ORIENTED MEASURES WITH CONTINUOUS DENSITIES AND THE BANG–BANG PRINCIPLE

Raphaël CERF – Carlo MARICONDA

Ecole Normale Supérieure, Paris – Università di Padova

February 1993

Abstract. We introduce the notion of an oriented measure.
For such a measure μ, given ν in $L^1([a,b])$, $0<\nu<1$, there exist two sets $E \subset [a,b]$ whose characteristic functions have less than n discontinuity points and such that $\int \nu \, d\mu = \mu(E)$.
Given a solution x to the control problem

$$L(x) = x^{(n)} + a_{n-1}(t)x^{(n-1)} + \cdots + a_1(t)x' + a_0(t) \in [\phi_1, \phi_2]$$

there exist two bang–bang solutions y, z having a contact of order n with x at a and b such that $y \leq x \leq z$.
Reachable sets of bang–bang constrained solutions are convex; an application to the calculus of variations yields a density result.

1991 *Mathematics Subject Classification.* 15A15, 28B20, 34H05, 49B10, 93C15.

Key words and phrases. oriented measure, range of a vector measure, Caccioppoli Global Inversion Theorem, Wronskian, generalized Rolle Theorem, bang–bang, control, reachable set, calculus of variations.

We are deeply grateful to Professor Arrigo Cellina for suggesting the initial problem, for his useful advices and for his warm encouragement; we also wish to thank Professors Jean–Pierre Aubin and Helena Frankowska who gave the authors the opportunity to meet together. We thank the D.M.I. of the E.N.S. for the technical support during the preparation of this paper. The C.N.R. partially supported Carlo Mariconda during this research.

Typeset by $\mathcal{AMS}$-TEX

Introduction

A classical theorem of Liapunov [8] states that given a finite dimensional vector measure μ on an interval $[a,b]$ which admits a density function $f = (f_1, \cdots, f_n)$ and given a measurable function ν defined on $[a,b]$ with values in $[0,1]$, there exists a measurable subset E of $[a,b]$ such that

$$(*) \qquad \forall i \in \{1, \cdots, n\} \qquad \int_a^b f_i \chi_E = \int_a^b f_i \nu.$$

However the proofs of this theorem are not constructive and thus do not give any information about the set E.
Halkin [9] showed that if for each vector $p \in \mathbb{R}^n$ the set

$$\left\{ t \in [a,b] : p \cdot f(t) > 0 \right\}$$

(where $\cdot$ is the usual scalar product) is a finite (respectively countable) union of intervals then there exists a set E satisfying $(*)$ which is a finite (resp. countable) union of intervals. As far as we know this condition has not been applied apart the case of piecewise analytical functions [9,10,12].
The results we present here are based on the following new

Orientation condition Δ. We say that n real functions $f_1, \cdots, f_n$ verify condition Δ on an interval $[a,b]$ if for each k in $\{1, \cdots, n\}$, the determinant

$$\begin{vmatrix} f_1(x_1) & f_1(x_2) & \cdots & f_1(x_k) \\ f_2(x_1) & f_2(x_2) & \cdots & f_2(x_k) \\ \vdots & \vdots & \ddots & \vdots \\ f_k(x_1) & f_k(x_2) & \cdots & f_k(x_k) \end{vmatrix}$$

is not equal to zero whenever the $x_i \in [a,b]$ are distinct and its sign is constant on the k–uples $(x_1, \cdots, x_k)$ such that $a \le x_1 < x_2 < \cdots < x_k \le b$.

A measure μ whose components $\mu_1, \cdots, \mu_n$ admit continuous density functions $f_1, \cdots, f_n$ which satisfy the orientation condition Δ is said to be oriented.
Although this condition implies Halkin's one, it possesses various advantages:
- it allows to build a set E satisfying $(*)$ whose characteristic function has at most n points of discontinuity;
- in the case where $0 < \nu < 1$ there exist exactly two such sets E_1 and E_2 and in addition the associated characteristic functions χ_{E_1} and χ_{E_2} have exactly n discontinuity points; moreover, one set is a neighbourhood of a whereas the other is not.

We give two proofs of this result, neither of which uses the traditional convexity–extremal points arguments. Both use algebraic tricks directly related to condition Δ; the first one is based on the Implicit Function Theorem and the second one on Caccioppoli Global Inversion Theorem.

Consequence of our theorem is that if the interval $[a, b]$ can be partitioned as a finite (respectively countable) union of intervals on which the orientation condition Δ holds then we can build a set E satisfying $(*)$ which is a finite (resp. countable) union of intervals.

We also point out an operational criterion which ensures the validity of the orientation condition Δ: if $f_1, \cdots, f_n$ are of class $\mathcal{C}^{n-1}$ on $[a, b]$ it is enough that the Wronskians $W(f_1), \cdots, W(f_1, \cdots, f_n)$ do not vanish on $[a, b]$ for Δ to hold.

This allows us to formulate a new result concerning bang–bang solutions to linear control systems described by a generic linear differential equation

$$L(x) = x^{(n)} + a_{n-1}(t)x^{(n-1)} + \cdots + a_1(t)x' + a_0(t) \in [\phi_1, \phi_2]$$

where ϕ_1 and ϕ_2 belong to L^1. More precisely we show that given a solution x to the above problem there exist two bang–bang solutions y and z (i.e. $L(y), L(z) \in \{\phi_1, \phi_2\}$) such that

$$\begin{array}{cc} \forall t \in [a, b] & y(t) \leq x(t) \leq z(t) \\ \forall k \in \{0, \cdots, n-1\} & y^{(k)}(a) = x^{(k)}(a) = z^{(k)}(a), \quad y^{(k)}(b) = x^{(k)}(b) = z^{(k)}(b) \end{array}$$

and $L(D)y$ and $L(D)z$ are of the form $\chi_E\phi_1 + (1 - \chi_E)\phi_2$ where the set E is a finite union of intervals, i.e. y and z are solutions associated to relay controls. The relay principle was studied by Andreini and Bacciotti in [4] under the strong assumption that $\phi_1, \phi_2, a_0, \cdots, a_{n-1}$ be analytical. In order to apply our Liapunov's type theorem we explicit the solutions to

$$L(x) = \nu \in [0, 1], \qquad x(a) = \cdots = x^{(n-1)}(a) = 0$$

through the integral representation formulas

$$\forall k \in \{0, \cdots, n-1\} \qquad x^{(k)}(t) = \int_a^t \frac{\partial^k R}{\partial t^k}(t, s)\nu(s)\, ds$$

where $R(t, s)$ is the resolvent of the operator L. Our Wronskian criterion then applies directly to the functions

$$R(b, \cdot),\ \frac{\partial R}{\partial t}(b, \cdot),\ \cdots,\ \frac{\partial^{n-1} R}{\partial t^{n-1}}(b, \cdot)$$

and thus our main theorem yields a bang–bang solution

$$y(t) = \int_a^t R(t, s)\chi_E(s)\, ds$$

satisfying the required tangency conditions; moreover the set E is a finite union of intervals which does not contain the point a.
Surprisingly the same Wronskian conditions allow us to apply an extended version of Pólya's generalized Rolle theorem for linear differential operators of order n and functions whose n–th derivative are only piecewise continuous. We obtain that if $0 < \nu < 1$ then the graphs of x and y do not intersect. Since $y^{(n)}(a) < x^{(n)}(a)$ then $y < x$ on the whole interval $]a, b[$.
We give two applications of this result.

- The reachable set of solutions which are constrained by a given obstacle and subject to prescribed initial conditions coincides with the reachable set of bang–bang solutions submit to the same conditions, so that this last one is convex.
- We consider the problem of minimizing the integral functionals

$$I(x, u) = \int_a^b f(t, x(t), u(t))\, dt$$

where $x : [a, b] \to \mathbb{R}^n$ is such that $x^{(k)}(a)$, $x^{(k)}(b)$ $(0 \le k \le n-1)$ are fixed and u is a control belonging to $U(t, x) \subset R^n$. The classical approach to obtain existence of a minimum is to impose conditions in order to have the lower semicontinuity of I with respect to u (for instance convexity of $u \mapsto f(t, x, u)$).

Recently in an effort to provide existence criteria other than convexity in u some sufficient conditions have been given: for problems of the calculus of variations ($x' = u$ in the above setting) and for maps of the form $f(t, x, x') = g(t, x) + h(t, x')$, existence of solutions has been obtained by requiring that the real map $x \mapsto g(t, x)$ be monotonic [11] or, for x in $\mathbb{R}^n$, that the same function be concave [5]. Optimal control problems escaping to convexity conditions have been handled in [14].
It has been proved further in [6] that there exists a dense subset $\mathcal{D}$ of $\mathcal{C}(\mathbb{R})$ such that, for g in it, the problem

$$\text{minimize } \int_a^b g(x(t))\, dt + \int_a^b h(x'(t))\, dt \quad : \quad x(a) = x_0, x(b) = x_1$$

admits a solution for every lower semicontinuous h satisfying growth conditions.
Our theorem gives a straightforward generalization of the above result.
Let us remark that the elementary case $n = 1$ of our n–dimensional Liapunov's type theorem appeared as a technical tool in [1, Lemma 3.4]; the case $n = 2$ was handled in our previous paper [7] with very different techniques which are not applicable to higher dimensions.
This work deals only with measures having continuous densities; the general case will be treated in a forthcoming paper.

One of the two proofs of theorem 1 relies on the following powerful but not enough appreciated

Caccioppoli Global Inversion Theorem. *Let E be an arcwise connected metric space, F be a simply connected metric space, f be a proper map from E with values in F. If f is a local homeomorphism at each point of E then f is a global homeomorphism between E and F.*

Proof. The proof and several applications of this theorem can be found in [2,3]. $\square$

Let us introduce some notations.
Let A be an $n \times n$ matrix with real coefficients. By $\det A$ or $|A|$ we denote its determinant. For each $i, j \in \{1, \cdots, n\}$, by A_{ij} we mean the $(n-1) \times (n-1)$ matrix obtained by removing the i–th row and the j–th column from A. Surprisingly, the following simple algebraic trick will play an essential role in the existence part of the proof of theorem 1 which does not involve Caccioppoli Theorem.

