,,$$

then it follows from equation (7.110) in [Gri99] that there exist positive constants $C_2(p)$ and $C_3(p)$ such that, for all $m \geqslant k \geqslant 1$,

$$\mathbb{P}_p\left(L_{k,\,m}\right) \;\geqslant\; 1 - C_2 m e^{-C_3 k}\,. \tag{3}$$

Define the rectangles

$$\begin{aligned}
R_1 &= \mathbb{Z}^2 \cap \Big]\frac{n}{2},\, n\Big[\times [-n,\, n[\,, \\
R_2 &= \mathbb{Z}^2 \cap [-n,\, n[\times \Big]\frac{n}{2},\, n\Big[\,, \\
R_3 &= \mathbb{Z}^2 \cap \Big[-n,\, -\frac{n}{2}\Big[\times [-n,\, n[\,, \\
R_4 &= \mathbb{Z}^2 \cap [-n,\, n[\times \Big[-n,\, -\frac{n}{2}\Big[\,,
\end{aligned}$$

which are represented in figure 5. Following a classical argument (see the proof of theorem 7.61 in [Gri99]), we consider the events

$$\mathcal{E}_n \;=\; \Big\{ \text{There exists an open path in } \Lambda(2n)\backslash\Lambda(n) \text{ containing } \Lambda(n) \text{ in its interior} \Big\}$$

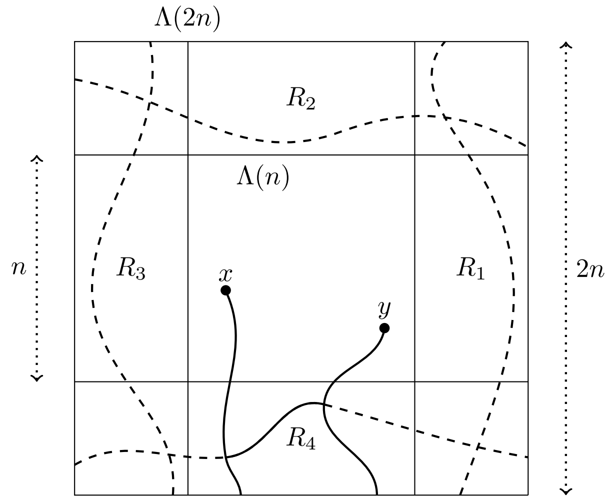

Figure 5: If each of the four rectangles is crossed by an open path in its long direction, then $\Lambda(n)$ is surrounded by an open path in $\Lambda(2n)$, and thus any two vertices x and y in the box $\Lambda(n)$ cannot be connected to $\partial\Lambda(2n)$ without being connected to each other by an open path inside $\Lambda(2n)$.

and

$$\mathcal{F}_n \;=\; \Big\{ \text{Each of the rectangles } R_1,\ R_2,\ R_3,\ R_4 \text{ is crossed by an open path in its long direction} \Big\} .$$

As illustrated on figure 5, we have the inclusion $\mathcal{E}_n \supset \mathcal{F}_n$. In addition, by the FKG inequality, we have that

$$\mathbb{P}_p\left(\mathcal{F}_n\right) \;\geqslant\; \mathbb{P}_p\left(L_{\lfloor n/2 \rfloor,\, 2n}\right)^4 .$$

In combination with (3), this yields

$$\mathbb{P}_p\left(\mathcal{E}_n\right) \;\geqslant\; \mathbb{P}_p\left(\mathcal{F}_n\right) \;\geqslant\; \mathbb{P}_p\left(L_{\lfloor n/2 \rfloor,\, 2n}\right)^4 \;\geqslant\; \left(1 - 2C_2 n e^{-C_3 \lfloor n/2 \rfloor}\right)^4 \;\geqslant\; 1 - 8C_2 n e^{-C_3 \lfloor n/2 \rfloor} .$$

Yet if the event $\mathcal{E}_n$ occurs, then all the vertices of $\Lambda(n)$ which are connected by an open path to the boundary of $\Lambda(2n)$ must be connected to each other inside $\Lambda(2n)$, which implies that $|C_{max}\left(\Lambda(2n)\right)| \geqslant |C_\infty \cap \Lambda(n)|$. Therefore, we have the inclusion

$$\mathcal{E}_n \cap \left\{ |C_\infty \cap \Lambda(n)| > \frac{\theta(p) n^2}{2} \right\} \;\subset\; \left\{ \big|\, C_{max}\left(\Lambda(2n)\right) \big| > \frac{\theta(p) n^2}{2} \right\} .$$

Considering complementary events leads to

$$\begin{aligned} \mathbb{P}_p\left(|C_{max}\left(\Lambda(2n)\right)| \leqslant \frac{\theta(p) n^2}{2} \right) \;&\leqslant\; 1 - \mathbb{P}_p\left(\mathcal{E}_n\right) + \mathbb{P}_p\left(|C_\infty \cap \Lambda(n)| \leqslant \frac{\theta(p) n^2}{2} \right) \\ &\leqslant\; 8C_2 n e^{-C_3 \lfloor n/2 \rfloor} + e^{-Ln} \\ &\leqslant\; e^{-L' n} \end{aligned}$$

for a certain constant $L' > 0$, which concludes the proof. □

4.3 Lower bound on the partition function

We show the following inequality on the normalization constant Z_n of our model:

Lemma 7. *For any real number a such that $0 < a < d$, we have*

$$\liminf_{n\to\infty} \frac{\ln Z_n}{(\ln n) n^{a(d-1)/d}} \;>\; -\infty\,.$$

Proof. The partition function can be rewritten as

$$Z_n \;=\; \sum_{\omega\in\{0,1\}^{\mathbb{E}_n}} \mathbb{P}_{p_n(\omega)}(\omega) \;=\; \sum_{b=1}^{n^d} \sum_{\substack{\omega\in\{0,1\}^{\mathbb{E}_n}\\ |C_{max}(\omega)|=b}} \mathbb{P}_{\varphi_n(b)}(\omega) \;=\; \sum_{b=1}^{n^d} \mathbb{P}_{\varphi_n(b)}\big(|C_{max}|=b\big)\,. \tag{4}$$

To get a lower bound on Z_n, we are going to define a monotone coupling of the distributions $\mathbb{P}_{\varphi_n(b)}$ for $b \in \big\{0,\,\dots,\,n^d\big\}$.

Construction of the coupling: Write $\mathbb{E}_n = \{e_1,\,\dots,\,e_r\}$ with $r = |\mathbb{E}_n|$. Consider a collection of i.i.d. random variables

$$(X_{b,e})_{b\in\{0,\,\dots,\,n^d-1\},\,e\in\mathbb{E}_n}$$

with Bernoulli law of parameter $\exp(-1/n^a)$. For $b_0 \in \big\{0,\,\dots,\,n^d\big\}$, define a random configuration

$$\omega(b_0)\;:\; e\in\mathbb{E}_n \;\longmapsto\; \min_{0\leqslant b<b_0} X_{b,e}\,.$$

Hence, for $b_0 \in \big\{0,\,\dots,\,n^d\big\}$ and $e\in\mathbb{E}_n$, we see that

$$\mathbb{P}\Big(\omega(b_0)(e)=1\Big) \;=\; \prod_{b=0}^{b_0-1} \mathbb{P}\big(X_{b,e}=1\big) \;=\; \exp\left(-\frac{b_0}{n^a}\right) \;=\; \varphi_n(b_0)\,,$$

therefore the configuration $\omega(b_0)$ has distribution $\mathbb{P}_{\varphi_n(b_0)}$. What's more, configurations are coupled in such a way that

$$\mathbb{1}_{\mathbb{E}_n} = \omega(0) \;\geqslant\; \omega(1) \;\geqslant\; \dots \;\geqslant\; \omega(n^d)\,.$$

When going from the configuration $\omega(b)$ to the configuration $\omega(b+1)$, a certain number or edges are closed (these are the edges e such that $\omega(b)(e)=1$ and $X_{b,e}=0$). In order to control the edge closures one by one, we define intermediate configurations. For $b\in\big\{0,\,\dots,\,n^d-1\big\}$ and $s_0\in\{0,\,\dots,\,r\}$, we set

$$\omega(b,\,s_0)\;:\; e_s\in\mathbb{E}_n \;\longmapsto\; \begin{cases} \omega(b+1)(e_s) & \text{if } s\leqslant s_0\,,\\ \omega(b)(e_s) & \text{otherwise.}\end{cases}$$

In this way, we have $\omega(b,\,0)=\omega(b)$ and for $s\geqslant 1$, the configuration $\omega(b,\,s)$ is obtained from the configuration $\omega(b,\,s-1)$ by closing the edge e_s if $X_{b,e_s}=0$, and by keeping everything unchanged if $X_{b,e_s}=1$. For $s=r=|\mathbb{E}_n|$, all edges have been updated, so $\omega(b,\,r)=\omega(b+1)$. The configurations are therefore coupled in such a way that

$$(b,\,s)\leqslant(b',\,s') \implies \omega(b,\,s)\geqslant\omega(b',\,s')\,,$$

where we use the lexicographic order on $\{0,\,\dots,\,n^d-1\}\times\{0,\,\dots,\,r\}$. With this construction, equation (4) becomes

$$Z_n \;=\; \sum_{b=1}^{n^d}\mathbb{P}\Big(\,|C_{max}(\omega(b))|=b\Big) \;=\; \mathbb{P}\Big(\exists b\in\big\{1,\,\dots,\,n^d\big\}\quad |C_{max}\,(\omega(b))|=b\Big)\,. \tag{5}$$

Hence, the partition function Z_n is equal to the probability that the non-increasing function $b\mapsto |C_{max}\,(\omega(b))|$ admits a fixed point. This leads us to consider an instant $b=B$ situated before this

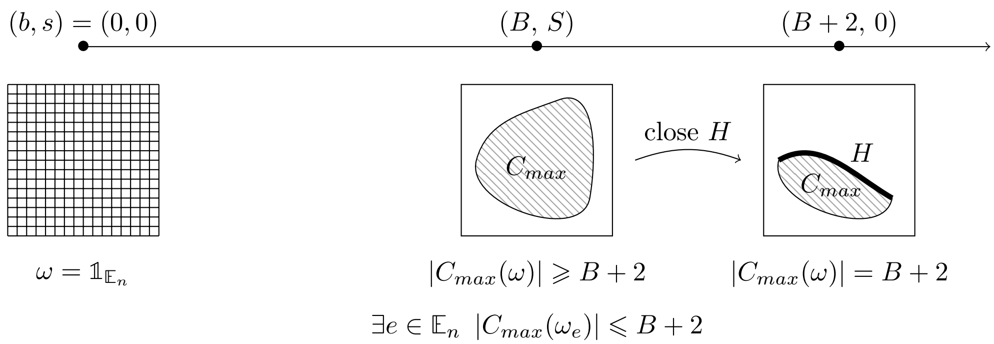

Figure 6: Sketch of the proof: if $\mathcal{E}$ occurs, i.e., between the instants $(B,\, S)$ and $(B+2,\, 0)$, the edges H are closed but no other edges of C_{max} is closed, then the largest cluster in the configuration $\omega(B+2, 0)$ contains $B+2$ vertices.

function goes under the first bisector, and to see what is needed on the variables $X_{b,e}$ for this function to actually cross the bisector at the instant $b = B+2$.

