ORIENTED MEASURES

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Abstract. A vector measure $\mu = (\mu_1, \cdots, \mu_n)$ defined on $[a, b]$ is oriented if for each k–tuple of disjoint measurable sets $(A_1, \cdots, A_k)$ such that $A_1 < \cdots < A_k$ the determinant

$$\begin{vmatrix} \mu_1(A_1) & \cdots & \mu_1(A_k) \\ \vdots & \ddots & \vdots \\ \mu_k(A_1) & \cdots & \mu_k(A_k) \end{vmatrix}$$

is positive. We study the range $\mathcal{R}$ of an oriented measure:

$$\overset{\circ}{\mathcal{R}} = \{\, \mu(E) : \chi_E \text{ has } n \text{ discontinuity points} \,\}\,,$$
$$\partial\mathcal{R} = \{\, \mu(E) : \chi_E \text{ has less than } n-1 \text{ discontinuity points} \,\}\,.$$

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1. Introduction

A theorem of Lyapunov states that the range $\mathcal{R}$ of a non–atomic vector measure μ on $[a,b]$

$$\mathcal{R} = \{\, \mu(A) : A \text{ measurable subset of } [a,b] \,\}$$

coincides with the convex set

$$\left\{ \int_a^b \rho \, d\mu : 0 \leq \rho \leq 1 \right\}.$$

However for a given ρ, $0 \leq \rho \leq 1$, the usual proofs based on convexity–extreme points arguments [4,5] do not give any information about the existence of a "nice" set E such that

$$\mu(E) = \int_a^b \rho \, d\mu.$$

Consider for instance the two–dimensional vector measure $\mu(A) = (|A|, |A| + 2|A \cap B|)$ where B is a borelian subset of $[a,b]$ and $|\quad|$ denotes the Lebesgue measure. For each set E, the equality $\mu(E) = \mu(B)$ implies $B = E$.
When the measure μ admits a density f, Halkin [3] showed that if for each vector $p \in \mathbb{R}^n$ the set

$$\{\, t \in [a,b] : p \cdot f(t) > 0 \,\}$$

(where $\cdot$ is the usual scalar product) is a finite (respectively countable) union of intervals then there exists a set E which is a finite (resp. countable) union of intervals.
In our paper [2] we introduced the stronger orientation condition Δ:
we say that n real functions $f_1, \cdots, f_n$ verify condition Δ on an interval $[a,b]$ if for each k in $\{1, \cdots, n\}$, the determinant

$$\begin{vmatrix} f_1(x_1) & f_1(x_2) & \cdots & f_1(x_k) \\ f_2(x_1) & f_2(x_2) & \cdots & f_2(x_k) \\ \vdots & \vdots & \ddots & \vdots \\ f_k(x_1) & f_k(x_2) & \cdots & f_k(x_k) \end{vmatrix}$$

is not equal to zero whenever the x_i's in $[a,b]$ are distinct and its sign is constant on the k–tuples $(x_1, \cdots, x_k)$ such that $a \leq x_1 < x_2 < \cdots < x_k \leq b$.
We showed that if a measure μ admits a density function whose components are continuous and satisfy the orientation condition Δ then the set E may be built in such a way that its characteristic function has at most n discontinuity points. Moreover, if $0 < \rho < 1$ there exist two such sets E_1 and E_2 whose characteristic functions χ_{E_1} and χ_{E_2} have exactly n

discontinuity points (one set is a neighbourhood of a whereas the other is not).
Our proofs relied upon the fact that the map

$$(\alpha_1, \cdots, \alpha_n) \longmapsto \int_{\alpha_1}^{\alpha_2} f(x)\,dx + \int_{\alpha_3}^{\alpha_4} f(x)\,dx + \cdots$$

is differentiable and has an invertible Jacobian whenever $a < \alpha_1 < \cdots < \alpha_n < b$.
We also showed that whenever a function x satisfies $x(0) = \cdots = x^{(n-2)}(0) = 0$ and $x^{(n-1)}(0) = 1$ then the n functions $(x^{(n-1)}, \cdots, x', x)$ verify Δ on a neighbourhood of 0.
We applied these results to the study of reachable sets of constrained bang–bang solutions and to non–convex problems of the calculus of variations.
In this paper we deal with measures which are not necessarily absolutely continuous with respect to the Lebesgue measure.

Oriented measure. If $A_1, \cdots, A_k$ are k measurable sets of $[a, b]$, by $A_1 < \cdots < A_k$ we mean that for all k–tuple $(x_1, \cdots, x_k)$ of $A_1 \times \cdots \times A_k$ we have $x_1 < \cdots < x_k$. A measure $\mu = (\mu_1, \cdots, \mu_n)$ is said to be oriented if for each k–tuple of measurable sets $A_1, \cdots, A_k$ such that $A_1 < \cdots < A_k$ the determinant

$$\begin{vmatrix} \mu_1(A_1) & \cdots & \mu_1(A_k) \\ \vdots & \ddots & \vdots \\ \mu_k(A_1) & \cdots & \mu_k(A_k) \end{vmatrix}$$

is positive.

In this more general framework we give a new proof of the results stated in [2].
We carry out a deep study of the range $\mathcal{R}$ of the measure:
• for each point q of its interior $\overset{\circ}{\mathcal{R}}$ there exist exactly two distinct "dual" sets E_1, E_2 whose characteristic functions have n discontinuity points such that $\mu(E_1) = q = \mu(E_2)$;
• the set $\overset{\circ}{\mathcal{R}}$ coincides with

$$\left\{ \int_a^b \rho\,d\mu : 0 < \rho < 1 \right\}$$

so that the above set is open;
• the set $\mathcal{R}$ is strictly convex;
• a point $\mu(E)$ belongs to the boundary $\partial\mathcal{R}$ of $\mathcal{R}$ if and only if the characteristic function of E has less than $n-1$ discontinuity points;
• finally we give a recursive decomposition of the boundary $\partial\mathcal{R}$.

2. Oriented measures

Throughout the paper we will work with an interval $[a, b]$ equipped with the Lebesgue σ–field $\mathcal{L}$. Measurable will mean measurable with respect to this σ–field. A negligible set is a measurable set of Lebesgue measure zero. A vector measure on $[a,b]$ is a countably additive set function defined on the Lebesgue σ–field with values in $\mathbb{R}^n$ for some integer n.

Notation. If $A_1, \cdots, A_k$ are k measurable sets of $[a,b]$, by $A_1 < \cdots < A_k$ we mean that $A_1, \cdots, A_k$ have non–zero Lebesgue measure and for all k–tuple $(x_1, \cdots, x_k)$ of $A_1 \times \cdots \times A_k$ we have $x_1 < \cdots < x_k$.
Let $\mu = (\mu_1, \cdots, \mu_k)$ be a vector measure. If ρ belongs to $L^1_\mu([a,b])$, we note

$$\mu_i(\rho) = \int_a^b \rho \, d\mu_i \,, \qquad \mu(\rho) = \int_a^b \rho \, d\mu = \big(\mu_1(\rho), \cdots, \mu_k(\rho)\big).$$

Definition 2.1. A vector measure $\mu = (\mu_1, \cdots, \mu_n)$ on $[a,b]$ is said to be oriented on $[a,b]$ if it is non–atomic and if for each k in $\{1, \cdots, n\}$ and for each k–tuple of measurable sets $A_1, \cdots, A_k$ such that $A_1 < \cdots < A_k$ the determinant

$$\begin{vmatrix} \mu_1(A_1) & \cdots & \mu_1(A_k) \\ \vdots & \ddots & \vdots \\ \mu_k(A_1) & \cdots & \mu_k(A_k) \end{vmatrix}$$

is positive.

