

The distribution of the quasispecies for a Moran model on the sharp peak landscape

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December 11, 2013

Abstract

We consider the Moran model on the sharp peak landscape, in the asymptotic regime studied in [3], where a quasispecies is formed. We find explicitly the distribution of this quasispecies.

1 Introduction

In his paper [9], Eigen introduced the model of quasispecies to describe the evolution of a population of macromolecules which is subject to two main forces: mutation and selection. The model was developed further in a series of papers by Eigen and Schuster [11, 12, 13], and analysed in great detail by Eigen, McCaskill and Schuster in [10]. A major conclusion is that this kind of evolutionary process, rather than selecting a single dominant species, is more likely to select a master sequence (the macromolecule with the highest fitness) along with a cloud of mutants that closely resemble the master sequence. Hence the name quasispecies. One other major discovery that Eigen made on this model was the existence of an error threshold allowing a quasispecies to form: if the mutation rate exceeds the error threshold, then the population evolves towards a totally random state, whereas if the mutation rate is below the error threshold, a quasispecies can be formed.

Even if Eigen's original goal was to explain the behaviour of a population of macromolecules, the theory of quasispecies rapidly extended to other areas of biology. In particular, experimental studies support the validity of the model in virology [7]. Some RNA viruses are known to have very high mutation rates, like the HIV virus, and this is a factor of resistance against conventional drugs. A promising strategy to combat this kind of viruses consists in developing mutagenic drugs that would increase the mutation rate

beyond the error threshold, in order to induce an error catastrophe [2, 23]. This strategy has successfully been applied to several types of RNA viruses [4]. Moreover, several similarities have been observed between the evolution of cancer cell populations and RNA viruses, in particular, the possibility of inducing an error catastrophe [22].

Two important features of Eigen's model are its deterministic nature (the model is based on a system of differential equations derived from certain chemical and physical laws) and the fact that the population is considered to be infinite. When dealing with simple macromolecules, these assumptions are quite natural. Nevertheless, they become unrealistic if we want to apply this model to population genetics, and they are two of the major drawbacks when applying it to virus populations, as pointed out by Wilke [24]. On one hand, we have to take into account the stochastic nature of the evolution of a finite population. The higher the complexity of the individuals, the harder it is to explain the replication and mutation schemes via chemical reactions. This fact, together with the widely recognised role of randomness in evolutionary processes strongly suggest a stochastic approach to the matter. On the other hand, when dealing with populations of complex individuals, the amount of possible genotypes largely exceeds the size of the population. Therefore, if we want to use Eigen's model in population genetics, a finite and stochastic version of the model is called for.

The interest of a finite stochastic counterpart to Eigen's model is not new. Eigen, McCaskill and Schuster already emphasise the importance of developing such a model [10], so does Wilke in the more recent paper [24]. Several researchers have pursued this task. Demetrius, Schuster and Sigmund [5] introduce stochasticity into Eigen's model using branching processes. McCaskill [15] also develops a stochastic version of Eigen's model. Nowak and Schuster [18] use birth and death Markov processes to give a finite stochastic version of Eigen's model on the sharp peak landscape. Alves and Fontanari [1] study the dependence of the error threshold on the population size for the single sharp peak replication landscape. Saakian, Deem and Hu [20] compute the variance of the mean fitness in a finite population model in order to control how it approximates the infinite population model. Deem, Muñoz and Park [19] use a field theoretic representation in order to derive analytical results. Other recent papers introduce finite stochastic models that approach Eigen's model asymptotically when the population size goes to ∞ , like Musso [17] or Dixit, Srivastava, Vishnoi [6].

In [3], Cerf studies a population of size m of chromosomes of length ℓ over

an alphabet \mathcal{A} of cardinality κ evolving according to a Moran model [16]. The mutation probability per locus is q . Only the sharp peak landscape is considered: the master sequence, which we denote by w^* , replicates with rate $\sigma > 1$, while all the other sequences replicate with rate 1. In the asymptotic regime where

$$\begin{aligned} \ell &\rightarrow +\infty, & m &\rightarrow +\infty, & q &\rightarrow 0, \\ \ell q &\rightarrow a, & \frac{m}{\ell} &\rightarrow \alpha, \end{aligned}$$

a critical curve is obtained in the parameter space (a, α) , which is given by $\alpha\phi(a) = \ln \kappa$. If $\alpha\phi(a) < \ln \kappa$, then the population is totally random, i.e., the fraction of the master sequence in a population at equilibrium converges to 0. On the contrary, if $\alpha\phi(a) > \ln \kappa$, then a quasispecies is formed, i.e., at equilibrium, the population contains a positive fraction of the master sequence, which in the asymptotic regime presented above converges to $(\sigma e^{-a} - 1)/(\sigma - 1)$.

The aim of our article is to obtain the whole distribution of the quasispecies. As it is customary with this kind of models, we introduce Hamming classes with respect to the master sequence in the space \mathcal{A}^ℓ of sequences of length ℓ . We say that a chromosome $u \in \mathcal{A}^\ell$ belongs to the class $d \in \{0, \dots, \ell\}$ if it differs from the master sequence in exactly d characters, i.e.,

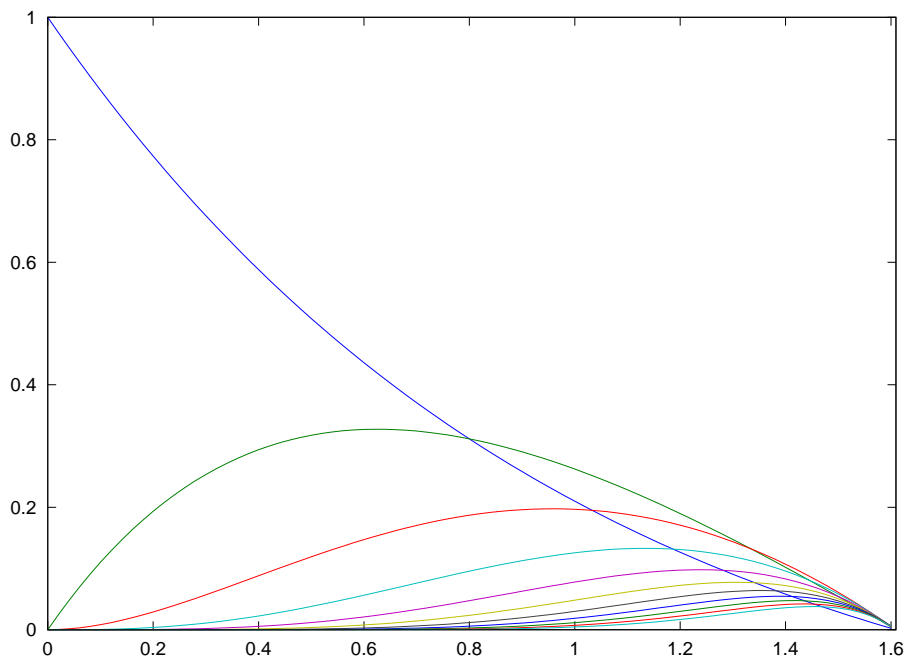
$$\text{card} \{ i \in \{0, \dots, \ell\} : w^*(i) \neq u(i) \} = d.$$

We study then the concentration of each of these classes in a population at equilibrium. For $k \geq 0$ fixed, in the above asymptotic regime, we recover the critical curve $\alpha\phi(a) = \ln \kappa$. If $\alpha\phi(a) < \ln \kappa$, then the fraction of the class k converges to 0, whereas if $\alpha\phi(a) > \ln \kappa$, then the fraction of the class k in a population at equilibrium converges to

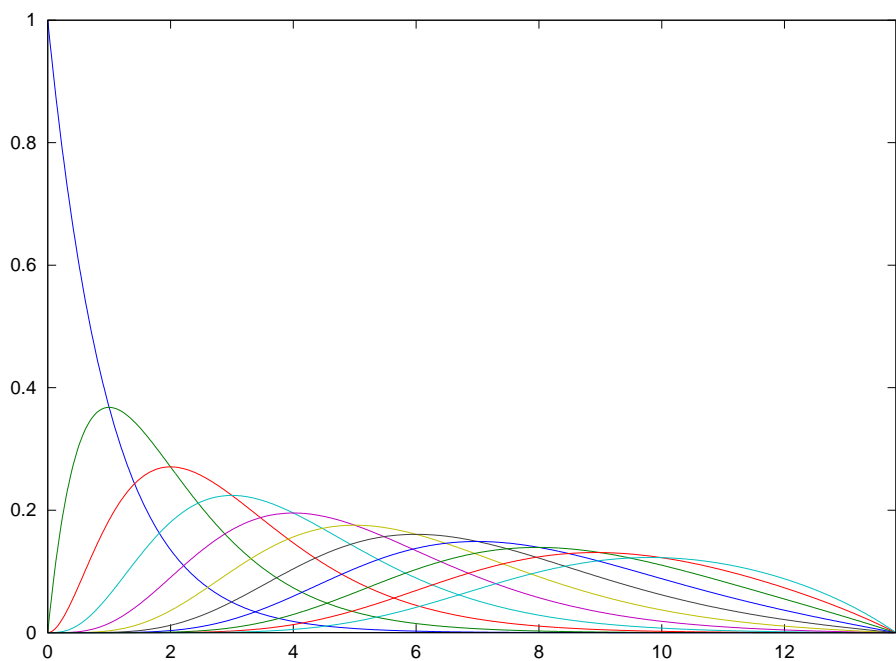
$$\rho_k^* = (\sigma e^{-a} - 1) \frac{a^k}{k!} \sum_{i \geq 1} \frac{i^k}{\sigma^i}.$$

We denote by $\mathcal{Q}(\sigma, a)$ the probability distribution which assigns mass ρ_k^* to k , for $k \geq 0$, and we call it the distribution of the quasispecies with parameters σ, a .

The article is organised as follows. First, we present our main result, along with a sketch of the proof, a brief discussion about the distribution of the quasispecies and some background material from [3]. The remaining sections are devoted to the proof.



Frequency of the master sequence and the first 10 classes for $\sigma = 5$



Frequency of the master sequence and the first 10 classes for $\sigma = 10^6$

2 Main Result

Let \mathcal{A} be a finite alphabet of cardinality κ and $\ell \geq 1$ an integer. We consider the space \mathcal{A}^ℓ of sequences of length ℓ over the alphabet \mathcal{A} . Elements of the space \mathcal{A}^ℓ represent the chromosome of an haploid individual. We consider a population of size m of individuals from \mathcal{A}^ℓ . The size of the population m is kept constant throughout the evolution.

When a reproduction occurs, the chromosome is subject to mutations. We suppose that mutations occur independently at random at each locus, with probability $q \in]0, 1 - 1/\kappa[$. If a mutation occurs, we replace the letter with a new one, chosen uniformly at random between the remaining $\kappa - 1$ letters of the alphabet \mathcal{A} . The mutation mechanism is encoded in a mutation matrix $(M(u, v), u, v \in \mathcal{A}^\ell)$, where $M(u, v)$ is the probability that the chromosome u is transformed into v by mutation. We have the following analytical expression for $M(u, v)$:

$$M(u, v) = \prod_{j=1}^{\ell} \left((1 - q)1_{u(j)=v(j)} + \frac{q}{\kappa - 1}1_{u(j) \neq v(j)} \right).$$

The only allowed transformations in a population consist of replacing a chromosome of the population with a new one. For a population $x \in (\mathcal{A}^\ell)^m$, $j \in \{1, \dots, m\}$, $u \in \mathcal{A}^\ell$, we denote by $x(j \leftarrow u)$ the population x where the j -th chromosome $x(j)$ has been replaced by u :

$$x(j \leftarrow u) = \begin{pmatrix} x(1) \\ \vdots \\ x(j-1) \\ u \\ x(j+1) \\ \vdots \\ x(m) \end{pmatrix}.$$

The replication mechanism is encoded in a fitness function:

$$A : \mathcal{A}^\ell \longrightarrow [0, +\infty[.$$

The discrete time Moran model is the Markov chain $(X_t)_{t \geq 0}$, whose transition

matrix is given by

$$\forall t \in \mathbb{N} \quad \forall x \in (\mathcal{A}^\ell)^m \quad \forall j \in \{1, \dots, \ell\} \quad \forall u \in \mathcal{A}^\ell \setminus \{x(j)\}$$

$$P(X_{t+1} = x(j \leftarrow u) \mid X_t = x) = \frac{1}{m^2 \lambda} \sum_{1 \leq i \leq m} A(x(i)) M(x(i), u),$$

where $\lambda > 0$ is a constant such that

$$\lambda \geq \max \{ A(u) : u \in \mathcal{A}^\ell \}.$$

The other non-diagonal coefficients of the matrix are null. The diagonal coefficients are arranged so that the matrix is stochastic, i.e., the sum over each row equals 1. We will only consider the sharp peak landscape. Let $\sigma > 1$ be a real number. There exists a particular sequence, called the master sequence or the wild type, and denoted by w^* , for which the replication rate is σ . The replication rate for all other sequences is 1. The fitness function is then given by

$$\forall u \in \mathcal{A}^\ell \quad A(u) = \begin{cases} 1 & \text{if } u \neq w^*, \\ \sigma & \text{if } u = w^*. \end{cases}$$

We denote by d_H the Hamming distance between two chromosomes:

$$\forall u, v \in \mathcal{A}^\ell \quad d_H(u, v) = \text{card} \{ i : 1 \leq i \leq m, u(i) \neq v(i) \}.$$

Let x be a population in $(\mathcal{A}^\ell)^m$. We fix an integer $K \geq 0$ and we look at the number $N^K(x)$ of chromosomes in x which are at distance K or less from the master sequence:

$$N^K(x) = \text{card} \{ i : 1 \leq i \leq m, d_H(x(i), w^*) \leq K \}.$$

Let $\phi : \mathbb{R}^+ \rightarrow \mathbb{R}^+ \cup \{+\infty\}$ be the map given by

$$\forall a < \ln \sigma \quad \phi(a) = \frac{\sigma(1 - e^{-a}) \ln \frac{\sigma(1 - e^{-a})}{\sigma - 1} + \ln(\sigma e^{-a})}{(1 - \sigma(1 - e^{-a}))},$$

and $\phi(a) = 0$ for $a \geq \ln \sigma$. Let $(\rho_k^*)_{k \geq 0}$ be the sequence given by

$$\forall k \geq 0 \quad \rho_k^* = (\sigma e^{-a} - 1) \frac{a^k}{k!} \sum_{i \geq 1} \frac{i^k}{\sigma^i}.$$

We have then the following result:

Theorem 2.1. *Suppose that*

$$\begin{aligned} \ell &\rightarrow +\infty, & m &\rightarrow +\infty, & q &\rightarrow 0, \\ \ell q &\rightarrow a \in]0, +\infty[, & \frac{m}{\ell} &\rightarrow \alpha \in [0, +\infty]. \end{aligned}$$

We have the following dichotomy:

- *If $\alpha \phi(a) < \ln \kappa$, then*

$$\forall K \geq 0 \quad \lim_{\substack{\ell, m \rightarrow \infty, q \rightarrow 0 \\ \ell q \rightarrow a, \frac{m}{\ell} \rightarrow \alpha}} \lim_{t \rightarrow \infty} E\left(\frac{1}{m} N^K(X_t)\right) = 0.$$

- *If $\alpha \phi(a) > \ln \kappa$, then*

$$\forall K \geq 0 \quad \lim_{\substack{\ell, m \rightarrow \infty, q \rightarrow 0 \\ \ell q \rightarrow a, \frac{m}{\ell} \rightarrow \alpha}} \lim_{t \rightarrow \infty} E\left(\frac{1}{m} N^K(X_t)\right) = \rho_0^* + \dots + \rho_K^*.$$

Furthermore, in both cases

$$\forall K \geq 0 \quad \lim_{\substack{\ell, m \rightarrow \infty, q \rightarrow 0 \\ \ell q \rightarrow a, \frac{m}{\ell} \rightarrow \alpha}} \lim_{t \rightarrow \infty} \text{Var}\left(\frac{1}{m} N^K(X_t)\right) = 0.$$

2.1 Sketch of proof

The state space of the Markov chain $(X_t)_{t \geq 0}$ has cardinality $\kappa^{\ell m}$, which is too big to work with. The first step in proving theorem 2.1 is to reduce the state space. We use a classical technique called lumping and we build a simpler process $(O_t)_{t \geq 0}$, called the occupancy process, whose state space is much smaller. The occupancy process $(O_t)_{t \geq 0}$ keeps track of the number of chromosomes in each of the $\ell + 1$ Hamming classes. This process will be the main subject of our study. In addition, the state space of the process $(O_t)_{t \geq 0}$ is naturally endowed with a partial order which allows us to use coupling and monotonicity arguments.

We then compare the time that the process $(O_t)_{t \geq 0}$ spends having at least a sequence in one of the Hamming classes $0, \dots, K$ (which we call the persistence time), with the time the process $(O_t)_{t \geq 0}$ spends having no sequences in any of the Hamming classes $0, \dots, K$ (which we call the discovery time). Asymptotically, when $\alpha \phi(a) > \ln \kappa$, the persistence time becomes negligible

with respect to the discovery time, whereas when $\alpha\phi(a) < \ln \kappa$, it is the discovery time that becomes negligible with respect to the persistence time. This already proves the first assertion in theorem 2.1.

The second statement of the theorem requires much more work. We build first a coupling to compare the occupancy process $(O_t)_{t \geq 0}$ with simpler processes: a lower process $(O_t^\ell)_{t \geq 0}$, and an upper process $(O_t^{K+1})_{t \geq 0}$. These processes stochastically bound the occupancy process $(O_t)_{t \geq 0}$, and they only keep track of the number of chromosomes in the Hamming classes $0, \dots, K$. The goal is to show that the invariant probability measures of the processes $(O_t^\ell)_{t \geq 0}$ and $(O_t^{K+1})_{t \geq 0}$ both converge to the Dirac mass at the point $(\rho_0^*, \dots, \rho_K^*)$. This is achieved by estimating the typical time that the processes spend inside and outside a neighbourhood of $(\rho_0^*, \dots, \rho_K^*)$. The time they spend inside a neighbourhood of $(\rho_0^*, \dots, \rho_K^*)$ is typically of exponential order in m , whereas the time they spend outside such a neighbourhood is typically of polynomial order in m . These estimates are enough to complete the proof of theorem 2.1. The strategy to obtain the estimates is as follows. The mutation probabilities $M(u, v)$ for $u > v$ go to 0 when $\ell \rightarrow \infty$, $q \rightarrow 0$ and $\ell q \rightarrow a$. Thanks to this fact we can build the processes $(O_t^\ell)_{t \geq 0}$ and $(O_t^{K+1})_{t \geq 0}$ in such a way that, for $0 \leq k \leq K$, the evolution of the Hamming classes $0, \dots, k$ does not depend on the remaining Hamming classes. We can then proceed to prove the estimates by induction. Both the initial case and the inductive step boil down to the study of birth and death Markov chains, for which explicit formulas are available.

2.2 The distribution of the quasispecies

Let a be such that $\sigma e^{-a} > 1$. As stated in theorem 2.1, the distribution of the quasispecies of parameters σ, a , is given by the sequence $(\rho_k^*)_{k \geq 0}$:

$$\forall k \geq 0 \quad \rho_k^* = (\sigma e^{-a} - 1) \frac{a^k}{k!} \sum_{i \geq 1} \frac{i^k}{\sigma^i}.$$

Nevertheless, we will carry out the proof by induction, and in our proof we will not find the sequence $(\rho_k^*)_{k \geq 0}$ in the above form, it will be given by the

following recurrence relation:

$$\rho_0^* = \frac{\sigma e^{-a} - 1}{\sigma - 1},$$

$$\rho_k^* = \frac{e^{-a}}{(\sigma - 1)\rho_0^* + 1 - e^{-a}} \left(\sigma \frac{a^k}{k!} \rho_0^* + \sum_{l=1}^{k-1} \frac{a^{k-l}}{(k-l)!} \rho_l^* \right), \quad k \geq 1.$$

We show next how to derive the explicit formula for $(\rho_k^*)_{k \geq 0}$ from the recurrence relation. Firstly, we remark that replacing ρ_0^* by its value,

$$\frac{e^{-a}}{(\sigma - 1)\rho_0^* + 1 - e^{-a}} = \frac{1}{\sigma - 1}.$$

Therefore, we can rewrite the recurrence relation as follows:

$$\rho_0^* = \frac{\sigma e^{-a} - 1}{\sigma - 1}$$

$$\rho_k^* = \frac{a^k}{k!} \rho_0^* + \frac{1}{\sigma} \sum_{l=1}^k \frac{a^{k-l}}{(k-l)!} \rho_l^*, \quad k \geq 1.$$

Let f be the generating function of the sequence $(\rho_k^*)_{k \geq 0}$:

$$f(X) = \sum_{k \geq 0} \rho_k^* X^k.$$

Let us consider the following formal series:

$$\frac{1}{\sigma} e^{aX} = \sum_{k \geq 0} \frac{1}{\sigma} \frac{a^k}{k!} X^k.$$

We do the Cauchy product of the two formal series above and we obtain:

$$f(X) \left(\frac{1}{\sigma} e^{aX} \right) = \sum_{k \geq 0} \left(\frac{1}{\sigma} \sum_{l=0}^k \frac{a^{k-l}}{(k-l)!} \rho_l^* \right) X^k.$$

Yet, thanks to the recurrence relation,

$$\frac{1}{\sigma} \sum_{l=0}^k \frac{a^{k-l}}{(k-l)!} \rho_l^* = \rho_k^* + \left(\frac{1}{\sigma} - 1 \right) \frac{a^k}{k!} \rho_0^*.$$

Thus,

$$f(X) \left(\frac{1}{\sigma} e^{aX} \right) = f(X) + \left(\frac{1}{\sigma} - 1 \right) \rho_0^* e^{aX}.$$

Replacing ρ_0^* with its value gives the following expression for f :

$$f(X) = (\sigma e^{-a} - 1) \frac{e^{aX}}{\sigma - e^{aX}}.$$

We remark that $f(1) = 1$, thus $(\rho_k^*)_{k \geq 0}$ is indeed a probability distribution on \mathbb{N} . We develop this last expression as follows:

$$\frac{e^{aX}}{\sigma - e^{aX}} = \sum_{i \geq 1} \left(\frac{e^{aX}}{\sigma} \right)^i = \sum_{i \geq 1} \frac{1}{\sigma^i} \sum_{k \geq 0} \frac{(aiX)^k}{k!} = \sum_{k \geq 0} \left(\sum_{i \geq 1} \frac{i^k}{\sigma^i} \right) \frac{a^k}{k!} X^k.$$

We obtain finally

$$\forall k \geq 0 \quad \rho_k^* = (\sigma e^{-a} - 1) \frac{a^k}{k!} \sum_{i \geq 1} \frac{i^k}{\sigma^i}.$$

We call this the probability distribution of the quasispecies with parameters σ , a and we denote it by $\mathcal{Q}(\sigma, a)$. A short calculation shows that the expectation and the variance of $\mathcal{Q}(\sigma, a)$ are given by:

$$\mathbb{E}(\mathcal{Q}) = \frac{\sigma a e^{-a}}{\sigma e^{-a} - 1}, \quad \text{Var}(\mathcal{Q}) = \frac{\sigma a e^{-a} (\sigma e^{-a} + a - 1)}{(\sigma e^{-a} - 1)^2}.$$

The graphs at the end of the introduction show the frequency of the master sequence and the first 10 Hamming classes for $\sigma = 5$ and $\sigma = 10^6$. The graphs closely resemble those obtained by solving the differential equations from Eigen's original model [10, 21].