Definition of the instant B**:** Still considering the lexicographic order, we define a pair of random variables

$$(B,\, S) \;=\; \min\Big\{(b,\, s) \in \big\{0,\, \dots,\, n^d - 2\big\} \times \{0,\, \dots,\, r\} \;:\quad \exists e \in \mathbb{E}_n \quad |C_{max}\left(\omega(b,\, s)_e\right)| \leqslant b+2\Big\}\,.$$

This minimum is well-defined because one always has $\big|C_{max}\left(\omega(n^d-2,\, 0)\right)\big| \leqslant n^d$. In addition, for every $(b_0,\, s_0)$, the event $\{(B,\, S) = (b_0,\, s_0)\}$ only depends on the variables $X_{b,\, e_s}$ for $(b,\, s) \leqslant (b_0,\, s_0)$, which means that $(B,\, S)$ is a stopping time for the filtration generated by the variables X_{b, e_s}. Also, closing one single edge cannot divide the size of the largest cluster by more than two, whence

$$|C_{max}\left(\omega(B,\, S)\right)| \;\leqslant\; 2(B+2)\,. \tag{6}$$

Let us prove that we also have

$$|C_{max}\left(\omega(B,\, S)\right)| \;\geqslant\; B+2\,. \tag{7}$$

We distinguish several cases.

• If $S \geqslant 1$, then the minimality of $(B,\, S)$ ensures that, for all $e \in \mathbb{E}_n$,

$$|C_{max}\left(\omega(B,\, S-1)_e\right)| \;>\; B+2\,.$$

Yet the configuration $\omega(B,\, S)$ is obtained from $\omega(B,\, S-1)$ by closing at most one edge, whence (7).

• If $S = 0$ and $B > 0$, then, $(B,\, S)$ being minimal, we have that

$$|C_{max}\left(\omega(B-1,\, r)\right)| \;>\; B-1+2 \;=\; B+1\,.$$

The configurations $\omega(B-1,\, r)$ and $\omega(B,\, 0)$ being identical, inequality (7) is also satisfied.

• The case $(B,\, S) = (0, 0)$ does not happen because all edges are open in the configuration $\omega(0, 0)$.

Construction of the happy event: Given (7), it follows from lemma 1 that there exists a (random) set of edges

$$H \;=\; H(B, \omega(B,\, S)) \;\subset\; \mathbb{E}\left[C_{max}\left(\omega(B,\, S)\right)\right]\,,$$

satisfying

$$|H| \;\leqslant\; K\, |C_{max}(\omega(B,\, S))|^{\frac{d-1}{d}} \tag{8}$$

and such that the largest connected component of the graph

$$\Big(C_{max}\left(\omega(B,\,S)\right),\,\mathbb{E}\left[C_{max}\left(\omega(B,\,S)\right)\right]\backslash H\Big)$$

contains exactly $B+2$ vertices. Note that we have defined $H = H(B,\,\omega(B,\,S))$ as a deterministic function of the variables B and $\omega(B,\,S)$, this will be useful later. The existence of an edge $e \in \mathbb{E}_n$ such that

$$|C_{max}\left(\omega(B,\,S)_e\right)| \;\leqslant\; B+2$$

entails that, in $\omega(B,\,S)$, there is at most one cluster containing strictly more than $B+2$ vertices. Thus, closing the edges of H is enough for the remaining largest cluster to contain $B+2$ vertices, i.e.

$$|C_{max}\left(\omega(B,\,S)_H\right)| \;=\; B+2\,.$$

Hence, closing the edges of H and no other edge of $\mathbb{E}\left[C_{max}\left(\omega(B,\,S)\right)\right]$ between the instants $(B,\,S)$ and $(B+2,\,0)$ ensures that $|C_{max}\left(\omega(B+2)\right)| = B+2$. But the edges $e_s \in H$ are note necessarily labelled with numbers $s > S$. It is therefore not generally possible to close all the edges of H between the instants $(B,\,S)$ and $(B+1,\,0)$. For this reason, the event we consider is the one in which no edge of $C_{max}(\omega(B,\,S))$ is closed between $(B,\,S)$ and $(B+1,\,0)$, and the edges of $C_{max}(\omega(B,\,S))$ which are closed between $(B+1,\,0)$ and $(B+2,0)$ are precisely the edges of H (or, at least the edges of H which were not already closed), that is to say

$$\mathcal{E} \;=\; \left\{\begin{array}{l} \forall s > S \quad e_s \in \mathbb{E}\left[C_{max}\left(\omega(B,\,S)\right)\right] \Rightarrow X_{B,\,e_s} = 1 \\ \forall e \in H \quad X_{B+1,\,e} = 0 \\ \forall e \in \mathbb{E}\left[C_{max}\left(\omega(B,\,S)\right)\right]\backslash H \quad X_{B+1,\,e} = 1 \end{array}\right\}.$$

If this event occurs, then in the configuration $\omega(B+2)$, all the edges of H are closed, the other edges of $\mathbb{E}\left[C_{max}\left(\omega(B,\,S)\right)\right]$ which were open in the configuration $\omega(B,\,S)$ remain open, and all the other clusters contain at most $B+2$ vertices, whence

$$\mathcal{E} \;\subset\; \Big\{\;|C_{max}\left(\omega(B+2)\right)| \;=\; |C_{max}\left(\omega(B,\,S)_H\right)| \;=\; B+2\Big\}.$$

Conditional probability of the happy event: Coming back to the expression (5) of the partition function, we find that

$$Z_n \;\geqslant\; \mathbb{P}\Big(\big|\,C_{max}\left(\omega(B+2)\right)\big| \;=\; B+2\Big) \;\geqslant\; \mathbb{P}\left(\mathcal{E}\right)\,. \tag{9}$$

Let $(b_0,\,s_0)$ and $\omega_0 : \mathbb{E}_n \to \{0,1\}$ be such that

$$\mathbb{P}\big(\mathcal{C}_{b_0,\,s_0,\,\omega_0}\big) \;>\; 0 \qquad \text{where} \qquad \mathcal{C}_{b_0,\,s_0,\,\omega_0} \;=\; \Big\{\,(B,\,S) = (b_0,\,s_0) \quad \text{and} \quad \omega(B,\,S) = \omega_0\Big\}.$$

Having defined H as a deterministic function of B and $\omega(B,\,S)$, we can consider the event

$$\widetilde{\mathcal{E}}_{b_0,\,s_0,\,\omega_0} \;=\; \left\{\begin{array}{l} \forall s > s_0 \quad e_s \in \mathbb{E}\left[C_{max}\left(\omega_0\right)\right] \Rightarrow X_{b_0,\,e_s} = 1 \\ \forall e \in H(b_0,\,\omega_0) \quad X_{b_0+1,\,e} = 0 \\ \forall e \in \mathbb{E}\left[C_{max}\left(\omega_0\right)\right]\backslash H(b_0,\,\omega_0) \quad X_{b_0+1,\,e} = 1 \end{array}\right\},$$

which satisfies

$$\mathbb{P}\big(\mathcal{E}\,|\,\mathcal{C}_{b_0,\,s_0,\,\omega_0}\big) \;=\; \mathbb{P}\left(\widetilde{\mathcal{E}}_{b_0,\,s_0,\,\omega_0}\,|\,\mathcal{C}_{b_0,\,s_0,\,\omega_0}\right)\,. \tag{10}$$

Now note that this event $\widetilde{\mathcal{E}}_{b_0,\, s_0,\, \omega_0}$ depends only on the variables $X_{b,\, e_s}$ with $(b,\, s) > (b_0,\, s_0)$, whereas the event $\mathcal{C}_{b_0,\, s_0,\, \omega_0}$ depends only on the variables $X_{b,\, e_s}$ with $(b,\, s) \leqslant (b_0,\, s_0)$. Thus, these two events are independent of each other, which allows us to write

$$\begin{aligned} \mathbb{P}\left(\widetilde{\mathcal{E}}_{b_0,\, s_0,\, \omega_0} \,|\, \mathcal{C}_{b_0,\, s_0,\, \omega_0}\right) \;&=\; \mathbb{P}\left(\widetilde{\mathcal{E}}_{b_0,\, s_0,\, \omega_0}\right) \\ &=\; \prod_{\substack{s > s_0 \\ e_s \in \mathbb{E}[C_{max}(\omega_0)]}} \mathbb{P}\big(\, X_{b_0,\, e_s} = 1 \,\big) \times \prod_{e \in H(b_0,\, \omega_0)} \mathbb{P}\big(\, X_{b_0+1,\, e} = 0 \,\big) \\ &\qquad \times \prod_{e \in \mathbb{E}[C_{max}(\omega_0)] \setminus H(b_0,\, \omega_0)} \mathbb{P}\big(\, X_{b_0+1,\, e} = 1 \,\big) \\ &\geqslant\; \left(e^{-1/n^a}\right)^{2|\mathbb{E}[C_{max}(\omega_0)]|} \left(1 - e^{-1/n^a}\right)^{|H(b_0,\, \omega_0)|} . \end{aligned}$$

Combining this with (10) yields

$$\mathbb{P}\big(\mathcal{E} \,|\, (B,\, S,\, \omega(B,\, S))\big) \;\geqslant\; \left(e^{-1/n^a}\right)^{2|\mathbb{E}[C_{max}(\omega(B,\, S))]|} \left(1 - e^{-1/n^a}\right)^{|H(B,\, \omega(B,\, S))|} . \qquad (11)$$

Yet, according to (6), we have

$$\big|\, \mathbb{E}\left[C_{max}\left(\omega(B,\, S)\right)\right] \big| \;\leqslant\; d\, |C_{max}(\omega(B,\, S))| \;\leqslant\; 2d(B+2) \,.$$

Furthermore, by concavity of $x \mapsto 1 - e^{-x}$, we get

$$1 - e^{-1/n^a} \;\geqslant\; \frac{1}{n^a}\left(1 - e^{-1}\right) \;\geqslant\; \frac{1}{2n^a} \,.$$

In addition, combining (8) and (6) leads to

$$|H| \;\leqslant\; K\, |C_{max}(\omega(B,\, S))|^{\frac{d-1}{d}} \;\leqslant\; K\big(2(B+2)\big)^{\frac{d-1}{d}} \;\leqslant\; 2K\big(B+2\big)^{\frac{d-1}{d}} \,.$$

Replacing all this in equation (11), we obtain

$$\mathbb{P}\big(\mathcal{E} \,|\, (B,\, S,\, \omega(B,\, S))\big) \;\geqslant\; \exp\left(-\frac{4d\,(B+2)}{n^a}\right) \left(\frac{1}{2n^a}\right)^{2K(B+2)^{\frac{d-1}{d}}} .$$

We take the conditional expectation with respect to B, and we deduce that

$$\mathbb{P}\big(\mathcal{E} \,|\, B\big) \;\geqslant\; \exp\left(-\frac{4d\,(B+2)}{n^a}\right) \left(\frac{1}{2n^a}\right)^{2K(B+2)^{\frac{d-1}{d}}} . \qquad (12)$$

Upper bound on B: We need a control on B in order to obtain a lower bound on $\mathbb{P}(\mathcal{E})$. Define

$$b_n \;=\; \left\lceil n^a \left(-\ln\left(\frac{p_c}{2}\right)\right)\right\rceil .$$

Lemma 6 implies that

$$\mathbb{P}_{p_c/2}\big(\, |C_{max}| \leqslant b_n \big) \;\overset{n \to \infty}{\longrightarrow}\; 1 \,.$$

This entails that, for n large enough,

$$\mathbb{P}_{p_c/2}\big(\, |C_{max}| \leqslant b_n \big) \;\geqslant\; \frac{1}{2} \,.$$