Lemma S. *Let $A = (a_{ij})_{1 \le i,j \le n}$ be an $n \times n$ matrix with real coefficients. Let $x_1, \cdots, x_n$ be such that*

$$\begin{cases} a_{1,1}x_1 + \cdots + a_{1,n-1}x_{n-1} + a_{1,n}x_n = 0 \\ a_{2,1}x_1 + \cdots + a_{2,n-1}x_{n-1} + a_{2,n}x_n = 0 \\ \vdots \quad \ddots \quad \vdots \quad \vdots \\ a_{n-1,1}x_1 + \cdots + a_{n-1,n-1}x_{n-1} + a_{n-1,n}x_n = 0 \end{cases}$$

If $\det A_{nn} \neq 0$ then

$$a_{n1}x_1 + \cdots + a_{nn}x_n = \frac{|A|}{|A_{nn}|}\, x_n.$$

Proof. Cramer rule applied to the above system yields

$$\forall i \in \{1, \cdots, n-1\} \qquad x_i = \frac{(-1)^{n+i}|A_{ni}|}{|A_{nn}|}\, x_n$$

so that

$$a_{n1}x_1 + \cdots + a_{nn}x_n = \frac{\sum_{i=1}^{n}(-1)^{n+i}a_{ni}|A_{ni}|}{|A_{nn}|}\, x_n = \frac{|A|}{|A_{nn}|}\, x_n$$

since $|A| = \sum_{i=1}^{n}(-1)^{n+i}a_{ni}|A_{ni}|$ is the development of the determinant of $|A|$ along the first row. $\square$

The main tool in the inductive proof of theorem 1 is the existence and uniqueness of maximal implicit functions passing through a prescribed point.

Lemma M. *Let Ω be an open subset of $\mathbb{R}^{n-1} \times [a, b]$ and F a continuously differentiable map from Ω into $\mathbb{R}^{n-1}$ such that $\frac{\partial F}{\partial(x_1, \cdots, x_{n-1})}$ is invertible everywhere. Let $(\bar{x}_1, \cdots, \bar{x}_n)$ verify $F(\bar{x}_1, \cdots, \bar{x}_n) = 0$. Then there exists a unique couple (I, Ψ) verifying Property P below such that I is maximal for the set inclusion with respect to this property.*

Property P. I is an interval containing $\bar{x}_n$, Ψ is a continuous map from I into $\mathbb{R}^{n-1}$, $\Psi(\bar{x}_n) = (\bar{x}_1, \cdots, \bar{x}_{n-1})$, $F(\Psi(x_n), x_n) = 0$ for every x_n in I.

Proof of Lemma. Suppose first (I, Ψ_I) and (J, Ψ_J) both satisfy property P. Put

$$Z = \{\, x_n \in I \cap J : \Psi_I(x_n) = \Psi_J(x_n) \,\}.$$

This set is not empty (since $\bar{x}_n \in Z$) and is closed because Ψ_I and Ψ_J are continuous. Let $x_n^* \in Z$ and

$$(x_1^*, \cdots, x_{n-1}^*) = \Psi_I(x_n^*) = \Psi_J(x_n^*)$$

so that $F(x_1^*, \cdots, x_n^*) = 0$. We have

$$\left| \frac{\partial F}{\partial(x_1, \cdots, x_{n-1})}(x_1^*, \cdots, x_n^*) \right| \neq 0$$

and we can thus apply the implicit function theorem at the point $(x_1^*, \cdots, x_n^*)$. There exist an open interval $]x_n^* - \epsilon, x_n^* + \epsilon[$, a neighbourhood $\mathcal{O}$ of $(x_1^*, \cdots, x_{n-1}^*)$ and a function ϕ from $]x_n^* - \epsilon, x_n^* + \epsilon[$ into $\mathcal{O}$ such that $\phi(x_n^*) = (x_1^*, \cdots, x_{n-1}^*)$ and

$$\forall x_n \in]x_n^* - \epsilon, x_n^* + \epsilon[\quad \forall (x_1, \cdots, x_{n-1}) \in \mathcal{O}$$
$$F(x_1, \cdots, x_n) = 0 \iff (x_1, \cdots, x_{n-1}) = \phi(x_n).$$

Thus for every x_n in $I \cap J \cap]x_n^* - \epsilon, x_n^* + \epsilon[$

$$\phi(x_n) = \Psi_I(x_n) = \Psi_J(x_n)$$

whence Z is also open. Since $I \cap J$ is connected then $Z = I \cap J$. Put

$$\mathcal{T} = \{(I, \Psi_I) \text{ satisfying property P}\}$$

and let

$$I_M = \bigcup_{(I, \Psi_I) \in \mathcal{T}} I.$$

The previous uniqueness property allows us to define a function Ψ_M on I_M such that $\Psi_M = \Psi_I$ on I. The couple (I_M, Ψ_M) solves our problem. $\square$

The orientation condition Δ and some related facts

Orientation condition Δ. We say that n real functions $f_1, \cdots, f_n$ verify condition Δ on an interval $[a,b]$ if for each k in $\{1, \cdots, n\}$, the determinant

$$\begin{vmatrix} f_1(x_1) & f_1(x_2) & \cdots & f_1(x_k) \\ f_2(x_1) & f_2(x_2) & \cdots & f_2(x_k) \\ \vdots & \vdots & \ddots & \vdots \\ f_k(x_1) & f_k(x_2) & \cdots & f_k(x_k) \end{vmatrix}$$

is not equal to zero whenever the $x_i \in [a,b]$ are distinct and its sign is constant on the k–uples $(x_1, \cdots, x_k)$ such that $a \le x_1 < x_2 < \cdots < x_k \le b$.

Example 1. For $n = 1$, condition Δ states that the function f_1 is positive. For $n = 2$, the functions f_1, f_2 satisfy Δ if and only if $f_1 > 0$ and f_2/f_1 is increasing.

Example 2. The functions $f_i(t) = t^{i-1}$ $(i \ge 1)$ satisfy condition Δ on $\mathbb{R}$ (the corresponding determinants are Vandermonde determinants).

Our interest in condition Δ relies on the following nice facts.

Lemma 1. *Let $f_1, \cdots, f_n$ be n measurable bounded functions satisfying Δ on $[a,b]$. Let $\nu_1, \cdots, \nu_n$ be n positive functions in $L^1([a,b])$. Then for each $(n-1)$–uple $(\gamma_1, \cdots, \gamma_{n-1})$ such that $a < \gamma_1 < \cdots < \gamma_{n-1} < b$ the determinant*

$$\begin{vmatrix} \int_a^{\gamma_1} f_1\nu_1 & \int_{\gamma_1}^{\gamma_2} f_1\nu_2 & \cdots & \int_{\gamma_{n-1}}^{b} f_1\nu_n \\ \int_a^{\gamma_1} f_2\nu_1 & \int_{\gamma_1}^{\gamma_2} f_2\nu_2 & \cdots & \int_{\gamma_{n-1}}^{b} f_2\nu_n \\ \vdots & \vdots & \ddots & \vdots \\ \int_a^{\gamma_1} f_n\nu_1 & \int_{\gamma_1}^{\gamma_2} f_n\nu_2 & \cdots & \int_{\gamma_{n-1}}^{b} f_n\nu_n \end{vmatrix}$$

is not equal to zero.

Proof. Since the determinant is a multilinear continuous form, we can write

$$\begin{vmatrix} \int_a^{\gamma_1} f_1\nu_1 & \cdots & \int_{\gamma_{n-1}}^{b} f_1\nu_n \\ \vdots & \ddots & \vdots \\ \int_a^{\gamma_1} f_n\nu_1 & \cdots & \int_{\gamma_{n-1}}^{b} f_n\nu_n \end{vmatrix} = \int_a^{\gamma_1} ds_1 \cdots \int_{\gamma_{n-1}}^{b} ds_n \begin{vmatrix} f_1(s_1)\nu_1(s_1) & \cdots & f_1(s_n)\nu_n(s_n) \\ \vdots & \ddots & \vdots \\ f_n(s_1)\nu_1(s_1) & \cdots & f_n(s_n)\nu_n(s_n) \end{vmatrix}$$

$$= \underset{[a,\gamma_1]\times[\gamma_1,\gamma_2]\times\cdots\times[\gamma_{n-1},b]}{\int\int\cdots\int} \nu_1(s_1)\nu_2(s_2)\cdots\nu_n(s_n)\,\omega(s_1,s_2,\cdots,s_n)\,ds_1\,ds_2\cdots ds_n$$

where

$$\omega(s_1,\cdots,s_n) = \begin{vmatrix} f_1(x_1) & \cdots & f_1(x_n) \\ \vdots & \ddots & \vdots \\ f_n(x_1) & \cdots & f_n(x_n) \end{vmatrix}.$$

However the function

$$(s_1,\cdots,s_n) \longmapsto \nu_1(s_1)\cdots\nu_n(s_n)\omega(s_1,\cdots,s_n)$$

is either positive a.e. or negative a.e. on the open non-empty domain

$$]a,\gamma_1[\times]\gamma_1,\gamma_2[\times\cdots\times]\gamma_{n-1},b[$$

so that its integral over $[a,\gamma_1]\times[\gamma_1,\gamma_2]\times\cdots\times[\gamma_{n-1},b]$ cannot vanish. □

Lemma 2. *Let $f_1,\cdots,f_m$ be m measurable bounded functions satisfying condition Δ on $[a,b]$. Let $\nu_1,\cdots,\nu_m$ be m positive functions in $L^1([a,b])$.*
Let $(\gamma_1,\cdots,\gamma_{m-1})$ be an $(m-1)$–uple such that $(\gamma_0 =)\, a < \gamma_1 < \cdots < \gamma_{m-1} < b\,(=\gamma_m)$. If $x_1,\cdots,x_m$ are m real numbers not all equal to zero then there exists k in $\{1,\cdots,m\}$ such that

$$\sum_{i=1}^{m} x_i \int_{\gamma_{i-1}}^{\gamma_i} f_k(s)\nu_i(s)\,ds \neq 0.$$

Proof. Assume

$$\forall k \in \{1,\cdots,m\} \qquad \sum_{i=1}^{m} x_i \int_{\gamma_{i-1}}^{\gamma_i} f_k(s)\nu_i(s)\,ds = 0.$$

