LARGE DEVIATIONS OF THE FINITE CLUSTER SHAPE FOR TWO–DIMENSIONAL PERCOLATION IN THE HAUSDORFF AND L^1 METRIC

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Abstract. We consider supercritical two–dimensional Bernoulli percolation. Conditionally on the event that the open cluster C containing the origin is finite, we prove that
• the laws of C/N satisfy a large deviations principle with respect to the Hausdorff metric,
• let $f(N)$ be a function from $\mathbb{N}$ to $\mathbb{R}$ such that $f(N)/\ln N \to +\infty$ and $f(N)/N \to 0$ as N goes to ∞; the laws of $\{\, x \in \mathbb{R}^2 : d(x, C) \le f(N)\,\}/N$ satisfy a large deviations principle with respect to the L^1 metric associated to the planar Lebesgue measure.
We link the second large deviations principle with the Wulff construction.

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1. Introduction

In this paper we continue to investigate questions related to large deviations principles for random sets. The simplest result of this kind, namely the analog of the Cramér theorem for random sets was proved in [3]. Our true goal is to obtain a formulation of the Wulff construction in dimension three. The main obstacle is to get rid of the skeleton coarse graining technique, which is essential for the current proofs of the Wulff construction in dimension two [1]. The work presented here is an intermediate step towards this goal. Here we prove large deviations principles for the finite cluster shape in the Hausdorff and L^1 metric. Although we rely here on the skeleton coarse graining technique, we think that the formulation of the large deviations principles themselves is robust and that they might be obtained with a very different strategy, which is likely to be implemented successfully in higher dimension. This program is completed in [4].

We consider Bernoulli bond percolation on the square lattice in which edges are independently open with probability p and closed with probability $1-p$. It is known that this model has a phase transition at $p_c = 1/2$: for $p < p_c$ the open clusters are finite and for $p > p_c$ there exists a unique infinite open cluster [11,13]. In the percolating phase where $p > p_c$, the probability that the origin belongs to a finite open cluster of diameter of order N goes to zero as N goes to infinity. Our aim is to study the shapes of the large finite open clusters. Let C be the open cluster containing the origin. Conditionally on the event that C is finite, we prove that

• as N goes to ∞, the laws of the random compact sets C/N satisfy a large deviations principle with respect to the Hausdorff metric;

• let $f(N)$ be a function from $\mathbb{N}$ to $\mathbb{R}$ such that $f(N)/\ln N \to +\infty$ and $f(N)/N \to 0$ as N goes to ∞; the laws of the random compact sets $\{\, x \in \mathbb{R}^2 : d(x,C) \leq f(N)\,\}/N$ (where $d(x,C)$ is the distance from x to C) satisfy a large deviations principle with respect to the L^1 metric associated to the planar Lebesgue measure.

The two rate functions are suitable surface energies and they coincide on regular sets. To build them, we first extract from the percolation model a direction dependent surface tension (as done in [1]). The value of the surface tension in a given direction characterizes the asymptotic exponential decay of the probability of seeing a small flat interface in this direction. The surface energy of a regular set is simply the linear integral of the surface tension along the boundary of the set. This surface energy then characterizes the asymptotic exponential decay of the probability that the rescaled open cluster containing the origin is close to the given set. We apply the second large deviations principle to estimate the asymptotic behavior of the probability that the random set $\{\, x : d(x,C) \leq f(N)\,\}$ has a finite area larger than N^2. This provides a link with the Wulff construction.

2. The model

We consider the square site lattice $\mathbb{Z}^2$, the dual square site lattice $(\mathbb{Z}^*)^2 = \mathbb{Z}^2 + (1/2, 1/2)$

and the plane $\mathbb{R}^2$. For $x = (x_1, x_2)$ in $\mathbb{Z}^2$ we set $x^* = x + (1/2, 1/2)$. We define the usual norms:

$$|x|_1 = |x_1| + |x_2|\,, \quad |x|_2 = \sqrt{x_1^2 + x_2^2}\,, \quad |x|_\infty = \max(|x_1|, |x_2|)\,.$$

On the plane $\mathbb{R}^2$ we will mostly use the Euclidean norm $|\ \ |_2$. For S a subset of $\mathbb{Z}^2$ we denote by $|S|$ its cardinality. We turn $\mathbb{Z}^2$ into a graph by adding edges between all pairs x, y of points of $\mathbb{Z}^2$ such that $|x-y|_1 = 1$. The set of all edges between nearest neighbor sites of $\mathbb{Z}^2$ is denoted by $\mathbb{E}^2$. A path in $(\mathbb{Z}^2, \mathbb{E}^2)$ is an alternating sequence $x_0, e_0, x_1, e_1, \cdots, e_{n-1}, x_n, \cdots$ of distinct vertices x_i and edges e_i, where e_i is the edge between x_i and x_{i+1} (we adopt here the definition of [11], which is slightly different from the one used in [1]). If the path terminates at some vertex x_n it is said to connect x_0 to x_n. A circuit in $(\mathbb{Z}^2, \mathbb{E}^2)$ is an alternating sequence $x_0, e_0, x_1, e_1, \cdots, e_{n-1}, x_n, e_n, x_0$ such that $x_0, e_0, x_1, e_1, \cdots, e_{n-1}, x_n$ is a path and e_n is the edge between x_n and x_0. Two paths are disjoint if they have no edges in common. The set of all edges between nearest neighbor sites of $(\mathbb{Z}^*)^2$ is denoted by $(\mathbb{E}^*)^2$ and we define analogously paths, circuits in $(\mathbb{E}^*)^2$. The nearest neighbor Bernoulli bond percolation model on the square lattice at density p is defined by independently choosing each edge of $\mathbb{E}^2$ to be open with probability p or closed with probability $1-p$. We denote by P the product probability measure on the configuration space $\Omega = \{0,1\}^{\mathbb{E}^2}$. Two subsets of sites S_1, S_2 of $\mathbb{Z}^2$ are connected in the configuration ω if there is a path of open edges in ω connecting a site of S_1 to a site of S_2. Let ω be a configuration. The open clusters in ω are the connected components of the graph having vertex set $\mathbb{Z}^2$ and the open edges of ω only. We will often consider an open cluster as a set of sites by looking only at its set of vertices. We define $C = C(\omega)$ to be the open cluster containing the origin in ω (which is reduced to $\{0\}$ if 0 is not connected to any other site by open edges). In dimension two, the model is self dual. A given edge e of $\mathbb{E}^2$ is closed if and only if the unique dual edge e^* of $(\mathbb{E}^*)^2$ which intersects e is open and vice–versa. Each finite open cluster of ω is surrounded by an innermost circuit of open dual edges in ω (see for instance [11, Proposition 9.2]). It is known that the model has a phase transition at $p_c = 1/2$: for $p < p_c$ the open clusters are finite and for $p > p_c$ there exists a unique infinite open cluster [11,13]. We work with a fixed value $p > p_c$.

We finally recall briefly two fundamental correlation inequalities. To a configuration ω, we associate the set $K(\omega) = \{\, e \in \mathbb{E}^2 : \omega(e) = 1 \,\}$. Let A, B be two events. The disjoint occurrence $A \circ B$ of A and B is the event

$$\left\{ \begin{matrix} \omega \text{ such that there exists a subset } H \text{ of } K(\omega) \text{ such that, if } \omega', \omega'' \text{ are the configu–} \\ \text{–rations determined by } K(\omega') = H, K(\omega'') = K(\omega) \setminus H, \text{ then } \omega' \in A \text{ and } \omega'' \in B \end{matrix} \right\}.$$

There is a natural order on Ω defined by the relation: $\omega_1 \le \omega_2$ if and only if all open edges in ω_1 are open in ω_2. An event is said to be increasing (respectively decreasing) if its characteristic function is non decreasing (respectively non increasing) with respect to this partial order. Suppose the two events A, B are both increasing (or both decreasing).

The Harris–FKG inequality [8,12] says that $P(A\cap B)\geq P(A)P(B)$.
The van den Berg–Kesten inequality [2] says that $P(A\circ B)\leq P(A)P(B)$.

3. Large deviations for the finite cluster shape

We denote by $\mathcal{B}$ the Borel σ–algebra of $\mathbb{R}^2$ and by $\mathcal{K}$ the collection of the compact subsets of $\mathbb{R}^2$. The Euclidean distance between two compact sets K_1,K_2 of $\mathbb{R}^2$ is

$$d(K_1,K_2)\,=\,\min\{\,|x_1-x_2|_2:x_1\in K_1,\,x_2\in K_2\,\}\,.$$

The r–neighborhood of a compact set K is the set

$$\mathcal{V}(K,r)\,=\,\{\,x\in\mathbb{R}^2:d(x,K)\leq r\,\}\,.$$

The diameter of a compact set K is $\operatorname{diam} K\,=\,\max\{\,|y-x|_2:x,y\in K\,\}$.
We endow $\mathcal{K}$ with the Hausdorff metric D_H:

$$\forall K_1,K_2\in\mathcal{K}\qquad D_H(K_1,K_2)\,=\,\max\{\max_{x_1\in K_1}d(x_1,K_2),\,\max_{x_2\in K_2}d(x_2,K_1)\}\,.$$

If K belongs to $\mathcal{K}$ and $\mathcal{U}$ is a subset of $\mathcal{K}$, we set also $D_H(K,\mathcal{U})=\inf\{\,D_H(K,U):U\in\mathcal{U}\,\}$. The metric space $(\mathcal{K},D_H)$ is complete. Let $\mathcal{K}_c$ be the subset of $\mathcal{K}$ consisting of connected sets i.e.

$$\mathcal{K}_c=\{\,K\subset\mathbb{R}^2:K\text{ compact and connected}\,\}\,.$$

We claim that $\mathcal{K}_c$ is a closed subspace of $(\mathcal{K},D_H)$. Indeed, let $(K_n)_{n\in\mathbb{N}}$ be a sequence of connected compact sets converging to K. Suppose K is not connected, so that there exist two open disjoint sets U,V such that $K\subset U\cup V$ and $K\cap U\neq\emptyset$, $K\cap V\neq\emptyset$. For n sufficiently large, we will also have $K_n\subset U\cup V$, $K_n\cap U\neq\emptyset$, $K_n\cap V\neq\emptyset$, which is absurd since K_n is connected.