Given that

$$\varphi_n(b_n) \leqslant \varphi_n\Big(n^a\Big(-\ln\Big(\frac{p_c}{2}\Big)\Big)\Big) = p_c/2\,,$$

we deduce that, for n large enough,

$$\begin{aligned}\mathbb{P}\big(B \leqslant b_n\big) &\geqslant \mathbb{P}\big(\,|C_{max}\,(\omega(b_n))| \leqslant b_n + 2\big)\\ &= \mathbb{P}_{\varphi_n(b_n)}\big(\,|C_{max}| \leqslant b_n + 2\big)\\ &\geqslant \mathbb{P}_{p_c/2}\big(\,|C_{max}| \leqslant b_n + 2\big)\\ &\geqslant \frac{1}{2}\,.\end{aligned}$$

Therefore, we can find $\kappa \geqslant 2$ such that, for all $n \geqslant 1$,

$$\mathbb{P}\big(B \leqslant \kappa n^a\big) \;\geqslant\; \frac{1}{2}\,. \tag{13}$$

Conclusion: Combining (13) with (12) gives

$$\begin{aligned}\mathbb{P}(\mathcal{E}) &\geqslant \mathbb{P}\big(\mathcal{E} \cap \{B \leqslant \kappa n^a\}\big)\\ &= \mathbb{P}\big(B \leqslant \kappa n^a\big)\mathbb{P}\big(\mathcal{E} \,|\, B \leqslant \kappa n^a\big)\\ &\geqslant \frac{1}{2}\exp\Big(-\frac{4d(\kappa n^a + 2)}{n^a} - 2K(\kappa n^a + 2)^{\frac{d-1}{d}}\ln(2n^a)\Big)\\ &\geqslant \frac{1}{2}\exp\Big(-8d\kappa - 4K\kappa(\ln 2)n^{a(d-1)/d} - 4K\kappa a(\ln n)n^{a(d-1)/d}\Big)\,.\end{aligned}$$

Now recall inequality (9) to deduce that

$$\liminf_{n\to\infty}\frac{\ln Z_n}{(\ln n)n^{a(d-1)/d}} \;\geqslant\; -4K\kappa a \;>\; -\infty\,,$$

which is the required lower bound. □

4.4 Proof of the convergence result

We are now in position to prove theorem 1 in the case of the first model.

Proof of theorem 1, case $F_n = |C_{max}|$. Let ε be such that $0 < \varepsilon < \min(p_c,\, 1 - p_c)$. We start with an upper bound on the left tail of the law of p_n. To this end, define

$$b_n^- \;=\; \big\lceil n^a\big(-\ln(p_c - \varepsilon)\big)\big\rceil\,.$$

We start by writing

$$\begin{aligned}
\mu_n\big(p_n \leqslant p_c-\varepsilon\big) \;&=\; \frac{1}{Z_n}\sum_{\omega\in\{0,1\}^{\mathbb{E}_n}} \mathbb{1}_{\{p_n(\omega)\leqslant p_c-\varepsilon\}}\mathbb{P}_{p_n(\omega)}(\omega)\\
&=\; \frac{1}{Z_n}\sum_{\omega\in\{0,1\}^{\mathbb{E}_n}} \mathbb{1}_{\big\{|C_{max}(\omega)|\geqslant b_n^-\big\}}\mathbb{P}_{p_n(\omega)}(\omega)\\
&=\; \frac{1}{Z_n}\sum_{b=b_n^-}^{n^d}\sum_{\omega\in\{0,1\}^{\mathbb{E}_n}} \mathbb{1}_{\{|C_{max}(\omega)|=b\}}\mathbb{P}_{p_n(\omega)}(\omega)\\
&=\; \frac{1}{Z_n}\sum_{b=b_n^-}^{n^d}\mathbb{P}_{\varphi_n(b)}\Big(\,|C_{max}|=b\Big)\\
&\leqslant\; \frac{1}{Z_n}\sum_{b=b_n^-}^{n^d}\mathbb{P}_{\varphi_n(b)}\Big(\,|C_{max}|\geqslant b_n^-\Big)\,.
\end{aligned}$$

Yet, the event $\big\{\,|C_{max}|\geqslant b_n^-\,\big\}$ is an increasing event, thus for all $b\geqslant b_n^-$, we have

$$\mathbb{P}_{\varphi_n(b)}\Big(\,|C_{max}|\geqslant b_n^-\Big) \;\leqslant\; \mathbb{P}_{p_c-\varepsilon}\Big(\,|C_{max}|\geqslant b_n^-\Big)\,.$$

Therefore, we get

$$\mu_n\big(p_n \leqslant p_c-\varepsilon\big) \;\leqslant\; \frac{n^d}{Z_n}\mathbb{P}_{p_c-\varepsilon}\Big(\,|C_{max}|\geqslant (-\ln(p_c-\varepsilon))n^a\Big)\,.$$

We can now use the results of lemmas 5 and 7 to obtain constants L, $L'>0$ such that, for all $n\geqslant 1$,

$$\mu_n\big(p_n \leqslant p_c-\varepsilon\big) \;\leqslant\; n^d\exp\Big(L(\ln n)n^{a(d-1)/d}-L'n^a\Big)\,.$$

Noting that $a>a(d-1)/d$, we find that

$$\limsup_{n\to\infty}\frac{1}{n^a}\ln\mu_n\big(p_n \leqslant p_c-\varepsilon\big) \;<\; 0\,.$$

We now deal with the right tail of the law of p_n by setting

$$b_n^+ \;=\; \big\lfloor n^a\big(-\ln(p_c+\varepsilon)\big)\big\rfloor$$

and by writing

$$\begin{aligned}
\mu_n\big(p_n \geqslant p_c+\varepsilon\big) \;&=\; \frac{1}{Z_n}\sum_{\omega\in\{0,1\}^{\mathbb{E}_n}} \mathbb{1}_{\{p_n(\omega)\geqslant p_c+\varepsilon\}}\mathbb{P}_{p_n(\omega)}(\omega)\\
&=\; \frac{1}{Z_n}\sum_{\omega\in\{0,1\}^{\mathbb{E}_n}} \mathbb{1}_{\big\{|C_{max}(\omega)|\leqslant b_n^+\big\}}\mathbb{P}_{p_n(\omega)}(\omega)\\
&=\; \frac{1}{Z_n}\sum_{b=0}^{b_n^+}\sum_{\omega\in\{0,1\}^{\mathbb{E}_n}} \mathbb{1}_{\{|C_{max}(\omega)|=b\}}\mathbb{P}_{p_n(\omega)}(\omega)\\
&=\; \frac{1}{Z_n}\sum_{b=0}^{b_n^+}\mathbb{P}_{\varphi_n(b)}\Big(\,|C_{max}|=b\Big)\\
&\leqslant\; \frac{1}{Z_n}\sum_{b=0}^{b_n^+}\mathbb{P}_{\varphi_n(b)}\Big(\,|C_{max}|\leqslant b_n^+\Big)\\
&\leqslant\; \frac{1}{Z_n}\sum_{b=0}^{b_n^+}\mathbb{P}_{p_c+\varepsilon}\Big(\,|C_{max}|\leqslant b_n^+\Big)\\
&\leqslant\; \frac{n^d}{Z_n}\mathbb{P}_{p_c+\varepsilon}\Big(\,|C_{max}|\leqslant b_n^+\Big)\,.
\end{aligned}$$

According to lemmas 6 and 7, there exist constants L, $L'>0$ such that, for all $n\geqslant 1$,

$$\mu_n\big(p_n \geqslant p_c+\varepsilon\big) \;\leqslant\; n^d\exp\Big(L(\ln n)n^{a(d-1)/d}-L'n^{d-1}\Big)\,,$$

which implies that

$$\limsup_{n\to\infty}\,\frac{1}{n^{d-1}}\ln\mu_n\big(p_n \geqslant p_c+\varepsilon\big) \;<\; 0\,.$$

This completes the proof of the estimate (1), which entails the convergence of p_n to p_c under the law μ_n. □

4.5 A variant on the torus

One can define a similar model on the torus of side n, which boils down to considering periodic boundary conditions on the box $\Lambda(n)$. Clusters on the torus are at least as big as in the box, so the exponential decay in the supercritical phase for the model defined on the torus immediately follows from lemma 6. The analog of lemma 5 can be proved by noting that the size of the cluster of the origin in the torus is stochastically dominated by the size of the cluster of the origin in a configuration on all $\mathbb{Z}^d$. The same proof for the lower bound on the partition function applies in the case of the torus, by adapting our geometrical lemma to extend it to subgraphs of the torus. We therefore have the same convergence of p_n to p_c when $n\to\infty$ for this alternative model.

5 The number of vertices connected to the boundary

We prove here theorem 1 in the case of the model defined with $|\mathcal{M}_n|$, which is the number of vertices connected by an open path to the boundary of the box $\partial\Lambda(n)$.

5.1 Exponential decay in the subcritical phase

Following the same method as for the first model, we start with an upper bound on the law of $|\mathcal{M}_n|$ in the subcritical regime.

Lemma 8. *For any $a > d-1$, we have the upper bound*

$$\forall p < p_c \quad \forall A > 0 \qquad \limsup_{n\to\infty} \frac{1}{n^a} \ln \mathbb{P}_p\Big(|\mathcal{M}_n| > An^a \Big) \ < \ 0 \,.$$

Proof. Take $a > d-1$, $p < p_c$ and $A > 0$. Write $\partial\Lambda(n) = \{x_1, \dots, x_t\}$ with $t = |\partial\Lambda(n)|$. If A and B are two events, then $A \circ B$ denotes the disjoint occurrence of these two events, which is defined in section 2.3 of [Gri99]. Let $\omega : \mathbb{E}_n \to \{0,1\}$ be a configuration such that $|\mathcal{M}_n(\omega)| > An^a$. Define, for $i \in \{1, \dots, t\}$,

$$n_i \ = \ \Big| C_{\Lambda(n)}(x_i) \setminus \bigcup_{j<i} C_{\Lambda(n)}(x_j) \Big| \ = \ \begin{cases} 0 \text{ if there exists } j < i \text{ such that } x_i \overset{\omega}{\longleftrightarrow} x_j \,, \\ \big| C_{\Lambda(n)}(x_i) \big| \text{ otherwise.} \end{cases}$$

We have that

$$\sum_{i=1}^t n_i \ = \ \Bigg| \bigcup_{i=1}^t C_{\Lambda(n)}(x_i) \Bigg| \ = \ |\mathcal{M}_n(\omega)| \ > \ An^a \,,$$

and

$$\omega \ \in \ \big\{ \big| C_{\Lambda(n)}(x_1) \big| \geqslant n_1 \big\} \circ \dots \circ \big\{ \big| C_{\Lambda(n)}(x_t) \big| \geqslant n_t \big\} \ .$$

Indeed, if $n_i = 0$, then the event $\big\{ \big| C_{\Lambda(n)}(x_i) \big| \geqslant n_i \big\}$ is trivial, whereas if we have $n_i > 0$ and $n_j > 0$ for some $i \neq j$, then the vertices x_i and x_j must belong to disjoint clusters. Whence the inclusion

$$\Big\{ \ |\mathcal{M}_n| > An^a \Big\} \ \subset \bigcup_{\substack{0 \leqslant n_1, \dots, n_t \leqslant n^d \\ n_1 + \dots + n_t > An^a}} \big\{ \big| C_{\Lambda(n)}(x_1) \big| \geqslant n_1 \big\} \circ \dots \circ \big\{ \big| C_{\Lambda(n)}(x_t) \big| \geqslant n_t \big\} \,.$$