Then the determinant whose elements are the coefficients of $x_1,\cdots,x_m$ in the above system

$$\begin{vmatrix} \int_a^{\gamma_1} f_1\nu_1 & \int_{\gamma_1}^{\gamma_2} f_1\nu_2 & \cdots & \int_{\gamma_{m-1}}^{b} f_1\nu_m \\ \int_a^{\gamma_1} f_2\nu_1 & \int_{\gamma_1}^{\gamma_2} f_2\nu_2 & \cdots & \int_{\gamma_{m-1}}^{b} f_2\nu_m \\ \vdots & \vdots & \ddots & \vdots \\ \int_a^{\gamma_1} f_m\nu_1 & \int_{\gamma_1}^{\gamma_2} f_m\nu_2 & \cdots & \int_{\gamma_{m-1}}^{b} f_m\nu_m \end{vmatrix}$$

is necessarily equal to zero, thus contradicting Lemma 1. □

Lemma 3. *Let $f_1, \cdots, f_n$ be n measurable bounded functions satisfying condition Δ on the interval $[a, b]$. Let $\alpha_1, \cdots, \alpha_n$ be such that $(\alpha_0 =) a < \alpha_1 < \cdots < \alpha_n < b (= \alpha_{n+1})$. Then, given a positive ϵ, there exist $n+1$ positive real numbers $\lambda_0, \cdots, \lambda_n$ such that*

$$\forall l \in \{0, \cdots, n\} \qquad 0 < \lambda_l < \epsilon \qquad \textit{and}$$
$$\forall k \in \{1, \cdots, n\} \qquad \sum_{l=0}^{n} (-1)^l \lambda_l \int_{\alpha_l}^{\alpha_{l+1}} f_k = 0.$$

Proof. Consider the $n \times n$ linear system

$$\begin{cases} \lambda_0 \int_a^{\alpha_1} f_1 - \lambda_1 \int_{\alpha_1}^{\alpha_2} f_1 + \cdots + (-1)^{n-1} \lambda_{n-1} \int_{\alpha_{n-1}}^{\alpha_n} f_1 = (-1)^{n-1} \lambda_n \int_{\alpha_n}^{b} f_1 \\ \lambda_0 \int_a^{\alpha_1} f_2 - \lambda_1 \int_{\alpha_1}^{\alpha_2} f_2 + \cdots + (-1)^{n-1} \lambda_{n-1} \int_{\alpha_{n-1}}^{\alpha_n} f_2 = (-1)^{n-1} \lambda_n \int_{\alpha_n}^{b} f_2 \\ \vdots \qquad \ddots \qquad \vdots \qquad \vdots \qquad \vdots \\ \lambda_0 \int_a^{\alpha_1} f_n - \lambda_1 \int_{\alpha_1}^{\alpha_2} f_n + \cdots + (-1)^{n-1} \lambda_{n-1} \int_{\alpha_{n-1}}^{\alpha_n} f_n = (-1)^{n-1} \lambda_n \int_{\alpha_n}^{b} f_n \end{cases}$$

where λ_n is a parameter. The determinant of the system is

$$\omega_n = (-1)^{\frac{n(n-1)}{2}} \begin{vmatrix} \int_a^{\alpha_1} f_1 & \cdots & \int_{\alpha_{n-1}}^{\alpha_n} f_1 \\ \vdots & \ddots & \vdots \\ \int_a^{\alpha_1} f_n & \cdots & \int_{\alpha_{n-1}}^{\alpha_n} f_n \end{vmatrix}.$$

By condition Δ, its sign is $(-1)^{\frac{n(n-1)}{2}}$. Moreover, for each i in $\{0, \cdots, n-1\}$,

$$\omega_i = \begin{vmatrix} \int_a^{\alpha_1} f_1 & \cdots & (-1)^{i-2} \int_{\alpha_{i-2}}^{\alpha_{i-1}} f_1 & (-1)^{n-1} \int_{\alpha_n}^{b} f_1 & (-1)^i \int_{\alpha_i}^{\alpha_{i+1}} f_1 & \cdots & (-1)^{n-1} \int_{\alpha_{n-1}}^{\alpha_n} f_1 \\ \int_a^{\alpha_1} f_2 & \cdots & (-1)^{i-2} \int_{\alpha_{i-2}}^{\alpha_{i-1}} f_2 & (-1)^{n-1} \int_{\alpha_n}^{b} f_2 & (-1)^i \int_{\alpha_i}^{\alpha_{i+1}} f_2 & \cdots & (-1)^{n-1} \int_{\alpha_{n-1}}^{\alpha_n} f_2 \\ \vdots & \ddots & \vdots & \vdots & \vdots & \ddots & \vdots \\ \int_a^{\alpha_1} f_n & \cdots & (-1)^{i-2} \int_{\alpha_{i-2}}^{\alpha_{i-1}} f_n & (-1)^{n-1} \int_{\alpha_n}^{b} f_n & (-1)^i \int_{\alpha_i}^{\alpha_{i+1}} f_n & \cdots & (-1)^{n-1} \int_{\alpha_{n-1}}^{\alpha_n} f_n \end{vmatrix}$$

i.e. $$\omega_i = (-1)^{\frac{n(n-1)}{2}} \begin{vmatrix} \int_a^{\alpha_1} f_1 & \cdots & \int_{\alpha_{i-2}}^{\alpha_{i-1}} f_1 & \int_{\alpha_i}^{\alpha_{i+1}} f_1 & \cdots & \int_{\alpha_n}^{b} f_1 \\ \int_a^{\alpha_1} f_2 & \cdots & \int_{\alpha_{i-2}}^{\alpha_{i-1}} f_2 & \int_{\alpha_i}^{\alpha_{i+1}} f_2 & \cdots & \int_{\alpha_n}^{b} f_2 \\ \vdots & \ddots & \vdots & \vdots & \ddots & \vdots \\ \int_a^{\alpha_1} f_n & \cdots & \int_{\alpha_{i-2}}^{\alpha_{i-1}} f_n & \int_{\alpha_i}^{\alpha_{i+1}} f_n & \cdots & \int_{\alpha_n}^{b} f_n \end{vmatrix}$$

Thus λ_i which by Cramer formula equals $\lambda_n \omega_i / \omega_n$ has, by condition Δ and lemma 1, the sign of λ_n; choosing λ_n such that

$$0 < \lambda_n < \min(\frac{\omega_n}{\omega_0}\epsilon, \cdots, \frac{\omega_n}{\omega_{n-1}}\epsilon, \epsilon)$$

we obtain an $(n+1)$–uple which solves the problem. □

We give now a criterion for the fulfilment of the orientation condition Δ.
If $f_1, \cdots, f_{k+1}$ are of class $\mathcal{C}^k$ on $[a, b]$ we will denote their Wronskian by

$$W(f_1, \cdots, f_{k+1})(t) = \begin{vmatrix} f_1(t) & \cdots & f_{k+1}(t) \\ \vdots & \ddots & \vdots \\ f_1^{(k)}(t) & \cdots & f_{k+1}^{(k)}(t) \end{vmatrix}.$$

Proposition 1. *Let $h_1, \cdots, h_n \in \mathcal{C}^{n-1}([a, b])$ be such that*

$$\forall t \in [a, b] \qquad W(h_1)(t) \neq 0, \cdots, W(h_1, \cdots, h_n)(t) \neq 0.$$

Then $h_1, \cdots, h_n$ satisfy the orientation condition Δ.

Proof. By [13, Theorem V], for each k–uple $(t_1, \cdots, t_k)$ such that $a \leq t_1 < \cdots < t_k \leq b$, there exists $\xi \in]t_1, t_k[$ such that $W(h_1(\xi), \cdots, h_k(\xi))$ has the same sign as the determinant

$$\begin{vmatrix} h_1(t_1) & h_1(t_2) & \cdots & h_1(t_k) \\ h_2(t_1) & h_2(t_2) & \cdots & h_2(t_k) \\ \vdots & \vdots & \ddots & \vdots \\ h_k(t_1) & h_k(t_2) & \cdots & h_k(t_k) \end{vmatrix}.$$

It follows that the above determinant does not vanish and by continuity, it keeps a constant sign on the connected set of the k–uples $(t_1, \cdots, t_k)$ such that $a \leq t_1 < \cdots < t_k \leq b$. □

Remark 1. It is easy to prove that if $h_1, \cdots, h_n$ satisfy the orientation condition Δ on $[a, b]$ and are of class $\mathcal{C}^{n-1}$ then $W(h_1), \cdots, W(h_1, \cdots, h_n)$ are either non–negative or non–positive on the whole interval $[a, b]$.

Remark 2. For $n = 2$, the Wronskian conditions on f_1, f_2 state exactly that $f_1 > 0$ and $f_1 f_2' - f_1' f_2 > 0$ whence f_2/f_1 is strictly monotonic. However these conditions are not necessary for property Δ to hold (a function may be strictly monotonic without having a positive derivative).

The range of a finite dimensional oriented measure

In this section we study the range of a finite dimensional measure μ whose components $\mu_1, \cdots, \mu_n$ admit continuous density functions $f_1, \cdots, f_n$ which satisfy the orientation condition Δ: such a measure is said to be oriented.

Theorem 1. *Let $\nu \in L^1([a,b])$ be such that $0 \le \nu \le 1$. Let $f_1, \cdots, f_n$ be n real valued continuous functions on $[a,b]$ satisfying condition Δ on $[a,b]$.*
Then there exist a n–uple $\alpha = (\alpha_1, \cdots, \alpha_n)$ and a n–uple $\beta = (\beta_1, \cdots, \beta_n)$ such that

$$a \le \alpha_1 \le \cdots \le \alpha_n \le b, \quad a \le \beta_1 \le \cdots \le \beta_n \le b$$

and if we define

$$E_\alpha^- = \bigcup_{\substack{0 \le i \le n \\ i \text{ odd}}} [\alpha_i, \alpha_{i+1}], \quad E_\beta^+ = \bigcup_{\substack{0 \le i \le n \\ i \text{ even}}} [\beta_i, \beta_{i+1}]$$

(where $\beta_0 = a, \alpha_{n+1} = \beta_{n+1} = b$)
then we have

$$(*) \qquad \forall k \in \{1, \cdots, n\} \quad \int_a^b f_k(s) \chi_{E_\alpha^-}(s)\, ds = \int_a^b f_k(s) \nu(s)\, ds = \int_a^b f_k(s) \chi_{E_\beta^+}(s)\, ds.$$

If in addition $0 < \nu < 1$ then $(\alpha_1, \cdots, \alpha_n)$ and $(\beta_1, \cdots, \beta_n)$ are unique and verify

$$a < \alpha_1 < \cdots < \alpha_n < b, \quad a < \beta_1 < \cdots < \beta_n < b.$$

Remark. This theorem has already been proved for $n = 2$ in [7], but the orientation condition Δ was not formulated in such a precise way (see remark 1 after proposition 1).

Example. There exist a non–oriented measure μ on an interval, a measurable subset A which is not a finite union of intervals such that for every measurable subset E

$$\mu(A) = \mu(E) \quad \Rightarrow \quad A = E \text{ a.e.}$$

Consider for instance the measure $\mu = (\mu_1, \mu_2)$ whose density functions are

$$f_1(t) = 1, \qquad f_2(t) = 1 + t \sin(1/t)$$

and the set $A = \{\, t \in [0,1] : t \sin(1/t) > 0 \,\}$ (in this case the measure μ is positive but condition Δ is not fulfilled).