2.3 The occupancy process

The occupancy process $(O_t)_{t \geq 0}$ will be the main subject of our study, it is obtained from the original process $(X_t)_{t \geq 0}$ via lumping, as in section 6.3 of [3]. Let $\mathcal{P}_{\ell+1}^m$ be the set of the ordered partitions of the integer m in at most $\ell + 1$ parts:

$$\mathcal{P}_{\ell+1}^m = \{ (o(0), \dots, o(\ell)) \in \mathbb{N}^{\ell+1} : o(0) + \dots + o(\ell) = m \}.$$

A partition $(o(0), \dots, o(\ell))$ is interpreted as an occupancy distribution, which corresponds to a population with $o(l)$ individuals in the Hamming class l , for $0 \leq l \leq \ell$. Since we are working with a Moran model, only a chromosome can

change classes at a time, i.e., the only possible transitions for the occupancy process $(O_t)_{t \geq 0}$ are of the form

$$o \longrightarrow o(k \rightarrow l), \quad 0 \leq k, l \leq \ell,$$

where $o(k \rightarrow l)$ is the occupancy distribution obtained by transferring a chromosome from the Hamming class k to the class l , i.e.,

$$\forall h \in \{0, \dots, \ell\} \quad o(k \rightarrow l)(h) = \begin{cases} o(h) & \text{if } h \neq k, l, \\ o(k) - 1 & \text{if } h = k, \\ o(l) + 1 & \text{if } h = l. \end{cases}$$

We will work with a discrete time occupancy process $(O_t)_{t \geq 0}$, whose transition matrix is given by

$$\forall o \in \mathcal{P}_{\ell+1}^m \quad \forall k, l \in \{0, \dots, \ell\}, \quad k \neq l, \\ p_{\mathcal{O}}(o, o(k \rightarrow l)) = \frac{o(k) \sum_{h=0}^{\ell} o(h) A_H(h) M_H(h, l)}{m \sum_{h=0}^{\ell} o(h) A_H(h)},$$

where A_H is the lumped fitness function, defined as follows

$$\forall b \in \{0, \dots, \ell\} \quad A_H(b) = \begin{cases} \sigma & \text{if } b = 0, \\ 1 & \text{if } b \geq 1, \end{cases}$$

and M_H is the lumped mutation matrix: for $b, c \in \{0, \dots, \ell\}$ the coefficient $M_H(b, c)$ is given by

$$\sum_{\substack{0 \leq k \leq \ell - b \\ 0 \leq l \leq b \\ k - l = c - b}} \binom{\ell - b}{k} \binom{b}{l} \left(p \left(1 - \frac{1}{\kappa}\right)\right)^k \left(1 - p \left(1 - \frac{1}{\kappa}\right)\right)^{\ell - b - k} \left(\frac{p}{\kappa}\right)^l \left(1 - \frac{p}{\kappa}\right)^{b - l}.$$

3 Stochastic bounds

In this section we will build a lower process $(O_t^\ell)_{t \geq 0}$ and an upper process $(O_t^{K+1})_{t \geq 0}$ in order to bound stochastically the occupancy process $(O_t)_{t \geq 0}$. The space $\mathcal{P}_{\ell+1}^m$ of the occupancy distributions is endowed with a natural

order \preceq . If o, o' are two occupancy distributions we write that $o \preceq o'$ if and only if

$$\forall l \in \{0, \dots, \ell\} \quad o(0) + \dots + o(l) \leq o'(0) + \dots + o'(l).$$

We will construct the lower process $(O_t^\ell)_{t \geq 0}$ and the upper process $(O_t^{K+1})_{t \geq 0}$ in such a way that for any $o \in \mathcal{P}_{\ell+1}^m$, if $O_0^\ell = O_0 = O_0^{K+1} = o$, then

$$\forall t \geq 0 \quad O_t^\ell \preceq O_t \preceq O_t^{K+1}.$$

The processes $(O_t^\ell)_{t \geq 0}$ and $(O_t^{K+1})_{t \geq 0}$ will be much simpler than the occupancy process $(O_t)_{t \geq 0}$.

3.1 The lower process

We start by building the lower process $(O_t^\ell)_{t \geq 0}$. First of all, let us explain loosely the dynamics of the lower process $(O_t^\ell)_{t \geq 0}$. As long as there is no master sequence present in the population, the lower process $(O_t^\ell)_{t \geq 0}$ evolves exactly as the original process $(O_t)_{t \geq 0}$. When a master sequence appears, all the chromosomes in the Hamming classes $K+1, \dots, \ell$ are sent to the class ℓ . As long as the master sequence is present in the population, a mutation to any of the classes $K+1, \dots, \ell$ is directly sent to the class ℓ . Furthermore, every mutation from a Hamming class to a lower class is also sent to the class ℓ . To make this construction rigorous, we will modify the coupling map Φ_O defined in section 7.1 of [3]. To do so, we will also use the maps \mathcal{M}_H and \mathcal{S}_O defined in the section 7.1 of [3]. We take \mathcal{R} to be the set

$$\mathcal{R} = [0, 1] \times \{0, \dots, m\}^2 \times [0, 1]^\ell,$$

and we define a map

$$\underline{\Phi}_O : \mathcal{P}_{\ell+1}^m \times \mathcal{R} \longrightarrow \mathcal{P}_{\ell+1}^m,$$

as follows. Let $r = (s, i, j, u_1, \dots, u_\ell) \in \mathcal{R}$ and $o \in \mathcal{P}_{\ell+1}^m$. We take $l = \mathcal{S}_O(o, s)$ and k the only index in $\{0, \dots, \ell\}$ such that

$$o(0) + \dots + o(k-1) < j \leq o(0) + \dots + o(k).$$

We define the map $\underline{\Phi}_O$ by:

$$\underline{\Phi}_O(o, r) = \begin{cases} o(k \rightarrow \ell) & \text{if } \mathcal{M}_H(l, u_1, \dots, u_\ell) < l, \\ o(k \rightarrow \mathcal{M}_H(l, u_1, \dots, u_\ell)) & \text{otherwise.} \end{cases}$$

From this construction we see that

$$\forall r \in \mathcal{R} \quad \forall o \in \mathcal{P}_{\ell+1}^m \quad \underline{\Phi}_O(o, r) \preceq \Phi_O(o, r).$$

We define a map $\pi_\ell : \mathcal{P}_{\ell+1}^m \rightarrow \mathcal{P}_{\ell+1}^m$ by setting, for $o \in \mathcal{P}_{\ell+1}^m$ and $l \in \{0, \dots, \ell\}$,

$$\pi_\ell(o)(l) = \begin{cases} o(l) & \text{if } 0 \leq l \leq K, \\ 0 & \text{if } K < l < \ell, \\ m - (o(0) + \dots + o(K)) & \text{if } l = \ell. \end{cases}$$

This map satisfies

$$\forall o \in \mathcal{P}_{\ell+1}^m \quad \pi_\ell(o) \preceq o.$$

We denote by \mathcal{W}^* the set of the occupancy distributions having at least one master sequence, i.e.,

$$\mathcal{W}^* = \{ o \in \mathcal{P}_{\ell+1}^m : o(0) \geq 1 \},$$

and we denote by \mathcal{N} the set of the occupancy distributions having no master sequence, i.e.,

$$\mathcal{N} = \{ o \in \mathcal{P}_{\ell+1}^m : o(0) = 0 \}.$$

Let us define

$$o_{\text{exit}}^\ell = (0, \dots, 0, m), \quad o_{\text{enter}}^\ell = (1, 0, \dots, 0, m-1).$$

We define a lower map Φ_O^ℓ by setting for $o \in \mathcal{P}_{\ell+1}^m$ and $r \in \mathcal{R}$,

$$\Phi_O^\ell(o, r) = \begin{cases} \Phi_O(o, r) & \text{if } o \in \mathcal{N} \text{ and } \Phi_O(o, r) \notin \mathcal{W}^*, \\ o_{\text{enter}}^\ell & \text{if } o \in \mathcal{N} \text{ and } \Phi_O(o, r) \in \mathcal{W}^*, \\ \pi_\ell(\Phi_O(\pi_\ell(o), r)) & \text{if } o \in \mathcal{W}^* \text{ and } \Phi_O(\pi_\ell(o), r) \notin \mathcal{N}, \\ o_{\text{exit}}^\ell & \text{if } o \in \mathcal{W}^* \text{ and } \Phi_O(\pi_\ell(o), r) \in \mathcal{N}. \end{cases}$$

The next proposition compares the map Φ_O^ℓ to the map Φ_O .

Proposition 3.1. *For all $r \in \mathcal{R}$ and for all $o \in \mathcal{P}_{\ell+1}^m$,*

$$\Phi_O^\ell(o, r) \preceq \Phi_O(o, r).$$

Proof. Let us take $r \in \mathcal{R}$ and $o \in \mathcal{P}_{\ell+1}^m$. We consider the four following cases:

- If $o \in \mathcal{N}$ and $\Phi_O(o, r) \notin \mathcal{W}^*$, then

$$\Phi_O^\ell(o, r) = \Phi_O(o, r).$$

- If $o \in \mathcal{N}$ and $\Phi_O(o, r) \in \mathcal{W}^*$, we have

$$\Phi_O^\ell(o, r) = o_{\text{enter}}^\ell \quad \text{and} \quad \Phi_O(o, r)(0) = 1,$$

the inequality holds since for all $o \in \mathcal{P}_{\ell+1}^m$ with $o(0) = 1$, we have $o_{\text{enter}}^\ell \preceq o$.

- If $o \in \mathcal{W}^*$ and $\underline{\Phi}_O(\pi_\ell(o), r) \notin \mathcal{N}$, since the mapping $\underline{\Phi}_O$ is lower than Φ_O , we have

$$\underline{\Phi}_O(\pi_\ell(o), r) \preceq \Phi_O(\pi_\ell(o), r).$$

Also $\pi_\ell(o) \preceq o$, and ϕ_O is monotone, so that

$$\pi_\ell(\underline{\Phi}_O(\pi_\ell(o), r)) \preceq \underline{\Phi}_O(\pi_\ell(o), r) \preceq \Phi_O(\pi_\ell(o), r) \preceq \Phi_O(o, r).$$

- If $o \in \mathcal{W}^*$ and $\underline{\Phi}_O(\pi_\ell(o), r) \in \mathcal{N}$, we have $\Phi_O^\ell(o) = o_{\text{exit}}^\ell$, we remark then that for all $o \in \mathcal{P}_{\ell+1}^m$, $o_{\text{exit}}^\ell \preceq o$.

We finally conclude that $\Phi_O^\ell(o, r) \preceq \Phi_O(o, r)$, for all $o \in \mathcal{P}_{\ell+1}^m$ and for all $r \in \mathcal{R}$. \square

We define next the lower process $(O_t^\ell)_{t \geq 0}$. Let

$$R_n = (I_n, J_n, S_n, U_{n,1}, \dots, U_{n,\ell}), \quad n \geq 1,$$

be an i.i.d. sequence of random vectors with values in \mathcal{R} , as defined in [3]. The components of R_n are independent random variables with uniform distribution on their corresponding spaces. Let $o \in \mathcal{P}_{\ell+1}^m$ be the starting point of the process. We set $O_0^\ell = o$ and

$$\forall n \geq 1 \quad O_n^\ell = \Phi_O^\ell(O_{n-1}^\ell, R_n).$$

Proposition 3.2. *We suppose that the processes $(O_t^\ell)_{t \geq 0}$ and $(O_t)_{t \geq 0}$ have the same starting occupancy distribution o . We have then*

$$\forall t \geq 0 \quad O_t^\ell \preceq O_t.$$

The proof is similar to the proof of proposition 8.1 of [3].

3.2 Dynamics of the lower process

We study now the dynamics of the lower process $(O_t^\ell)_{t \geq 0}$ in \mathcal{W}^* . For the process $(O_t^\ell)_{t \geq 0}$, the states in the set

$$\mathcal{T}^\ell = \{ o \in \mathcal{P}_{\ell+1}^m : o(0) \geq 1 \text{ and } o(0) + \dots + o(K) + o(\ell) < m \}$$

are transient, and the states in the set $\mathcal{N} \cup (\mathcal{W}^* \setminus \mathcal{T}^\ell)$ form a recurrent class. We will therefore focus on the dynamics of the process $(O_t^\ell)_{t \geq 0}$ restricted to $\mathcal{W}^* \setminus \mathcal{T}^\ell$. Since

$$\mathcal{W}^* \setminus \mathcal{T}^\ell = \{ o \in \mathcal{P}_{\ell+1}^m : o(0) \geq 1 \text{ and } o(0) + \dots + o(K) + o(\ell) = m \},$$

a state in $\mathcal{W}^* \setminus \mathcal{T}^\ell$ is completely determined by the occupation numbers of the classes $0, \dots, K$. The process $(O_t^\ell)_{t \geq 0}$ always enters the set $\mathcal{W}^* \setminus \mathcal{T}^\ell$ at o_{enter}^ℓ . For $i \in \{0, \dots, K\}$, we denote by w_i the vector of \mathbb{N}^{K+1} given by:

$$\forall l \in \{0, \dots, K\} \quad w_i(l) = \begin{cases} 1 & \text{if } l = i, \\ 0 & \text{otherwise.} \end{cases}$$

The only possible transitions for the Hamming classes $0, \dots, K$ of the process $(O_t^\ell)_{t \geq 0}$ starting from a state in $\mathcal{W}^* \setminus \mathcal{T}^\ell$ are

$$(o(0), \dots, o(K)) \longrightarrow (o(0), \dots, o(K)) - w_i \\ \text{if } 1 \leq o(i), \quad 0 \leq i \leq K,$$

$$(o(0), \dots, o(K)) \longrightarrow (o(0), \dots, o(K)) + w_i \\ \text{if } o(0) + \dots + o(K) \leq m - 1, \quad 0 \leq i \leq K,$$

$$(o(0), \dots, o(K)) \longrightarrow (o(0), \dots, o(K)) - w_i + w_j \\ \text{if } 1 \leq o(i), \quad 0 \leq i, j \leq K, \quad i \neq j.$$

The process $(O_t^\ell)_{t \geq 0}$ always exits the set $\mathcal{W}^* \setminus \mathcal{T}^\ell$ at o_{exit}^ℓ . If the process $(O_t^\ell)_{t \geq 0}$ starts from a state in $\mathcal{W}^* \setminus \mathcal{T}^\ell$, until the time of exit from $\mathcal{W}^* \setminus \mathcal{T}^\ell$, the dynamics of $(O_t^\ell(0), \dots, O_t^\ell(K))$ is that of a Markov chain on the state space

$$\mathcal{E}_K = \{ z \in \mathbb{N}^{K+1} : z_0 + \dots + z_K \leq m \}.$$

Let us compute the associated transition probabilities. Let $z \in \mathcal{E}_K$.

- For $0 \leq i \leq K$ and $0 < z_0 + \dots + z_K < m$,

$$p(z, z + w_i) = \frac{m - \sum_{l=0}^K z_l}{m((\sigma - 1)z_0 + m)} \times \left(\sigma z_0 M_H(0, i) + \sum_{l=1}^i z_l M_H(l, i) \right).$$

- For $0 \leq i \leq K$ and $1 \leq z_i$,

$$p(z, z - w_i) = \frac{z_i}{m((\sigma - 1)z_0 + m)} \times \left(\sigma z_0 \left(1 - \sum_{h=0}^K M_H(0, h) \right) + \sum_{l=1}^K z_l \left(1 - \sum_{h=l}^K M_H(l, h) \right) + m - \sum_{l=0}^K z_l \right).$$

- For $0 \leq i, j \leq K$, $i \neq j$ and $1 \leq z_i$,

$$p(z, z - w_i + w_j) = \frac{z_i}{m((\sigma - 1)z_0 + m)} \times \left(\sigma z_0 M_H(0, j) + \sum_{l=1}^j z_l M_H(l, j) \right).$$

The other non-diagonal coefficients of the matrix are null. The diagonal coefficients are arranged so that the matrix is stochastic, i.e., the sum over each row equals 1.

Since we are interested in the dynamics of $(O_t^\ell)_{t \geq 0}$ in $\mathcal{W}^* \setminus \mathcal{T}^\ell$, the transition probabilities starting from a point in $\{z \in \mathcal{E}_K : z_0 = 0\}$ are not relevant. Moreover, the law of the exit point from $\{z \in \mathcal{E}_K : z_0 \geq 1\}$ is also not relevant, what matters is the law of the exit time. Therefore we will modify the matrix p into another stochastic matrix p^ℓ such that:

- Starting from $\{z \in \mathcal{E}_K : z_0 = 0\}$, there is a jump with probability 1 to $(1, 0, \dots, 0)$.
- The law of the exit time from $\{z \in \mathcal{E}_K : z_0 \geq 1\}$ is unchanged, but the exit point is $(0, \dots, 0)$ with probability 1.

More precisely, we define the matrix p^ℓ as follows:

- For $z, z' \in \mathcal{E}_K$ with $z_0 = 0$,

$$\begin{aligned} p^\ell(z, (1, 0, \dots, 0)) &= 1, \\ p^\ell(z, z') &= 0 \quad \text{if } z' \neq (1, 0, \dots, 0). \end{aligned}$$

- For $z, z' \in \mathcal{E}_K$ with $z_0 = 1$ and $z'_0 = 0$,

$$\begin{aligned} p^\ell(z, (0, \dots, 0)) &= p(z, z - w_0) + \sum_{i=1}^K p(z, z - w_0 + w_i), \\ p^\ell(z, z') &= 0 \quad \text{if } z'_0 = 0 \text{ and } z' \neq (0, \dots, 0). \end{aligned}$$

Finally, $p^\ell(z, z') = p(z, z')$ for all remaining $z, z' \in \mathcal{E}_K$.

Let $(Z_t^\ell)_{t \geq 0}$ be a Markov chain with state space \mathcal{E}_K , starting at the point $z^\ell = (1, 0, \dots, 0)$ and having for transition matrix p^ℓ . Since $(O_t^\ell)_{t \geq 0}$ always enters $\mathcal{W}^* \setminus \mathcal{T}^\ell$ at o_{enter}^ℓ and its dynamics inside $\mathcal{W}^* \setminus \mathcal{T}^\ell$ is the same as the dynamics of the chain $(Z_t^\ell)_{t \geq 0}$, we will rely on this Markov chain to compute the desired estimates.

3.3 The upper process

We build now the upper process $(O_t^{K+1})_{t \geq 0}$. First of all, let us explain loosely the dynamics of the upper process $(O_t^{K+1})_{t \geq 0}$. As long as there is no master sequence present in the population, the upper process $(O_t^{K+1})_{t \geq 0}$ evolves exactly as the original process $(O_t)_{t \geq 0}$. When a master sequence appears, all the chromosomes in the Hamming classes $K+1, \dots, \ell$ are sent to the class $K+1$. As long as the master sequence is present in the population, a mutation to any of the classes $K+1, \dots, \ell$ is directly sent to the class $K+1$. Furthermore, for all $c < b$, the mutation probability from the Hamming class b to the Hamming class c is taken to be equal to $M_H(c+1, c)$. To make this construction rigorous, we modify the mutation probabilities $(M_H(b, c), 0 \leq b, c \leq \ell)$ and we define new mutation probabilities $(M_H^{K+1}(b, c), 0 \leq b, c \leq \ell)$ for the process $(O_t^{K+1})_{t \geq 0}$. Let us set for $b \in \{0, \dots, K+1\}$ and $c \in \{0, \dots, \ell\}$,

$$M_H^{K+1}(b, c) = \begin{cases} M_H(c+1, c) & \text{if } 0 \leq c < b \leq K+1, \\ M_H(b, c) & \text{if } b \leq c \leq K, \\ 0 & \text{if } c \in \{K+2, \dots, \ell\}. \end{cases}$$

The coefficient $M_H^{K+1}(b, K+1)$ is adjusted so that each row adds up to 1, i.e., we take, for $b \in \{0, \dots, K+1\}$,

$$\begin{aligned} M_H^{K+1}(b, K+1) &= 1 - \sum_{h=0}^K M_H^{K+1}(b, h) = \\ &= 1 - \sum_{h=0}^{b-1} M_H(h+1, h) - \sum_{h=b}^K M_H(b, h). \end{aligned}$$

Moreover, for $b \in \{K+2, \dots, \ell\}$, we set

$$\forall c \in \{0, \dots, \ell\} \quad M_H^{K+1}(b, c) = M_H(b, c).$$

We must verify that $(M_H^{K+1}(b, c), 0 \leq b, c \leq \ell)$ is a stochastic matrix, i.e., that all entries of the matrix are non-negative and that each row adds up to 1. Since $(M_H(b, c), 0 \leq b, c \leq \ell)$ is already a stochastic matrix, these conditions are satisfied for the rows $M_H^{K+1}(b, \cdot)$, $b \in \{K+2, \dots, \ell\}$. For the first $K+2$ rows, the only thing left to verify is that the coefficient $M_H^{K+1}(b, K+1)$ is non-negative, in other words, that

$$\forall b \in \{0, \dots, K+1\} \quad \sum_{h=0}^{b-1} M_H(h+1, h) + \sum_{h=b}^K M_H(b, h) \leq 1.$$

However, we are interested in the asymptotic regime

$$\begin{aligned} \ell &\rightarrow +\infty, & m &\rightarrow +\infty, & q &\rightarrow 0, \\ \ell q &\rightarrow a, & \frac{m}{\ell} &\rightarrow \alpha. \end{aligned}$$

Thus, it is enough to verify the preceding inequalities for ℓ, m big enough and q small enough. The mutation probabilities have the following limits:

$$\forall b, c \geq 0 \quad \lim_{\substack{\ell, m \rightarrow \infty \\ q \rightarrow 0, \ell q \rightarrow a}} M_H(b, c) = \begin{cases} 0 & \text{if } 0 \leq c < b, \\ \frac{a^{c-b}}{(c-b)!} e^{-a} & \text{if } 0 \leq b \leq c. \end{cases}$$

We deduce that, for $b \in \{0, \dots, K+1\}$,

$$\begin{aligned} \lim_{\substack{\ell, m \rightarrow \infty \\ q \rightarrow 0, \ell q \rightarrow a}} \sum_{h=0}^{b-1} M_H(h+1, h) + \sum_{h=b}^K M_H(b, h) &= \\ \sum_{h=b}^K \frac{a^{h-b}}{(h-b)!} e^{-a} &= \sum_{k=0}^{K-b} \frac{a^k}{k!} e^{-a} < 1. \end{aligned}$$

Therefore, for ℓ, m big enough and q small enough, the modified mutation matrix $(M_H^{K+1}(b, c), 0 \leq b, c \leq \ell)$ is indeed stochastic. We build now two maps

$$\mathcal{M}'_H, \mathcal{M}_H^{K+1} : \{0, \dots, \ell\} \times [0, 1] \longrightarrow \{0, \dots, \ell\}$$

in order to couple the mutation mechanisms of the processes $(O_t)_{t \geq 0}$ and $(O_t^{K+1})_{t \geq 0}$. Naturally, this coupling will allow us to compare these processes. Let $b \in \{0, \dots, \ell\}$ and $u \in [0, 1]$. We define $\mathcal{M}'_H(b, u)$ to be the only index $c \in \{0, \dots, \ell\}$ such that

$$M_H(b, 0) + \dots + M_H(b, c-1) < u \leq M_H(b, 0) + \dots + M_H(b, c).$$

Likewise, we define $\mathcal{M}_H^{K+1}(b, u)$ to be the only index $c \in \{0, \dots, \ell\}$ such that

$$M_H^{K+1}(b, 0) + \dots + M_H^{K+1}(b, c-1) < u \leq M_H^{K+1}(b, 0) + \dots + M_H^{K+1}(b, c).$$

Lemma 3.3. *The map \mathcal{M}'_H is above the map \mathcal{M}_H^{K+1} in the following sense:*

$$\forall b \in \{0, \dots, \ell\} \quad \forall u \in [0, 1] \quad \mathcal{M}'_H(b, u) \geq \mathcal{M}_H^{K+1}(b, u).$$

Proof. Since $M_H(b, c) \leq M_H(c+1, c)$ for $b > c$, it follows from the definition of the matrix $(M_H^{K+1}(b, c), 0 \leq b, c \leq \ell)$ that

$$\forall b, c \in \{0, \dots, \ell\} \quad \sum_{h=0}^c M_H(b, h) \leq \sum_{h=0}^c M_H^{K+1}(b, h).$$

These inequalities imply the desired result. \square

We have also the following result:

Lemma 3.4. *The map \mathcal{M}'_H is non-decreasing with respect to its first argument, i.e.,*

$$\forall b, c \in \{0, \dots, \ell\} \quad \forall u \in [0, 1] \quad b \leq c \Rightarrow \mathcal{M}'_H(b, u) \leq \mathcal{M}'_H(c, u).$$

Proof. We consider the map

$$\mathcal{M}_H : \{0, \dots, \ell\} \times [0, 1]^\ell \rightarrow \{0, \dots, \ell\}$$

defined in section 7.1 of [3] by

$$\forall b \in \{0, \dots, \ell\} \quad \forall u_1, \dots, u_\ell \in [0, 1]^\ell$$

$$\mathcal{M}_H(b, u_1, \dots, u_\ell) = b - \sum_{k=1}^b 1_{u_k < p/\kappa} + \sum_{k=b+1}^{\ell} 1_{u_k > 1-p(1-1/\kappa)}.$$

The interest of this map lies in the following fact: if U_1, \dots, U_ℓ are i.i.d. uniform random variables taking values on the interval $[0, 1]$, then for all $b \in \{0, \dots, \ell\}$, the law of $\mathcal{M}_H(b, U_1, \dots, U_\ell)$ is given by the b -th row of the mutation matrix M_H , i.e.,

$$\forall c \in \{0, \dots, \ell\} \quad P(\mathcal{M}_H(b, U_1, \dots, U_\ell) = c) = M_H(b, c).$$

Moreover, we know thanks to lemma 7.1 of [3] that the map \mathcal{M}_H is non-decreasing with respect to the Hamming class, i.e., for all $b, c \in \{0, \dots, \ell\}$ and $u_1, \dots, u_\ell \in [0, 1]$,

$$b \leq c \Rightarrow \mathcal{M}_H(b, u_1, \dots, u_\ell) \leq \mathcal{M}_H(c, u_1, \dots, u_\ell).$$

Take $a, b, h \in \{0, \dots, \ell\}$ with $a \leq b$ and let U_1, \dots, U_ℓ be i.i.d. uniform random variables on $[0, 1]$. Thanks to the properties of the map \mathcal{M}_H , we have

$$M_H(a, 0) + \dots + M_H(a, h) = P(\mathcal{M}_H(a, U_1, \dots, U_\ell) \leq h) \geq P(\mathcal{M}_H(b, U_1, \dots, U_\ell) \leq h) = M_H(b, 0) + \dots + M_H(b, h).$$

This implies the desired result. \square

Let us define

$$\mathcal{R}' = [0, 1] \times \{0, \dots, \ell\}^2 \times [0, 1].$$

We build next two coupling maps

$$\Phi'_O, \bar{\Phi}_O : \mathcal{P}_{\ell+1}^m \times \mathcal{R}' \longrightarrow \mathcal{P}_{\ell+1}^m.$$

Take $r = (s, i, j, u) \in \mathcal{R}'$, and $o \in \mathcal{P}_{\ell+1}^m$. We set $l = \mathcal{S}_O(o, s)$ and we set k to be the only index in $\{0, \dots, \ell\}$ such that

$$o(0) + \dots + o(k-1) < j \leq o(0) + \dots + o(k).$$

The maps Φ'_O et $\bar{\Phi}_O$ are defined by:

$$\begin{aligned} \Phi'_O(o, r) &= o(k \rightarrow \mathcal{M}'_H(l, u)), \\ \bar{\Phi}_O(o, r) &= o(k \rightarrow \mathcal{M}^{K+1}_H(l, u)). \end{aligned}$$

We have thanks to lemma 3.3 that

$$\forall r \in \mathcal{R}' \quad \forall o \in \mathcal{P}_{\ell+1}^m \quad \Phi'_O(o, r) \preceq \bar{\Phi}_O(o, r).$$

Let

$$R'_n = (S'_n, I'_n, J'_n, U'_n), \quad n \geq 1$$

be an i.i.d. sequence of random vectors with values in \mathcal{R}' , the random variables S'_n, I'_n, J'_n, U'_n being independent and having the uniform law in their corresponding spaces. We also take the sequence $(R'_n)_{n \geq 1}$ to be independent of the sequence $(R_n)_{n \geq 1}$ defined in section 3.1. We build the process $(O_t)_{t \geq 0}$ with the help of the sequence $(R'_n)_{n \geq 1}$. Let $o \in \mathcal{P}_{\ell+1}^m$ be the starting point of the process, we set $O_0 = o$ and

$$\forall n \geq 1 \quad O_n = \Phi'_O(O_{n-1}, R'_n).$$

The next lemma shows that the process $(O_t)_{t \geq 0}$ is monotone.