We consider also the metric D_λ on $\mathcal{B}$ associated to the planar Lebesgue measure:

$$\forall B_1,B_2\in\mathcal{B}\qquad D_\lambda(B_1,B_2)\,=\,\int|\chi_{B_1}-\chi_{B_2}|\,d\lambda\,=\,\lambda(B_1\Delta B_2)\,,$$

where λ is the Lebesgue measure on $\mathbb{R}^2$ and Δ is the symmetric difference operator. In fact D_λ is not a metric on $\mathcal{B}$, but rather on the set of equivalence classes modulo Lebesgue negligible sets.

We define an equivalence relation on $\mathcal{B}$ by: B_1 is equivalent to B_2 if and only if B_1 is a translate of B_2 (i.e. there exists x in $\mathbb{R}^2$ such that $B_1=x+B_2$). The class of an element B of $\mathcal{B}$ is denoted by $\overline{B}$ and we denote by $\overline{\mathcal{B}}$ the quotient set of the equivalence classes associated to this relation. If $\mathcal{U}$ is a subset of $\mathcal{B}$ we define $\overline{\mathcal{U}}=\{\,\overline{U}:U\in\mathcal{U}\,\}$. The set $\overline{\mathcal{K}}$

(respectively $\overline{\mathcal{B}}$) is endowed with the quotient metric $\overline{D}_H$ (respectively $\overline{D}_\lambda$) associated to D_H (respectively D_λ): for any $\overline{K}_1, \overline{K}_2$ in $\overline{\mathcal{K}}$, we define

$$\overline{D}_H(\overline{K}_1, \overline{K}_2) = \inf_{x_1, x_2 \in \mathbb{R}^2} D_H(K_1 + x_1, K_2 + x_2) = D_H(K_1, \overline{K}_2)$$

and we proceed analogously to define $\overline{D}_\lambda$ from D_λ.

In the supercritical regime where $1 > p > p_c = 1/2$, the origin belongs to a finite open cluster with probability strictly between 0 and 1. We denote by $\widehat{P}$ the measure P conditioned on this event i.e. $\widehat{P}(\cdot) = P(\cdot / |C| < \infty)$. Under $\widehat{P}$, the open cluster C containing the origin is a random compact set. We next state our large deviations principles, using the Freidlin–Wentzell presentation [9]. The rate functions σg_H and σg_λ are built in the next section.

Theorem 3.1. *Under $\widehat{P}$, the family of the laws of $(\overline{C}/N)_{N \in \mathbb{N}}$ on the space $\overline{\mathcal{K}}$ equipped with the Hausdorff metric $\overline{D}_H$ satisfies a large deviations principle with good rate function σg_H and speed N:*
(i) upper bound: $\forall u \geq 0 \quad \forall \delta > 0 \quad \forall \alpha > 0 \quad \exists N_0 \quad \forall N \geq N_0$

$$\widehat{P}\Big(\overline{D}_H(\overline{C}/N, \Phi_H(u)) \geq \delta\Big) \;\leq\; \exp -Nu(1-\alpha)\,,$$

where $\Phi_H(u) \;=\; \{\,\overline{K} \in \overline{\mathcal{K}}_c : \sigma g_H(\overline{K}) \leq u\,\}\,.$
(ii) lower bound: $\forall \overline{K} \in \overline{\mathcal{K}}_c \quad \forall \delta > 0 \quad \forall \alpha > 0 \quad \exists N_0 \quad \forall N \geq N_0$

$$\widehat{P}\Big(\overline{D}_H(\overline{C}/N, \overline{K}) \leq \delta\Big) \;\geq\; \exp -N\sigma g_H(\overline{K})(1+\alpha)\,.$$

Remark. The rate function σg_H is infinite on the set $\overline{\mathcal{K}} \setminus \overline{\mathcal{K}}_c$; hence the non connected compact sets do not intervene in the statement of the large deviations principle.

We recall that the r–neighborhood of K is $\mathcal{V}(K, r) = \{\, x \in \mathbb{R}^2 : d(x, K) \leq r\,\}$.

Theorem 3.2. *Let $f(N)$ be a function from $\mathbb{N}$ to $\mathbb{R}$ such that*

$$\lim_{N \to \infty} f(N)/\ln N \;=\; +\infty\,, \qquad \lim_{N \to \infty} f(N)/N \;=\; 0\,.$$

Under $\widehat{P}$, the family of the laws of $(\overline{\mathcal{V}}(C, f(N))/N)_{N \in \mathbb{N}}$ on the space $\overline{\mathcal{B}}$ equipped with the metric $\overline{D}_\lambda$ satisfies a large deviations principle with good rate function σg_λ and speed N:
(i) upper bound: $\forall u \geq 0 \quad \forall \delta > 0 \quad \forall \alpha > 0 \quad \exists N_0 \quad \forall N \geq N_0$

$$\widehat{P}\Big(\overline{D}_\lambda(\overline{\mathcal{V}}(C, f(N))/N, \Phi_\lambda(u)) \geq \delta\Big) \;\leq\; \exp -Nu(1-\alpha)\,,$$

where $\Phi_\lambda(u) \,=\, \{\, \overline{B} \in \overline{\mathcal{B}} : \sigma g_\lambda(\overline{B}) \leq u \,\}$.
(ii) lower bound: $\forall \overline{B} \in \overline{\mathcal{B}} \quad \forall \delta > 0 \quad \forall \alpha > 0 \quad \exists N_0 \quad \forall N \geq N_0$

$$\widehat{P}\Big(\overline{D}_\lambda(\overline{\mathcal{V}}(C, f(N))/N, \overline{B}) \leq \delta\Big) \;\geq\; \exp - N\sigma g_\lambda(\overline{B})(1+\alpha)\,.$$

Remark. The sets $\Phi_H(u)$ and $\Phi_\lambda(u)$ are translation invariant so that for any K in $\mathcal{K}$, we have for instance $\overline{D}_H(\overline{K}, \Phi_H(u)) = \inf\{\, D_H(K, K') : \overline{K'} \in \Phi_H(u)\,\} = D_H(K, \Phi_H(u))$ (notice that $\Phi_H(u)$ should be understood as a set of elements of $\mathcal{K}$ in the last expression).

Remark. The Freidlin–Wentzell formulation of the large deviations principle is equivalent to the more classical one. For instance the result of theorem 3.1 can be rewritten as follows. For any Borel subset $\overline{\mathcal{U}}$ of $\overline{\mathcal{K}}_c$,

$$-\inf\{\, \sigma g_H(\overline{U}) : \overline{U} \in \text{ interior}(\overline{\mathcal{U}})\,\} \;\leq\; \liminf_{N\to\infty} \frac{1}{N} \ln P(\overline{C}/N \in \overline{\mathcal{U}})$$
$$\leq\; \limsup_{N\to\infty} \frac{1}{N} \ln P(\overline{C}/N \in \overline{\mathcal{U}}) \;\leq\; -\inf\{\, \sigma g_H(\overline{U}) : \overline{U} \in \text{ closure}(\overline{\mathcal{U}})\,\}$$

where interior($\overline{\mathcal{U}}$) and closure($\overline{\mathcal{U}}$) are the interior and the closure of $\overline{\mathcal{U}}$ with respect to the Hausdorff metric on $\overline{\mathcal{K}}$ associated to $\overline{D}_H$.

We next show how Theorem 3.2 is related to the two dimensional Wulff construction for the percolation model. We define the Wulff constant $\omega(p)$ at density p by

$$\omega(p) \;=\; \inf\{\, g_p(\partial K) : K \in \mathcal{K}_c^J,\, \lambda(K) \geq 1\,\}\,.$$

where ∂K is the topological boundary of K and $\mathcal{K}_c^J$ is the class of the connected compact sets K such that $\mathbb{R}^2 \setminus K$ has a finite number of bounded components, the boundaries of which are disjoint Jordan curves. The variational problem defining the Wulff constant is a problem of the isoperimetric type. Up to a translation, there exists a unique set in $\mathcal{K}_c^J$ realizing the above minimum, described by the Wulff construction

$$W \;=\; \{\, x \in \mathbb{R}^2 : x \cdot y \leq g(y) \text{ for all } y \text{ in } \mathbb{R}^2 \,\}$$

(where g is the direction–dependent surface tension of the model and $x \cdot y$ is the usual scalar product between x, y in $\mathbb{R}^2$). See [1] and the references therein for more details.

Corollary 3.3. *Let $f(N)$ be a function from $\mathbb{N}$ to $\mathbb{R}$ such that*

$$\lim_{N\to\infty} f(N)/\ln N \;=\; +\infty\,, \qquad \lim_{N\to\infty} f(N)/N \;=\; 0\,.$$

We have

$$\lim_{N\to\infty}\frac{1}{N}\ln\widehat{P}\Big(\lambda(\mathcal{V}(C,f(N)))\geq N^2\Big) \;=\; -\omega(p)\sigma(p)\,.$$

Proof. Clearly, we have the equality $\big\{\,\lambda(\mathcal{V}(C,f(N)))\geq N^2\,\big\}=\big\{\,\lambda(\mathcal{V}(C,f(N))/N)\geq 1\,\big\}$. Let $\mathcal{A}=\{\,B\in\mathcal{B}:\lambda(B)\geq 1\,\}$. Since $(\mathcal{V}(C,f(N))/N)_{N\in\mathbb{N}}$ satisfies a large deviations principle, we have also

$$-\inf\{\,\sigma g_\lambda(B):B\in\mathrm{int}_\lambda\mathcal{A}\,\}\;\leq\;\liminf_{N\to\infty}\frac{1}{N}\ln\widehat{P}\Big(\,\lambda(\mathcal{V}(C,f(N))/N)\geq 1\,\Big)\,,$$

$$\limsup_{N\to\infty}\frac{1}{N}\ln\widehat{P}\Big(\,\lambda(\mathcal{V}(C,f(N))/N)\geq 1\,\Big)\;\leq\;-\inf\{\,\sigma g_\lambda(B):B\in\mathrm{adh}_\lambda\mathcal{A}\,\}$$

where $\mathrm{int}_\lambda\mathcal{A}$ is the interior of $\mathcal{A}$ with respect to the D_λ metric, $\mathrm{adh}_\lambda\mathcal{A}$ is its closure. We have $\mathrm{adh}_\lambda\mathcal{A}=\mathcal{A}$ and $\mathrm{int}_\lambda\mathcal{A}=\{\,B\in\mathcal{B}:\lambda(B)>1\,\}$. Clearly, for any $\epsilon>0$,

$$\inf\{\,g_\lambda(B):B\in\mathcal{B},\lambda(B)>1\,\}\;\leq\;\inf\{\,g_\lambda((1+\epsilon)K):K\in\mathcal{K}^J_c,\,\lambda(K)\geq 1\,\}\;=\;(1+\epsilon)\omega(p)\,.$$