We will deal only with the situation where $0 < \nu < 1$: the fact that the number of intervals corresponding to ν does not depend on ν together with a classical approximation argument yields the general case (this is done explicitly in the proof of theorem 5).
We will give two proofs of the theorem. The first one relies on an induction whereas the second one is based on Caccioppoli Global Inversion Theorem. The following lemma will be used in both proofs.

Lemma. *Assume* $0 < \nu < 1$ *and let* l *be an integer smaller than* n. *Then if the* l*–uple* $\alpha = (\alpha_1, \cdots, \alpha_l)$ *(respectively* $\beta = (\beta_1, \cdots, \beta_l)$*) and its corresponding set* E^-_α *(respectively* E^+_β*) satisfy* $(*)$ *with* $a \leq \alpha_1 \leq \cdots \leq \alpha_l \leq b$ *(respectively* $a \leq \beta_1 \leq \cdots \leq \beta_l \leq b$*) then* $l = n$ *and* $a < \alpha_1 < \cdots < \alpha_l < b$ *(resp.* $a < \beta_1 < \cdots < \beta_l < b$*).*

Proof of the lemma. We first show that under the above assumption there exists a m–uple $\gamma = (\gamma_1, \cdots, \gamma_m)$, $m \leq l$, such that $a < \gamma_1 < \cdots < \gamma_m < b$ and either E^-_γ or E^+_γ satisfy $(*)$. Assume for instance there exists $i \in \{0, \cdots, l\}$ such that $\alpha_i = \alpha_{i+1}$ (where possibly $\alpha_0 = a$ and $\alpha_{l+1} = b$). We have the following cases:

- $i = 0$ so that $a = \alpha_1$. Put $m = l - 1$, $\gamma = (\alpha_2, \cdots, \alpha_l)$; then E^+_γ satisfies $(*)$.
- $0 < i < l$. Put $m = l - 2$, $\gamma = (\alpha_1, \cdots, \alpha_{i-1}, \alpha_{i+2}, \cdots, \alpha_n)$; then E^-_γ satisfies $(*)$.
- $i = l$. Put $m = l - 1$, $\gamma = (\alpha_1, \cdots, \alpha_{l-1})$; then E^-_γ satisfies $(*)$.

If two components of the m–uple γ are equal we iterate the above operation on γ until after a finite number of steps we obtain an uple having distinct components and whose one of the associated sets satisfies $(*)$.
We are thus led to prove the result for a l–uple α such that $a < \alpha_1 < \cdots < \alpha_l < b$, similar arguments hold for a l–uple of type β. Suppose $l < n$. Then by $(*)$ we have

$$\forall k \in \{1, \cdots, n\} \qquad \sum_{\substack{0 \leq i \leq l \\ i \text{ even}}} \int_{\alpha_i}^{\alpha_{i+1}} f_k(s)\nu(s)\, ds - \sum_{\substack{0 \leq i \leq l \\ i \text{ odd}}} \int_{\alpha_i}^{\alpha_{i+1}} f_k(s)(1 - \nu(s))\, ds = 0$$

(where $\alpha_0 = a, \alpha_{l+1} = b$).
We restrict our attention on the first $l + 1$ equations of the above system i.e. k belongs to $\{1, \cdots, l + 1\}$. Application of Lemma 2 with $m = l + 1$ and $\gamma_1 = \alpha_1, \cdots, \gamma_l = \alpha_l$

$$x_i = (-1)^{i+1}, \qquad \nu_i = \begin{cases} \nu & i \text{ odd} \\ 1 - \nu & i \text{ even} \end{cases}, \qquad 1 \leq i \leq l + 1$$

shows that these equations cannot hold simultaneously, thus yielding a contradiction. $\square$

First proof of the theorem. Consider the case $n=1$. Let $f_1 \in \mathcal{C}([a,b])$ satisfy Δ i.e. f_1 does not vanish on $[a,b]$. Since f_1 has a constant sign on $[a,b]$ there exist unique real numbers α, β in $[a,b]$ such that

$$\int_\alpha^b f_1(s)\,ds = \int_a^b f_1(s)\nu(s)\,ds = \int_a^\beta f_1(s)\,ds.$$

Clearly $E_\alpha^- = [\alpha, b]$ and $E_\beta^+ = [a,\beta]$ satisfy $(*)$.
Assume the theorem is true at rank $n-1$.
Let $f_1, \cdots, f_n, \nu$ be functions satisfying the hypothesis of the theorem. By the induction assumption there exist $(n-1)$–uples $(\bar\alpha_1, \cdots, \bar\alpha_{n-1})$ and $(\bar\beta_1, \cdots, \bar\beta_{n-1})$ satisfying $(*)$. Define for each k in $\{1, \cdots, n\}$ and n–uple $(\alpha_1, \cdots, \alpha_n)$ such that $a \le \alpha_1 \le \cdots \le \alpha_n \le b$

$$F_k(\alpha_1, \cdots, \alpha_n) = \sum_{\substack{0 \le i \le n \\ i \text{ odd}}} \int_{\alpha_i}^{\alpha_{i+1}} f_k(s)\,ds - \int_a^b f_k(s)\nu(s)\,ds$$

and put

$$\mathcal{S} = \{\, (\alpha_1, \cdots, \alpha_n) \in \mathbb{R}^n : a \le \alpha_1 < \alpha_2 < \cdots < \alpha_n \le b, \\ \forall k \in \{1, \cdots, n-1\} \quad F_k(\alpha_1, \cdots, \alpha_n) = 0 \,\}.$$

The set $\mathcal{S}$ is not empty: $(\bar\alpha_1, \cdots, \bar\alpha_{n-1}, b)$ and $(a, \bar\beta_1, \cdots, \bar\beta_{n-1})$ belong to $\mathcal{S}$.

i) Existence of $(\alpha_1, \cdots, \alpha_n)$.
Let D be the open subset of $\mathbb{R}^{n-1} \times [a,b]$ defined by

$$D = \{\, (\alpha_1, \cdots, \alpha_n) \in \mathbb{R}^n : a < \alpha_1 < \cdots < \alpha_n \le b \,\}$$

and define $F : D \to \mathbb{R}^{n-1}$ by

$$F(\alpha_1, \cdots, \alpha_n) = (F_1(\alpha_1, \cdots, \alpha_n), \cdots, F_{n-1}(\alpha_1, \cdots, \alpha_n)).$$

The map F is $\mathcal{C}^1$ on D and its jacobian matrix is

$$\text{Jac } F(\alpha_1, \cdots, \alpha_n) = \begin{pmatrix} -f_1(\alpha_1) & +f_1(\alpha_2) & \cdots & (-1)^n f_1(\alpha_n) \\ -f_2(\alpha_1) & +f_2(\alpha_2) & \cdots & (-1)^n f_2(\alpha_n) \\ \vdots & \vdots & \ddots & \vdots \\ -f_{n-1}(\alpha_1) & +f_{n-1}(\alpha_2) & \cdots & (-1)^n f_{n-1}(\alpha_n) \end{pmatrix}$$

We see that

$$\left| \frac{\partial F}{\partial(\alpha_1, \cdots, \alpha_{n-1})}(\alpha_1, \cdots, \alpha_n) \right| = (-1)^{\frac{n(n-1)}{2}} \begin{vmatrix} f_1(\alpha_1) & \cdots & f_1(\alpha_{n-1}) \\ \vdots & \ddots & \vdots \\ f_{n-1}(\alpha_1) & \cdots & f_{n-1}(\alpha_{n-1}) \end{vmatrix}$$

which by the orientation condition Δ does not vanish and keeps a constant sign when $\alpha_1 < \alpha_2 < \cdots < \alpha_{n-1}$. Consider the equation

$$(\dagger) \qquad F\left((\alpha_1, \cdots, \alpha_{n-1}), \alpha_n\right) = 0.$$

Let $(\xi_1, \cdots, \xi_n) \in D$ verify $(\dagger)$ i.e.

$$(\xi_1, \cdots, \xi_n) \in \mathcal{S} \setminus \left\{(a, \bar{\beta}_1, \cdots, \bar{\beta}_{n-1})\right\}.$$

Such a point exists: for instance $(\bar{\alpha}_1, \cdots, \bar{\alpha}_{n-1}, b)$. We apply the implicit function theorem at $(\xi_1, \cdots, \xi_n)$. There exists an open interval I containing ξ_n, an open neighbourhood U of $(\xi_1, \cdots, \xi_{n-1})$, a continuous function

$$\psi : \begin{array}{ccc} I & \longrightarrow & U \\ \alpha_n & \longmapsto & (\alpha_1(\alpha_n), \cdots, \alpha_{n-1}(\alpha_n)) \end{array}$$

such that

$$\begin{array}{l} \forall\, ((\eta_1, \cdots, \eta_{n-1}), \eta_n) \in D \cap (U \times I) \\ \qquad F((\eta_1, \cdots, \eta_n)) = (c_1, \cdots, c_{n-1}) \iff (\eta_1, \cdots, \eta_{n-1}) = \psi(\eta_n). \end{array}$$

Moreover, ψ is $\mathcal{C}^1$ and we have

$$\alpha_j'(\alpha_n) = \frac{\begin{vmatrix} f_1(\alpha_1) & \cdots & f_1(\alpha_{j-1}) & f_1(\alpha_{j+1}) & \cdots & f_1(\alpha_n) \\ \vdots & \ddots & \vdots & \vdots & \ddots & \vdots \\ f_{n-1}(\alpha_1) & \cdots & f_{n-1}(\alpha_{j-1}) & f_{n-1}(\alpha_{j+1}) & \cdots & f_{n-1}(\alpha_n) \end{vmatrix}}{\begin{vmatrix} f_1(\alpha_1) & \cdots & f_1(\alpha_{n-1}) \\ \vdots & \ddots & \vdots \\ f_{n-1}(\alpha_1) & \cdots & f_{n-1}(\alpha_{n-1}) \end{vmatrix}}$$

so that $\alpha_j'(\alpha_n) > 0$ on I and the functions α_j are increasing. Lemma M yields a maximal interval I_M on which ψ can be extended. Let $\xi_n^* = \inf I_M$. The functions $\alpha_1, \cdots, \alpha_{n-1}$ being increasing on I_M, they admit limits

$$\xi_j^* = \lim_{\substack{\eta_n \to \xi_n^* \\ \eta_n > \xi_n^*}} \alpha_j(\eta_n).$$