Lemma 3.5. *The coupling map Φ'_O is non-decreasing with respect to the occupancy distribution, i.e.,*

$$\forall o, o' \in \mathcal{P}_{\ell+1}^m \quad \forall r \in \mathcal{R}' \quad o \preceq o' \Rightarrow \Phi'_O(o, r) \preceq \Phi'_O(o', r).$$

The proof is very similar to that of lemma 7.5 in [3], so we do not include it here. We build next the upper process $(O_t^{K+1})_{t \geq 0}$. We define a map $\pi_{K+1} : \mathcal{P}_{\ell+1}^m \rightarrow \mathcal{P}_{\ell+1}^m$ as follows: for $o \in \mathcal{P}_{\ell+1}^m$ and $l \in \{0, \dots, \ell\}$,

$$\pi_{K+1}(o)(l) = \begin{cases} o(l) & \text{if } 0 \leq l \leq K, \\ m - (o(0) + \dots + o(K)) & \text{if } l = K + 1, \\ 0 & \text{if } K + 2 \leq l \leq \ell. \end{cases}$$

This map satisfies

$$\forall o \in \mathcal{P}_{\ell+1}^m \quad o \preceq \pi_{K+1}(o).$$

We also define

$$o_{\text{exit}}^{K+1} = (0, m, 0, \dots, 0), \quad o_{\text{enter}}^{K+1} = (1, m - 1, 0, \dots, 0).$$

We build an upper map Φ_O^{K+1} by setting for $o \in \mathcal{P}_{\ell+1}^m$ and $r \in \mathcal{R}'$,

$$\Phi_O^{K+1}(o, r) = \begin{cases} \Phi'_O(o, r) & \text{if } o \in \mathcal{N} \text{ and } \Phi'_O(o, r) \notin \mathcal{W}^* \\ o_{\text{enter}}^{K+1} & \text{if } o \in \mathcal{N} \text{ and } \Phi'_O(o, r) \in \mathcal{W}^* \\ \pi_{K+1}(\overline{\Phi}_O(\pi_{K+1}(o), r)) & \text{if } o \in \mathcal{W}^* \text{ and } \overline{\Phi}_O(\pi_{K+1}(o), r) \notin \mathcal{N} \\ o_{\text{exit}}^{K+1} & \text{if } o \in \mathcal{W}^* \text{ and } \overline{\Phi}_O(\pi_{K+1}(o), r) \in \mathcal{N} \end{cases}$$

A proof similar to that of proposition 3.1 shows that:

Proposition 3.6. *For all $r \in \mathcal{R}'$ and for all $o \in \mathcal{P}_{\ell+1}^m$,*

$$\Phi'_O(o, r) \preceq \Phi_O^{K+1}(o, r).$$

We define an upper process $(O_t^{K+1})_{t \geq 0}$ with the help of the i.i.d. sequence $(R'_n)_{n \geq 1}$ and the map Φ_O^{K+1} . Let $o \in \mathcal{P}_{\ell+1}^m$ be the starting point of the process, we set $O_0^{K+1} = o$ and

$$\forall n \geq 1 \quad O_n^{K+1} = \Phi_O^{K+1}(O_{n-1}^{K+1}, R'_n).$$

Proposition 3.7. *Suppose that the processes $(O_t)_{t \geq 0}$, $(O_t^{K+1})_{t \geq 0}$, have the same starting occupancy distribution o . We have then*

$$\forall t \geq 0 \quad O_t \preceq O_t^{K+1}.$$

See proposition 8.1 of [3] for a detailed proof.

3.4 Dynamics of the upper process

We will now study the dynamics of the upper process $(O_t^{K+1})_{t \geq 0}$ in \mathcal{W}^* . For the process $(O_t^{K+1})_{t \geq 0}$, the states in the set

$$\mathcal{T}^{K+1} = \{ o \in \mathcal{P}_{\ell+1}^m : o(0) \geq 1 \text{ and } o(0) + \dots + o(K) + o(K+1) < m \}$$

are transient, and the states in the set $\mathcal{N} \cup (\mathcal{W}^* \setminus \mathcal{T}^{K+1})$ form a recurrent class. We will therefore focus on the dynamics of the process $(O_t^{K+1})_{t \geq 0}$ restricted to $\mathcal{W}^* \setminus \mathcal{T}^{K+1}$. Since

$$\mathcal{W}^* \setminus \mathcal{T}^{K+1} = \{ o \in \mathcal{P}_{\ell+1}^m : o(0) \geq 1 \text{ and } o(0) + \dots + o(K) + o(K+1) = m \},$$

a state in $\mathcal{W}^* \setminus \mathcal{T}^{K+1}$ is completely determined by the occupation numbers of the classes $0, \dots, K$. The process $(O_t^{K+1})_{t \geq 0}$ always enters the set $\mathcal{W}^* \setminus \mathcal{T}^{K+1}$ at o_{enter}^{K+1} . The only possible transitions for the Hamming classes $0, \dots, K$ of the process $(O_t^{K+1})_{t \geq 0}$ starting from a state in $\mathcal{W}^* \setminus \mathcal{T}^{K+1}$ are

$$(o(0), \dots, o(K)) \longrightarrow (o(0), \dots, o(K)) - w_i \\ \text{if } 1 \leq o(i), \quad 0 \leq i \leq K,$$

$$(o(0), \dots, o(K)) \longrightarrow (o(0), \dots, o(K)) + w_i \\ \text{if } o(0) + \dots + o(K) \leq m - 1, \quad 0 \leq i \leq K,$$

$$(o(0), \dots, o(K)) \longrightarrow (o(0), \dots, o(K)) - w_i + w_j \\ \text{if } 1 \leq o(i), \quad 0 \leq i, j \leq K, \quad i \neq j.$$

The process $(O_t^{K+1})_{t \geq 0}$ always exits the set $\mathcal{W}^* \setminus \mathcal{T}^{K+1}$ at o_{exit}^{K+1} . If the process $(O_t^{K+1})_{t \geq 0}$ starts from a state in $\mathcal{W}^* \setminus \mathcal{T}^{K+1}$, until the time of exit from $\mathcal{W}^* \setminus \mathcal{T}^{K+1}$, the dynamics of $(O_t^{K+1}(0), \dots, O_t^{K+1}(K))_{t \geq 0}$ is that of a Markov chain on the state space

$$\mathcal{E}_K = \{ z \in \mathbb{N}^{K+1} : z_0 + \dots + z_K \leq m \}.$$

Let us compute the associated transition probabilities. Let $z \in \mathcal{E}_K$.

- For $0 \leq i \leq K$ and $0 < z_0 + \dots + z_K < m$,

$$p(z, z + w_i) = \frac{m - \sum_{l=0}^K z_l}{m((\sigma - 1)z_0 + m)} \\ \times \left(\sigma z_0 M_H^{K+1}(0, i) + \sum_{l=1}^K z_l M_H^{K+1}(l, i) + \left(m - \sum_{l=0}^K z_l \right) M_H^{K+1}(K+1, i) \right).$$

- For $0 \leq i \leq K$ and $1 \leq z_i$,

$$p(z, z - w_i) = \frac{z_i}{m((\sigma - 1)z_0 + m)} \times \left(\sigma z_0 \left(1 - \sum_{h=0}^K M_H^{K+1}(0, h) \right) + \sum_{l=1}^K z_l \left(1 - \sum_{h=0}^K M_H^{K+1}(l, h) \right) + \left(m - \sum_{l=0}^K z_l \right) \left(1 - \sum_{h=0}^K M_H^{K+1}(K+1, h) \right) \right).$$

- For $0 \leq i, j \leq K$, $i \neq j$ and $1 \leq z_i$,

$$p(z, z - w_i + w_j) = \frac{z_i}{m((\sigma - 1)z_0 + m)} \times \left(\sigma z_0 M_H^{K+1}(0, j) + \sum_{l=1}^K z_l M_H^{K+1}(l, j) + \left(m - \sum_{l=0}^K z_l \right) M_H^{K+1}(K+1, j) \right).$$

The other non-diagonal coefficients of the matrix are null. The diagonal coefficients are arranged so that the matrix is stochastic, i.e., the sum over each row equals 1.

Since we are interested in the dynamics of $(O_t^{K+1})_{t \geq 0}$ in $\mathcal{W}^* \setminus \mathcal{T}^{K+1}$, the transition probabilities starting from a point in $\{z \in \mathcal{E}_K : z_0 = 0\}$ are not relevant. Moreover, the law of the exit point from $\{z \in \mathcal{E}_K : z_0 \geq 1\}$ is also not relevant, what matters is the law of the exit time. Therefore we will modify the matrix p into another stochastic matrix p^{K+1} such that:

- Starting from $\{z \in \mathcal{E}_K : z_0 = 0\}$, there is a jump with probability 1 to $(1, m - 1, 0, \dots, 0)$.
- The law of the exit time from $\{z \in \mathcal{E}_K : z_0 \geq 1\}$ is unchanged, but the exit point is $(0, \dots, 0)$ with probability 1.

More precisely, we define the matrix p^{K+1} as follows:

- For $z, z' \in \mathcal{E}_K$ with $z_0 = 0$,

$$\begin{aligned} p^{K+1}(z, (1, m - 1, \dots, 0)) &= 1, \\ p^{K+1}(z, z') &= 0 \quad \text{if } z' \neq (1, m - 1, \dots, 0). \end{aligned}$$

- For $z, z' \in \mathcal{E}_K$ with $z_0 = 1$ and $z'_0 = 0$,

$$\begin{aligned} p^{K+1}(z, (0, \dots, 0)) &= p(z, z - w_0) + \sum_{i=1}^K p(z, z - w_0 + w_i), \\ p^{K+1}(z, z') &= 0 \quad \text{if } z'_0 = 0 \text{ and } z' \neq (0, \dots, 0). \end{aligned}$$

Finally, $p^{K+1}(z, z') = p(z, z')$ for all remaining $z, z' \in \mathcal{E}_K$.

Let $(Z_t^{K+1})_{t \geq 0}$ be a Markov chain with state space \mathcal{E}_K , starting at the point $z^{K+1} = (1, m-1, 0, \dots, 0)$ and having for transition matrix p^{K+1} . Since the process $(O_t^{K+1})_{t \geq 0}$ always enters $\mathcal{W}^* \setminus \mathcal{T}^{K+1}$ at o_{enter}^{K+1} and its dynamics inside $\mathcal{W}^* \setminus \mathcal{T}^{K+1}$ is the same as the dynamics of the chain $(Z_t^{K+1})_{t \geq 0}$, we will rely on this Markov chain to compute the desired estimates.

3.5 Bounds on the invariant measure

We denote by $\mu_O^\ell, \mu_O, \mu_O^{K+1}$ the invariant probability measures of $(O_t^\ell)_{t \geq 0}, (O_t)_{t \geq 0}, (O_t^{K+1})_{t \geq 0}$. Let ν_K be the image measure of μ_O through the map

$$o \in \mathcal{P}_{\ell+1}^m \mapsto \frac{1}{m}(o(0) + \dots + o(K)) \in [0, 1].$$

For any function $f : [0, 1] \rightarrow \mathbb{R}$,

$$\begin{aligned} \int_{[0,1]} f d\nu_K &= \int_{\mathcal{P}_{\ell+1}^m} f\left(\frac{o(0) + \dots + o(K)}{m}\right) d\mu_O(o) \\ &= \lim_{t \rightarrow \infty} E\left(f\left(\frac{O_t(0) + \dots + O_t(K)}{m}\right)\right). \end{aligned}$$

We fix a non-decreasing function $f : [0, 1] \rightarrow \mathbb{R}$ such that $f(0) = 0$. Thanks to proposition 3.2 we have the following inequality:

$$\forall t \geq 0 \quad f\left(\frac{O_t^\ell(0) + \dots + O_t^\ell(K)}{m}\right) \leq f\left(\frac{O_t(0) + \dots + O_t(K)}{m}\right).$$

Moreover, thanks to proposition 3.7,

$$\forall t \geq 0 \quad f\left(\frac{O_t(0) + \dots + O_t(K)}{m}\right) \leq f\left(\frac{O_t^{K+1}(0) + \dots + O_t^{K+1}(K)}{m}\right).$$

We take the expectations and we send t to ∞ , and we obtain

$$\begin{aligned} \int_{\mathcal{P}_{\ell+1}^m} f\left(\frac{o(0) + \dots + o(K)}{m}\right) d\mu_O^\ell(o) \\ \leq \int_{[0,1]} f d\nu_K \leq \\ \int_{\mathcal{P}_{\ell+1}^m} f\left(\frac{o(0) + \dots + o(K)}{m}\right) d\mu_O^{K+1}(o). \end{aligned}$$

Our next goal is to find estimates of the above integrals. The strategy is the same for the lower and upper integrals. Let θ be either $K + 1$ or ℓ and let us study the invariant probability measure μ_O^θ . We will rely on a renewal result. Let $(X_t)_{t \geq 0}$ be a discrete time Markov chain taking values in a finite space \mathcal{E} . We suppose that $(X_t)_{t \geq 0}$ is irreducible and aperiodic and we call μ its invariant probability measure.

Proposition 3.8. *Let \mathcal{W}^* be a subset of \mathcal{E} and let e be a point in $\mathcal{E} \setminus \mathcal{W}^*$. Let f be a function from \mathcal{E} to \mathbb{R} . We define*

$$\tau^* = \inf \{ t \geq 0 : X_t \in \mathcal{W}^* \}, \quad \tau = \inf \{ t \geq \tau^* : X_t = e \}.$$

We have

$$\int_{\mathcal{E}} f(x) d\mu(x) = \frac{1}{E(\tau | X_0 = e)} E \left(\int_0^\tau f(X_s) ds \mid X_0 = e \right).$$

The proof is standard and similar to the proof of proposition 8.2 in [3], so we omit it. We apply this renewal result to the process $(O_t^\theta)_{t \geq 0}$ restricted to $\mathcal{N} \cup (\mathcal{W}^* \setminus \mathcal{T}^\theta)$, the set $\mathcal{W}^* \setminus \mathcal{T}^\theta$, the occupancy distribution o_{exit}^θ and the function $o \mapsto f((o(0) + \dots + o(K))/m)$. Set

$$\begin{aligned} \tau^* &= \inf \{ t \geq 0 : O_t^\theta \in \mathcal{W}^* \}, \\ \tau &= \inf \{ t \geq \tau^* : O_t^\theta = o_{\text{exit}}^\theta \}. \end{aligned}$$

We then have

$$\begin{aligned} & \int_{\mathcal{P}_{\ell+1}^m} f \left(\frac{o(0) + \dots + o(K)}{m} \right) d\mu_O^\theta(o) \\ &= \frac{E \left(\int_0^\tau f \left(\frac{O_s^\theta(0) + \dots + O_s^\theta(K)}{m} \right) ds \mid O_0^\theta = o_{\text{exit}}^\theta \right)}{E(\tau | O_0^\theta = o_{\text{exit}}^\theta)} \\ &= \frac{E \left(\int_0^{\tau^*} f \left(\frac{O_s^\theta(0) + \dots + O_s^\theta(K)}{m} \right) ds \mid O_0^\theta = o_{\text{exit}}^\theta \right)}{E(\tau | O_0^\theta = o_{\text{exit}}^\theta)} \\ & \quad + \frac{E \left(\int_{\tau^*}^\tau f \left(\frac{O_s^\theta(0) + \dots + O_s^\theta(K)}{m} \right) ds \mid O_0^\theta = o_{\text{exit}}^\theta \right)}{E(\tau | O_0^\theta = o_{\text{exit}}^\theta)}. \end{aligned}$$

As long as $(O_t^\theta)_{t \geq 0}$ is in $\mathcal{W}^* \setminus \mathcal{T}^\theta$, the dynamics of $(O_t^\theta(0), \dots, O_t^\theta(K))_{t \geq 0}$ is that of the Markov chain $(Z_t^\theta)_{t \geq 0}$ defined at the end of the sections 3.2 and

3.4. Suppose that $(Z_t^\theta)_{t \geq 0}$ starts from z^θ , where z^θ is the point of \mathcal{E}_K given by

$$z^\theta = \begin{cases} z^{K+1} = (1, 0, \dots, 0) & \text{if } \theta = K + 1, \\ z^\ell = (1, m - 1, 0, \dots, 0) & \text{if } \theta = \ell. \end{cases}$$

Let τ_0 be the first time that the coordinate 0 becomes null, i.e.,

$$\tau_0 = \inf \{ n \geq 0 : Z_n^\theta(0) = 0 \}.$$

Since the process $(O_t^\theta)_{t \geq 0}$ always enters the set $\mathcal{W}^* \setminus \mathcal{T}^\theta$ at o_{enter}^θ , the law of τ_0 is the same as the law of $\tau - \tau^*$ whenever the process $(O_t^\theta)_{t \geq 0}$ starts from o_{exit}^θ . We conclude that the laws of $((O_t^\theta(0), \dots, O_t^\theta(K)), \tau^* \leq t \leq \tau)$ and $(Z_t^\theta, 0 \leq t \leq \tau_0)$ are the same. In particular,

$$E(\tau - \tau^* \mid O_0^\theta = o_{\text{exit}}^\theta) = E(\tau_0 \mid Z_0^\theta = z^\theta),$$

and also

$$E\left(\int_{\tau^*}^{\tau} f\left(\frac{O_s^\theta(0) + \dots + O_s^\theta(K)}{m}\right) ds \mid O_0^\theta = o_{\text{exit}}^\theta\right) = E\left(\int_0^{\tau_0} f\left(\frac{Z_s^\theta(0) + \dots + Z_s^\theta(K)}{m}\right) ds \mid Z_0^\theta = z^\theta\right).$$

The formula for the invariant measure μ_O^θ can then be written as follows:

$$\begin{aligned} \int_{\mathcal{P}_{\ell+1}^m} f\left(\frac{o(0) + \dots + o(K)}{m}\right) d\mu_O^\theta(o) &= \\ &= \frac{E\left(\int_0^{\tau^*} f\left(\frac{O_s^\theta(0) + \dots + O_s^\theta(K)}{m}\right) ds \mid O_0^\theta = o_{\text{exit}}^\theta\right)}{E(\tau^* \mid O_0^\theta = o_{\text{exit}}^\theta) + E(\tau_0 \mid Z_0^\theta = z^\theta)} \\ &+ \frac{E\left(\int_0^{\tau_0} f\left(\frac{Z_s^\theta(0) + \dots + Z_s^\theta(K)}{m}\right) ds \mid Z_0^\theta = z^\theta\right)}{E(\tau^* \mid O_0^\theta = o_{\text{exit}}^\theta) + E(\tau_0 \mid Z_0^\theta = z^\theta)}. \end{aligned}$$

The process $(Z_t^\theta)_{t \geq 0}$ always enters the set $\{z \in \mathcal{E}_K : z_0 \geq 1\}$ at z^θ , and always exits the set $\{z \in \mathcal{E}_K : z_0 \geq 1\}$ at $(0, \dots, 0)$. In order to write the previous formula in terms of the invariant probability measure of $(Z_t^\theta)_{t \geq 0}$, we apply the renewal result stated in proposition 3.8 to the process $(Z_t^\theta)_{t \geq 0}$, the set $\{z \in \mathcal{E}_K : z_0 \geq 1\}$, the point 0 and the map $z \mapsto f((z_0 + \dots + z_K)/m)$. We set

$$\begin{aligned} \tau_1 &= \inf \{ t \geq 0 : Z_t^\theta(0) \geq 1 \}, \\ \tau'_0 &= \inf \{ t \geq \tau_1 : Z_t^\theta = 0 \}, \end{aligned}$$

and we denote by ν^θ the invariant probability measure of the process $(Z_t^\theta)_{t \geq 0}$. We have

$$\int_{\mathcal{E}_K} f\left(\frac{z_0 + \cdots + z_K}{m}\right) d\nu^\theta(z) = \frac{E\left(\int_0^{\tau'_0} f\left(\frac{Z_s^\theta(0) + \cdots + Z_s^\theta(K)}{m}\right) ds \mid Z_0^\theta = 0\right)}{E(\tau'_0 \mid Z_0^\theta = 0)}.$$

Since $p^\theta(0, z^\theta) = 1$, the Markov property yields

$$\int_{\mathcal{E}_K} f\left(\frac{z_0 + \cdots + z_K}{m}\right) d\nu^\theta(z) = \frac{E\left(\int_0^{\tau_0} f\left(\frac{Z_s^\theta(0) + \cdots + Z_s^\theta(K)}{m}\right) ds \mid Z_0^\theta = z^\theta\right)}{1 + E(\tau_0 \mid Z_0^\theta = z^\theta)}.$$

Reporting back in the formula for μ_O^θ , we get:

$$\begin{aligned} \int_{\mathcal{P}_{\ell+1}^m} f\left(\frac{o(0) + \cdots + o(K)}{m}\right) d\mu_O^\theta(o) &= \\ &= \frac{E\left(\int_0^{\tau^*} f\left(\frac{O_s^\theta(0) + \cdots + O_s^\theta(K)}{m}\right) ds \mid O_0^\theta = o_{\text{exit}}^\theta\right)}{E(\tau^* \mid O_0^\theta = o_{\text{exit}}^\theta) + E(\tau_0 \mid Z_0^\theta = z^\theta)} \\ &+ \frac{1 + E(\tau_0 \mid Z_0^\theta = z^\theta)}{E(\tau^* \mid O_0^\theta = o_{\text{exit}}^\theta) + E(\tau_0 \mid Z_0^\theta = z^\theta)} \int_{\mathcal{E}_K} f\left(\frac{z_0 + \cdots + z_K}{m}\right) d\nu^\theta(z). \end{aligned}$$

In the sequel, we will try to estimate each of the terms appearing on the right-hand side of this formula.