The set $\{\,K\in\mathcal{K}^J_c:g_\lambda(K)<\infty,\,\lambda(K)>1\,\}$ is dense into the set $\{\,B\in\mathcal{B}:g_\lambda(B)<+\infty,\,\lambda(B)>1\,\}$ for the D_λ metric, and therefore

$$\begin{aligned}\inf\{\,g_\lambda(B):B\in\mathcal{B},\lambda(B)\geq 1\,\}\;&\geq\;\inf\{\,g_\lambda((1-\epsilon)B):B\in\mathcal{B},\,g_\lambda(B)<+\infty,\,\lambda(B)>1\,\}\\&=\;\inf\{\,g_\lambda((1-\epsilon)K):K\in\mathcal{K}^J_c,\,\lambda(K)>1\,\}\;\geq\;(1-\epsilon)\omega(p)\,.\end{aligned}$$

Letting ϵ go to zero, we get

$$\inf\{\,g_\lambda(B):B\in\mathrm{int}_\lambda\mathcal{A}\,\}\;=\;\inf\{\,g_\lambda(B):B\in\mathrm{adh}_\lambda\mathcal{A}\,\}\;=\;\omega(p)$$

which implies the claim of the corollary. □

4. Construction of the rate functions

The aim of this section is to build the rate functions σg_H and σg_λ and to study some of their properties. An important issue is to obtain good rate functions which are lower semicontinuous and have compact level sets [6]. For x^*,y^* two dual sites we denote by $x^*\leftrightarrow y^*$ the event that x^* is connected to y^* by a path of open dual edges. We sum up in the next proposition several properties of the surface tension (see [1] for the proofs).

Proposition 4.1. *The limit*

$$\sigma(p)\;=\;-\lim_{n\to\infty}\frac{1}{n}\ln P(0^*\leftrightarrow(n,0)^*)$$

exists and is positive for $p > p_c$. Furthermore, there exists a convex continuous function g_p from $\mathbb{R}^2$ to $\mathbb{R}$ satisfying the following property: for any x in $\mathbb{Q}^2$, for any k in $\mathbb{Z}$ such that kx is in $\mathbb{Z}^2$,

$$\sigma(p)g_p(x) \,=\, -\lim_{n\to\infty}\frac{1}{nk}\ln P(0^*\leftrightarrow(nkx)^*)\,.$$

In addition, for any x in $\mathbb{Z}^2$, we have

$$P(0^*\leftrightarrow x^*)\,\leq\,\exp -\sigma(p)g_p(x)\,.$$

The function g (we drop the subscript p in the sequel) is homogeneous i.e.

$$\forall t\in\mathbb{R}\quad\forall x\in\mathbb{R}^2\qquad g(tx)=|t|g(x)\,,$$

it is symmetric with respect to the axis and the diagonals i.e.

$$\forall x_1,x_2\in\mathbb{Z}^2\qquad g(x_1,x_2)=g(-x_1,x_2)=g(x_1,-x_2)=g(x_2,x_1)\,,$$

and it defines a norm equivalent to the Euclidean norm:

$$\forall x\in\mathbb{R}^2\qquad \frac{1}{\sqrt{2}}|x|_2\,\leq|x|_\infty\,\leq\,g(x)\,\leq\,|x|_1\leq\,\sqrt{2}|x|_2\,.$$

In the subsequent definitions we will use the function g introduced in proposition 4.1. We recall that a planar Jordan curve γ is a closed simple curve of the plane $\mathbb{R}^2$. By γ we mean either the image of the curve (that is the set of the points of $\mathbb{R}^2$ belonging to the curve) or a parametrization of the curve, that is a continuous map γ from $[0,1]$ to $\mathbb{R}^2$ such that $\gamma(0)=\gamma(1)$. The Jordan curve theorem says that the complement of a Jordan curve γ consists of two components, each of which has γ as its boundary, one which is unbounded (the exterior of γ, denoted by $\operatorname{ext}\gamma$) and one which is bounded (the interior of γ, denoted by $\operatorname{int}\gamma$).
The topological boundary of a compact set K is denoted by ∂K. We say that an element K of $\mathcal{K}_c$ is regular if $\mathbb{R}^2\setminus K$ has a finite number of bounded components and the boundaries of the components of $\mathbb{R}^2\setminus K$ are disjoint Jordan curves. The boundary of the unbounded component of $\mathbb{R}^2\setminus K$ is called the external boundary of K, the boundaries of the bounded components of $\mathbb{R}^2\setminus K$ are called the inner boundaries of K. We denote by $\mathcal{K}_c^J$ the class of the regular connected compact sets. For K in $\mathcal{K}_c^J$ with external boundary γ_0 and inner boundaries $\gamma_1,\cdots,\gamma_r$ (so that $\partial K=\gamma_0\cup\cdots\cup\gamma_r$), we define its g–perimeter by

$$g(\partial K)\,=\,\sum_{i=0}^r g(\gamma_i)\,,$$

where for γ a Jordan curve, we set

$$g(\gamma) \;=\; \sup_{0<t_1<\cdots<t_l<1} \sum_j g(\gamma(t_{j+1})-\gamma(t_j))$$

(the supremum being taken over all finite subdivisions of $[0,1]$). A Jordan curve γ is said to be rectifiable if $g(\gamma)$ is finite. Remark that the g–perimeter of a regular compact set K depends only on its boundary ∂K. We recall next a useful tool introduced in [1].

Definition 4.2. *Let A be a subset of $\mathbb{R}^2$. Its one dimensional g–Hausdorff measure $\mu_g(A)$ is*

$$\mu_g(A)=\lim_{\epsilon\to 0}\Big(\inf\ \Big\{2\sum_{i\in I}\epsilon_i\Big\}\Big)\,,$$

the infimum being taken over all countable families $(\epsilon_i)_{i\in I}$ such that $0<\epsilon_i\leq\epsilon$ for all i in I and there exists a family $(x_i)_{i\in I}$ of points of $\mathbb{R}^2$ with $A\subset\bigcup_{i\in I}\{\,x: g(x-x_i)<\epsilon_i\,\}$.

Lemma 4.3. *Let γ be a rectifiable curve. We have $\mu_g(\gamma)\leq g(\gamma)$. If γ is self avoiding then $\mu_g(\gamma)=g(\gamma)$.*

This lemma is proved in [1, Lemma A.2]. We next state three lemmas which are the keys in order to prove the lower semicontinuity of g with respect to the Hausdorff metric.

Lemma 4.4. *Let A be a subset of $\mathbb{R}^2$ and let $(\gamma_n)_{n\in\mathbb{N}}$ be a sequence of Jordan curves such that for any x belonging to A, $\lim_{n\to\infty} d(x,\gamma_n)\;=\;0$. Then $\liminf_{n\to\infty} g(\gamma_n)\;\geq\;\mu_g(A)$.*

Proof. We need only to consider the case where $\liminf_{n\to\infty} g(\gamma_n)$ is finite and A is bounded. Passing to a subsequence, we may assume that the sequence $(g(\gamma_n))_{n\in\mathbb{N}}$ converges. Now the lengths of all the curves γ_n are bounded; we can choose a parametrization of the curves on the interval $[0,1]$ which is proportional to the arc length. With these parametrizations, the functions $t\in[0,1]\mapsto\gamma_n(t)$ are Lipschitz and the Lipschitz constants are uniformly bounded so that the functions (γ_n) form an equicontinuous family. Moreover the functions (γ_n) are uniformly bounded. In fact,

$$\sup_{n,t}|\gamma_n(t)|_2\;\leq\;\sup_{x\in A}|x|_2\;+\;\sup_n d(A,\gamma_n)\;+\;\sqrt{2}\sup_n g(\gamma_n)\,.$$

By the Ascoli theorem, the sequence $(\gamma_n)_{n\in\mathbb{N}}$ admits a subsequence which converges uniformly to a curve γ' (notice that γ' is not necessarily self avoiding). Fix a subdivision $t_0=0<t_1<\cdots<t_l<1=t_{l+1}$ of $[0,1]$. We have

$$\lim_{n\to\infty} g(\gamma_n)\;\geq\;\lim_{n\to\infty}\sum_{i=0}^{l} g(\gamma_n(t_{i+1})-\gamma_n(t_i))\;\geq\;\sum_{i=0}^{l} g(\gamma'(t_{i+1})-\gamma'(t_i))$$

whence, by taking the supremum of the righthand side with respect to all possible subdivisions, we get $\lim_{n\to\infty} g(\gamma_n) \geq g(\gamma')$. Moreover the uniform convergence of γ_n to γ' implies that $D_H(\gamma_n, \gamma')$ converges to 0. Let x belong to A. By hypothesis $\lim_{n\to\infty} d(x, \gamma_n) = 0$ whence $d(x, \gamma') = 0$ and x belongs to γ'. Therefore $A \subset \gamma'$, and applying lemma 4.3 we get $g(\gamma') \geq \mu_g(\gamma') \geq \mu_g(A)$. $\square$

Lemma 4.5. *Let O be a connected open set with compact closure. There exists a sequence $(O_n)_{n\in\mathbb{N}}$ of increasing connected open subsets of O such that:*

$$\forall n \in \mathbb{N} \quad \forall x \in O_n \quad d(x, \partial O) > 1/n \qquad \textit{and} \qquad \bigcup_{n\in\mathbb{N}} O_n = O\,.$$

Proof. Let n belong to $\mathbb{N}$. We define a relation $\mathcal{R}_n$ on the points of O by: $x\mathcal{R}_n y$ if and only if there exists a continuous path $\gamma : [0,1] \to O$ such that $\gamma(0) = x$, $\gamma(1) = y$ and $d(\gamma(t), \partial O) > 1/n$ for all t in $[0,1]$. For any pair x, y of points of O there exists n_0 such that $x\mathcal{R}_n y$ for all n larger than n_0. In fact, O is an open connected subset of $\mathbb{R}^2$ and is thus arcwise connected. Thus there exists a continuous path $\gamma : [0,1] \to O$ such that $\gamma(0) = x$, $\gamma(1) = y$. Since $\gamma([0,1])$ does not intersect ∂O and is compact, the distance $d(\gamma([0,1]), \partial O)$ is positive. It follows that $x\mathcal{R}_n y$ as soon as $d(\gamma([0,1]), \partial O) > 1/n$. Let us fix a point x_0 in O and let $C(x_0, n)$ be its equivalence class for the relation $\mathcal{R}_n$. Then $(C(x_0, n))_{n\in\mathbb{N}}$ is an increasing sequence of open connected sets satisfying the requirements of the lemma. $\square$