Note that, for all $i \in \{1, \dots, t\}$, the event $\big\{ \big| C_{\Lambda(n)}(x_i) \big| \geqslant n_i \big\}$ is an increasing event, thus by the BK inequality, we have

$$\begin{aligned} \mathbb{P}_p\Big(|\mathcal{M}_n| > An^a \Big) \ &\leqslant \sum_{\substack{0 \leqslant n_1, \dots, n_t \leqslant n^d \\ n_1 + \dots + n_t > An^a}} \prod_{i=1}^t \mathbb{P}_p\Big(\big| C_{\Lambda(n)}(x_i) \big| \geqslant n_i \Big) \\ &\leqslant \sum_{\substack{0 \leqslant n_1, \dots, n_t \leqslant n^d \\ n_1 + \dots + n_t > An^a}} \prod_{i=1}^t \mathbb{P}_p\Big(|C(0)| \geqslant n_i \Big) \,. \end{aligned}$$

Furthermore, according to theorem 6.75 in [Gri99], for $p < p_c$, there exists a constant $\lambda(p) > 0$ such that, for all $n \geqslant 1$,

$$\mathbb{P}_p\big(|C(0)| \geqslant n \big) \ \leqslant \ e^{-n\lambda(p)} \,,$$

which is also true if $n = 0$. It follows that

$$\begin{aligned} \mathbb{P}_p\Big(|\mathcal{M}_n| > An^a \Big) \ &\leqslant \sum_{\substack{0 \leqslant n_1, \dots, n_t \leqslant n^d \\ n_1 + \dots + n_t > An^a}} \prod_{i=1}^t \exp\big(-\lambda(p) n_i \big) \\ &\leqslant \sum_{0 \leqslant n_1, \dots, n_t \leqslant n^d} \exp\big(-\lambda(p) An^a \big) \\ &\leqslant \ \big(n^d + 1 \big)^t \exp\big(-\lambda(p) An^a \big) \\ &= \ \exp\big(|\partial\Lambda(n)| \ln(n^d+1) - \lambda(p) An^a \big) \ . \end{aligned}$$

To conclude, note that

$$|\partial\Lambda(n)|\ln(n^d+1) \;=\; O\left((\ln n)n^{d-1}\right) \;=\; o(n^a)\,.$$

This completes the proof of the lemma. □

5.2 Exponential decay in the supercritical phase

We show now an upper bound in the supercritical regime.

Lemma 9. *For all $a<d$, we have*

$$\forall p>p_c\quad \forall A>0 \qquad \limsup_{n\to\infty}\frac{1}{n^{d-1}}\ln\mathbb{P}_p\Big(\,|\mathcal{M}_n|<An^a\Big)\;<\;0\,.$$

Proof. As in the proof of lemma 6, we show that

$$\forall p>p_c\quad \forall A>0 \qquad \limsup_{n\to\infty}\frac{1}{n^{d-1}}\ln\mathbb{P}_p\left(|\mathcal{M}_n|\leqslant\frac{\theta(p)n^d}{2}\right)\;<\;0\,.$$

For $d\geqslant 3$, the result follows from theorem 1.2 of [Pis96], which proves it for p larger than $\widehat{p_c}$, which was proved to be equal to p_c in [GM90]. In dimension $d=2$, the claim follows from theorem 6.1 in [ACC90]. □

5.3 Lower bound on the partition function

We establish now a lower bound on the normalization constant Z_n.

Lemma 10. *For any real a such that $0<a<d$, there exists $L>0$ such that, for all $n\geqslant 1$,*

$$Z_n\;\geqslant\;\exp\Big(-Ln^{d-a}-L(\ln n)n^{a(d-1)/d}\Big)\,. \tag{14}$$

Heuristics of the proof: We wish to apply the same technique as in the proof for the case of the largest cluster (section 4.3), by constructing a decreasing coupling between the distributions $\mathbb{P}_{\varphi_n(b)}$ for b varying from 0 (all edges open) to n^d (almost all edges closed). We monitor the evolution of the variable $|\mathcal{M}_n|$ until an instant $b=B'$ when $|\mathcal{M}_n|$ is of order B'. Then we find a set of edges $H\subset\mathbb{E}_n$ whose closure would lead to $|\mathcal{M}_n|=B'+2$ at the instant $B'+2$.

The hurdle is that, in order to find such a set H which is not too big (and thus whose closure is likely enough), we need a control on the size of the clusters which are connected to the boundary of the box at the instant B'. To obtain such a control, the idea is to monitor first the evolution of the size of the clusters connected to the boundary, to wait for an instant B when these clusters have become small enough, and then to define the instant B' in a way which ensures that it is later than B.

Proof. **Sketch of the proof:** We first define a decreasing coupling of configurations $(\omega(b,\,s))_{b,\,s}$, and we consider the first instant $(B,\,S)$ when any of the clusters connected to the boundary of the box contains at most $2B+3$ vertices. We will show that, at this instant, we have $|\mathcal{M}_n(\omega)|\geqslant B+2$. Next we will construct a second instant $(B',\,S')\geqslant(B,\,S)$ and a set of edges H such that, if the only edges of $\mathcal{M}_n$ which are closed between $(B',\,S')$ and $(B'+2,\,0)$ are the edges of H, then we have $|\mathcal{M}_n(\omega(B'+2))|=B'+2$. We will call this scenario the "happy event", and our aim is to obtain a lower bound on its probability. To this end, we will show that, with sufficiently high probability, we have $B=O(n^a)$, which implies that, from the instant $(B,\,S)$ onwards, any of the clusters connected to the boundary contains $O(n^a)$ vertices. This control will allow us to show that it is possible to find H small enough to ensure that the happy event is likely enough.

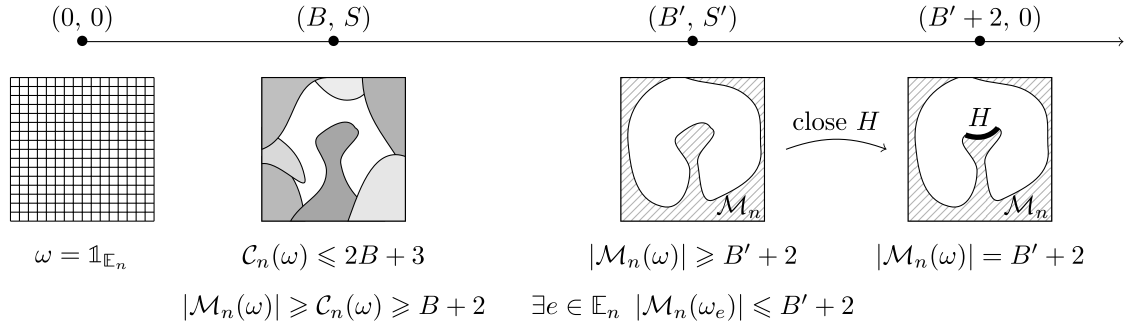

Figure 7: Illustration for the sketch of the proof of lemma 10.

Construction of the coupling and definition of B: Take $n \geqslant 2$. We construct the coupling as in the proof of lemma 7. Write $\mathbb{E}_n = \{e_1, \dots, e_r\}$ with $r = |\mathbb{E}_n|$, and consider a collection of i.i.d. random variables

$$(X_{b,e})_{b \in \{0, \dots, n^d - 1\},\, e \in \mathbb{E}_n}$$

all distributed with a Bernoulli law of parameter $\exp(-1/n^a)$. We set, for $b_0 \in \{0, \dots, n^d\}$,

$$\omega(b_0) \ : \ e \in \mathbb{E}_n \ \longmapsto \ \min_{0 \leqslant b < b_0} X_{b,e} \, ,$$

and for $b \in \{0, \dots, n^d - 1\}$ and $s_0 \in \{0, \dots, r\}$, we define

$$\omega(b, s_0) \ : \ e_s \in \mathbb{E}_n \ \longmapsto \ \begin{cases} \omega(b+1)(e_s) & \text{if } s \leqslant s_0 \, , \\ \omega(b)(e_s) & \text{otherwise.} \end{cases}$$

For a configuration $\omega : \mathbb{E}_n \to \{0, 1\}$, we denote

$$\mathcal{C}_n(\omega) \ = \ \max_{v \in \partial \Lambda(n)} \left| C_{\Lambda(n)}(v, \omega) \right| \, ,$$

the size of the largest cluster connected to the boundary of the box in the configuration ω. We consider the pair of random variables

$$(B, S) \ = \ \min \Big\{ (b, s) \in \big\{0, \dots, n^d - 2\big\} \times \{0, \dots, r\} \ : \quad \mathcal{C}_n\left(\omega(b, s)\right) \leqslant 2b + 3 \Big\} \, ,$$

which is well-defined because $\mathcal{C}_n(\omega(n^d - 2, 0)) \leqslant n^d$. Let us show that, at this instant (B, S), we have

$$\mathcal{C}_n\left(\omega(B, S)\right) \ \geqslant \ B + 2 \, . \tag{15}$$

We distinguish several cases.
• If $S \geqslant 1$ then, (B, S) being minimal, we have $\mathcal{C}_n\left(\omega(B, S - 1)\right) \geqslant 2B + 4$. To obtain (15), note that closing a single edge can at most divide $\mathcal{C}_n$ by a factor two.
• If $B \neq 0$ and $S = 0$ then, by minimality of (B, S), we have that

$$\mathcal{C}_n\left(\omega(B - 1, r)\right) \ \geqslant \ 2(B - 1) + 4 \ = \ 2B + 2 \ \geqslant \ B + 2 \, ,$$

which implies inequality (15), because the configurations $\omega(B - 1, r)$ and $\omega(B, 0)$ are identical.
• The case $(B, S) = (0, 0)$ never occurs because we have $\mathcal{C}_n(\omega(0, 0)) = n^d > 3$.
We have shown that (15) holds, which entails

$$\left|\mathcal{M}_n\left(\omega(B, S)\right)\right| \ \geqslant \ \mathcal{C}_n\left(\omega(B, S)\right) \ \geqslant \ B + 2 \, . \tag{16}$$

Construction of the second instant B': We now consider

$$(B',\, S') \;=\; \min\Big\{ (b,\, s) \geqslant (B,\, S) \;:\; \quad \exists e \in \mathbb{E}_n \quad |\mathcal{M}_n\,(\omega(b,\, s)_e)| \;\leqslant\; b+2 \Big\}\,.$$

The fact that $B \leqslant n^d - 2$ and $\big|\mathcal{M}_n\,\big(\omega(n^d-2,\, r)\big)\big| \leqslant n^d$ ensures that $(B',\, S')$ is well-defined and that we have $B' \leqslant n^d - 2$. Let us show, by distinguishing several cases, that

$$|\mathcal{M}_n\,(\omega(B',\, S'))| \;\geqslant\; B'+2\,. \tag{17}$$

- If $(B',\, S') = (B,\, S)$, then the claim follows from (16).
- If $(B',\, S') > (B,\, S)$ and $S' = 0$, then the minimality of $(B',\, S')$ implies that

$$B'-1+2 \;<\; |\mathcal{M}_n\,(\omega(B'-1,\, r))| \;=\; |\mathcal{M}_n\,(\omega(B',\, S'))|\,.$$

- Else if $(B',\, S') > (B,\, S)$ and $S' \neq 0$, then by minimality of $(B',\, S')$, we know that for all $e \in \mathbb{E}_n$,

$$|\mathcal{M}_n\,(\omega(B',\, S'-1)_e)| \;>\; B'+2\,,$$

which entails in particular that $|\mathcal{M}_n\,(\omega(B',\, S'))| > B'+2$, because the configuration $\omega(B',\, S')$ is obtained from $\omega(B',\, S'-1)$ by closing at most one edge.
We conclude that (17) holds in all cases.