4 Induction and estimates

In this section we estimate the integral

$$\int_{\mathcal{E}_K} f\left(\frac{z_0 + \cdots + z_K}{m}\right) d\nu^\theta(z).$$

Suppose that $\sigma e^{-a} > 1$ and let $\rho^* = (\rho_0^*, \dots, \rho_K^*)$ be the point of $[0, 1]^{K+1}$ given by

$$\rho_k^* = (\sigma e^{-a} - 1) \frac{a^k}{k!} \sum_{i \geq 1} \frac{i^k}{\sigma^i}, \quad 0 \leq k \leq K.$$

Let $f : \mathcal{P}_{\ell+1}^m \rightarrow \mathbb{R}$ be a non-decreasing continuous function with $f(0) = 0$. We consider the following asymptotic regime:

$$\begin{aligned} \ell &\rightarrow +\infty, & m &\rightarrow +\infty, & q &\rightarrow 0, \\ \ell q &\rightarrow a, & \frac{m}{\ell} &\rightarrow \alpha. \end{aligned}$$

Our goal is to prove that in this regime

$$\lim_{\substack{\ell, m \rightarrow \infty, q \rightarrow 0 \\ \ell q \rightarrow a, \frac{m}{\ell} \rightarrow \alpha}} \int_{\mathcal{E}_K} f\left(\frac{z_0 + \dots + z_K}{m}\right) d\nu^\theta(z) = f(\rho_0^* + \dots + \rho_K^*).$$

In order to prove this convergence, we estimate some hitting times associated to the Markov chain $(Z_t^\theta)_{t \geq 0}$. Let us define for $0 \leq k \leq K$ and for $\delta > 0$,

$$U_k(\delta) = \left\{ z \in \mathcal{E}_K : \left| \frac{z_i}{m} - \rho_i^* \right| < \delta, 0 \leq i \leq k \right\}.$$

We also define, for any subset $A \subset \mathcal{E}_K$ the hitting time of A :

$$\tau(A) = \inf\{t \geq 0 : Z_t^\theta \in A\}.$$

Theorem 4.1. *Let $\delta > 0$. For all $0 \leq k \leq K$, there exist positive real numbers $\alpha_k, \alpha'_k, \beta_k, \beta'_k$ (all of them depending on δ), such that for ℓ, m big enough and for q small enough, we have:*

- For all $z \in \mathcal{E}_K$,

$$P(\tau(U_k(\delta)) \geq m^{\alpha_k} \mid Z_0^\theta = z) \leq \exp(-\alpha'_k m).$$

- For all $z \in U_k(\delta)$,

$$P(\tau(U_k(2\delta)^c) \leq \exp(\beta_k m) \mid Z_0^\theta = z) \leq \exp(-\beta'_k m).$$

We will prove this theorem by induction on k . The strategy is as follows. From the definition of $U_k(\delta)$ we see that

$$U_0(\delta) \supset U_1(\delta) \supset \dots \supset U_K(\delta),$$

and that in order to know whether the process $(Z_t^\theta)_{t \geq 0}$ is in $U_k(\delta)$, it is enough to check the first $k + 1$ coordinates, $Z_t^\theta(0), \dots, Z_t^\theta(k)$. The process $(Z_t^\theta)_{t \geq 0}$ has been built in such a way that the dynamics of $(Z_t^\theta(k))_{t \geq 0}$ does not depend on the coordinates $k + 1, \dots, K$. The case $k = 0$ boils down to the study of birth and death Markov chains, which are very similar to the ones studied in section 9 of [3]. If the estimates hold at rank $k - 1$, then we know that the process $(Z_t^\theta)_{t \geq 0}$ spends almost all of its time inside the set $U_{k-1}(\delta)$. As long as the process is in $U_{k-1}(\delta)$, we can bound stochastically the dynamics of $(Z_t^\theta(k))_{t \geq 0}$ with a pair of birth and death chains, which can be studied with the same techniques as the birth and death chains from the case $k = 0$. To begin with, we show that the process $(Z_t^\theta(k))_{t \geq 0}$ does not depend on the coordinates $k + 1, \dots, K$. Let us take k in $\{0, \dots, K\}$, $z = (z_0, \dots, z_K)$ a point in \mathcal{E}_K , and let us compute the following transition probabilities:

$$P(Z_{t+1}^\theta(k) = z_k + 1 \mid Z_t^\theta = z) = p^\theta(z, z + w_k) + \sum_{\substack{0 \leq i \leq K \\ i \neq k}} p^\theta(z, z + w_k - w_i),$$

$$P(Z_{t+1}^\theta(k) = z_k - 1 \mid Z_t^\theta = z) = p^\theta(z, z - w_k) + \sum_{\substack{0 \leq i \leq K \\ i \neq k}} p^\theta(z, z - w_k + w_i).$$

In the case of the lower process $(Z_t^\ell)_{t \geq 0}$, we obtain

$$p^\ell(z, z + w_k) + \sum_{\substack{0 \leq i \leq K \\ i \neq k}} p^\ell(z, z + w_k - w_i) =$$

$$\frac{m - z_k}{m((\sigma - 1)z_0 + m)} \times \left(\sigma z_0 M_H(0, k) + \sum_{l=1}^k z_l M_H(l, k) \right),$$

$$p^\ell(z, z - w_k) + \sum_{\substack{0 \leq i \leq K \\ i \neq k}} p^\ell(z, z - w_k + w_i) = \frac{z_k}{m((\sigma - 1)z_0 + m)}$$

$$\times \left(\sigma z_0 (1 - M_H(0, k)) + \sum_{l=1}^k z_l (1 - M_H(l, k)) + m - \sum_{l=0}^k z_l \right).$$

In the case of the upper process $(Z_t^{K+1})_{t \geq 0}$, we obtain

$$p^{K+1}(z, z + w_k) + \sum_{\substack{0 \leq i \leq K \\ i \neq k}} p^{K+1}(z, z + w_k - w_i) = \frac{m - z_k}{m((\sigma - 1)z_0 + m)}$$

$$\times \left(\sigma z_0 M_H(0, k) + \sum_{l=1}^k z_l M_H(l, k) + \left(m - \sum_{l=0}^k z_l \right) M_H(k + 1, k) \right),$$

$$\begin{aligned}
p^{K+1}(z, z - w_k) + \sum_{\substack{0 \leq i \leq K \\ i \neq k}} p^{K+1}(z, z - w_k + w_i) = \\
\frac{z_k}{m((\sigma - 1)z_0 + m)} \left(\sigma z_0(1 - M_H(0, k)) + \sum_{l=1}^k z_l(1 - M_H(l, k)) \right. \\
\left. + \left(m - \sum_{l=0}^k z_l \right) (1 - M_H(k + 1, k)) \right).
\end{aligned}$$

We observe that none of the above expressions depend on z_{k+1}, \dots, z_K . Moreover, if we define for $k \in \{0, \dots, K\}$,

$$M_H^\ell(\ell, k) = 0, \quad M_H^{K+1}(K + 1, k) = M_H(k + 1, k),$$

we can rewrite the above transition probabilities as follows:

$$\begin{aligned}
p^\theta(z, z + w_k) + \sum_{\substack{0 \leq i \leq K \\ i \neq k}} p^\theta(z, z + w_k - w_i) = \frac{m - z_k}{m((\sigma - 1)z_0 + m)} \\
\times \left(\sigma z_0 M_H(0, k) + \sum_{l=1}^k z_l M_H(l, k) + \left(m - \sum_{l=0}^k z_l \right) M_H^\theta(\theta, k) \right),
\end{aligned}$$

$$\begin{aligned}
p^\theta(z, z - w_k) + \sum_{\substack{0 \leq i \leq K \\ i \neq k}} p^\theta(z, z - w_k + w_i) = \\
\frac{z_k}{m((\sigma - 1)z_0 + m)} \left(\sigma z_0(1 - M_H(0, k)) + \sum_{l=1}^k z_l(1 - M_H(l, k)) \right. \\
\left. + \left(m - \sum_{l=0}^k z_l \right) (1 - M_H^\theta(\theta, k)) \right).
\end{aligned}$$

We first give some general results concerning birth and death Markov chains, which we then use to study both the initial case $k = 0$ and the inductive step.

4.1 Birth and death Markov chains

First of all we give some explicit formulas for mean hitting times of birth and death Markov chains. These formulas can be derived from the classical

formulas for finite state space Markov chains (see [14], chapter 6). Next, we apply these formulas to study the limiting behaviour of a family of birth and death Markov chains, under suitable hypotheses. Let $(Z_t)_{t \geq 0}$ be a birth and death Markov chain taking values in $\{0, \dots, m\}$ and having the following transition probabilities

$$\begin{aligned} P(Z_{t+1} = i+1 \mid Z_t = i) &= \delta_i, & 0 \leq i < m, \\ P(Z_{t+1} = i-1 \mid Z_t = i) &= \gamma_i, & 0 < i \leq m. \end{aligned}$$

Let us define

$$\pi(0) = 1, \quad \pi(i) = \frac{\delta_1 \cdots \delta_i}{\gamma_1 \cdots \gamma_i}, \quad 0 < i < m.$$

For a subset $A \subset \{0, \dots, m\}$, we define the hitting time of A by

$$\tau(A) = \inf \{t \geq 0 : Z_t \in A\}.$$

Let a, b be two points in $\{0, \dots, m\}$, with $a < b$. We have the following formula for the hitting time of b starting from a :

$$E(\tau(\{b\}) \mid Z_0 = a) = \sum_{i=a}^{b-1} \sum_{j=i}^{b-1} \frac{1}{\delta_i} \frac{\pi(i)}{\pi(j)}.$$

Likewise, we have the following formula for the hitting time of a starting from b :

$$E(\tau(\{a\}) \mid Z_0 = b) = \sum_{i=a}^{b-1} \sum_{j=i}^{b-1} \frac{1}{\gamma_{i+1}} \frac{\pi(i)}{\pi(j)}.$$

Let $a < i < b$ be three points in $\{0, \dots, m\}$. We have the following formulas for the exit point of $\{a, \dots, b\}$:

$$\begin{aligned} P(Z_{\tau(\{a,b\})} = a \mid Z_0 = i) &= \frac{\sum_{j=i}^{b-1} \frac{1}{\pi(j)}}{\sum_{j=a}^{b-1} \frac{1}{\pi(j)}}, \\ P(Z_{\tau(\{a,b\})} = b \mid Z_0 = i) &= \frac{\sum_{j=a}^{i-1} \frac{1}{\pi(j)}}{\sum_{j=a}^{b-1} \frac{1}{\pi(j)}}. \end{aligned}$$

We consider now a family of birth and death Markov chains $(Z_t)_{t \geq 0}$ depending on four parameters: $m, \ell \geq 1, q \in [0, 1], \delta' > 0$. For $m, \ell \geq 0, q \in [0, 1]$ and $\delta' > 0$, the process $(Z_t)_{t \geq 0}$ is a birth and death Markov chain with state space $\{0, \dots, m\}$ and transition probabilities given by:

$$\begin{aligned} P(Z_{t+1} = i + 1 \mid Z_t = i) &= \delta_i, & 0 \leq i < m, \\ P(Z_{t+1} = i - 1 \mid Z_t = i) &= \gamma_i, & 0 < i \leq m, \end{aligned}$$

where δ_i, γ_i can depend on the parameters m, ℓ, q, δ' . We are interested in the asymptotic behaviour of $(Z_t)_{t \geq 0}$ when m, ℓ go to ∞ and q, δ' go to 0. We make the following assumptions.

Assumption 1. There exist a constant $C > 0$ (which can depend on ℓ, q, δ' but not on m) and an integer $k \geq 1$ such that for all m, ℓ large enough and q, δ' small enough

$$\begin{aligned} \forall i \in \{0, \dots, m-1\} \quad \delta_i &\geq \frac{C}{m^k}, \\ \forall i \in \{1, \dots, m\} \quad \gamma_i &\geq \frac{C}{m^k}. \end{aligned}$$

Assumption 2. For all $\rho \in [0, 1]$, we have the following large deviation estimate:

$$\lim_{\substack{\ell, m \rightarrow \infty \\ q \rightarrow 0}} \frac{1}{m} \ln \pi(\lfloor \rho m \rfloor) = f_{\delta'}(\rho),$$

where $f_{\delta'} : [0, 1] \rightarrow \mathbb{R}$ is a function for which there exists $\rho_1^* \leq \rho^* \leq \rho_2^*$ in $[0, 1]$ such that $f_{\delta'}$ is increasing over $[0, \rho_1^*[$ and decreasing over $] \rho_2^*, 1]$. Moreover, ρ^* does not depend on δ' and

$$\lim_{\delta' \rightarrow 0} \rho_1^* = \lim_{\delta' \rightarrow 0} \rho_2^* = \rho^*.$$

Assumption 3. For each $m, \ell \geq 1, q \in [0, 1], \delta' > 0$, there exists $r_1 \leq r_2$ in $[0, 1]$ such that

$$\begin{aligned} 1 \leq i \leq j \leq \lfloor r_1 m \rfloor &\implies \pi(i) \leq \pi(j), \\ \lfloor r_2 m \rfloor \leq i \leq j \leq m &\implies \pi(i) \geq \pi(j). \end{aligned}$$

Moreover,

$$\lim_{\substack{\ell, m \rightarrow \infty \\ q \rightarrow 0}} r_1 = \rho_1^*, \quad \lim_{\substack{\ell, m \rightarrow \infty \\ q \rightarrow 0}} r_2 = \rho_2^*.$$

These three assumptions will be verified by the birth and death Markov chains that we will derive later from the process $(Z_t^\theta)_{t \geq 0}$. With these assumptions in hand we can prove the following proposition. Let us define, for $\delta > 0$,

$$V(\delta) = \left\{ i \in \{0, \dots, m\} : \left| \frac{i}{m} - \rho^* \right| < \delta \right\}.$$

For any subset $A \subset \{0, \dots, m\}$ we define the hitting time of A by

$$\tau(A) = \inf\{t \geq 0 : Z_t \in A\}.$$

Proposition 4.2. *Suppose that the three assumptions are satisfied. Let $\delta > 0$. There exist positive real numbers $\alpha, \alpha', \beta, \beta'$ (depending on δ, δ') such that for m, ℓ large enough and q, δ' small enough*

$$\begin{aligned} \forall i \in \{0, \dots, m\} \quad & P(\tau(V(\delta)) \geq m^\alpha \mid Z_0 = i) \leq \exp(-\alpha' m), \\ \forall i \in V(\delta) \quad & P(\tau(V(2\delta)^c) \leq \exp(\beta m) \mid Z_0 = i) \leq \exp(-\beta' m). \end{aligned}$$

Proof. We begin by showing the first statement in the proposition. The cases $i \leq \rho^* m$ and $i \geq \rho^* m$ are dealt with in a similar way, thus, we will only show the result for $i \leq \rho^* m$. Let us call b the minimum of the discrete interval $V(\delta)$,

$$b = \lfloor (\rho^* - \delta)m \rfloor + 1.$$

From the formulas provided at the beginning of the section we obtain

$$E(\tau(V(\delta)) \mid Z_0 = i) = E(\tau(\{b\}) \mid Z_0 = i) = \sum_{j=i}^{b-1} \sum_{k=j}^{b-1} \frac{1}{\delta_j} \frac{\pi(j)}{\pi(k)}.$$

By assumptions 2 and 3, for m, ℓ big enough and q, δ' small enough, the point r_1 is in $V(\delta)$, so that $b \leq r_1$, thus

$$1 \leq j \leq k \leq b-1 \implies \frac{\pi(j)}{\pi(k)} \leq 1.$$

Furthermore, by assumption 1, for m, ℓ large enough and q, δ' small enough,

$$\forall j \in \{0, \dots, m-1\} \quad \delta_j \geq \frac{C}{m^k}.$$

It follows that

$$E(\tau(V(\delta)) \mid Z_0 = 0) \leq \frac{1}{C} m^{k+2}.$$

Let $a > k + 2$. By the Markov inequality,

$$P(\tau(V(\delta)) > m^a \mid Z_0 = i) \leq \frac{1}{C} m^{-(a-k-2)}.$$

We will next estimate for $n \geq 1$,

$$P(\tau(V(\delta)) \geq nm^a \mid Z_0 = i).$$

We decompose this probability according to the possible states of the process at time $(n-1)m^a$:

$$\begin{aligned} P(\tau(V(\delta)) \geq nm^a \mid Z_0 = i) &= \\ &\sum_{j < b} P(Z_{(n-1)m^a} = j, \tau(V(\delta)) \geq (n-1)m^a, \tau(V(\delta)) \geq nm^a \mid Z_0 = i) \\ &= \sum_{j < b} P(Z_{(n-1)m^a} = j, \tau(V(\delta)) \geq (n-1)m^a \mid Z_0 = i) \\ &\quad \times P(\tau(V(\delta)) \geq nm^a \mid Z_0 = i, Z_{(n-1)m^a} = j, \tau(V(\delta)) \geq (n-1)m^a). \end{aligned}$$

Thanks to the Markov property,

$$\begin{aligned} P(\tau(V(\delta)) \geq nm^a \mid Z_0 = i, Z_{(n-1)m^a} = j, \tau(V(\delta)) \geq (n-1)m^a) \\ = P(\tau(V(\delta)) \geq nm^a - (n-1)m^a \mid Z_0 = j) \leq \frac{1}{C} m^{-(a-k-2)}. \end{aligned}$$

Therefore, for all $n \geq 1$,

$$\begin{aligned} P(\tau(V(\delta)) \geq nm^a \mid Z_0 = i) &\leq \\ &\frac{1}{C} m^{-(a-k-2)} P(\tau(V(\delta)) \geq (n-1)m^a \mid Z_0 = i). \end{aligned}$$

We iterate this procedure for the times $(n-2)m^a, \dots, 2m^a, m^a$ and we obtain

$$\begin{aligned} P(\tau(V(\delta)) \geq nm^a \mid Z_0 = i) &\leq \left(\frac{1}{C} m^{-(a-k-2)} \right)^n \\ &= \exp \left(-n((a-k-2) \ln m + \ln C) \right). \end{aligned}$$

Thus, setting $n = m$, we obtain the desired result with $\alpha = k + 3$ and $\alpha' = (\alpha - k - 3) \ln m_0 + \ln C$ for m_0 large enough so that $\alpha' > 0$.

We show next the second statement of the proposition. Let $t > 0$, $i \in V(\delta)$, and let us first estimate the value of

$$P(\tau(V(2\delta)^c) \leq t \mid Z_0 = i).$$

Let θ be the last time the process $(Z_t)_{t \geq 0}$ visits the set $V(\delta)$ before time $\tau(V(2\delta)^c)$, i.e.,

$$\theta = \max \{ s < \tau(V(2\delta)^c) : Z_s \in V(\delta) \}.$$

We denote by b and c the extreme points of the discrete interval $V(\delta)$,

$$b = \lfloor (\rho^* - \delta)m \rfloor + 1, \quad c = \lfloor (\rho^* + \delta)m \rfloor.$$

Likewise, we denote by b' and c' the extreme points of the discrete interval $V(2\delta)$,

$$b' = \lfloor (\rho^* - 2\delta)m \rfloor + 1, \quad c' = \lfloor (\rho^* + 2\delta)m \rfloor.$$

We have then

$$\begin{aligned} P(\tau(V(2\delta)^c) \leq t \mid Z_0 = i) &= \sum_{s < t} P(\theta = s, \tau(V(2\delta)^c) \leq t \mid Z_0 = i) \\ &= \sum_{s < t} \left(P(\theta = s, Z_s = b, \tau(V(2\delta)^c) \leq t \mid Z_0 = i) \right. \\ &\quad \left. + P(\theta = s, Z_s = c, \tau(V(2\delta)^c) \leq t \mid Z_0 = i) \right). \end{aligned}$$

Let us consider the first term within the parenthesis. By the Markov property,

$$\begin{aligned} &P(\theta = s, Z_s = b, \tau(V(2\delta)^c) \leq t \mid Z_0 = i) \\ &= P\left(\begin{array}{l} Z_s = b, Z_{s+1} = b-1, \tau(V(2\delta)^c) \leq t \\ Z_r \notin V(\delta) \text{ for } s < r \leq \tau(V(2\delta)^c) \end{array} \mid Z_0 = i \right) \\ &\leq P\left(\begin{array}{l} Z_r \notin V(\delta) \text{ for } r \leq \tau(V(2\delta)^c) \\ \tau(V(2\delta)^c) \leq t - s - 1 \end{array} \mid Z_0 = b-1 \right) \\ &\leq P(Z_{\tau(V(2\delta)^c \cup \{b\})} \in V(2\delta)^c \mid Z_0 = b-1) \\ &= P(Z_{\tau(\{b'-1, b\})} = b' - 1 \mid Z_0 = b-1). \end{aligned}$$

We now use the formulas provided at the beginning of the section:

$$P(Z_{\tau(\{b'-1, b\})} = b' - 1 \mid Z_0 = b-1) = \frac{1}{\frac{\pi(b-1)}{b-1} \sum_{i=b'-1}^{\infty} \frac{1}{\pi(i)}}.$$

Therefore,

$$P(Z_{\tau(\{b'-1, b\})} = b' - 1 \mid Z_0 = b-1) \leq \frac{\pi(b'-1)}{\pi(b-1)}.$$

Let $\varepsilon > 0$ and let ℓ, m be large enough and q small enough so that

$$\begin{aligned} \left| \frac{1}{m} \ln \pi(b-1) - f_{\delta'}(\rho^* - \delta) \right| &< \frac{\varepsilon}{2}, \\ \left| \frac{1}{m} \ln \pi(b'-1) - f_{\delta'}(\rho^* - 2\delta) \right| &< \frac{\varepsilon}{2}. \end{aligned}$$

We have then

$$\begin{aligned} \frac{\pi(b'-1)}{\pi(b-1)} &= \exp \left(m \left(\frac{1}{m} \ln \pi(b'-1) - \frac{1}{m} \ln \pi(b-1) \right) \right) \\ &\leq \exp \left(-m(f_{\delta'}(\rho^* - \delta) - f_{\delta'}(\rho^* - 2\delta) - \varepsilon) \right). \end{aligned}$$

Thanks to assumption 2, we can choose ε and δ' small enough so that

$$\varepsilon < f_{\delta'}(\rho^* - \delta) - f_{\delta'}(\rho^* - 2\delta).$$

We choose β_1 as follows:

$$\beta_1 = f_{\delta'}(\rho^* - \delta) - f_{\delta'}(\rho^* - 2\delta) - \varepsilon.$$

We have $\beta_1 > 0$ and

$$P(\theta = s, Z_{s+1} = b-1, \tau(V(2\delta)^c) \leq t \mid Z_0 = i) \leq \exp(-\beta_1 m).$$

The term

$$P(\theta = s, Z_{s+1} = c+1, \tau(V(2\delta)^c) \leq t \mid Z_0 = i)$$

is dealt with in a similar fashion, thus obtaining $\beta_2 > 0$ such that

$$P(\theta = s, Z_{s+1} = c+1, \tau(V(2\delta)^c) \leq t \mid Z_0 = i) \leq \exp(-\beta_2 m).$$

It follows that

$$P(\tau(V(2\delta)^c) \leq t \mid Z_0 = i) \leq t \left(\exp(-\beta_1 m) + \exp(-\beta_2 m) \right).$$

We choose $\beta < \min(\beta_1, \beta_2)$ and for m_0 large enough

$$\beta' = \min(\beta_1 - \beta, \beta_2 - \beta) - \frac{1}{m_0} \ln 2.$$

We apply the previous inequality at time $t = \exp(\beta m)$ and we obtain the desired result. \square

4.2 The initial case

We study here the case $k = 0$. We will use the results from the previous section to study the birth and death Markov chains $(Z_t^\theta(0))_{t \geq 0}$, $\theta = k + 1$ or $\theta = \ell$. These processes are very similar to the upper and lower birth and death chains studied in [3]. In particular, the analysis in chapter 9 of [3] is still valid for $(Z_t^\theta(0))_{t \geq 0}$. The process $(Z_t^\theta(0))_{t \geq 0}$ is a birth and death Markov chain taking values on $\{0, \dots, m\}$ and having the following transition probabilities

$$\begin{aligned} \delta_0 &= 1, \\ \delta_i &= \frac{\sigma i(m-i)M_H(0,0) + (m-i)^2 M_H^\theta(\theta,0)}{m(\sigma i + m - i)}, \quad 1 \leq i \leq m-1, \\ \gamma_i &= \frac{\sigma i^2(1 - M_H(0,0)) + i(m-i)(1 - M_H^\theta(\theta,0))}{m(\sigma i + m - i)}, \quad 1 \leq i \leq m, \end{aligned}$$

where $M_H^\ell(\ell, 0) = 0$ and $M_H^{K+1}(K+1, 0) = M_H(1, 0)$. The mutation probabilities M_H^θ depend on the parameters ℓ and q . In particular, the process $(Z_t^\theta(0))_{t \geq 0}$ belongs to the class of birth and death Markov chains studied in the previous section, even if there is no parameter δ' . We show next that $(Z_t^\theta(0))_{t \geq 0}$ fulfils the three assumptions made in section 4.1. From the expressions given for δ_i and γ_i we see that for all $\ell, m \geq 1, q \in [0, 1]$,

$$\begin{aligned} \forall i \in \{0, \dots, m-1\} \quad \delta_i &\geq \frac{M_H(0,0)}{m^2}, \\ \forall i \in \{1, \dots, m\} \quad \gamma_i &\geq \frac{1 - M_H(0,0)}{m^2}. \end{aligned}$$

Define a function $\phi :]0, 1[\times [0, 1[\times]0, 1[\rightarrow]0, +\infty[$ by

$$\phi(\beta, \varepsilon, \rho) = \frac{(1-\rho)(\sigma\beta\rho + (1-\rho)\varepsilon)}{\rho(\sigma(1-\beta)\rho + (1-\rho)(1-\varepsilon))},$$

and let $\rho(\beta, \varepsilon)$ be the only positive root of the equation $\phi(\beta, \varepsilon, \rho) = 1$, i.e.,

$$\rho(\beta, \varepsilon) = \frac{1}{2(\sigma-1)} \left(\sigma\beta - 1 - \varepsilon + \sqrt{(\sigma\beta - 1 - \varepsilon)^2 + 4\varepsilon(\sigma-1)} \right).$$

As shown in [3], we have:

$$\begin{aligned} 1 \leq i \leq j \leq \lfloor \rho(\beta, \varepsilon)m \rfloor &\implies \pi(i) \leq \pi(j), \\ \lfloor \rho(\beta, \varepsilon)m \rfloor \leq i \leq j \leq m &\implies \pi(i) \geq \pi(j). \end{aligned}$$

We have also the following result:

Proposition 4.3. *Let $a \in]0, +\infty[$. For $\rho \in [0, 1]$, we have*

$$\lim_{\substack{\ell, m \rightarrow \infty \\ q \rightarrow 0, \ell q \rightarrow a}} \frac{1}{m} \ln \pi(\lfloor \rho m \rfloor) = \int_0^\rho \ln \phi(\exp(-a), 0, s) ds.$$

