A Curie-Weiss Model of Self-Organized Criticality

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Abstract

We try to design a simple model exhibiting self-organized criticality, which is amenable to a rigorous mathematical analysis. To this end, we modify the generalized Ising Curie-Weiss model by implementing an automatic control of the inverse temperature. For a class of symmetric distributions whose density satisfies some integrability conditions, we prove that the sum S_n of the random variables behaves as in the typical critical generalized Ising Curie-Weiss model. The fluctuations are of order $n^{3/4}$ and the limiting law is $C\exp(-\lambda x^4)\,dx$ where C and λ are suitable positive constants.

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1 Introduction

In their famous article [2], Per Bak, Chao Tang and Kurt Wiesenfeld showed that certain complex systems are naturally attracted by critical points, without any external intervention. These systems exhibit the phenomenon of self-organized criticality.

Self-organized criticality can be observed empirically or simulated on a computer in various models. However the mathematical analysis of these models turns out to be extremely difficult. Even models whose definition is seemingly simple, such as those describing the dynamics of a sandpile, are poorly understood. Other challenging models are the models for forest fires [14], which are built with the help of the classical percolation process. Some simple models of evolutions also lead to critical behaviours [6].

Our goal here is to design a model exhibiting self-organized criticality, which is as simple as possible, and which is amenable to a rigorous mathematical analysis. The most widely studied model in statistical mechanics, which exhibits a phase transition and presents critical states, is the Ising model. Its mean field version is called the Ising Curie-Weiss model (see for instance [9]). It has been extended to real-valued spins by Richard S. Ellis and Charles M. Newman [10], in the so called generalized Ising Curie-Weiss model. This model is our starting point and we will modify it in order to build a system of interacting random variables, which exhibits a phenomenon of self-organized criticality.

Let us first recall the definition and some results on the generalized Ising Curie-Weiss model. Let ρ be a symmetric probability measure on $\mathbb{R}$ with positive variance σ^2 and such that

$$\forall t \geq 0 \qquad \int_{\mathbb{R}} \exp(tx^2)\, d\rho(x) < \infty$$

The generalized Ising Curie-Weiss model associated to ρ and the inverse temperature $\beta > 0$ is defined through an infinite triangular array of real-valued random variables $(X_n^k)_{1 \leq k \leq n}$ such that, for all $n \geq 1$, $(X_n^1, \dots, X_n^n)$ has the distribution

$$d\mu_{n,\rho,\beta}(x_1, \dots, x_n) = \frac{1}{Z_n(\beta)} \exp\left(\frac{\beta}{2} \frac{(x_1 + \cdots + x_n)^2}{n}\right) \prod_{i=1}^{n} d\rho(x_i)$$

where $Z_n(\beta)$ is a normalization. For any $n \geq 1$, we set $S_n = X_n^1 + \cdots + X_n^n$. When $\rho = (\delta_{-1} + \delta_1)/2$, we recover the classical Ising Curie-Weiss model.

We denote by L the Log-Laplace of ρ (see section 6). Richard S. Ellis and Theodor Eisele have shown in [8] that, if $L^{(3)}(t) \leq 0$ for any $t \geq 0$, then there exists a map m which is null on $]0, 1/\sigma^2]$, real analytic and positive on $]1/\sigma^2, +\infty[$, and such that

$$\frac{S_n}{n} \underset{n\to\infty}{\overset{\mathcal{L}}{\longrightarrow}} \begin{cases} \delta_0 & \text{if} \quad \beta \leq 1/\sigma^2 \\ \frac{1}{2}(\delta_{-m(\beta)} + \delta_{m(\beta)}) & \text{if} \quad \beta > 1/\sigma^2 \end{cases}$$

The point $1/\sigma^2$ is a critical value and the function m cannot be extended analytically around $1/\sigma^2$. The main theorem of [10] states that, if $\beta < 1/\sigma^2$, then, under $\mu_{n,\rho,\beta}$,

$$\frac{S_n}{\sqrt{n}} \underset{n\to\infty}{\overset{\mathcal{L}}{\longrightarrow}} \mathcal{N}\left(0, \frac{\sigma^2}{1 - \beta\sigma^2}\right)$$

Moreover, for $\beta = 1/\sigma^2$, there exists $k \in \mathbb{N}\backslash\{0,1\}$ and $\lambda > 0$ such that, under $\mu_{n,\rho,\beta}$,

$$\frac{S_n}{n^{1-1/2k}} \xrightarrow[n\to\infty]{\mathcal{L}} C_{k,\lambda} \exp\left(-\lambda \frac{s^{2k}}{(2k)!}\right) ds$$

where $C_{k,\lambda}$ is a normalization. This is a consequence of [10] and some properties of m explained in [8] implying that the function $s \longmapsto L(s\sqrt{\beta}) - s^2/2$ has a unique maximum at 0 whenever $\beta \leq 1/\sigma^2$ (see [12] for the details).

We will transform the previous probability distribution in order to obtain a model which presents a phenomenon of self-organized criticality, i.e., a model which evolves towards the critical state $\beta = 1/\sigma^2$ of the previous model. More precisely, the critical generalized Ising Curie-Weiss model is the model where $(X_n^1, \dots, X_n^n)$ has the distribution

$$\frac{1}{Z_n} \exp\left(\frac{(x_1 + \cdots + x_n)^2}{2n\sigma^2}\right) \prod_{i=1}^n d\rho(x_i)$$

We search an automatic control of the inverse temperature β, which would be a function of the random variables in the model, so that, when n goes to $+\infty$, β converges towards the critical value of the model. We start with the following observation : if $(Y_n)_{n\in\mathbb{N}}$ is a sequence of independent random variables with identical distribution ρ, then, by the law of large numbers,

$$\frac{Y_1^2 + \cdots + Y_n^2}{n} \underset{n\to\infty}{\longrightarrow} \sigma^2 \quad \text{a.s}$$

Thus we are tempted to « replace β by $n\,(x_1^2 + \cdots + x_n^2)^{-1}$ » in the distribution

$$\frac{1}{Z_n} \exp\left(\frac{\beta}{2}\frac{(x_1 + \cdots + x_n)^2}{n}\right) \prod_{i=1}^n d\rho(x_i)$$

Hence the model we consider in this paper is given by the distribution

$$\frac{1}{Z_n} \exp\left(\frac{1}{2}\frac{(x_1 + \cdots + x_n)^2}{x_1^2 + \cdots + x_n^2}\right) \prod_{i=1}^n d\rho(x_i)$$

These considerations suggest that this model should evolve spontaneously towards a critical state. We will prove rigorously that our model indeed exhibits a phenomenon of self-organized criticality.

Our main result states that, if ρ has an even density satisfying some integrability conditions, then, asymptotically, the sum S_n of the random variables behaves as in the typical critical generalized Ising Curie-Weiss model : if μ_4 denotes the fourth moment of ρ, then

$$\frac{\mu_4^{1/4} S_n}{\sigma^2 n^{3/4}} \xrightarrow[n\to\infty]{\mathcal{L}} \left(\frac{4}{3}\right)^{1/4} \Gamma\left(\frac{1}{4}\right)^{-1} \exp\left(-\frac{s^4}{12}\right) ds$$

In section 2 we define properly our model. We state our main results and the strategy for proving them in section 3. Next we split the proofs in the remaining sections (4-10).

