THE LOW-TEMPERATURE EXPANSION OF THE WULFF CRYSTAL IN THE 3D ISING MODEL

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Abstract. We compute the expansion of the surface tension of the 3D random cluster model for $q \geq 1$ in the limit where p goes to 1. We also compute the asymptotic shape of a plane partition of n as n goes to ∞. This same shape determines the Wulff crystal to order $o(\varepsilon)$ in the 3D Ising model (and more generally in the 3D random cluster model for $q \geq 1$) at temperature ε.

1. Introduction

The three-dimensional Ising model on the cubic lattice is one of the most challenging models of modern statistical physics. Many qualitative properties of the model are understood but despite a great deal of attention by physicists, very few exact, quantitative results have been obtained.

In this paper we study the low-temperature expansion of both the surface tension and the Wulff shape in the three-dimensional Ising model, and more generally the three-dimensional random cluster (FK percolation) model. At temperature ε, we obtain an expansion of the surface tension and Wulff shape to order ε.

A closely related probabilistic model which we study is the random plane partition. This is a three-dimensional analogue of a random integer partition; we prove the existence of, and compute a formula for, the asymptotic limit shape for a "3D Young diagram" of a plane partition of n as $n \to \infty$. This is accompanied by a large deviation principle. This limit

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shape turns out to be the same as the asymptotic Wulff shape for the three-dimensional Ising model (taken near a corner and appropriately scaled).

Here are more precise statements of our results.

1.1. The surface tension. We consider the FK percolation (random cluster) model in the three dimensional cubic lattice $\mathbb{L}^3$ with parameters p,q. In a finite box $B_M = [-M,M]^3 \cap \mathbb{L}^3$ this is the probability measure $\Phi_M^{p,q}$ on subsets of edges of B_M, such that for C a set of edges, $\Phi_M^{p,q}(C)$ is proportional to $(p/(1-p))^{|C|} q^{n_C}$ where n_C is the number of connected components of C. We are interested in the case where $q \geq 1$ and p is close to 1. From [18], we know that there exists a value $p_0(q) < 1$ depending on q such that, for any p in $(p_0(q), 1]$, there exists a unique infinite volume FK measure $\Phi_\infty^{p,q}$ on $\mathbb{L}^3$ corresponding to the parameters p,q.

We fix $q \geq 1$. Let p belong to $(p_0(q), 1]$ and let ν be a vector in the unit sphere S^2. Let A be a unit square orthogonal to ν, let $\mathrm{cyl}\, A$ be the cylinder $A + \mathbb{R}\nu$. We define the surface tension $\tau(\nu, p)$ depending on the direction ν and the parameter p as

$$\tau(\nu,p) \;=\; \lim_{n\to\infty} -\frac{1}{n^2} \ln \Phi_\infty^{p,q} \begin{pmatrix} \text{inside } n\mathrm{cyl}\, A \text{ there exists a finite set of closed edges } E \text{ which cuts} \\ n\mathrm{cyl}\, A \text{ in at least two unbounded components and the edges of } E \text{ at} \\ \text{distance less than 6 from } \partial n\mathrm{cyl}\, A \text{ are at distance less than 6 from } nA \end{pmatrix}$$

The existence of the limit follows from the FKG property of the measure $\Phi_\infty^{p,q}$ and a classical subadditivity argument (see [6] for a detailed proof). We know that for a fixed value of p sufficiently close to 1, the map $\tau(\cdot, p) : \nu \in S^2 \mapsto \tau(\nu, p)$ is strictly positive, continuous, invariant under the isometries which leave $\mathbb{Z}^3$ invariant. Furthermore it satisfies the *weak simplex inequality*, that is, the homogeneous extension of $\tau(\cdot, p)$ to $\mathbb{R}^3$ is convex.

Theorem 1.1. *We have the following expansion of $\tau(\nu, p)$ as p goes to 1: for any $q \geq 1$, uniformly over ν in S^2, as p goes to 1,*

$$\tau(\nu,p) \;=\; |\nu|_1 \ln(\frac{1}{1-p}) - |\nu|_1 \mathrm{ent}(\nu) + o(1)$$

where for any $\nu = (a,b,c)$ in the unit sphere S^2 we set

$$|\nu|_1 = |a| + |b| + |c| \,, \qquad \mathrm{ent}(\nu) = \frac{1}{\pi} L\left(\pi \frac{|a|}{|\nu|_1}\right) + \frac{1}{\pi} L\left(\pi \frac{|b|}{|\nu|_1}\right) + \frac{1}{\pi} L\left(\pi \frac{|c|}{|\nu|_1}\right)$$

and L is the Lobachevsky function given by

$$\forall x \in [0,\pi] \qquad L(x) = -\int_0^x \ln(2\sin t)\, dt$$

Note that the first two terms of this expansion are independent of q. When $q = 1$, we have the surface tension of the Bernoulli percolation model [6]; when $q = 2$, it is the surface tension of the Ising model [4,7], and when q is an integer larger than 2, it is the surface tension of the q-state Potts model [8].

1.2. The Wulff crystal. We denote by $\mathcal{W}_\tau$ the Wulff set associated to the surface tension $\tau = \tau(\nu, p)$, called also the crystal of τ,

$$\mathcal{W}_\tau \;=\; \{\, x \in \mathbb{R}^3 : x \cdot w \leq \tau(w) \text{ for all } w \text{ in } S^2 \,\}\,. \tag{1}$$

Since τ is continuous and bounded away from 0, its crystal $\mathcal{W}_\tau$ is convex, closed, bounded and contains the origin 0 in its interior [15, Proposition 3.5].

The surface energy $\mathcal{I}(A)$ of a set A having a smooth boundary ∂A is defined to be the surface integral of τ on the boundary of A, that is

$$\mathcal{I}(A) \;=\; \int_{\partial A} \tau(\nu_A(x))\, d\mathcal{H}^2(x)$$

where $\nu_A(x)$ is the exterior normal vector to ∂A at x and $\mathcal{H}^2$ is the two dimensional Hausdorff measure in $\mathbb{R}^3$. The Wulff Theorem asserts that, up to dilations and translations, the Wulff crystal $\mathcal{W}_\tau$ is the unique solution to the isoperimetric problem associated to the surface energy $\mathcal{I}$, that is, it is the unique set enclosing a fixed volume and minimizing the surface energy $\mathcal{I}$. In the case where τ is constant, we have the classical isoperimetric problem and the Wulff crystal is an Euclidean ball. In the case of anisotropic functions τ, the first attempts to solve this problem are due to Wulff, at the turn of the century [37]. Later Dinghas [12] proved that, among convex polyhedra, the Wulff crystal $\mathcal{W}_\tau$ is the solution to the problem. Taylor obtained general existence and uniqueness results in the framework of the Geometric Measure Theory [33,34,35]. Recently, Fonseca and Müller reworked and slightly enhanced these results using the theory of the Caccioppoli sets [15,16].

We recall that the Hausdorff distance between two compact sets A_1, A_2 is defined by

$$d_H(A_1, A_2) \;=\; \max\left\{ \sup_{a_1 \in A_1} \inf_{a_2 \in A_2} |a_1 - a_2|_2,\; \sup_{a_2 \in A_2} \inf_{a_1 \in A_1} |a_2 - a_1|_2 \right\}.$$

A direct consequence of Theorem 1.1 is that we can compute an approximation of the Wulff crystal $\mathcal{W}_\tau$ as p goes to 1. Here is a Corollary to Theorem 1.1:

Corollary 1.2. *Let $\mathcal{W}_\sigma$ be the Wulff crystal associated to the function σ defined by*

$$\forall \nu \in S^2 \qquad \sigma(\nu, p) = -|\nu|_1 \ln(1-p) - |\nu|_1 \mathrm{ent}(\nu)\,.$$

Then, for any $q \geq 1$,

$$\lim_{p \to 1} d_H(\mathcal{W}_\sigma, \mathcal{W}_\tau) \;=\; 0\,.$$

Remark. For p close to 1, the diameter of these Wulff crystals is of order $-\ln(1-p)$, which tends to infinity as $p \to 1$. If we rescale the Wulff crystal to have volume 1, then the corollary gives an approximation to order $o(1/\log(1-p))$.

Proof. The functions τ and σ are the support functions of the crystals $\mathcal{W}_\tau$ and $\mathcal{W}_\sigma$. By an identity due to Rådström [27] and Hörmander [21], we have

$$d_H(\mathcal{W}_\sigma, \mathcal{W}_\tau) = \sup_{\nu \in S^2} |\sigma(\nu, p) - \tau(\nu, p)| .$$

The result follows then directly from the uniformity of the asymptotic expansion of Theorem 1.1. $\square$

The following theorem determines W_τ to order $o(1)$ in the Hausdorff metric.

Theorem 1.3. *Let $\varepsilon = -1/\ln(1-p)$ and let T_ε be the map $T_\varepsilon(x) = \frac{1}{\varepsilon}(1,1,1) - x$. The boundary of the corner of $T_\varepsilon(\mathcal{W}_\tau)$ converges (in the Hausdorff topology on compact subsets of $(\mathbb{R}^+)^3$) to the surface*

$$S_0 = \{(f(A,B,C) - \ln A, f(A,B,C) - \ln B, f(A,B,C) - \ln C) \mid A, B, C > 0\}$$

where for A, B, C positive,

$$f(A,B,C) = \frac{1}{4\pi^2} \int_{[0,2\pi]} \int_{[0,2\pi]} \ln |A + Be^{iu} + Ce^{iv}| \, du \, dv .$$

The definition of the topology for which this convergence is proved is given in section 3.2. Note that this theorem and the previous Corollary do not give any control on the size of the *facets* of the true crystal. They only determine the shape of the crystal to order $o(1)$.

A plot of S_0 is shown in Figure 1.

1.3. Plane partitions. A plane partition of a positive integer n is a collection of positive integers $\{p_{i,j}\}_{1 \le i,j < \infty}$ indexed by pairs (i,j) of positive integers with the properties

$$\sum_{i,j} p_{i,j} = n \quad \text{and} \quad \forall i,j \in \mathbb{N} \quad p_{i,j} \ge p_{i+1,j} \, , \quad p_{i,j} \ge p_{i,j+1} \, .$$

To a plane partition $\{p_{i,j}\}$ we associate a **3D Young diagram** by putting a column of unit cubes of height $p_{i,j}$ over the unit square centered at $(-1/2, -1/2, 0) + (i, j, 0)$ in the horizontal plane. For convenience, we consider also that the axis planes belong to the 3D Young diagram. The 3D Young diagram provides a way to view a plane partition as a stack of cubes, see Figure 3 for an example. The exposed surface of the stack of cubes and the axis planes will be called the *surface* of the 3D Young diagram.

Let now $Y_n^=$ (respectively $Y_n^\le$) be a 3D Young diagram chosen randomly with the uniform distribution over the set of all 3D Young diagrams associated to n (respectively to integers less than or equal to n).

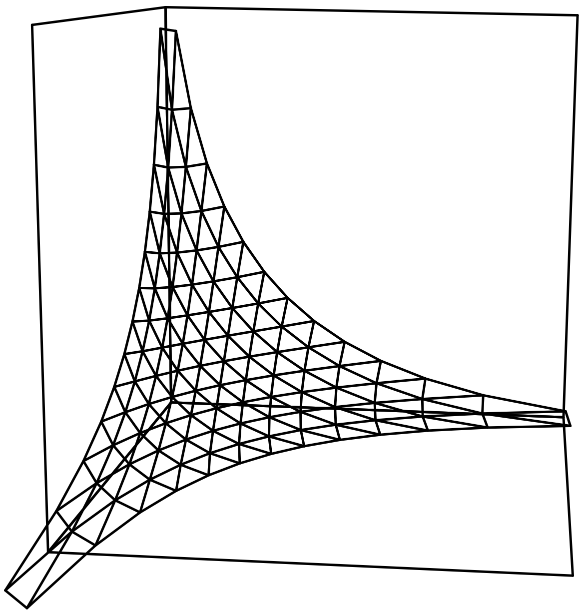

figure 1: The surface S_0

The asymptotic form of $Y_n^=$ for large n is obtained as the solution of a variational problem for minimizing a certain surface energy while enclosing a fixed volume. The surface energy is exactly that given by the function $-|\nu|_1 \mathrm{ent}(\nu)$. We prove:

Theorem 1.4. *The surface $(\zeta(3)/4)^{-1/3} S_0$ is the asymptotic shape of (the surface of) a random rescaled 3D Young diagram $n^{-1/3} Y_n^=$ or $n^{-1/3} Y_n^\leq$.*

See the precise statement in Theorem 3.4. Moreover we prove a large deviation principle for $Y_n^=$ and $Y_n^\leq$.

1.4. Organization. The remainder of the paper is organized as follows. In section 2 we prove Theorem 1.3 using the expansion of Theorem 1.1. This is a straightforward computation using the Wulff construction. We also compute the volume under the surface S_0 and we study its smoothness.

In section 3 we derive a large deviation principle for 3D Young diagrams and use it to prove Theorem 1.4. This section relies on the results of [10]. In that paper the authors deal with a four-parameter family of "domino tilings"; in the current paper we need a specialization of their results, when one of the parameters is set to 0. Specifically, the four

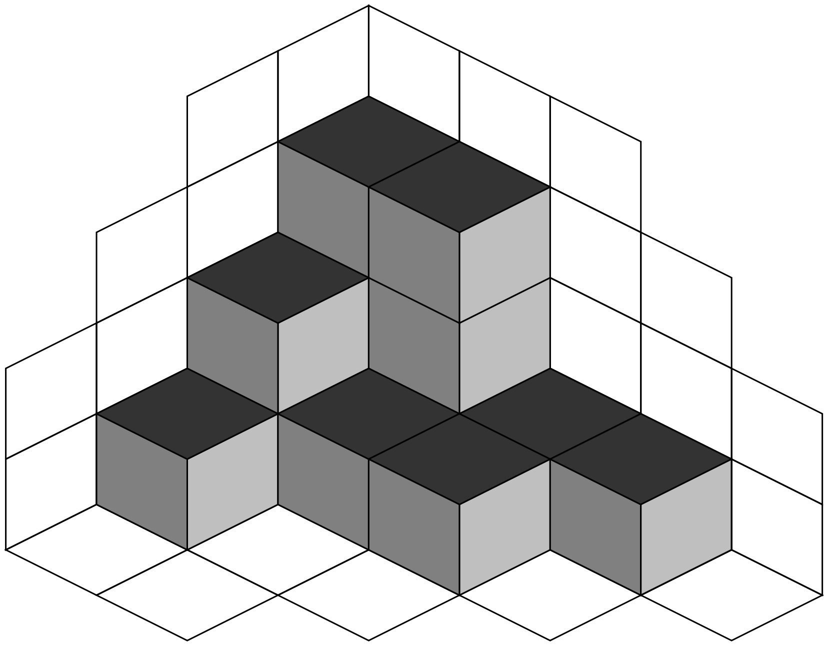

figure 2: 3D Young diagram of a plane partition.

parameters a, b, c, d are edge activities in a dimer model on $\mathbb{Z}^2$. The edge weights a, b, c, d are staggered, with a and b alternating on horizontal edges and c and d alternating on vertical edges of $\mathbb{Z}^2$, in such a way that around each lattice square all four activities occur. As noted in [10], when one of the activities, say d, is zero the model becomes isomorphic to a dimer model on a honeycomb lattice, with activities a, b, c. This is dual to the lozenge-tiling model which is the model we need in this paper.