A detailed proof is provided in [3]. Let us define, for $\sigma e^{-a} > 1$,

$$\rho_0^* = \rho(e^{-a}, 0) = \frac{\sigma e^{-a} - 1}{\sigma - 1}.$$

The function

$$\rho \mapsto \int_0^\rho \ln \phi(e^{-a}, 0, s) ds$$

is increasing on $]0, \rho_0^*[$, and decreasing on $]\rho_0^*, 1[$. We also have

$$\lim_{\substack{\ell, m \rightarrow \infty \\ q \rightarrow 0, \ell q \rightarrow a}} \rho(M_H(0, 0), M_H^\theta(\theta, 0)) = \rho_0^*.$$

Since all three assumptions are verified we can apply proposition 4.2 to the process $(Z_t^\theta(0))_{t \geq 0}$. Let $\delta > 0$ and define

$$V_0(\delta) = \left\{ i \in \{0, \dots, m\} : \left| \frac{i}{m} - \rho_0^* \right| < \delta \right\}.$$

We also define, for any subset $A \subset \{0, \dots, m\}$ the hitting time of A :

$$\tau(A) = \inf \{ t \geq 0 : Z_t^\theta(0) \in A \}.$$

Corollary 4.4. *Let $\delta > 0$. There exist positive real numbers $\alpha_0, \alpha'_0, \beta_0, \beta'_0$ (depending on δ), such that for ℓ, m big enough and q small enough,*

$$\begin{aligned} \forall i \in \{0, \dots, m\} \quad & P(\tau(V_0(\delta)) \geq m^{\alpha_0} \mid Z_0^\theta(0) = i) \leq \exp(-\alpha'_0 m), \\ \forall i \in V_0(\delta) \quad & P(\tau(V_0(2\delta)^c) \leq \exp(\beta_0 m) \mid Z_0^\theta(0) = i) \leq \exp(-\beta'_0 m). \end{aligned}$$

4.3 The inductive step

We perform now the inductive step. Let us fix an integer $k \in \{0, \dots, K\}$. We suppose that the result of theorem 4.1 is true at rank $k - 1$, and we prove that it remains true at rank k . We recall that, for $\delta > 0$, the subset $U_k(\delta) \subset \mathcal{E}_K$ is defined by

$$U_k(\delta) = \left\{ z \in \mathcal{E}_K : \left| \frac{z_i}{m} - \rho_i^* \right| < \delta, \quad 0 \leq i \leq k \right\}.$$

The hitting time of a subset $A \subset \mathcal{E}_K$ is defined by

$$\tau(A) = \inf\{t \geq 0 : Z_t^\theta \in A\}.$$

We will prove that for $\delta > 0$, there exist positive real numbers $\alpha_k, \alpha'_k, \beta_k, \beta'_k$ (depending on δ, δ') such that for ℓ, m large enough and q small enough,

$$\begin{aligned} \forall z \in \mathcal{E}_K \quad & P(\tau(U_k(\delta)) \geq m^{\alpha_k} \mid Z_0^\theta = z) \leq \exp(-\alpha'_k m), \\ \forall z \in U_k(\delta) \quad & P(\tau(U_k(2\delta)^c) \leq \exp(\beta_k m) \mid Z_0^\theta = z) \leq \exp(-\beta'_k m). \end{aligned}$$

Let $\delta' > 0$. By the induction hypothesis, the process $(Z_t^\theta)_{t \geq 0}$ spends most of its time inside the set $U_{k-1}(2\delta')$. Therefore, we will study the dynamics of the k th coordinate $(Z_t^\theta(k))_{t \geq 0}$ when Z_t^θ is in $U_{k-1}(2\delta')$. Conditionally on $(Z_t^\theta(0), \dots, Z_t^\theta(k-1))_{t \geq 0}$, the process $(Z_t^\theta(k))_{t \geq 0}$ can be seen as a birth and death process having time-dependent transition probabilities. The classical formula for birth and death processes cannot be applied directly to the process $(Z_t^\theta(k))_{t \geq 0}$. Our goal is to get rid of this time dependence. We will build a process $(Z_t)_{t \geq 0}$ whose conditional law given $(Z_t^\theta(0), \dots, Z_t^\theta(k-1))_{t \geq 0}$ is the same as the law of $(Z_t^\theta(k))_{t \geq 0}$. We will then realize a coupling between the process $(Z_t)_{t \geq 0}$ and a pair of birth and death Markov chains, a lower one $(Z_t^L)_{t \geq 0}$ and an upper one $(Z_t^U)_{t \geq 0}$. The key point is that the transition probabilities of $(Z_t^L)_{t \geq 0}$ and $(Z_t^U)_{t \geq 0}$ are not time dependent any more. The coupling only works as long as the process $(Z_t^\theta)_{t \geq 0}$ is in the set $U_{k-1}(2\delta')$. Since $(Z_t^\theta)_{t \geq 0}$ spends most of its time inside $U_{k-1}(2\delta')$, the coupling will allow us to obtain the desired estimates. For $i \in \{0, \dots, K\}$, we define a pair of maps $\delta_i, \gamma_i : [0, 1]^k \times \{0, \dots, m\} \rightarrow \{0, \dots, m\}$ as follows. Let $\rho = (\rho_0, \dots, \rho_{k-1}) \in [0, 1]^k$.

- We set $\delta_m(\rho) = 0$ and for $0 \leq i < m$,

$$\begin{aligned} \delta_i(\rho) = \frac{1 - i/m}{(\sigma - 1)\rho_0 + 1} & \left(\sigma \rho_0 M_H(0, k) + \sum_{l=1}^{k-1} \rho_l M_H(l, k) \right. \\ & \left. + \frac{i}{m} M_H(k, k) + \left(1 - \sum_{l=0}^{k-1} \rho_l - \frac{i}{m} \right) M_H(\theta, k) \right). \end{aligned}$$

- We set $\gamma_0(\rho) = 0$ and for $0 < i \leq m$,

$$\begin{aligned} \gamma_i(\rho) = \frac{i/m}{(\sigma - 1)\rho_0 + 1} & \left(\sigma \rho_0 (1 - M_H(0, k)) + \sum_{l=1}^{k-1} \rho_l (1 - M_H(l, k)) \right. \\ & \left. + \frac{i}{m} (1 - M_H(k, k)) + \left(1 - \sum_{l=0}^{k-1} \rho_l - \frac{i}{m} \right) (1 - M_H(\theta, k)) \right). \end{aligned}$$

These maps allow us to express the transition probabilities of $(Z_t^\theta(k))_{t \geq 0}$ in the following way. For $z = (z_0, \dots, z_k) \in \mathcal{E}_K \setminus \{0\}$ and $i = z_k$,

$$\begin{aligned} P(Z_{t+1}^\theta(k) = i + 1 \mid Z_t^\theta = z) &= \delta_i\left(\frac{z_0}{m}, \dots, \frac{z_{k-1}}{m}\right), \\ P(Z_{t+1}^\theta(k) = i - 1 \mid Z_t^\theta = z) &= \gamma_i\left(\frac{z_0}{m}, \dots, \frac{z_{k-1}}{m}\right). \end{aligned}$$

The process $(Z_t^\theta(k))_{t \geq 0}$ is not well suited to build a coupling, therefore, we build another process $(Z_t)_{t \geq 0}$ whose conditional law given the trajectories $(Z_t^\theta(0), \dots, Z_t^\theta(k-1))_{t \geq 0}$ is the same as the law of $(Z_t^\theta(k))_{t \geq 0}$. We define a map

$$C : [0, 1]^k \times \{0, \dots, m\} \times [0, 1] \longrightarrow [0, 1]$$

by setting for $\rho \in [0, 1]^k$, $i \in \{0, \dots, m\}$ and $u \in [0, 1]$,

$$C(\rho, i, u) = i - \mathbf{1}_{u < \gamma_i(\rho)} + \mathbf{1}_{u > 1 - \delta_i(\rho)}.$$

The map C is defined so that if U is a uniform random variable on $[0, 1]$, then for $\rho \in [0, 1]^k$ and $i \in \{0, \dots, m\}$,

$$P(C(\rho, i, U) = i + 1) = \delta_i(\rho), \quad P(C(\rho, i, U) = i - 1) = \gamma_i(\rho).$$

Lemma 4.5. *For m large enough, the map C is non-decreasing with respect to the argument i :*

$$\begin{aligned} \forall \rho \in [0, 1]^k \quad \forall i, j \in \{0, \dots, m\} \quad \forall u \in [0, 1] \\ i \leq j \implies C(\rho, i, u) \leq C(\rho, j, u). \end{aligned}$$

Proof. It is enough to show that the result holds for $j = i + 1$. We have

$$\begin{aligned} C(\rho, i + 1, u) - C(\rho, i, u) = \\ 1 - (\mathbf{1}_{u < \gamma_{i+1}(\rho)} - \mathbf{1}_{u < \gamma_i(\rho)}) + (\mathbf{1}_{u > 1 - \delta_{i+1}(\rho)} - \mathbf{1}_{u > 1 - \delta_i(\rho)}). \end{aligned}$$

This quantity is negative if and only if

$$\gamma_i(\rho) \leq u < \gamma_{i+1}(\rho) \quad \text{and} \quad 1 - \delta_i(\rho) < u \leq 1 - \delta_{i+1}(\rho).$$

This can only happen if $\delta_i(\rho) + \gamma_{i+1}(\rho) > 1$. However, taking $y \in [0, 1]$ such

that $i = \lfloor ym \rfloor$, we see that

$$\begin{aligned} \lim_{m \rightarrow \infty} \delta_i(\rho) + \gamma_{i+1}(\rho) &= \\ &= \frac{1}{(\sigma - 1)\rho_0 + 1} \left(\sigma \rho_0 \left((1 - y)M_H(0, k) + y(1 - M_H(0, k)) \right) \right. \\ &\quad + \sum_{l=1}^{k-1} \rho_l \left((1 - y)M_H(l, k) + y(1 - M_H(l, k)) \right) \\ &\quad + y \left((1 - y)M_H(k, k) + y(1 - M_H(k, k)) \right) \\ &\quad \left. + \left(1 - \sum_{l=0}^{k-1} \rho_l - y \right) \left((1 - y)M_H(\theta, k) + y(1 - M_H(\theta, k)) \right) \right), \end{aligned}$$

Yet, for $u \in]0, 1[$ and $v \in [0, 1]$ we have

$$(1 - u)v + u(1 - v) < v + 1 - v = 1.$$

Thus,

$$\begin{aligned} \lim_{m \rightarrow \infty} \delta_i(\rho) + \gamma_{i+1}(\rho) &< \\ &= \frac{1}{(\sigma - 1)\rho_0 + 1} \left(\sigma \rho_0 + \sum_{l=1}^{k-1} \rho_l + y + \left(1 - \sum_{l=0}^{k-1} \rho_l - y \right) \right) = 1, \end{aligned}$$

as required. \square

Let $(U_n)_{n \geq 1}$ be an i.i.d. sequence of uniform random variables on $[0, 1]$. We define the process $(Z_t)_{t \geq 0}$ using the process $(Z_t^\theta)_{t \geq 0}$ and the sequence $(U_n)_{n \geq 1}$. Let $z = (z_0, \dots, z_K) \in \mathcal{E}_K \setminus \{0\}$ be the starting point of the process $(Z_t^\theta)_{t \geq 0}$. We take $Z_0 = z_k$ and

$$\forall n \geq 1 \quad Z_n = C \left(\frac{Z_{n-1}^\theta(0)}{m}, \dots, \frac{Z_{n-1}^\theta(k-1)}{m}, Z_{n-1}, U_n \right).$$

From this construction, conditionally on $(Z_t^\theta(0), \dots, Z_t^\theta(k-1))_{t \geq 0}$, both processes $(Z_t)_{t \geq 0}$ and $(Z_t^\theta(k))_{t \geq 0}$ have the same law.

Let us fix $\delta' > 0$. We build next a lower birth and death Markov chain $(Z_t^L)_{t \geq 0}$ and an upper one $(Z_t^U)_{t \geq 0}$ in order to bound stochastically the process $(Z_t)_{t \geq 0}$ when $(Z_t^\theta)_{t \geq 0}$ is in $U_{k-1}(2\delta')$. Let $W_{k-1}(\delta')$ be the subset of $[0, 1]^k$ given by

$$W_{k-1}(\delta') = \{ \rho \in [0, 1]^k : |\rho_i - \rho_i^*| < \delta', \quad 0 \leq i < k \}.$$

In particular, we have

$$W_{k-1}(\delta') \cap \frac{\mathbb{Z}^k}{m} = U_{k-1}(\delta').$$

We define $(Z_t^L)_{t \geq 0}$ to be a birth and death Markov chain on the state space $\{0, \dots, m\}$, having the following transition probabilities:

$$\begin{aligned} \delta_i^L &= \min\{\delta_i(\rho) : \rho \in W_{k-1}(2\delta')\}, & 0 \leq i \leq m, \\ \gamma_i^L &= \max\{\gamma_i(\rho) : \rho \in W_{k-1}(2\delta')\}, & 0 \leq i \leq m. \end{aligned}$$

Likewise, we define $(Z_t^U)_{t \geq 0}$ to be a birth and death Markov chain on the state space $\{0, \dots, m\}$ having the following transition probabilities:

$$\begin{aligned} \delta_i^U &= \max\{\delta_i(\rho) : \rho \in W_{k-1}(2\delta')\} & 0 \leq i \leq m, \\ \gamma_i^U &= \min\{\gamma_i(\rho) : \rho \in W_{k-1}(2\delta')\} & 0 \leq i \leq m. \end{aligned}$$

The processes $(Z_t^L)_{t \geq 0}$ and $(Z_t^U)_{t \geq 0}$ are well defined, since if ρ_i and ρ'_i are the points that maximise the functions $\gamma_i(\rho)$ and $\delta_i(\rho)$ respectively, we have

$$\begin{aligned} \delta_i^L + \gamma_i^L &= \delta_i^L + \gamma_i(\rho_i) \leq \delta_i(\rho_i) + \gamma_i(\rho_i) \leq 1, \\ \delta_i^U + \gamma_i^U &= \delta_i(\rho'_i) + \gamma_i^U \leq \delta_i(\rho'_i) + \gamma_i(\rho'_i) \leq 1. \end{aligned}$$

In order to couple these processes we define the maps

$$C^L, C^U : \{0, \dots, m\} \times [0, 1] \longrightarrow [0, 1]$$

by setting for $i \in \{0, \dots, m\}$ and $u \in [0, 1]$,

$$\begin{aligned} C^L(i, u) &= i - 1_{u < \gamma_i^L} + 1_{u > 1 - \delta_i^L}, \\ C^U(i, u) &= i - 1_{u < \gamma_i^U} + 1_{u > 1 - \delta_i^U}. \end{aligned}$$

The maps C^L, C^U are built so that if U is a uniform random variable on $[0, 1]$, then

$$\begin{aligned} P(C^L(i, U) = i + 1) &= \delta_i^L & \text{and} & & P(C^L(i, U) = i - 1) &= \gamma_i^L, \\ P(C^U(i, U) = i + 1) &= \delta_i^U & \text{and} & & P(C^U(i, U) = i - 1) &= \gamma_i^U. \end{aligned}$$

The definition of the transition probabilities $\delta_i^L, \gamma_i^L, \delta_i^U, \gamma_i^U$ implies that the map C^L is below the map C and the map C^U is above the map C , i.e.,

$$\begin{aligned} \forall \rho \in W_{k-1}(2\delta') \quad \forall i \in \{0, \dots, m\} \quad \forall u \in [0, 1] \\ C^L(i, u) \leq C(\rho, i, u) \leq C^U(i, u). \end{aligned}$$

We define the processes $(Z_t^L)_{t \geq 0}$, $(Z_t^U)_{t \geq 0}$ with the help of the same sequence $(U_n)_{n \geq 1}$ that was used to define $(Z_t)_{t \geq 0}$. Let $i \in \{0, \dots, m\}$ be the starting point of the processes. We set $Z_0^L = Z_0^U = i$ and

$$\forall n \geq 1 \quad Z_n^L = C^L(Z_{n-1}^L, U_n), \quad Z_n^U = C^U(Z_{n-1}^U, U_n).$$

Let $\tau(U_{k-1}(2\delta')^c)$ be the exit time from the set $U_{k-1}(2\delta')$ for the process $(Z_t^\theta)_{t \geq 0}$:

$$\tau(U_{k-1}(2\delta')^c) = \inf \{ t \geq 0 : Z_t^\theta \notin U_{k-1}(2\delta') \}.$$

Proposition 4.6. *Let $z = (z_0, \dots, z_K) \in U_{k-1}(2\delta')$ be the starting point of the process $(Z_t^\theta)_{t \geq 0}$. If $Z_0^L = Z_0 = Z_0^U = z_k$, then*

$$\forall n \in [0, \tau(U_{k-1}(2\delta')^c)] \quad Z_n^L \leq Z_n \leq Z_n^U.$$

Proof. We will show the inequality by induction on $n \in \mathbb{N}$. For $n = 0$ we have equality $Z_0^L = Z_0 = Z_0^U$. Suppose that the inequality holds at time $n < \tau(U_{k-1}(2\delta')^c)$, i.e., $Z_n^L \leq Z_n \leq Z_n^U$ and $(Z_n^\theta(0)/m, \dots, Z_n^\theta(k-1)/m)$ is in the set $W_{k-1}(2\delta')$. We then have

$$\begin{aligned} Z_{n+1}^L &= C^L(Z_n^L, U_{n+1}), \\ Z_{n+1} &= C\left(\frac{Z_n^\theta(0)}{m}, \dots, \frac{Z_n^\theta(k-1)}{m}, Z_n, U_{n+1}\right), \\ Z_{n+1}^U &= C^U(Z_n^U, U_{n+1}). \end{aligned}$$

Lemma 4.5 and the induction hypothesis together imply that

$$\begin{aligned} &C\left(\frac{Z_n^\theta(0)}{m}, \dots, \frac{Z_n^\theta(k-1)}{m}, Z_n^L, U_{n+1}\right) \\ &\leq C\left(\frac{Z_n^\theta(0)}{m}, \dots, \frac{Z_n^\theta(k-1)}{m}, Z_n, U_{n+1}\right) \leq \\ &C\left(\frac{Z_n^\theta(0)}{m}, \dots, \frac{Z_n^\theta(k-1)}{m}, Z_n^U, U_{n+1}\right). \end{aligned}$$

Since the map C^L is below C and the map C^U is above C , we have

$$\begin{aligned} C^L(Z_n^L, U_{n+1}) &\leq C\left(\frac{Z_n^\theta(0)}{m}, \dots, \frac{Z_n^\theta(k-1)}{m}, Z_n^L, U_{n+1}\right), \\ C^U(Z_n^U, U_{n+1}) &\geq C\left(\frac{Z_n^\theta(0)}{m}, \dots, \frac{Z_n^\theta(k-1)}{m}, Z_n^U, U_{n+1}\right). \end{aligned}$$

Combining the above inequalities we obtain $Z_{n+1}^L \leq Z_{n+1} \leq Z_{n+1}^U$ and the induction step is completed. \square

Let $\delta > 0$ and define

$$V_k(\delta) = \left\{ i \in \{0, \dots, m\} : \left| \frac{i}{m} - \rho_k^* \right| < \delta \right\}.$$

We define the hitting time of a subset $A \subset \{0, \dots, m\}$ for the processes $(Z_t^L)_{t \geq 0}$, $(Z_t^U)_{t \geq 0}$ as follows

$$\begin{aligned} \tau^L(A) &= \inf \{ t \geq 0 : Z_t^L \in A \}, \\ \tau^U(A) &= \inf \{ t \geq 0 : Z_t^U \in A \}. \end{aligned}$$

The following result will help to finish the proof of the induction step for theorem 4.1. We recall that the definition of $(Z_t^L)_{t \geq 0}$ and $(Z_t^U)_{t \geq 0}$ depends on the parameter $\delta' > 0$.

Proposition 4.7. *Let $\delta > 0$. There exist positive real numbers $\alpha, \alpha', \beta, \beta'$ (depending on δ, δ') such that for ℓ, m large enough and q, δ' small enough:*

- For all $i \in \{0, \dots, m\}$,

$$\begin{aligned} P(\tau^L(V_k(\delta)) \geq m^\alpha \mid Z_0^L = i) &\leq \exp(-\alpha' m), \\ P(\tau^U(V_k(\delta)) \geq m^\alpha \mid Z_0^U = i) &\leq \exp(-\alpha' m). \end{aligned}$$

- For all $i \in V_k(\delta)$,

$$\begin{aligned} P(\tau^L(V_k(2\delta)^c) \leq \exp(\beta m) \mid Z_0^L = i) &\leq \exp(-\beta' m), \\ P(\tau^U(V_k(2\delta)^c) \leq \exp(\beta m) \mid Z_0^U = i) &\leq \exp(-\beta' m). \end{aligned}$$

We prove this proposition in the next section. The proof is the same for both the lower and the upper birth and death chain, we will therefore show the result for the process $(Z_t^L)_{t \geq 0}$ only. We show now how to complete the inductive step with the help of this result.

Let $\delta, \delta' > 0$ with $2\delta' < \delta$ and let $z^0 \in \mathcal{E}_K$ be the starting point of the process. Thanks to the induction hypothesis, there exist positive real numbers $\alpha_{k-1}, \alpha'_{k-1}, \beta_{k-1}, \beta'_{k-1}$ (depending on δ') such that for ℓ, m large enough and q small enough,

$$\begin{aligned} \forall z \in \mathcal{E}_K \quad P(\tau(U_{k-1}(\delta')) \geq m^{\alpha_{k-1}} \mid Z_0^\theta = z) &\leq \exp(-\alpha'_{k-1} m), \\ \forall z \in U_{k-1}(\delta') \quad P(\tau(U_{k-1}(2\delta')^c) \leq \exp(\beta_{k-1} m) \mid Z_0^\theta = z) &\leq \exp(-\beta'_{k-1} m). \end{aligned}$$

Let $\alpha_k > \alpha_{k-1}$, we have

$$\begin{aligned} P(\tau(U_k(\delta)) \geq m^{\alpha_k} \mid Z_0^\theta = z^0) \\ = P(\tau(U_{k-1}(\delta')) \geq m^{\alpha_{k-1}}, \tau(U_k(\delta)) \geq m^{\alpha_k} \mid Z_0^\theta = z^0) \\ + P(\tau(U_{k-1}(\delta')) < m^{\alpha_{k-1}}, \tau(U_k(\delta)) \geq m^{\alpha_k} \mid Z_0^\theta = z^0). \end{aligned}$$

By the induction hypothesis the first term in the sum is bounded above by $\exp(-\alpha'_{k-1}m)$. We use the Markov property to control the second term:

$$\begin{aligned} P(\tau(U_{k-1}(\delta')) < m^{\alpha_{k-1}}, \tau(U_k(\delta)) \geq m^{\alpha_k} \mid Z_0^\theta = z^0) \\ = \sum_{\substack{t < m^{\alpha_{k-1}} \\ z \in U_{k-1}(\delta')}} P(\tau(U_{k-1}(\delta')) = t, Z_t^\theta = z, \tau(U_k(\delta)) \geq m^{\alpha_k} \mid Z_0^\theta = z^0) \\ = \sum_{\substack{t < m^{\alpha_{k-1}} \\ z \in U_{k-1}(\delta')}} P(\tau(U_{k-1}(\delta')) = t, Z_t^\theta = z \mid Z_0^\theta = z^0) \\ \quad \times P(\tau(U_k(\delta)) \geq m^{\alpha_k} - t \mid Z_0^\theta = z). \end{aligned}$$

Let m be large enough so that $m^{\alpha_k} - m^{\alpha_{k-1}} < \exp(\beta_{k-1}m)$. For $t < m^{\alpha_{k-1}}$,

$$\begin{aligned} P(\tau(U_k(\delta)) \geq m^{\alpha_k} - t \mid Z_0^\theta = z) &\leq P(\tau(U_k(\delta)) \geq m^{\alpha_k} - m^{\alpha_{k-1}} \mid Z_0^\theta = z) \\ &= P(\tau(U_{k-1}(2\delta')^c) \leq \exp(\beta_{k-1}m), \tau(U_k(\delta)) \geq m^{\alpha_k} - m^{\alpha_{k-1}} \mid Z_0^\theta = z) \\ &\quad + P(\tau(U_{k-1}(2\delta')^c) > \exp(\beta_{k-1}m), \tau(U_k(\delta)) \geq m^{\alpha_k} - m^{\alpha_{k-1}} \mid Z_0^\theta = z). \end{aligned}$$

By the induction hypothesis, the first term in the sum is bounded above by $\exp(-\beta'_{k-1}m)$. For the second term we have:

$$\begin{aligned} P(\tau(U_{k-1}(2\delta')^c) > \exp(\beta_{k-1}m), \tau(U_k(\delta)) \geq m^{\alpha_k} - m^{\alpha_{k-1}} \mid Z_0^\theta = z) \\ \leq P(\tau(U_k(\delta)) \geq m^{\alpha_k} - m^{\alpha_{k-1}} \mid \tau(U_{k-1}(2\delta')^c) > \exp(\beta_{k-1}m), Z_0^\theta = z). \end{aligned}$$

Since $\exp(\beta_{k-1}m) > m^{\alpha_k} - m^{\alpha_{k-1}}$ and $2\delta' < \delta$, conditionally on the event $\tau(U_{k-1}(2\delta')^c) > \exp(\beta_{k-1}m)$, the event $\tau(U_k(\delta)) \geq m^{\alpha_k} - m^{\alpha_{k-1}}$ depends only on $Z_t^\theta(k)$. Moreover, by proposition 4.6,

$$\forall t \in \{0, \dots, \exp(\beta_{k-1}m)\} \quad Z_t^L \leq Z_t^\theta(k) \leq Z_t^U.$$

Therefore,

$$\begin{aligned} P(\tau(U_k(\delta)) \geq m^{\alpha_k} - m^{\alpha_{k-1}} \mid \tau(U_{k-1}(2\delta')^c) > \exp(\beta_{k-1}m), Z_0^\theta = z) \\ \leq P(\tau^L(V_k(\delta)) \geq m^{\alpha_k} - m^{\alpha_{k-1}} \mid Z_0^L = z_k) \\ \quad + P(\tau^U(V_k(\delta)) \geq m^{\alpha_k} - m^{\alpha_{k-1}} \mid Z_0^U = z_k). \end{aligned}$$

Let $\alpha > 0$ be given by proposition 4.7. Choosing α_k large enough so that $m^{\alpha_k} - m^{\alpha_{k-1}} > m^\alpha$, this last expression is bounded by $2 \exp(-\alpha' m)$ (by proposition 4.7), and this yields the desired bound for the hitting time of $U_k(\delta)$.