2 The model

Let ρ be a probability measure on $\mathbb{R}$, which is not the Dirac mass at 0. We consider an infinite triangular array of real-valued random variables $(X_n^k)_{1\leq k\leq n}$ such that for all $n \geq 1$, $(X_n^1, \ldots, X_n^n)$ has the distribution $\widetilde{\mu}_{n,\rho}$, where

$$d\widetilde{\mu}_{n,\rho}(x_1,\ldots,x_n) = \frac{1}{Z_n}\exp\left(\frac{1}{2}\frac{(x_1+\cdots+x_n)^2}{x_1^2+\cdots+x_n^2}\right)\mathbb{1}_{\{x_1^2+\cdots+x_n^2>0\}}\prod_{i=1}^n d\rho(x_i)$$

with

$$Z_n = \int_{\mathbb{R}^n}\exp\left(\frac{1}{2}\frac{(x_1+\cdots+x_n)^2}{x_1^2+\cdots+x_n^2}\right)\mathbb{1}_{\{x_1^2+\cdots+x_n^2>0\}}\prod_{i=1}^n d\rho(x_i)$$

We define $S_n = X_n^1 + \cdots + X_n^n$ and $T_n = (X_n^1)^2 + \cdots + (X_n^n)^2$.

The indicator function in the density of the distribution $\widetilde{\mu}_{n,\rho}$ helps to avoid any problem of definition if $\rho(\{0\})$ is positive, since, if $\rho(\{0\}) > 0$, the event $\{x_1^2 + \cdots + x_n^2 = 0\}$ may occur with positive probability. We notice that, unlike the generalized model, our model is defined for any probability measure. Indeed $x \longmapsto x^2$ is a convex function, therefore

$$\forall (x_1,\ldots,x_n)\in\mathbb{R}^n \qquad \left(\sum_{i=1}^n x_i\right)^2 = n^2\left(\sum_{i=1}^n \frac{x_i}{n}\right)^2 \leq n\sum_{i=1}^n x_i^2$$

Thus for any $n \geq 1$, $1 \leq Z_n \leq e^{n/2} < +\infty$.

If we choose $\rho = (\delta_{-1} + \delta_1)/2$, we obtain the classical Ising Curie-Weiss model at the critical value.

3 Convergence theorems

We state here our main results.

By the classical law of large numbers, if ρ is centered and has variance σ^2, then, under $\rho^{\otimes n}$, $(S_n/n, T_n/n)$ converges in probability towards $(0, \sigma^2)$. The next theorem shows that, under the law $\widetilde{\mu}_{n,\rho}$, given certain conditions, $(S_n/n, T_n/n)$ also converges in probability to $(0, \sigma^2)$

Theorem 1. *Let ρ be a symmetric probability measure on $\mathbb{R}$ with positive variance σ^2 and such that the function*

$$\Lambda : (u,v) \longmapsto \ln\int_{\mathbb{R}} e^{uz+vz^2}\, d\rho(z)$$

is finite in an open neighbourhood of $(0,0)$. We suppose that one of the following conditions holds :

(a) *ρ has a density*

(b) *ρ is the sum of a finite number of Dirac masses*

(c) *There exists $c > 0$ such that $\rho(]0,c[) = 0$*

(d) *$\rho(\{0\}) < 1/\sqrt{e}$*

Then, under $\widetilde{\mu}_{n,\rho}$, $(S_n/n, T_n/n)$ converges in probability towards $(0, \sigma^2)$.

By the classical central limit theorem, under $\rho^{\otimes n}$, $S_n/\sqrt{n}$ converges in distribution to a normal distribution with mean zero and variance σ^2. The following theorem, shows that, given certain conditions, under $\widetilde{\mu}_{n,\rho}$, $S_n/n^{3/4}$ converges towards a specific distribution.

Theorem 2. *Let ρ be a probability measure on $\mathbb{R}$ with a density f satisfying :*

(a) *f is even*

(b) *There exists $v_0 > 0$ such that*

$$\int_{\mathbb{R}} e^{v_0 z^2} f(z)\, dz < +\infty$$

(c) *There exists $p \in]1,2]$ such that*

$$\int_{\mathbb{R}^2} f^p(x+y) f^p(y) |x|^{1-p}\, dx\, dy < +\infty$$

Let σ^2 be the variance of ρ and let μ_4 be the fourth moment of ρ. We have

$$\frac{\mu_4^{1/4} S_n}{\sigma^2 n^{3/4}} \xrightarrow[n\to\infty]{\mathcal{L}} \left(\frac{4}{3}\right)^{1/4} \Gamma\left(\frac{1}{4}\right)^{-1} \exp\left(-\frac{s^4}{12}\right)\, ds$$

The convergence can equivalently be rewritten as

$$\frac{S_n}{n^{3/4}} \xrightarrow[n\to\infty]{\mathcal{L}} \left(\frac{4\mu_4}{3\sigma^8}\right)^{1/4} \Gamma\left(\frac{1}{4}\right)^{-1} \exp\left(-\frac{\mu_4}{12\sigma^8} s^4\right)\, ds$$

We prove this convergence in section 10.

The following corollary is a version of theorem 2 with an hypothesis which is weaker but easier to check.

Corollary 3. *Let ρ be a probability measure on $\mathbb{R}$ with an even and bounded density f such that*

$$\exists\, v_0 > 0 \qquad \int_{\mathbb{R}} e^{v_0 z^2}\, d\rho(z) < +\infty$$

Let σ^2 be the variance of ρ and let μ_4 be the fourth moment of ρ. Then

$$\frac{\mu_4^{1/4} S_n}{\sigma^2 n^{3/4}} \xrightarrow[n\to\infty]{\mathcal{L}} \left(\frac{4}{3}\right)^{1/4} \Gamma\left(\frac{1}{4}\right)^{-1} \exp\left(-\frac{s^4}{12}\right)\, ds$$

Proof. We check that the hypothesis of the corollary imply the condition (c) of theorem 2. We have

$$\begin{aligned}
&\int_{\mathbb{R}^2} f^{3/2}(x+y) f^{3/2}(y) |x|^{-1/2}\, dx\, dy \\
&\quad = \int_{[-1,1]^2} \frac{f^{3/2}(x+y) f^{3/2}(y)}{|x|^{1/2}}\, dx\, dy + \int_{([-1,1]^2)^c} \frac{f^{3/2}(x+y) f^{3/2}(y)}{|x|^{1/2}}\, dx\, dy \\
&\quad \leq \left(\sup_{[-2,2]} |f|\right)^3 \int_{[-1,1]^2} \frac{1}{|x|^{1/2}}\, dx\, dy + \int_{([-1,1]^2)^c} f^{3/2}(x+y) f^{3/2}(y)\, dx\, dy \\
&\quad \leq \left(\sup_{[-2,2]} |f|\right)^3 \int_{[-1,1]} \frac{2}{|x|^{1/2}}\, dx + \left(\int_{\mathbb{R}} |f(x)|^{3/2}\, dx\right)^2
\end{aligned}$$

The second inequality is obtained by applying Fubini's theorem. The first term is finite and the second too because, by the Cauchy-Schwarz inequality,

$$\begin{aligned}\int_{\mathbb{R}} |f(x)|^{3/2}\,dx &\leq \|f\|_\infty \int_{\mathbb{R}} f(x)^{1/2}\,dx = \|f\|_\infty \int_{\mathbb{R}} e^{v_0x^2/2} f(x)^{1/2} e^{-v_0x^2/2}\,dx \\ &\leq \|f\|_\infty \left(\int_{\mathbb{R}} e^{v_0x^2} f(x)\,dx\right)^{1/2} \left(\int_{\mathbb{R}} e^{-v_0x^2}\,dx\right)^{1/2} < +\infty\end{aligned}$$

Thus, with $p = 3/2 \in]1,2]$, the function $(x,y) \longmapsto f^p(x+y)f^p(y)|x|^{1-p}$ is integrable. □

For instance, if ρ has a bounded support and a density which is even and continuous on it, then the hypothesis of the theorem are fulfilled.