Sections 4 and 5 below are devoted to the proof of Theorem 1.1.

2. Proof of Theorem 1.3.

To determine $\mathcal{W}_\sigma$, we need to find for each direction $\nu \in S^2$

$$r(\nu) = \min_{w \in S^2} \frac{\sigma(w)}{\nu \cdot w}\,.$$

Then $r(\nu)$ will be the radius of $\mathcal{W}_\sigma$ in direction ν. If we extend homogeneously σ to a function on $\mathbb{R}^3 \setminus \{0\}$ by setting $\sigma(\nu) = |\nu|_2 \sigma(\nu/|\nu|_2)$ for $\nu \in \mathbb{R}^3 \setminus \{0\}$, we have

$$r(\nu) = \min_{w \in \mathbb{R}^3 \setminus \{0\}} \frac{\sigma(w)}{\nu \cdot w}\,. \tag{2}$$

From Theorem 1.1, setting $w = (x, y, z)$, we see that σ is of the form

$$\sigma(x, y, z) = (|x| + |y| + |z|)(\frac{1}{\varepsilon} - \mathrm{ent}(x, y, z))$$

where $\varepsilon = -1/\ln(1-p)$ and

$$\mathrm{ent}(x,y,z) = \frac{1}{\pi} L\left(\pi \frac{|x|}{|w|_1}\right) + \frac{1}{\pi} L\left(\pi \frac{|y|}{|w|_1}\right) + \frac{1}{\pi} L\left(\pi \frac{|z|}{|w|_1}\right).$$

By symmetry we need only to work in the positive orthant

$$O_+ = \{\, (x,y,z) \in (\mathbb{R}^+)^3, x \geq 0, y \geq 0, z \geq 0,\ x+y+z > 0 \,\}.$$

Letting $\nu = (a,b,c)$ we have from (2)

$$r(a,b,c) = \min_{x,y,z \in O_+} \frac{(x+y+z)(1-\varepsilon \mathrm{ent}(x,y,z))}{\varepsilon(ax+by+cz)}.$$

Setting derivatives with respect to x, y and z respectively equal to zero we find three equations for the minimum (any two of which suffice to determine the minimum):

$$\begin{aligned}
(ax+by+cz)(1-\varepsilon\mathrm{ent}-\varepsilon(x+y+z)\mathrm{ent}_x) - a(x+y+z)(1-\varepsilon\mathrm{ent}) = 0\\
(ax+by+cz)(1-\varepsilon\mathrm{ent}-\varepsilon(x+y+z)\mathrm{ent}_y) - b(x+y+z)(1-\varepsilon\mathrm{ent}) = 0\\
(ax+by+cz)(1-\varepsilon\mathrm{ent}-\varepsilon(x+y+z)\mathrm{ent}_z) - c(x+y+z)(1-\varepsilon\mathrm{ent}) = 0.
\end{aligned}$$

Rather than solving for (x,y,z) as a function of (a,b,c), these can be solved for (a,b,c) in terms of (x,y,z). The solution $\nu = (a,b,c)$ is only defined up to a constant multiple, which is chosen so that $\sigma(w) = \nu \cdot w$. Using the fact that $x\mathrm{ent}_x + y\mathrm{ent}_y + z\mathrm{ent}_z = 0$ since ent depends only on the direction of ν and not the length, the solution is

$$\begin{aligned}
a &= \varepsilon^{-1} - \mathrm{ent} - (x+y+z)\mathrm{ent}_x\\
b &= \varepsilon^{-1} - \mathrm{ent} - (x+y+z)\mathrm{ent}_y\\
c &= \varepsilon^{-1} - \mathrm{ent} - (x+y+z)\mathrm{ent}_z.
\end{aligned}$$

The interpretation of this solution is that when $\nu = (a,b,c)$ is of the above form, then $w = (x,y,z)$ minimizes $\sigma(w)/(\nu \cdot w)$ (and the minimum value is 1). Therefore as (x,y,z) runs over S^2, (a,b,c) gives a parametric representation of $\partial\mathcal{W}_\sigma$.

From section 1, we have a parametric representation of $\mathcal{W}_\sigma$ in the positive orthant, given by

$$\partial\mathcal{W}_\sigma = \big\{(\varepsilon^{-1} - \mathrm{ent} - (x+y+z)\mathrm{ent}_x, \varepsilon^{-1} - \mathrm{ent} - (x+y+z)\mathrm{ent}_y, \varepsilon^{-1} - \mathrm{ent} - (x+y+z)\mathrm{ent}_z) \\ \mid\ (x,y,z) \in O_+\big\}.$$

If we translate by $(-\varepsilon^{-1}, -\varepsilon^{-1}, -\varepsilon^{-1})$ so that the corner of the cube (to which $\mathcal{W}_\sigma$ tends as $p \to 1$) is at the origin in $\mathbb{R}^3$, then the surface $\partial\mathcal{W}_\sigma - (\varepsilon^{-1}, \varepsilon^{-1}, \varepsilon^{-1})$ is -1 times the fixed surface S_0 defined by the parametric equation

$$S_0 = \{(\text{ent} + (x+y+z)\text{ent}_x, \text{ent} + (x+y+z)\text{ent}_y, \text{ent} + (x+y+z)\text{ent}_z) \mid x, y, z \in O_+\}.$$

Recall

$$\text{ent}(x,y,z) = \frac{1}{\pi}\left(L\left(\frac{\pi x}{x+y+z}\right) + L\left(\frac{\pi y}{x+y+z}\right) + L\left(\frac{\pi z}{x+y+z}\right)\right),$$

where

$$L(x) = -\int_0^x \ln(2\sin t)dt$$

is the Lobachevsky function. We have $L'(x) = -\ln(2\sin x)$. Setting

$$\theta_x = \frac{\pi x}{x+y+z}, \quad \theta_y = \frac{\pi y}{x+y+z}, \quad \theta_z = \frac{\pi z}{x+y+z},$$

a short computation gives

$$(x+y+z)\,\text{ent}_x(x,y,z) = \left(\frac{\theta_x}{\pi}\ln\sin\theta_x + \frac{\theta_y}{\pi}\ln\sin\theta_y + \frac{\theta_z}{\pi}\ln\sin\theta_z\right) - \ln\sin\theta_x\,.$$

Similar expressions hold for $(x+y+z)\text{ent}_y$ and $(x+y+z)\text{ent}_z$. Note that ent and $(x+y+z)\text{ent}_x$ depend only on the direction of (x,y,z), not on its length. From [10] we have the identity

$$\text{ent}(x,y,z) + \left(\frac{\theta_x}{\pi}\ln\sin\theta_x + \frac{\theta_y}{\pi}\ln\sin\theta_y + \frac{\theta_z}{\pi}\ln\sin\theta_z\right) =$$

$$= \frac{1}{4\pi^2}\int_0^{2\pi}\int_0^{2\pi} \ln\left|\sin\theta_x + e^{iu}\sin\theta_y + e^{iv}\sin\theta_c\right| du\ dv.$$

Replacing $(\sin\theta_x, \sin\theta_y, \sin\theta_z)$ with (A,B,C) satisfying $A:B:C = \sin\theta_x : \sin\theta_y : \sin\theta_z$, S_0 can be written as in the statement of the theorem. This completes the proof.

2.1. Properties of S_0. When $A \geq B + C$, we have $f(A,B,C) = \ln A$. The part of the surface S_0 described by these parameters is given by the set of points

$$(0, \ln(A/B), \ln(A/C))$$

as A, B, C vary while satisfying $A \geq B+C$ and $B, C > 0$. This set consists of the points in the yz plane lying above the curve parametrized by $(0, \ln \frac{B+C}{B}, \ln \frac{B+C}{C})$, which is the curve $\{(0, y, z) \mid e^{-y} + e^{-z} = 1\}$. In particular the "curved" part of S_0 intersects the yz plane in the curve $\{(0, y, z) \mid e^{-y} + e^{-z} = 1\}$, the xy plane in the curve $\{(x, y, 0) \mid e^{-x} + e^{-y} = 1\}$, and the xz-plane in the curve $\{(x, 0, z) \mid e^{-x} + e^{-z} = 1\}$. Surprisingly, each of these curves is, up to scale, the boundary of the asymptotic Wulff crystal of the *two-dimensional* Ising model when the temperature goes to 0 (and the asymptotic shape of the 2D Young diagram of a uniform partition of n), see [9,36, 32].

Another property of the surface S_0 is that it is C^1 but not C^2 at the points where it touches the axis planes. For example, when $(A, B, C) = (2 - \delta, 1, 1)$ we have

$$f(2 - \delta, 1, 1) = \ln 2 - \frac{\delta}{2} + \frac{2}{3\pi}\delta^{3/2} + O(\delta^2),$$

so that the intersection of S_0 with the plane $y = z$ is a curve which, near $x = 0$, is parametrized by

$$\begin{aligned}(f(2 - \delta, 1, 1) - \ln(2 - \delta), f(2 - \delta, 1, 1), f(2 - \delta, 1, 1)) &= \\ \Big(\frac{2}{3\pi}\delta^{3/2} + O(\delta^2), \ln 2 - \frac{\delta}{2} + O(\delta), \ln 2 - \frac{\delta}{2} + O(\delta)\Big),\end{aligned}$$

so that $x = (c_1 - c_2 y)^{3/2}$ near $x = 0$ for constants c_1, c_2. The actual curve is shown in Figure 2.

Notice that the facets of the Wulff crystal in the Ising model still exist for fixed small temperature [5,24].

We finally compute the volume under S_0.

Proposition 2.1. *The volume under the surface S_0 is equal to $\zeta(3)/4$.*

Proof. This volume is $\int_{\mathbb{R}^2} z dx dy$, where (x, y, z) are given by the parametric equation of S_0 in the canonical basis. We proceed in two steps. First change coordinates from x, y, z to A, B, C. Here we let A, B be the independent variables and fix $C = 1$; the curved part of S_0 corresponds to pairs A, B for which $\{A, B, 1\}$ satisfies the triangle inequality. We have the identity ([10])

$$\frac{\partial}{\partial A} f(A, B, C) = \frac{\theta_A}{A\pi}$$

where θ_A is the angle opposite edge A in a triangle of edge lengths A, B, C. Similar expressions hold for B and C. We compute

$$dx = df(A, B, 1) - d\ln A = \Big(\frac{\theta_A}{\pi A} - \frac{1}{A}\Big)dA + \frac{\theta_B}{\pi B}dB,$$

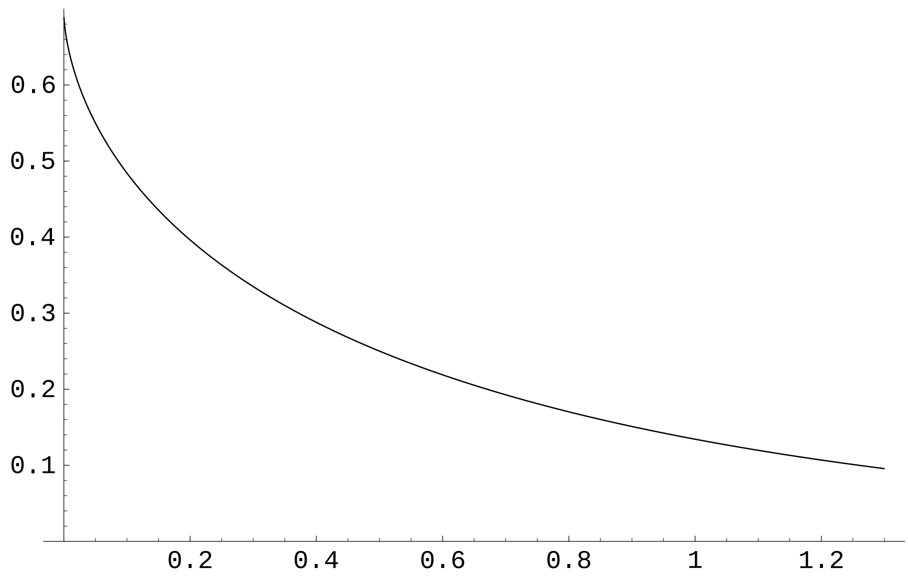

figure 3: The intersection of the surface with the plane $y = z$;
the horizontal axis is the x-axis and the vertical is the y-coordinate (or z-coordinate).

$$dy = df(A, B, 1) - d\ln B = \frac{\theta_A}{\pi A}dA + (\frac{\theta_B}{\pi B} - \frac{1}{B})dB,$$

yielding

$$dxdy = \frac{\theta_C}{\pi AB}dAdB$$

where we used $\theta_A + \theta_B + \theta_C = \pi$. Changing coordinates again, to θ_A, θ_B (where $\theta_C = \pi - \theta_A - \theta_B$) we have $A = \sin\theta_A / \sin\theta_C$ and $B = \sin\theta_B / \sin\theta_C$. A short computation gives

$$\frac{\theta_C}{\pi}\frac{dA\ dB}{AB} = \frac{\theta_C}{\pi}d\ln A\ d\ln B = \frac{\theta_C}{\pi}d\theta_A\ d\theta_B.$$

Now

$$f(A, B, 1) = \frac{1}{\pi}(L(\theta_A) + L(\theta_B) + L(\theta_C)) + \frac{\theta_A}{\pi}\ln A + \frac{\theta_B}{\pi}\ln B.$$

We use the expansion

$$L(\theta) = \frac{1}{2}\sum_{n=1}^{\infty}\frac{\sin(2n\theta)}{n^2}.$$

We have

$$\int_0^\pi \int_0^{\pi-\theta_A} \sin(2n\theta_A)\frac{\theta_C}{\pi} d\theta_B d\theta_A = \frac{\pi}{4n}$$

with the same expression holding for $\sin(2n\theta_B)$; and

$$\int_0^\pi \int_0^{\pi-\theta_A} \sin(2n\theta_C)\frac{\theta_C}{\pi} d\theta_B d\theta_A = 0.$$

In particular

$$\int_0^\pi \int_0^{\pi-\theta_A} \frac{1}{\pi}(L(\theta_A)+L(\theta_B)+L(\theta_C))\frac{\theta_C}{\pi} d\theta_B d\theta_A = \frac{1}{4}\sum_{n=1}^{\infty}\frac{1}{n^3} = \frac{1}{4}\zeta(3).$$

The integral of the terms

$$\frac{\theta_A}{\pi}\ln\frac{\sin\theta_A}{\sin\theta_C} + \frac{\theta_B}{\pi}\ln\frac{\sin\theta_B}{\sin\theta_C}$$

is easily shown to be zero. In conclusion we have

$$\int_0^\pi \int_0^{\pi-\theta_A} f(A,B,1)\frac{\theta_C}{\pi} d\theta_B d\theta_A = \frac{1}{4}\zeta(3). \quad \square$$

3. Plane partitions

3.1. Monotone sets and entropy. Our goal is to derive a large deviation principle and a law of large numbers for the rescaled random 3D Young diagrams $n^{-1/3}Y_n^=$ and $n^{-1/3}Y_n^\leq$. To this end, we need first to define our topological framework and to embed our random objects in a continuous space.