In order to show the bound on the exit time of $U_k(2\delta)$, we argue in a similar way. Let $z^0 \in U_k(\delta)$ be the starting point of the process. Let β_{k-1} be given by the induction hypothesis and let $\beta_k > 0$. We have

$$\begin{aligned} P(\tau(U_k(2\delta)^c) \leq \exp(\beta_k m) \mid Z_0^\theta = z^0) &= \\ P(\tau(U_{k-1}(2\delta)^c) \leq \exp(\beta_{k-1} m), \tau(U_k(2\delta)^c) \leq \exp(\beta_k m) \mid Z_0^\theta = z^0) &+ \\ + P(\tau(U_{k-1}(2\delta)^c) > \exp(\beta_{k-1} m), \tau(U_k(2\delta)^c) \leq \exp(\beta_k m) \mid Z_0^\theta = z^0). \end{aligned}$$

By the induction hypothesis, the first term in the sum is bounded above by $\exp(-\beta'_{k-1} m)$. For the second term we have:

$$\begin{aligned} P(\tau(U_{k-1}(2\delta)^c) > \exp(\beta_{k-1} m), \tau(U_k(2\delta)^c) \leq \exp(\beta_k m) \mid Z_0^\theta = z^0) &\leq \\ \leq P(\tau(U_k(2\delta)^c) \leq \exp(\beta_k m) \mid Z_0^\theta = z^0, \tau(U_{k-1}(2\delta)^c) > \exp(\beta_{k-1} m)). \end{aligned}$$

Let β be given by proposition 4.7, and $\beta_k > 0$ such that $\beta_k < \beta_{k-1} \wedge \beta$. Then, conditionally on $\tau(U_{k-1}(2\delta)^c) > \exp(\beta_{k-1} m)$, the event $\tau(U_k(\delta)) \leq \exp(\beta_k m)$ only depends on $Z_t^\theta(k)$. Since $\tau(U_{k-1}(2\delta)^c) > \exp(\beta_{k-1} m) > \exp(\beta_k m)$, by proposition 4.6 we have

$$\forall t \in \{0, \dots, \exp(\beta_{k-1} m)\} \quad Z_t^L \leq Z_t^\theta(k) \leq Z_t^U.$$

Therefore,

$$\begin{aligned} P(\tau(U_k(2\delta)^c) \leq \exp(\beta_k m) \mid Z_0^\theta = z^0, \tau(U_{k-1}(2\delta)^c) > \exp(\beta_{k-1} m)) &\leq \\ \leq P(\tau^L(V_k(2\delta)^c) \leq \exp(\beta_k m) \mid Z_0^L = z_k^0) &+ \\ + P(\tau^U(V_k(2\delta)^c) \leq \exp(\beta_k m) \mid Z_0^U = z_k^0) &\leq 2 \exp(-\beta' m). \end{aligned}$$

This completes the induction step.

4.4 Dynamics of $(Z_t^L)_{t \geq 0}$

We study here the dynamics of the process $(Z_t^L)_{t \geq 0}$ in order to prove proposition 4.7. First of all, we look for the points ρ in $W_{k-1}(\delta')$ that minimise and maximise the functions $\delta_i(\rho)$ and $\gamma_i(\rho)$. Since we have

$$\forall l \in \{1, \dots, k-1\} \quad M_H(l, k) \geq M_H(\theta, k),$$

the function $\delta_i(\rho_0, \dots, \rho_{k-1})$ is non-decreasing with respect to the variables $\rho_1, \dots, \rho_{k-1}$. Likewise, $\gamma_i(\rho_0, \dots, \rho_{k-1})$ is non-increasing with respect to the variables $\rho_1, \dots, \rho_{k-1}$. Therefore, for all $i \in \{0, \dots, m\}$,

$$\begin{aligned}\delta_i^L &= \min_{\rho_0: |\rho_0 - \rho_0^*| < \delta'} \delta_i(\rho_0, \rho_1^* - \delta', \dots, \rho_{k-1}^* - \delta'), \\ \gamma_i^L &= \max_{\rho_0: |\rho_0 - \rho_0^*| < \delta'} \gamma_i(\rho_0, \rho_1^* - \delta', \dots, \rho_{k-1}^* - \delta').\end{aligned}$$

Let us take the partial derivatives of $\delta_i(\rho)$ and $\gamma_i(\rho)$ with respect to ρ_0 :

$$\begin{aligned}\frac{\partial \delta_i(\rho_0, \dots, \rho_{k-1})}{\partial \rho_0} &= \frac{1 - i/m}{((\sigma - 1)\rho_0 + 1)^2} \left(\sigma \left(M_H(0, k) - M_H(\theta, k) \right) \right. \\ &\quad \left. - (\sigma - 1) \sum_{l=1}^{k-1} \rho_l \left(M_H(l, k) - M_H(\theta, k) \right) - (\sigma - 1) \frac{i}{m} \left(M_H(k, k) - M_H(\theta, k) \right) \right),\end{aligned}$$

$$\begin{aligned}\frac{\partial \gamma_i(\rho_0, \dots, \rho_{k-1})}{\partial \rho_0} &= \frac{i/m}{((\sigma - 1)\rho_0 + 1)^2} \left(-\sigma \left(M_H(0, k) - M_H(\theta, k) \right) + \right. \\ &\quad \left. (\sigma - 1) \sum_{l=1}^{k-1} \rho_l \left(M_H(l, k) - M_H(\theta, k) \right) + (\sigma - 1) \frac{i}{m} \left(M_H(k, k) - M_H(\theta, k) \right) \right).\end{aligned}$$

The sign of these partial derivatives does not depend on ρ_0 . In particular, for fixed $\rho_1, \dots, \rho_{k-1}$, the functions $\delta_i(\rho)$ and $\gamma_i(\rho)$ are monotone with respect to ρ_0 . Furthermore, the partial derivatives above have opposite signs, thus

$$\begin{aligned}\frac{\partial \delta_i(\rho_0, \dots, \rho_{k-1})}{\partial \rho_0} = 0 &\iff \frac{\partial \gamma_i(\rho_0, \dots, \rho_{k-1})}{\partial \rho_0} = 0 \\ &\iff \sigma \left(M_H(0, k) - M_H(\theta, k) \right) - (\sigma - 1) \sum_{l=1}^{k-1} \rho_l \left(M_H(l, k) - M_H(\theta, k) \right) \\ &\quad - (\sigma - 1) \frac{i}{m} \left(M_H(k, k) - M_H(\theta, k) \right) = 0.\end{aligned}$$

We suppose that

$$\ell \rightarrow +\infty, \quad m \rightarrow +\infty, \quad q \rightarrow 0,$$

in such a way that

$$\ell q \rightarrow a \in]0, +\infty[.$$

We have the following limits for the mutation probabilities:

$$\lim_{\substack{\ell \rightarrow \infty, q \rightarrow 0 \\ \ell q \rightarrow a}} M_H(l, k) = \begin{cases} \frac{a^{k-l}}{(k-l)!} e^{-a} & \text{if } l \leq k, \\ 0 & \text{if } l = \theta. \end{cases}$$

For ℓ large enough and q small enough, the coefficient $M_H(l, k) - M_H(\theta, k)$ is positive. Since the equation $\partial \delta_i(\rho_0, \dots, \rho_{k-1}) / \partial \rho_0 = 0$ is linear with respect to i , we conclude that there exists an $i^* \in \{0, \dots, m\}$ (depending on $m, \rho_1^*, \dots, \rho_{k-1}^*, \delta'$) such that:

- If $0 \leq i \leq i^*$, the function $\rho_0 \mapsto \delta_i(\rho_0, \rho_1^* - \delta', \dots, \rho_{k-1}^* - \delta')$ is non-increasing, the function $\rho_0 \mapsto \gamma_i(\rho_0, \rho_1^* - \delta', \dots, \rho_{k-1}^* - \delta')$ is non-decreasing, and

$$\begin{aligned} \delta_i^L &= \delta_i(\rho_0^* + \delta', \rho_1^* - \delta', \dots, \rho_{k-1}^* - \delta'), \\ \gamma_i^L &= \gamma_i(\rho_0^* + \delta', \rho_1^* - \delta', \dots, \rho_{k-1}^* - \delta'). \end{aligned}$$

- If $i^* < i \leq m$, the function $\rho_0 \mapsto \delta_i(\rho_0, \rho_1^* - \delta', \dots, \rho_{k-1}^* - \delta')$ is non-decreasing, the function $\rho_0 \mapsto \gamma_i(\rho_0, \rho_1^* - \delta', \dots, \rho_{k-1}^* - \delta')$ is non-increasing, and

$$\begin{aligned} \delta_i^L &= \delta_i(\rho_0^* - \delta', \rho_1^* - \delta', \dots, \rho_{k-1}^* - \delta'), \\ \gamma_i^L &= \gamma_i(\rho_0^* - \delta', \rho_1^* - \delta', \dots, \rho_{k-1}^* - \delta'). \end{aligned}$$

From the definition of $\delta_i(\rho)$, $\gamma_i(\rho)$ we deduce that, for $m \geq 2$,

$$\begin{aligned} \forall i \in \{0, \dots, m-1\} \quad \delta_i^L &\geq \frac{\sigma(\rho_0^* - \delta') M_H(0, k)}{m((\sigma-1)(\rho_0^* + \delta') + 1)} \geq \frac{c}{m}, \\ \forall i \in \{1, \dots, m\} \quad \gamma_i^L &\geq \frac{\sigma(\rho_0^* - \delta')(1 - M_H(0, k))}{m((\sigma-1)(\rho_0^* + \delta') + 1)} \geq \frac{c}{m}, \end{aligned}$$

where c is a positive constant depending on k, δ' but not on m .

We study now the products $\pi(i)$, which are defined by

$$\pi(0) = 1, \quad \pi(i) = \frac{\delta_1 \cdots \delta_i}{\gamma_1 \cdots \gamma_i}, \quad 1 < i < m.$$

We study first the ratio $\delta_i(\rho) / \gamma_i(\rho)$. For $0 < i < m$, we have

$$\frac{\delta_i(\rho)}{\gamma_i(\rho)} = \phi\left(M_H(0, k), \dots, M_H(k, k), M_H(\theta, k), \rho_0, \dots, \rho_{k-1}, \frac{i}{m}\right),$$

where the function $\phi :]0, 1]^{k+1} \times [0, 1[\times]0, 1[\times]0, 1[\longrightarrow]0, +\infty[$ is given by:

$$\forall \beta \in]0, 1]^{k+1} \quad \forall \varepsilon \in [0, 1[\quad \forall \rho \in]0, 1]^k \quad \forall \eta \in]0, 1[$$

$$\phi(\beta, \varepsilon, \rho, \eta) = \frac{(1 - \eta) \left(\sigma \rho_0 \beta_0 + \sum_{l=1}^{k-1} \rho_l \beta_l + \eta \beta_k + \left(1 - \sum_{l=0}^{k-1} \rho_l - \eta \right) \varepsilon \right)}{\eta \left(\sigma \rho_0 (1 - \beta_0) + \sum_{l=1}^{k-1} \rho_l (1 - \beta_l) + \eta (1 - \beta_k) + \left(1 - \sum_{l=0}^{k-1} \rho_l - \eta \right) (1 - \varepsilon) \right)}.$$

In order to understand the behaviour of the products $\pi(i)$, it is enough to know whether the value of ϕ is larger or smaller than 1. The equation $\phi(\beta, \varepsilon, \rho, \eta) = 1$ is linear with respect to η , its only root being

$$\eta(\beta, \varepsilon, \rho) = \frac{\sigma \rho_0 \beta_0 + \sum_{l=1}^{k-1} \rho_l \beta_l + \left(1 - \sum_{l=0}^{k-1} \rho_l \right) \varepsilon}{(\sigma - 1) \rho_0 + 1 - \beta_k + \varepsilon}.$$

Therefore,

$$\begin{aligned} \phi(\beta, \varepsilon, \rho, \eta) &> 1 && \text{if } \eta < \eta(\beta, \varepsilon, \rho), \\ \phi(\beta, \varepsilon, \rho, \eta) &< 1 && \text{if } \eta > \eta(\beta, \varepsilon, \rho). \end{aligned}$$

Moreover, the function $\phi(\beta, \varepsilon, \rho, \eta)$ is continuous and non-decreasing with respect to the variables β, ε . Take $\psi :]0, 1]^{k+1} \times [0, 1[\times]0, 1[\longrightarrow]0, +\infty[$ to be the function defined by:

$$\forall \beta \in]0, 1]^{k+1} \quad \forall \varepsilon \in [0, 1[\quad \forall \eta \in]0, 1[$$

$$\psi(\beta, \varepsilon, \eta) = \begin{cases} \phi(\beta, \varepsilon, \rho_0^* + \delta', \rho_1^* - \delta', \dots, \rho_{k-1}^* - \delta', \eta) & \text{if } \eta \leq i^*/m, \\ \phi(\beta, \varepsilon, \rho_0^* - \delta', \rho_1^* - \delta', \dots, \rho_{k-1}^* - \delta', \eta) & \text{if } \eta > i^*/m. \end{cases}$$

We have the following large deviation estimates for the products $\pi(i)$.

Proposition 4.8. *Let $a \in]0, +\infty[$. For $\eta \in [0, 1]$, we have*

$$\lim_{\substack{\ell, m \rightarrow \infty \\ q \rightarrow 0, \ell q \rightarrow a}} \frac{1}{m} \ln \pi(\lfloor \eta m \rfloor) = \int_0^\eta \ln \psi \left(e^{-a} \frac{a^k}{k!}, \dots, e^{-a}, 0, s \right) ds.$$

The proof is very similar to that of proposition 9.1 of [3], so we omit it. Let us define

$$\begin{aligned}\rho^- &= \min \left\{ \eta \left(e^{-a} \frac{a^k}{k!}, \dots, e^{-a}, 0, \rho_0^* + \delta', \rho_1^* - \delta', \dots, \rho_{k-1}^* - \delta' \right), \right. \\ &\quad \left. \eta \left(e^{-a} \frac{a^k}{k!}, \dots, e^{-a}, 0, \rho_0^* - \delta', \rho_1^* - \delta', \dots, \rho_{k-1}^* - \delta' \right) \right\}, \\ \rho^+ &= \max \left\{ \eta \left(e^{-a} \frac{a^k}{k!}, \dots, e^{-a}, 0, \rho_0^* + \delta', \rho_1^* - \delta', \dots, \rho_{k-1}^* - \delta' \right), \right. \\ &\quad \left. \eta \left(e^{-a} \frac{a^k}{k!}, \dots, e^{-a}, 0, \rho_0^* - \delta', \rho_1^* - \delta', \dots, \rho_{k-1}^* - \delta' \right) \right\}.\end{aligned}$$

From the definitions, we see that

$$\begin{aligned}\psi \left(e^{-a} \frac{a^k}{k!}, \dots, e^{-a}, 0, \eta \right) &> 1 \quad \text{for } \eta < \rho^-, \\ \psi \left(e^{-a} \frac{a^k}{k!}, \dots, e^{-a}, 0, \eta \right) &< 1 \quad \text{for } \eta > \rho^+.\end{aligned}$$

In particular, the function

$$\eta \mapsto \int_0^\eta \ln \psi \left(e^{-a} \frac{a^k}{k!}, \dots, e^{-a}, 0, s \right) ds$$

is non-decreasing on $]0, \rho^-[$ and non-increasing on $] \rho^+, 1[$. Furthermore, when δ' goes to 0, the points ρ^- and ρ^+ converge to ρ_k^* :

$$\lim_{\delta' \rightarrow 0} \rho^- = \lim_{\delta' \rightarrow 0} \rho^+ = \rho_k^*.$$

We also define

$$\begin{aligned}\eta^- &= \\ \min \left\{ \eta \left(M_H(0, k), \dots, M_H(k, k), M_H(\theta, k), \rho_0^* + \delta', \rho_1^* - \delta', \dots, \rho_{k-1}^* - \delta' \right), \right. \\ &\quad \left. \eta \left(M_H(0, k), \dots, M_H(k, k), M_H(\theta, k), \rho_0^* - \delta', \rho_1^* - \delta', \dots, \rho_{k-1}^* - \delta' \right) \right\}, \\ \eta^+ &= \\ \max \left\{ \eta \left(M_H(0, k), \dots, M_H(k, k), M_H(\theta, k), \rho_0^* + \delta', \rho_1^* - \delta', \dots, \rho_{k-1}^* - \delta' \right), \right. \\ &\quad \left. \eta \left(M_H(0, k), \dots, M_H(k, k), M_H(\theta, k), \rho_0^* - \delta', \rho_1^* - \delta', \dots, \rho_{k-1}^* - \delta' \right) \right\}.\end{aligned}$$

We then have

$$\begin{aligned} 1 \leq i \leq j \leq \eta^- m &\implies \pi(i) \leq \pi(j), \\ \eta^+ m \leq i \leq j \leq m &\implies \pi(i) \geq \pi(j), \end{aligned}$$

and the situation between $\eta^- m$ and $\eta^+ m$ is somewhat more delicate. Anyhow, when $\ell, m \rightarrow \infty$, $q \rightarrow 0$ and $\ell q \rightarrow a$, we have

$$\lim_{\substack{\ell, m \rightarrow \infty \\ q \rightarrow 0, \ell q \rightarrow a}} \eta^- = \rho^-, \quad \lim_{\substack{\ell, m \rightarrow \infty \\ q \rightarrow 0, \ell q \rightarrow a}} \eta^+ = \rho^+.$$

For $\delta > 0$, we set

$$V_k(\delta) = \left\{ i \in \{0, \dots, m\} : \left| \frac{i}{m} - \rho_k^* \right| < \delta \right\},$$

and we define the hitting time of a subset $A \subset \{0, \dots, m\}$ by

$$\tau^L(A) = \inf \{ t \geq 0 : Z_t^L \in A \}.$$

We recall that the definition of $(Z_t^L)_{t \geq 0}$ depends on the parameter $\delta' > 0$. The above results show that the birth and death Markov chain $(Z_t^L)_{t \geq 0}$ verifies assumptions 1,2,3 of section 4.1, we can therefore apply proposition 4.2 to the process $(Z_t^L)_{t \geq 0}$ and we obtain the following result.

Corollary 4.9. *Let $\delta > 0$. There exist positive real numbers $\alpha_k, \alpha'_k, \beta_k, \beta'_k$ (depending on δ, δ') such that for ℓ, m large enough and q, δ' small enough,*

$$\begin{aligned} \forall i \in \{0, \dots, m\} \quad P(\tau^L(V_k(\delta)) \geq m^{\alpha_k} \mid Z_0^L = i) &\leq \exp(-\alpha'_k m), \\ \forall i \in V_k(\delta) \quad P(\tau^L(V_k(2\delta)^c) < \exp(\beta_k m) \mid Z_0^L = i) &\leq \exp(-\beta'_k m). \end{aligned}$$

Thus the estimates of proposition 4.7 for the lower process $(Z_t^L)_{t \geq 0}$ are proved.

4.5 Convergence

In this section we will prove that when $\sigma e^{-a} > 1$, the invariant probability measure ν^θ converges to the Dirac mass at ρ^* . Let a such that $\sigma e^{-a} > 1$. Let ρ^* be the point of $[0, 1]^{K+1}$ given by:

$$\forall k \geq 0 \quad \rho_k^* = (\sigma e^{-a} - 1) \frac{a^k}{k!} \sum_{i \geq 1} \frac{i^k}{\sigma^i}.$$

We consider the asymptotic regime

$$\begin{aligned} \ell &\rightarrow +\infty, & m &\rightarrow +\infty, & q &\rightarrow 0, \\ \ell q &\rightarrow a, & \frac{m}{\ell} &\rightarrow \alpha. \end{aligned}$$

Theorem 4.10. *For every continuous and increasing function $f : \mathcal{P}_{\ell+1}^m \rightarrow \mathbb{R}$ such that $f(0) = 0$, we have*

$$\lim_{\substack{\ell, m \rightarrow \infty, q \rightarrow 0 \\ \ell q \rightarrow a, \frac{m}{\ell} \rightarrow \alpha}} \int_{\mathcal{E}_K} f\left(\frac{z_0 + \cdots + z_K}{m}\right) d\nu^\theta(z) = f(\rho_0^* + \cdots + \rho_K^*).$$

Proof. Let $\delta > 0$ and let us define

$$U_K(\delta) = \left\{ z \in \mathcal{E}_K : \left| \frac{z_k}{m} - \rho_k^* \right| < \delta, \quad 0 \leq k \leq K \right\}.$$

We define two sequences of stopping times $(T_n)_{n \geq 0}$ and $(T_n^*)_{n \geq 1}$ as follows. Let $T_0 = 0$ and set

$$\begin{aligned} T_1^* &= \inf \{ t \geq 0 : Z_t^\theta \in U_K(\delta) \}, & T_1 &= \inf \{ t \geq T_1^* : Z_t^\theta \notin U_K(2\delta) \}, \\ &\vdots & &\vdots \\ T_k^* &= \inf \{ t \geq T_{k-1} : Z_t^\theta \in U_K(\delta) \}, & T_k &= \inf \{ t \geq T_k^* : Z_t^\theta \notin U_K(2\delta) \}, \\ &\vdots & &\vdots \end{aligned}$$

The ergodic theorem for Markov chains implies that

$$\int_{\mathcal{E}_K} f\left(\frac{z_0 + \cdots + z_K}{m}\right) d\nu^\theta(z) = \lim_{t \rightarrow \infty} \frac{1}{t} E\left(\sum_{i=0}^t f\left(\frac{Z_i^\theta(0) + \cdots + Z_i^\theta(K)}{m}\right)\right).$$

Let $t \geq 0$. We decompose this last sum as follows:

$$\begin{aligned} \sum_{i=0}^t f\left(\frac{Z_i^\theta(0) + \cdots + Z_i^\theta(K)}{m}\right) &= \sum_{n \geq 1} \sum_{i=T_{n-1} \wedge t}^{T_n^* \wedge t-1} f\left(\frac{Z_i^\theta(0) + \cdots + Z_i^\theta(K)}{m}\right) \\ &\quad + \sum_{n \geq 1} \sum_{i=T_n^* \wedge t}^{T_n \wedge t-1} f\left(\frac{Z_i^\theta(0) + \cdots + Z_i^\theta(K)}{m}\right). \end{aligned}$$

The function f is continuous. Let $\varepsilon > 0$ and let us choose δ small enough so that

$$\forall z \in U_K(\delta) \quad \left| f\left(\frac{z_0 + \cdots + z_K}{m}\right) - f(\rho_0^* + \cdots + \rho_K^*) \right| < \frac{\varepsilon}{2}.$$

We have then

$$\left| E\left(\sum_{i=0}^t f\left(\frac{Z_i^\theta(0) + \dots + Z_i^\theta(K)}{m} \right) \right) - tf(\rho_0^* + \dots + \rho_K^*) \right| \leq \sum_{n \geq 1} 2f(1)E(T_n^* \wedge t - T_{n-1} \wedge t) + \frac{t\varepsilon}{2}.$$

Next we study the expression

$$\sum_{n \geq 1} (T_n^* \wedge t - T_{n-1} \wedge t).$$

Let us define

$$N(t) = \max \{ n \geq 0 : T_n \leq t \}.$$

We can rewrite the previous sum as follows

$$\sum_{n \geq 1} (T_n^* \wedge t - T_{n-1} \wedge t) \leq \sum_{n=1}^{N(t)} (T_n^* - T_{n-1}) + (t - T_{N(t)}).$$

We study now the random variable $N(t)$. Let $n \in \mathbb{N}$, $b > 0$ and $z \in \mathcal{E}_K$. More precisely, we seek estimates on the following probability:

$$P(N(n \exp(bm)/2) \geq n \mid Z_0^\theta = z).$$

From the definition of $N(t)$, it follows that $N(t) \geq n$ if and only if $T_n \leq t$. Thus

$$P(N(n \exp(bm)/2) \geq n \mid Z_0^\theta = z) = P(T_n \leq n \exp(bm)/2 \mid Z_0^\theta = z).$$