Remark that $\xi_n^* < b$ since $\xi_n \leq b$.
By continuity

$$F(\xi_1^*, \cdots, \xi_n^*) = (0, \cdots, 0).$$

We claim that $\xi_1^* = a$.
Suppose $\xi_1^* > a$. By the maximality of I_M, $(\xi_1^*, \cdots, \xi_n^*)$ belongs to $\bar{D} \setminus D$ so that there exists $i \in \{1, \cdots, n-2\}$ such that $\xi_i^* = \xi_{i+1}^*$.
The $(n-1)$–uple $(\xi_1^*, \cdots, \xi_{i-1}^*, \xi_{i+2}^*, \cdots, \xi_n^*, b)$ and its associated set $E_{\xi^*}^-$ satisfy

$$\forall k \in \{1, \cdots, n-1\} \qquad \int_a^b f_k(s)\chi_{E_{\xi^*}^-}(s)\, ds = \int_a^b f_k(s)\nu(s)\, ds$$

so that the induction hypothesis implies

$$a < \xi_1^* < \cdots < \xi_{i-1}^* < \xi_{i+2}^* < \cdots < \xi_n^* < b < b$$

which is absurd.
Since $\xi_1^* = a$, the $(n-1)$–uple $(\xi_1^*, \cdots, \xi_{n-1}^*)$ is the one given by the theorem at rank $n-1$ so that $\xi_i^* = \bar{\beta}_{i-1}$ for each i in $\{2, \cdots, n\}$.
Thus for each point $(\xi_1, \cdots, \xi_n)$ of $\mathcal{S} \setminus \{(a, \bar{\beta}_1, \cdots, \bar{\beta}_{n-1})\}$ there exists a continuous arc in $\mathcal{S}$ joining $(\xi_1, \cdots, \xi_n)$ to $(a, \bar{\beta}_1, \cdots, \bar{\beta}_{n-1})$. This proves that $\mathcal{S}$ is arcwise connected.
At this stage we prove that $F_n(a, \bar{\beta}_1, \cdots, \bar{\beta}_{n-1})$ and $F_n(\bar{\alpha}_1, \cdots, \bar{\alpha}_{n-1}, b)$ have opposite signs. Since $F(\bar{\alpha}_1, \cdots, \bar{\alpha}_{n-1}, b) = 0$ then for each k in $\{1, \cdots, n-1\}$

$$-\sum_{\substack{0 \le i \le n-1 \\ i \text{ even}}} \int_{\bar{\alpha}_i}^{\bar{\alpha}_{i+1}} f_k(s)\nu(s)\, ds \;+ \sum_{\substack{0 \le i \le n-1 \\ i \text{ odd}}} \int_{\bar{\alpha}_i}^{\bar{\alpha}_{i+1}} f_k(s)(1-\nu(s))\, ds = 0$$

(where $\bar{\alpha}_0 = a, \bar{\alpha}_n = b$).
Put for k, j in $\{1, \cdots, n\}$

$$x_j^\alpha = (-1)^j, \quad a_{kj}^\alpha = \int_{\bar{\alpha}_{j-1}}^{\bar{\alpha}_j} f_k \nu_j^\alpha, \quad A^\alpha = \left(a_{kj}^\alpha\right)_{1 \le k,j \le n}$$

where

$$\nu_j^\alpha = \begin{cases} \nu & \text{if } j \text{ is odd} \\ 1-\nu & \text{if } j \text{ is even} \end{cases}$$

so that the above equations become

$$\forall k \in \{1, \cdots, n-1\} \qquad \sum_{j=1}^n a_{kj}^\alpha x_j^\alpha = 0.$$

Since $F_n(\bar{\alpha}_1, \cdots, \bar{\alpha}_{n-1}, b) = \sum_{j=1}^n a_{nj}^\alpha x_j^\alpha$, application of Lemma S gives

$$F_n(\bar{\alpha}_1, \cdots, \bar{\alpha}_{n-1}, b) = \frac{|A^\alpha|}{|A_{nn}^\alpha|}(-1)^n.$$

Similarly if we define for k, j in $\{1, \cdots, n\}$ ($\bar{\beta}_0 = a, \bar{\beta}_n = b$)

$$x_j^\beta = (-1)^{j+1}, \quad a_{kj}^\beta = \int_{\bar{\beta}_{j-1}}^{\bar{\beta}_j} f_k \nu_j^\beta, \quad A^\beta = \left(a_{kj}^\beta\right)_{1 \leq k, j \leq n}$$

where

$$\nu_j^\beta = \begin{cases} \nu & \text{if } j \text{ is even} \\ 1 - \nu & \text{if } j \text{ is odd} \end{cases}$$

then we have

$$F_n(a, \bar{\beta}_1, \cdots, \bar{\beta}_{n-1}) = \sum_{j=1}^n a_{nj}^\beta x_j^\beta = \frac{|A^\beta|}{|A_{nn}^\beta|}(-1)^{n+1}.$$

By condition Δ on $f_1, \cdots, f_n$ and Lemma 1, $|A^\alpha|$ and $|A^\beta|$ have the same sign, as do $|A_{nn}^\alpha|$ and $|A_{nn}^\beta|$. It follows that $F_n(a, \bar{\beta}_1, \cdots, \bar{\beta}_{n-1})$ and $F_n(\bar{\alpha}_1, \cdots, \bar{\alpha}_{n-1}, b)$ have opposite signs. Moreover the set $\mathcal{S}$ is connected, the map F_n is continuous on $\mathcal{S}$ and thus must vanish at a point $(\alpha_1, \cdots, \alpha_n)$ of $\mathcal{S}$. By the very definition of $\mathcal{S}$ we have also

$$\forall k \in \{1, \cdots, n-1\} \qquad F_k(\alpha_1, \cdots, \alpha_n) = 0$$

so that $(\alpha_1, \cdots, \alpha_n)$ solves the problem.

ii) Uniqueness of $(\alpha_1, \cdots, \alpha_n)$

Let $(\alpha_1, \cdots, \alpha_n)$ be in $\mathcal{S}$ with $a < \alpha_1 < \cdots < \alpha_n < b$ and build (I_M, ψ) as in the existence part. The maximal interval I_M is in fact $[\bar{\beta}_{n-1}, b]$ so that $(\alpha_1, \cdots, \alpha_n)$ belongs to a continuous path in $\mathcal{S}$ joining $(a, \bar{\beta}_1, \cdots, \bar{\beta}_{n-1})$ and $(\bar{\alpha}_1, \cdots, \bar{\alpha}_{n-1}, b)$. By local unicity of ψ near $(\bar{\alpha}_1, \cdots, \bar{\alpha}_{n-1}, b)$, the arc does not depend on $(\alpha_1, \cdots, \alpha_n)$ (recall that we apply the Implicit Function Theorem on the space $\mathbb{R}^{n-1} \times [a, b]$ and that b is an interior point of the topological space $[a, b]$). For each $\alpha_n \in]\bar{\beta}_{n-1}, b[$, we have

$$\begin{aligned}
\frac{d}{d\alpha_n} F_n(\psi(\alpha_n), \alpha_n) &= \sum_{i=1}^{n} \frac{\partial F_n}{\partial \alpha_i} \alpha_i'(\alpha_n) \\
&= \sum_{i=1}^{n} (-1)^i f_n(\alpha_i) \frac{\begin{vmatrix} f_1(\alpha_1) & \cdots & f_1(\alpha_{i-1}) & f_1(\alpha_{i+1}) & \cdots & f_1(\alpha_n) \\ \vdots & \ddots & \vdots & \vdots & \ddots & \vdots \\ f_{n-1}(\alpha_1) & \cdots & f_{n-1}(\alpha_{i-1}) & f_{n-1}(\alpha_{i+1}) & \cdots & f_{n-1}(\alpha_n) \end{vmatrix}}{\begin{vmatrix} f_1(\alpha_1) & \cdots & f_1(\alpha_{n-1}) \\ \vdots & \ddots & \vdots \\ f_{n-1}(\alpha_1) & \cdots & f_{n-1}(\alpha_{n-1}) \end{vmatrix}} \\
&= (-1)^n \frac{\begin{vmatrix} f_1(\alpha_1) & \cdots & f_1(\alpha_n) \\ \vdots & \ddots & \vdots \\ f_n(\alpha_1) & \cdots & f_n(\alpha_n) \end{vmatrix}}{\begin{vmatrix} f_1(\alpha_1) & \cdots & f_1(\alpha_{n-1}) \\ \vdots & \ddots & \vdots \\ f_{n-1}(\alpha_1) & \cdots & f_{n-1}(\alpha_{n-1}) \end{vmatrix}}
\end{aligned}$$

Thus F_n is strictly monotonic along the arc joining $(a, \bar{\beta}_1, \cdots, \bar{\beta}_{n-1})$ and $(\bar{\alpha}_1, \cdots, \bar{\alpha}_{n-1}, b)$ so that F_n vanishes only for one value α_n. Since this path is unique then the n–uple $(\alpha_1, \cdots, \alpha_n)$ is unique.

Existence and uniqueness of a n–uple β corresponding to ν at rank n follows from the fact that it coincides with the n–uple α corresponding to $1 - \nu$. $\square$

Second proof of the theorem.
We only deal with n–uples α, similar arguments hold for n–uples β.
Let

$$\Omega = \left\{ (\alpha_1, \cdots, \alpha_n) \in \mathbb{R}^n : a < \alpha_1 < \cdots < \alpha_n < b \right\}$$

and

$$F = \left\{ \left(\int_a^b f_1 \nu, \cdots, \int_a^b f_n \nu \right) : \nu \in L^1([a,b]), \quad 0 < \nu < 1 \right\}.$$

Put for each $(\alpha_1, \cdots, \alpha_n) \in \Omega$

$$\theta(\alpha_1, \cdots, \alpha_n) = \left(\int_a^b f_1 \chi_{E_\alpha^-}, \cdots, \int_a^b f_n \chi_{E_\alpha^-} \right)$$

where
$$E_\alpha^- = \bigcup_{\substack{0 \le i \le n \\ i \text{ odd}}} [\alpha_i, \alpha_{i+1}] \qquad (\alpha_{n+1} = b).$$

We first show that θ takes its values in F.
Let $(\alpha_1, \cdots, \alpha_n)$ in Ω; applying lemma 3 to $(f_1, \cdots, f_n)$, $(\alpha_1, \cdots, \alpha_n)$ and $\epsilon = 1/4$, we obtain an $(n+1)$–uple $(\lambda_0, \cdots, \lambda_n)$ such that:

$$\forall l \in \{0, \cdots, n\} \qquad 0 < \lambda_l < \epsilon \qquad \text{and}$$
$$\forall k \in \{1, \cdots, n\} \qquad \sum_{l=0}^{n} (-1)^l \lambda_l \int_{\alpha_l}^{\alpha_{l+1}} f_k = 0.$$

Put

$$\nu = \sum_{\substack{0 \le i \le n \\ i \text{ even}}} \lambda_i \chi_{[\alpha_i, \alpha_{i+1}]} + \sum_{\substack{0 \le i \le n \\ i \text{ odd}}} (1 - \lambda_i) \chi_{[\alpha_i, \alpha_{i+1}]}.$$

By construction we have $0 < \nu < 1$ and

$$\forall k \in \{1, \cdots, n\} \qquad \int_a^b f_k \nu = \int_a^b f_k \chi_{E_\alpha^-}$$

so that $\theta(\alpha_1, \cdots, \alpha_n)$ belongs to F.
The purpose of what follows is to show that the map $\theta : \Omega \to F$ satisfies the hypotheses of Caccioppoli Theorem.
1) Obviously Ω is arcwise connected.
2) The set F, being convex, is simply connected.