We end this section by explaining the strategy for proving these results.

We denote by ν_ρ the law of (Z, Z^2) where Z is a random variable with distribution ρ. By proposition 4, a possible approach to obtain a limit law for (S_n, T_n), correctly renormalized, under $\widetilde{\mu}_{n,\rho}$, is to compute the density of ν_ρ^{*n} for n large enough, when ρ has a specific density. We will use this approach in the section 5 in the case of Gaussian distributions.

A more robust approach to obtain a limit law for (S_n, T_n), correctly renormalized, under $\widetilde{\mu}_{n,\rho}$, is to use the theory of large deviations. We denote by $\widetilde{\nu}_{n,\rho}$ the law of $(S_n/n, T_n/n)$ under $\rho^{\otimes n}$ and by $\theta_{n,\rho}$ the law of $(S_n/n, T_n/n)$ under $\widetilde{\mu}_{n,\rho}$. Proposition 4, presented in the next section, states that, if A is a subset of $\mathbb{R}^2$, then

$$\begin{aligned}\theta_{n,\rho}(A) = \widetilde{\mu}_{n,\rho}\left(\left(\frac{S_n}{n}, \frac{T_n}{n}\right) \in A\right) &= \frac{1}{Z_n} \int_A \exp\left(\frac{nx^2}{2y}\right) \mathbb{1}_{\{y>0\}}\,d\widetilde{\nu}_{n,\rho}(x,y) \\ &= \frac{\displaystyle\int_A \exp\left(\frac{nx^2}{2y}\right) \mathbb{1}_{\{y>0\}}\,d\widetilde{\nu}_{n,\rho}(x,y)}{\displaystyle\int_{\mathbb{R}^2} \exp\left(\frac{nx^2}{2y}\right) \mathbb{1}_{\{y>0\}}\,d\widetilde{\nu}_{n,\rho}(x,y)}\end{aligned}$$

By convexity of $t \longmapsto t^2$, we have $S_n^2 \leq nT_n$ for any $n \geq 1$. We define

$$\Delta = \{\,(x,y) \in \mathbb{R}^2 : x^2 \leq y\,\} \qquad \text{and} \qquad \Delta^* = \Delta \backslash \{(0,0)\}$$

Hence we have $\widetilde{\nu}_{n,\rho}(\Delta^c) = 0$. Therefore

$$\theta_{n,\rho}(A) = \frac{\displaystyle\int_{A\cap\Delta^*} \exp\left(\frac{nx^2}{2y}\right)\,d\widetilde{\nu}_{n,\rho}(x,y)}{\displaystyle\int_{\Delta^*} \exp\left(\frac{nx^2}{2y}\right)\,d\widetilde{\nu}_{n,\rho}(x,y)}$$

For $n \geq 1$, under $\rho^{\otimes n}$,

$$\left(\frac{S_n}{n}, \frac{T_n}{n}\right) = \frac{1}{n}\sum_{i=1}^n (X_i, X_i^2)$$

where $(X_n, X_n^2)_{n\geq 1}$ is a sequence of independent and identically distributed random variables with distribution ν_ρ. Cramér's theorem implies that $(\widetilde{\nu}_{n,\rho})_{n\geq 1}$

satisfies a weak large deviations principle with speed n, governed by the rate function

$$I : (x,y) \longmapsto \sup_{(u,v)\in\mathbb{R}^2} (xu + yv - \Lambda(u,v))$$

where for any $(u,v) \in \mathbb{R}^2$,

$$\Lambda(u,v) = \ln \int_{\mathbb{R}^2} e^{us+vt}\, d\nu_\rho(s,t) = \ln \int_{\mathbb{R}} e^{uz+vz^2}\, d\rho(z)$$

We note that $\Lambda(u,v)$ can be equal to $+\infty$. If we suppose that the function Λ is finite in an open neighbourhood of $(0,0)$, then I is a good rate function and $(\widetilde{\nu}_{n,\rho})_{n\geq 1}$ satisfies a large deviations principle with speed n, governed by I (see the section 19 of [5] for these results).

Here is a classical heuristic : as n goes to $+\infty$, the distribution $\theta_{n,\rho}$ concentrates exponentially fast on the minima of the function

$$G = I - F - \inf_{\Delta^*} (I - F)$$

where F denotes the map $(x,y) \longmapsto x^2/(2y)$. Thus, if G has a unique minimum at $(x_0,y_0) \in \Delta^*$, then, under $\widetilde{\mu}_{n,\rho}$, $(S_n/n, T_n/n)$ converges in probability to (x_0,y_0). Moreover, for n large enough, $\widetilde{\nu}_{n,\rho}$ can roughly be approximated by the distribution $C_n \exp(-nI(x,y))\, dx\, dy$ where C_n is a renormalization constant. Thus, for each bounded continuous function h and $\alpha, \beta > 0$,

$$\mathbb{E}_{\widetilde{\mu}_n}\left(h\left(\frac{S_n - nx_0}{n^{1-\alpha}}\right)\right) \approx \frac{\displaystyle\int_{\Delta^*} h((x-x_0)n^\alpha)\exp\left(-nG(x,y)\right)\, dx\, dy}{\displaystyle\int_{\Delta^*} \exp\left(-nG(x,y)\right)\, dx\, dy}$$

$$\approx \frac{\displaystyle\int_{\Delta^*} h(x)\exp\left(-nG\left(xn^{-\alpha}+x_0, yn^{-\beta}+y_0\right)\right)\, dx\, dy}{\displaystyle\int_{\Delta^*} \exp\left(-nG\left(xn^{-\alpha}+x_0, yn^{-\beta}+y_0\right)\right)\, dx\, dy}$$

We use then Laplace method. The key point is the study of the function G in the neighbourhood of its minimum (x_0,y_0). We have to find four values $a \in \mathbb{N}$, $b \in \mathbb{N}$, $A > 0$ and $B > 0$ such that, uniformly on a neighbourhood of (x_0,y_0),

$$-nG\left(xn^{-\alpha}+x_0, yn^{-\beta}+y_0\right) \underset{n\to\infty}{\longrightarrow} -Ax^\alpha - By^\beta$$

After computing the law of $(S_n/n, T_n/n)$ in section 4 and giving some general results on the Cramér transform in section 6, we study the minima of $I - F$ in section 7. Next we state a variant of Varadhan's lemma in section 8, which helps us to prove theorems 1 and 2 respectively in sections 9 and 10.