Construction of the happy event: We now wish to define a set of edges H that we want to be closed between the configuration $\omega(B',\, S')$ and the configuration $\omega(B'+2,\, 0)$ in order to have

$$|\mathcal{M}_n(\omega(B'+2))| \;=\; B'+2\,.$$

By definition of $(B',\, S')$, we can consider an edge $e \in \mathbb{E}_n$ such that

$$|\mathcal{M}_n\,(\omega(B',\, S')_e)| \leqslant B'+2\,. \tag{18}$$

We choose this edge e minimal (for the order $e_1,\, \dots,\, e_r$ we have considered on $\mathbb{E}_n$) among the edges satisfying (18), which ensures that e depends only on B' and $\omega(B',\, S')$. We then construct the set H by distinguishing two cases depending on whether this inequality (18) is strict or not.

- In case of equality in (18), we take $H = \{e\}$.
- Assume that equation (18) is a strict inequality. It follows from (17) that

$$|\mathcal{M}_n\,(\omega(B',\, S')_e)| \;<\; |\mathcal{M}_n\,(\omega(B',\, S'))|\,,$$

which means that closing the edge e changes the number of vertices connected to the boundary of the box. Consequently, one end of the edge e, say v, must be disconnected from $\partial\Lambda(n)$ when closing e in the configuration $\omega(B',\, S')$. Write C_v for the cluster of v in the configuration $\omega(B',\, S')_e$. We have, using (17),

$$|\mathcal{M}_n\,(\omega(B',\, S')_e)| \;=\; |\mathcal{M}_n\,(\omega(B',\, S'))| - |C_v| \;\geqslant\; B'+2-|C_v|\,.$$

Combining this with the (strict) inequality (18) yields

$$1 \;\leqslant\; B'+2-|\mathcal{M}_n(\omega(B',\, S')_e)| \;\leqslant\; |C_v|\,.$$

Applying lemma 1 to the graph $(C_v,\, \mathbb{E}\,[C_v])$ and the vertex v, we can choose a set $H \subset \mathbb{E}\,[C_v]$ satisfying

$$|H| \;\leqslant\; K\,|C_v|^{\frac{d-1}{d}} \tag{19}$$

and such that the cluster of v in the graph $(C_v,\, \mathbb{E}\,[C_v] \setminus H)$ contains exactly $B'+2-|\mathcal{M}_n(\omega(B',\, S')_e)|$ vertices. We then have

$$|\mathcal{M}_n\,(\omega(B',\, S')_H)| \;=\; B'+2\,.$$

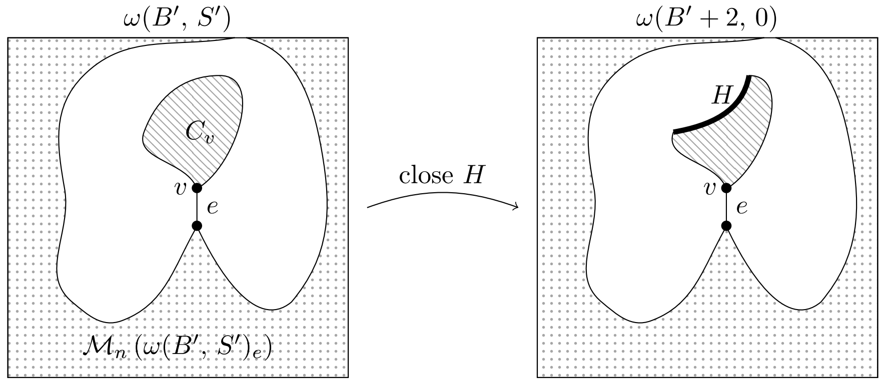

Figure 8: If (18) is a strict inequality, then closing the edge e in the configuration $\omega(B',\, S')$ changes the number $|\mathcal{M}_n|$ of vertices connected to the boundary of the box. This means that one end of the edge e, say v, happens to be disconnected from the boundary when e is closed. We then choose a subset H of the edges of the cluster C_v which is disconnected by the closure of e, such that closing all the edges of H and no other edges of $\mathbb{E}\,[\mathcal{M}_n]$ between $(B',\, S')$ and $(B'+2,\, 0)$ implies $|\mathcal{M}_n\,(\omega(B'+2))| = B'+2$.

The edge e (and thus the vertex v) depends only on B' and $\omega(B',\, S')$, thus we can choose such a set H which also depends only on B' and $\omega(B',\, S')$. Besides, given that the vertex v was connected to the boundary $\partial\Lambda(n)$ in the configuration $\omega(B',\, S')$, we have the following control over $|C_v|$:

$$|C_v| \;=\; \big|C_{\Lambda(n)}\,(v,\, \omega(B',\, S')_e)\big| \;\leqslant\; \big|C_{\Lambda(n)}\,(v,\, \omega(B',\, S'))\big| \;\leqslant\; \mathcal{C}_n\,(\omega(B',\, S'))\;. \tag{20}$$

Note now that $\mathcal{C}_n(\omega)$ is a decreasing function of ω and that, by definition, $(B',\, S') \geqslant (B,\, S)$, whence

$$\mathcal{C}_n\,(\omega(B',\, S')) \;\leqslant\; \mathcal{C}_n\,(\omega(B,\, S)) \;\leqslant\; 2B+3\,. \tag{21}$$

Combining (20) and (21), we get

$$|C_v| \;\leqslant\; 2B+3\,, \tag{22}$$

and therefore the upper bound (19) becomes

$$|H| \;\leqslant\; K\big(2B+3\big)^{\frac{d-1}{d}}\;. \tag{23}$$

To sum up these two cases, we have defined a (random) set of edges $H \subset \mathbb{E}_n$ whose size is controlled by (23) and which satisfies

$$|\mathcal{M}_n\,(\omega(B',\, S')_H)| \;=\; B'+2\,.$$

Therefore, if the edges of H and no other edges of $\mathbb{E}\,[\mathcal{M}_n\,(\omega(B',\, S'))]$ are closed between the configurations $\omega(B',\, S')$ and $\omega(B'+2,\, 0)$, then we have

$$|\mathcal{M}_n\,(\omega(B'+2))| \;=\; B'+2\,. \tag{24}$$

This leads us to consider the event

$$\mathcal{E} \;=\; \left\{\begin{array}{l} \forall s > S' \quad e_s \in \mathbb{E}\,[\mathcal{M}_n\,(\omega(B',\, S'))] \Rightarrow X_{B',\, e_s} = 1 \\ \forall e \in H \quad X_{B'+1,\, e} = 0 \\ \forall e \in \mathbb{E}\,[\mathcal{M}_n\,(\omega(B',\, S'))] \setminus H \quad X_{B'+1,\, e} = 1 \end{array}\right\}$$

which, if it occurs, implies (24). Also,

$$Z_n \;=\; \sum_{b=1}^{n^d} \mathbb{P}_{\varphi_n(b)}\big(\,|\mathcal{M}_n| = b\big) \;=\; \mathbb{P}\Big(\exists b \in \big\{1,\, \ldots,\, n^d\big\}\;\, |\mathcal{M}_n\,(\omega(b))| = b\Big) \;\geqslant\; \mathbb{P}\,(\mathcal{E})\;.$$

Conditional probability of the happy event: As in the proof of lemma 7, we consider $(b_0,\, b_0',\, s_0')$ and $\omega_0 : \mathbb{E}_n \to \{0,1\}$ such that

$$\mathbb{P}\big(\mathcal{C}_{b_0,\, b_0',\, s_0',\, \omega_0}\big) \;>\; 0 \quad \text{where} \quad \mathcal{C}_{b_0,\, b_0',\, s_0',\, \omega_0} \;=\; \Big\{\, (B,\, B',\, S') = (b_0,\, b_0',\, s_0') \quad \text{and} \quad \omega(B',\, S') = \omega_0 \,\Big\}\,.$$

This event $\mathcal{C}_{b_0,\, b_0',\, s_0',\, \omega_0}$ depends only on the variables $X_{b,\, e_s}$ with $(b,\, s) \leqslant (b_0',\, s_0')$ and, conditionally on this event, the event $\mathcal{E}$ depends only on the variables $X_{b,\, e_s}$ for $(b_0',\, s_0') < (b,\, s) < (b_0' + 2,\, 0)$. What's more, the set H depends only on B' and $\omega(B',\, S')$, which allows us to write $H = H(B',\, \omega(B',\, S')$. Therefore, we have

$$\begin{aligned}
\mathbb{P}\big(\mathcal{E} \,|\, \mathcal{C}_{b_0,\, b_0',\, s_0',\, \omega_0}\big) \;&=\; \prod_{\substack{s > s_0' \\ e_s \in \mathbb{E}[\mathcal{M}_n(\omega_0)]}} \mathbb{P}\big(\, X_{b_0',\, e_s} = 1 \,\big) \times \prod_{e \in H(b_0',\, \omega_0)} \mathbb{P}\big(\, X_{b_0'+1,\, e} = 0 \,\big) \\
&\qquad \times \prod_{e \in \mathbb{E}[\mathcal{M}_n(\omega_0)] \setminus H(b_0',\, \omega_0)} \mathbb{P}\big(\, X_{b_0'+1,\, e} = 1 \,\big) \\
&\geqslant\; \Big(e^{-1/n^a}\Big)^{2|\mathbb{E}[\mathcal{M}_n(\omega_0)]|} \Big(1 - e^{-1/n^a}\Big)^{|H(b_0',\, \omega_0)|} \\
&\geqslant\; \exp\left(-\frac{2\,|\mathbb{E}_n|}{n^a}\right)\left(\frac{1}{2n^a}\right)^{|H(b_0',\, \omega_0)|} \\
&\geqslant\; \exp\left(-2dn^{d-a} - 2a(\ln n)\,|H(b_0',\, \omega_0)|\right)\,.
\end{aligned}$$

Using the upper bound (23) leads to

$$\mathbb{P}\Big(\mathcal{E} \,|\, (B,\, B',\, S',\, \omega(B',\, S'))\Big) \;\geqslant\; \exp\Big(-2dn^{d-a} - 2Ka(\ln n)\,(2B+3)^{\frac{d-1}{d}}\Big)\,.$$

Taking the conditional expectation with respect to B, we obtain

$$\mathbb{P}\big(\mathcal{E} \,|\, B\big) \;\geqslant\; \exp\Big(-2dn^{d-a} - 2Ka(\ln n)\,(2B+3)^{\frac{d-1}{d}}\Big)\,. \tag{25}$$

Upper bound on B: It follows from lemma 5 that, for n large enough,

$$\mathbb{P}_{p_c/2}\left(|C_{max}| \leqslant n^a \left(-\ln\left(\frac{p_c}{2}\right)\right)\right) \;\geqslant\; \frac{1}{2}\,.$$

Since $\mathcal{C}_n \leqslant |C_{max}|$, using the same technique as in section 4.3, we deduce that, for n large enough,

$$\mathbb{P}\left(B \leqslant n^a \left(-\ln\left(\frac{p_c}{2}\right)\right)\right) \;\geqslant\; \frac{1}{2}\,.$$

Therefore, we can find $\kappa \geqslant 2$ such that, for all $n \geqslant 1$,

$$\mathbb{P}\big(B \leqslant \kappa n^a\big) \;\geqslant\; \frac{1}{2}\,. \tag{26}$$

Conclusion: Combining (25) and (26) yields

$$\begin{aligned}
\mathbb{P}\,(\mathcal{E}) \;&\geqslant\; \mathbb{P}\big(B \leqslant \kappa n^a\big)\,\mathbb{P}\Big(\mathcal{E} \,|\, B \leqslant \kappa n^a\Big) \\
&\geqslant\; \frac{1}{2}\exp\Big(-2dn^{d-a} - 2Ka(\ln n)\,(2\kappa n^a + 3)^{\frac{d-1}{d}}\Big) \\
&\geqslant\; \frac{1}{2}\exp\Big(-2dn^{d-a} - 8K\kappa a(\ln n) n^{a(d-1)/d}\Big)\,,
\end{aligned}$$

which is the required lower bound. □

5.4 Proof of the convergence result

Once we have established the lower bound for Z_n (lemma 10) and the results of exponential decay in the subcritical (lemma 8) and supercritical phases (lemma 9), the proof of the second case of theorem 1 is identical to the proof for the model with C_{max}, which is given in detail in section 4.4.