We consider the space $\mathcal{E}$ consisting of closed subsets E of $(\mathbb{R}^+)^3$ having finite volume ($\mathcal{L}^3(E)<\infty$) and satisfying the following monotonicity property: for any $(x,y,z)\in(\mathbb{R}^+)^3$,

$$(x,y,z)\in E \quad\Rightarrow\quad [0,x]\times\{y\}\times\{z\}\subset E,\quad \{x\}\times[0,y]\times\{z\}\subset E,\quad \{x\}\times\{y\}\times[0,z]\subset E\,.$$

For convenience, we impose also that the axis planes are included in E. Let P_{111} be the plane containing the origin and orthogonal to the vector $(1,1,1)$, i.e.,

$$P_{111} \;=\; \{\,(a,b,c)\in\mathbb{R}^3 : a+b+c=0\,\}\,.$$

The boundary of an element E of $\mathcal{E}$ can be conveniently parametrized by looking at the height of E over the plane P_{111}, that is, to E we associate the function $f_E : P_{111} \mapsto \mathbb{R}^+$ defined by

$$\forall x \in P_{111} \qquad f_E(x) \,=\, \sup\{\, t \in \mathbb{R}^+ : x + t(1,1,1) \in E \,\}\,.$$

The set E can be recovered from its height function f_E, indeed

$$E \,=\, \{\, (a,b,c) \in (\mathbb{R}^+)^3 : a+b+c \leq f_E(\pi_{111}(a,b,c))\,\}$$

where π_{111} is the projection on P_{111} parallel to the direction $(1,1,1)$. The monotonicity condition satisfied by the set E implies that

$$\forall x,y \in P_{111} \qquad f_E(y) \,\geq\, f_E(x) - |x-y|_2$$

hence the height function f_E belongs to the space $\mathrm{Lip}_1(P_{111}, \mathbb{R}^+)$ of the maps from P_{111} to $\mathbb{R}^+$ which are Lipschitz with Lipschitz constant 1.

In particular the boundary of an element E of $\mathcal{E}$ admits a Lipschitz parametrization. By Rademacher's Theorem, a Lipschitz function is differentiable almost everywhere with respect to the Lebesgue measure, hence the set E admits a tangent plane at $(x, f_E(x))$ for λ almost all x, where λ is the planar Lebesgue measure in the plane P_{111}. We denote by $\nu_E^{\mathrm{classic}}(x)$ the normal vector at $(x, f_E(x))$. Let $\mathrm{axis}(\cdot)$ be the height function associated to the axis planes, that is

$$\forall x \in P_{111} \qquad \mathrm{axis}(x) \,=\, \inf\{\, t : x + t(1,1,1) \in (\mathbb{R}^+)^3 \,\}\,.$$

We define the domain of E as

$$\mathrm{dom}(E) \,=\, \{\, x \in P_{111} : f_E(x) > \mathrm{axis}(x) \,\}$$

and the entropy of E as

$$\mathrm{ent}(E) \,=\, \int_{\mathrm{dom}(E)} \mathrm{ent}(\nu_E^{\mathrm{classic}}(x))\, d\lambda(x)$$

where ent is the function appearing in Theorem 1.1; notice that in the integral we do not take into account the boundary points lying in the axis planes.

A more flexible way to define the entropy is to work in the space of functions having bounded variation. Since f is Lipschitz, it is absolutely continuous and it belongs to the space $BV_{\mathrm{loc}}(P_{111})$, so that its distributional derivative is the Radon measure having density ∇f_E with respect to λ, the Lebesgue measure in P_{111}. As in [31, formula (15)], to the entropy function $|\nu|_1 \mathrm{ent}(\nu)$ we associate the convex set $\mathcal{S}$ defined by

$$\mathcal{S} \,=\, \{\, x \in (\mathbb{R}^+)^3 : \forall \nu \in (\mathbb{R}^+)^3 \quad x \cdot \nu \geq |\nu|_1 \mathrm{ent}(\nu) \,\}\,.$$

Since $|\nu|_1 \mathrm{ent}(\nu)$ is concave, we have the dual relation

$$\forall \nu \in (\mathbb{R}^+)^3 \qquad |\nu|_1 \mathrm{ent}(\nu) \,=\, \inf_{x \in \mathcal{S}} x \cdot \nu \,.$$

Let $\mathcal{C}_0^1(P_{111}, \mathcal{S})$ be the set of the C^1 vector fields with compact support in P_{111} and taking values in $\mathcal{S}$. Like in [6, Chapter 6], it can be shown that

$$\mathrm{ent}(E) \,=\, \inf \Big\{ \int_{\mathrm{dom}(E)} g(x) \cdot \nabla f_E(x) \, d\lambda(x) : g \in \mathcal{C}_0^1(P_{111}, \mathcal{S}) \Big\}$$

and by the generalized Gauss–Green formula,

$$\mathrm{ent}(E) \,=\, \inf \Big\{ \int_{\mathrm{dom}(E)} f_E(x) \, \mathrm{div}\, g(x) \, d\lambda(x) : g \in \mathcal{C}_0^1(P_{111}, \mathcal{S}) \Big\} \,.$$

From this last expression, it is obvious that the entropy ent is upper semicontinuous in the topology $L^1_{\mathrm{loc}}(P_{111})$, as well as in the stronger topology $L^1(P_{111})$.

Lemma 3.1. *The function* ent *is upper semicontinuous in the topology* $L^1_{\mathrm{loc}}(P_{111})$.

Another way to prove this lemma is to rely on the specific fact that we work with the set of functions $\mathrm{Lip}_1(P_{111}, \mathbb{R}^+)$. On $\mathrm{Lip}_1(P_{111}, \mathbb{R}^+)$, the topology $L^1_{\mathrm{loc}}(P_{111})$ agrees with the topology of uniform convergence on compact subsets. Let $(f_n)_{n \in \mathbb{N}}$ be a sequence of functions belonging to $\mathrm{Lip}_1(P_{111}, \mathbb{R}^+)$ and converging in $L^1_{\mathrm{loc}}(P_{111})$ (and therefore uniformly on compact subsets of P_{111}) towards f. By [10, Lemma 2.1], for any compact set K,

$$\limsup_{n \to \infty} \int_K \mathrm{ent}(\nabla f_n) \, d\lambda \,\leq\, \int_K \mathrm{ent}(\nabla f) \, d\lambda \,.$$

We can write P_{111} as the union $P_{111} = \bigcup_{m \in \mathbb{N}} K_m$ where the sets K_m, $m \in \mathbb{N}$, are compact and satisfy

$$\forall m_1, m_2 \in \mathbb{N}, \quad m_1 \neq m_2, \qquad \lambda(K_{m_1} \cap K_{m_2}) = 0 \,.$$

Thus

$$\begin{aligned} \limsup_{n \to \infty} \mathrm{ent}(f_n) \,&=\, \limsup_{n \to \infty} \sum_{m \in \mathbb{N}} \int_{K_m} \mathrm{ent}(\nabla f_n) \, d\lambda \\ &\leq\, \sum_{m \in \mathbb{N}} \limsup_{n \to \infty} \int_{K_m} \mathrm{ent}(\nabla f_n) \, d\lambda \,\leq\, \sum_{m \in \mathbb{N}} \int_{K_m} \mathrm{ent}(\nabla f) \, d\lambda \,=\, \mathrm{ent}(f) \,. \end{aligned}$$

3.2. Topology on $\mathcal{E}$. We endow $\mathcal{E}$ with the topology $\mathcal{H}$ of Hausdorff convergence on compact sets, that is, the topology whose basis elements are

$$\Big\{ F \in \mathcal{E} : d_H(F \cap K, E \cap K) < \epsilon \Big\}, \qquad \epsilon > 0 , \quad E \in \mathcal{E} , \quad K \text{ compact subset of } (\mathbb{R}^+)^3$$

where d_H is the Hausdorff metric. For other possible topologies on $\mathcal{E}$, as well as their relationships with the topology $\mathcal{H}$, see [30]. It is likely that our results hold with a finer topology, for instance the topology L^1. If we express the topology $\mathcal{H}$ on the space $\mathrm{Lip}_1(P_{111}, \mathbb{R}^+)$ through the map

$$E \in \mathcal{E} \mapsto f_E \in \mathrm{Lip}_1(P_{111}, \mathbb{R}^+) ,$$

the corresponding functional topology is the topology of uniform convergence over compact subsets of P_{111}. This topology is metrizable: if $(K_n)_{n\in\mathbb{N}}$ is an increasing sequence of compact sets such that $(\mathbb{R}^+)^3 = \bigcup_{n\in\mathbb{N}} K_n$, setting for E, F in $\mathcal{E}$

$$\mathrm{dist}(E,F) \;=\; \sum_{n\in\mathbb{N}} 2^{-n} \min\Big(\frac{d_H(E\cap K_n, F\cap K_n)}{\mathrm{diam}\, K_n}, 1 \Big)$$

we get a metric compatible with this topology.

A standard diagonal argument shows that for any $\alpha > 0$, the subset

$$\mathcal{E}_\alpha \;=\; \{\, E \in \mathcal{E} : \mathcal{L}^3(E) \leq \alpha \,\}$$

is compact. Indeed, let $(E_n)_{n\in\mathbb{N}}$ be a sequence in $\mathcal{E}_\alpha$ and let $(f_{E_n})_{n\in\mathbb{N}}$ be the associated height functions in $\mathrm{Lip}_1(P_{111}, \mathbb{R}^+)$. Since

$$\forall n \in \mathbb{N} \qquad \alpha \;\geq\; \mathcal{L}^3(E_n) \;\geq\; \Big(f_{E_n}(0)/\sqrt{3} \Big)^3$$

the sequence $(f_{E_n}(0))_{n\in\mathbb{N}}$ is bounded. The functions f_{E_n}, $n \in \mathbb{N}$, being Lipschitz, by a diagonal argument, we can extract a subsequence (which we redenote by f_{E_n}) which converges uniformly on any compact subset of P_{111} towards a function f. Let E be the element of $\mathcal{E}$ associated to f. Then the sequence $(E_n)_{n\in\mathbb{N}}$ converges towards E with respect to the topology $\mathcal{H}$ and

$$\mathcal{L}^3(E) \;=\; \int_{P_{111}} |f(x) - \mathrm{axis}(x)| \, d\lambda(x) \;\leq\; \liminf_{n\to\infty} \int_{P_{111}} |f_{E_n}(x) - \mathrm{axis}(x)| \, d\lambda(x) \;\leq\; \alpha \,.$$

The topology $\mathcal{H}$ on $\mathcal{E}$ is identical to the topology $L^1_{\mathrm{loc}}(P_{111})$, hence the entropy ent is upper semicontinuous when $\mathcal{E}$ is endowed with the topology $\mathcal{H}$.

3.3. Large deviation principle. We are now ready to state our large deviation principle. Notice that the rescaled 3D Young diagrams $n^{-1/3}Y_n^=$ and $n^{-1/3}Y_n^\leq$ belong to the space

$$\mathcal{E}_1 \,=\, \{\, E \in \mathcal{E} : \mathcal{L}^3(E) \leq 1 \,\}\,.$$

Theorem 3.2. *The sequences $(n^{-1/3}Y_n^=)_{n\in\mathbb{N}}$ and $(n^{-1/3}Y_n^\leq)_{n\in\mathbb{N}}$ satisfy a large deviation principle in $\mathcal{E}_1$ endowed with the topology $\mathcal{H}$, with speed $n^{2/3}$, governed by the good rate function $\mathcal{I}$ defined by*

$$\forall E \in \mathcal{E}_1 \qquad \mathcal{I}(E) \,=\, \sup_{F\in\mathcal{E}_1} \mathrm{ent}(F) - \mathrm{ent}(E)\,,$$

i.e., for $*$ equal either to $=$ or to $\leq$, for any open subset $\mathcal{O}$ of $(\mathcal{E}_1,\mathcal{H})$,

$$-\inf\{\,\mathcal{I}(E) : E \in\, \mathcal{O}\,\} \,\leq\, \liminf_{n\to\infty} \,\frac{1}{n^{2/3}} \ln P\big(n^{-1/3}Y_n^* \in \mathcal{O}\big)\,, \tag{3}$$

and, for any closed subset $\mathcal{F}$ of $(\mathcal{E}_1,\mathcal{H})$,

$$\limsup_{n\to\infty} \,\frac{1}{n^{2/3}} \ln P\big(n^{-1/3}Y_n^* \in \mathcal{F}\big) \,\leq\, -\inf\{\,\mathcal{I}(E) : E \in \mathcal{F}\,\} \tag{4.}$$

Proof. For $n\in\mathbb{N}$, we denote by $N(n)$ the number of 3D Young diagrams associated to n. We deal first with the large deviations lower bound for $(Y_n^=)_{n\in\mathbb{N}}$. Let E belong to $\mathcal{E}_1$ and let f_E be the associated height function. By density, we can assume that $\mathcal{L}^3(E) < 1$.

For an $M > 0$ we set $E_M = E\cap[0,M]^3$ and $R_M^* = \pi_{111}([0,M]^3)$. There is a one to one correspondance between random lozenge tilings (with step size $n^{-1/3}$) of the region R_M^*, and 3D Young diagrams contained in $[0,M]^3$. Each such 3D Young diagram has an associated height function h which agrees with axis$(\cdot)$ outside of R_M^*. We are interested in 3D Young diagrams whose height function h lies close to f_{E_M}. The volume of the 3D Young diagram with height function h is less than

$$\mathcal{L}^3(E_M) + \lambda(R_M^*) \sup_{x\in R_M^*} |f_{E_M}(x) - h(x)|\,.$$

Now let $\varepsilon > 0$ and take M large enough that

$$\mathrm{ent}(f_{E_M}) \geq \mathrm{ent}(E) - \varepsilon.$$

Since the entropy ent vanishes in the axis directions, we have $\mathcal{I}(E_M) \leq \mathcal{I}(E)$. Since $\mathcal{L}^3(E) < 1$, there exists $\delta > 0$ small enough so that

$$\mathcal{L}^3(E_M) + \delta\lambda(R_M^*) < 1\,.$$

By [10, Theorem 4.3], for n large enough, the number of height functions h over R_M^* corresponding to random tilings such that $\sup_{x\in R_M^*}|f_{E_M}(x)-h(x)|<\delta$ is larger than

$$\exp\Big(n^{2/3}(\mathrm{ent}(f_{E_M})-\epsilon)\Big)\,.$$

Each such height function corresponds to a 3D Young diagram associated to an integer m with $m\leq n$. By adding a column of $n-m$ cubes along the z axis we obtain a 3D Young diagram y_n associated to n which still satisfies $d_H(n^{-1/3}y_n\cap[0,M]^3,E_M)<2\delta$ for n large enough, say $n^{-1/3}<\delta$ (notice that we use here the fact that the axis planes belong to the set E). Thus, using the inequality $\mathrm{ent}(f_{E_M})\geq\mathrm{ent}(E)-\varepsilon$,

$$\frac{1}{n^{2/3}}\ln P\Big(d_H(n^{-1/3}Y_n^{=}\cap[0,M]^3,E_M)<2\delta\Big)\,\geq\,\mathrm{ent}(E)-2\epsilon-\frac{1}{n^{2/3}}\ln N(n)\,. \tag{5}$$