4 Computation of the law of $(S_n/n, T_n/n)$

In this section we compute the laws of (S_n, T_n) and $(S_n/n, T_n/n)$ under $\widetilde{\mu}_{n,\rho}$.

Proposition 4. *We denote by ν_ρ the law of (Z, Z^2) where Z is a random variable with the distribution ρ. Under $\widetilde{\mu}_{n,\rho}$, the law of (S_n, T_n) is*

$$\frac{1}{Z_n} \exp\left(\frac{x^2}{2y}\right) \mathbb{1}_{\{y>0\}} \, d\nu_\rho^{*n}(x,y)$$

We denote by $\widetilde{\nu}_{n,\rho}$ the law of $(S_n/n, T_n/n)$ under $\rho^{\otimes n}$. Under $\widetilde{\mu}_{n,\rho}$, the law of $(S_n/n, T_n/n)$ is

$$\frac{1}{Z_n} \exp\left(\frac{nx^2}{2y}\right) \mathbb{1}_{\{y>0\}} \, d\widetilde{\nu}_{n,\rho}(x,y)$$

Proof. Let $f : \mathbb{R}^2 \longrightarrow \mathbb{R}$ be a bounded measurable function. We have

$$\mathbb{E}_{\tilde{\mu}_{n,\rho}}(f(S_n, T_n)) = \frac{1}{Z_n} \int_{\mathbb{R}^n} f(x_1 + \cdots + x_n, x_1^2 + \cdots + x_n^2) \exp\left(\frac{1}{2} \frac{(x_1 + \cdots + x_n)^2}{x_1^2 + \cdots + x_n^2}\right) \mathbb{1}_{\{x_1^2 + \cdots + x_n^2 > 0\}} \prod_{i=1}^n d\rho(x_i)$$

We define

$$h : (x,y) \in \mathbb{R}^2 \longmapsto f(x,y) \exp\left(\frac{x^2}{2y}\right) \mathbb{1}_{\{y>0\}}$$

We have then

$$\begin{aligned}
\mathbb{E}_{\tilde{\mu}_{n,\rho}}(f(S_n, T_n)) &= \frac{1}{Z_n} \int_{\mathbb{R}^n} h(x_1 + \cdots + x_n, x_1^2 + \cdots + x_n^2) \prod_{i=1}^n d\rho(x_i) \\
&= \frac{1}{Z_n} \int_{\mathbb{R}^n} h((x_1, x_1^2) + \cdots + (x_n, x_n^2)) \prod_{i=1}^n d\rho(x_i) \\
&= \frac{1}{Z_n} \int_{\mathbb{R}^{2n}} h(z_1 + \cdots + z_n) \prod_{i=1}^n d\nu_\rho(z_i) \\
&= \frac{1}{Z_n} \int_{\mathbb{R}^2} h(z) \, d\nu_\rho^{*n}(z)
\end{aligned}$$

Hence the announced law of (S_n, T_n), under $\widetilde{\mu}_{n,\rho}$. Moreover, we have for any $(x_1, \ldots, x_n) \in \mathbb{R}^n$,

$$\frac{(x_1 + \cdots + x_n)^2}{x_1^2 + \cdots + x_n^2} = n \frac{((x_1 + \cdots + x_n)/n)^2}{(x_1^2 + \cdots + x_n^2)/n}$$

Hence

$$\mathbb{E}_{\tilde{\mu}_{n,\rho}}\left(f\left(\frac{S_n}{n}, \frac{T_n}{n}\right)\right) = \frac{1}{Z_n} \int_{\mathbb{R}^2} f(x,y) \exp\left(\frac{nx^2}{2y}\right) \mathbb{1}_{\{y>0\}} \, d\widetilde{\nu}_{n,\rho}(x,y)$$

This ends the proof of the proposition. □

5 The Gaussian case

In this section, we prove theorem 2 when ρ is the Normal law $\mathcal{N}(0,\sigma^2)$ with mean 0 and variance σ^2. We use the method of residue to compute the characteristic function of ν_ρ^{*n} and a Fourier inversion formula to get its density. We finish the proof with Laplace method.

For simplicity, we assume that $\sigma^2=1$. We just write ν^{*n} for ν_ρ^{*n} and we denote by Φ_n its characteristic function. For $(u,v)\in\mathbb{R}^2$,

$$\Phi_n(u,v)=(\Phi_1(u,v))^n=\left(\mathbb{E}(e^{iuZ+ivZ^2})\right)^n=\left(\int_{\mathbb{R}} e^{iux+ivx^2}e^{-x^2/2}\,\frac{dx}{\sqrt{2\pi}}\right)^n$$

We need some preliminary results.

The Gamma distribution with shape $k>0$ and scale $\theta>0$, denoted by $\Gamma(k,\theta)$, is the probability distribution with density function

$$x\longmapsto\frac{x^{k-1}e^{-x/\theta}}{\Gamma(k)\,\theta^k}\mathbb{1}_{x>0}$$

with respect to the Lebesgue measure on $\mathbb{R}$, where Γ denotes the gamma function defined by

$$\forall z>0\qquad \Gamma(z)=\int_0^{+\infty}x^{z-1}e^{-x}dx$$

For $k>0$ and $\theta>0$, the characteristic function of the distribution $\Gamma(k,\theta)$ is

$$u\in\mathbb{R}\longmapsto(1-\theta iu)^{-k}$$

The complex logarithm function (or the principle value of complex logarithm), denoted by Log, is defined on $\Omega=\mathbb{C}\backslash]-\infty,0]$ by

$$\forall z=x+iy\in\Omega\qquad \mathrm{Log}(z)=\frac{1}{2}\ln(x^2+y^2)+2i\arctan\left(\frac{y}{x+\sqrt{x^2+y^2}}\right)$$

If $\alpha\in\mathbb{C}$ and $z\in\Omega$, then the α-exponentiation of z is defined by

$$z^\alpha=\exp(\alpha\mathrm{Log}(z))$$

We can now prove the following key lemma :

Lemma 5. *Let $t\in\mathbb{R}$ and $\zeta\in\mathbb{C}$ such that $\mathfrak{Re}(\zeta)>0$. Then*

$$\int_{\mathbb{R}}e^{itx-\zeta x^2/2}\,dx=\sqrt{\frac{2\pi}{\mathfrak{Re}(\zeta)}}\exp\left(-\frac{t^2}{2\zeta}\right)\left(1+i\,\frac{\mathfrak{Im}(\zeta)}{\mathfrak{Re}(\zeta)}\right)^{-1/2}$$

Proof. Let $t\in\mathbb{R}$ and $\zeta=a+ib\in\mathbb{C}$ such that $\mathfrak{Re}(\zeta)>0$. We define

$$K(t,\zeta)=\int_{\mathbb{R}}e^{itx-\zeta x^2/2}\,dx$$

We factorize :

$$ixt-\frac{1}{2}\zeta x^2=-\frac{1}{2}\zeta\left(x-\frac{it}{\zeta}\right)^2-\frac{t^2}{2\zeta}=-\frac{1}{2}\zeta\left(x-\frac{tb}{|\zeta|}-i\frac{ta}{|\zeta|}\right)^2-\frac{t^2}{2\zeta}$$