Lemma 4.6. *Let K belong to $\mathcal{K}_c^J$ and let $(K_n)_{n\in\mathbb{N}}$ be a sequence in $\mathcal{K}_c^J$ converging to K for the Hausdorff metric D_H. Let γ_0 (respectively γ_0^n) be the external boundary of K (resp. K_n) and let $\gamma_1, \cdots, \gamma_r$ (resp. $\gamma_1^n, \cdots, \gamma_{\phi(n)}^n$) be the inner boundaries of K (resp. K_n). Then for each i in $\{0 \cdots r\}$, there exists a sequence of integers $(\rho_i(n))_{n\in\mathbb{N}}$ such that $0 \leq \rho_i(n) \leq \phi(n)$ for all n and*

$$\forall x \in \gamma_i \qquad \lim_{n\to\infty} d(x, \gamma_{\rho_i(n)}^n) \;=\; 0\,.$$

Proof. We denote by $B_0, \cdots, B_r$ the components of $\mathbb{R}^2 \setminus K$, so that $\gamma_i = \partial B_i$ for each i in $\{0 \cdots r\}$. Let ϵ be a positive real number. By lemma 4.5, there exists a positive δ (smaller than ϵ) and connected open sets B_i^δ for each i in $\{0 \cdots r\}$ such that $B_i^\delta \subset B_i$ and

$$\forall x \in \partial B_i \quad d(x, B_i^\delta) \;\leq\; \epsilon\,, \qquad \forall x \in B_i^\delta \quad d(x, \partial B_i) > \delta\,.$$

Let n_0 be such that $D_H(K, K_n) < \delta$ for $n \geq n_0$. For $n \geq n_0$, K_n is included in $\mathbb{R}^2 \setminus B_i^\delta$ so that none of the curves γ_j^n, $0 \leq j \leq \phi(n)$, intersect B_i^δ. Therefore B_i^δ is included in a component of $\mathbb{R}^2 \setminus K_n$. Either $B_i^\delta \subset \operatorname{ext}\gamma_0^n$ and we set $m = 0$ or there exists j in

$\{1\cdots\phi(n)\}$ such that $B_i^\delta \subset \operatorname{int}\gamma_j^n$ and we set $m=j$. Let x belong to $\partial B_i=\gamma_i$. There exists y in B_i^δ such that $d(x,y)\leq\epsilon$ and z in K_n such that $d(x,z)<\epsilon$. In particular, y does not belong to K_n so that the segment $[y,z]$ intersects γ_m^n and $d(x,\gamma_m^n)<\epsilon$. Thus we have proved that

$$\forall i\in\{0\cdots r\}\qquad \lim_{n\to\infty}\quad \inf_{1\leq m\leq\phi(n)}\ \sup_{x\in\gamma_i}\ d(x,\gamma_m^n)\;=\;0$$

which is the desired result. $\square$

Proposition 4.7. *The g–perimeter is lower semicontinuous on the space $(\mathcal{K}_c^J,D_H)$ i.e. for any K in $\mathcal{K}_c^J$, any sequence $(K_n)_{n\in\mathbb{N}}$ in $\mathcal{K}_c^J$ converging to K for the Hausdorff metric D_H, we have $\liminf_{n\to\infty} g(\partial K_n)\;\geq\; g(\partial K)$.*

Remark. It is essential to work only with connected sets to have the lower semicontinuity of g with respect to the Hausdorff metric.

Proof. Let K belong to $\mathcal{K}_c^J$ and let $(K_n)_{n\in\mathbb{N}}$ be a sequence in $\mathcal{K}_c^J$ converging to K. Let γ_0 (respectively γ_0^n) be the external boundary of K (resp. K_n) and let $\gamma_1,\cdots,\gamma_r$ (resp. $\gamma_1^n,\cdots,\gamma_{\phi(n)}^n$) be the inner boundaries of K (resp. K_n). We apply lemma 4.6:

$$\forall i\in\{0\cdots r\}\quad \exists(\rho_i(n))_{n\in\mathbb{N}}\quad \forall x\in\gamma_i\qquad \lim_{n\to\infty} d(x,\gamma_{\rho_i(n)}^n)\;=\;0\,.$$

Passing again to a subsequence, we may impose that for all i,j in $\{0\cdots r\}$

$$\text{either}\quad [\,\forall n\in\mathbb{N}\quad \rho_i(n)=\rho_j(n)\,]\quad\text{or}\quad [\,\forall n\in\mathbb{N}\quad \rho_i(n)\neq\rho_j(n)\,]\,.$$

We define an equivalence relation $\sim$ on $\{0\cdots r\}$ by: $i\sim j$ if $\rho_i(n)=\rho_j(n)$ for all $n\in\mathbb{N}$. Let π be an equivalence class for this relation. Denoting by $\rho_\pi(n)$ the common value $\rho_i(n)$ for i in π, we have

$$\forall i\in\pi\quad \forall x\in\gamma_i\quad \lim_{n\to\infty} d(x,\gamma_{\rho_\pi(n)}^n)\;=\;0$$

and lemma 4.4 implies that $\liminf_{n\to\infty} g(\gamma_{\rho_\pi(n)}^n)\;\geq\; g(\bigcup_{i\in\pi}\gamma_i)\;=\;\sum_{i\in\pi} g(\gamma_i)$ (recall that the curves γ_i are disjoint since K is regular). Let $\mathcal{P}$ be the set of the equivalence classes of the relation $\sim$. We have $g(\partial K_n)\;\geq\;\sum_{\pi\in\mathcal{P}} g(\gamma_{\rho_\pi(n)}^n)$ whence

$$\liminf_{n\to\infty} g(\partial K_n)\;\geq\;\sum_{\pi\in\mathcal{P}}\liminf_{n\to\infty} g(\gamma_{\rho_\pi(n)}^n)\;\geq\;\sum_{\pi\in\mathcal{P}}\sum_{i\in\pi} g(\gamma_i)\;=\;\sum_{i=0}^r g(\gamma_i)\;=\;g(\partial K)\,.\quad\square$$

Proposition 4.8. *The g–perimeter is lower semicontinuous on the space $(\mathcal{K}_c^J, D_\lambda)$ i.e. for any K in $\mathcal{K}_c^J$, any sequence $(K_n)_{n\in\mathbb{N}}$ in $\mathcal{K}_c^J$ converging to K for the metric D_λ, we have* $\liminf_{n\to\infty} g(\partial K_n) \geq g(\partial K)$.

Proof. The lower semicontinuity of g in this case is a consequence of the functional definition of the perimeter. Let $B = \{ x \in \mathbb{R}^2 : g(x) \leq 1 \}$ be the unit ball of the norm induced by g. The usual scalar product between x, y in $\mathbb{R}^2$ is denoted by $x \cdot y$. The Wulff shape associated to the convex function g is the polar set of B i.e.

$$W = \{ x \in \mathbb{R}^2 : x \cdot y \leq g(y) \text{ for all } y \text{ in } \mathbb{R}^2 \} = \{ x \in \mathbb{R}^2 : x \cdot y \leq 1 \text{ for all } y \text{ in } B \}.$$

By duality, the polar set of W is in turn B [15]. As a consequence, for any x in $\mathbb{R}^2$, we have $g(x) = \sup_{y\in W} x \cdot y$. Let γ be a Jordan curve and let K be the closure of the interior of γ. We have then (see for instance [10, Definition 1.1])

$$g(\gamma) = \int g(\nabla \chi_K)\, d\lambda = \sup \Big\{ \int \chi_K \operatorname{div}\phi \, d\lambda : \phi \text{ of class } C^1 \text{ from } \mathbb{R}^2 \text{ to } W \Big\}.$$

This formula implies the lower semicontinuity of g with respect to the metric D_λ [10, Theorem 1.9]. One could also propose an argument similar to [14, 6.1.3, Lemma 1]. □

We extend the g–perimeter to $\mathcal{K}$ on one hand and to $\mathcal{B}$ on the other hand by setting,

$$\forall K \in \mathcal{K} \quad g_H(K) = \inf \Big\{ \liminf_{n\to\infty} g(\partial K_n) : (K_n)_{n\in\mathbb{N}} \in (\mathcal{K}_c^J)^{\mathbb{N}},\ \lim_{n\to\infty} D_H(K, K_n) = 0 \Big\},$$
$$\forall B \in \mathcal{B} \quad g_\lambda(B) = \inf \Big\{ \liminf_{n\to\infty} g(\partial K_n) : (K_n)_{n\in\mathbb{N}} \in (\mathcal{K}_c^J)^{\mathbb{N}},\ \lim_{n\to\infty} D_\lambda(B, K_n) = 0 \Big\}.$$

By propositions 4.7, 4.8, the g–perimeter was lower semicontinuous on the set $\mathcal{K}_c^J$, hence the previous definition makes sense i.e. g_H and g_λ coincide and are equal to g on $\mathcal{K}_c^J$. However these two maps do not agree in general on $\mathcal{K}$. For instance, if we consider a segment $[x, y]$, we have $g_H([x, y]) = 2g(y - x)$ whereas $g_\lambda([x, y]) = 0$. Notice that a segment $[x, y]$ does not belong to $\mathcal{K}_c^J$. Also if K is not connected then $g_H(K)$ is infinite (in this case K is not the limit in the Hausdorff metric of a sequence of connected sets) but $g_\lambda(K)$ might be finite. We next prove that g_H and g_λ are subadditive.

Proposition 4.9. *For any K_1, K_2 in $\mathcal{K}_c$ such that $K_1 \cap K_2 \neq \emptyset$, we have $g_H(K_1 \cup K_2) \leq g_H(K_1) + g_H(K_2)$. For any B_1, B_2 in $\mathcal{B}$, we have $g_\lambda(B_1 \cup B_2) \leq g_\lambda(B_1) + g_\lambda(B_2)$.*

Remark. The condition $K_1 \cap K_2 \neq \emptyset$ ensures that $K_1 \cup K_2$ is connected.