3) The map θ is a local homeomorphism at each point of Ω. In fact, θ is differentiable at each point $(\alpha_1, \cdots, \alpha_n)$ of Ω and its jacobian is

$$\det \operatorname{Jac} \theta(\alpha_1, \cdots, \alpha_n) = (-1)^{\frac{n(n+1)}{2}} \begin{vmatrix} f_1(\alpha_1) \cdots & f_1(\alpha_n) \\ \vdots \quad \ddots & \vdots \\ f_n(\alpha_1) \cdots & f_n(\alpha_n) \end{vmatrix}$$

does not vanish on Ω by condition Δ.

4) Finally θ is proper. Let K be a compact subset of F and let $\alpha^k = (\alpha_1^k, \cdots, \alpha_n^k)$ be a sequence of points in $\theta^{-1}(K)$. Since the sequence $(\theta(\alpha^k))_{k\in\mathbb{N}}$ is contained in $K \subset F$ then by compactness we may assume that there exists $\nu^* \in L^1([a,b])$, $0 < \nu^* < 1$ such that

$$(\circ) \qquad \lim_{k\to\infty} \theta(\alpha^k) = \left(\int_a^b f_1 \nu^*, \cdots, \int_a^b f_n \nu^* \right).$$

The closure $\bar{\Omega} = \{ (\alpha_1, \cdots, \alpha_n) \in \mathbb{R}^n : a \leq \alpha_1 \leq \cdots \leq \alpha_n \leq b \}$ of Ω is compact and therefore $(\alpha^k)_{k\in\mathbb{N}}$ admits a subsequence which converges to $\alpha^* = (\alpha_1^*, \cdots, \alpha_n^*) \in \bar{\Omega}$. By $(\circ)$ we have

$$\forall k \in \{1, \cdots, n\} \qquad \int_a^b f_k \chi_{E^-_{\alpha^*}} = \int_a^b f_k \nu^*$$

and the initial lemma implies $a < \alpha_1^* < \cdots < \alpha_n^* < b$. Thus α^* belongs to Ω and $\theta^{-1}(K)$ is compact.

By Caccioppoli Theorem, θ is a global homeomorphism. $\square$

As a consequence of theorem 1, we deduce the following

Theorem 2. *Let ν be a measurable function on $[a,b]$ such that $0 \leq \nu \leq 1$. Let $f_1, \cdots, f_n$ be n continuous functions on $[a,b]$. Assume that the interval $[a,b]$ is a finite (respectively countable) union of intervals on which the orientation condition Δ for $f_1, \cdots, f_n$ holds. Then there exists a set E which is finite (resp. countable) union of intervals such that*

$$\forall k \in \{1, \cdots, n\} \qquad \int_a^b f_k \chi_E = \int_a^b f_k \nu.$$

Proof. Under the hypothesis of the theorem, there exists a finite (respectively countable) family of disjoint open intervals $(I_j)_{j \in J}$ included in $[a,b]$ such that $[a,b] \setminus \bigcup_{j \in J} I_j$ is a negligeable set (with respect to Lebesgue measure) and the functions $f_1, \cdots, f_n$ satisfy condition Δ on each interval I_j, $j \in J$. We apply theorem 1 to $f_1, \cdots, f_n$ and ν on the interval I_j: there exists a set E_j included in I_j whose characteristic function has less than n discontinuity points such that

$$\forall k \in \{1, \cdots, n\} \quad \int_{I_j} f_k \chi_{E_j} = \int_{I_j} f_k \nu.$$

The set $E = \bigcup_{j \in J} E_j$ solves the problem. $\square$

Proposition 1 shows that the hypotheses of the theorem are fulfilled as soon as
- $f_1, \cdots, f_n$ are of class $\mathcal{C}^{n-1}$ on $[a,b]$,
- the set $Z = \{\, t \in [a,b] : \exists k \in \{1, \cdots, n\} \quad W(f_1, \cdots, f_k)(t) = 0 \,\}$ is finite (respectively is negligeable).

This result weakens Halkin's condition [9] that the interval is a countable union of intervals on which the functions $f_1, \cdots, f_n$ are analytical.

We consider a linear differential operator

$$L(D) = D^n + a_{n-1}(t)D^{n-1} + \cdots + a_1(t)D + a_0(t)$$

where D is the derivative operator $D = d/dt$ and $a_0, \cdots, a_{n-1}$ are n real–valued continuous functions on an interval $[a, b]$.

A generalized Rolle Theorem.
If $f_1, \cdots, f_{k+1}$ are of class $\mathcal{C}^k$ on $[a, b]$ we will denote their Wronskian by

$$W(f_1, \cdots, f_{k+1})(t) = \begin{vmatrix} f_1(t) & \cdots & f_{k+1}(t) \\ \vdots & \ddots & \vdots \\ f_1^{(k)}(t) & \cdots & f_{k+1}^{(k)}(t) \end{vmatrix}.$$

Definition (see [13]**).** The operator L possesses property W on $[a, b]$ if there exist $n-1$ functions $h_1, \cdots, h_{n-1}$ satisfying

$$\forall i \in \{1, \cdots, n-1\} \qquad L(D)(h_i) = 0 \quad \text{on } [a, b]$$
$$\forall t \in [a, b] \qquad W(h_1)(t) > 0, \cdots, W(h_1, \cdots, h_{n-1})(t) > 0.$$

We will use the fact that property W always holds locally: for each fixed t_0 in $[a, b]$, the $n-1$ solutions $h_1, \cdots, h_{n-1}$ to the $n-1$ Cauchy problems $(1 \leq i \leq n-1)$

$$L(D)(h_i) = 0, \quad h_i^{(k)}(t_0) = \delta(i-1, k) \quad 0 \leq k \leq n-1$$

(where $\delta(j, k) = 0$ if $j \neq k$ and $\delta(j, k) = 1$ if $j = k$)
are such that

$$\forall i \in \{1, \cdots, n-1\} \qquad W(h_1, \cdots, h_i)(t_0) = 1.$$

Therefore the inequalities

$$W(h_1)(t) > 0, \cdots, W(h_1, \cdots, h_{n-1})(t) > 0$$

hold in a neighbourhood of t_0.
The interest of property W is that it allows us to decompose the linear differential operator L into a "product" of differential expressions of the first order.

Theorem 3(see [13]**).** *Let the linear differential operator $L(D)$ possess property W on $[a,b]$. Then there exist $n+1$ functions $u_0,\cdots,u_n$ such that for each i in $\{0,\cdots,n\}$, u_i is of class $\mathcal{C}^{n-i}$ on $[a,b]$ and*

$$\forall y\in\mathcal{C}^n([a,b]) \qquad L\left(\frac{d}{dt}\right)y=u_n\frac{d}{dt}u_{n-1}\frac{d}{dt}u_{n-2}\cdots u_2\frac{d}{dt}u_1\frac{d}{dt}u_0y.$$

As a consequence of this decomposition, we derive a generalized Rolle theorem.
We say that the function f has N zeroes on $[a,b]$ if there exist l distinct points $t_1,\cdots,t_l$ and l positive integers $m_1,\cdots,m_l$ such that $m_1+\cdots+m_l=N$, f is at least m_k-1 times differentiable at t_k $(1\le k\le l)$ and

$$\forall k\in\{1,\cdots,l\} \quad \forall i\in\{0,\cdots,m_k-1\} \qquad f^{(i)}(t_k)=0.$$

Theorem 4. *Let the differential operator $L(D)$ possess property W on $[a,b]$. Let f be a piecewise $\mathcal{C}^n$ function of class $\mathcal{C}^{n-1}$ defined on $[a,b]$ and k be the number of discontinuity points of $f^{(n)}$ in $]a,b[$. If f vanishes at $(n+1)+k$ points in the interval $[a,b]$ then there exists ξ in $]a,b[$ such that $L(D)f(\xi)=0$.*

Proof. Let $t_1<\cdots<t_k$ be the distinct zeroes of f in $[a,b]$ with multiplicities $m_1,\cdots,m_k$. By applying Rolle Theorem successively on the intervals $[t_1,t_2],\cdots,[t_{k-1},t_k]$, we obtain $k-1$ points $t_1^1,\cdots,t_{k-1}^1$ such that $t_1<t_1^1<t_2<\cdots<t_{k-1}<t_{k-1}^1<t^k$ and

$$\forall i\in\{1,\cdots,k-1\} \qquad \frac{d}{dt}(u_0f)(t_i^1)=0.$$

Taking into account the multiple zeroes of f we see that $D(u_0f)$ admits $n+k$ zeroes on $[a,b]$. At step $n-1$, this process yields the existence of $k+2$ zeroes for the function

$$g=u_{n-1}\frac{d}{dt}u_{n-2}\cdots u_2\frac{d}{dt}u_1\frac{d}{dt}u_0f.$$

Either one of these zeroes is double or g possesses $k+2$ distinct roots: in this situation, at least two of them must lie in one of the $k+1$ intervals on which g is $\mathcal{C}^1$ and Rolle Theorem yields a zero of Dg. In both cases, we obtain the existence of a zero of the function $u_nD(g)=L(D)(f)$. □

We will use the following straightforward corollary of this theorem.