Thus

$$e^{t^2/2\zeta} K(t,\zeta) = \int_{\mathbb{R}} e^{-\zeta(x - tb/|\zeta| - ita/|\zeta|)^2/2}\, dx$$

The change of variables $y = x - tb/|\zeta|$ gives us

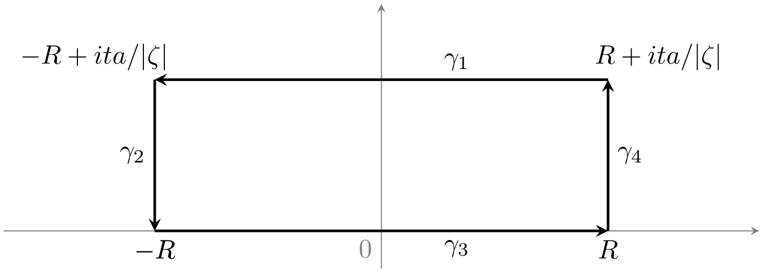
$$e^{t^2/2\zeta} K(t,\zeta) = \int_{\mathbb{R}} e^{-\zeta(y - ita/|\zeta|)^2/2}\, dy = -\lim_{R\to+\infty} \int_{\gamma_1} e^{-\zeta z^2/2}\, dz$$

where the last integral is the contour integral of the entire function $z \longmapsto e^{-\zeta z^2/2}$, along the segment γ_1 in the complex plane with end points $R + ita/|\zeta|$ and $-R + ita/|\zeta|$.

Let γ be the rectangle in the complex plane joining successively the points $R + ita/|\zeta|$, $-R + ita/|\zeta|$, $-R$ and R. We apply the residue theorem :

$$\int_\gamma e^{-\zeta z^2/2}\, dz = 0$$

since $z \longmapsto \exp(-\zeta z^2/2)$ has no pole. We denote γ_1, γ_2, γ_3 and γ_4 the successive edges of the rectangle γ.



$$\int_{\gamma_3} e^{-\zeta z^2/2}\, dz = \int_{-R}^{R} e^{-\zeta x^2/2}\, dx \underset{R\to+\infty}{\longrightarrow} \int_{\mathbb{R}} e^{-\zeta x^2/2}\, dx = 2\int_0^{+\infty} e^{-\zeta x^2/2}\, dx$$

We make the change of variables $y = x^2$ on $]0, +\infty[$:

$$\begin{aligned} 2\int_0^{+\infty} e^{-\zeta x^2/2}\, dx = \int_0^{+\infty} e^{-\zeta y/2} \frac{dy}{\sqrt{y}} &= \int_0^{+\infty} e^{-iby/2} e^{-ay/2} \frac{dy}{\sqrt{y}} \\ &= \sqrt{\frac{2}{a}}\, \Gamma\left(\frac{1}{2}\right) \left(1 + i\frac{b}{a}\right)^{-1/2} \end{aligned}$$

since we recognize, up to a normalization factor, the characteristic function of the Gamma distribution with shape $1/2$ and scale $2/a$. Moreover we have

$$\begin{aligned} \left| \int_{\gamma_4} e^{-\zeta z^2/2}\, dz \right| &= \left| \int_0^1 \exp\left(-\frac{\zeta}{2} \left(R + \frac{iat}{|\zeta|} x \right)^2 \right) \frac{iat}{|\zeta|}\, dx \right| \\ &\leq \frac{a|t|}{|\zeta|} \int_0^1 \exp\left(-\frac{aR^2}{2} + \frac{Ratbx}{|\zeta|} + \frac{a}{2} \left(\frac{atx}{|\zeta|} \right)^2 \right) dx \\ &\leq \frac{a|t|}{|\zeta|} \exp\left(-\frac{aR^2}{2} + \frac{Ra|tb|}{|\zeta|} + \frac{a}{2} \left(\frac{at}{|\zeta|} \right)^2 \right) \underset{R\to+\infty}{\longrightarrow} 0 \end{aligned}$$

Likewise

$$\int_{\gamma_2} e^{-\zeta z^2/2}\, dz \underset{R\to+\infty}{\longrightarrow} 0$$

Letting R go to $+\infty$, we conclude that

$$\sqrt{\frac{2}{a}}\,\Gamma\left(\frac{1}{2}\right)\left(1+i\frac{b}{a}\right)^{-1/2} + 0 - e^{t^2/2\zeta}K(t,\zeta) + 0 = 0$$

Since $\Gamma(1/2) = \sqrt{\pi}$, we obtain the identity stated in the lemma. □

Proposition 6. *If $\rho = \mathcal{N}(0,1)$ then the characteristic function Φ_n of the distribution ν_ρ^{*n} is*

$$(u,v) \in \mathbb{R}^2 \longmapsto \exp\left(-\frac{n}{2}\left(\frac{u^2}{1-2iv} + \mathrm{Log}(1-2iv)\right)\right)$$

Proof. Let $(u,v)\in\mathbb{R}^2$. Setting $\zeta = 1-2iv \in \{\, z\in\mathbb{C} : \mathfrak{Re}(z) > 0\,\}$, we have

$$\begin{aligned}\Phi_n(u,v) = (\Phi_1(u,v))^n &= \left(\int_{\mathbb{R}} e^{iux+ivx^2} e^{-x^2/2}\,\frac{dx}{\sqrt{2\pi}}\right)^n \\ &= \frac{1}{(2\pi)^{n/2}}\left(\int_{\mathbb{R}} e^{iux-\zeta x^2/2}\,dx\right)^n\end{aligned}$$

Lemma 5 implies that

$$\Phi_n(u,v) = \frac{1}{(2\pi)^{n/2}}\left(\sqrt{2\pi}\exp\left(-\frac{u^2}{2(1-2iv)}\right)(1-2iv)^{-1/2}\right)^n$$

and the proposition is proved. □

Once we know the characteristic function Φ_n of the law ν^{*n}, a Fourier inversion formula gives us its density. We first have to check that Φ_n is integrable with respect to the Lebesgue measure on $\mathbb{R}^2$.

Let $(u,v)\in\mathbb{R}^2$. Since $(1-2iv)^{-1} = (1+2iv)/(1+4v^2)$, we have

$$\mathfrak{Re}\left(\frac{u^2}{1-2iv} + \mathrm{Log}(1-2iv)\right) = \frac{u^2}{1+4v^2} + \ln(\sqrt{1+4v^2})$$

It follows that

$$\begin{aligned}\int_{\mathbb{R}^2} |\Phi_n(u,v)|\,du\,dv &= \int_{\mathbb{R}^2}\exp\left(-\frac{nu^2}{2(1+4v^2)}\right)(1+4v^2)^{-n/4}\,du\,dv \\ &= \int_{\mathbb{R}}(1+4v^2)^{-n/4}\left(\int_{\mathbb{R}}\exp\left(-\frac{nu^2}{2(1+4v^2)}\right)du\right)dv \\ &= \int_{\mathbb{R}}(1+4v^2)^{-n/4}\sqrt{\frac{2\pi(1+4v^2)}{n}}\,dv \\ &= \sqrt{\frac{2\pi}{n}}\int_{\mathbb{R}}(1+4v^2)^{-(n-2)/4}\,dv\end{aligned}$$

where we used Fubini's theorem in the third integral. The function

$$v \longmapsto (1+4v^2)^{-(n-2)/4}$$

is continuous on $\mathbb{R}$ and integrable in the neighbourhood of $+\infty$ and $-\infty$ if and only if $n > 4$.