Proof. Suppose first that K_1, K_2 belong to $\mathcal{K}_c^J$. Since $\partial(K_1 \cup K_2)$ is included in $\partial K_1 \cup \partial K_2$, we have $g(\partial(K_1 \cup K_2)) \leq g(\partial K_1) + g(\partial K_2)$ (we recall that g is a Hausdorff measure, see lemma 4.3). We consider now the case of arbitrary elements K_1, K_2 of $\mathcal{K}_c$ such that $K_1 \cap K_2 \neq \emptyset$. Let $(K_n^1)_{n\in\mathbb{N}}$, $(K_n^2)_{n\in\mathbb{N}}$ be sequences in $\mathcal{K}_c^J$ such that

$$\lim_{n\to\infty} D_H(K_n^1, K_1) \,=\, 0\,, \quad \lim_{n\to\infty} g(\partial K_n^1) \,=\, g_H(K_1)\,,$$
$$\lim_{n\to\infty} D_H(K_n^2, K_2) \,=\, 0\,, \quad \lim_{n\to\infty} g(\partial K_n^2) \,=\, g_H(K_2)\,.$$

Let x belong to $K_1 \cap K_2$ and let ϵ be a positive real number. For n large enough, K_n^1 and K_n^2 intersect the closed ball $B(x,\epsilon)$ of center x and radius ϵ. Thus the set $K_n^1 \cup K_n^2 \cup B(x,\epsilon)$ is connected and belongs to $\mathcal{K}_c^J$ whence

$$g_H(K_n^1 \cup K_n^2 \cup B(x,\epsilon)) \,\leq\, g_H(K_n^1) + g_H(K_n^2) + g_H(B(x,\epsilon))\,.$$

Letting n go to infinity, we obtain, using the lower semicontinuity of g_H,

$$g_H(K_1 \cup K_2 \cup B(x,\epsilon)) \,\leq\, g_H(K_1) + g_H(K_2) + g_H(B(x,\epsilon))\,.$$

As ϵ goes to zero, $K_1 \cup K_2 \cup B(x,\epsilon)$ converges to $K_1 \cup K_2$ for the Hausdorff metric D_H and $g_H(B(x,\epsilon))$ goes to zero. Using again the lower semicontinuity of g_H, we get $g_H(K_1 \cup K_2) \leq g_H(K_1) + g_H(K_2)$. The proof is similar for the function g_λ. □

We finally define g_H and g_λ on the quotient sets $\overline{\mathcal{K}}$ and $\overline{\mathcal{B}}$:

$$\forall \overline{K} \in \overline{\mathcal{K}} \quad g_H(\overline{K}) = g_H(K)\,, \quad \forall \overline{B} \in \overline{\mathcal{B}} \quad g_\lambda(\overline{B}) = g_\lambda(B)\,,$$

which makes sense since g_H and g_λ are translation invariant on $\mathcal{K}$ and $\mathcal{B}$.

Proposition 4.10. *The maps $g_H : K \in (\mathcal{K}, D_H) \mapsto g_H(K)$ and $g_\lambda : B \in (\mathcal{B}, D_\lambda) \mapsto g_\lambda(B)$ are lower semicontinuous. For t positive the level set $\{\, \overline{K} \in \overline{\mathcal{K}}_c : g_H(\overline{K}) \leq t \,\}$ is compact in $(\overline{\mathcal{K}}_c, \overline{D}_H)$, the level set $\{\, \overline{B} \in \overline{\mathcal{B}} : g_\lambda(\overline{B}) \leq t \,\}$ is compact in $(\overline{\mathcal{B}}, \overline{D}_\lambda)$.*

Proof. The lower semicontinuity of g_H and g_λ are mere consequences of their definition, together with propositions 4.7, 4.8. Let $(\overline{K}_n)_{n\in\mathbb{N}}$ be a sequence in $\overline{\mathcal{K}}_c$ such that $g_H(\overline{K}_n) \leq t$ for all n in $\mathbb{N}$. For each n we can assume that the origin belongs to K_n (since K_n is defined up to a translation). Since the diameter of an element of $\mathcal{K}_c$ is bounded by its g_H–perimeter (up to a multiplicative constant), there exists a bounded set B such that

$$K \in \mathcal{K}_c, \quad 0 \in K, \quad g_H(K) \leq t \quad \Longrightarrow \quad K \subset B\,.$$

Thus the sets K_n are subsets of B. For any compact set K_0, the subset $\{\, K \in \mathcal{K} : K \subset K_0 \,\}$ of $\mathcal{K}$ is itself compact with respect to the metric D_H [5, Theorem II-5]. Hence $(K_n)_{n\in\mathbb{N}}$ admits a subsequence converging for the metric D_H; the same subsequence of $(\overline{K}_n)_{n\in\mathbb{N}}$ converges for the metric $\overline{D}_H$.

Let $(\overline{B}_n)_{n\in\mathbb{N}}$ be a sequence in $\overline{\mathcal{B}}$ such that $g_\lambda(\overline{B}_n) \le t$ for all n in $\mathbb{N}$. If infinitely many of the sets B_n are Lebesgue negligible we can extract a subsequence converging to the empty set for $\overline{D}_\lambda$. If not, passing to a subsequence, we may suppose that none of them is negligible. Let U be a neighborhood of the origin. We can choose the sets B_n such that $\lambda(B_n \cap U)$ is positive for all n (since B_n is defined up to a translation and $\lambda(B_n) > 0$). Since the norm defined by g is equivalent to the Euclidean norm (see proposition 4.1), then the Euclidean perimeters of the sets B_n are uniformly bounded i.e. $\sup_n \int |\nabla \chi_{B_n}|\, d\lambda$ is finite. We claim that there exists a bounded set B such that $B_n \setminus B$ is negligible for all n. In fact, if E is an element of $\mathcal{B}$ and U, V are two subsets of E of positive Lebesgue measure, then $g_\lambda(E) \ge d(U,V)/\sqrt{2}$ (any compact in $\mathcal{K}_c^J$ approximating E has to visit U and V so that its diameter is larger than $d(U,V)$). By Theorem 6.1.4 of [14], we can extract a subsequence of $(B_n)_{n\in\mathbb{N}}$ which converges with respect to the metric D_λ. We can equivalently apply Theorem 1.19 of [10]: the sequence of functions $(\chi_{B_n})_{n\in\mathbb{N}}$ is a sequence of functions in $L^1(B)$ whose BV–norms are uniformly bounded, hence it admits a converging subsequence. □

5. Proofs of the large deviations principles

The proofs of the large deviations upper bounds rely on the coarse–graining procedure introduced in [1,7].

Geometrical constructions. Let M be a positive real number. We successively define the notion of M–skeleton of a dual circuit and M–skeleton of a connected set of sites.
Let γ be a dual Jordan curve (that is the image of γ is a dual circuit) of diameter larger than M. The M–skeleton of γ, denoted by $\gamma^{(M)}$, is defined as follows. We set $s_0 = \gamma(0)$, $t_0 = 0$, and for $n \ge 0$,

$$t_{n+1} = \inf\{\, t > t_n : \gamma(t) \in (\mathbb{Z}^*)^2,\ |\gamma(t) - \gamma(t_n)|_2 \ge M \,\}, \quad s_{n+1} = \gamma(t_{n+1})\,.$$

By l we denote the largest index n such that t_n is finite. The M–skeleton of γ is then the ordered sequence of dual sites $\gamma^{(M)} = [s_0, s_1, \cdots, s_l]$.
Let A be a connected set of sites of $\mathbb{Z}^2$ (i.e. two arbitrary points of A are connected by a path of edges in A). Let

$$\widetilde{A} = \bigcup_{x\in A} \{\, y : |x-y|_\infty \le 1/2 \,\}\,.$$

The set $\widetilde{A}$ is an element of $\mathcal{K}_c^J$. The external boundary γ_0 of $\widetilde{A}$ and the inner boundaries of $\widetilde{A}$ are dual circuits. The M–skeleton $\mathcal{S}(A)$ of A is empty if the diameter of $\widetilde{A}$ is less than M; otherwise it is the union of the M–skeleton of γ_0 and the M–skeletons of the inner boundaries of $\widetilde{A}$ whose diameters are larger than M, say $\gamma_1, \cdots, \gamma_r$. We denote the

M–skeletons of these circuits by $\gamma_i^{(M)} = [s_0^i, \cdots, s_{l_i}^i]$ for i in $\{0 \cdots r\}$, and we define

$$I(\mathcal{S}(A)) \,=\, \sum_{i=0}^{r} \left(\sum_{j=0}^{l_i} g(s_{j+1}^i - s_j^i) \right),$$

(with the convention that $s_{l_i+1}^i = s_0^i$) and also $\operatorname{card} \mathcal{S}(A) = l_0 + \cdots + l_r$+r.

Lemma 5.1. *There exist two positive constants a_0, a_1 such that for any M–skeleton $\mathcal{S}$ we have $a_0\, M \,card\mathcal{S} \,\leq\, I(\mathcal{S}) \,\leq\, a_1\, M\, card\mathcal{S}$.*

Proof. This lemma is a mere consequence of the fact that the norm defined by g is equivalent to the Euclidean norm $|\ \ |_2$ (see proposition 4.1). □

To A we associate an approximate shape $A^{(M)}$ following the method introduced in [7, section 2.10]. If the M–skeleton of A is empty then $A^{(M)} = \{a\}$ where a is any point of A. Otherwise, we draw all the segments $[s_j^i, s_{j+1}^i]$, $0 \leq i \leq r$, $0 \leq j \leq l_i$, and we denote by $L(\mathcal{S}(A))$ the union of these polygonal lines. The set $\mathbb{R}^2 \setminus L(\mathcal{S}(A))$ splits up into a collection of connected components with exactly one unbounded component. A component is called a minus component if any path that connects its interior points with points of the unbounded component and intersects the polygonal lines of the skeleton $L(\mathcal{S}(A))$ in a finite number of points, intersects them in an odd number of points. The set $A^{(M)}$ is the closure of the union of all the minus components. By construction, $A^{(M)}$ is connected.

Lemma 5.2. *For any connected set of sites A, any positive M, we have $g_H(A^{(M)}) \,\leq\, I(\mathcal{S}(A))$ and $g_\lambda(A^{(M)}) \,\leq\, I(\mathcal{S}(A))$.*

Remark. We believe that these inequalities are in fact equalities (but we won't use it).

Proof. Let $(A_k)_{1\leq k\leq m}$ be the minus components of $\mathbb{R}^2 \setminus L(\mathcal{S}(A))$. By proposition 4.9, $g_H(A^{(M)}) \,\leq\, \sum_{k=1}^m g_H(A_k)$. However, for each k, the set A_k is connected, and ∂A_k is an union of polygonal lines included in the union of the segments $[s_j^i, s_{j+1}^i]$, $0 \leq i \leq r$, $0 \leq j \leq l_i$. Moreover, the definition of minus components implies that, for $k \neq k'$, $\partial A_k \cap \partial A_{k'}$ is finite (if it were infinite, it would contain a non–trivial segment, and this segment would separate two distinct minus components, which is absurd). It follows that

$$\sum_{k=1}^{m} g_H(A_k) \,=\, \sum_{k=1}^{m} g(\partial A_k) \,\leq\, \sum_{i=0}^{r} \left(\sum_{j=0}^{l_i} g(s_{j+1}^i - s_j^i) \right) \,=\, I(\mathcal{S}(A))\,.$$

The proof for $g_\lambda(A^{(M)})$ is similar. □

The next two lemmas estimate the distance between A and $A^{(M)}$.