Corollary. *Let the differential operator $L(D)$ possess property W on $[a,b]$. Let f be a piecewise $\mathcal{C}^n$ function of class $\mathcal{C}^{n-1}$ defined on $[a,b]$, $f^{(n)}$ having at most n discontinuity points. Moreover, assume that*

$$\forall i\in\{0,\cdots,n-1\} \qquad f^{(i)}(a)=f^{(i)}(b)=0.$$

If $L(D)(f)$ does not vanish then f has no roots in $]a,b[$.

Let us remark that property W is essential for theorem 4 to hold.

Example. Let $f(t) = \sin t - \alpha t$ where α is chosen so that f admits three zeroes on the interval $[\pi/2, 3\pi]$. If we set $L(D) = D^2 + 1$ we have $L(D)(f)(t) = -\alpha t$ which does not vanish on $[\pi/2, 3\pi]$. However, it is easy to check that the operator L possesses property W on the interval $[a, b]$ if and only if $b - a < \pi$.

The resolvent and property Δ.

Definition. We say that $R : [a, b] \times [a, b] \mapsto \mathbb{R}$ is the resolvent of the operator L if for each fixed s in $[a, b]$ the function $t \mapsto R(t, s)$ solves the Cauchy problem

$$L(D)y = 0 \qquad y(s) = \cdots = y^{(n-2)}(s) = 0, \quad y^{(n-1)}(s) = 1.$$

As it is well known, R is of class $\mathcal{C}^{n+k}$ on $[a, b] \times [a, b]$ whenever the functions $a_0, \cdots, a_{n-1}$ are of class $\mathcal{C}^k$.

Proposition 2. *Let $R(t, s)$ be a function in $\mathcal{C}^{2n-2}([a, b] \times [a, b])$ satisfying*

$$\forall s \in [a, b] \qquad R(s, s) = \frac{\partial R}{\partial t}(s, s) = \cdots = \frac{\partial^{n-2} R}{\partial t^{n-2}}(s, s) = 0, \qquad \frac{\partial^{n-1} R}{\partial t^{n-1}}(s, s) = 1.$$

Then, for each t_0 in $[a, b]$, there exists $\delta > 0$ such that for every t in $[t_0 - \delta, t_0 + \delta] \cap [a, b]$, the functions

$$h_i^t(s) = \frac{\partial^{n-i} R}{\partial t^{n-i}}(t, s) \qquad 1 \le i \le n$$

satisfy condition Δ on the interval $[t_0 - \delta, t_0 + \delta] \cap [a, b]$.

Proof. For each $t_0 \in [a, b]$ and for each $k \in \{1, \cdots, n\}$, we have $W(h_1^{t_0}(t_0), \cdots, h_k^{t_0}(t_0)) = 1$; in fact

$$W(h_1^{t_0}(t_0), \cdots, h_k^{t_0}(t_0)) = \begin{vmatrix} \frac{\partial^{n-1} R}{\partial t^{n-1}}(t_0, t_0) & \cdots & \frac{\partial^{n-k} R}{\partial t^{n-k}}(t_0, t_0) \\ \vdots & \ddots & \vdots \\ \frac{\partial^{n+k-2} R}{\partial t^{n+k-2}}(t_0, t_0) & \cdots & \frac{\partial^{n-1} R}{\partial t^{n-1}}(t_0, t_0) \end{vmatrix} = \begin{vmatrix} 1 & 0 & \cdots & \cdots & 0 \\ * & 1 & 0 & \cdots & 0 \\ \vdots & \ddots & \ddots & \ddots & \vdots \\ * & \cdots & * & 1 & 0 \\ * & \cdots & \cdots & * & 1 \end{vmatrix} = 1.$$

By continuity, there exists $\delta > 0$ such that

$$\forall t, s \in [t_0 - \delta, t_0 + \delta] \cap [a, b] \quad \forall k \in \{1, \cdots, n\} \qquad W(h_1^t(s), \cdots, h_k^t(s)) > 0.$$

Proposition 1 yields the conclusion. $\square$

We consider the n–dimensional linear control system

$$\text{(P)} \qquad L(D)x = x^{(n)} + a_{n-1}(t)x^{(n-1)} + \cdots + a_1(t)x' + a_0(t)x \in [\phi_1, \phi_2] \text{ a.e. on } [a,b]$$

where the n functions $a_0, \cdots, a_{n-1}$ belong to $\mathcal{C}^{n-2}([a,b])$ and ϕ_1, ϕ_2 in $L^1([a,b])$ verify $\phi_1 \leq \phi_2$. The function y is said to be a bang–bang solution to (P) if it solves (P) and moreover

$$L(D)y \in \{\phi_1, \phi_2\}.$$

Given a solution x to (P), existence of a bang–bang solution y satisfying

$$\forall k \in \{0, \cdots, n-1\} \qquad y^{(k)}(a) = x^{(k)}(a), \quad y^{(k)}(b) = x^{(k)}(b)$$

has been proven for instance by Cesari [8] and Olech [12].

Theorem 5. *Let x in $W^{n,1}([a,b])$ be a solution to the control problem (P). Then there exist two bang–bang solutions y and z satisfying the tangency conditions*

$$\forall k \in \{0, \cdots, n-1\} \qquad y^{(k)}(a) = x^{(k)}(a) = z^{(k)}(a), \quad y^{(k)}(b) = x^{(k)}(b) = z^{(k)}(b)$$

and the inequalities

$$\forall t \in [a,b] \qquad y(t) \leq x(t) \leq z(t).$$

Moreover $L(D)y$ and $L(D)z$ are of the form $\chi_E \phi_1 + (1 - \chi_E)\phi_2$ where the set E is a finite union of intervals i.e. y, z are solutions associated to relay controls (see [4]*).*

Proof. We will only prove the existence of the function y; similar arguments hold for z. Let $R(t,s) \in \mathcal{C}^{2n-2}([a,b] \times [a,b])$ be the resolvent of the operator L. By proposition 1, there exists $\delta > 0$ such that the functions

$$h_i^t(s) = \frac{\partial^{n-i} R}{\partial t^{n-i}}(t,s) \qquad 1 \leq i \leq n$$

satisfy condition Δ on $[a, a+\delta]$ for each t in $[a, a+\delta]$. Choosing δ small enough, we may assume that the operator L possesses property W on $[a, a+\delta]$.

Suppose first that conditions W and Δ hold in the whole interval $[a,b]$.
It is not restrictive to assume $\phi_1 = 0$, $\phi_2 = \phi \geq 0$, $x(a) = x'(a) = \cdots = x^{(n-1)}(a) = 0$. In fact, let x satisfy $L(D)x \in [\phi_1, \phi_2]$. Then, if we set

$$x_a(t) = x(a) + \frac{x'(a)}{1!}(t-a) + \cdots + \frac{x^{(n-1)}(a)}{(n-1)!}(t-a)^{n-1}$$

the function $\tilde{x}$ defined by $\tilde{x} = x - x_a$ verifies

$$L(D)\tilde{x} \in [\psi_1, \psi_2], \qquad \forall k \in \{0, \cdots, n-1\} \quad \tilde{x}^{(k)}(a) = 0$$

where $\psi_i = \phi_i - L(D)x_a$, $i = 1, 2$. Clearly the function $\bar{x}$ defined by

$$\bar{x}(t) = \tilde{x}(t) - \int_a^t R(t,s)\psi_1(s)\,ds$$

satisfies

$$Ł(D)\bar{x} = L(D)\tilde{x} - \psi_1 \in [0, \psi_2 - \psi_1], \qquad \forall k \in \{0, \cdots, n-1\} \quad \bar{x}^{(k)}(a) = 0.$$

If we assume that the theorem holds in this situation, there exists a function $\bar{y}$ such that $\bar{y}^{(n)}$ has at most n first–kind discontinuity points and

$$Ł(D)\bar{y} \in \{0, \psi_2 - \psi_1\}, \qquad \forall k \in \{0, \cdots, n-1\} \quad \bar{y}^{(k)}(a) = 0, \quad \bar{y}^{(k)}(b) = \bar{x}^{(k)}(b),$$

$$\forall t \in [a,b] \qquad \bar{y}(t) \leq \bar{x}(t).$$

It is now easy to check that the function y defined by

$$y(t) = \bar{y}(t) + \int_a^t R(t,s)\psi_1(s)\,ds + x_a(t)$$

solves our problem.
We assume now

$$0 \leq \rho \leq \phi, \quad L(D)x = \rho, \quad \forall k \in \{0, \cdots, n-1\} \quad x^{(k)}(a) = 0$$

so that, with the notations of proposition 2, we have

$$\forall k \in \{0, \cdots, n-1\} \qquad x^{(k)}(t) = \int_a^t \frac{\partial^k R}{\partial t^k}(t,s)\rho(s)\,ds = \int_a^t h_{n-k}^t(s)\rho(s)\,ds.$$

Let $(\rho_m)_{m\in\mathbb{N}}$ and $(\phi_m)_{m\in\mathbb{N}}$ be two sequences of continuous functions such that

$$\forall t \in [a,b] \qquad 0 < \rho_m(t) < \phi_m(t), \qquad \rho_m \xrightarrow{L^1} \rho, \qquad \phi_m \xrightarrow{L^1} \phi$$

and set

$$x_m(t) = \int_a^t R(t,s)\rho_m(s)\,ds.$$

Clearly,

$$\forall k \in \{0, \cdots, n-1\} \qquad x_m^{(k)}(t) = \int_a^t \frac{\partial^k R}{\partial t^k}(t,s)\rho_m(s)\,ds = \int_a^t h_{n-k}^t(s)\rho_m(s)\,ds.$$

Since each ϕ_m is positive then the functions

$$f_i(s) = h_i^b(s)\phi_m(s) = \frac{\partial^{n-i} R}{\partial t^{n-i}}(b,s)\phi_m(s) \quad 1 \le i \le n$$

satisfy condition Δ on $[a,b]$. Then by theorem 1 applied to $f_1, \cdots, f_n$ and $\nu = \rho_m/\phi_m$, corresponding to each m there exists a unique n–uple $(\alpha_1^m, \cdots, \alpha_n^m)$ such that

$$(\alpha_0^m =)\, a < \alpha_1^m < \cdots < \alpha_n^m < b\, (= \alpha_{n+1}^m)$$

and if we set

$$E_m^- = \bigcup_{\substack{i \text{ odd} \\ 0 \le i \le n}} [\alpha_i^m, \alpha_{i+1}^m], \qquad y_m(t) = \int_a^t R(t,s)\phi_m(s)\chi_{E_m^-}(s)\,ds$$

then we have

$$\forall i \in \{1, \cdots, n\} \qquad \int_a^b f_i(s)\chi_{E_m^-}(s)\,ds = \int_a^b f_i(s)\nu(s)\,ds$$

i.e.