We deal next with the large deviations upper bound for $(Y_n^{\leq})_{n\in\mathbb{N}}$. Let E belong to $\mathcal{E}_1$. For $M>0$, we set $E_M=E\cap[0,M]^3$. Notice that the map

$$t\in\mathbb{R}^+\mapsto\mathcal{H}^2(E\cap\{x=t\})\in\mathbb{R}^+$$

decreases monotonically to 0 as t increases to ∞ (here $\mathcal{H}^2$ is the two dimensional Hausdorff measure in $\mathbb{R}^3$ and $\{x=t\}$ is the plane consisting of the points whose first coordinate is equal to t). Let $\epsilon>0$. We choose M_1 large enough so that $\mathcal{H}^2(E\cap\{x=M_1\})<\epsilon$. We proceed similarly for the two other axis directions to get M_2,M_3. Let also M_4 be such that $\mathcal{I}(E_{M_4})\geq\mathcal{I}(E)-\epsilon$. Let $M=\max(M_1,M_2,M_3,M_4)$. Since E is closed, by the dominated convergence Theorem,

$$\lim_{\delta\to0}\mathcal{H}^2(\{\,y\in[0,M]^3:d(y,E)<\delta\}\cap\{x=M\})\;=\;\mathcal{H}^2(E_M\cap\{x=M\})$$

and there exists $\delta_1>0$ such that

$$\forall F\in\mathcal{E}\qquad d_H(E_M,F\cap[0,M]^3)<\delta_1\quad\Rightarrow\quad\mathcal{H}^2(F\cap(\{M\}\times[0,M]^2))<2\epsilon\,.$$

We proceed similarly for the two other axis directions to get δ_2,δ_3. By [10, Theorem 4.3], there exists $\delta_4>0$ such that, for n sufficiently large, the number of tilings of $R_M^*=\pi_{111}([0,M]^3)$, whose corresponding height function h satisfies

$$\sup_{x\in R_M^*}|f_{E_M}(x)-h(x)|<\,\delta_4$$

is less than $\exp n^{2/3}(\mathrm{ent}(E_M)+\epsilon)$. We set $\delta=\min(\delta_1,\delta_2,\delta_3,\delta_4)$. We have then

$$\begin{gathered}\forall F\in\mathcal{E}\qquad d_H(E_M,F\cap[0,M]^3)<\delta\quad\Rightarrow\quad\mathcal{H}^2(F\cap(\{M\}\times[0,M]^2))<2\epsilon\,,\\ \mathcal{H}^2(F\cap([0,M]\times\{M\}\times[0,M]))<2\epsilon\,,\quad\mathcal{H}^2(F\cap([0,M]^2\times\{M\}))<2\epsilon\,.\end{gathered}$$

We will now compute an upper bound on the number of 3D Young diagrams y_n such that

$$d_H(n^{-1/3}y_n \cap [0,M]^3, E_M) < \delta\,.$$

Let y_n be such a 3D Young diagram. To y_n we associate a height function over R_M^* given by $f_{n^{-1/3}y_n}|R_M^*$ (the restriction of $f_{n^{-1/3}y_n}$ to R_M^*) as well as the three 3D Young diagrams $y_n^{i,M}$, $1 \le i \le 3$, corresponding to $y_n \setminus [0, n^{1/3}M]^3$. More precisely,

$$y_n^{1,M} \;=\; y_n \setminus \Big([0, n^{1/3}M] \times (\mathbb{R}^+)^2\Big) - (n^{1/3}M, 0, 0)$$

and $y_n^{2,M}$, $y_n^{3,M}$ are defined analogously, considering the two other axis directions. The number of possible configurations for $n^{-1/3}y_n \cap [0,M]^3$ is bounded by the number of corresponding tilings, that is, by $\exp \frac{1}{2}n^{2/3}(\mathrm{ent}(E_M)+\epsilon)$. The number of possible configurations for $y_n^{i,M}$, $1 \le i \le 3$, is estimated with the help of the following lemma.

Lemma 3.3. *There exists a constant $c > 0$ such that for any $\epsilon > 0$, any $n \in \mathbb{N}$, the number of 3D Young diagrams associated to an integer less than n, with step size $n^{-1/3}$ and having less than $\epsilon n^{2/3}$ cubes intersecting one of the axis planes is less than* $\exp(c\epsilon^{1/4}n^{2/3})$.

Proof. Recall that the number of ordinary (2D) partitions of n is at most $c^{\sqrt{n}}$ for a constant c. Let $Z_{n,\varepsilon}$ be the set of 3D Young diagrams whose intersection with the xy-plane has area less than $\varepsilon n^{2/3}$. Let $K = \sqrt{\varepsilon}n^{1/3}$. For $Y \in Z_{n,\varepsilon}$, let

$$Y^{(1)} = Y \cap \{(x,y,z) | 0 \le y \le K\}$$

and

$$Y^{(2)} = Y \cap \{(x,y,z) | y \ge K \text{ and } 0 \le x \le K\}.$$

Then $Y = Y^{(1)} \cup Y^{(2)}$ and $Y^{(1)}$ and $Y^{(2)}$ are disjoint. Now $Y^{(1)}$ is made up of the K 2D-Young diagrams

$$Y^{(1)}(i) = \{(x,y,z) \in Y^{(1)} | i \le y < i+1\}\,, \qquad 0 \le i < K$$

and similarly $Y^{(2)}$ is made up of

$$Y^{(2)}(j) = \{(x,y,z) \in Y^{(2)} | j \le x < j+1\} \qquad 0 \le j < K\,.$$

Let M_1 be the volume of $Y^{(1)}$, and let m_i be the volume of $Y^{(1)}(i)$. Given the volumes $m_1, \ldots, m_K$, the number of choices for the $Y^{(1)}(i)$ is at most

$$c^{\sqrt{m_1}+\cdots+\sqrt{m_K}} \le c^{\sqrt{K}\sqrt{M_1}}.$$

The number of choices for the volumes $m_1, \dots, m_K$ is at most the number of partitions of M_1, which is at most $c^{\sqrt{n}}$. Similar bounds hold for $Y^{(2)}$. Therefore the total number of elements of $Z_{n,\varepsilon}$ is at most

$$|Z_{n,\varepsilon}| \le c_2^{\sqrt{n}+\sqrt{Kn}} \le c_3^{\varepsilon^{1/4} n^{2/3}}$$

for a constant c_3. This completes the proof. $\square$

We conclude that: $\forall E \in \mathcal{E}_1 \quad \forall \epsilon > 0 \quad \exists M, \delta > 0 \quad \exists N \quad \forall n > N$

(6)

$$P\big(d_H(n^{-1/3} Y_n^{\le} \cap [0,M]^3, E_M) < \delta\big) \le \Big(\sum_{1 \le k \le n} N(k)\Big)^{-1} \exp n^{2/3}(\mathrm{ent}(E) + 2\epsilon + 3c\epsilon^{1/4})$$

Inequality (5) implies on one hand that

$$\liminf_{n\to\infty} \frac{1}{n^{2/3}} \ln N(n) \ \ge\ \max\{\, \mathrm{ent}(E) : E \in \mathcal{E}_1 \,\}\,.$$

On the other hand, let $\epsilon > 0$; to each set E in $\mathcal{E}_1$ consider the neighborhood

$$\{\, F \in \mathcal{E}_1 : d_H(F \cap [0,M]^3, E_M) < \delta \,\}$$

where M, δ are chosen as in inequality (6). The space $\mathcal{E}_1$ being compact, we can extract from this covering a finite subcover, associated to (E_i, M_i, δ_i), $i \in I$. For n large enough, inequality (6) is satisfied for each set E_i, $i \in I$, hence,

$$\begin{aligned} 1 = P\big(n^{-1/3} Y_n^{\le} \mathcal{E}_1\big) &\le \sum_{i\in I} P\big(d_H(n^{-1/3} Y_n^{\le} \cap [0,M_i]^3, E_{M_i}) < \delta_i\big) \\ &\le \Big(\sum_{1 \le k \le n} N(k)\Big)^{-1} \sum_{i \in I} \exp n^{2/3}(\mathrm{ent}(E_i) + 2\epsilon + 3c\epsilon^{1/4}) \end{aligned}$$

whence

$$\limsup_{n\to\infty} \frac{1}{n^{2/3}} \ln N(n) \ \le\ \max_{i\in I}\ \mathrm{ent}(E_i) + 2\epsilon + 3c\epsilon^{1/4}\,.$$

Sending ϵ to 0, we conclude that

$$\lim_{n\to\infty} \frac{1}{n^{2/3}} \ln N(n) \ =\ \max\{\, \mathrm{ent}(E) : E \in \mathcal{E}_1 \,\}\,. \tag{7}$$

Now the large deviations lower bound (3) for $Y_n^{=}$ follows directly from inequality (5) and equality (7). The large deviations upper bound (4) for $Y_n^{\le}$ follows directly from inequality (6), equality (7) and the compactness of $\mathcal{E}_1$. Because $N(n)$ increases with n, we have $\sum_{1\leq k\leq n} N(k) \leq nN(n)$ and for any $E\in\mathcal{E}_1$, $M,\delta>0$ and $n\in\mathbb{N}$,

$$P\big(d_H(n^{-1/3}Y_n^{=}\cap[0,M]^3,E_M)<\delta\big)\ \leq\ n\,P\big(d_H(n^{-1/3}Y_n^{\leq}\cap[0,M]^3,E_M)<\delta\big)\,.$$

Therefore the large deviations upper bound for $Y_n^{\leq}$ implies large deviations upper bound for $Y_n^{=}$, while the large deviations lower bound for $Y_n^{=}$ implies the large deviations lower bound for $Y_n^{\leq}$. $\square$

Our large deviation principle implies automatically a law of large numbers for the random rescaled 3D Young diagrams.

We recall that

$$\mathcal{S}\ =\ \{\, x\in\mathbb{R}^3 : \forall\nu\in S^2\quad x\cdot\nu\geq|\nu|_1\mathrm{ent}(\nu)\,\}$$

and that the boundary of $\mathcal{S}$ is the surface S_0.

Theorem 3.4. *The set $(\zeta(3)/4)^{-1/3}((\mathbb{R}^+)^3\setminus\mathcal{S})$ is the asymptotic shape of a random rescaled 3D Young diagram: for any $M,\delta>0$, we have*

$$\limsup_{n\to\infty}\frac{1}{n^{2/3}}\ln P\big(d_H(n^{-1/3}Y_n^{=}\cap[0,M]^3,[0,M]^3\setminus(\zeta(3)/4)^{-1/3}\mathcal{S})\geq\delta\big)\ <\ 0\,.$$

Proof. The solution of the variational problem

$$\text{minimize}\quad\mathcal{I}(E)\text{ over }E\in\mathcal{E}_1$$

which is of course equivalent to the problem

$$\text{maximize}\quad\mathrm{ent}(E)\text{ over }E\in\mathcal{E}_1$$

is given by a slight variant of the famous Wulff isoperimetric theorem. The unique solution is the adequate dilation of $((\mathbb{R}^+)^3\setminus\mathcal{S})$ which encloses a volume 1 (see [31]). Since the volume under S_0 is $(\frac{1}{4}\zeta(3))$, it is the set $(\frac{1}{4}\zeta(3))^{-1/3}((\mathbb{R}^+)^3\setminus\mathcal{S})$. The large deviation principle for $Y_n^{=}$ implies that

$$\begin{aligned}\limsup_{n\to\infty}\frac{1}{n^{2/3}}\ln P\big(d_H(n^{-1/3}Y_n^{=}\cap[0,M]^3,[0,M]^3\setminus\mathcal{S})\geq\delta\big)\ \leq\\ -\inf\{\,\mathcal{I}(E):E\in\mathcal{E}_1,\ (d_H(E,[0,M]^3\setminus\mathcal{S})\geq\delta\,\}\end{aligned}$$

and the righthand side is strictly negative. $\square$

4. Preliminaries for Theorem 1.1

In this section we introduce first the notation and we give some basic definitions. In the second part, we recall some basic properties of FK (or random cluster) measures.

4.1. Notation. The cardinality of a set A is denoted by $|A|$. The symmetric difference between two sets A_1, A_2 is denoted by $A_1 \Delta A_2$. We denote by d_p the metric associated with the p-norm, i.e., $d_p(x,y) = |x-y|_p$ for any x, y in $\mathbb{R}^3$. We will only use the $1, 2$ and ∞ norms. The d_p distance between two subsets E_1 and E_2 of $\mathbb{R}^3$ is

$$d_p(E_1, E_2) \,=\, \inf\{\, |x_1 - x_2|_p : x_1 \in E_1,\, x_2 \in E_2 \,\}\,.$$

The r–neighborhood of $E \subset \mathbb{R}^3$ with respect to the d_2 metric is the set

$$\mathcal{V}(E, r) \,=\, \{\, x \in \mathbb{R}^3 : d_2(x, E) < r \,\}\,.$$

We will usually work with the Euclidean distance d_2 on the continuous space $\mathbb{R}^3$ and with the distance d_1 or d_∞ on the discrete lattice $\mathbb{Z}^3$. The unit sphere of $\mathbb{R}^3$ is denoted by S^2. We denote by $\mathcal{H}^2$ the standard 2–dimensional Hausdorff measure.

We turn $\mathbb{Z}^3$ into a graph with vertex set $\mathbb{Z}^3$ and edge set

$$\mathbb{E}^1 = \{\, \{x,y\} : x, y \in \mathbb{Z}^3,\, |x-y|_1 = 1 \,\}\,.$$

This graph is called the three dimensional cubic lattice and is denoted by $\mathbb{L}^1$. Let D be a subset of $\mathbb{R}^3$. An edge $\{x,y\}$ of $\mathbb{E}^1$ is said to be included in D if both sites x, y belong to D. We denote by $\mathbb{E}^1(D)$ the set of the edges of $\mathbb{E}^1$ included in D. For D a subset of $\mathbb{Z}^3$, the graph $(D, \mathbb{E}^1(D))$ will be often identified with its vertex set D. Let A be a subset of $\mathbb{Z}^3$. We define its inner vertex boundary,

$$\partial^{in}_\infty A \,=\, \{\, x \in A : \exists\, y \in A^c \quad d_\infty(x,y) = 1\}.$$

The set $A \subset \mathbb{Z}^3$ is said to be connected or $\mathbb{L}^1$-connected (respectively $\mathbb{L}^\infty$-connected) if any two of its points are connected by a path $x_0, x_1, \ldots, x_n$ of points of A with $d_1(x_i, x_{i+1}) = 1$, $0 \le i < n$ (respectively $d_\infty(x_i, x_{i+1}) = 1$). Note that $\mathbb{L}^1$-connectedness implies $\mathbb{L}^\infty$-connectedness.

The object dual to the edge $e = \{x,y\}$ of $\mathbb{E}^1$ is the unit square orthogonal to e centered at $(x+y)/2$, also called the *plaquette* associated to e. Two edges in $\mathbb{E}^1$ are said to be adjacent if (and only if) their corresponding plaquettes meet along a unit segment. A set of edges $E \subset \mathbb{E}^1$ is said to be connected if any two edges are connected by a path of adjacent edges.