Proposition 7. *If $\rho = \mathcal{N}(0,1)$ and $n \geq 5$ then ν_ρ^{*n} has the density*

$$(x,y) \in \mathbb{R}^2 \longmapsto \left(\sqrt{2^n \pi n}\, \Gamma\left(\frac{n-1}{2}\right)\right)^{-1} \exp\left(-\frac{y}{2}\right) \left(y - \frac{x^2}{n}\right)^{(n-3)/2} \mathbb{1}_{x^2<ny}$$

with respect to the Lebesgue measure on $\mathbb{R}^2$.

Proof. We have seen that, if $n \geq 5$, then Φ_n is integrable on $\mathbb{R}^2$. The Fourier inversion formula implies that ν_ρ^{*n} has the density

$$f_n : (x,y) \longmapsto \frac{1}{(2\pi)^2} \int_{\mathbb{R}^2} e^{-ixu-iyv}\, \Phi_n(u,v)\, du\, dv$$

with respect to the Lebesgue measure on $\mathbb{R}^2$. Let $(x,y) \in \mathbb{R}^2$. By Fubini's theorem,

$$\begin{aligned} f_n(x,y) &= \frac{1}{(2\pi)^2} \int_{\mathbb{R}} \frac{e^{-iyv}}{(1-2iv)^{n/2}} \left(\int_{\mathbb{R}} \exp\left(-ixu - \frac{nu^2}{2(1-2iv)}\right) du \right) dv \\ &= \frac{1}{(2\pi)^2} \int_{\mathbb{R}} \frac{e^{-iyv}}{(1-2iv)^{n/2}} K\left(-x, \frac{n}{1-2iv}\right) dv \end{aligned}$$

where K is defined by

$$\forall a > 0 \qquad \forall (t,b) \in \mathbb{R}^2 \qquad K(t, a+ib) = \int_{\mathbb{R}} e^{itz-(a+ib)z^2/2}\, dz$$

Lemma 5 implies that for any $v \in \mathbb{R}$,

$$\begin{aligned} K\left(-x, \frac{n}{1-2iv}\right) &= \sqrt{\frac{2\pi(1+4v^2)}{n}} \exp\left(-\frac{x^2(1-2iv)}{2n}\right) (1+2iv)^{-1/2} \\ &= \sqrt{\frac{2\pi}{n}} \exp\left(-\frac{x^2(1-2iv)}{2n}\right) \left(\frac{1+4v^2}{1+2iv}\right)^{1/2} \\ &= \sqrt{\frac{2\pi}{n}} \exp\left(-\frac{x^2(1-2iv)}{2n}\right) (1-2iv)^{1/2} \end{aligned}$$

Thus

$$\begin{aligned} f_n(x,y) &= \frac{1}{(2\pi)^2} \sqrt{\frac{2\pi}{n}} \int_{\mathbb{R}} \exp\left(-iyv - \frac{x^2(1-2iv)}{2n}\right) (1-2iv)^{-(n-1)/2}\, dv \\ &= \frac{1}{\sqrt{2\pi n}} \exp\left(-\frac{x^2}{2n}\right) \frac{1}{2\pi} \int_{\mathbb{R}} \exp\left(-iv\left(y - \frac{x^2}{n}\right)\right) (1-2iv)^{-(n-1)/2}\, dv \end{aligned}$$

Therefore $\sqrt{2\pi n}\exp(x^2/2n) f_n(x,y)$ is the inverse Fourier transform of the distribution $\Gamma((n-1)/2, 2)$ taken at the point $y - x^2/n$. Hence

$$\begin{aligned} \sqrt{2\pi n} \exp\left(\frac{x^2}{2n}\right) f_n(x,y) = \left(\Gamma\left(\frac{n-1}{2}\right) 2^{(n-1)/2}\right)^{-1} &\left(y - \frac{x^2}{n}\right)^{(n-3)/2} \\ &\times \exp\left(-\frac{y}{2} + \frac{x^2}{2n}\right) \mathbb{1}_{y>x^2/n} \end{aligned}$$

Finally

$$f_n(x,y) = \left(\sqrt{2\pi n}\,\Gamma\left(\frac{n-1}{2}\right)2^{(n-1)/2}\right)^{-1}\left(y-\frac{x^2}{n}\right)^{(n-3)/2}\exp\left(-\frac{y}{2}\right)\mathbb{1}_{x^2<ny}$$

The proposition is proved. □

This previous result and proposition 4 imply that, for $n \geq 5$, under $\widetilde{\mu}_{n,\rho}$, the law of (S_n, T_n) on $\mathbb{R}^2$ is

$$C_n^{-1}\exp\left(\frac{x^2}{2y}-\frac{y}{2}\right)\left(y-\frac{x^2}{n}\right)^{(n-3)/2}\mathbb{1}_{x^2<ny}\,dx\,dy$$

where $C_n = Z_n\sqrt{2^n\pi n}\,\Gamma((n-1)/2)$.

Let $\alpha, \beta \in]0,1]$, $n \geq 5$ and f a bounded measurable function. The change of variables $(x,y) \longmapsto (n^\alpha x, n^\beta y)$ yields

$$\mathbb{E}_{\tilde{\mu}_n}\left(f\left(\frac{S_n}{n^\alpha},\frac{T_n}{n^\beta}\right)\right) = \frac{n^{\alpha+\beta}}{C_n}\int_{\mathbb{R}^2} f(x,y)\exp\left(\frac{n^{2\alpha-\beta}x^2}{2y}-\frac{n^\beta y}{2}\right) \times \left(n^\beta y - n^{2\alpha-1}x^2\right)^{(n-3)/2}\mathbb{1}_{n^{2\alpha}x^2<n^{\beta+1}y}\,dx\,dy$$

Factorizing by $n^{(n-3)/2}$, we notice that all the terms in the integral are functions of $x^2/n^{2-2\alpha}$ and $y/n^{1-\beta}$. We obtain the following proposition.

Proposition 8. *Let $\alpha, \beta \in]0,1]$. If $\rho = \mathcal{N}(0,1)$ and $n \geq 5$ then, under $\widetilde{\mu}_{n,\rho}$, the distribution of $(S_n/n^\alpha, T_n/n^\beta)$ is*

$$\frac{n^{\alpha+\beta}n^{(n-3)/2}}{C_n}\exp\left(-n\psi\left(\frac{x^2}{n^{2-2\alpha}},\frac{y}{n^{1-\beta}}\right)\right)\varphi\left(\frac{x^2}{n^{2-2\alpha}},\frac{y}{n^{1-\beta}}\right) \times \mathcal{X}\left(\frac{x^2}{n^{2-2\alpha}},\frac{y}{n^{1-\beta}}\right)dx\,dy$$

where $\mathcal{X}$ is the indicator function of the set

$$D^+ = \{\,(x,y)\in\mathbb{R}^2 : y > x \geq 0\,\}$$

and ψ and φ are the functions defined on D^+ by

$$\psi : (x,y) \longmapsto \frac{1}{2}\left(-\frac{x}{y}+y-\ln(y-x)\right)$$

$$\varphi : (x,y) \longmapsto (y-x)^{-3/2}$$

We give next some properties of the map ψ in order to determine which values of α and β to choose.