$$(*) \qquad \forall k \in \{0, \cdots, n-1\} \qquad \int_a^b \frac{\partial^k R}{\partial t^k}(b,s)\phi_m(s)\chi_{E_m^-}(s)\,ds = \int_a^b \frac{\partial^k R}{\partial t^k}(b,s)\rho_m(s)\,ds$$

so that

$$\forall k \in \{0, \cdots, n-1\} \qquad D^k(y_m - x_m)(a) = D^k(y_m - x_m)(b) = 0$$

and $L(D)(y_m - x_m) = \phi_m\chi_{E_m^-} - \rho_m$ does not vanish on $[a,b]$; since L possesses property W on $[a,b]$, the corollary to theorem 4 then implies that

$$(**) \qquad \forall t \in]a,b[\qquad y_m(t) \neq x_m(t).$$

Since by construction

$$y_m^{(n)}(a) = L(D)y_m(a) - \big(a_{n-1}(a)D^{n-1}y_m(a) + \cdots + a_1(a)Dy_m(a) + a_0(a)y_m(a)\big) = 0$$

and analogously $x_m^{(n)}(a) = \rho(a) > 0$ then by continuity $y_m < x_m$ in a neighbourhood $]a, a+\epsilon[$ of a, which together with $(**)$, yields the global inequality

$$(***) \qquad \forall t \in]a,b[\qquad y_m(t) < x_m(t).$$

By compactness we may assume

$$\forall i \in \{1, \cdots, n\} \qquad \lim_{m\to\infty} \alpha_i^m = \alpha_i.$$

Clearly

$$(\alpha_0 =)\, a \leq \alpha_1 \leq \cdots \leq \alpha_n \leq b\, (= \alpha_{n+1}).$$

Put

$$E^- = \bigcup_{\substack{i \text{ odd} \\ 0 \leq i \leq n}} [\alpha_i, \alpha_{i+1}], \qquad y(t) = \int_a^t R(t,s)\phi(s)\chi_{E^-}(s)\, ds.$$

Obviously, for all k in $\{0, \cdots, n-1\}$, $y^{(k)}(a) = 0$. Moreover, by passing through the limit in $(*)$ and $(***)$, we obtain

$$x^{(k)}(b) = \int_a^b \frac{\partial^k R}{\partial t^k}(b,s)\rho(s)\, ds = \int_a^b \frac{\partial^k R}{\partial t^k}(b,s)\phi(s)\chi_{E^-}(s)\, ds = y^{(k)}(b)$$

for all k in $\{0, \cdots, n-1\}$ and

$$\forall t \in [a,b] \qquad y(t) \leq x(t).$$

The function y solves our problem.

The general case. Since the functions

$$h_i^t(s) = \frac{\partial^{n-i} R}{\partial t^{n-i}}(t,s) \qquad 1 \leq i \leq n$$

satisfy locally condition Δ and the operator $L(D)$ possesses property W then by compactness there exists a subdivision

$$c_0 = a < c_1 < \cdots < c_l < b = c_{l+1}$$

of $[a,b]$ such that properties W and Δ hold on each interval $I_j = [c_j, c_{j+1}]$, $0 \leq j \leq l$. Let x solve $L(D)x = \rho$, $\phi_1 \leq \rho \leq \phi_2$; by the first case, for each j in $\{0, \cdots, l\}$ there exists a function y_j in $W^{n,1}(I_j)$ such that $y_j^{(n)}$ has only a finite number of first–kind discontinuities satisfying

$$\begin{array}{rl} & L(D)y_j \in \{\phi_1, \phi_2\} \text{ on } I_j \\ \forall k \in \{0, \cdots, n-1\} & y_j^{(k)}(c_j) = x^{(k)}(c_j), \quad y_j^{(k)}(c_{j+1}) = x^{(k)}(c_{j+1}) \end{array}$$

and

$$\forall t \in I_j \qquad y_j(t) \leq x_j(t).$$

The function $y \in W^{n,1}([a,b])$ obtained by glueing together the functions $y_0, \cdots, y_l$ is a solution to our problem. □

Remark. The proof of the theorem shows that if there exist n solutions $h_1, \cdots, h_n$ to $L(D)y = 0$ on $[a,b]$ satisfying

$$W(h_1) > 0, \cdots, W(h_1, \cdots, h_n) > 0 \quad \text{on } [a,b]$$

then the resolvent of the operator L satisfies condition Δ on $[a,b]$ and therefore the bang–bang solutions y and z can be built in such a way that $L(D)y$ and $L(D)z$ are of the form $\chi_E\phi_1 + (1-\chi_E)\phi_2$ where the characteristic function of the set E has less than n discontinuity points on $[a,b]$.
For instance, this is the case when $L(D) = D^n$ (see example 2 following the definition of condition Δ) or when

$$L(D) = D^n + a_{n-1}D^{n-1} + \cdots + a_1 D + a_0$$

and the algebraic equation

$$X^n + a_{n-1}X^{n-1} + \cdots + a_1 X + a_0 = 0$$

has n distinct real roots.

The reachable set of bang–bang constrained solutions

Let $a_0, \cdots, a_{n-1} \in \mathcal{C}^{n-2}([a,b])$ and ϕ_1, ϕ_2 in $L^1([a,b])$ verify $\phi_1 \leq \phi_2$. Consider the control problem (P_a)

$$L(D)x = x^{(n)} + a_{n-1}(t)x^{(n-1)} + \cdots + a_1(t)x' + a_0(t)x \in [\phi_1, \phi_2] \text{ a.e. on } [a,b]$$

with the initial conditions

$$\forall k \in \{0, \cdots, n-1\} \qquad x^{(k)}(a) = x_k$$

where $x_0, \cdots, x_{n-1}$ are n fixed real numbers. Let c be an arbitrary function defined on $I = [a,b]$ and consider the reachable sets $\mathcal{X}_I^c$ and $\mathcal{Y}_I^c$ associated to (P_a) defined by

$$\mathcal{X}_I^c = \{(x(b), x'(b), \cdots, x^{(n-1)}(b)) : \forall t \in I \ x(t) \leq c(t), \ x \text{ solution to } (P_a)\},$$
$$\mathcal{Y}_I^c = \{(y(b), y'(b), \cdots, y^{(n-1)}(b)) : \forall t \in I \ y(t) \leq c(t), \ y \text{ bang–bang solution to } (P_a)\}.$$

Then theorem 5 yields the following result.

Theorem 6. *The sets $\mathcal{X}_I^c$ and $\mathcal{Y}_I^c$ coincide; in particular, the reachable set associated to bang–bang constrained solutions $\mathcal{Y}_I^c$ is convex.*

An Application to the Calculus of Variations

Theorem 7. *Let $a_0, \cdots, a_{n-1} \in \mathcal{C}^{n-2}([a,b])$, $\phi_1, \phi_2 \in L^1([a,b])$ verify $\phi_1 \leq \phi_2$ and let L be the linear differential operator of order n defined by*

$$L(D) = D^n + a_{n-1}(t)D^{n-1} + \cdots + a_1(t)D + a_0(t).$$

Let $x_0^1, \cdots, x_{n-1}^1$ and $x_0^2, \cdots, x_{n-1}^2$ be $2n$ fixed real numbers.
Then there exists a dense subset $\mathcal{D}$ of $\mathcal{C}(\mathbb{R})$ for the uniform convergence such that for g in $\mathcal{D}$ the problem

$$\min \left\{ \int_a^b g(x(t))\, dt + \int_a^b h(L(D)x(t))\, dt \ : \ x \in W^{n,1}([a,b]), \right.$$
$$\left. \forall k \in \{0, \cdots, n-1\} \quad x^{(k)}(a) = x_k^1, \quad x^{(k)}(b) = x_k^2 \right\}$$

admits at least one solution for every lower semicontinuous function h satisfying the growth condition $h(u) \geq c\psi(|u|)$, ψ being l.s.c. and convex, $\lim_{r \to +\infty} \psi(r)/r = +\infty$.

Proof. With our theorem 5 and the preceding application, the proof is a direct adaptation of the proof given in [6] for the case $L(D) = D$. $\square$

References

1. M. Amar–A. Cellina, *On passing to the limit for non convex variational problems*, preprint S.I.S.S.A. **176/91/M** (1991).
2. A. Ambrosetti, *Global inversion theorems and applications to nonlinear problems*, Conferenze del seminario di matematica dell'Università di Bari **168** (79), 211–232.
3. A. Ambrosetti–G. Prodi, *Analisi Non Lineare, I Quaderno*, Scuola Normale Superiore di Pisa, 1973.
4. A. Andreini–A. Bacciotti, *Bang–bang and relay principles for time–dependent constraint sets*, J.O.T.A **31 no. 2** (1980), 207–216.
5. A. Cellina–G. Colombo, *On a classical problem of the calculus of variations without convexity assumptions*, Ann. Inst. Henri Poincaré **7** (1990), 97–106.
6. A. Cellina–C. Mariconda, *The existence question in the calculus of variations: a density result*, Proc. A.M.S. (1993) (to appear).
7. R. Cerf–C. Mariconda, *On bang–bang constrained solutions of a control system*, preprint S.I.S.S.A. Ref. 166/92/M (1992).
8. L. Cesari, *Optimization–theory and applications*, Springer–Verlag, New–York, 1983.
9. H. Halkin, *On a generalization of a theorem of Lyapounov*, Jour. Math. Anal. Appl. **10** (1965), 325-329.
10. H. Halkin, *A generalization of La Salle's bang–bang principle*, J. S.I.A.M. Control Ser. A **2** (1965), 199-202.
11. P. Marcellini, *Alcune osservazioni sull'esistenza del minimo di integrali del calcolo delle variazioni senza ipotesi di convessitá*, Rendiconti di Matematica (2) **13** (1980), 271–281.
12. C. Olech, *Extremal solutions of a control system*, J.Diff.Eq. **2** (1966), 74–101.
13. G. Pólya, *On the mean–value theorem corresponding to a given liner homogeneous differential equation*, Trans. A.M.S. **24** (1922), 312–324.
14. J. P. Raymond, *Existence theorems in optimal control problems without convexity assumptions*, J.O.T.A. **67 no. 1** (1990), 109–132.

R. Cerf, Département de Mathématiques et d'Informatique, Ecole Normale Supérieure, 45 rue d'Ulm, 75005 Paris–France

C. Mariconda, Dipartimento di Matematica pura e applicata, Università di Padova, 7 via Belzoni, 35100 Padova–Italy

E-mail address: Raphael.Cerf@ens.fr -- mariconda@pdmat1.unipd.it