Let A, B, D be subsets of $\mathbb{R}^3$ with $A \cap D \cap B = \emptyset$. A set of edges $E \subset \mathbb{E}^1$ is said to *separate* A and B in D if there is no path in the graph $(\mathbb{Z}^3 \cap D, \mathbb{E}^1(D) \setminus E)$ connecting a vertex of A and a vertex of B. The set E *separates* ∞ *in* D if the graph $(\mathbb{Z}^3 \cap D, \mathbb{E}^1(D) \setminus E)$ has at least two infinite components.

One is naturally lead to work simultaneously with the two metrics $\mathbb{L}^1$ and $\mathbb{L}^\infty$ for topological reasons. Here we will use the following result.

Lemma 4.1. *If A is a $\mathbb{L}^\infty$ connected set of vertices and R is a $\mathbb{L}^1$ connected component of A^c, then $\partial_\infty^{in} R$ is $\mathbb{L}^1$ connected.*

Proof. From R, we construct a three dimensional manifold M with boundary as follows. The manifold M is the ε-neighborhood of the union of those edges whose vertices are both in R and the set of (solid) unit cubes all of whose 8 vertices are in R. The boundary of M consists of closed oriented two dimensional manifolds. Now each boundary component of M divides $\mathbb{R}^3$ into an outside and an inside: this is a classical theorem of algebraic topology (essentially the generalized Schoenflies' Theorem, see [29]). Note that two boundary components of M are at d_∞-distance at least $1 + 2\varepsilon$.

Let x_1, x_2 be two vertices in $\partial_\infty^{in} R$ and x_1', x_2' points of ∂M to which they are closest. We claim that x_1', x_2' are on the same boundary component of M. This is because (using connectedness of A) there is a path in $\mathbb{R}^d$ from x_1' to x_2' which does not pass through the interior of M. Since x_1', x_2' are on the same boundary component of M, there exists a path on this boundary component from x_1' to x_2'. This path can be pushed onto an $\mathbb{L}^1$-path on the underlying edges close to the boundary, whose vertices belong to $\partial_\infty^{in} R$. $\square$

A detailed proof of a similar result has been done by Kesten [22, Lemma 2.23]; see also [11, Lemma 2.1].

4.2. FK percolation. We give here a short account of FK measures; we refer to [18,26] for a more detailed exposition. For $E \subset \mathbb{E}^1$ with $E \neq \varnothing$, we write $\Omega(E)$ for the set $\{0,1\}^E$; its elements are called edge configurations in E. The natural projections are given by $\omega \in \Omega(E) \mapsto \omega(e) \in \{0,1\}$, where $e \in E$. An edge e is called open in the configuration ω if $\omega(e) = 1$, and closed otherwise. For $A \subset \mathbb{Z}^3$, let $\Omega_A = \Omega(\mathbb{E}^1(A))$, the set of the configurations within A (recall that $\mathbb{E}^1(A)$ denotes the set of edges between sites in A). We set also $\Omega = \Omega_{\mathbb{Z}^3}$. Given $\omega \in \Omega$ and $E \subset \mathbb{E}^1$, we denote by $\omega(E)$ the restriction of ω to $\Omega(E)$. Given $\omega \in \Omega$, we denote by $\mathcal{O}(\omega)$ the set of the edges of $\mathbb{E}^1$ which are open in the configuration ω. The connected components of the graph $(\mathbb{Z}^3, \mathcal{O}(\omega))$ are called ω-clusters. The path $\gamma = (x_1, e_1, x_2, \dots)$ is said to be ω-open if all the edges e_i belong to $\mathcal{O}(\omega)$. An edge $e = \{x, y\}$ is said to be wired in the configuration ω if there exists an ω–open path joining the endvertices x, y of e which does not use the edge e itself. Let $\omega \in \Omega$ and $V \subset \mathbb{Z}^3$ be a finite subset of $\mathbb{Z}^3$. The *open clusters in* V are the connected components of the random graph $(V, \mathcal{O}(\omega(\mathbb{E}^1(V))))$. The number of open clusters in V is denoted by $cl(\omega)$. Let $\mathcal{F}_V$ be the σ-field with atoms $\{\omega\}$, $\omega \in \Omega(V)$. For fixed $p \in [0,1]$ and $q \geq 1$, the FK measure with parameters (p,q) is a probability measure $\Phi_V^{p,q}$ on $\mathcal{F}_V$, defined by the formula

$$(8) \qquad \forall \omega \in \Omega_V \qquad \Phi_V^{p,q}[\{\omega\}] = \frac{1}{Z_V^{p,q}} \Big(\prod_{e \in E} p^{\omega(e)} (1-p)^{1-\omega(e)} \Big) q^{cl(\omega)}$$

where $Z_V^{p,q}$ is the appropriate normalization factor.

There exists $p_0(q)$ in $(0,1)$ such that, for any p in $(p_0(q),1)$, the weak limit

$$\Phi^{p,q}_\infty \;=\; \lim_{V\to\mathbb{Z}^3} \Phi^{p,q}_V$$

exists and is the unique FK measure in infinite volume corresponding to the parameters p,q [18, proof of Theorem 5.3]. We will work in this regime throughout the paper. By conditioning on the wiring status of the endvertices of a fixed edge, we obtain the following estimates for the probabilities of this edge to be open or closed:

$$\forall e\in\mathbb{E}^1 \qquad \Phi^{p,q}_\infty(e\text{ is open}) \;\geq\; \frac{p}{p+q(1-p)}\,,\quad \Phi^{p,q}_\infty(e\text{ is closed}) \;\geq\; 1-p\,.$$

There is a partial order $\preceq$ in Ω given by $\omega\preceq\omega'$ if and only if $\omega(e)\leq\omega'(e)$ for every $e\in\mathbb{E}^1$. A function $f:\Omega\to\mathbb{R}$ is called *increasing* if $f(\omega)\leq f(\omega')$ whenever $\omega\preceq\omega'$. An event is called increasing if its characteristic function is increasing. A property of crucial importance is that for $q\geq 1$, $p>p_0(q)$, $\Phi^{p,q}_\infty$ satisfies the FKG inequality, i.e., for all $\mathcal{F}$-measurable bounded increasing functions f,g, we have (see [18])

$$\Phi^{p,q}_\infty(fg) \;\geq\; \Phi^{p,q}_\infty(f)\,\Phi^{p,q}_\infty(g)\,.$$

Lemma 4.2. *Let $p>p_0(q)$. Let F be a fixed finite set of edges and let N be an integer with $N\leq|F|$. Then*

$$\Phi^{p,q}_\infty\big(\textit{the edges of } F \textit{ are closed, at least } N \textit{ edges of } F \textit{ are wired}\big) \;\leq\; \Big(\frac{1-p}{p}\Big)^{|F|} q^{|F|-N}$$

Proof. We denote by $\mathcal{E}$ the event

$$\mathcal{E} \;=\; \big\{\text{the edges of } F \text{ are closed, at least } N \text{ edges of } F \text{ are wired}\big\}\,.$$

Since we work in the region where there is uniqueness of the infinite volume FK measure, we have

$$\Phi^{p,q}_\infty\big(\mathcal{E}\big) \;=\; \lim_{\Lambda\to\mathbb{Z}^3} \Phi^{p,q}_\Lambda\big(\mathcal{E}\big)\,.$$

Let Λ be a box containing all the edges of F. To a configuration η in $\Omega(\Lambda)$, we associate the configuration $\overline{\eta}$ obtained by changing the states of all the edges of F and keeping the remaining edges unchanged. If at least N edges of F are wired in the configuration η, then $cl(\overline{\eta})\geq cl(\eta)+N-|F|$. This follows by induction using the fact that edges in F are closed. Therefore

$$\Phi^{p,q}_\Lambda(\mathcal{E}) \;\leq\; \frac{\displaystyle\sum_{\eta\in\mathcal{E}\cap\Omega(\Lambda)}\Big(\prod_{e\in\mathbb{E}^1(\Lambda)} p^{\eta(e)}(1-p)^{1-\eta(e)}\Big)\,q^{cl(\eta)}}{\displaystyle\sum_{\eta\in\mathcal{E}\cap\Omega(\Lambda)}\Big(\prod_{e\in\mathbb{E}^1(\Lambda)} p^{\overline{\eta}(e)}(1-p)^{1-\overline{\eta}(e)}\Big)\,q^{cl(\overline{\eta})}} \;\leq\; \Big(\frac{1-p}{p}\Big)^{|F|} q^{|F|-N}$$

Letting Λ grow to $\mathbb{Z}^3$, we obtain the inequality stated in the Lemma. $\square$

Lemma 4.3. *Let $p > \max(p_0(q), 1/2)$. Let E, F, G be three finite sets of edges, with $G \subset F$. Then*

$$\Phi^{p,q}_\infty\big(\text{the edges of } E \cup (F \setminus G) \text{ are closed, the edges of } G \text{ are open}\big) \le$$
$$\Big(2q\frac{1-p}{p}\Big)^{|F \setminus G|} \Phi^{p,q}_\infty\big(\text{the edges of } E \text{ are closed, the edges of } F \text{ are open}\big)\,.$$

Proof. We denote by $\mathcal{E}$ and $\mathcal{F}$ the events

$$\begin{aligned}
\mathcal{E} \ &= \ \big\{\,\text{the edges of } E \cup (F \setminus G) \text{ are closed, the edges of } G \text{ are open}\,\big\}\,,\\
\mathcal{F} \ &= \ \big\{\,\text{the edges of } E \text{ are closed, the edges of } F \text{ are open}\,\big\}\,.
\end{aligned}$$

Since we work in the region where there is uniqueness of the infinite volume FK measure, we have

$$\Phi^{p,q}_\infty\big(\mathcal{E}\big) \ = \ \lim_{\Lambda\to\mathbb{Z}^3} \Phi^{p,q}_\Lambda\big(\mathcal{E}\big)\,, \qquad \Phi^{p,q}_\infty\big(\mathcal{F}\big) \ = \ \lim_{\Lambda\to\mathbb{Z}^3} \Phi^{p,q}_\Lambda\big(\mathcal{F}\big)\,.$$

Let Λ be a box containing all the edges of $E \cup F$. To a configuration η in $\Omega(\Lambda)$, we associate the configuration $\overline{\eta}$ obtained by opening all the edges of $F \setminus G$ and keeping the remaining edges unchanged. We have $cl(\overline{\eta}) \ge cl(\eta) - |F \setminus G|$. Therefore

$$\begin{aligned}
\Phi^{p,q}_\Lambda(\mathcal{E}) \ &= \ \frac{1}{Z^{p,q}_\Lambda} \sum_{\eta\in\mathcal{E}\cap\Omega(\Lambda)} \Big(\prod_{e\in\mathbb{E}^1(\Lambda)} p^{\eta(e)}(1-p)^{1-\eta(e)}\Big)\, q^{cl(\eta)}\\
&= \ \frac{1}{Z^{p,q}_\Lambda} \sum_{\rho\in\mathcal{F}\cap\Omega(\Lambda)} \ \sum_{\eta\in\mathcal{E}\cap\Omega(\Lambda),\,\overline{\eta}=\rho} \Big(\prod_{e\in\mathbb{E}^1(\Lambda)} p^{\eta(e)}(1-p)^{1-\eta(e)}\Big)\, q^{cl(\eta)}\\
&\le \ \frac{1}{Z^{p,q}_\Lambda} \sum_{\rho\in\mathcal{F}\cap\Omega(\Lambda)} \Big(2q\frac{1-p}{p}\Big)^{|F \setminus G|} \Big(\prod_{e\in\mathbb{E}^1(\Lambda)} p^{\rho(e)}(1-p)^{1-\rho(e)}\Big)\, q^{cl(\rho)}\\
&\le \ \Big(2q\frac{1-p}{p}\Big)^{|F \setminus G|} \Phi^{p,q}_\Lambda(\mathcal{F})\,.
\end{aligned}$$

Letting Λ grow to $\mathbb{Z}^3$, we obtain the inequality stated in the Lemma. $\square$

5. Proof of Theorem 1.1

Let ν belong to S^2 and let A be a unit square orthogonal to ν. Let $\mathrm{cyl}\, A$ be the cylinder $A + \mathbb{R}\nu$. Let n belong to $\mathbb{N}$. We define $\mathcal{E}(n, A, \nu)$ as the collection of all the subsets $E \subset \mathbb{E}^1$ such that

- E is a finite connected subset of $n\mathrm{cyl}\, A$.
- E separates $n\mathrm{cyl}\, A$ in at least two unbounded components.

• The edges of E at distance less than 6 from $\partial n\mathrm{cyl}\, A$ are at distance less than 6 from nA.
We define next the event

$$W(n, A, \nu) \,=\, \{\, \text{there exists } E \text{ in } \mathcal{E}(n, A, \nu) \text{ such that all the edges of } E \text{ are closed} \,\}\,.$$

The only difference between the collections of edges in $\mathcal{E}(n, A, \nu)$ and those realizing the event appearing in the definition of the surface tension is the additional connectedness constraint. The next lemma shows however that this constraint is irrelevant to compute the surface tension.

Lemma 5.1. *Let A be a square in $\mathbb{R}^3$. Let E be a finite set of edges which separates ∞ in cyl A and such that the edges of E at distance less than 6 from ∂cyl A are at distance less than 6 from A. Then there exists a connected subset E^* of E which separates ∞ in cyl A.*

Proof. Let Y be the union of plaquettes associated to edges in E. Note that this set disconnects $\mathrm{cyl}\, A \setminus \big(\mathcal{V}(\partial \mathrm{cyl}\, A, 6) \cap \mathcal{V}(A, 6)\big)$ in a topological sense (no continuous path joins the top to the bottom). This is because any continuous path avoiding a set of plaquettes can be pushed onto a lattice path avoiding the same set of plaquettes. Let X be the set $X = \mathcal{V}(Y, \varepsilon) \cup \big(\mathcal{V}(\partial \mathrm{cyl}\, A, 6) \cap \mathcal{V}(A, 6)\big)$. We locally modify X at each vertex as follows: let x be a vertex of a plaquette of Y. If Y contains several plaquettes with vertex x which are not connected locally (in the sense of not being connected via a chain of adjacent plaquettes containing x), separate X near x so that locally, different components of Y correspond to different components of X: one can do this by pushing each local component of X slightly off of x. Note that this does not destroy the property of X of disconnecting $\mathrm{cyl}\, A \setminus \big(\mathcal{V}(\partial \mathrm{cyl}\, A, 6) \cap \mathcal{V}(A, 6)\big)$.