Remark. If μ is oriented then μ_1 is a positive measure which assigns positive values to sets of positive Lebesgue measure. In particular, if I is a non–trivial interval, then $\mu(I)$ is non–zero.

Remark. If μ is oriented and $I_1, \cdots, I_n$ are n disjoint non–trivial intervals, then the vectors $\mu(I_1), \cdots, \mu(I_n)$ form a basis of $\mathbb{R}^n$.

A very important fact concerning oriented measures is that their characteristic property carries on from sets to positive functions.

Notation. If ρ is a function its support is the set supp $\rho = \{\, x : \rho(x) \neq 0 \,\}$.

Theorem 2.2. *Let $\mu = (\mu_1, \cdots, \mu_n)$ be an oriented measure. If $\rho_1, \cdots, \rho_n$ are n μ integrable non–negative functions such that supp $\rho_1 < \cdots <$ supp ρ_n then the determinant*

$$\begin{vmatrix} \mu_1(\rho_1) & \cdots & \mu_1(\rho_n) \\ \vdots & \ddots & \vdots \\ \mu_n(\rho_1) & \cdots & \mu_n(\rho_n) \end{vmatrix}$$

is positive.

Let us first state a preparatory lemma.

Lemma 2.3. *Let $\mu = (\mu_1, \cdots, \mu_n)$ be a vector measure and $\rho_1, \cdots, \rho_n$ be n measurable μ–integrable functions. The determinant*

$$\begin{vmatrix} \displaystyle\int \rho_1 \, d\mu_1 & \cdots & \displaystyle\int \rho_n \, d\mu_1 \\ \vdots & \ddots & \vdots \\ \displaystyle\int \rho_1 \, d\mu_n & \cdots & \displaystyle\int \rho_n \, d\mu_n \end{vmatrix}$$

is equal to

$$\int \cdots \int \rho_1(s_1) \cdots \rho_n(s_n) \, d\Big(\sum_{\sigma \in \mathcal{S}_n} \epsilon(\sigma) \mu_{\sigma(1)} \otimes \cdots \otimes \mu_{\sigma(n)} \Big)(s_1, \cdots, s_n).$$

Proof of the lemma. The identity is obviously true whenever $\rho_1, \cdots, \rho_n$ are characteristic functions. The monotone class theorem yields the result. $\square$

Proof of theorem 2.2. We apply the lemma. The domain of integration of the n–fold integral is reduced to supp $\rho_1 \times \cdots \times$ supp ρ_n.

We first prove that the measure $\hat{\mu} = \sum_{\sigma \in \Sigma_n} \epsilon(\sigma) \mu_{\sigma(1)} \otimes \cdots \otimes \mu_{\sigma(n)}$ is positive on the product space (supp $\rho_1, \mathcal{L}) \times \cdots \times$ (supp $\rho_n, \mathcal{L})$ equipped with the product σ–field (where $\mathcal{L}$ denotes the one–dimensional Lebesgue σ–field). Notice that the product σ–field $\mathcal{L}^{\otimes n}$ does not coincide in general with the n–dimensional Lebesgue σ–field (i.e. the completion of the n–dimensional Borel σ–field).

Consider first the case of a subset of supp $\rho_1 \times \cdots \times$ supp ρ_n which is a product set $A_1 \times \cdots \times A_n$ (where the A_i's are measurable). Necessarily, each A_i is a subset of supp ρ_i. If none of the A_i's is negligible, then we have $A_1 < \cdots < A_n$ and $\hat{\mu}(A_1 \times \cdots \times A_n) = \det[\mu(A_1), \cdots, \mu(A_n)]$ is positive by definition.

Suppose now some of the A_i's are negligible. For each index i, $1 \leq i \leq n$, there exists a decreasing sequence $(B^i_m)_{m \in \mathbb{N}}$ of non–negligible measurable subsets of supp ρ_i having an empty intersection (this is a consequence of the fact that supp ρ_i is not negligible). Now for each m we have $A_1 \cup B^1_m < \cdots < A_n \cup B^n_m$ whence $\hat{\mu}(A_1 \cup B^1_m \times \cdots \times A_n \cup B^n_m)$ is positive. By the continuity of the measure μ we have

$$\hat{\mu}(A_1 \times \cdots \times A_n) \; = \; \lim_{m \to \infty} \hat{\mu}(A_1 \cup B^1_m \times \cdots \times A_n \cup B^n_m)$$

so that $\hat{\mu}(A_1 \cdots A_n)$ is non–negative. It follows that $\hat{\mu}$ is non–negative on the boolean algebra of the finite (disjoint) union of product sets: its unique extension to the σ–field $\mathcal{L}^{\otimes n}$ generated by these products is also non–negative.

The function $(s_1, \cdots, s_n) \mapsto \rho_1(s_1) \cdots \rho_n(s_n)$ is positive everywhere on this set and is measurable with respect to the σ–field $\mathcal{L}^{\otimes n}$: thus the integral $\int \rho_1(s_1) \cdots \rho_n(s_n)\, d\hat{\mu}(s_1, \cdots, s_n)$ is positive. $\square$

Remark. If μ is absolutely continuous with respect to the Lebesgue measure then Lyapunov theorem yields an alternative proof of theorem 2.2. In fact

$$\forall k \in \{1, \cdots, n\} \quad \exists E_k \subset \text{supp } \rho_k \qquad \mu(\rho_k) \;=\; \mu(E_k).$$

Necessarily $\mu(E_k)$ is non–zero for each k (see remark after definition 2.1) and the absolute continuity hypothesis on μ implies that the E_k's are not negligible.
It follows that $E_1 < \cdots < E_n$ and $\det[\mu(\rho_1), \cdots, \mu(\rho_n)] = \det[\mu(E_1), \cdots, \mu(E_n)] > 0$.

We shall denote by Γ_k the subset

$$\Gamma_k \;=\; \{\, (x_1, \cdots, x_k) \in \mathbb{R}^k : a \le x_1 \le \cdots \le x_k \le b \,\}.$$

Definition 2.4. The measure μ is said to be locally oriented if for each n–tuple x of Γ_n there exists a neighbourhood $V = V_1 \times \cdots \times V_n$ of x such that for each k–tuple of measurable sets $A_1 < \cdots < A_k$ satisfying $A_1 \times \cdots \times A_k \subset V_1 \times \cdots \times V_k$, the determinant

$$\begin{vmatrix} \mu_1(A_1) & \cdots & \mu_1(A_k) \\ \vdots & \ddots & \vdots \\ \mu_k(A_1) & \cdots & \mu_k(A_k) \end{vmatrix}$$

is positive.