6 The distribution of the cluster sizes

The goal of this section is to prove the remaining part of theorem 1, namely the case of $F_n = B_n^b$, along with theorem 2. To this end, we fix two real numbers a and b such that

$$\frac{5d}{6} < a < d \qquad \text{and} \qquad 0 < b < \frac{2a}{d} - \frac{5}{3}\,. \tag{27}$$

Recall the definition, for $\omega : \mathbb{E}_n \to \{0,1\}$, of

$$B_n^b(\omega) \;=\; \Big|\Big\{x \in \Lambda(n),\ \big|C_{\Lambda(n)}(x,\omega)\big| \geqslant n^b\Big\}\Big|\,. \tag{28}$$

Consider also, for $\omega : \mathbb{E}^d \to \{0,1\}$ and $X \subset \mathbb{Z}^d$,

$$A_n^b(X) \;=\; A_n^b(X,\omega) \;=\; \Big|\Big\{x \in X,\ |C(x,\omega)| \geqslant n^b\Big\}\Big|\,. \tag{29}$$

6.1 Preliminary results

We begin with three elementary lemmas.

Lemma 11. *For all $\eta > 0$ and $r < \eta/2$, and for any $p, q \in [0,1]$ such that $\eta < q < 1-\eta$ and $|p-q| < r$, we have*

$$\forall n \geqslant 1 \quad \forall \omega \in \{0,1\}^{\mathbb{E}_n} \quad \mathbb{P}_p(\omega) \;\geqslant\; \mathbb{P}_q(\omega)\exp\left(-\frac{2dn^d r}{\eta}\right).$$

Proof. Take η, r, p, q, n and ω satisfying the aforementioned conditions. We write

$$\begin{aligned}
\mathbb{P}_p(\omega) \;&=\; \mathbb{P}_q(\omega)\left(\frac{p}{q}\right)^{o(\omega)}\left(\frac{1-p}{1-q}\right)^{|\mathbb{E}_n|-o(\omega)}\\
&\geqslant\; \mathbb{P}_q(\omega)\left(1-\frac{r}{q}\right)^{o(\omega)}\left(1-\frac{r}{1-q}\right)^{|\mathbb{E}_n|-o(\omega)}\\
&\geqslant\; \mathbb{P}_q(\omega)\left(1-\frac{r}{\eta}\right)^{|\mathbb{E}_n|}\\
&\geqslant\; \mathbb{P}_q(\omega)\exp\left(|\mathbb{E}_n|\ln\left(\frac{2r}{\eta}\left(\frac{1}{2}\right)+\left(1-\frac{2r}{\eta}\right)\times 1\right)\right)\\
&\geqslant\; \mathbb{P}_q(\omega)\exp\left(|\mathbb{E}_n|\,\frac{2r}{\eta}\ln\left(\frac{1}{2}\right)\right)\\
&\geqslant \mathbb{P}_q(\omega)\exp\left(-\frac{2r\,|\mathbb{E}_n|}{\eta}\right),
\end{aligned}$$

which completes the proof, noting that $|\mathbb{E}_n| \leqslant dn^d$. □

Lemma 12. *We have the following bounds on B_n^b:*

$$\forall n \geqslant 1 \quad \forall \omega \in \{0,1\}^{\mathbb{E}^d} \quad A_n^b(\Lambda(n),\omega) - 4dn^{b+d-1} \;\leqslant\; B_n^b(\omega) \;\leqslant\; A_n^b(\Lambda(n),\omega)\,. \tag{30}$$

Proof. The right inequality comes from the fact that, for any $x \in \Lambda(n)$, we have $C_{\Lambda(n)}(x) \subset C(x)$. To get the left inequality, note that, if $x \in \Lambda(n - 2\lceil n^b \rceil)$, then the events

$$\left\{ \left| C_{\Lambda(n)}(x) \right| \geqslant n^b \right\} \qquad \text{and} \qquad \left\{ |C(x)| \geqslant n^b \right\}$$

coincide, whence

$$B_n^b(\omega) \;\geqslant\; A_n^b\left(\Lambda\left(n - 2\lceil n^b \rceil\right), \omega\right) \;\geqslant\; A_n^b(\Lambda(n), \omega) - \Big(|\Lambda(n)| - \left|\Lambda\left(n - 2\lceil n^b \rceil\right)\right| \Big).$$

Noting that

$$|\Lambda(n)| - \left|\Lambda\left(n - 2\lceil n^b \rceil\right)\right| \;\leqslant\; 2dn^{d-1}\lceil n^b \rceil \;\leqslant\; 4dn^{b+d-1}$$

then concludes the proof. □

Lemma 13. *On the one hand, we have the following upper bound on $\mathbb{E}_p\left[A_n^b(\Lambda(m))\right]$ in the subcritical phase:*

$$\forall p < p_c \quad \exists \lambda(p) > 0 \quad \forall n, m \geqslant 1 \qquad \mathbb{E}_p\big[A_n^b(\Lambda(m))\big] \;\leqslant\; m^d \exp\left(-n^b \lambda(p)\right). \tag{31}$$

On the other hand, we have the following lower bound in the supercritical phase:

$$\forall p > p_c \quad \forall n, m \geqslant 1 \qquad \mathbb{E}_p\big[A_n^b(\Lambda(m))\big] \;\geqslant\; m^d \theta(p). \tag{32}$$

Proof. We write

$$\mathbb{E}_p\big[A_n^b(\Lambda(m))\big] \;=\; \sum_{x \in \Lambda(m)} \mathbb{P}_p\big(|C(x)| \geqslant \lceil n^b \rceil \big) \;=\; m^d \mathbb{P}_p\big(|C(0)| \geqslant \lceil n^b \rceil \big).$$

The upper bound (31) follows immediately from theorem 6.75 in [Gri99] which states that

$$\forall p < p_c \quad \exists \lambda(p) > 0 \quad \forall n \geqslant 1 \quad \mathbb{P}_p\big(|C(0)| \geqslant n \big) \;\leqslant\; e^{-n\lambda(p)}.$$

In addition to this, note that

$$\mathbb{P}_p\big(|C(0)| \geqslant \lceil n^b \rceil \big) \;\geqslant\; \mathbb{P}_p\big(|C(0)| = \infty \big) \;=\; \theta(p),$$

which proves the lower bound (32). □

6.2 Exponential decay in the subcritical phase

We prove again an exponential decay in the subcritical regime.

Lemma 14. *We have the upper bound*

$$\forall p < p_c \quad \forall A > 0 \qquad \limsup_{n \to \infty} \frac{1}{n^{2a-bd-d}} \ln \mathbb{P}_p\Big(B_n^b > An^a \Big) \;<\; 0.$$

Proof. Take $A > 0$ and $p < p_c$. Given the upper bound $B_n^b \leqslant A_n^b$ in (30), it is enough to show the result with A_n^b instead of B_n^b. We therefore deal with A_n^b in the rest of the proof. We divide $\Lambda(n)$ in boxes (all cubic boxes except maybe the boxes on the boundaries), as represented on figure 9. Let us explain this construction. Set

$$N \;=\; \lceil 2n^b \rceil \qquad \text{and} \qquad M \;=\; 3 + \left\lfloor \frac{n}{2N} \right\rfloor.$$

We can decompose $\mathbb{Z}^d$ in the partition

$$\mathbb{Z}^d \;=\; \bigsqcup_{z \in \mathbb{Z}^d} \Big(Nz + \Lambda(N) \Big) \;=\; \bigsqcup_{\substack{x \in \{0,1\}^d \\ y \in \mathbb{Z}^d}} \Big(N(x + 2y) + \Lambda(N) \Big).$$

We define, for $x \in \{0,1\}^d$ and $y \in \mathbb{Z}^d$, the box

$$\mathcal{B}_{x,y} \;=\; \Big(N(x+2y) + \Lambda(N)\Big) \cap \Lambda(n)\,.$$

These boxes have the following properties:

(i) If $z \in \mathcal{B}_{x,y}$ and $z' \in \mathcal{B}_{x,y'}$ are in two boxes carrying the same label x but different labels $y \neq y'$, then we have $\|z - z'\|_\infty > N$.

(ii) The boxes $\mathcal{B}_{x,y}$ with $y \notin \Lambda(M)$ are empty. Indeed, for $x \in \{0,1\}^d$, $y \in \mathbb{Z}^d$ and $z \in \Lambda(N)$, we have

$$y \notin \Lambda(M) \;\Rightarrow\; \|y\|_\infty \geqslant \frac{M}{2} \geqslant \frac{n}{4N} + 1 \;\Rightarrow\; \|N(x+2y)+z\|_\infty \geqslant -N + \frac{n}{2} + 2N - \frac{N}{2} > \frac{n}{2}$$
$$\Rightarrow\; N(x+2y)+z \notin \Lambda(n).$$

(iii) Conversely, if $y \in \Lambda(M-5)$, then the box $\mathcal{B}_{x,y}$ is full, i.e., it contains N^d vertices. Indeed, for $x \in \{0,1\}^d$, $y \in \mathbb{Z}^d$ and $z \in \Lambda(N)$, we have

$$y \in \Lambda(M-5) \;\Rightarrow\; \|y\|_\infty \leqslant \frac{M-5}{2} \leqslant \frac{n}{4N} - 1 \;\Rightarrow\; \|N(x+2y)+z\|_\infty \leqslant N + \frac{n}{2} - 2N + \frac{N}{2} < \frac{n}{2}$$
$$\Rightarrow\; N(x+2y)+z \in \Lambda(n).$$

Thus we obtain a partition of the box $\Lambda(n)$,

$$\Lambda(n) \;=\; \bigsqcup_{x\in\{0,1\}^d} \mathcal{D}_x \qquad \text{where} \qquad \mathcal{D}_x \;=\; \bigsqcup_{y\in\Lambda(M)} \mathcal{B}_{x,y}\,,$$

with each set $\mathcal{D}_x$ corresponding to a hatching pattern on figure 9. This allows us to write

$$\begin{aligned}
\mathbb{P}_p\Big(A_n^b(\Lambda(n)) > An^a\Big) \;&=\; \mathbb{P}_p\left(\sum_{x\in\{0,1\}^d} A_n^b(\mathcal{D}_x) > An^a\right)\\
&\leqslant\; \mathbb{P}_p\left(\exists x \in \{0,1\}^d \quad A_n^b(\mathcal{D}_x) > \frac{An^a}{2^d}\right)\\
&\leqslant\; \sum_{x\in\{0,1\}^d} \mathbb{P}_p\left(A_n^b(\mathcal{D}_x) > \frac{An^a}{2^d}\right). \qquad (33)
\end{aligned}$$

For any $x \in \{0,1\}^d$, the upper bound (31) yields

$$\mathbb{E}_p\big[A_n^b(\mathcal{D}_x)\big] \;\leqslant\; \mathbb{E}_p\big[A_n^b(\Lambda(n))\big] \;\leqslant\; n^d \exp\big(-\lambda(p)n^b\big) \;=\; o\left(n^a\right)\,.$$

This implies that, for n large enough,

$$\forall x \in \{0,1\}^d \qquad \mathbb{E}_p\big[A_n^b(\mathcal{D}_x)\big] \;\leqslant\; \frac{An^a}{2^{d+1}}\,.$$