Now ∂X consists of closed two-manifolds and by the generalized Schoenflies theorem [29], at least one component C of $\partial X \cap \mathrm{cyl}\, A$ disconnects $\mathrm{cyl}\, A \setminus \big(\mathcal{V}(\partial \mathrm{cyl}\, A, 6) \cap \mathcal{V}(A, 6)\big)$. Let E^* be the set of edges corresponding to plaquettes in the ε-neighborhood of C. Then E^* is connected (in the sense that any two plaquettes are connected by a path of adjacent plaquettes) and separates ∞ in cyl A. $\square$

Corollary 5.2. *We have*

$$\tau(\nu, p) \,=\, \lim_{n\to\infty} -\frac{1}{n^2} \ln \Phi^{p,q}_{\infty}(W(n, A, \nu))\,.$$

We define $f(n, A, \nu)$ to be the minimal number of edges of a set in $\mathcal{E}(n, A, \nu)$, that is,

$$f(n, A, \nu) \,=\, \min \{\, |E| : E \in \mathcal{E}(n, A, \nu) \,\}$$

and $\phi(n, A, \nu)$ to be the number of elements in $\mathcal{E}(n, A, \nu)$ having minimal cardinality, i.e.,

$$\phi(n, A, \nu) \,=\, \big|\, \{\, E \in \mathcal{E}(n, A, \nu) : |E| = f(n, A, \nu) \,\} \,\big|\,.$$

Lemma 5.3. *For any ν in S^2, any unit square A orthogonal to ν, we have*

$$\lim_{n\to\infty}\frac{1}{n^2}f(n,A,\nu)\;=\;|\nu|_1\,,$$
$$\lim_{n\to\infty}\frac{1}{n^2|\nu|_1}\ln\phi(n,A,\nu)\;=\;\mathrm{ent}(\nu)$$

where $\mathrm{ent}(\nu)$ *is the function defined in Theorem 1.1.*

Proof. For the first statement, note that the projection of A onto the xy-axis has area $|\nu_z|$, the absolute value of the z-component of ν. Similarly the projections onto the xz and yz axes have areas $|\nu_y|$ and $|\nu_x|$ respectively. Replacing edges in E by their plaquettes, we clearly need at least $n^2|\nu_z|+O(n)$ plaquettes of type xy, $n^2|\nu_y|+O(n)$ plaquettes of type xz, and $n^2|\nu_x|+O(n)$ plaquettes of type yz for E to disconnect $n\mathrm{cyl}\,A$. It is also easy to see that these are enough: take the set X of unit lattice cubes intersecting nA; the upper boundary of X is an element of $\mathcal{E}(n,A,\nu)$ with the required number of plaquettes.

For the second statement, see [10, Theorem 4.1]. In particular setting $d=0$ in the entropy formula of [10] yields the desired entropy formula for lozenge tilings. $\square$

Let now $q\geq 1$ be fixed and let p belong to $(p_0(q),1]$. We set

$$\Delta(n,p)\;=\;\frac{1}{n^2}\ln\Phi^{p,q}_{\infty}(W(n,A,\nu))-\frac{1}{n^2}f(n,A,\nu)\ln(1-p)\,.$$

Lemma 5.3 and the definition of the surface tension imply immediately that

$$\lim_{n\to\infty}\Delta(n,p)\;=\;-\tau(\nu,p)-|\nu|_1\ln(1-p)\,.$$

Thus the asymptotic expansion of Theorem 1.1 is equivalent to saying that

$$\lim_{\substack{p\to1\\p<1}}\;\lim_{n\to\infty}\;\Delta(n,p)\;=\;|\nu|_1\,\mathrm{ent}(\nu)\,.$$

We will prove the even stronger statement

$$\lim_{\substack{(n,p)\to(\infty,1)\\n<\infty,p<1}}\Delta(n,p)\;=\;|\nu|_1\,\mathrm{ent}(\nu)\,.$$

To this end, we study separately the infimum and the supremum limits of $\Delta(n,p)$.

Lemma 5.4. *For any ν in S^2, any unit square A orthogonal to ν, we have*

$$\liminf_{\substack{(n,p)\to(\infty,1)\\ n<\infty,p<1}} \quad \Delta(n,p) \;\geq\; |\nu|_1 \,\mathrm{ent}(\nu)\,.$$

Proof. Let $D(n,A,\nu)$ be the set of the edges which are at distance less than 6 from the boundary of $n\mathrm{cyl}\,A$ and at distance less than 6 from nA. Let $E\in\mathcal{E}(n,A,\nu)$. Let F be the set of edges which share a vertex with an edge of E but which are not in E. Let $W(E)$ be the event: all the edges of E are closed and all the edges in

$$E^* \;=\; (n\mathrm{cyl}\,A)\cap\big[(D(n,A,\nu)\setminus E)\cup F\big]$$

are open. Whenever $W(E)$ occurs, the set E is the unique element of $\mathcal{E}(n,A,\nu)$ realizing the event $W(n,A,\nu)$. Therefore the events

$$W(E),\quad E\in\mathcal{E}(n,A,\nu),\quad |E|=f(n,A,\nu)$$

for different E's are pairwise disjoint and

$$\Phi^{p,q}_\infty(W(n,A,\nu))\;\geq\;\sum_{\substack{E\in\mathcal{E}(n,A,\nu)\\ |E|=f(n,A,\nu)}}\Phi^{p,q}_\infty(W(E))\,.$$

We fix now a set of edges E in $\mathcal{E}(n,A,\nu)$ such that $|E|=f(n,A,\nu)$ and we compute a lower bound on $\Phi^{p,q}_\infty(W(E))$ as follows:

$$\Phi^{p,q}_\infty(W(E))\;=\;\Phi^{p,q}_\infty(\text{the edges of } E \text{ are closed, the edges of } E^* \text{ are open})\;=$$
$$\Phi^{p,q}_\infty(\text{the edges of } E \text{ are closed}\,|\,\text{the edges of } E^* \text{ are open})\,\Phi^{p,q}_\infty(\text{the edges of } E^* \text{ are open})\,.$$

There exists a constant c such that $|D(n,A,\nu)|\leq cn$, hence

$$|E^*|\leq cn+5|E|\leq cn+5f(n,A,\nu)$$

and, using the FKG inequality with the lower bound on the probability of an edge to be open,

$$\Phi^{p,q}_\infty(\text{the edges of } E^* \text{ are open})\geq\prod_{e\in E^*}\Phi^{p,q}_\infty(e\text{ open})\geq\Big(\frac{p}{p+q(1-p)}\Big)^{cn+5f(n,A,\nu)}.$$

On the other hand, if we define

$$\partial^{vert}E\;=\;\{\,x\in\mathbb{Z}^3:\exists y,z\in\mathbb{Z}^3,\ \{x,y\}\in\mathbb{E}^1\setminus E,\ \{x,z\}\in E\,\}$$

we have

$\Phi^{p,q}_\infty$(the edges of E are closed | the edges of E^* are open) $\geq$

$\Phi^{p,q}_\infty$(the edges of E are closed | the vertices of $\partial^{\,vert} E$ are wired) $\geq (1-p)^{f(n,A,\nu)}$

so that

$$\Phi^{p,q}_\infty(W(E)) \,\geq\, (1-p)^{f(n,A,\nu)} \Big(\frac{p}{p+q(1-p)}\Big)^{cn+5f(n,A,\nu)}\,.$$

Plugging this estimate in the initial sum, we get

$$\Phi^{p,q}_\infty(W(n,A,\nu)) \,\geq\, \phi(n,A,\nu)(1-p)^{f(n,A,\nu)} \Big(\frac{p}{p+q(1-p)}\Big)^{cn+5f(n,A,\nu)}$$

and for $n\in\mathbb{N}$, $p\in(p_0(q),1]$,

$$\Delta(n,p) \,\geq\, \frac{1}{n^2}\ln\phi(n,A,\nu) + \big(\frac{c}{n}+\frac{5}{n^2}f(n,A,\nu)\big)\ln\Big(\frac{p}{p+q(1-p)}\Big)$$

Sending (n,p) to $(\infty,1)$ and using Lemma 5.3, we obtain the claim of Lemma 5.4. $\square$

We turn now to the study of the supremum limit of $\Delta(n,p)$. We consider the general case where ν belongs to the positive orthant

$$O_+ = \{\,(x,y,z)\in(\mathbb{R}^+)^3,\ x+y+z>0\,\}\,.$$

The first step is to reduce the problem to collections of edges which are close to the minimal ones. To this end, we relax the definitions with the help of an additional parameter α representing the allowed fraction of additional edges. For α in $[0,\infty]$ (the values 0 and ∞ are not excluded), we define successively

$$\begin{aligned}\mathcal{E}(n,A,\nu,\alpha) \,&=\, \{\,E\in\mathcal{E}(n,A,\nu): |E|\leq f(n,A,\nu)+\alpha n^2\,\}\,,\\ \phi(n,A,\nu,\alpha) \,&=\, |\mathcal{E}(n,A,\nu,\alpha)|\,.\end{aligned}$$

We define the event

$$W(n,A,\nu,\alpha) \,=\, \left\{\begin{array}{l}\text{there exists } E \text{ in } \mathcal{E}(n,A,\nu,\alpha) \text{ such}\\ \text{that all the edges of } E \text{ are closed}\end{array}\right\}$$

For the particular values $\alpha=0,\infty$, the previous definitions yield

$$\mathcal{E}(n,A,\nu,\infty)=\mathcal{E}(n,A,\nu)\,,\quad W(n,A,\nu,\infty)=W(n,A,\nu)\,,\quad \phi(n,A,\nu,0)=\phi(n,A,\nu)\,.$$

We set finally

$$\Delta(n,p,\alpha) \,=\, \frac{1}{n^2}\ln\Phi^{p,q}_\infty(W(n,A,\nu,\alpha)) - \frac{1}{n^2}f(n,A,\nu)\ln(1-p)\,.$$

We first show that the study can be reduced to $\Delta(n,p,\alpha)$.

Lemma 5.5. *For any positive α, we have*

$$\limsup_{\substack{(n,p)\to(\infty,1)\\ n<\infty,p<1}} \Delta(n,p,\alpha) \;=\; \limsup_{\substack{(n,p)\to(\infty,1)\\ n<\infty,p<1}} \Delta(n,p)\,.$$

Proof. Clearly, for any $\alpha>0$, any $n\in\mathbb{N}$ and $p<1$, we have $\Delta(n,p,\alpha)\le\Delta(n,p)$ and therefore

$$\limsup_{\substack{(n,p)\to(\infty,1)\\ n<\infty,p<1}} \Delta(n,p,\alpha) \;\le\; \limsup_{\substack{(n,p)\to(\infty,1)\\ n<\infty,p<1}} \Delta(n,p)\,.$$

Let us prove the converse inequality. There exists a constant $c_0\ge 1$ such that, for any $m\in\mathbb{N}$,

$$(9)\quad \Big|\Big\{\,F\subset\mathbb{E}^1 : F \text{ connected}, |F|=m,\ F \text{ contains an edge with } 0 \text{ as vertex}\Big\}\Big| \;\le\; c_0^m\,.$$

Let $D(n,A,\nu)$ be the set of the edges which are at distance less than 6 from the boundary of $n\operatorname{cyl}A$ and at distance less than 6 from nA. We have

$$\begin{aligned}
&\Phi^{p,q}_\infty(W(n,A,\nu))-\Phi^{p,q}_\infty(W(n,A,\nu,\alpha))\\
&\qquad\le\ \Phi^{p,q}_\infty\left(\begin{array}{c}\text{there is a connected set } E \text{ of closed edges such that}\\ E\cap D(n,A,\nu)\neq\varnothing \text{ and } |E|\ge f(n,A,\nu)+\alpha n^2\end{array}\right)\\
&\qquad\le\ |D(n,A,\nu)|\sum_{m\ge f(n,A,\nu)+\alpha n^2}\ \sum_F\ \Phi^{p,q}_\infty\big(\text{all the edges of } F \text{ are closed}\big)
\end{aligned}$$

where the last summation extends over all connected sets of edges F such that $|F|=m$ and F contains an edge with 0 as endvertex. Using (9) and applying Lemma 4.2, we get

$$\Phi^{p,q}_\infty(W(n,A,\nu))-\Phi^{p,q}_\infty(W(n,A,\nu,\alpha))\ \le\ |D(n,A,\nu)|\sum_{m\ge f(n,A,\nu)+\alpha n^2}\Big(c_0\frac{q(1-p)}{p}\Big)^m\,.$$

There exists a constant c_1 such that $|D(n,A,\nu)|\le c_1n^2$. For p sufficiently close to 1, so that $c_0q(1-p)/p<1/2$, we have thus

$$\Phi^{p,q}_\infty(W(n,A,\nu))-\Phi^{p,q}_\infty(W(n,A,\nu,\alpha))\ \le\ 2c_1n^2\Big(c_0\frac{q(1-p)}{p}\Big)^{f(n,A,\nu)+\alpha n^2}\,.$$

Using the inequality $\Phi^{p,q}_\infty(W(n,A,\nu,\alpha))\ge(1-p)^{f(n,A,\nu)}$, we obtain

$$\begin{aligned}
\Delta(n,p)-\Delta(n,p,\alpha)\ &=\ \frac{1}{n^2}\ln\Big(1+\frac{\Phi^{p,q}_\infty(W(n,A,\nu))-\Phi^{p,q}_\infty(W(n,A,\nu,\alpha))}{\Phi^{p,q}_\infty(W(n,A,\nu,\alpha))}\Big)\\
&\le\ 2c_1\Big(c_0\frac{q(1-p)}{p}\Big)^{f(n,A,\nu)+\alpha n^2}(1-p)^{-f(n,A,\nu)}\,.
\end{aligned}$$

There exists n_1 such that $f(n,A,\nu)\leq 2n^2|\nu|_1$ for $n\geq n_1$ and

$$\Delta(n,p)-\Delta(n,p,\alpha)\ \leq\ 2c_1\bigg(\Big(c_0\frac{q}{p}\Big)^{2|\nu|_1}\Big(c_0\frac{q(1-p)}{p}\Big)^{\alpha}\bigg)^{n^2}\,.$$

Let $p_1>p_0(q)$ be such that

$$\Big(c_0\frac{q}{p_1}\Big)^{2|\nu|_1}\Big(c_0\frac{q(1-p_1)}{p_1}\Big)^{\min(\alpha,1)}\ <\ \frac{1}{2}\,.$$

For any $n\geq n_1$, any p in $(p_1,1)$, we have

$$\Delta(n,p)-\Delta(n,p,\alpha)\ \leq\ 2c_1 2^{-n^2}$$

which implies the desired inequality on the supremum limits. $\square$

We next estimate the supremum limit of $\Delta(n,p,\alpha)$ with the help of $\phi(n,A,\nu,\alpha)$.