As a curiosity, we prove the following

Proposition 2.5. *A locally oriented measure on $[a, b]$ is oriented on $[a, b]$.*

Proof. Let μ be a locally oriented measure. The compact set Γ_n can be covered by a finite family of open sets $(V_i)_{i \in \Upsilon}$ where $V_i = I_1^i \times \cdots \times I_n^i$ and $(I_k^i)_{\substack{i \in \Upsilon \\ 1 \le k \le n}}$ are subintervals of $[a, b]$ in such a way that for each k–tuple of measurable sets $A_1 < \cdots < A_k$ satisfying $A_1 \times \cdots \times A_k \subset V_i$ for some $i \in \Upsilon$, the determinant formed with the first k components of the vectors $\mu(A_1), \cdots, \mu(A_k)$ is positive.
Let $(J_l)_{l \in \Sigma}$ be the finite family of the atoms of the algebra generated by the sets $(I_k^i, i \in \Upsilon, 1 \le k \le n)$ (thus the J_l's are exactly the sets of the form $\bigcap_{i,k : x \in I_k^i} I_k^i$ for some $x \in [a, b]$).
Let us remark that for each $(l_1, \cdots, l_k)$ in Σ^k, the product $J_{l_1} \times \cdots \times J_{l_k}$ is contained in some product $I_1^{i_0} \times \cdots \times I_k^{i_0}$. In fact

$$J_{l_1} \times \cdots \times J_{l_k} \subset \bigcup_{i \in \Upsilon} I_1^i \times \cdots \times I_k^i$$

so that there exits i_0 such that $J_{l_1} \times \cdots \times J_{l_k} \cap I_1^{i_0} \times \cdots \times I_k^{i_0}$ is not empty. It follows that $J_{l_1} \cap I_1^{i_0} \neq \emptyset, \cdots, J_{l_k} \cap I_k^{i_0} \neq \emptyset$ and by the very construction of the sets J_l's we obtain $J_{l_1} \subset I_1^{i_0}, \cdots, J_{l_k} \subset I_k^{i_0}$. We denote by $\hat{\mu}_k$ the measure $\hat{\mu}_k = \sum_{\sigma \in \Sigma_k} \epsilon(\sigma) \mu_{\sigma(1)} \otimes \cdots \otimes \mu_{\sigma(k)}$. Let $(A_1, \cdots, A_k)$ be a k–tuple of measurable sets such that $A_1 < \cdots < A_k$. The product $A_1 \times \cdots \times A_k$ is the disjoint union of the sets $(A_1 \times \cdots \times A_k) \cap (J_{l_1} \times \cdots \times J_{l_k})$ when $(l_1, \cdots, l_k)$ varies in Σ^k. Let now $(l_1, \cdots, l_k)$ belong to Σ^k. Either $(A_1 \times \cdots \times A_k) \cap (J_{l_1} \times \cdots \times J_{l_k})$ is empty (and thus has a zero $\hat{\mu}_k$ measure) or it is not empty and necessarily, $J_{l_1} < \cdots < J_{l_k}$. Proceeding as in the proof of theorem 2.2, we show that $\hat{\mu}_k$ is a positive measure on the set $(J_{l_1} \times \cdots \times J_{l_k})$ whence $\hat{\mu}_k((A_1 \times \cdots \times A_k) \cap (J_{l_1} \times \cdots \times J_{l_k}))$ is non–negative. Since the set $A_1 \times \cdots \times A_k$ is not negligible, at least one of these sets is not negligible. Let $(A_1 \times \cdots \times A_k) \cap (J_{l_1} \times \cdots \times J_{l_k})$ be such a set. It's a subset of one of the V_i's and moreover $(A_1 \cap J_{l_1}) < \cdots < (A_k \cap J_{l_k})$ whence $\hat{\mu}_k((A_1 \cap J_{l_1}) \times \cdots \times (A_k \cap J_{l_k}))$ is positive. Thus $\hat{\mu}_k(A_1 \times \cdots \times A_k)$ is positive. $\square$

3. Oriented measures with densities

Orientation condition Δ. We say that n real functions $f_1, \cdots, f_n$ verify condition Δ on an interval $[a, b]$ if for each k in $\{1, \cdots, n\}$, the determinant

$$\begin{vmatrix} f_1(x_1) & \cdots & f_1(x_k) \\ f_2(x_1) & \cdots & f_2(x_k) \\ \vdots & \ddots & \vdots \\ f_k(x_1) & \cdots & f_k(x_k) \end{vmatrix}$$

is positive whenever the x_i's in $[a, b]$ are such that $a \leq x_1 < x_2 < \cdots < x_k \leq b$.

Remark. In our previous paper [2], we didn't impose the sign of the above determinant to be positive. When dealing with continuous functions, a connectedness argument shows immediately that the sign is constant on the set Γ_k. In our present framework (at the measure level), we find it convenient to work with this slightly more restrictive condition.

Examples. For $n = 1$, condition Δ states that the function f_1 is positive; for $n = 2$, the functions f_1, f_2 satisfy Δ if and only if f_1 is positive and f_2/f_1 is strictly increasing.
The functions $f_i(t) = t^{i-1}$ $(i \geq 1)$ satisfy condition Δ on $\mathbb{R}$ (the corresponding determinants are Vandermonde determinants).

Proposition 3.1. *Let $f_1, \cdots, f_n$ be n functions in $L^1([a, b])$ satisfying the orientation condition Δ on $[a, b]$. Let μ_i be the measure on $[a, b]$ whose density with respect to the Lebesgue measure is f_i. Then the measure $\mu = (\mu_1, \cdots, \mu_n)$ is oriented.*

Proof. Let $A_1 < \cdots < A_k$ be k measurable sets of $[a, b]$. Since the determinant is a multilinear continuous form, we can write

$$\begin{vmatrix} \int_{A_1} f_1 & \cdots & \int_{A_k} f_1 \\ \vdots & \ddots & \vdots \\ \int_{A_1} f_k & \cdots & \int_{A_k} f_k \end{vmatrix} = \int \cdots \int_{A_1 \times \cdots \times A_k} \begin{vmatrix} f_1(s_1) & \cdots & f_1(s_k) \\ f_2(s_1) & \cdots & f_2(s_k) \\ \vdots & \ddots & \vdots \\ f_k(s_1) & \cdots & f_k(s_k) \end{vmatrix} ds_1 \cdots ds_k.$$

By condition Δ, the integrand is positive on $A_1 \times \cdots \times A_k$. $\square$

If $f_1, \cdots, f_k$ are of class $\mathcal{C}^{k-1}$ on $[a, b]$ we will denote their Wronskian by $W(f_1, \cdots, f_k)$. The following operational criterion for the fulfilment of the orientation condition Δ has been used in [2].

Proposition 3.2. *Let $f_1, \cdots, f_n \in \mathcal{C}^{n-1}([a, b])$ be such that*

$$\forall t \in [a, b] \qquad W(f_1)(t) > 0, \cdots, W(f_1, \cdots, f_n)(t) > 0.$$

Then $f_1, \cdots, f_n$ satisfy the orientation condition Δ on $[a, b]$.

4. Notations and preliminary lemmas

Let us introduce some notations.
If $u_1, \cdots, u_n$ are vectors of $\mathbb{R}^n$, their determinant is sometimes denoted by $\det[u_1, \cdots, u_n]$. Let A be a $n \times n$ matrix with real coefficients; by $\det A$ or $|A|$ we denote its determinant. For each $i, j \in \{1, \cdots, n\}$, by A_{ij} we mean the $(n-1) \times (n-1)$ matrix obtained by removing the i–th row and the j–th column from A. Surprisingly, the following simple algebraic trick will play an essential role in the existence part of the proof of theorem 1.