Lemma 9. *The map ψ has a unique minimum at $(0,1)$ and $\psi(0,1) = 1/2$. The map ψ is $\mathcal{C}^2$ on D^+ and it satisfies :*
⋆ In the neighbourhood of $(0,1)$,

$$\psi(x,y) - \frac{1}{2} = \frac{1}{4}(x^2 + (y-1)^2) + o(\|x, y-1\|^2)$$

$\star$ *There exists* $\delta > 0$ *such that for all* $(x, y) \in D^+$,

$$|x| < \delta,\ |y-1| < \delta \quad \Longrightarrow \quad \psi(x,y) - \frac{1}{2} \geq \frac{1}{8}(x^2 + (y-1)^2)$$

$\star$ $\inf\{\,\psi(x,y) : |x| \geq \delta \quad \textit{or} \quad |y-1| \geq \delta\,\} > 1/2$

The map φ *is bounded by* 1*, it converges to* 1 *when* (x, y) *goes to* $(0, 1)$ *and*

$$\int_{\mathbb{R}^2} e^{-2\psi(x^2,y)} \varphi(x^2, y) \mathbb{1}_{x^2<y}\, dx\, dy \;<\; +\infty$$

Proof. The map ψ is $\mathcal{C}^2$ on D^+ and, for fixed $y > 0$,

$$\frac{\partial \psi}{\partial x}(x,y) = \frac{1}{2}\left(-\frac{1}{y} + \frac{1}{y-x}\right) \geq 0$$

Equality holds if and only if $x = 0$. Thus $x \longmapsto \psi(x, y)$ is increasing on $]0, y[$ and $\psi(0, y) = (y - \ln(y))/2$. Hence for any $(x, y) \in D^+$,

$$\psi(x,y) > \frac{1}{2}(y - \ln(y)) > \frac{1}{2} = \psi(0,1)$$

with equality if and only if $(x, y) = (0, 1)$. Therefore $(0, 1)$ is the unique minimum of ψ. In the neighbourhood of $(0, 0)$,

$$\begin{aligned}\psi(x, 1+h) &= \frac{1}{2}(-x(1-h+o(h^2)) + 1 + h - (h - x - \frac{1}{2}(h-x)^2 + o((h-x)^2))\\ &= \frac{1}{2} + \frac{h^2}{4} + \frac{x^2}{4} + o(\|x,h\|^2)\end{aligned}$$

Thus in the neighbourhood of $(0, 1)$,

$$\psi(x,y) - \frac{1}{2} = \frac{1}{4}(x^2 + (y-1)^2) + o(\|x, y-1\|^2)$$

It follows that there exists $\delta > 0$ such that for $(x, y) \in D^+$, if $|x| < \delta$ and $|y - 1| < \delta$, then,

$$\psi(x,y) - \frac{1}{2} \geq \frac{1}{8}(x^2 + (y-1)^2)$$

Moreover, if $|y - 1| \geq \delta$ and $x \in\ [0, y[$, then

$$\psi(x,y) \geq \frac{1}{2}\min\{\,1-\delta-\ln(1-\delta)\,,\,1+\delta-\ln(1+\delta)\,\} > \frac{1}{2}$$

We suppose that $\delta < 1$, otherwise we reduce δ. If $x \geq \delta$ and $y > x$, then

$$2\psi(x,y) \geq -\frac{\delta}{y} + y - \ln(y-\delta) > \inf_{y>\delta}\left(-\frac{\delta}{y} + y - \ln(y-\delta)\right) > 1$$

since $\delta \neq 0$. Therefore

$$\inf\{\,\psi(x,y) : |x| \geq \delta \quad \text{or} \quad |y-1| \geq \delta\,\} > 1/2$$

Finally the map φ is bounded by 1, it converges to 1 when (x,y) goes to $(0,1)$ and

$$\int_{\mathbb{R}^2} e^{-2\psi(x^2,y)}\varphi(x^2,y)\mathbb{1}_{x^2<y}\,dx\,dy \leq e\left(\int_{\mathbb{R}} e^{-x^2}\,dx\right)\left(\int_0^{+\infty}\frac{e^{-y}}{\sqrt{y}}\,dy\right) < +\infty$$

where we used the change of variables $(x,y)\longmapsto(x,y-x^2)$. □

Thus, for fixed (x,y), when n goes to $+\infty$,

$$\psi\left(\frac{x^2}{n^{2-2\alpha}},\frac{y}{n^{1-\beta}}\right)-\frac{1}{2}\sim\frac{x^4}{4}n^{3-4\alpha}+\frac{n}{4}\left(\frac{y}{n^{1-\beta}}-1\right)^2$$

Hence we take $\alpha=3/4$ and $\beta=1$. We prove now the following theorem :

Theorem 10. *For $\rho=\mathcal{N}(0,1)$, under $\widetilde{\mu}_{n,\rho}$,*

$$\frac{S_n}{n^{3/4}}\underset{n\to+\infty}{\overset{\mathcal{L}}{\longrightarrow}}\frac{e^{-x^4/4}dx}{\int_{\mathbb{R}}e^{-y^4/4}\,dy}\qquad\text{and}\qquad\frac{T_n}{n}\underset{n\to+\infty}{\overset{\mathcal{L}}{\longrightarrow}}1$$

Proof. Let $n\in\mathbb{N}$ and let $f:\mathbb{R}^2\longrightarrow\mathbb{R}$ be a continuous bounded function. By proposition 8, we have

$$\mathbb{E}_{\tilde{\mu}_n}\left(f\left(\frac{S_n}{n^{3/4}},\frac{T_n}{n}\right)\right)=\frac{n^{7/4}n^{(n-3)/2}}{C_n}\int_{\mathbb{R}^2}f(x,y)\exp\left(-n\psi\left(\frac{x^2}{\sqrt{n}},y\right)\right)\\ \times\varphi\left(\frac{x^2}{\sqrt{n}},y\right)\mathbb{1}_{\sqrt{n}y>x^2}\,dx\,dy$$

Let δ be as in lemma 9. We denote

$$A_n=\int_{x^2<\delta\sqrt{n}}\int_{|y-1|<\delta}f(x,y)\exp\left(-n\psi\left(\frac{x^2}{\sqrt{n}},y\right)\right)\varphi\left(\frac{x^2}{\sqrt{n}},y\right)\mathbb{1}_{\sqrt{n}y>x^2}\,dx\,dy$$

The change of variables $(x,y)\longmapsto(x,y/\sqrt{n}+1)$ gives

$$\sqrt{n}e^{n/2}A_n=\int_{x^2<\delta\sqrt{n}}\int_{|y|<\delta\sqrt{n}}f\left(x,\frac{y}{\sqrt{n}}+1\right)\exp\left(-n\psi\left(\frac{x^2}{\sqrt{n}},\frac{y}{\sqrt{n}}+1\right)\right)\\ \exp\left(\frac{n}{2}\right)\varphi\left(\frac{x^2}{\sqrt{n}},\frac{y}{\sqrt{n}}+1\right)\mathbb{1}_{y+\sqrt{n}>x^2}\,dx\,dy$$