Lemma 5.6. *For any positive α, we have*

$$\limsup_{\substack{(n,p)\to(\infty,1)\\ n<\infty,p<1}}\ \Delta(n,p,\alpha)\ \leq\ \limsup_{n\to\infty}\frac{1}{n^2}\ln\phi(n,A,\nu,\alpha)\,+\,3\alpha\ln q\,.$$

Proof. Using the symmetry of the lattice, we need only to consider vectors ν whose three coordinates are non–negative. To avoid unessential discussions, we suppose also that all three coordinates of ν are strictly positive. Let as usual A be a unit square orthogonal to ν; for simplicity we suppose that A is centered at the origin. The main technical problem to get the correct upper bound on $\Delta(n,p,\alpha)$ is to show that, whenever p is close to 1 and the event $W(n,A,\nu,\alpha)$ occurs, most of the closed edges realizing the event have their endvertices wired. Let $D(n,A,\nu)$ be the set of the edges which are at distance less than 6 from the boundary of $n\,\mathrm{cyl}\,A$ and at distance less than 6 from nA. Let us define

$$P_1(n,A)\ =\\ \big\{\,(x,y)\in\mathbb{Z}\times\mathbb{Z}:\exists\, z\in\mathbb{Z}\quad d_2((x,y,z),nA)\leq 1,\,\forall z\in\mathbb{Z}\quad d_2((x,y,z),D(n,A,\nu))\geq 2\,\big\}$$

and for E a set of edges and x,y in $\mathbb{Z}\times\mathbb{Z}$,

$$\pi_1(E,x,y)\ =\ \big\{\,e\in E:\exists\, z\in\mathbb{Z}\quad e=\{(x,y,z),(x,y,z+1)\}\,\big\}\,.$$

Let (x,y) belong to $P_1(n,A)$. Then the set of edges $\{(x,y,z),(x,y,z+1)\}$, $z\in\mathbb{Z}$, contains a finite path of edges linking the two connected components of

$$\{\, w\in\mathbb{R}^3 : d_2(w,n\partial\mathrm{cyl}\,A) < 6 \leq d_2(w,nA)\,\}\,.$$

Let E be a set of edges in $\mathcal{E}(n,A,\nu)$. Since by definition E contains no edge having an endvertex in the above set, necessarily at least one edge of the previous path belongs to E. We define next

$$T_1(E)\,=\,\bigcup_{(x,y)\in P_1(n,A)}\pi_1(E,x,y)\,,\qquad T_1^*(E)\,=\,\bigcup_{\substack{(x,y)\in P_1(n,A)\\ |\pi_1(E,x,y)|=1}}\pi_1(E,x,y)\,.$$

We define the analogous quantities related to the two other directions parallel to the axis, $P_i(n,A)$, $\pi_i(E,x,y)$, $T_i(E)$, $T_i^*(E)$, $1\leq i\leq 3$.

We proceed now to estimating $\Phi^{p,q}_\infty(W(n,A,\nu,\alpha))$. We write

$$\Phi^{p,q}_\infty(W(n,A,\nu,\alpha))\,\leq\sum_{E\in\mathcal{E}(n,A,\nu,\alpha)}\Phi^{p,q}_\infty\big(\text{the edges of } E \text{ are closed}\big)\,. \tag{10}$$

Let $V(E)$ be the set of the vertices belonging to an edge of E. Since E is connected, then $V(E)$ is $\mathbb{L}^1$–connected and therefore $\mathbb{L}^\infty$–connected. Let R be the unbounded $\mathbb{L}^1$ component of $V(E)^c$. By Lemma 4.1, $\partial^{\,in}_\infty R$ is $\mathbb{L}^1$ connected. Let F be the edges having an endvertex in $\partial^{\,in}_\infty R$. Let us denote by $\mathcal{E}$ the event

$$\mathcal{E}\,=\,\{\,\text{the edges of } E \text{ are closed, the edges of } F \text{ are open}\}\,.$$

We have, using Lemma 4.3,

$$\begin{aligned}
&\Phi^{p,q}_\infty\big(\text{the edges of } E \text{ are closed}\big)\\
&\quad\leq\sum_{G\subset F}\Phi^{p,q}_\infty\big(\text{the edges of } E\cup(F\setminus G) \text{ are closed, the edges of } G \text{ are open}\big)\\
&\quad\leq\sum_{G\subset F}\Big(2q\frac{1-p}{p}\Big)^{|F\setminus G|}\Phi^{p,q}_\infty(\mathcal{E})\,\leq\sum_{0\leq N\leq|F|}\ \sum_{\substack{G\subset F\\ |G|=N}}\Big(2q\frac{1-p}{p}\Big)^{|F|-N}\Phi^{p,q}_\infty(\mathcal{E})\\
&\quad\leq\sum_{N\leq|F|}\binom{|F|}{N}\Big(2q\frac{1-p}{p}\Big)^{|F|-N}\Phi^{p,q}_\infty(\mathcal{E})\,=\,\Big(1+2\frac{q}{p}(1-p)\Big)^{|F|}\Phi^{p,q}_\infty(\mathcal{E})\,. \qquad (11)
\end{aligned}$$

Yet $|E|\leq f(n,A,\nu)+\alpha n^2$, thus for n large enough, $|V(E)|\leq 2(2|\nu|_1+\alpha)n^2$ and $|F|\leq c_1n^2$ with some positive constant c_1 so that

$$\Phi^{p,q}_\infty\big(\text{the edges of } E \text{ are closed}\big)\,\leq\,\Big(1+2\frac{q}{p}(1-p)\Big)^{c_1n^2}\Phi^{p,q}_\infty(\mathcal{E})\,.$$

On the event $\mathcal{E}$, all the edges of F are open, hence all the vertices of $\partial_\infty^{in} R$ are connected by open paths. Moreover, the endvertices of any edge in $T_1^*(E)\cup T_2^*(E)\cup T_3^*(E)$ are connected by an open edge of F to a vertex of $\partial_\infty^{in} R$. In conclusion, on the event $\mathcal{E}$, all the edges in $T_1^*(E)\cup T_2^*(E)\cup T_3^*(E)$ have their endvertices wired. Whenever $(x,y)\in P_1(n,A)$ and $|\pi_1(E,x,y)|\neq 1$, the line $\{(x,y,z): z\in\mathbb{Z}\}$ contains at least two edges of E, therefore

$$\sum_{1\leq i\leq 3} |T_i^*(E)| + 2(|P_i(n,A)| - |T_i^*(E)|) \,\leq\, |E|$$

whence

$$\sum_{1\leq i\leq 3} |T_i^*(E)| \,\geq\, 2\sum_{1\leq i\leq 3} |P_i(n,A)| - |E| \,\geq\, f(n,A,\nu) - 2\alpha n^2\,,$$

the last inequality being valid for n large enough, since

$$\lim_{n\to\infty} \frac{1}{n^2} \sum_{1\leq i\leq 3} |P_i(n,A)| \,=\, |\nu|_1\,.$$

Thus at least $f(n,A,\nu) - 2\alpha n^2$ edges of E have their endvertices wired together. By Lemma 4.2, we have

$$\begin{aligned}
&\Phi_\infty^{p,q}(\mathcal{E})\\
&\leq \Phi_\infty^{p,q}\big(\text{the edges of } E \text{ are closed, at least } f(n,A,\nu) - 2\alpha n^2 \text{ edges of } E \text{ are wired}\big)\\
&\qquad\qquad\leq \Big(\frac{1-p}{p}\Big)^{f(n,A,\nu)} q^{3\alpha n^2}\,.
\end{aligned}$$

Plugging this estimate in (10) and (11), we obtain

$$\Phi_\infty^{p,q}(W(n,A,\nu,\alpha)) \,\leq\, \phi(n,A,\nu,\alpha)\Big(1 + 2\frac{q}{p}(1-p)\Big)^{c_1 n^2} \Big(\frac{1-p}{p}\Big)^{f(n,A,\nu)} q^{3\alpha n^2}$$

whence

$$\Delta(n,p,\alpha) \,\leq\, \frac{1}{n^2}\ln\phi(n,A,\nu,\alpha) + c_1\ln\big(1 + 2q(1-p)/p\big) + 3\alpha\ln q - \frac{f(n,A,\nu)}{n^2}\ln p$$

Taking the supremum limit as $(n,p)\to(\infty,1)$ yields the desired result. $\square$

We finally prove that the entropy is continuous with respect to α at $\alpha = 0$.

Lemma 5.7. *For any ν in S^2, we have*

$$\lim_{\substack{\alpha\to 0\\ \alpha>0}}\ \limsup_{n\to\infty}\ \frac{1}{n^2}\ln\phi(n,A,\nu,\alpha)\ =\ |\nu|_1\mathrm{ent}(\nu)\,.$$

Proof. Using the symmetry of the lattice, we need only to consider vectors ν whose three coordinates are non–negative. To avoid unessential discussions, we suppose also that all three coordinates of ν are strictly positive. Let as usual A be a unit square orthogonal to ν; for simplicity we suppose that A is centered at the origin. Let P_{111} be the plane containing the origin and orthogonal to the vector $(1,1,1)$, i.e.,

$$P_{111}\ =\ \{\,(x,y,z)\in\mathbb{R}^3 : x+y+z=0\,\}\,.$$

Let also π_{111} be the projection on P_{111} parallel to the direction $(1,1,1)$. Let D be the parallelogram $D=\pi_{111}(nA)$. Let k be an integer. We tile D with k^2 translates of D/k, which we denote by D_i, $1\le i\le k^2$. Let $E\in\mathcal{E}(n,A,\nu,\alpha)$. For n sufficiently large, we have $|\nu|_1n^2-\alpha n^2\le f(n,A,\nu)\le|\nu|_1n^2+\alpha n^2$ hence $|\nu|_1n^2-\alpha n^2\le|E|\le|\nu|_1n^2+2\alpha n^2$. Let $E^*\in\mathcal{E}(n,A,\nu)$ be such that $|E^*|=f(n,A,\nu)$. Let a^*,b^*,c^* (respectively a,b,c) be the number of the edges of E^* (respectively E) parallel to the first, second and third axis respectively. We have

$$\max(|a-a^*|,|b-b^*|,|c-c^*|)\ \le\ 3\alpha n^2\,,$$
$$\lim_{n\to\infty}\frac{1}{n^2}(a^*,b^*,c^*)\ =\ \nu\,.$$

The plaquette associated to an edge e is denoted by $p(e)$. We say that a parallelogram D_i is good if the π_{111} projection of the collection of the plaquettes associated to E above D_i is one to one in the following sense:

$$\forall e_1,e_2\in E,\quad e_1\ne e_2,\quad \mathcal{H}^2(\pi_{111}(p(e_1))\cap\pi_{111}(p(e_2))\cap D_i)=0$$

where $\mathcal{H}^2$ is the two dimensional Hausdorff measure. We denote by $I(E)$ the set of the indices of the good parallelograms. The area of the π_{111} projection is the same for the three types of plaquettes; call this area H. Since E belongs to $\mathcal{E}(n,A,\nu,\alpha)$, we have

$$\mathcal{H}^2\big(\pi_{111}\big(\bigcup_{e\in E}p(e)\big)\big)\ge n^2|\nu|_1H-O(n)$$

so that for n large enough, the number of bad parallelograms is less than $3\alpha n^2$ and therefore $|I(E)|\ge k^2-3\alpha n^2$. If B is a subset of P_{111}, we say that an edge e is above B if $\pi_{111}(p(e))\cap B\ne\varnothing$. Let $\mathcal{B}(E)$ be the edges of E which are above good parallelograms, i.e.,

$$\mathcal{B}(E)\ =\ \big\{\,e\in E:\exists\, i\in I(E)\quad \pi_{111}(p(e))\cap D_i\ne\varnothing\,\big\}\,.$$

Let next $\mathcal{F}(E)$ be the edges of E which are above the boundaries of the parallelograms D_i, $1 \leq i \leq k^2$, i.e.,

$$\mathcal{F}(E) = \big\{ e \in E : \exists\, i \in \{1, \cdots, k^2\} \quad \pi_{111}(p(e)) \cap \partial D_i \neq \varnothing \big\} .$$

We have $|\mathcal{F}(E)| = k^2 O(n/k) = O(kn)$. Let finally $\mathcal{M}(E) = (E \backslash \mathcal{B}(E)) \cup \mathcal{F}(E)$. Let i belong to $I(E)$. The edges of E which are above D_i cut the cylinder of basis D_i and direction ν in two infinite components and it has thus cardinality larger than $|\nu|_1 n^2/k^2 - O(n/k)$. Therefore

$$|\mathcal{B}(E)| \,\geq\, (|\nu|_1(n^2/k^2) - O(n/k))|I(E)| - 4|\mathcal{F}(E)| \,\geq\, |\nu|_1 n^2 - O(nk) - |\nu|_1 3\alpha n^4/k^2$$

and

$$|\mathcal{M}(E)| \,\leq\, 2\alpha n^2 + O(nk) + |\nu|_1 3\alpha n^4/k^2 .$$

Since $\mathcal{M}(E)$ is a connected set of edges intersecting $D(n, A, \nu)$, the total number of possible configurations for $\mathcal{M}(E)$ is less than

$$\exp\Big(O\big(2\alpha n^2 + O(nk) + |\nu|_1 3\alpha n^4/k^2\big)\Big) .$$

We next estimate the number of possible configurations for $\mathcal{B}(E)$ once $\mathcal{M}(E)$ is given. For each i in $I(E)$, let $\mathcal{B}_i$ be the edges of $\mathcal{B}(E)$ which are above D_i and let a_i, b_i, c_i be the number of the edges of $\mathcal{B}_i$ parallel to the first, second and third axis respectively. The number of possible choices for $\mathcal{B}_i$ corresponding to a fixed value of a_i, b_i, c_i is estimated with the help of the following lemma.

Lemma 5.8. *Let R be a parallelogram in P_{111}. Let a, b, c belong to $\mathbb{N}$. Let $\mathcal{B}(a, b, c, R)$ be the set of the collections $\mathcal{B}$ of edges above R such that*
• *$\mathcal{B}$ is connected and at least one edge of $\mathcal{B}$ is at distance less than one from P_{111}*
• *the π_{111} projection of the collection of the plaquettes associated to $\mathcal{B}$ is one to one and covers R*
• *the number of the edges of $\mathcal{B}$ parallel to the first, second and third axis are equal to a, b, c.*
For any $\varepsilon > 0$, there exists $n(\varepsilon)$ such that:

$$\forall n \geq n(\varepsilon) \quad \forall (a, b, c) \in \mathbb{N}^3 \qquad |\mathcal{B}(a, b, c, nR)| \,\leq\, \exp\big((a + b + c)(\mathrm{ent}(a, b, c) + \varepsilon)\big) .$$

Remark. Notice that whenever $\mathcal{B}(a, b, c, nR)$ is not empty, then $a + b + c$ is of order $|\nu|_1 n^2$.