Lemma 4.1. *Let $A = (a_{ij})_{1 \le i,j \le n}$ be an $n \times n$ matrix with real coefficients. Let $x_1, \cdots, x_n$ be such that*

$$\begin{cases} a_{1,1}x_1 + \cdots + a_{1,n-1}x_{n-1} + a_{1,n}x_n = 0 \\ a_{2,1}x_1 + \cdots + a_{2,n-1}x_{n-1} + a_{2,n}x_n = 0 \\ \vdots \qquad \ddots \qquad \vdots \qquad \vdots \\ a_{n-1,1}x_1 + \cdots + a_{n-1,n-1}x_{n-1} + a_{n-1,n}x_n = 0 \end{cases}$$

If $\det A_{nn} \neq 0$ then

$$a_{n1}x_1 + \cdots + a_{nn}x_n = \frac{|A|}{|A_{nn}|}\, x_n.$$

Proof. Cramer rule applied to the above system yields

$$\forall i \in \{1, \cdots, n-1\} \qquad x_i = \frac{(-1)^{n+i}|A_{ni}|}{|A_{nn}|}\, x_n$$

so that

$$a_{n1}x_1 + \cdots + a_{nn}x_n = \frac{\sum_{i=1}^{n}(-1)^{n+i}a_{ni}|A_{ni}|}{|A_{nn}|}\, x_n = \frac{|A|}{|A_{nn}|}\, x_n$$

since $\sum_{i=1}^{n}(-1)^{n+i}a_{ni}|A_{ni}|$ is the development of the determinant $|A|$ along the first row. $\square$

The next lemmas involve strongly the notion of oriented measure.

Lemma 4.2. *Let F and G be two distinct subsets of $[a,b]$ which are the union of l and m disjoint closed intervals*

$$F = \bigcup_{i=1}^{l} I_i\,, \quad G = \bigcup_{j=1}^{m} J_j$$

and let $\mu = (\mu_1, \cdots, \mu_n)$ be an oriented measure. Assume $\mu(F) = \mu(G)$.
Then $n < l + m$; moreover if $\partial F \cap \partial G \neq \emptyset$ then $n < l + m - 1$.

Proof. Let us first remark that the symmetric difference

$$(I_1 \cup \cdots \cup I_l)\, \Delta\, (J_1 \cup \cdots \cup J_m) = \Big(\bigcup_{i,j}(I_i \cup J_j)\Big) \setminus \Big(\bigcup_{i,j}(I_i \cap J_j)\Big)$$

is the union of at most $l+m$ non–trivial intervals and that whenever at least two intervals have a common boundary point then this number is smaller than $l + m - 1$. Since the intervals $I_1, \cdots, I_l$ are disjoint, as well as $J_1, \cdots, J_m$, we have

$$(I_1 \cup \cdots \cup I_l) \cup (J_1 \cup \cdots \cup J_m) \setminus (I_1 \cap J_1) \;=\; (I_1 \cup J_1) \setminus (I_1 \cap J_1) \cup (I_2 \cup \cdots \cup I_l) \cup (J_2 \cup \cdots \cup J_m).$$

Now, the set $(I_2 \cup \cdots \cup I_l) \cup (J_2 \cup \cdots \cup J_m)$ is a union of at most $l+m-2$ disjoint intervals. Either $I_1 \cap J_1 = \emptyset$ or $I_1 \cap J_1 \neq \emptyset$ and $(I_1 \cup J_1)$ is an interval. In both cases $(I_1 \cup J_1) \setminus (I_1 \cap J_1)$ is the union of at most two intervals (at most one if I_1 and J_1 have a boundary point in common). A straightforward induction gives the result.
Since the sets F and G are distinct, $F \Delta G$ is not empty. Let $A_1 < \cdots < A_p$ be the connected components of $F \Delta G$. For k in $\{1, \cdots, p\}$ we have

$$A_k = (A_k \cap F) \cup (A_k \cap G)\,,$$
$$(A_k \cap F) \cap (A_k \cap G) \subset A_k \cap (F \cap G) \subset (F \Delta G) \cap (F \cap G) = \emptyset\,;$$

the set A_k being connected, either $A_k \subset F \setminus G$ or $A_k \subset G \setminus F$. Put

$$\lambda_k = \begin{cases} +1 & \text{if} \quad A_k \subset F \setminus G \\ -1 & \text{if} \quad A_k \subset G \setminus F \end{cases}$$

so that the equality $\mu(F) = \mu(G)$ can be rewritten as

$$\begin{cases} \lambda_1\mu_1(A_1) + \cdots + \lambda_p\mu_1(A_p) = 0 \\ \quad\vdots \qquad\qquad \ddots \qquad\qquad \vdots \\ \lambda_1\mu_n(A_1) + \cdots + \lambda_p\mu_n(A_p) = 0 \end{cases}$$

Suppose $n \geq p$; the first p equations imply that the determinant

$$\begin{vmatrix} \mu_1(A_1) & \cdots & \mu_1(A_p) \\ \vdots & \ddots & \vdots \\ \mu_p(A_1) & \cdots & \mu_p(A_p) \end{vmatrix}$$

vanishes, which contradicts the fact that μ is oriented. $\square$

The following notations will be used throughout the paper.

Notations 4.3. We shall denote by Γ_k the set

$$\Gamma_k = \{\, (\gamma_1, \cdots, \gamma_k) \in \mathbb{R}^k : a \leq \gamma_1 \leq \cdots \leq \gamma_k \leq b \,\}.$$

To each k–tuple $\gamma = (\gamma_1, \cdots, \gamma_k)$ belonging to Γ_k we associate the two sets

$$E^-_\gamma = \bigcup_{\substack{0 \leq i \leq k \\ i \text{ odd}}} [\gamma_i, \gamma_{i+1}]\,, \quad E^+_\gamma = \bigcup_{\substack{0 \leq i \leq k \\ i \text{ even}}} [\gamma_i, \gamma_{i+1}]$$

where by convention $\gamma_0 = a$, $\gamma_{k+1} = b$.

Lemma 4.4 (Uniqueness). *Let μ be a n–dimensional oriented measure on $[a, b]$. Assume the n–tuples $\gamma = (\gamma_1, \cdots, \gamma_n)$ and $\delta = (\delta_1, \cdots, \delta_n)$ of Γ_n satisfy $\mu(E^-_\gamma) = \mu(E^-_\delta)$ (respectively $\mu(E^+_\gamma) = \mu(E^+_\delta)$). Then $E^-_\gamma = E^-_\delta$ (resp. $E^+_\gamma = E^+_\delta$).*

Proof. Assume E^-_γ, E^-_δ are distinct and $\mu(E^-_\gamma) = \mu(E^-_\delta)$.
Now, two possible cases may occur according to the parity of n.
• If $n = 2r$ the sets E^-_γ and E^-_δ are the union of at most r intervals. Lemma 4.2 implies $n < r + r$ which is absurd.
• If $n = 2r + 1$ the sets E^-_γ and E^-_δ are the union of at most $r + 1$ intervals. However b is a common boundary point. Lemma 4.2 implies $n < (r + 1) + (r + 1) - 1$ which is absurd.
The dual case $\mu(E^+_\gamma) = \mu(E^+_\delta)$ can be treated similarly. $\square$

The following essential lemma will be used repeatedly.