Let us define for $i \geq 1$,

$$Y_i = T_i - T_{i-1}, \quad Y_i^* = T_i - T_i^*.$$

By theorem 4.1, there exist positive real numbers β et β' such that, for all $i \geq 1$,

$$\begin{aligned} P(Y_i \leq \exp(\beta m) \mid Z_0^\theta = z) &\leq P(Y_i^* \leq \exp(\beta m) \mid Z_0^\theta = z) \\ &= \sum_{z' \in U_K(\delta)} P(Y_i^* \leq \exp(\beta m) \mid Z_0^\theta = z, Z_{T_i^*}^\theta = z') P(Z_{T_i^*}^\theta = z' \mid Z_0^\theta = z) \\ &= \sum_{z' \in U_K(\delta)} P(T_1 \leq \exp(\beta m) \mid Z_0^\theta = z') P(Z_{T_i^*}^\theta = z' \mid Z_0^\theta = z) \\ &\leq \exp(-\beta' m). \end{aligned}$$

Let us define the following Bernoulli random variables:

$$\forall i \geq 1, \quad \varepsilon_i = 1_{Y_i^* \leq \exp(\beta m)}.$$

Notice that

$$T_n = Y_1 + \cdots + Y_n \geq Y_1^* + \cdots + Y_n^*.$$

If $T_n \leq n \exp(\beta m)/2$, then there exist $n/2$ indices in $\{1, \dots, n\}$ such that $Y_i^* \leq \exp(\beta m)$. Therefore,

$$T_n \leq \frac{1}{2}n \exp(\beta m) \implies \varepsilon_1 + \cdots + \varepsilon_n \geq \frac{n}{2}.$$

Thus,

$$\begin{aligned} P\left(N(n \exp(\beta m)/2) \geq n \mid Z_0^\theta = z\right) &= \\ P\left(T_n \leq n \exp(\beta m)/2 \mid Z_0^\theta = z\right) &\leq P\left(\varepsilon_1 + \cdots + \varepsilon_n \geq n/2\right). \end{aligned}$$

Let $\lambda \geq 0$, thanks to Chebyshev's exponential inequality we have

$$P\left(\varepsilon_1 + \cdots + \varepsilon_n \geq n/2\right) \leq \exp\left(-\lambda/2 + \ln E\left(\exp(\lambda \varepsilon_1/n) \cdots \exp(\lambda \varepsilon_n/n)\right)\right).$$

Since $\varepsilon_1, \dots, \varepsilon_{n-1}$ are measurable with respect to $(Z_t^\theta, 0 \leq t \leq T_n^*)$,

$$\begin{aligned} E\left(\exp(\lambda \varepsilon_1/n) \cdots \exp(\lambda \varepsilon_n/n)\right) &= \\ &= E\left(E\left(\exp(\lambda \varepsilon_1/n) \cdots \exp(\lambda \varepsilon_n/n) \mid Z_t^\theta, 0 \leq t \leq T_n^*\right)\right) \\ &= E\left(\exp(\lambda \varepsilon_1/n) \cdots \exp(\lambda \varepsilon_{n-1}/n) E\left(\exp(\lambda \varepsilon_n/n) \mid Z_t^\theta, 0 \leq t \leq T_n^*\right)\right). \end{aligned}$$

Thanks to the strong Markov property, the above conditional expectation can be rewritten as follows:

$$E\left(\exp(\lambda \varepsilon_n/n) \mid Z_t^\theta, 0 \leq t \leq T_n^*\right) = E\left(\exp(\lambda \varepsilon_1/n) \mid Z_0^\theta = Z_{\tau_n^*}^\theta\right).$$

Yet, for all $z' \in U_K(\delta)$,

$$E\left(\exp(\lambda \varepsilon_1/n) \mid Z_0^\theta = z'\right) \leq \exp\left(-\beta' m + \frac{\lambda}{n}\right) + 1 - \exp(-\beta' m).$$

We iterate this procedure and we obtain

$$E\left(\exp(\lambda \varepsilon_1/n) \cdots \exp(\lambda \varepsilon_n/n)\right) \leq \left(\exp\left(-\beta' m + \frac{\lambda}{n}\right) + 1 - \exp(-\beta' m)\right)^n.$$

The change of variables $\lambda \rightarrow n\lambda$ yields

$$\begin{aligned} P(\varepsilon_1 + \dots + \varepsilon_n \geq n/2) \\ \leq \exp\left(-n\left(\lambda/2 - \ln(\exp(-\beta'm + \lambda) + 1 - \exp(-\beta'm))\right)\right). \end{aligned}$$

Let $\Lambda^*(t)$ be the Cramér transform of the Bernoulli law with parameter $p = \exp(-\beta'm)$:

$$\Lambda^*(t) = \sup_{\lambda \geq 0} \left(\lambda t - \ln(pe^\lambda + 1 - p) \right).$$

Optimising the previous inequality over $\lambda \geq 0$, we obtain

$$P(\varepsilon_1 + \dots + \varepsilon_n \geq n/2) \leq \exp(-n\Lambda^*(1/2)).$$

We can rewrite the function $\Lambda^*(t)$ as

$$\Lambda^*(t) = t \ln \frac{t}{p} + (1-t) \ln \frac{1-t}{1-p}.$$

In our particular case, for m large enough,

$$\Lambda^*(1/2) = \frac{1}{2} \ln \frac{\exp(\beta'm)}{2} + \frac{1}{2} \ln \frac{\exp(\beta'm)}{2(\exp(\beta'm) - 1)} \geq c(m),$$

where $c(m)$ is a positive constant depending on m but not on n . It follows that for m large enough,

$$\forall n \geq 1 \quad P\left(N(n \exp(\beta m)/2) \geq n \mid Z_0^\theta = z\right) \leq \exp(-nc(m)).$$

Let $t \geq 0$. We seek next an upper bound for the expectation

$$E\left(\sum_{k=1}^{N(t)} (T_k^* - T_{k-1}) + (t - T_{N(t)})\right).$$

The sum inside the parenthesis is at most t , therefore, for $n \geq 1$

$$\begin{aligned} E\left(\sum_{k=1}^{N(t)} (T_k^* - T_{k-1}) + (t - T_{N(t)})\right) &\leq \\ E\left(\left(\sum_{k=1}^{N(t)} (T_k^* - T_{k-1}) + (t - T_{N(t)})\right) 1_{N(t) < n}\right) &+ tP(N(t) \geq n). \end{aligned}$$

Let

$$n_t = \min \left\{ n \in \mathbb{N} : t \leq \frac{n \exp(\beta m)}{2} \right\}.$$

On one hand, the analysis of the random variable $N(n \exp(\beta m)/2)$ shows that taking $n = n_t$, the second term is bounded by

$$\frac{n_t}{2} e^{\beta m} P(N(n_t e^{\beta m}/2) \geq n_t) \leq \frac{n_t}{2} \exp(\beta m - n_t c(m)),$$

which goes to 0 when t goes to ∞ . On the other hand, we can bound the first term thanks to theorem 4.1:

$$\begin{aligned} & E \left(\left(\sum_{k=1}^{N(t)} (T_k^* - T_{k-1}) + (t - T_{N(t)}) \right) 1_{N(t) < n_t} \right) \\ & \leq E \left(\sum_{k=1}^{n_t+1} (T_k^* - T_{k-1}) \right) = \sum_{k=1}^{n_t+1} E(T_k^* - T_{k-1}) \leq (n_t + 1) \frac{m^\alpha}{1 - \exp(-\alpha' m)}, \end{aligned}$$

where $\alpha, \alpha' > 0$. We combine the above inequalities, and we obtain for m large enough and for all $t > 0$,

$$\begin{aligned} & \frac{1}{t} E \left(\sum_{n \geq 1} (T_n^* \wedge t - T_{n-1} \wedge t) \right) \\ & \leq \frac{2}{(n_t - 1) \exp(\beta m)} \left(\frac{n_t}{2} \exp(\beta m - n_t c(m)) + (n_t + 1) \frac{m^\alpha}{1 - \exp(-\alpha' m)} \right). \end{aligned}$$

When t goes to ∞ this expression goes to $2m^\alpha / \exp(\beta m)(1 - \exp(-\alpha' m))$, which in turn goes to 0 with m . We deduce that, for m large enough, there exists $t_m > 0$ such that for all $t \geq t_m$,

$$\left| \frac{1}{t} E \left(\sum_{i=0}^t f \left(\frac{Z_i^\theta(0) + \dots + Z_i^\theta(K)}{m} \right) \right) - f(\rho_0^* + \dots + \rho_K^*) \right| < \varepsilon.$$

Thus,

$$\lim_{\substack{\ell, m \rightarrow \infty, q \rightarrow 0 \\ \ell q \rightarrow a, \frac{m}{\ell} \rightarrow \alpha}} \int_{\mathcal{E}_K} f \left(\frac{z_0 + \dots + z_K}{m} \right) d\nu^\theta(z) = f(\rho_0^* + \dots + \rho_K^*).$$

□

5 The neutral phase

Throughout this section we take a such that $\sigma e^{-a} < 1$. Let μ_O be the invariant probability measure of the process $(O_t)_{t \geq 0}$ and let ν_K be the image measure of μ_O through the map

$$o \in \mathcal{P}_{\ell+1}^m \mapsto \frac{1}{m}(o(0) + \cdots + o(K)) \in [0, 1].$$

We will prove that when $\sigma e^{-a} < 1$, the probability measure ν_K converges to the Dirac mass at 0. In order to prove the convergence of ν_K to δ_0 , we slightly modify the occupancy process $(O_t)_{t \geq 0}$. Let \mathcal{W}_K^* be the set of the occupancy distributions having at least one individual in the Hamming classes $0, \dots, K$, i.e.,

$$\mathcal{W}_K^* = \{ o \in \mathcal{P}_{\ell+1}^m : o(0) + \cdots + o(K) \geq 1 \},$$

and let \mathcal{N}_K be the set of the occupancy distributions having no individuals in the Hamming classes $0, \dots, K$, i.e.,

$$\mathcal{N}_K = \{ o \in \mathcal{P}_{\ell+1}^m : o(0) + \cdots + o(K) = 0 \}.$$

Let us define the occupancy distribution o_{exit} by:

$$\forall l \in \{0, \dots, \ell\} \quad o_{\text{exit}}(l) = \begin{cases} m & \text{if } l = K + 1, \\ 0 & \text{otherwise.} \end{cases}$$

Let Φ_O be the coupling map defined in section 7.1 of [3]. We define a new coupling map Φ_Θ by setting for $o \in \mathcal{P}_{\ell+1}^m$ and $r \in \mathcal{R}$,

$$\Phi_\Theta(o, r) = \begin{cases} o_{\text{exit}} & \text{if } o \in \mathcal{W}_K^* \text{ and } \Phi_O(o, r) \in \mathcal{N}_K, \\ \Phi_O(o, r) & \text{otherwise.} \end{cases}$$

Since for all $o \in \mathcal{N}_K$ we have $o \preceq o_{\text{exit}}$, the map Φ_Θ is above the map Φ_O in the following sense:

$$\forall r \in \mathcal{R} \quad \forall o \in \mathcal{P}_{\ell+1}^m \quad \Phi_O(o, r) \preceq \Phi_\Theta(o, r).$$

Thus, we can build an upper process $(\Theta_t)_{t \geq 0}$ with the help of the i.i.d. sequence of random vectors $(R_n)_{n \geq 0}$, such that if the processes $(O_t)_{t \geq 0}$, $(\Theta_t)_{t \geq 0}$ both start from the same occupancy distribution o , then

$$\forall t \geq 0 \quad O_t \preceq \Theta_t.$$

Let μ_Θ be the invariant probability measure of the process $(\Theta_t)_{t \geq 0}$. We fix a non-decreasing function $f : [0, 1] \rightarrow \mathbb{R}$ such that $f(0) = 0$. We have

$$\int_{[0,1]} f d\nu_K \leq \int_{\mathcal{P}_{\ell+1}^m} f\left(\frac{o(0) + \dots + o(K)}{m}\right) d\mu_\Theta(o).$$

We apply now the renewal result of proposition 3.8 to the process $(\Theta_t)_{t \geq 0}$, the set \mathcal{W}_K^* , the function $o \mapsto f((o(0) + \dots + o(K))/m)$, and the occupancy distribution o_{exit} . Set

$$\begin{aligned} \tau_K^* &= \inf\{t \geq 0 : \Theta_t \in \mathcal{W}_K^*\}, \\ \tau_K &= \inf\{t \geq \tau_K^* : \Theta_t = o_{\text{exit}}\}. \end{aligned}$$

We have then

$$\begin{aligned} \int_{\mathcal{P}_{\ell+1}^m} f\left(\frac{o(0) + \dots + o(K)}{m}\right) d\mu_\Theta(o) &= \\ &= \frac{E\left(\int_0^{\tau_K} f\left(\frac{\Theta_s(0) + \dots + \Theta_s(K)}{m}\right) ds \mid \Theta_0 = o_{\text{exit}}\right)}{E(\tau_K \mid \Theta_0 = o_{\text{exit}})}. \end{aligned}$$

Since $f(0) = 0$, we have

$$\int_0^{\tau_K} f\left(\frac{\Theta_s(0) + \dots + \Theta_s(K)}{m}\right) ds = \int_{\tau_K^*}^{\tau_K} f\left(\frac{\Theta_s(0) + \dots + \Theta_s(K)}{m}\right) ds.$$

Moreover, since f is non-decreasing,

$$\begin{aligned} \int_{\mathcal{P}_{\ell+1}^m} f\left(\frac{o(0) + \dots + o(K)}{m}\right) d\mu_\Theta(o) &\leq \\ &\leq \frac{f(1)E(\tau_K - \tau_K^* \mid \Theta_0 = o_{\text{exit}})}{E(\tau_K^* \mid \Theta_0 = o_{\text{exit}}) + E(\tau_K - \tau_K^* \mid \Theta_0 = o_{\text{exit}})}. \end{aligned}$$

Our aim is to show that the ratio of the right-hand side of this expression converges to 0. We call τ_K^* the discovery time of the first $K + 1$ Hamming classes, and $\tau_K - \tau_K^*$ the persistence time of the first $K + 1$ Hamming classes. Let $o \in \mathcal{N}_K$. Starting from any occupancy distribution o , the processes $(O_t)_{t \geq 0}$ and $(\Theta_t)_{t \geq 0}$ both have the same dynamics until time τ_K . Thus, it

is enough to estimate the expectations of τ_K^* and $\tau_K - \tau_K^*$ for the original occupancy process $(O_t)_{t \geq 0}$. With a slight abuse of notation, we set

$$\begin{aligned}\tau_K^* &= \inf\{t \geq 0 : O_t \in \mathcal{W}_K^*\}, \\ \tau_K &= \inf\{t \geq \tau_K^* : O_t = o_{\text{exit}}\}.\end{aligned}$$

We deduce from the last inequality that

$$\int_{[0,1]} f d\nu_K \leq \frac{f(1)E(\tau_K - \tau_K^* | O_0 = o_{\text{exit}})}{E(\tau_K^* | O_0 = o_{\text{exit}}) + E(\tau_K - \tau_K^* | O_0 = o_{\text{exit}})}.$$

We estimate next the expectations appearing in the above inequality.

Proposition 5.1. *For all $o \in \mathcal{N}_K$, we have*

$$\lim_{\substack{\ell, m \rightarrow \infty, q \rightarrow 0 \\ \ell q \rightarrow a, \frac{m}{\ell} \rightarrow \alpha}} \frac{1}{\ell} \ln E(\tau_K^* | O_0 = o) = \ln \kappa.$$

Proof. Let τ^* be the discovery time of the master sequence, i.e.,

$$\tau^* = \inf\{t \geq 0 : O_t(0) \geq 1\}.$$

Obviously, $\tau^* \geq \tau_K^*$. Moreover, thanks to propositions 10.3 and 10.6 of [3], we know that for any occupancy distribution o such that $o(0) = 0$, we have

$$\lim_{\substack{\ell, m \rightarrow \infty, q \rightarrow 0 \\ \ell q \rightarrow a, \frac{m}{\ell} \rightarrow \alpha}} \frac{1}{\ell} \ln E(\tau^* | O_0 = o) = \ln \kappa.$$

We immediately deduce the upper bound for τ_K^* :

$$\forall o \in \mathcal{N}_K \quad \limsup_{\substack{\ell, m \rightarrow \infty, q \rightarrow 0 \\ \ell q \rightarrow a, \frac{m}{\ell} \rightarrow \alpha}} \frac{1}{\ell} \ln E(\tau_K^* | O_0 = o) \leq \ln \kappa.$$

We next prove the lower bound following the strategy proposed in [3] to estimate the discovery time of the master sequence. We work with the distance process $(D_t)_{t \geq 0}$ introduced in chapter 7 of [3]. We set

$$\begin{aligned}\mathcal{N}_K &= \{d \in \{0, \dots, \ell\}^m : d(i) > K, 1 \leq i \leq m\}, \\ \mathcal{W}_K^* &= \{d \in \{0, \dots, \ell\}^m : \exists i \in \{1, \dots, m\} \text{ such that } d(i) \leq K\}.\end{aligned}$$

Let

$$\tau_K^* = \inf \{ t \geq 0 : D_t \in \mathcal{W}_K^* \}.$$

The law of the discovery time τ_K^* is the same for the distance process and for the occupancy process. Let $b \in \{0, \dots, \ell\}$ and let us denote by $(b)^m$ the column vector whose entries are all b :

$$(b)^m = \begin{pmatrix} b \\ \vdots \\ b \end{pmatrix}.$$

On the sharp peak landscape the distance process restricted to the set \mathcal{N} is monotone (corollary 8.6 of [3]). Thus,

$$\forall d \in \mathcal{N}_K \quad E(\tau_K^* | D_0 = d) \geq E(\tau_K^* | D_0 = (K+1)^m).$$

In order to estimate this last expectation we use lemmas 11.4 and 11.5 of [3], which we state next. We denote by \mathcal{B} the binomial law with parameters ℓ and $1 - 1/\kappa$.

Lemma 5.2. *For $b \leq \ell/2$, we have*

$$\frac{1}{\kappa^\ell} \left(\frac{\ell}{2b} \right)^b \leq \mathcal{B}(b) \leq \frac{\ell^b}{\kappa^{\ell-b}}.$$

Lemma 5.3. *For $\rho \in [0, 1]$, we have*

$$\lim_{\ell \rightarrow \infty} \frac{1}{\ell} \ln \mathcal{B}(\lfloor \rho \ell \rfloor) = -(1 - \rho) \ln(\kappa(1 - \rho)) - \rho \ln \frac{\kappa \rho}{\kappa - 1}.$$

The following result is a variation of lemma 10.15 of [3]. The proof is similar and we omit it.

Lemma 5.4. *For $b \in \{K+1, \dots, \ell\}$, we have*

$$\forall n \geq 0 \quad P(\tau_K^* \leq n | D_0 = (b)^m) \leq nm \frac{\mathcal{B}(0) + \dots + \mathcal{B}(K)}{\mathcal{B}(b)}.$$

We show next the lower bound on the hitting time τ_K^* :

$$\liminf_{\substack{\ell, m \rightarrow \infty, q \rightarrow 0 \\ \ell q \rightarrow a, \frac{m}{\ell} \rightarrow \alpha}} \frac{1}{\ell} \ln E(\tau_K^* | D_0 = (K+1)^m) \geq \ln \kappa.$$

Let \mathcal{E} be the event given by

$$\mathcal{E} = \{ \forall n \leq m\ell^{3/4} \quad \forall l \leq \ln \ell \quad U_{n,l} > p/\kappa \}.$$

If the event \mathcal{E} happens, then until time $m\ell^{3/4}$, none of the mutation events in the process $(D_n)_{n \geq 0}$ can create a chromosome in one of the classes $0, \dots, K$. Indeed, on \mathcal{E} ,

$$\begin{aligned} \forall b \in \{K+1, \dots, \ell\} \quad \forall n \leq m\ell^{3/4} \\ \mathcal{M}_H(b, U_{n,1}, \dots, U_{n,\ell}) &\geq \mathcal{M}_H(K+1, U_{n,1}, \dots, U_{n,\ell}) \\ &\geq K+1 + \sum_{l=K+2}^{\ell} 1_{U_{n,l} > 1-p(1-1/\kappa)} \geq K+1. \end{aligned}$$

Thus, on the event \mathcal{E} , we have $\tau_K^* \geq m\ell^{3/4}$. The probability of \mathcal{E} is

$$P(\mathcal{E}) = \left(1 - \frac{p}{\kappa}\right)^{m\ell^{3/4} \ln \ell}.$$

Let $\varepsilon > 0$. Let us suppose that the process starts at $(K+1)^m$ and let us estimate the probability

$$\begin{aligned} P(\tau_K^* > \kappa^{\ell(1-\varepsilon)}) &\geq P(\tau_K^* > \kappa^{\ell(1-\varepsilon)}, \mathcal{E}) \\ &\geq P(\forall t \in \{m\ell^{3/4}, \dots, \kappa^{\ell(1-\varepsilon)}\} \quad D_t \in \mathcal{N}_K, \mathcal{E}) \\ &= \sum_{d \in \mathcal{N}_K} P(\forall t \in \{m\ell^{3/4}, \dots, \kappa^{\ell(1-\varepsilon)}\} \quad D_t \in \mathcal{N}_K, D_{m\ell^{3/4}} = d, \mathcal{E}) \\ &\geq \sum_{d \geq (\ln \ell)^m} P(\forall t \in \{m\ell^{3/4}, \dots, \kappa^{\ell(1-\varepsilon)}\} \quad D_t \in \mathcal{N}_K \mid D_{m\ell^{3/4}} = d, \mathcal{E}) \\ &\quad \times P(D_{m\ell^{3/4}} = d, \mathcal{E}). \quad (\Sigma_1) \end{aligned}$$

Using the Markov property we obtain

$$\begin{aligned} P(\forall t \in \{m\ell^{3/4}, \dots, \kappa^{\ell(1-\varepsilon)}\} \quad D_t \in \mathcal{N}_K \mid D_{m\ell^{3/4}} = d, \mathcal{E}) \\ = P(\forall t \in \{0, \dots, \kappa^{\ell(1-\varepsilon)} - m\ell^{3/4}\} \quad D_t \in \mathcal{N}_K \mid D_0 = d) \\ = P(\tau_K^* > \kappa^{\ell(1-\varepsilon)} - m\ell^{3/4} \mid D_0 = d) \geq P(\tau_K^* > \kappa^{\ell(1-\varepsilon)} \mid D_0 = d). \end{aligned}$$

In the neutral case the distance process is monotone (corollary 8.6 of [3]). Thus, for all $d \geq (\ln \ell)^m$,

$$P(\tau_K^* > \kappa^{\ell(1-\varepsilon)} \mid D_0 = d) \geq P(\tau_K^* > \kappa^{\ell(1-\varepsilon)} \mid D_0 = (\ln \ell)^m).$$

Therefore, we can rewrite inequality (Σ_1) as follows:

$$P(\tau_K^* > \kappa^{\ell(1-\varepsilon)}) \geq P(\tau_K^* > \kappa^{\ell(1-\varepsilon)} \mid D_0 = (\ln \ell)^m) P(D_{m\ell^{3/4}} \geq (\ln \ell)^m, \mathcal{E}). \quad (\circ)$$

Since the distance process is monotone, we have

$$P(D_{m\ell^{3/4}} \geq (\ln \ell)^m, \mathcal{E} \mid D_0 = (K+1)^m) \geq P(D_{m\ell^{3/4}} \geq (\ln \ell)^m, \mathcal{E} \mid D_0 = (1)^m).$$

We also have the following estimate (section 10.5 of [3]):

$$P(D_{m\ell^{3/4}} \geq (\ln \ell)^m, \mathcal{E} \mid D_0 = (1)^m) \geq \left(1 - m \exp\left(-\frac{1}{3}(\ln \ell)^2\right)\right) \left(1 - \frac{p}{\kappa}\right)^{m\ell^{3/4} \ln \ell}. \quad (\nabla)$$

We estimate next

$$P(\tau_K^* > \kappa^{\ell(1-\varepsilon)} \mid D_0 = (\ln \ell)^m).$$

Let $\varepsilon' > 0$. We have

$$\begin{aligned} & P(\tau_K^* > \kappa^{\ell(1-\varepsilon)} \mid D_0 = (\ln \ell)^m) \\ &= P(\tau_K^* > m\ell^2, D_t \in \mathcal{N}_K \text{ for } m\ell^2 \leq t \leq \kappa^{\ell(1-\varepsilon)} \mid D_0 = (\ln \ell)^m) \\ &= \sum_{d \in \mathcal{N}_K} P\left(\tau_K^* > m\ell^2, D_{m\ell^2} = d \mid D_0 = (\ln \ell)^m\right) \\ &\geq \sum_{d \geq (\ell_\kappa(1-\varepsilon'))^m} P\left(D_t \in \mathcal{N}_K \text{ for } m\ell^2 \leq t \leq \kappa^{\ell(1-\varepsilon)} \mid \tau_K^* > m\ell^2, D_{m\ell^2} = d\right) \\ &\quad \times P(\tau_K^* > m\ell^2, D_{m\ell^2} = d \mid D_0 = (\ln \ell)^m). \quad (\Sigma_3) \end{aligned}$$