Consequently, we have

$$\mathbb{P}_p\left(A_n^b(\mathcal{D}_x) > \frac{An^a}{2^d}\right) \;\leqslant\; \mathbb{P}_p\left(A_n^b(\mathcal{D}_x) - \mathbb{E}_p\big[A_n^b(\mathcal{D}_x)\big] > \frac{An^a}{2^{d+1}}\right)\,. \qquad (34)$$

Note now that the above-mentioned property (i) entails that, for $z \in \mathcal{B}_{x,y}$ and $z' \in \mathcal{B}_{x,y'}$ with $y \neq y'$, the events $\big\{\,|C(z)| \geqslant n^b\big\}$ and $\big\{\,|C(z')| \geqslant n^b\big\}$ are independent of each other. The following sum is thus a sum of independent variables:

$$\forall x \in \{0,1\}^d \qquad A_n^b(\mathcal{D}_x) \;=\; \sum_{y\in\Lambda(M)} A_n^b(\mathcal{B}_{x,y})\,.$$

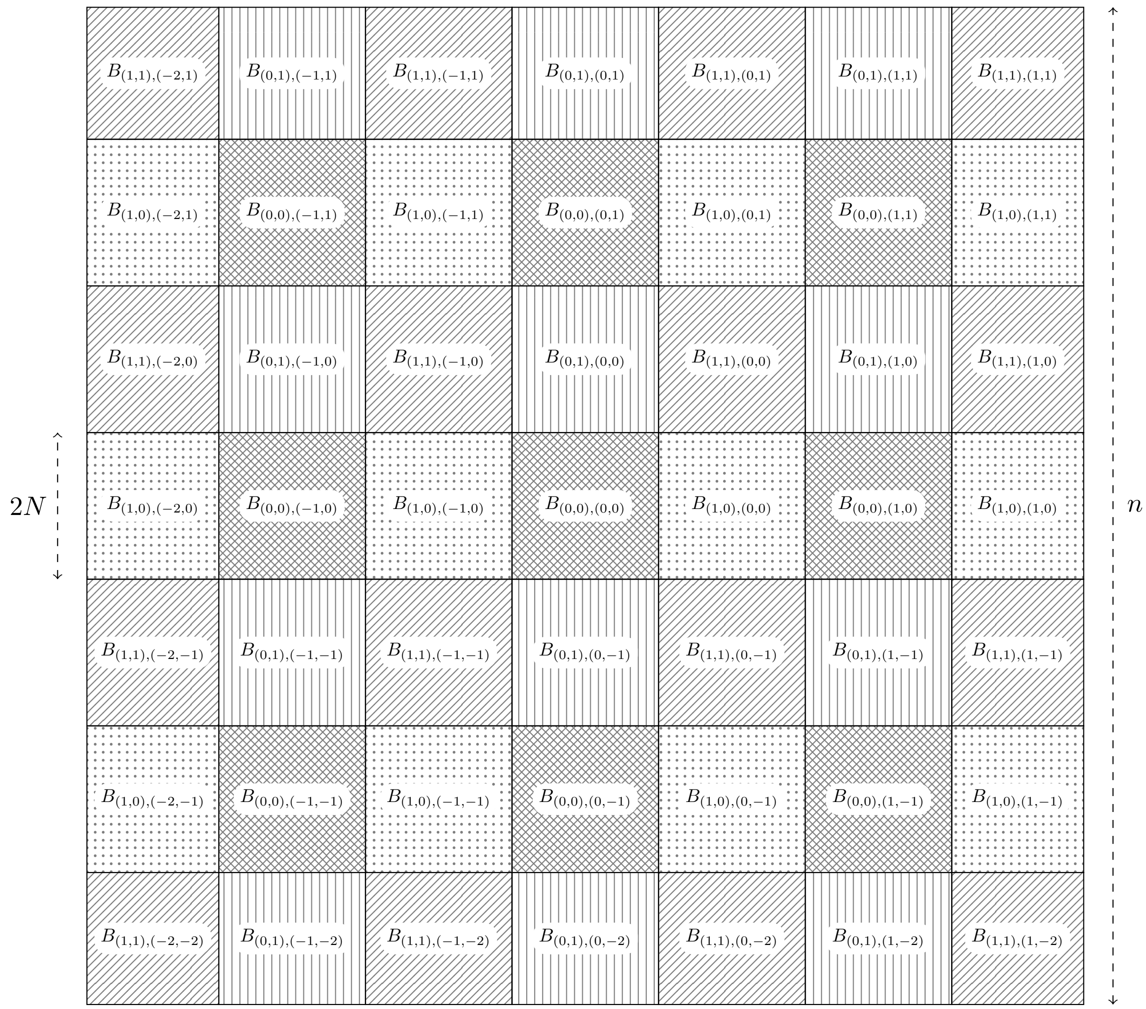

Figure 9: Partition of $\Lambda(n)$ in 2^d subsets $\mathcal{D}_x$ each corresponding to a hatching pattern, and each $\mathcal{D}_x$ being in turn partitioned into boxes $\mathcal{B}_{x,y}$ with $y \in \Lambda(M)$.

What's more, the variables $A_n^b(\mathcal{B}_{x,y})$ take their values in $\{0, \ldots, N^d\}$, and the number of variables involved in the sum is $|\Lambda(M)|$. Therefore, applying Hoeffding's inequality (see [Hoe63]) yields that, for n large enough,

$$\mathbb{P}_p\left(A_n^b(\mathcal{D}_x) - \mathbb{E}_p\big[A_n^b(\mathcal{D}_x)\big] > \frac{An^a}{2^{d+1}}\right) \leqslant \exp\left(-\frac{2A^2 n^{2a}}{2^{2d+2}\,|\Lambda(M)|\,N^{2d}}\right).$$

Yet, from the definition of M and N, we have that, as $n \to \infty$,

$$\frac{n^{2a}}{|\Lambda(M)|\,N^{2d}} \sim \frac{n^{2a}}{N^{2d}}\left(\frac{2N}{n}\right)^d \sim \frac{2^d n^{2a-d}}{N^d} \sim n^{2a-bd-d},$$

whence, for n large enough,

$$\mathbb{P}_p\left(A_n^b(\mathcal{D}_x) - \mathbb{E}_p\big[A_n^b(\mathcal{D}_x)\big] > \frac{An^a}{2^{d+1}}\right) \leqslant \exp\left(-\frac{A^2}{2^{2d+2}} n^{2a-bd-d}\right). \tag{35}$$

Combining (33), (34) and (35), we obtain that, for n large enough,

$$\mathbb{P}_p\Big(A_n^b > An^a\Big) \;\leqslant\; 2^d \exp\left(-\frac{A^2}{2^{2d+2}}n^{2a-bd-d}\right),$$

which, given the upper bound in (30), implies the desired inequality. □

6.3 Exponential decay in the supercritical phase

We now deal with the deviations in the regime $p > p_c$.

Lemma 15. *We have the upper bound*

$$\forall p > p_c \quad \forall A > 0 \qquad \limsup_{n\to\infty} \frac{1}{n^{d-bd}} \ln \mathbb{P}_p\Big(B_n^b < An^a\Big) \;<\; 0\,.$$

Proof. We use the same partition of $\Lambda(n)$ in boxes as in the proof of lemma 14. Using the lower bound in (30), we have

$$\begin{aligned}\mathbb{P}_p\Big(B_n^b < An^a\Big) \;&\leqslant\; \mathbb{P}_p\Big(A_n^b(\Lambda(n)) < An^a + 4dn^{b+d-1}\Big)\\ &\leqslant\; \mathbb{P}_p\left(\forall x\in\{0,1\}^d \;\; A_n^b(\mathcal{D}_x) < An^a + 4dn^{b+d-1}\right)\\ &\leqslant\; \mathbb{P}_p\Big(A_n^b(\mathcal{D}_0) < An^a + 4dn^{b+d-1}\Big)\,. \end{aligned} \tag{36}$$

According to the property (iii) of the partition in boxes, the set $\mathcal{D}_0$ contains at least $(M-5)^d$ boxes of cardinality N^d. In combination with (32), this yields

$$\mathbb{E}_p\left[A_n^b(\mathcal{D}_0)\right] \;\geqslant\; (M-5)^d N^d\theta(p) \;\sim\; M^dN^d\theta(p) \;\sim\; \left(\frac{n}{2N}\right)^d N^d\theta(p) \;\sim\; \frac{n^d}{2^d}\theta(p)\,.$$

Since $a < d$ and $b < 1$, this implies that, for n large enough,

$$An^a + 4dn^{b+d-1} \;\leqslant\; \frac{n^d}{2^{d+2}}\theta(p) \;\leqslant\; \frac{1}{2}\mathbb{E}_p\left[A_n^b(\mathcal{D}_0)\right]\,. \tag{37}$$

Inequality (36) therefore becomes

$$\mathbb{P}_p\Big(B_n^b < An^a\Big) \;\leqslant\; \mathbb{P}_p\left(A_n^b(\mathcal{D}_0) - \mathbb{E}_p\left[A_n^b(\mathcal{D}_0)\right] < -\frac{n^d}{2^{d+2}}\theta(p)\right)\,.$$

We apply once again Hoeffding's inequality, which gives us

$$\mathbb{P}_p\Big(B_n^b < An^a\Big) \;\leqslant\; \exp\left(-\frac{2n^{2d}\theta(p)^2}{2^{2d+4}\left|\Lambda(M)\right| N^{2d}}\right)\,.$$

Yet, we have

$$\frac{n^{2d}}{\left|\Lambda(M)\right| N^{2d}} \;\sim\; \frac{n^{2d}}{N^{2d}}\left(\frac{n}{2N}\right)^d \;\sim\; \frac{2^dn^d}{N^d} \;\sim\; n^{d-bd}\,,$$

which completes the proof of the lemma. □

6.4 Lower bound on the partition function

It remains to prove a lower bound on the normalization constant Z_n. This is done in the following lemma:

Lemma 16. *For all real parameters a and b satisfying the conditions (27), we have*

$$\liminf_{n\to\infty} \frac{\ln Z_n}{n^{2d/3}} \;>\; -\infty\,. \tag{38}$$

Proof. The function

$$f_n \;:\; p \;\longmapsto\; \varphi_n\left(\mathbb{E}_p\left[B_n^b\right]\right) \;=\; \exp\left(-\frac{\mathbb{E}_p\left[B_n^b\right]}{n^a}\right)$$

is continuous and decreasing on $[0,1]$, and it takes its values in $[0,1]$. Thus it admits a unique fixed point in this interval, which we denote by q_n. On the one hand, for $p<p_c$, combining the upper bound in (30) with the inequality (31), we get

$$f_n(p) \;\geqslant\; \varphi_n\left(\mathbb{E}_p\left[A_n^b(\Lambda(n))\right]\right) \;\geqslant\; \exp\left(-n^{d-a}e^{-\lambda(p)n^b}\right) \;\stackrel{n\to\infty}{\longrightarrow}\; 1\,.$$

On the other hand, for $p>p_c$, it follows from the lower bound in (30), the inequality (32) and the conditions (27) that

$$f_n(p) \;\leqslant\; \exp\left(-\frac{\mathbb{E}_p\left[A_n^b(\Lambda(n))\right]-4dn^{b+d-1}}{n^a}\right) \;\leqslant\; \exp\Big(-n^{d-a}\theta(p)+4dn^{b-a+d-1}\Big) \;\stackrel{n\to\infty}{\longrightarrow}\; 0\,.$$