Lemma 4.5. *Let $\mu = (\mu_1, \cdots, \mu_n)$ be an oriented measure on the interval $[a,b]$ and $I_0 < I_1 < \cdots < I_n$ be $n+1$ subintervals of $[a,b]$. Then, given a positive ϵ, there exist $n+1$ positive real numbers $\lambda_0, \cdots, \lambda_n$ such that*

$$\forall l \in \{0, \cdots, n\} \quad 0 < \lambda_l < \epsilon \quad \textit{and} \quad \sum_{l=0}^{n} (-1)^l \lambda_l \, \mu(I_l) = 0.$$

Proof. Consider the $n \times n$ linear system

$$\lambda_0 \mu(I_0) - \lambda_1 \mu(I_1) + \cdots + (-1)^{n-1} \lambda_{n-1} \mu(I_{n-1}) = (-1)^{n-1} \lambda_n \mu(I_n).$$

where λ_n is a parameter. The determinant of the system is

$$\omega_n = (-1)^{\frac{n(n-1)}{2}} \det[\mu(I_0), \cdots, \mu(I_{n-1})].$$

The measure μ being oriented, ω_n is not zero. Moreover, for each i in $\{0, \cdots, n-1\}$,

$$\omega_i = \begin{vmatrix} \mu_1(I_0) & \cdots & (-1)^{i-2}\mu_1(I_{i-2}) & (-1)^{n-1}\mu_1(I_n) & (-1)^i \mu_1(I_i) & \cdots & (-1)^{n-1}\mu_1(I_{n-1}) \\ \mu_2(I_0) & \cdots & (-1)^{i-2}\mu_2(I_{i-2}) & (-1)^{n-1}\mu_2(I_n) & (-1)^i \mu_2(I_i) & \cdots & (-1)^{n-1}\mu_2(I_{n-1}) \\ \vdots & \ddots & \vdots & \vdots & \vdots & \ddots & \vdots \\ \mu_n(I_0) & \cdots & (-1)^{i-2}\mu_n(I_{i-2}) & (-1)^{n-1}\mu_n(I_n) & (-1)^i \mu_n(I_i) & \cdots & (-1)^{n-1}\mu_n(I_{n-1}) \end{vmatrix}$$

i.e.
$$\omega_i = (-1)^{\frac{n(n-1)}{2}} \det[\mu(I_0), \cdots, \mu(I_{i-2}), \mu(I_i), \cdots, \mu(I_n)].$$

By Cramer formula, λ_i equals $\lambda_n \omega_i / \omega_n$. The measure μ being oriented ω_i and ω_n have the same sign so that λ_i is positive whenever λ_n is positive. Choosing λ_n such that

$$0 < \lambda_n < \min(\frac{\omega_n}{\omega_0}\epsilon, \cdots, \frac{\omega_n}{\omega_{n-1}}\epsilon, \epsilon)$$

we obtain an $(n+1)$–tuple which solves the problem. $\square$

5. Main result

The statement of the main result involves the notations 4.3.

Theorem 5.1. *Let μ be an oriented measure on $[a,b]$ and let ρ be a measurable function defined on $[a,b]$ with values in $[0,1]$.*
There exist two n–tuples $\alpha=(\alpha_1,\cdots,\alpha_n)$ and $\beta=(\beta_1,\cdots,\beta_n)$ in Γ_n such that

$$\mu(E_\alpha^-)=\int_a^b \rho\, d\mu=\mu(E_\beta^+). \tag{$*$}$$

If in addition $0<\rho<1$ then α and β in Γ_n satisfying ($*$) are unique and verify

$$a<\alpha_1<\cdots<\alpha_n<b\,,\quad a<\beta_1<\cdots<\beta_n<b.$$

Remark. The measure μ being non–atomic we don't care about boundary points of intervals and we might write $\mu(\alpha,\beta)$ for the measure of the interval $\mu([\alpha,\beta])$.

Proof. We consider first the case $0<\rho<1$ and we prove the result by induction on n.
• n=1. The measure μ being oriented on $[a,b]$, the maps $\alpha\mapsto\mu([\alpha,b])$ and $\beta\mapsto\mu([a,\beta])$ are continuous and strictly monotonic on $[a,b]$. It follows that there exist unique real numbers α_1 and β_1 such that

$$\mu([\alpha_1,b])=\int_a^b \rho\, d\mu=\mu([a,\beta_1]).$$

• Assume the result is true at rank $n-1$. We deal only with the n–tuple β: existence of the n–tuple α corresponding to ρ at rank n follows from the fact that it coincides with the n–tuple β corresponding to $1-\rho$.
Define for each k in $\{1,\cdots,n\}$

$$\mu_k(\rho)=\int_a^b \rho\, d\mu_k$$

and for each n–tuple β in Γ_n

$$\theta_k(\beta)=\mu_k(E_\beta^+).$$

The inductive assumption yields the existence of two $(n-1)$–tuples $\bar\alpha=(\bar\alpha_1,\cdots,\bar\alpha_{n-1})$ and $\bar\beta=(\bar\beta_1,\cdots,\bar\beta_{n-1})$ such that

$$a<\bar\alpha_1<\cdots<\bar\alpha_{n-1}<b\,,\quad a<\bar\beta_1<\cdots<\bar\beta_{n-1}<b$$

and for each k in $\{1,\cdots,n-1\}$

$$\begin{aligned}\theta_k(a,\bar\alpha_1,\cdots,\bar\alpha_{n-1})&=\sum_{\substack{0\le i\le n-1\\ i\ \text{odd}}}\mu_k(\bar\alpha_i,\bar\alpha_{i+1})=\mu_k(\rho)\,,\\ \theta_k(\bar\beta_1,\cdots,\bar\beta_{n-1},b)&=\sum_{\substack{0\le i\le n-1\\ i\ \text{even}}}\mu_k(\bar\beta_i,\bar\beta_{i+1})=\mu_k(\rho)\,.\end{aligned} \tag{$**$}$$

Put

$$\mathcal{S} = \left\{ \beta = (\beta_1, \cdots, \beta_n) \in \Gamma_n : \beta_1 \leq \bar{\beta}_1, \quad \forall k \in \{1, \cdots, n-1\} \quad \theta_k(\beta) = \mu_k(\rho) \right\}.$$

Since $(\bar{\beta}_1, \cdots, \bar{\beta}_{n-1}, b)$ and $(a, \bar{\alpha}_1, \cdots, \bar{\alpha}_{n-1})$ belong to $\mathcal{S}$, the set $\mathcal{S}$ is not empty.
We show now that

either $\qquad \theta_n(\bar{\beta}_1, \cdots, \bar{\beta}_{n-1}, b) < \mu_n(\rho) < \theta_n(a, \bar{\alpha}_1, \cdots, \bar{\alpha}_{n-1})$

or $\qquad \theta_n(a, \bar{\alpha}_1, \cdots, \bar{\alpha}_{n-1}) < \mu_n(\rho) < \theta_n(\bar{\beta}_1, \cdots, \bar{\beta}_{n-1}, b)$.