The Markov property and the monotonicity of the process $(D_t)_{t \geq 0}$ give for $d \geq (\ell_\kappa(1-\varepsilon'))^m$,

$$\begin{aligned} & P(D_t \in \mathcal{N}_K \text{ for } m\ell^2 \leq t \leq \kappa^{\ell(1-\varepsilon)} \mid \tau_K^* > m\ell^2, D_{m\ell^2} = d) \\ &= P(\forall t \in \{0, \dots, \kappa^{\ell(1-\varepsilon)} - m\ell^2\} \quad D_t \in \mathcal{N}_K \mid D_0 = d) \\ &= P(\tau_K^* > \kappa^{\ell(1-\varepsilon)} - m\ell^2 \mid D_0 = d) \geq P(\tau_K^* > \kappa^{\ell(1-\varepsilon)} \mid D_0 = d) \\ &\geq P(\tau_K^* > \kappa^{\ell(1-\varepsilon)} \mid D_0 = (\ell_\kappa(1-\varepsilon'))^m). \end{aligned}$$

Reporting back into the inequality (Σ_3) ,

$$P\left(\tau_K^* > \kappa^{\ell(1-\varepsilon)} \mid D_0 = (\ln \ell)^m\right) \geq P\left(\tau_K^* > \kappa^{\ell(1-\varepsilon)} \mid D_0 = (\ell_\kappa(1-\varepsilon'))^m\right) \\ \times P\left(\tau_K^* > m\ell^2, D_{m\ell^2} \geq (\ell_\kappa(1-\varepsilon'))^m \mid D_0 = (\ln \ell)^m\right). \quad (\heartsuit)$$

Estimation of $P(\tau_K^* > m\ell^2, D_{m\ell^2} \geq (\ell_\kappa(1-\varepsilon'))^m \mid D_0 = (\ln \ell)^m)$. We write

$$P\left(\tau_K^* > m\ell^2, D_{m\ell^2} \geq (\ell_\kappa(1-\varepsilon'))^m \mid D_0 = (\ln \ell)^m\right) \geq \quad (\natural) \\ P\left(D_{m\ell^2} \geq (\ell_\kappa(1-\varepsilon'))^m \mid D_0 = (\ln \ell)^m\right) - P\left(\tau_K^* \leq m\ell^2 \mid D_0 = (\ln \ell)^m\right).$$

We control the last term by applying lemma 5.4 with $n = m\ell^2$ and $b = \ln \ell$:

$$P\left(\tau_K^* \leq m\ell^2 \mid D_0 = (\ln \ell)^m\right) \leq (m\ell)^2 \frac{\mathcal{B}(0) + \cdots + \mathcal{B}(K)}{\mathcal{B}(\ln \ell)}.$$

Using lemma 5.2 we get

$$\frac{\mathcal{B}(0) + \cdots + \mathcal{B}(K)}{\mathcal{B}(\ln \ell)} \leq \frac{1 - (\ell\kappa)^{K+1}}{1 - \ell\kappa} \left(\frac{2 \ln \ell}{\ell}\right)^{\ln \ell}.$$

Thus,

$$P\left(\tau_K^* \leq m\ell^2 \mid D_0 = (\ln \ell)^m\right) \leq (m\ell)^2 \frac{1 - (\ell\kappa)^{K+1}}{1 - \ell\kappa} \left(\frac{2 \ln \ell}{\ell}\right)^{\ln \ell}. \quad (\clubsuit)$$

An estimate from section 10.5 of [3] will help us control the other term: there exists a constant $c(\varepsilon') > 0$ such that for ℓ large enough, we have

$$P\left(D_{m\ell^2} \geq (\ell_\kappa(1-\varepsilon'))^m \mid D_0 = (\ln \ell)^m\right) \geq 1 - m \exp\left(-\frac{1}{2}c(\varepsilon')\ell\right).$$

This inequality together with estimates (\natural) and (\clubsuit) give

$$P\left(\tau_K^* > m\ell^2, D_{m\ell^2} \geq (\ell_\kappa(1-\varepsilon'))^m \mid D_0 = (\ln \ell)^m\right) \geq \quad (\clubsuit) \\ 1 - m \exp\left(-\frac{1}{2}c(\varepsilon')\ell\right) - (m\ell)^2 \frac{1 - (\ell\kappa)^{K+1}}{1 - \ell\kappa} \left(\frac{2 \ln \ell}{\ell}\right)^{\ln \ell}.$$

Estimation of $P(\tau_K^* > \kappa^{\ell(1-\varepsilon)} \mid D_0 = (\ell_\kappa(1-\varepsilon'))^m)$. We use the inequality of lemma 5.4 with $n = \kappa^{\ell(1-\varepsilon)}$ and $b = \ell_\kappa(1-\varepsilon')$:

$$P\left(\tau_K^* \leq \kappa^{\ell(1-\varepsilon)} \mid D_0 = (\ell_\kappa(1-\varepsilon'))^m\right) \leq \kappa^{\ell(1-\varepsilon)} m \frac{\mathcal{B}(0) + \cdots + \mathcal{B}(K)}{\mathcal{B}(\ell_\kappa(1-\varepsilon'))}.$$

For ε' small enough, the large deviation estimates of lemma 5.3 imply the existence of a constant $c(\varepsilon, \varepsilon') > 0$ such that, for ℓ large enough,

$$P(\tau_K^* \leq \kappa^{\ell(1-\varepsilon)} \mid D_0 = (\ell\kappa(1-\varepsilon'))^m) \leq \exp(-c(\varepsilon, \varepsilon')\ell). \quad (\spadesuit)$$

Plugging the inequalities () and () into () we obtain

$$P(\tau_K^* > \kappa^{\ell(1-\varepsilon)} \mid D_0 = (\ln \ell)^m) \geq \left(1 - \exp(-c(\varepsilon, \varepsilon')\ell)\right) \quad (\Delta) \\ \left(1 - m \exp\left(-\frac{1}{2}c(\varepsilon')\ell\right) - (m\ell)^2 \frac{1 - (\ell\kappa)^{K+1}}{1 - \ell\kappa} \left(\frac{2 \ln \ell}{\ell}\right)^{\ln \ell}\right).$$

We now use the inequalities (), (∇), (Δ) to conclude that, for ℓ large enough,

$$P(\tau_K^* > \kappa^{\ell(1-\varepsilon)} \mid D_0 = (K+1)^m) \geq \\ \left(1 - m \exp\left(-\frac{1}{3}(\ln \ell)^2\right)\right) \left(1 - \frac{p}{\kappa}\right)^{m\ell^{3/4} \ln \ell} \left(1 - \exp(-c(\varepsilon, \varepsilon')\ell)\right) \\ \times \left(1 - m \exp\left(-\frac{1}{2}c(\varepsilon')\ell\right) - (m\ell)^2 \frac{1 - (\ell\kappa)^{K+1}}{1 - \ell\kappa} \left(\frac{2 \ln \ell}{\ell}\right)^{\ln \ell}\right).$$

Moreover, thanks to the Markov inequality,

$$E(\tau_K^* \mid D_0 = (K+1)^m) \geq \kappa^{\ell(1-\varepsilon)} P(\tau_K^* \geq \kappa^{\ell(1-\varepsilon)} \mid D_0 = (K+1)^m).$$

We finally deduce that

$$\liminf_{\substack{\ell, m \rightarrow \infty, q \rightarrow 0 \\ \ell q \rightarrow a, \frac{m}{\ell} \rightarrow \alpha}} \frac{1}{\ell} \ln E(\tau_K^* \mid D_0 = (K+1)^m) \geq (1-\varepsilon) \ln \kappa.$$

Sending ε to 0 gives the desired lower bound. \square

We estimate next $E(\tau_K - \tau_K^* \mid O_0 = o_{\text{exit}})$. Let $\phi :]0, +\infty[\rightarrow [0, +\infty]$ be the function defined in [3] by setting $\phi(a) = 0$ if $a \geq \ln \sigma$ and

$$\forall a < \ln \sigma \quad \phi(a) = \frac{\sigma(1 - e^{-a}) \ln \frac{\sigma(1 - e^{-a})}{\sigma - 1} + \ln(\sigma e^{-a})}{(1 - \sigma(1 - e^{-a}))}.$$

Proposition 5.5. *We have*

$$\lim_{\substack{\ell, m \rightarrow \infty, q \rightarrow 0 \\ \ell q \rightarrow a, \frac{m}{\ell} \rightarrow \alpha}} \frac{1}{m} \ln E(\tau_K - \tau_K^* \mid O_0 = o_{\text{exit}}) = \phi(a).$$

Proof. For any subset $E \subset \mathcal{P}_{\ell+1}^m$ we define the hitting time of E by

$$\tau(E) = \inf \{ t \geq 0 : O_t \in E \}.$$

Let us define also the following occupancy distributions:

$$\begin{aligned} o_1 &= (1, m-1, 0, \dots, 0), & o_2 &= (1, 0, \dots, 0, m-1), \\ \forall l \in \{0, \dots, \ell\} \quad o_3(l) &= \begin{cases} 1 & \text{if } l = K, \\ m-1 & \text{if } l = \ell, \\ 0 & \text{otherwise.} \end{cases} \end{aligned}$$

Thanks to the monotonicity of the process $(O_t)_{t \geq 0}$ we have

$$E(\tau(\mathcal{N}_K) | O_0 = o_3) \leq E(\tau_K - \tau_K^* | O_0 = o_{\text{exit}}) \leq E(\tau(\mathcal{N}_K) | O_0 = o_1).$$

We prove first the lower bound, i.e.,

$$\liminf_{\substack{\ell, m \rightarrow \infty, q \rightarrow 0 \\ \ell q \rightarrow a, \frac{m}{\ell} \rightarrow \alpha}} \frac{1}{m} \ln E(\tau(\mathcal{N}_K) | O_0 = o_3) \geq \phi(a).$$

Indeed, we have

$$\begin{aligned} E(\tau(\mathcal{N}_K) | O_0 = o_3) &\geq \\ &E(\tau(\mathcal{N}_K) | O_0 = o_3, O_1 = o_2) P(O_1 = o_2 | O_0 = o_3). \end{aligned}$$

On one hand, thanks to the Markov property and the monotonicity of the process

$$E(\tau(\mathcal{N}_K) | O_0 = o_3, O_1 = o_2) \geq 1 + E(\tau(\mathcal{N}_K) | O_0 = o_2).$$

Since $\tau(\mathcal{N}_K) \geq \tau(\mathcal{N})$, thanks to the estimate for $\tau(\mathcal{N})$ (corollary 9.2 of [3]), we obtain

$$\begin{aligned} \liminf_{\substack{\ell, m \rightarrow \infty, q \rightarrow 0 \\ \ell q \rightarrow a, \frac{m}{\ell} \rightarrow \alpha}} \frac{1}{m} \ln E(\tau(\mathcal{N}_K) | O_0 = o_2) &\geq \\ &\liminf_{\substack{\ell, m \rightarrow \infty, q \rightarrow 0 \\ \ell q \rightarrow a, \frac{m}{\ell} \rightarrow \alpha}} \frac{1}{m} \ln E(\tau(\mathcal{N}) | O_0 = o_2) \geq \phi(a). \end{aligned}$$

On the other hand, we have

$$\begin{aligned} P(O_1 = o_2 | O_0 = o_3) &= \frac{1}{m^2} (M_H(K, 0) + (m-1)M_H(\ell, 0)) \\ &\geq \frac{1}{m^2} M_H(K, 0) = \frac{1}{m^2} \left(1 - p\left(1 - \frac{1}{\kappa}\right)\right)^{\ell-K} \left(\frac{p}{\kappa}\right)^K. \end{aligned}$$

Thus,

$$\liminf_{\substack{\ell, m \rightarrow \infty, q \rightarrow 0 \\ \ell q \rightarrow a, \frac{m}{\ell} \rightarrow \alpha}} \frac{1}{m} \ln P(O_1 = o_2 \mid O_0 = o_3) \geq 0.$$

The above inequalities give the desired lower bound:

$$\liminf_{\substack{\ell, m \rightarrow \infty, q \rightarrow 0 \\ \ell q \rightarrow a, \frac{m}{\ell} \rightarrow \alpha}} \frac{1}{m} \ln E(\tau(\mathcal{N}_K) \mid O_0 = o_3) \geq \phi(a).$$

We prove next the upper bound:

$$\limsup_{\substack{\ell, m \rightarrow \infty, q \rightarrow 0 \\ \ell q \rightarrow a, \frac{m}{\ell} \rightarrow \alpha}} \frac{1}{m} \ln E(\tau(\mathcal{N}_K) \mid O_0 = o_1) \leq \phi(a).$$

To alleviate the notation we denote by P_o and E_o the probability and the expectation for the process $(O_t)_{t \geq 0}$ starting from the occupancy distribution o . Let o' be the occupancy distribution given by $o' = (0, m, 0, \dots, 0)$. We have

$$E_{o_1}(\tau(\mathcal{N}_K)) = E_{o_1}(\tau(\mathcal{N})) + E_{o_1}(\tau(\mathcal{N}_K) - \tau(\mathcal{N})).$$

Yet, thanks to the strong Markov property and the monotonicity of the process, and since $\tau(\mathcal{N}_K) \geq \tau(\mathcal{N})$,

$$\begin{aligned} E_{o_1}(\tau(\mathcal{N}_K) - \tau(\mathcal{N})) &= \sum_{o \in \mathcal{N}} E_o(\tau(\mathcal{N}_K)) P_{o_1}(O_{\tau(\mathcal{N})} = o) \\ &\leq \sum_{o \in \mathcal{N}} E_{o'}(\tau(\mathcal{N}_K)) P_{o_1}(O_{\tau(\mathcal{N})} = o) = E_{o'}(\tau(\mathcal{N}_K)). \end{aligned}$$

We develop this last expectation as follows:

$$E_{o'}(\tau(\mathcal{N}_K)) = E_{o'}(\tau(\mathcal{W}^* \cup \mathcal{N}_K)) + E_{o'}(\tau(\mathcal{N}_K) - \tau(\mathcal{W}^* \cup \mathcal{N}_K)).$$

Yet,

$$\begin{aligned} E_{o'}(\tau(\mathcal{N}_K) - \tau(\mathcal{W}^* \cup \mathcal{N}_K)) &= \sum_{o \in \mathcal{W}^*} E_o(\tau(\mathcal{N}_K)) P_{o'}(O_{\tau(\mathcal{W}^* \cup \mathcal{N}_K)} = o) \\ &\leq \sum_{o \in \mathcal{W}^*} E_{o_1}(\tau(\mathcal{N}_K)) P_{o'}(O_{\tau(\mathcal{W}^* \cup \mathcal{N}_K)} = o) \\ &= E_{o_1}(\tau(\mathcal{N}_K)) P_{o'}(\tau(\mathcal{W}^*) < \tau(\mathcal{N}_K)). \end{aligned}$$

Thus,

$$E_{o'}(\tau(\mathcal{N}_K)) \leq E_{o'}(\tau(\mathcal{W}^* \cup \mathcal{N}_K)) + E_{o_1}(\tau(\mathcal{N}_K))P_{o'}(\tau(\mathcal{W}^*) < \tau(\mathcal{N}_K)).$$

Therefore,

$$E_{o_1}(\tau(\mathcal{N}_K)) \leq \frac{1}{P_{o'}(\tau(\mathcal{N}_K) < \tau(\mathcal{W}^*))} \left(E_{o_1}(\tau(\mathcal{N})) + E_{o'}(\tau(\mathcal{W}^* \cup \mathcal{N}_K)) \right).$$

We estimate next the three terms appearing on the right-hand side of this formula. We control the first expectation with the help of the estimate for the persistence time of the master sequence (corollary 9.2 of [3]) and we obtain:

$$\limsup_{\substack{\ell, m \rightarrow \infty, q \rightarrow 0 \\ \ell q \rightarrow a, \frac{m}{\ell} \rightarrow \alpha}} \frac{1}{m} \ln E_{o_1}(\tau(\mathcal{N})) \leq \phi(a).$$

We bound from below the probability in the denominator using the estimates on the discovery time τ_K^* . On the event

$$\mathcal{E} = \left\{ \forall n \leq m\ell^{3/4} \quad \forall l \leq \ln \ell \quad U_{n,l} > p/\kappa \right\},$$

if $D_{m\ell^{3/4}} \geq (\ln \ell)^m$, we have $\tau(\mathcal{N}_K) < \tau(\mathcal{W}^*)$. Therefore, using (∇) ,

$$P_{o'}(\tau(\mathcal{N}_K) < \tau(\mathcal{W}^*)) \geq P(D_{m\ell^{3/4}} \geq (\ln \ell)^m, \mathcal{E} \mid D_0 = (1)^m) \geq \left(1 - m \exp\left(-\frac{1}{3}(\ln \ell)^2\right)\right) \left(1 - \frac{p}{\kappa}\right)^{m\ell^{3/4} \ln \ell}.$$

It remains to estimate the expectation $E_{o'}(\tau(\mathcal{W}^* \cup \mathcal{N}_K))$. We estimate first, for $n \geq 0$, the probability $P_{o'}(\tau(\mathcal{W}^* \cup \mathcal{N}_K) > n)$. We have

$$\begin{aligned} P_{o'}(\tau(\mathcal{W}^* \cup \mathcal{N}_K) > n) &= \sum_{o \in \mathcal{W}_K^* \setminus \mathcal{W}^*} P_{o'}(\tau(\mathcal{W}^* \cup \mathcal{N}_K) > n, O_{n-1} = o, \tau(\mathcal{W}^* \cup \mathcal{N}_K) > n-1) \\ &= \sum_{o \in \mathcal{W}_K^* \setminus \mathcal{W}^*} P_{o'}(\tau(\mathcal{W}^* \cup \mathcal{N}_K) > n \mid O_{n-1} = o, \tau(\mathcal{W}^* \cup \mathcal{N}_K) > n-1) \\ &\quad \times P_{o'}(O_{n-1} = o, \tau(\mathcal{W}^* \cup \mathcal{N}_K) > n-1). \end{aligned}$$

Thanks to the Markov property and the monotonicity of the process, we have for all $o \in \mathcal{W}_K^* \setminus \mathcal{W}^*$,

$$\begin{aligned} P_{o'}(\tau(\mathcal{W}^* \cup \mathcal{N}_K) > n \mid O_{n-1} = o, \tau(\mathcal{W}^* \cup \mathcal{N}_K) > n-1) &= \\ P_o(\tau(\mathcal{W}^* \cup \mathcal{N}_K) > 1) &= 1 - P_o(\tau(\mathcal{W}^* \cup \mathcal{N}_K) = 1) \\ \leq 1 - P_o(\tau(\mathcal{W}^*) = 1) &\leq 1 - P(\tau(\mathcal{W}^*) = 1 \mid O_0 = o_3) \\ &\leq 1 - \frac{M_H(K, 0)}{m^2}. \end{aligned}$$

Thus,

$$P_{\sigma'}(\tau(\mathcal{W}^* \cup \mathcal{N}_K) > n) \leq \left(1 - \frac{M_H(K, 0)}{m^2}\right) P_{\sigma'}(\tau(\mathcal{W}^* \cup \mathcal{N}_K) > n - 1).$$

We iterate this inequality and we obtain

$$P_{\sigma'}(\tau(\mathcal{W}^* \cup \mathcal{N}_K) > n) \leq \left(1 - \frac{M_H(K, 0)}{m^2}\right)^n.$$

Finally,

$$\begin{aligned} E_{\sigma'}(\tau(\mathcal{W}^* \cup \mathcal{N}_K)) &= \sum_{n \geq 0} P_{\sigma'}(\tau(\mathcal{W}^* \cup \mathcal{N}_K) > n) \\ &\leq \sum_{n \geq 0} \left(1 - \frac{M_H(K, 0)}{m^2}\right)^n = \frac{m^2}{M_H(K, 0)} = \frac{m^2}{\left(1 - p\left(1 - \frac{1}{\kappa}\right)\right)^{\ell-K} \left(\frac{p}{\kappa}\right)^K}. \end{aligned}$$

We put together the above inequalities and we obtain the desired upper bound. \square

Let $\phi :]0, +\infty[\rightarrow [0, +\infty]$ be the function defined by $\phi(a) = 0$ if $a \geq \ln \sigma$ and

$$\forall a < \ln \sigma \quad \phi(a) = \frac{\sigma(1 - e^{-a}) \ln \frac{\sigma(1 - e^{-a})}{\sigma - 1} + \ln(\sigma e^{-a})}{(1 - \sigma(1 - e^{-a}))}.$$

From the estimates obtained in this section we conclude that for $\alpha \in [0, +\infty[$ or $\alpha = +\infty$,

$$\lim_{\substack{\ell, m \rightarrow \infty, q \rightarrow 0 \\ \ell q \rightarrow a, \frac{m}{\ell} \rightarrow \alpha}} \frac{E(\tau_K - \tau_K^* \mid O_0 = o_{\text{exit}})}{E(\tau_K^* \mid O_0 = o_{\text{exit}})} = \begin{cases} 0 & \text{if } \alpha \phi(a) < \ln \kappa \\ +\infty & \text{if } \alpha \phi(a) > \ln \kappa \end{cases}$$

In particular, if $\alpha \phi(a) < \ln \kappa$, we have

$$\lim_{\substack{\ell, m \rightarrow \infty, q \rightarrow 0 \\ \ell q \rightarrow a, \frac{m}{\ell} \rightarrow \alpha}} \int_{[0,1]} f d\nu_K = 0.$$

6 Synthesis

Let us look at the formula given at the end of section 3.5:

$$\begin{aligned} \int_{\mathcal{P}_{\ell+1}^m} f\left(\frac{o(0) + \dots + o(K)}{m}\right) d\mu_O^\theta(o) = \\ \frac{E\left(\int_0^{\tau^*} f\left(\frac{O_s^\theta(0) + \dots + O_s^\theta(K)}{m}\right) ds \mid O_0^\theta = o_{\text{exit}}^\theta\right)}{E(\tau^* \mid O_0^\theta = o_{\text{exit}}^\theta) + E(\tau_0 \mid Z_0^\theta = z^\theta)} \\ + \frac{1 + E(\tau_0 \mid Z_0^\theta = z^\theta)}{E(\tau^* \mid O_0^\theta = o_{\text{exit}}^\theta) + E(\tau_0 \mid Z_0^\theta = z^\theta)} \int_{\mathcal{E}_K} f\left(\frac{z_0 + \dots + z_K}{m}\right) d\nu^\theta(z). \end{aligned}$$

The integral appearing in the first term of the right-hand side is bounded as follows:

$$0 \leq E\left(\int_0^{\tau^*} f\left(\frac{O_s^\theta(0) + \dots + O_s^\theta(K)}{m}\right) ds\right) \leq f(1)E(\tau^* \mid O_0^\theta = o_{\text{exit}}^\theta).$$

The stopping times τ^* and τ_0 are the same as the discovery time of the master sequence and the persistence time of the master sequence from [3]. As shown in [3], the following estimates hold:

$$\begin{aligned} \lim_{\substack{\ell, m \rightarrow \infty \\ q \rightarrow 0, \ell q \rightarrow a}} \frac{1}{\ell} \ln E(\tau^* \mid O_0^\theta = o_{\text{exit}}^\theta) &= \ln \kappa, \\ \lim_{\substack{\ell, m \rightarrow \infty \\ q \rightarrow 0, \ell q \rightarrow a}} \frac{1}{m} \ln E(\tau_0 \mid Z_0^\theta = z^\theta) &= \phi(a), \end{aligned}$$

where the function $\phi : \mathbb{R}^+ \rightarrow \mathbb{R}^+ \cup \{+\infty\}$ is given by $\phi(a) = 0$ if $a \geq \ln \sigma$, and

$$\forall a < \ln \sigma \quad \phi(a) = \frac{\sigma(1 - e^{-a}) \ln \frac{\sigma(1 - e^{-a})}{\sigma - 1} + \ln(\sigma e^{-a})}{(1 - \sigma(1 - e^{-a}))}.$$

Therefore,

$$\lim_{\substack{\ell, m \rightarrow \infty, q \rightarrow 0 \\ \ell q \rightarrow a, \frac{m}{\ell} \rightarrow \alpha}} \frac{E(\tau_0 \mid Z_0^\theta = z^\theta)}{E(\tau^* \mid O_0^\theta = o_{\text{exit}}^\theta)} = \begin{cases} 0 & \text{si } \alpha \phi(a) < \ln \kappa, \\ +\infty & \text{si } \alpha \phi(a) > \ln \kappa. \end{cases}$$

This, together with the results from the previous section and section 4.5 imply that

- If $\alpha\phi(a) < \ln \kappa$, then

$$\lim_{\substack{\ell, m \rightarrow \infty, q \rightarrow 0 \\ \ell q \rightarrow a, \frac{m}{\ell} \rightarrow \alpha}} \int_{\mathcal{P}_{\ell+1}^m} f\left(\frac{o(0) + \dots + o(K)}{m}\right) d\mu_O^\theta(o) = 0.$$

- If $\alpha\phi(a) > \ln \kappa$, then

$$\lim_{\substack{\ell, m \rightarrow \infty, q \rightarrow 0 \\ \ell q \rightarrow a, \frac{m}{\ell} \rightarrow \alpha}} \int_{\mathcal{P}_{\ell+1}^m} f\left(\frac{o(0) + \dots + o(K)}{m}\right) d\mu_O^\theta(o) = f(\rho_0^* + \dots + \rho_K^*).$$

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