We conclude that q_n converges to p_c when n tends to infinity. Since q_n is never equal to 0 nor to 1, and given that $0<p_c<1$, there exists $\eta>0$ such that

$$\forall n\geqslant 1 \qquad \eta \;<\; q_n \;<\; 1-\eta\,.$$

Consider now

$$r_n \;=\; \frac{1}{n^a}\sqrt{2\mathrm{Var}_{q_n}\left[B_n^b\right]}\,,$$

where Var_{q_n} denotes the variance under the law $\mathbb{P}_{q_n}$. The key step of our proof is to bound this variance, using the fact that, if two vertices $x,y\in\Lambda(n)$ are such that $|x-y|_1\geqslant 2n^b$, then the events $\left\{\left|C_{\Lambda(n)}(x)\right|\geqslant n^b\right\}$ and $\left\{\left|C_{\Lambda(n)}(y)\right|\geqslant n^b\right\}$ are independent of each other. This independence property allows us to write

$$\begin{aligned}
\mathrm{Var}_{q_n}\left[B_n^b\right] \;&=\; \mathrm{Var}_{q_n}\left[\sum_{x\in\Lambda(n)} \mathbb{1}_{\left\{\left|C_{\Lambda(n)}(x)\right|\geqslant n^b\right\}}\right]\\
&=\; \sum_{x,y\in\Lambda(n)} \mathrm{Cov}\left[\mathbb{1}_{\left\{\left|C_{\Lambda(n)}(x)\right|\geqslant n^b\right\}},\; \mathbb{1}_{\left\{\left|C_{\Lambda(n)}(y)\right|\geqslant n^b\right\}}\right]\\
&=\; \sum_{\substack{x,y\in\Lambda(n)\\ |x-y|_1<2n^b}} \mathrm{Cov}\left[\mathbb{1}_{\left\{\left|C_{\Lambda(n)}(x)\right|\geqslant n^b\right\}},\; \mathbb{1}_{\left\{\left|C_{\Lambda(n)}(y)\right|\geqslant n^b\right\}}\right]\\
&\leqslant\; \left|\left\{(x,y)\in\Lambda(n)^2 \;:\; |x-y|_1<2n^b\right\}\right|\\
&\leqslant\; 4^d n^{bd+d}\,.
\end{aligned}$$

It follows that

$$r_n \;\leqslant\; 2^{d+1}n^{bd/2+d/2-a} \;\leqslant\; \frac{2^{d+1}}{n^{d/3}}\,,$$

where we have used condition (27). This implies that r_n converges to 0 and therefore that $r_n < \eta/2$ for n large enough. We are now in position to get the desired lower bound on the partition function. Thanks to lemma 11, we have, for n large enough,

$$\begin{aligned} Z_n \;&\geqslant \sum_{\substack{\omega\in\{0,1\}^{\mathbb{E}_n}\\ |p_n(\omega)-q_n|<r_n}} \mathbb{P}_{p_n(\omega)}(\omega) \\ &\geqslant \sum_{\substack{\omega\in\{0,1\}^{\mathbb{E}_n}\\ |p_n(\omega)-q_n|<r_n}} \mathbb{P}_{q_n}(\omega)\exp\left(-\frac{2dn^d r_n}{\eta}\right) \\ &\geqslant \mathbb{P}_{q_n}\Big(|p_n-q_n|<r_n\Big)\exp\left(-\frac{2^{d+2}dn^{2d/3}}{\eta}\right). \end{aligned} \tag{39}$$

We then use the definition of p_n and q_n, along with the Lipschitz continuity of the exponential function on $]-\infty,0]$ with Lipschitz constant 1, to obtain that, for n large enough,

$$\begin{aligned} \mathbb{P}_{q_n}\Big(|p_n-q_n|<r_n\Big) \;&=\; \mathbb{P}_{q_n}\left(\left|\varphi_n\left(B_n^b\right)-\varphi_n\left(\mathbb{E}_{q_n}\left[B_n^b\right]\right)\right|<r_n\right) \\ &\geqslant\; \mathbb{P}_{q_n}\left(\left|\frac{B_n^b}{n^a}-\frac{\mathbb{E}_{q_n}\left[B_n^b\right]}{n^a}\right|<r_n\right) \\ &=\; \mathbb{P}_{q_n}\Big(\left|B_n^b-\mathbb{E}_{q_n}\left[B_n^b\right]\right|<r_n n^a\Big) \\ &=\; \mathbb{P}_{q_n}\Big(\left(B_n^b-\mathbb{E}_{q_n}\left[B_n^b\right]\right)^2<2\mathrm{Var}_{q_n}\left[B_n^b\right]\Big) \\ &\geqslant\; \frac{1}{2}, \end{aligned} \tag{40}$$

where we have used Chebyshev's inequality in the last step. Combining (39) and (40), we get

$$\liminf_{n\to\infty}\frac{\ln Z_n}{n^{2d/3}} \;\geqslant\; -\frac{2^{d+2}d}{\eta} \;>\; -\infty,$$

which concludes our proof. □

6.5 Proof of the convergence result

Given the upper bounds proven in the subcritical and supercritical phases and the lower bound on the partition function, the proof of the convergence result of theorem 1 for the distribution of the cluster sizes is the same as the proof for the case of the first model, which is detailed in section 4.4. To deal with the only difference, namely the exponents on n, we use (27) to obtain

$$2a-bd-d \;>\; \frac{5d}{3}-d \;=\; \frac{2d}{3} \qquad\text{and}\qquad d-bd \;>\; d-2a+\frac{5d}{3} \;>\; \frac{2d}{3},$$

which yields the desired result.

6.6 A control on the convergence speed

We make now the previous arguments more precise, in order to obtain an estimate on the convergence speed of p_n towards the critical point p_c. We fix for all this part some real numbers a, b and c such that

$$\frac{5d}{6}<a<d, \qquad 0<b<\frac{2a}{d}-\frac{5}{3} \qquad\text{and}\qquad 0\;<\;c\;<\;\min\left(\frac{b}{2\gamma},\,\frac{1-b}{\beta},\,\frac{d-a}{\beta}\right),$$

and we assume that there exist real numbers $\beta > 0$, $\gamma > 0$ such that

$$\limsup_{\substack{p\to p_c \\ p>p_c}} \frac{\ln \theta(p)}{\ln(p-p_c)} \;\leqslant\; \beta \qquad \text{and} \qquad \liminf_{\substack{p\to p_c \\ p<p_c}} \frac{\ln \chi(p)}{\ln(p_c-p)} \;\geqslant\; -\gamma\,.$$

We also choose β' and γ' such that

$$\beta \;<\; \beta' \;<\; \min\left(\frac{1-b}{c},\, \frac{d-a}{c}\right) \qquad \text{and} \qquad \gamma \;<\; \gamma' \;<\; \frac{b}{2c}\,.$$

Therefore, we can find $\varepsilon_0 > 0$ such that, for $0 < \varepsilon < \varepsilon_0$,

$$\theta\left(p_c+\varepsilon\right) \;\geqslant\; \varepsilon^{\beta'} \qquad \text{and} \qquad \chi\left(p_c-\varepsilon\right) \;\leqslant\; \frac{1}{\varepsilon^{\gamma'}}\,.$$

6.6.1 Subcritical phase

Lemma 17. *We have the upper bound*

$$\forall \varepsilon>0 \quad \forall A>0 \qquad \limsup_{n\to\infty} \frac{1}{n^{2a-bd-d}} \ln \mathbb{P}_{p_c-\varepsilon/n^c}\Big(B_n^b > An^a\Big) \;<\; 0\,.$$

Proof. Take $A > 0$ and $0 < \varepsilon < p_c$. Without loss of generality, we can assume that $\varepsilon < \varepsilon_0$. We repeat the proof of lemma 14, but replacing p with $p_c - \varepsilon/n^c$. To control $\mathbb{E}_{p_c-\varepsilon/n^c}\big[A_n^b(\mathcal{D}_x)\big]$, the upper bound (31) is no longer sufficient, because we need to specify the dependence in n of $\lambda(p_c - \varepsilon/n^c)$. Thus, we use another inequality provided by the same theorem 6.75 in [Gri99], which states that

$$\forall p < p_c \qquad \forall n > \chi(p)^2 \qquad \mathbb{P}_p\big(\,|C(0)| \geqslant n\big) \;\leqslant\; 2\exp\left(-\frac{n}{2\chi(p)^2}\right)\,. \tag{41}$$

With our choice of γ', we have that

$$\chi\left(p_c - \frac{\varepsilon}{n^c}\right)^2 \;\leqslant\; \frac{n^{2\gamma' c}}{\varepsilon^{2\gamma'}} \;=\; o\left(n^b\right)\,,$$

hence the condition $n^b \geqslant \chi\left(p_c - \varepsilon/n^c\right)^2$ is satisfied for n large enough. This allows us to apply (41) to get, for all $x \in \{0,1\}^d$ and for n large enough,

$$\begin{aligned}
\mathbb{E}_{p_c-\varepsilon/n^c}\big[A_n^b(\mathcal{D}_x)\big] \;&\leqslant\; \mathbb{E}_{p_c-\varepsilon/n^c}\big[A_n^b\left(\Lambda(n)\right)\big] \\
&\leqslant\; n^d \mathbb{P}_{p_c-\varepsilon/n^c}\left(|C(0)| \geqslant \lceil n^b \rceil\right) \\
&\leqslant\; 2n^d \exp\left(-\frac{n^b}{2\chi\left(p_c - \dfrac{\varepsilon}{n^c}\right)^2}\right) \\
&\leqslant\; 2n^d \exp\left(-\frac{\varepsilon^{2\gamma'} n^{b-2\gamma' c}}{2}\right) \;=\; o\left(n^a\right)\,.
\end{aligned}$$

Having shown this, the rest of the proof is identical to the proof of lemma 14. $\square$

6.6.2 Supercritical phase

Lemma 18. *We have the upper bound*

$$\forall \varepsilon>0 \quad \forall A>0 \qquad \limsup_{n\to\infty} \frac{1}{n^{d-bd-2c}} \ln \mathbb{P}_{p_c+\varepsilon/n^c}\Big(B_n^b < An^a\Big) \;<\; 0\,.$$

Proof. The proof is identical to the proof of lemma 15, replacing p with $p_c + \varepsilon/n^c$, and using, to show (37), the fact that

$$An^a + 4dn^{b+d-1} \;=\; o\left(n^d \theta(p_c + \varepsilon/n^c)\right)$$

which follows from the inequalities $a < d - c\beta'$ and $b + d - 1 < d - c\beta'$. $\square$

6.6.3 Conclusion

Combining the lower bound on Z_n obtained in lemma 16 and the results of lemmas 17 and 18, we get the convergence of $n^c(p_n - p_c)$ to 0 with exactly the same technique as in section 4.4.

6.7 An alternative model with cluster diameters

To deal with the variant obtained by replacing B_n^b with the function

$$\widetilde{B}_n^b : \omega \longmapsto \left| \left\{ x \in \Lambda(n),\ x \overset{\omega}{\longleftrightarrow} \left(x + \partial\Lambda\left(\lceil n^b \rceil \right) \right) \cap \Lambda(n) \right\} \right| ,$$

we can use exactly the same technique, replacing A_n^b with

$$\widetilde{A}_n^b : (X, \omega) \ \longmapsto \ \left| \left\{ x \in X,\ x \overset{\omega}{\longleftrightarrow} \left(x + \partial\Lambda\left(\lceil n^b \rceil \right) \right) \right\} \right| .$$

The only significant difference is that, instead of using theorem 6.75 of [Gri99], we use the theorem 5.4 therein, which states that

$$\forall p < p_c \quad \exists\, \psi(p) > 0 \quad \forall n \geqslant 1 \qquad \mathbb{P}_p\left(0 \overset{\omega}{\longleftrightarrow} \partial\Lambda(n) \right) \ \leqslant \ e^{-n\psi(p)} .$$

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