Lemma 5.3. *For any connected set of sites A, any positive M, we have $D_H(A, A^{(M)}) \leq 2M + 3$.*

Proof. If the M–skeleton of A is empty, the result holds. Otherwise let γ_0 be the external circuit of $\widetilde{A}$ and let $\gamma_1, \cdots, \gamma_r$ be the inner boundaries of $\widetilde{A}$ of diameter larger than M. It follows from the construction of the skeleton that $D_H(\gamma_i, \gamma_i^{(M)}) \leq M + 1$ for all i in $\{0 \cdots r\}$. Let x belong to A.

- If $d(x, \gamma_0 \cup \cdots \cup \gamma_r) \leq M + 1$ then $d(x, A^{(M)}) \leq 2M + 2$.
- If $d(x, \gamma_0 \cup \cdots \cup \gamma_r) > M + 1$ then x is in a minus component so that x belongs to $A^{(M)}$.

Conversely, let x belong to $A^{(M)}$.

- If $d(x, \gamma_0^{(M)} \cup \cdots \cup \gamma_r^{(M)}) \leq M + 1$ then $d(x, A) \leq 2M + 3$.
- If $d(x, \gamma_0^{(M)} \cup \cdots \cup \gamma_r^{(M)}) > M + 1$ then x is inside γ_0 and outside of $\gamma_1, \cdots, \gamma_r$ so that $d(x, A) \leq M + 1$. □

Lemma 5.4. *There exists a positive constant a_2 such that for any connected set of sites A, any M larger than 1, we have*

$$\forall t > 2M + 3 \qquad D_\lambda(\mathcal{V}(A, t), A^{(M)}) \leq a_2 M^{-1} (t + M)^2 I(\mathcal{S}(A)) .$$

Proof. For t larger than $2M+3$, we have by lemma 5.3 that $A^{(M)} \subset \mathcal{V}(A, t) \subset \mathcal{V}(A^{(M)}, 2t)$. Therefore $D_\lambda(\mathcal{V}(A, t), A^{(M)}) \leq \lambda(\mathcal{V}(A^{(M)}, 2t) \setminus A^{(M)})$. Any point in $\mathcal{V}(A^{(M)}, 2t) \setminus A^{(M)}$ is at distance at most $2t + M + 1$ from the skeleton of A. Using lemma 5.1, it follows that

$$\lambda\big(\mathcal{V}(A^{(M)}, 2t) \setminus A^{(M)}\big) \leq \pi(2t + M + 1)^2 \mathrm{card}\, \mathcal{S}(A) \leq \frac{4\pi}{a_0} M^{-1} (t + M)^2 I(\mathcal{S}(A))$$

which gives the desired result. □

Two probabilistic estimates.

Lemma 5.5. *For any M–skeleton $\mathcal{S}$, we have $\widehat{P}(\mathcal{S}(C) = \mathcal{S}) \leq \exp -\sigma I(\mathcal{S})$.*

Proof. We suppose that the skeleton $\mathcal{S}$ is the union of the M–skeletons $\gamma_i^{(M)} = [s_0^i, \cdots, s_{l_i}^i]$, $0 \leq i \leq r$. The event $\{\mathcal{S}(C) = \mathcal{S}\}$ is included in the disjoint occurrence of events $\{s_0^0 \leftrightarrow s_1^0\} \circ \cdots \circ \{s_{l_0-1}^0 \leftrightarrow s_{l_0}^0\} \circ \{s_{l_0}^0 \leftrightarrow s_0^0\} \circ \{s_0^1 \leftrightarrow s_1^1\} \circ \cdots \circ \{s_{l_1-1}^1 \leftrightarrow s_{l_1}^1\} \circ \{s_{l_1}^1 \leftrightarrow s_0^1\} \circ \cdots \circ \{s_0^r \leftrightarrow s_1^r\} \circ \cdots \circ \{s_{l_r-1}^r \leftrightarrow s_{l_r}^r\} \circ \{s_{l_r}^r \leftrightarrow s_0^r\}$. An application of the van den Berg–Kesten inequality yields the result. □

Lemma 5.6. *There exist two positive constants a_3, a_4 such that*

$$\forall d \qquad \widehat{P}(diam\, C \geq d) \leq a_3 \exp -a_4 d .$$

Proof. If the diameter of C is n, there exist two dual sites x^*, y^* such that $|x^*-y^*|_2 \geq n$, $|x^*|_2 \leq n$, $|y^*|_2 \leq n$ and there is an open dual path from x^* to y^*. Therefore, setting $a = \inf_{x:|x|_2=1} \sigma g(x)$, we have $\widehat{P}(\operatorname{diam} C = n) \leq \pi^2(n+1)^4 \exp -an$ so that

$$\widehat{P}(\operatorname{diam} C \geq d) \leq \sum_{n=d}^{+\infty} \pi^2 (n+1)^4 \exp -an$$

from which the desired conclusion follows easily. □

Proof of the upper bound (i) of theorem 3.1**.** Let c be a large positive real number, to be chosen later. We set $M = c \ln N$. We build the approximate shape $C^{(M)}$. By lemma 5.3, we have $D_H(C, C^{(M)}) \leq 2M+3$. For N large enough, so that $(2M+3)/N < \delta/2$, we have therefore

$$\widehat{P}(D_H(C/N, \Phi_H(u)) \geq \delta) \leq \widehat{P}(D_H(C^{(M)}/N, \Phi_H(u)) \geq \delta/2)\,.$$

However, $C^{(M)}/N$ is an element of $\mathcal{K}_c$; its distance to the set $\Phi_H(u)$ is positive, thus $\sigma g_H(C^{(M)}) > uN$. Moreover lemma 5.2 implies that $g_H(C^{(M)}) \leq I(\mathcal{S}(C))$. Thus

$$\widehat{P}(D_H(C/N, \Phi_H(u)) \geq \delta) \leq \widehat{P}(\sigma I(\mathcal{S}(C)) \geq uN)\,.$$

The next lemma shows that we get the desired upper bound if we choose c larger than a_5/α.

Lemma 5.7. *Suppose that $M \geq c \ln N$. There exist a positive constant a_5 and an integer N_0 such that for $N \geq N_0$, for any $u \geq 0$,*

$$\widehat{P}(\sigma I(\mathcal{S}(C)) \geq uN) \leq \exp -uN(1 - a_5/c)\,.$$

Proof. Let a be such that $a > u/a_4$ (a_4 is the constant appearing in lemma 5.6). We have

$$\widehat{P}(\sigma I(\mathcal{S}(C)) \geq uN) \leq \widehat{P}(\sigma I(\mathcal{S}(C)) \geq uN, \operatorname{diam} C \leq aN) + \widehat{P}(\operatorname{diam} C > aN).$$

By lemma 5.6, $\widehat{P}(\operatorname{diam} C > aN) \leq a_3 \exp -a_4 aN \leq a_3 \exp -uN$.
We estimate now the term

$$\widehat{P}(\sigma I(\mathcal{S}(C)) \geq uN,\, \operatorname{diam} C \leq aN) = \sum_n \sum_{\mathcal{S} \in \mathcal{A}(n,u,a,N)} \widehat{P}(\mathcal{S}(C) = \mathcal{S})$$

where $\mathcal{A}(n,u,a,N)$ is the set of M–skeletons $\mathcal{S}$ such that $\sigma I(\mathcal{S}) \geq uN$, $\operatorname{card} \mathcal{S} = n$, and there exists a connected set of sites containing the origin of diameter less than aN whose

M–skeleton is $\mathcal{S}$ (it is the set of the M–skeletons of cardinality n which are the M–skeleton of a cluster C satisfying $\sigma I(\mathcal{S}(C)) \geq uN$, $\operatorname{diam} C \leq aN$, $0 \in C$). The number of M–skeletons in $\mathcal{A}(n,u,a,N)$ is at most $\pi^n(aN+1)^{2n}n^n \leq \exp 8n\ln N$ for N large enough (of course we have that $n \leq \pi(aN+1)^2$ since all the points of the M–skeleton are at a distance at most aN from the origin). Let b be such that $8 < \sigma a_0 b$ (the constant a_0 is the one appearing in lemma 5.1). For $\mathcal{S}$ in $\mathcal{A}(n,u,a,N)$, we write, using lemma 5.1,

$$\sigma I(\mathcal{S}) = \sigma I(\mathcal{S})(1-b/c) + (b/c)\sigma I(\mathcal{S}) \geq uN(1-b/c) + \sigma a_0 b\, n\ln N$$

and we use the estimate of lemma 5.5 to get

$$\begin{aligned}
\widehat{P}(\sigma I(\mathcal{S}(C)) \geq uN, \operatorname{diam} C \leq aN) \;&\leq \sum_n \sum_{\mathcal{S}\in\mathcal{A}(n,u,a,N)} \exp -\sigma I(\mathcal{S}) \\
&\leq \sum_n \sum_{\mathcal{S}\in\mathcal{A}(n,u,a,N)} \exp\Big(-uN(1-b/c) - \sigma a_0 b\, n\ln N\Big) \\
&\leq \exp\Big(-uN(1-b/c)\Big)\sum_n \exp\Big((8-\sigma a_0 b)\, n\ln N\Big) \\
&\leq \exp -uN(1-a_5/c)
\end{aligned}$$

for any $a_5 > b$ and N large enough, since $\sigma a_0 b > 8$. $\square$

Proof of the upper bound (i) of theorem 3.2. We set $M = f(N)$. We build the approximate shape $C^{(M)}$. We have

$$\begin{aligned}
&\widehat{P}(D_\lambda(\mathcal{V}(C,f(N))/N, \Phi_\lambda(u)) \geq \delta) \;\leq \\
&\qquad\qquad \widehat{P}(D_\lambda(\mathcal{V}(C,f(N))/N, C^{(M)}/N) \geq \delta/2) \,+\, \widehat{P}(D_\lambda(C^{(M)}/N, \Phi_\lambda(u)) \geq \delta/2)\,.
\end{aligned}$$