The equalities $(**)$ yield for each k in $\{1, \cdots, n-1\}$

$$\sum_{\substack{0 \leq i \leq n-1 \\ i \text{ even}}} \int_{\bar{\beta}_i}^{\bar{\beta}_{i+1}} (1-\rho)\, d\mu_k - \sum_{\substack{0 \leq i \leq n-1 \\ i \text{ odd}}} \int_{\bar{\beta}_i}^{\bar{\beta}_{i+1}} \rho\, d\mu_k = 0.$$

Put for k, j in $\{1, \cdots, n\}$

$$x_j^\beta = (-1)^{j+1}, \quad a_{kj}^\beta = \int_{\bar{\beta}_{j-1}}^{\bar{\beta}_j} \rho_j^\beta\, d\mu_k, \quad A^\beta = \left(a_{kj}^\beta\right)_{1 \leq k,j \leq n}$$

where

$$\rho_j^\beta = \begin{cases} \rho & \text{if } j \text{ is even,} \\ 1-\rho & \text{if } j \text{ is odd.} \end{cases}$$

With these notations the above equalities become

$$\forall k \in \{1, \cdots, n-1\} \qquad \sum_{j=1}^n a_{kj}^\beta x_j^\beta = 0.$$

Since the measure μ is oriented then the determinant $|A_{nn}^\beta|$ does not vanish by theorem 2.2.
We are thus in the position to apply lemma 4.1:

$$\theta_n(\bar{\beta}_1, \cdots, \bar{\beta}_{n-1}, b) - \mu_n(\rho) = \sum_{j=1}^n a_{nj}^\beta x_j^\beta = \frac{|A^\beta|}{|A_{nn}^\beta|}(-1)^{n+1}.$$

Similarly if we define for k, j in $\{1, \cdots, n\}$

$$x_j^\alpha = (-1)^j, \quad a_{kj}^\alpha = \int_{\bar{\alpha}_{j-1}}^{\bar{\alpha}_j} \rho_j^\alpha\, d\mu_k, \quad A^\alpha = \left(a_{kj}^\alpha\right)_{1 \leq k,j \leq n}$$

where

$$\rho_j^\alpha = \begin{cases} \rho & \text{if } j \text{ is odd,} \\ 1-\rho & \text{if } j \text{ is even,} \end{cases}$$

we have

$$\theta_n(a, \bar{\alpha}_1, \cdots, \bar{\alpha}_{n-1}) - \mu_n(\rho) = \frac{|A^\alpha|}{|A^\alpha_{nn}|}(-1)^n.$$

The measure μ being oriented, the determinants $|A^\alpha|$ and $|A^\beta|$ have the same sign, as do $|A^\alpha_{nn}|$ and $|A^\beta_{nn}|$. It follows that $\theta_n(\bar{\beta}_1, \cdots, \bar{\beta}_{n-1}, b) - \mu_n(\rho)$ and $\theta_n(a, \bar{\alpha}_1, \cdots, \bar{\alpha}_{n-1}) - \mu_n(\rho)$ have opposite signs.
At this stage, we prove that the set $\mathcal{S}$ is the graph of a continuous function, this will imply that $\mathcal{S}$ is connected.
Let β_1 belong to $[a, \bar{\beta}_1]$. We are looking for a $(n-1)$–tuple $(\beta_2, \cdots, \beta_n)$ satisfying for each k in $\{1, \cdots, n-1\}$

$$\mu_k(a, \beta_1) + \sum_{\substack{2 \le i \le n \\ i \text{ even}}} \mu_k(\beta_i, \beta_{i+1}) = \mu_k(\rho) = \mu_k(a, \bar{\beta}_1) + \sum_{\substack{2 \le i \le n-1 \\ i \text{ even}}} \mu_k(\bar{\beta}_i, \bar{\beta}_{i+1})$$

or equivalently

$$\forall k \in \{1, \cdots, n-1\} \qquad \sum_{\substack{2 \le i \le n \\ i \text{ even}}} \mu_k(\beta_i, \beta_{i+1}) = \mu_k(\beta_1, \bar{\beta}_1) + \sum_{\substack{2 \le i \le n-1 \\ i \text{ even}}} \mu_k(\bar{\beta}_i, \bar{\beta}_{i+1})$$

Suppose first $\beta_1 = \bar{\beta}_1$. The above equations become

$$\forall k \in \{1, \cdots, n-1\} \qquad \sum_{\substack{2 \le i \le n \\ i \text{ even}}} \mu_k(\beta_i, \beta_{i+1}) = \sum_{\substack{2 \le i \le n-1 \\ i \text{ even}}} \mu_k(\bar{\beta}_i, \bar{\beta}_{i+1}).$$

We put $\beta = (\beta_2, \cdots, \beta_{n-1}, \beta_n)$ and $\hat{\beta} = (\bar{\beta}_2, \cdots, \bar{\beta}_{n-1}, b)$.
If n is odd then

$$E^-_\beta = [\beta_2, \beta_3] \cup \cdots \cup [\beta_{n-1}, \beta_n], \quad E^-_{\hat{\beta}} = [\bar{\beta}_2, \bar{\beta}_3] \cup \cdots \cup [\bar{\beta}_{n-1}, b];$$

if n is even then

$$E^-_\beta = [\beta_2, \beta_3] \cup \cdots \cup [\beta_n, b], \quad E^-_{\hat{\beta}} = [\bar{\beta}_2, \bar{\beta}_3] \cup \cdots \cup [\bar{\beta}_{n-2}, \bar{\beta}_{n-1}].$$

In both cases the preceding formulae can be rewritten as

$$\forall k \in \{1, \cdots, n-1\} \qquad \mu_k(E^-_\beta) = \mu_k(E^-_{\hat{\beta}});$$

lemma 4.4 implies that $E^-_\beta = E^-_{\bar\beta}$. Since in addition $\bar\beta_2 < \cdots < \bar\beta_{n-1} < b$ then necessarily $\beta_2 = \bar\beta_2, \cdots, \beta_{n-1} = \bar\beta_{n-1}, \beta_n = b$.
Suppose now $\beta < \bar\beta_1$. Since $\beta_1 < \bar\beta_1 < \cdots < \bar\beta_{n-1} < b$ then lemma 4.5 yields the existence of n real numbers $\lambda_1, \cdots, \lambda_n$ in $]0, 1/2[$ such that for each k in $\{1, \cdots, n-1\}$

$$-\lambda_1 \mu_k(\beta_1, \bar\beta_1) + \sum_{1 \le i \le n-1} (-1)^{i+1} \lambda_{i+1} \mu_k(\bar\beta_i, \bar\beta_{i+1}) = 0.$$

The function

$$\tilde\rho = (1-\lambda_1)\chi_{[\beta_1, \bar\beta_1]} + \sum_{\substack{1 \le i \le n-1 \\ i \text{ odd}}} \lambda_{i+1} \chi_{[\bar\beta_i, \bar\beta_{i+1}]} + \sum_{\substack{2 \le i \le n-1 \\ i \text{ even}}} (1-\lambda_{i+1}) \chi_{[\bar\beta_i, \bar\beta_{i+1}]}$$

satisfies $0 < \tilde\rho < 1$ on $[\beta_1, b]$ and for each k in $\{1, \cdots, n-1\}$

$$\int_{\beta_1}^b \tilde\rho \, d\mu_k = \mu_k(\beta_1, \bar\beta_1) + \sum_{\substack{2 \le i \le n-1 \\ i \text{ even}}} \mu_k(\bar\beta_i, \bar\beta_{i+1}).$$

We are thus led to find a $(n-1)$–tuple $(\beta_2, \cdots, \beta_n)$ such that $(\beta_1 \le)\beta_2 \le \cdots \le \beta_n (\le b)$ and for each k in $\{1, \cdots, n-1\}$

$$\sum_{\substack{2 \le i \le n \\ i \text{ even}}} \mu_k(\beta_i, \beta_{i+1}) = \int_{\beta_1}^b \tilde\rho \, d\mu_k,$$

or equivalently, if we put $\tilde\beta = (\beta_2, \cdots, \beta_n)$,

$$\forall k \in \{1, \cdots, n-1\} \qquad \mu_k(E^-_{\tilde\beta}) = \int_{\beta_1}^b \tilde\rho \, d\mu_k.$$