Proof. This follows from [10, Theorem 1.1]: partition $\mathcal{B}(a, b, c, nR)$ into sets having the same boundary configurations (the edges which project to an $O(1)$ neighborhood of ∂nR). The size of each element of the partition is determined by the entropy formula of [10, Theorem 1.1]. The boundary configuration of largest entropy is the one whose boundary edges approximate a plane of slope (a,b,c). The size of this element of the partition is less than $\exp((a+b+c)(\mathrm{ent}(a,b,c)+\varepsilon))$ uniformly over (a,b,c) for $a+b+c$ sufficiently large. Since the number of elements of the partition is at most exponential in the length of the boundary, whereas a,b,c are quadratic, summing over all elements of the partition gives the same bound up to a lower order error. $\square$

We apply Lemma 5.8 to the parallelogram D_i. For any $\varepsilon>0$, there exists ρ_0 such that for any n,k such that $k/n<\rho_0$, for each i in $I(E)$, the number of possible choices for $\mathcal{B}_i$ corresponding to a fixed value of a_i,b_i,c_i is less than $\exp\big((a_i+b_i+c_i)(\mathrm{ent}(a_i,b_i,c_i)+\varepsilon)\big)$. Letting

$$a_I=\sum_{i\in I}a_i\,,\quad b_I=\sum_{i\in I}b_i\,,\quad c_I=\sum_{i\in I}c_i\,,$$

we have

$$\max(|a-a_I|,|b-b_I|,|c-c_I|)\,\leq\,|\mathcal{M}(E)|$$

whence

$$\max(|a^*-a_I|,|b^*-b_I|,|c^*-c_I|)\,\leq\,3\alpha n^2+O(nk)+|\nu|_1 3\alpha n^4/k^2\,.$$

The concavity of $\nu\in\mathbb{R}^3\mapsto|\nu|_1\mathrm{ent}(\nu)$ yields

$$\sum_{i\in I}(a_i+b_i+c_i)\mathrm{ent}(a_i,b_i,c_i)\,\leq\,(a_I+b_I+c_I)\mathrm{ent}(a_I+b_I+c_I)$$

and we conclude that the total number of possible configurations for $\mathcal{B}(E)$ once $\mathcal{M}(E)$, a_I, b_I, c_I are fixed is bounded above by

$$\binom{a_I+|I|-1}{|I|-1}\binom{b_I+|I|-1}{|I|-1}\binom{c_I+|I|-1}{|I|-1}\exp\Big((a_I+b_I+c_I)(\mathrm{ent}(a_I,b_I,c_I)+\varepsilon)\Big).$$

(Recall that $\binom{p+q-1}{q-1}$ is the number of ways to partition p identical elements into q labelled subsets.) We now choose $\alpha<\rho_0^4$ and $k=\alpha^{1/4}n$. We then have $|I|\leq\sqrt{\alpha}n^2$ which implies

$$\binom{a_I+|I|-1}{|I|-1}\leq\binom{|\nu|_1n^2(1+\sqrt{\alpha})}{\sqrt{\alpha}n^2}=\exp(n^2O(\sqrt{\alpha}\ln\sqrt{\alpha}))\,.$$

Therefore

$$\binom{a_I+|I|-1}{|I|-1}\binom{b_I+|I|-1}{|I|-1}\binom{c_I+|I|-1}{|I|-1}=\exp(O(\alpha^{1/4}n^2))\,.$$

Now

$$\max(|a^* - a_I|, |b^* - b_I|, |c^* - c_I|) \le (3\alpha + O(\alpha^{1/4}) + 9\sqrt{\alpha})n^2 = O(\alpha^{1/4}n^2)$$

so the number of possible values for a_I, b_I, c_I is bounded by $O(\alpha^{3/4}n^6)$. Moreover ent is continuous with respect to the direction; hence there exists $\rho_1 > 0$ such that, for $\alpha < \rho_1$ and n sufficiently large, we have

$$|a^* + b^* + c^* - |\nu|_1 n^2| \le \varepsilon n^2, \qquad |\mathrm{ent}(\nu) - \mathrm{ent}(a_I, b_I, c_I)| \le \varepsilon.$$

Putting together the previous estimates, we see that for $\alpha < \min(\rho_0^4, \rho_1)$, and n sufficiently large, the total number of possible choices for E is less than

$$O(\alpha^{3/4}n^6)\exp(O(\alpha^{1/4}n^2))\exp\Big((|\nu|_1 + \varepsilon)n^2(\mathrm{ent}(\nu) + 2\varepsilon)\Big).$$

Since ε and α can be chosen arbitrarily small, the desired estimate on $\phi(n, A, \nu, \alpha)$ follows. $\square$

Lemmas 5.5, 5.6, 5.7 together imply the following result, which implies the expansion stated in Theorem 1.1.

Corollary 5.9. *For any ν in S^2, any unit square A orthogonal to ν, we have*

$$\limsup_{\substack{(n,p)\to(\infty,1)\\ n<\infty, p<1}} \Delta(n,p) \le |\nu|_1 \,\mathrm{ent}(\nu).$$

It remains to prove that the expansion is uniform with respect to ν in S^2. For $\epsilon > 0$, let T_ϵ be the symmetry in $\mathbb{R}^3$ defined by

$$\forall x \in \mathbb{R}^3 \qquad T_\epsilon(x) = \frac{e}{\epsilon} - x \qquad \text{where}\, e = (1,1,1).$$

Since $\tau(\nu, p)$ is the support function of $\mathcal{W}_\tau$, we have for any ν in $(\mathbb{R}^+)^3$

$$|\nu|_2 \tau(\frac{\nu}{|\nu|_2}, p) = \sup_{w \in \mathcal{W}_\tau} w \cdot \nu = \sup_{w \in T_\epsilon(\mathcal{W}_\tau)} T_\epsilon(w)\cdot\nu = \frac{e\cdot\nu}{\epsilon} + \sup_{w \in T_\epsilon(\mathcal{W}_{\tau(\nu,p)})} -w\cdot\nu.$$

Restricting this identity to $S^2 \cap (\mathbb{R}^+)^3$, we get

$$\forall \nu \in S^2 \cap (\mathbb{R}^+)^3 \qquad \tau(\nu, p) - \frac{|\nu|_1}{\epsilon} = \sup_{w \in T_\epsilon(\mathcal{W}_\tau)} -w\cdot\nu$$

which converges pointwise towards $-|\nu|_1 \mathrm{ent}(\nu)$. By homogeneity we see that

$$\nu \mapsto |\nu|_2 \tau(\frac{\nu}{|\nu|_2}, p) - \frac{|\nu|_1}{\epsilon}$$

is a sequence of convex functions from $(\mathbb{R}^+)^3$ to $\mathbb{R}$ which converges pointwise to the continuous function $-|\nu|_1 \mathrm{ent}(\nu)$. The uniformity stated in Theorem 1.1 follows from the following lemma.

Lemma 5.10. *Let $(f_n)_{n\in\mathbb{N}}$ be a sequence of convex functions from $(\mathbb{R}^+)^d$ to $\mathbb{R}$ which converges pointwise to a continuous function f. Then the convergence is uniform on any compact set included in $(\mathbb{R}^+)^d$.*

Remark. A classical result shows that the convergence is uniform on any compact set included in the interior of $(\mathbb{R}^+)^d$ (see for instance [28, Theorem 10.8]). For a related result concerning more complicated domains, see [19].

Proof. The proof is done by induction on the dimension d. In the case $d = 0$, $(\mathbb{R}^+)^0 = \{0\}$ and pointwise convergence implies uniform convergence! Suppose now that the result holds in dimension $d-1$ where d is a fixed integer, $d \geq 1$. Let $(f_n)_{n\in\mathbb{N}}$, f be functions from $(\mathbb{R}^+)^d$ to $\mathbb{R}$ satisfying the hypothesis of the lemma. Let $\epsilon > 0$ and $M > 0$. The function f is uniformly continuous on the compact set $[0, M]^d$, thus

$$\exists\, \delta > 0 \quad \forall x, y \in [0, M]^d \quad |x - y|_2 < 2\sqrt{d}\delta \quad \Rightarrow \quad |f(x) - f(y)| < \epsilon\,.$$

By the classical result quoted in the above remark, the sequence $(f_n)_{n\in\mathbb{N}}$ converges uniformly to f on $[\delta, M]^d$, hence

$$\exists\, N_1(\delta, M, \epsilon) \quad \forall n \geq N_1 \quad \forall x \in [\delta, M]^d \qquad |f_n(x) - f(x)| < \epsilon\,.$$

For each k in $\{1, \dots, d\}$, the restrictions of the functions $(f_n)_{n\in\mathbb{N}}$, f to the $d-1$ dimensional hyperplane $x_k = 0$ (x_k is the k–th coordinate in the canonical basis of $\mathbb{R}^d$) are convex functions from $(\mathbb{R}^+)^{d-1}$ to $\mathbb{R}$ which satisfy the induction hypothesis. Therefore we have uniform convergence on the set

$$D \,=\, [0, M]^d \cap \bigcup_{1\leq k\leq d} \{\, x \in \mathbb{R}^d : x_k = 0\,\}$$

that is,

$$\exists\, N_2(M, \epsilon) \quad \forall n \geq N_2 \quad \forall x \in D \qquad |f_n(x) - f(x)| < \epsilon\,.$$

Let now n be an integer larger than $\max(N_1, N_2)$. Let y belong to $[0, M]^d \setminus D \setminus [\delta, M]^d$. Let z_0 be the orthogonal projection of y on $[\delta, M]^d$ (that is, the point of $[\delta, M]^d$ closest to y). Let z_1 be the intersection of the line $(y\, z_0)$ and D and let z_2 be the point symmetric of y with respect to z_0. We have $|z_1 - z_0|_2 < \sqrt{d}\delta$. Since y belongs to the segment $[z_0, z_1]$, there exist $\alpha, \beta \geq 0$ such that $\alpha + \beta = 1$ and $y = \alpha z_0 + \beta z_1$. By convexity and the previous inequalities, we have then

$$f_n(y) \,\leq\, \alpha f_n(z_0) + \beta f_n(z_1) \,\leq\, \alpha f(z_0) + \beta f(z_1) + \epsilon \,\leq\, f(y) + 2\epsilon\,.$$

Finally, we have also $f_n(z_0) \,\leq\, (1/2)(f_n(y) + f_n(z_2))$ whence

$$f_n(y) \,\geq\, 2f_n(z_0) - f_n(z_2) \,\geq\, 2f(z_0) - 2\epsilon - f(z_2) - \epsilon \,\geq\, f(y) - 6\epsilon\,.$$

Thus we have uniform convergence over $[0, M]^d$ and the induction step is completed. $\square$

References

1. M. Aizenman, J.T. Chayes, L. Chayes, J. Fröhlich and L. Russo, *On a sharp transition from area law to perimeter law in a system of random surfaces*, Commun. Math. Phys. **92** (1983), 19–69.
2. K.S. Alexander, J.T. Chayes and L. Chayes, *The Wulff construction and asymptotics of the finite cluster distribution for two–dimensional Bernoulli percolation*, Commun. Math. Phys. **131** (1990), 1–50.
3. F. Almgren, *Questions and answers about area–minimizing surfaces and geometric measure theory*, Differential geometry. Part 1, R. Greene (ed.) et al., Proc. Symp. Pure Math. **54** (1993), 29–53.
4. T. Bodineau, *The Wulff construction in three and more dimensions.*, Commun. Math. Phys. **207** no.1 (1999), 197-229.
5. J. Bricmont, A. El Mellouki, J. Frohlich, *Random surfaces in statistical mechanics: roughening, rounding, wetting*, J. Stat. Phys. **42** no.5-6 (1986), 743-798.
6. R. Cerf, *Large deviations for three dimensional supercritical percolation*, Astérisque **267** (2000).
7. R. Cerf and A. Pisztora, *On the Wulff crystal in the Ising model*, Ann. Probab. **28 no.3** (2000), 945–1015.
8. R. Cerf and A. Pisztora, *Phase coexistence in Ising, Potts and percolation models*, Preprint (2000).
9. H. Cheng, T. T. Wu, Phys. Rev. **164** (1967), 719.
10. H. Cohn, R. Kenyon, J. Propp, *A variational principle for domino tilings*, J. AMS (??) (to appear).
11. J.D. Deuschel and A. Pisztora, *Surface order large deviations for high–density percolation*, Probab. Theory Relat. Fields **104 no.4** (1996), 467–482.
12. A. Dinghas, *Uber einen geometrischen Satz von Wulff für die Gleichgewichtsform von Kristallen*, Z. Kristallogr. **105** (1944), 304–314.
13. R.L. Dobrushin, R. Kotecký and S.B. Shlosman, *Wulff construction: a global shape from local interaction*, AMS translations series, Providence (Rhode Island), 1992.
14. R.L. Dobrushin and S.B. Shlosman, *Thermodynamic inequalities for the surface tension and the geometry of the Wulff construction*, Ideas and methods in quantum and statistical physics **2** (1992), S. Albeverio (ed.), Cambridge University Press, 461–483.
15. I. Fonseca, *The Wulff theorem revisited*, Proc. R. Soc. Lond. Ser. A **432 No.1884** (1991), 125–145.
16. I. Fonseca and S. Müller, *A uniqueness proof for the Wulff theorem*, Proc. R. Soc. Edinb. Sect. A **119 No.1/2** (1991), 125–136.
17. G.R. Grimmett, *Percolation*, Springer–Verlag, 1989.
18. G.R. Grimmett, *The stochastic random-cluster process and the uniqueness of random-cluster measures.*, Ann. Probab. **23** (1995), 1461–1510.
19. I. Ja. Guberman, *On the uniform convergence of convex functions in a closed domain*, Leningrad. Gos. Ped. Inst. Uv cen. Zap. **274** (1965), 7–37.
20. G.T. Herman and D. Webster, *Surfaces of organs in discrete three–dimensional space*, Mathematical aspects of computerized tomography, Oberwolfach, Lect. Notes Med. Inf. **8** (1981), 204–224.
21. L. Hörmander, *Sur la fonction d'appui des ensembles convexes dans un espace localement convexe*, Arkiv Matematik **3** (1954), 181–186.
22. H. Kesten, *Aspects of first passage percolation, Ecole d'été de probabilités de Saint–Flour, XIV–1984*, Lecture Notes in Math. **1180** (1986), 125–264.
23. A. Messager, S. Miracle–Solé and J. Ruiz, *Convexity properties of the surface tension and equilibrium crystals*, J. Stat. Phys. **67 nos.3/4** (1992), 449–469.
24. S. Miracle–Solé, *Surface tension, step free energy, and facets in the equilibrium crystal*, J. Stat. Phys. **79 nos.1/2** (1995), 183–214.
25. C.E. Pfister, *Large deviations and phase separation in the two–dimensional Ising model*, Helv. Phys. Acta **64 no.4** (1991), 953–1054.

26. A. Pisztora, *Surface order large deviations for Ising, Potts and percolation models*, Probab. Theory Relat. Fields **104 no.4** (1996), 427–466.
27. H. Rådström, *An embedding theorem for spaces of convex sets*, Proc. Amer. Math. Soc. **3** (1952), 165–169.
28. R.T. Rockafellar, *Convex analysis*, Princeton Univ. Press, New Jersey, 1970.
29. D. Rolfsen, *Knots and links*, Publish or Perish, Houston, 1990.
30. G. Salinetti and R. J.-B. Wets, *On the convergence of sequences of convex sets in finite dimensions*, Siam review **21 no.1** (1979), 18–33.
31. S. Shlosman, *Geometric variational problems of statistical mechanics and of combinatorics. Probabilistic techniques in equilibrium and nonequilibrium statistical physics.*, J. Math. Phys. **41 no.3** (2000), 1364-1370.
32. M. Szalay and P. Turán, *On some problems of the statistical theory of partitions with application to characters of the symmetric group. I,II,III*, Acta Math. Acad. Sci. **29** (1977), 361-379, 381-392, **32** (1978),129-155.
33. Jean E. Taylor, *Crystalline variational problems*, Bull. Am. Math. Soc. **84 no.4** (1978), 568–588.
34. Jean E. Taylor, *Existence and structure of solutions to a class of nonelliptic variational problems*, Symposia Mathematica **14 no.4** (1974), 499–508.
35. Jean E. Taylor, *Unique structure of solutions to a class of nonelliptic variational problems*, Proc. Symp. pure Math. **27** (1975), 419–427.
36. A. Vershik, *Statistical mechanics of combinatorial partitions and their limit shapes*, Func. Anal. Appl. **30, no. 2** (1996), 90-105.
37. G. Wulff, *Zur Frage der Geschwindigkeit des Wachsturms und der Auflösung der Kristallflächen*, Z. Kristallogr. **34** (1901), 449–530.

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