Lemma 5.4 implies that $D_\lambda(\mathcal{V}(C,f(N)),\, C^{(M)}) \leq 4a_2 f(N) I(\mathcal{S}(C))$ whence

$$\widehat{P}(D_\lambda(\mathcal{V}(C,f(N))/N, C^{(M)}/N) \geq \delta/2) \;\leq\; \widehat{P}(\sigma I(\mathcal{S}(C)) \geq \sigma N^2\delta/(8a_2 f(N)))\,.$$

In addition, if $D_\lambda(C^{(M)}/N, \Phi_\lambda(u)) \geq \delta/2$ then $g_\lambda(C^{(M)}) > Nu$ and by lemma 5.2, we have also $\sigma I(\mathcal{S}(C)) > Nu$, thus

$$\widehat{P}(D_\lambda(C^{(M)}/N, \Phi_\lambda(u)) \geq \delta/2) \;\leq\; \widehat{P}(\sigma I(\mathcal{S}(C)) \geq Nu)\,.$$

Combining the two previous inequalities, we obtain that

$$\begin{aligned}
&\widehat{P}(D_\lambda(\mathcal{V}(C,f(N))/N, \Phi_\lambda(u)) \geq \delta) \;\leq \\
&\qquad\qquad \widehat{P}(\sigma I(\mathcal{S}(C)) \geq \sigma N^2\delta/(8a_2 f(N))) \,+\, \widehat{P}(\sigma I(\mathcal{S}(C)) \geq Nu)\,.
\end{aligned}$$

The desired upper bound follows from lemma 5.7 and the hypothesis that as N goes to infinity, $f(N)/\ln N$ goes to infinity and $f(N)/N$ goes to zero. $\square$

Proof of the lower bound (ii) of theorem 3.1. Let r be positive and let x^*, y^* be two dual sites. The event that there exists an open dual path from x^* to y^* whose Hausdorff distance to the segment $[x^*, y^*]$ is less than r is denoted by $x^* \overset{r}{\longleftrightarrow} y^*$.

Lemma 5.8. *Let $\phi(n)$ be a function such that $\lim_{n\to\infty} \phi(n) = +\infty$. For any dual site x^*, we have*

$$\lim_{n\to\infty} \frac{1}{n} \ln P(0^* \overset{\phi(n)}{\longleftrightarrow} nx^*) \;=\; -\sigma g(x)\,.$$

Proof. Clearly $P(0^* \overset{\phi(n)}{\longleftrightarrow} nx^*) \le P(0^* \leftrightarrow nx^*)$ so that

$$\limsup_{n\to\infty} \frac{1}{n} \ln P(0^* \overset{\phi(n)}{\longleftrightarrow} nx^*) \;\le\; -\sigma g(x)\,.$$

Let ϵ be a real number in $]0,1[$. For any m in $\mathbb{N}$, there exists $N(m)$ such that

$$(1-\epsilon) P(0^* \leftrightarrow mx^*) \;\le\; P(0^* \overset{N(m)}{\longleftrightarrow} mx^*)\,.$$

Let n_0 be such that $\phi(n) \ge N(m)$ for $n \ge n_0$. Let $n = km + l$ be the Euclidean division of n by m. Then the event

$$\{\, 0^* \overset{N(m)}{\longleftrightarrow} mx^*,\, mx^* \overset{N(m)}{\longleftrightarrow} 2mx^*,\, \cdots (k-1)mx^* \overset{N(m)}{\longleftrightarrow} kmx^*,\, kmx^* \overset{N(m)}{\longleftrightarrow} nx^* \,\}$$

is included in the event $\{\, 0^* \overset{\phi(n)}{\longleftrightarrow} nx^* \,\}$. Using the Harris–FKG inequality and the translation invariance of the model, we obtain

$$P(0^* \overset{\phi(n)}{\longleftrightarrow} nx^*) \;\ge\; P(0^* \overset{N(m)}{\longleftrightarrow} mx^*)^k \, P(0^* \overset{N(m)}{\longleftrightarrow} lx^*)\,.$$

Thus

$$\frac{1}{n} \ln P(0^* \overset{\phi(n)}{\longleftrightarrow} nx^*) \;\ge\; \frac{1}{n} \left\lfloor \frac{n}{m} \right\rfloor \ln P(0^* \overset{N(m)}{\longleftrightarrow} mx^*) + \frac{1}{n} \ln P(0^* \overset{N(m)}{\longleftrightarrow} lx^*)$$

whence

$$\liminf_{n\to\infty} \frac{1}{n} \ln P(0^* \overset{\phi(n)}{\longleftrightarrow} nx^*) \;\ge\; \frac{1}{m} \ln P(0^* \leftrightarrow mx^*) + \frac{1}{m} \ln(1-\epsilon)\,.$$

Letting m go to infinity, we get the result. $\square$

Lemma 5.9. *Let γ be a Jordan curve and let δ be a positive real number. By Loop$^*(\gamma, \delta)$ (respectively Loop(γ, δ)) we denote the event that there exists a dual circuit L^* of open edges of $(\mathbb{E}^*)^2$ (resp. a circuit L of open edges of $\mathbb{E}^2$) such that $D_H(\gamma, L^*) < \delta$ (resp. $D_H(\gamma, L) < \delta$). Let α be any positive real number. There exists N_0 such that for $N \ge N_0$ we have*

$$P(\mathit{Loop}^*(N\gamma, N\delta)) \ge \exp -N\sigma g(\gamma)(1+\alpha), \qquad P(\mathit{Loop}(N\gamma, N\delta)) \ge \exp -N\alpha\,.$$

Proof. We do the proof for the dual circuit; the other case is similar, using the fact that the surface tension corresponding to paths in $\mathbb{E}^2$ is zero in the supercritical regime. Let $< P_0, \cdots, P_l >$ be a polygonal line such that

$$D_H(\gamma, < P_0, \cdots, P_l >) \, < \, \delta/2 \, , \qquad g(< P_0, \cdots, P_l >) \, \leq \, g(\gamma)(1 + \alpha/2)$$

(by $< P_0, \cdots, P_l >$ we denote the polygonal line consisting of the segments $[P_0, P_1]$, $\cdots$, $[P_{l-1}, P_l]$, $[P_l, P_0]$). We have that (with the convention $P_{l+1} = P_0$)

$$P(\text{Loop}^*(N\gamma, N\delta)) \, \geq \, P(NP_k^* \overset{N\delta/2-1}{\longleftrightarrow} NP_{k+1}^*, \, 0 \leq k \leq l)$$

(NP_k^* is the dual site closest to NP_k or any of them if there are several closest dual sites). Using the Harris–FKG inequality we obtain

$$P(\text{Loop}^*(N\gamma, N\delta)) \, \geq \, \prod_{k=0}^{l} P(NP_k^* \overset{N\delta/2-1}{\longleftrightarrow} NP_{k+1}^*) \, .$$

Lemma 5.8 implies that

$$\liminf_{N \to \infty} \frac{1}{N} \ln P(\text{Loop}^*(N\gamma, N\delta)) \, \geq \, -\sum_{k=0}^{l} \sigma g(P_{k+1} - P_k) \, \geq \, -\sigma g(\gamma)(1 + \alpha/2)$$

from which the desired result follows. □

Let us first remark that we need only to prove the lower bound for arbitrary small values of δ. That is we might suppose that δ is smaller than any value $\delta_0 > 0$, possibly depending on K. Moreover, for any K in $\mathcal{K}_c$ containing the origin, we have

$$\widehat{P}(\overline{D}_H(\overline{C}/N, \overline{K}) \leq \delta) \, \geq \, \widehat{P}(D_H(C/N, K) \leq \delta) \, \geq \, \widehat{P}(D_H(C/N, K \cup \{x : |x|_2 \leq \delta/2\}) \leq \delta')$$

for any δ' smaller than $\delta/2$. Therefore it is enough to prove the lower bound for a connected compact set K containing the origin in its interior and with a value of δ much smaller than $d(0, \partial K)$.

By the very definition of the g_H–perimeter, there exists a set K' belonging to $\mathcal{K}_c^J$ such that $D_H(K, K') < \delta/3$, $g_H(K') \leq g_H(K)(1 + \alpha/4)$. Let γ_0' be the external boundary of K' and let $\gamma_1', \cdots, \gamma_r'$ be its inner boundaries. Let O' be the interior of K'. Since K' belongs to $\mathcal{K}_c^J$, then O' is connected and K' is the closure of O'. We apply lemma 4.5 to O': there exists an open connected subset O'' of O' and a positive η such that, if we denote by K'' the closure of O'', $D_H(K', K'') < \delta/3$ and for any x in K'', $d(x, \partial K') > \eta$. We might in addition assume that the origin belongs to O''. By lemma 5.9 and the Harris–FKG inequality, for N sufficiently large,

$$P(\text{Loop}^*(N\gamma_0', N\eta/2), \cdots, \text{Loop}^*(N\gamma_r', N\eta/2)) \, \geq \, \prod_{k=0}^{r} P(\text{Loop}^*(N\gamma_k', N\eta/2))$$

$$\geq \, \prod_{k=0}^{r} \exp -N\sigma g(\gamma_k')(1 + \alpha/4) \, \geq \, \exp -N\sigma g_H(K)(1 + \alpha/4)^2 \, .$$

The measure P restricted to the set of edges included in NK'' is independent from the events $\text{Loop}^*(N\gamma'_0, N\eta/2)$, ... , $\text{Loop}^*(N\gamma'_r, N\eta/2)$ (these events depend only on the edges whose endpoints are at a distance less than $N\eta/2$ from the boundary of NK'). Let $\rho > 0$ and let γ be a Jordan curve such that

$$D_H(\gamma, K'') < \rho, \qquad 0 \in \gamma, \qquad \gamma \cap \partial K'' = \emptyset\,.$$

Such a curve exists because K'' is the closure of the open connected set O'' and also O'' contains the origin. Let ψ be such that $0 < \psi < d(\gamma, \partial K'')$. Clearly ψ is less than ρ. Let L be a circuit around the origin such that $d(0, L) > 2\rho N$, $|L| \leq 20\rho N$. For ρ sufficiently small, any connected set whose Hausdorff distance to $N\gamma$ is less than ρN intersects L (it has to meet the interior and the exterior of L). We have then

$$\begin{aligned}
&\widehat{P}(D_H(C/N, K) \leq \delta) \;\geq\; \widehat{P}(D_H(C/N, K') \leq 2\delta/3) \;\geq \\
&\quad P(\text{Loop}^*(N\gamma'_0, N\eta/2), \cdots, \text{Loop}^*(N\gamma'_r, N\eta/2), \text{Loop}(N\gamma, N\psi), 0 \leftrightarrow L, L \text{ is open}) \;\geq \\
&\quad P(\text{Loop}^*(N\gamma'_0, N\eta/2), \cdots, \text{Loop}^*(N\gamma'_r, N\eta/2)) P(\text{Loop}(N\gamma, N\psi))\, P(0 \leftrightarrow L) P(L \text{ open}) \\
&\qquad \geq\; \exp -N\sigma g_H(K)(1 + \alpha/4)^2\, P(\text{Loop}(N\gamma, N\psi))\, p^{40\rho N}\,.
\end{aligned}$$