Existence and uniqueness of $\tilde\beta$ follow from the inductive assumption at rank $n-1$.
In addition, since $0 < \tilde\rho < 1$ on $[\beta_1, b]$, we have $\beta_1 < \beta_2 < \cdots < \beta_n < b$.
We can thus define a map $\psi : [a, \bar\beta_1] \to \mathbb{R}^{n-1}$ such that for all n–tuple $(\beta_1, \cdots, \beta_n)$ in Γ_n

$$(\beta_1, \cdots, \beta_n) \in \mathcal{S} \iff (\beta_2, \cdots, \beta_n) = \psi(\beta_1).$$

Thus $\mathcal{S}$ is the graph of ψ.
By the continuity of the measure μ, the maps θ_k, $1 \le k \le n-1$, are continuous so that the set $\mathcal{S}$ is closed; moreover, the function ψ takes its values in the compact set $[a, b]^{n-1}$. It follows that ψ is continuous. Henceforth $\mathcal{S}$ is connected. As a consequence,

the map θ_n, being continuous on $\mathcal{S}$, reaches all the values between $\theta_n(\bar\beta_1, \cdots, \bar\beta_{n-1}, b)$ and $\theta_n(a, \bar\alpha_1, \cdots, \bar\alpha_{n-1})$. In particular, there exists a n–tuple β in $\mathcal{S}$ such that $\theta_n(\beta) = \mu_n(\rho)$. This n–tuple β solves the problem.
Since $\theta_n(a, \bar\alpha_1, \cdots, \bar\alpha_{n-1}) \neq \mu_n(\rho)$ and $\theta_n(\bar\beta_1, \cdots, \bar\beta_{n-1}, b) \neq \mu_n(\rho)$ then $a < \beta_1 < \bar\beta_1$ so that $a < \beta_1 < \beta_2 < \cdots < \beta_n < b$. Uniqueness of β follows from lemma 4.4.

Consider now the case $0 \leq \rho \leq 1$. Let $(\rho_m)_{m\in\mathbb{N}}$ be a sequence of measurable functions such that $0 < \rho_m < 1$ and ρ_m converges to ρ in $L^1_\mu([a,b])$. For each function ρ_m there exists a unique n-tuple β^m such that

$$\mu(E^+_{\beta^m}) = \int_a^b \rho_m \, d\mu.$$

By compactness, we may assume that β^m converges to some n–tuple β of Γ_n. Passing to the limit, we obtain $\mu(E^+_\beta) = \mu(\rho)$. $\square$

6. The range of an oriented measure

Let μ be an oriented measure on $[a,b]$. We denote by $\mathcal{R}$ the range of μ i.e.

$$\mathcal{R} = \{\, \mu(A) : A \text{ measurable subset of } [a,b]\,\}.$$

Lemma 6.1. *Let $\bar\rho$ be a measurable function on $[a,b]$, $0 \leq \bar\rho \leq 1$. Suppose there exist a non–trivial interval I of $[a,b]$ and a positive real number ϵ such that $\epsilon \leq \bar\rho \leq 1-\epsilon$ on I. Then the set*

$$\left\{ \int_a^b \rho \, d\mu : \rho = \nu\chi_I + \bar\rho,\, \nu \in L^1_\mu(I),\, |\nu| < \epsilon \right\}$$

is a neighbourhood of $\int_a^b \bar\rho \, d\mu$ in $\mathbb{R}^n$.

Proof. Let $I_1 < \cdots < I_n$ be n non–trivial subintervals of I. The measure μ being oriented, the vectors $\mu(I_1), \cdots, \mu(I_n)$ form a basis of $\mathbb{R}^n$. The map

$$\Lambda : (\lambda_1, \cdots, \lambda_n) \in \mathbb{R}^n \longmapsto \sum_{1\leq i\leq n} \lambda_i \, \mu(I_i) \in \mathbb{R}^n$$

is a linear isomorphism and is thus open. Let

$$V_\epsilon = \{\, (\lambda_1, \cdots, \lambda_n) : \max_{1\leq i\leq n} |\lambda_i| < \epsilon \,\}.$$

Since $\Lambda(V_\epsilon)$ is a neighbourhood of the origin and is contained in the set

$$\left\{ \int_I \nu \, d\mu : \nu \in L^1_\mu(I),\, |\nu| < \epsilon \right\},$$

then the conclusion follows. $\square$

Remark. The hypothesis $\epsilon \le \bar{\rho} \le 1-\epsilon$ implies that $\mu(\bar{\rho})$ belongs to the interior of $\mathcal{R}$.

Remark. The conclusion of lemma 6.1 does not hold for an arbitrary vector measure: consider for instance the n–dimensional Lebesgue measure.

Let $\theta : \Gamma_n \to \mathcal{R}$ be the function defined by $\theta(\gamma) = \mu(E_\gamma^-)$.
The interior of Γ_n is the set $\overset{\circ}{\Gamma}_n = \{ (\gamma_1, \cdots, \gamma_n) \in \mathbb{R}^n : a < \gamma_1 < \cdots < \gamma_n < b \}$.

Corollary 6.2. *The set $\theta(\overset{\circ}{\Gamma}_n)$ is contained in $\overset{\circ}{\mathcal{R}}$.*

Lemma 6.3. *The set $\theta(\overset{\circ}{\Gamma}_n)$ coincides with the set*

$$F = \left\{ \int_a^b \rho \, d\mu : 0 < \rho < 1 \right\}.$$

Proof. Existence part of theorem 5.1 implies that F is contained in $\theta(\overset{\circ}{\Gamma}_n)$.
Conversely, let $\gamma = (\gamma_1, \cdots, \gamma_n)$ belong to $\overset{\circ}{\Gamma}_n$; applying lemma 4.5 to μ, γ and $\epsilon < 1/2$, we obtain a $(n+1)$–tuple $(\lambda_0, \cdots, \lambda_n)$ such that

$$\forall i \in \{0, \cdots, n\} \qquad 0 < \lambda_i < \epsilon \quad \text{and} \quad \sum_{i=0}^{n} (-1)^i \lambda_i \, \mu(\gamma_i, \gamma_{i+1}) = 0.$$

Put

$$\rho = \sum_{\substack{0 \le i \le n \\ i \text{ even}}} \lambda_i \chi_{[\gamma_i, \gamma_{i+1}]} + \sum_{\substack{0 \le i \le n \\ i \text{ odd}}} (1 - \lambda_i) \chi_{[\gamma_i, \gamma_{i+1}]}.$$

By construction we have $0 < \rho < 1$ and

$$\int_a^b \rho \, d\mu = \mu(E_\gamma^-) = \theta(\gamma)$$

so that $\theta(\gamma)$ belongs to F. $\square$

We have the following

Theorem 6.4. *The range of θ coincides with $\mathcal{R}$; the map θ induces an homeomorphism from $\overset{\circ}{\Gamma}_n$ onto $\overset{\circ}{\mathcal{R}}$ and maps $\partial\Gamma_n$ onto $\partial\mathcal{R}$.*

Proof. The surjectivity of θ follows directly from theorem 5.1. Injectivity of the restriction of θ to $\overset{\circ}{\Gamma}_n$ is a consequence of the uniqueness part of theorem 5.1 together with lemma 6.3. We claim that $\theta(\overset{\circ}{\Gamma}_n)$ is open. Let γ belong to $\overset{\circ}{\Gamma}_n$. Lemma 4.5 allows as usual to find a piecewise constant function $\bar\rho$ such that $0 < \bar\rho < 1$ and $\mu(\bar\rho) = \theta(\gamma)$. Clearly there exist a positive ϵ and a subinterval I of $[a,b]$ on which $\epsilon \le \bar\rho \le 1-\epsilon$. Put

$$V_{\bar\rho}^{I,\epsilon} = \{\, \nu\chi_I + \bar\rho : \nu \in L^1_\mu(I),\ |\nu| < \epsilon \,\}.$$

Lemma 6.1 implies that the set

$$\mu(V_{\bar\rho}^{I,\epsilon}) = \left\{ \int_a^b \rho\, d\mu : \rho \in V_{\bar\rho}^{I,\epsilon} \right\}$$

is a neighbourhood of $\mu(\bar\rho)$ in $\mathbb{R}^n$. Since each element ρ of $V_{\bar\rho}^{I,\epsilon}$ satisfies $0 < \rho < 1$ then $\mu(V_{\bar\rho}^{I,\epsilon})$ is entirely contained in F. Moreover F coincides with $\theta(\mathring{\Gamma}_n)$ and thus $\theta(\mathring{\Gamma}_n)$ is a neighbourhood of $\theta(\gamma)$.
Now each open convex set in $\mathbb{R}^n$ is the interior of its closure; by lemma 6.3, the set $\theta(\mathring{\Gamma}_n)$ is convex and its closure is $\mathcal{R}$, whence $\theta(\mathring{\Gamma}_n) = \mathring{\mathcal{R}}$.
Finally we show that the map θ is proper (i.e. that the inverse image of a compact subset is compact). Let K be a compact subset of F and $(\gamma^m)_{m\in\mathbb{N}}$ be a sequence in $\theta^{-1}(K)$ such that $\theta(\gamma^m)$ converges to $\mu(\rho)$ for some ρ, $0 < \rho < 1$. Since the sequence $(\gamma^m)_{m\in\mathbb{N}}$ is contained in Γ_n, by compactness, we may assume that γ^m converges to γ in Γ_n. By the continuity of θ, we have

$$\theta(\gamma) = \mu(E_\gamma^-) = \int_a^b \rho\, d\mu.$$

Uniqueness part of theorem 5.1 implies that γ belongs to $\mathring{\Gamma}_n$.
The map θ is proper and thus closed. It follows that its inverse θ^{-1} is continuous.
The equality $\theta(\partial\Gamma_n) = \partial\mathcal{R}$ is a consequence of the inclusion $\theta(\mathring{\Gamma}_n) \subset \mathring{\mathcal{R}}$ and the fact that θ is one to one. $\square$

We refer to [7] for the definitions of classical notions associated with convex sets. We have the following

Theorem 6.5. *The range $\mathcal{R}$ of an oriented measure is strictly convex.*

Proof. Let $\mu(E), \mu(F)$ be two distinct points of $\mathcal{R}$. By theorem 5.1 we may assume that the sets E and F are finite unions of closed intervals. Let $\lambda \in]0,1[$ and put $\bar\rho = \lambda\chi_E + (1-\lambda)\chi_F$. Assume for instance $E \setminus F \neq \emptyset$. Then there exists a non–trivial interval I such that

$$\forall x \in I \qquad \bar\rho(x) = \lambda\chi_E(x) + (1-\lambda)\chi_F(x) = \lambda.$$

Put $\epsilon = \min(\lambda, 1-\lambda)$. Lemma 6.1 applied to $\bar\rho, I, \epsilon$ shows that $\mu(\bar\rho)$ belongs to $\mathring{\mathcal{R}}$. $\square$

Corollary 6.6. *Let E be a measurable subset of $[a,b]$. Then $\mu(E)$ belongs to the boundary of $\mathcal{R}$ if and only if there exists a set F which is a finite union of intervals such that χ_F has less than $n-1$ discontinuity points and $E\Delta F$ is μ–negligible (such a set has also a zero Lebesgue measure).*

Proof. We first remark that the family of the sets which are a finite union of intervals and whose characteristic function has less than $n-1$ discontinuity points coincides with the family $\{\, E_\gamma^- : \gamma \in \partial\Gamma_n \,\}$.
Theorem 6.4 shows that $\mu(F)$ belongs to $\partial\mathcal{R}$ whenever $F = E_\gamma^-$ for some $\gamma \in \partial\Gamma_n$.
Conversely let E be such that $\mu(E)$ belongs to $\partial\mathcal{R}$. Theorem 6.4 yields the existence of a n–tuple γ belonging to $\partial\Gamma_n$ such that $\mu(E_\gamma^-) = \mu(E)$; a consequence of theorem 6.5 is that $\mu(E)$ is an extreme point of $\mathcal{R}$. Olech Theorem [5, Th. 1] implies that $E\Delta E_\gamma^-$ is μ–negligible. $\square$

Our approach discloses the recursive structure of the boundary of the range of an oriented measure. For k belonging to $\{0,\cdots,n\}$ let

$$\mathcal{R}_k^- = \{\, \mu(E_\gamma^-) : \gamma \in \Gamma_k \,\}\,, \qquad \mathcal{R}_k^+ = \{\, \mu(E_\gamma^+) : \gamma \in \Gamma_k \,\}.$$

Notice that $\Gamma_0 = \emptyset$, $\mathcal{R}_0^- = \{0\}$, $\mathcal{R}_0^+ = \{\mu(a,b)\}$.

Proposition 6.7. *The function $\gamma \in \overset{\circ}{\Gamma}_k \longmapsto \mu(E_\gamma^-) \in \mathcal{R}_k^-$ (resp. $\gamma \in \overset{\circ}{\Gamma}_k \longmapsto \mu(E_\gamma^+) \in \mathcal{R}_k^+$) is a homeomorphism from $\overset{\circ}{\Gamma}_k$ onto its range which coincides with $\overset{\circ}{\mathcal{R}}{}_k^-$ (resp. $\overset{\circ}{\mathcal{R}}{}_k^+$).*

Proof. Injectivity follows directly from corollary 6.6. The rest of the proof uses the techniques of the proof of theorem 6.4. $\square$

Remark. For each k in $\{1,\cdots,n-1\}$, the set $\mathcal{R}_k \setminus \mathcal{R}_{k-1}$ is partitioned into two connected components $\overset{\circ}{\mathcal{R}}{}_k^-, \overset{\circ}{\mathcal{R}}{}_k^+$. However, for $k=n$, $\mathcal{R}_n^- = \mathcal{R}_n^+ = \mathcal{R}$.
These results yield the following

Proposition 6.8. *The boundary of the range $\mathcal{R}$ of an oriented n–dimensional measure admits the decomposition*

$$\partial\mathcal{R} = \overset{\circ}{\mathcal{R}}{}_{n-1}^- \cup \cdots \cup \overset{\circ}{\mathcal{R}}{}_1^- \cup \{0\} \cup \{\mu(a,b)\} \cup \overset{\circ}{\mathcal{R}}{}_1^+ \cup \cdots \cup \overset{\circ}{\mathcal{R}}{}_{n-1}^+.$$

Let T be the symmetry with respect to $\mu(a,b)/2$ (so that for each measurable subset A of $[a,b]$, $T(\mu(A)) = \mu([a,b]\setminus A)$). Then for each k belonging to $\{0,\cdots,n\}$ we have

$$T(\overset{\circ}{\mathcal{R}}{}_k^-) = \overset{\circ}{\mathcal{R}}{}_k^+, \qquad T(\mathcal{R}_k) = \mathcal{R}_k.$$

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