In the penultimate step we have used the independence between the occurrence of the events $\text{Loop}^*(N\gamma'_k, N\eta/2)$ and the configuration restricted to NK'', as well as the Harris–FKG inequality. By lemma 5.9, we have $\liminf_{N\to\infty} (1/N) \ln P(\text{Loop}(N\gamma, N\psi)) = 0$. We obtain thus

$$\liminf_{N\to\infty} \frac{1}{N} \ln \widehat{P}(D_H(C/N, K) \leq \delta) \;\geq\; -\sigma g_H(K)(1 + \alpha/4)^2 \;+\; 40\rho \ln p$$

and letting ρ go to zero we get the result. □

Proof of the lower bound (ii) of theorem 3.2**.** Let us first remark that we need only to prove the lower bound for arbitrary small values of δ. That is we might suppose that δ is smaller than any value $\delta_0 > 0$, possibly depending on B. Moreover, for any set B in $\mathcal{B}$, for any x in $\mathbb{R}^2$, we have

$$\widehat{P}(\overline{D}_\lambda(\overline{C}/N, \overline{B}) \leq \delta) \;\geq\; \widehat{P}(D_\lambda(C/N, B + x) \leq \delta)$$

so that we will prove a lower bound for the righthand side. By the very definition of the g_λ–perimeter, there exists a set K' belonging to $\mathcal{K}_c^J$ such that $D_\lambda(B, K') < \delta/4$, $g_\lambda(K') \leq g_\lambda(B)(1 + \alpha/4)$. Let γ'_0 be the external boundary of K' and let $\gamma'_1, \cdots, \gamma'_r$ be its inner boundaries. Let O' be the interior of K'. Since K' belongs to $\mathcal{K}_c^J$, then O' is connected and K' is the closure of O'. We apply lemma 4.5 to O': there exist an open connected set O'' and a positive η such that, if we denote by K'' the closure of O'', we have

$$D_\lambda(K',K'')<\delta/4,\qquad \forall x''\in K''\quad d(x'',\partial K')>\eta\,.$$

We might assume that the origin belongs to O'' (otherwise we simply translate the sets B,K',K'') and that η is so small that $D_\lambda(\mathcal{V}(K',\eta),K')<\delta/4$. By lemma 5.9 and the Harris–FKG inequality, for N sufficiently large,

$$P(\mathrm{Loop}^*(N\gamma_0',N\eta/2),\cdots,\mathrm{Loop}^*(N\gamma_r',N\eta/2))\,\geq\,\exp-N\sigma g_\lambda(K')(1+\alpha/4)\,.$$

The measure P restricted to the set of edges included in NK'' is independent from the events $\mathrm{Loop}^*(N\gamma_0',N\eta/2)$, ... , $\mathrm{Loop}^*(N\gamma_r',N\eta/2)$ (these events depend only on the edges whose endpoints are at a distance less than $N\eta/2$ from the boundary of NK'). Moreover, whenever these events happen and whenever N is large enough so that $f(N)/N<\eta/2$, we have that $\mathcal{V}(C,f(N))\subset\mathcal{V}(NK',\eta N)$. Let C'' be the open cluster containing the origin in the configuration restricted to NK''. Clearly C'' is a subset of C. Any set E such that: $E\subset\mathcal{V}(K',\eta)$, $D_\lambda(E\cap K'',K'')\leq\delta/4$ satisfies $D_\lambda(E,K')\leq 3\delta/4$. It follows that

$$\begin{aligned}&\widehat{P}(D_\lambda(\mathcal{V}(C,f(N))/N,B)\leq\delta)\,\geq\,\widehat{P}(D_\lambda(\mathcal{V}(C,f(N))/N,K')\leq 3\delta/4)\\&\geq\,P(\mathrm{Loop}^*(N\gamma_k',N\eta/2),\,0\leq k\leq r,\,D_\lambda(NK''\cap\mathcal{V}(C'',f(N)),NK'')\leq N^2\delta/4)\\&\qquad\geq\,\exp-N\sigma g_\lambda(B)(1+\alpha/4)^2\,P(D_\lambda(NK''\cap\mathcal{V}(C'',f(N)),NK'')\leq N^2\delta/4)\,.\end{aligned}$$

Let ρ be a positive real number smaller than $\delta/4$. Let L be a circuit around the origin such that $d(0,L)>N\sqrt{\rho/\pi}$, $|L|\leq 10N\sqrt{\rho/\pi}$. For ρ sufficiently small, so that the ball $\{x:|x|_2<20\sqrt{\rho/\pi}\}$ is included in K'', any connected set E satisfying $D_\lambda(E,NK'')<\rho N^2$ necessarily intersects the circuit L (it has to meet the interior and the exterior of L). For $t\geq 0$, we denote by $A(t)$ the event that there exists an open cluster E in NK'' such that $D_\lambda(NK''\cap\mathcal{V}(E,f(N)),NK'')<t$. This event is increasing, whence by the Harris–FKG inequality

$$\begin{aligned}&P(D_\lambda(NK''\cap\mathcal{V}(C'',f(N)),NK'')\leq N^2\delta/4)\,\geq\\&\qquad P(A(\rho N^2),\,0\leftrightarrow L, L\text{ is open})\,\geq\,P(A(\rho N^2))\,p^{20\sqrt{\rho/\pi}N}\,.\end{aligned}$$

We apply lemma 4.5 to O'': there exist an open connected set O''' and a positive μ such that, if we denote by K''' the closure of O''', we have

$$D_\lambda(K'',K''')<\rho,\qquad \forall x'''\in K''',\quad d(x''',\partial K'')>\mu\,.$$

Let A_0 be the event: there exists an open cluster C_0 in NK'' such that for any x''' in NK''', we have $d(x,C_0)\leq f(N)$. We have the inclusion $A_0\subset A(\rho N^2)$. Let A_1 be the event that there is no open dual path in NK'' of diameter larger than $f(N)$. We claim that for N large enough, we have also $A_1 \subset A_0$. Suppose indeed that the event A_1 occurs. We first show that there is inside NK'' an open cluster C_0 of diameter larger than $f(N)$ which intersects NK'''. Let y in $\mathbb{Z}^2$ and $r > 0$ be such that the ball $\{\, x : |x-y|_2 < rN \,\}$ is included in NK''' and let N be large enough so that $f(N) < \min\{rN/2, \mu N-2\}$. If all the open clusters intersecting the ball $\{\, x : |x-y|_2 < rN \,\}$ have a diameter smaller than $f(N)$, then the set $\{\, x : \exists z \ \ |y-z|_2 < rN/2,\ z \leftrightarrow x \,\}$ is included in the ball $\{\, x : |x-y|_2 < rN \,\}$. It is surrounded by an innermost circuit of open dual edges (see [11, Proposition 9.2]) of diameter at least $rN/2 > f(N)$, contradicting the occurrence of A_1. Thus there exists an open cluster C_0 in NK'' of diameter larger than $f(N)$ which intersects NK'''. We next show that this open cluster C_0 is such that $\mathcal{V}(C_0, f(N))$ contains NK'''. Suppose there exists x''' in NK''' such that $d(x''', C_0) > f(N)$. By [11, Proposition 9.2], there exists a circuit γ^* of dual edges surrounding C_0. This dual circuit γ^* separates the plane into two components, and both have a diameter larger or equal than $f(N)$: one contains C_0, the other contains $\{\, x : |x-x'''|_2 \leq f(N) \,\}$. Hence the diameter of γ^* is larger than $f(N)$. Moreover γ^* has to meet the set $\mathcal{V}(NK''', 2)$ because $C_0 \cap NK''' \neq \emptyset$, $x''' \in NK'''$ and γ^* separates C_0 and x''' (since NO''' is open and connected, and therefore arcwise connected, then there exists a path of edges in $\mathcal{V}(NK''', 2)$ connecting x''' and C_0; this path necessarily intersects γ^*). The edges of γ^* contained in NK'' are dual edges which are open, whereas the edges of γ^* not included in NK'' may be either open or closed (since we consider the configuration restricted to NK'' to define C_0 we have no information on the status of the edges not included in NK'').

• If all the edges of γ^* belong to NK'' then they are all open dual edges and γ^* is an open dual circuit of diameter larger than $f(N)$.

• If some of the edges of γ^* do not belong to NK'', then γ^* meets the complement of NK''. Since it also meets the set $\mathcal{V}(NK''', 2)$ and the distance between NK''' and the complement of NK'' is larger than μN, then it contains a dual path of diameter larger than $\mu N - 2$ whose edges are all included in NK'' and are therefore open.

We see finally that for N large enough so that $f(N) < \min\{rN/2, \mu N - 2\}$, the event A_1 is included in A_0. We estimate the probability of A_1 by

$$1 - P(A_1) \ \leq\ |NK'' \cap (\mathbb{Z}^*)^2|^2 \exp -af(N) \ \leq\ 16(\operatorname{diam} K'')^4 N^4 \exp -af(N)$$

(with $a = \inf_{x:|x|_2=1} \sigma g(x)$). Putting together the previous inequalities, we get

$$\widehat{P}(D_\lambda(\mathcal{V}(C, f(N))/N, B) \leq \delta) \ \geq \\ \big(\exp -N\sigma g_\lambda(B)(1+\alpha/4)^2\big)(1 - 16(\operatorname{diam} K'')^4 N^4 \exp -af(N))p^{20\sqrt{\rho/\pi}N}$$

We deduce from this inequality that

$$\liminf_{N\to\infty} \frac{1}{N} \ln \widehat{P}(D_\lambda(\mathcal{V}(C, f(N))/N, B) \leq \delta) \ \geq\ -\sigma g_\lambda(B)(1+\alpha/4)^2 \ +\ 20\sqrt{\rho/\pi}\ln p$$

and letting ρ go to zero we get the result. $\square$

Remark. We introduce the third set K''' in the last step of the preceding proof to ensure that an open dual path of diameter less than $f(N)$ cannot destroy the connection between two large regions (which might happen with K'').

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