Lemma 9 and the continuity of f imply that

$$n\psi\left(\frac{x^2}{\sqrt{n}},\frac{y}{\sqrt{n}}+1\right)-\frac{n}{2}\underset{n\to+\infty}{\longrightarrow}\frac{x^4}{4}+\frac{y^2}{4}$$

$$f\left(x,\frac{y}{\sqrt{n}}+1\right)\varphi\left(\frac{x^2}{\sqrt{n}},\frac{y}{\sqrt{n}}+1\right)\mathbb{1}_{y+\sqrt{n}>x^2}\mathbb{1}_{x^2<\delta\sqrt{n}}\mathbb{1}_{|y|<\delta\sqrt{n}}\underset{n\to+\infty}{\longrightarrow}f(x,1)$$

Moreover the function inside the integral is dominated by

$$(x,y)\longmapsto\|f\|_\infty\exp\left(-\frac{1}{8}(x^4+y^2)\right)$$

which is independent of n and integrable with respect to the Lebesgue measure on $\mathbb{R}^2$. By Lebesgue's dominated convergence theorem, we have

$$\sqrt{n}e^{n/2}A_n \underset{n\to+\infty}{\longrightarrow} \int_{\mathbb{R}^2} f(x,1)e^{-x^4/4}e^{-y^2/4}\,dx\,dy = \sqrt{4\pi}\int_{\mathbb{R}} f(x,1)e^{-x^4/4}\,dx$$

We define

$$\mathrm{B}_\delta = \{\, (x,y)\in D^+ : |x|<\delta,\ |y-1|<\delta \,\}$$

and

$$B_n = \int_{(x^2/\sqrt{n},y)\in \mathrm{B}_\delta^c} f(x,y)\exp\left(-n\psi\left(\frac{x^2}{\sqrt{n}},y\right)\right)\varphi\left(\frac{x^2}{\sqrt{n}},y\right)\mathbb{1}_{\sqrt{n}y>x^2}\,dx\,dy$$

Let $\varepsilon = \inf\{\,\psi(x,y) : (x,y)\in \mathrm{B}_\delta^c\}$,

$$|B_n| \le e^{-(n-2)\varepsilon}\|f\|_\infty \int_{\mathbb{R}^2}\exp\left(-2\psi\left(\frac{x^2}{\sqrt{n}},y\right)\right)\varphi\left(\frac{x^2}{\sqrt{n}},y\right)\mathbb{1}_{\sqrt{n}y>x^2}\,dx\,dy$$

The change of variables $(x,y)\longmapsto(xn^{1/4},y)$ yields

$$\sqrt{n}e^{n/2}|B_n| \le e^{2\varepsilon}\|f\|_\infty e^{-n(\varepsilon-1/2)}n^{3/4}\int_{\mathbb{R}^2} e^{-2\psi(x^2,y)}\varphi(x^2,y)\mathbb{1}_{x^2<y}\,dx\,dy$$

Lemma 9 guarantees that $\varepsilon>1/2$ and that the above integrable is finite. Therefore $\sqrt{n}e^{n/2}B_n$ goes to 0 as n goes to $+\infty$. Finally

$$\begin{aligned}\int_{\mathbb{R}^2} f(x,y)\exp\left(-n\psi\left(\frac{x^2}{\sqrt{n}},y\right)\right)\varphi\left(\frac{x^2}{\sqrt{n}},y\right)\mathbb{1}_{\sqrt{n}y>x^2}\,dx\,dy &= A_n+B_n\\ &\underset{+\infty}{=} \frac{e^{-n/2}}{\sqrt{n}}\left(\sqrt{4\pi}\int_{\mathbb{R}} f(x,1)e^{-x^4/4}\,dx + o(1)+o(1)\right)\\ &\underset{+\infty}{\sim} \sqrt{\frac{4\pi}{n}}e^{-n/2}\int_{\mathbb{R}} f(x,1)e^{-x^4/4}\,dx\end{aligned}$$

If $f=1$, we have

$$\frac{C_n}{n^{7/4}n^{(n-3)/2}} \underset{+\infty}{\sim} \sqrt{\frac{4\pi}{n}}e^{-n/2}\int_{\mathbb{R}} e^{-x^4/4}\,dx$$

Hence

$$\begin{aligned}\mathbb{E}_{\tilde{\mu}_n}\left(f\left(\frac{S_n}{n^{3/4}},\frac{T_n}{n}\right)\right) &\underset{n\to+\infty}{\longrightarrow} \int_{\mathbb{R}} f(x,1)\frac{e^{-x^4/4}\,dx}{\int_{\mathbb{R}} e^{-u^4/4}\,du}\\ &= \int_{\mathbb{R}^2} f(x,y)\left(\frac{e^{-x^4/4}\,dx}{\int_{\mathbb{R}} e^{-u^4/4}\,du}\otimes\delta_1(y)\right)\end{aligned}$$

This ends the proof of the theorem. □

A straightforward change of variables implies that, if $\rho=\mathcal{N}(0,\sigma^2)$, then, under $\widetilde{\mu}_{n,\rho}$,

$$\frac{S_n}{n^{3/4}} \underset{n\to+\infty}{\overset{\mathcal{L}}{\longrightarrow}} \frac{e^{-x^4/4\sigma^4}\,dx}{\int_{\mathbb{R}} e^{-y^4/4\sigma^4}\,dy} \qquad \text{and} \qquad \frac{T_n}{n} \underset{n\to+\infty}{\overset{\mathcal{L}}{\longrightarrow}} \sigma^2$$

We also get

$$\frac{S_n}{n} \xrightarrow[n\to+\infty]{\mathcal{L}} 0$$

Since $\mu_4 = 3\sigma^4$, we have

$$\frac{\mu_4}{12\sigma^8} = \frac{1}{4\sigma^4}$$

We have thus proved theorems 1 and 2 for a Gaussian distribution $\mathcal{N}(0,\sigma^2)$.

In the general case we cannot compute explicitly the distribution ν_ρ^{*n}. In the following sections, we deal with the more robust method we suggested in the heuristics of section 3.

6 General results on the Cramér transform

This section, which may be omitted on a first reading, presents some general results on the Cramér transform of a probability distribution in $\mathbb{R}^d$. Let ν be a probability measure on $\mathbb{R}^d$, $d \geq 1$. The Log-Laplace L of ν is defined in $\mathbb{R}^d$ by

$$\forall \lambda \in \mathbb{R}^d \qquad L(\lambda) = \ln \int_{\mathbb{R}^d} \exp\langle \lambda, z\rangle \, d\nu(z)$$

where $\langle\ ,\ \rangle$ denotes the inner product in $\mathbb{R}^d$. The Log-Laplace L is convex on $\mathbb{R}^d$ and takes its values in $]-\infty,+\infty]$. We denote by D_L the set where L is finite. In particular, if ν has a bounded support, then $D_L = \mathbb{R}^d$.

The set D_L is convex and contains 0 since $L(0) = 0$. If $\overset{o}{D}_L \neq \varnothing$ then L is $\mathcal{C}^\infty$ on $\overset{o}{D}_L$ and

$$\forall \lambda \in \overset{o}{D}_L,\ \alpha \in \mathbb{N}^d \qquad \frac{\partial^\alpha \exp(L)}{\partial \lambda_1^{\alpha_1} \dots \partial \lambda_d^{\alpha_d}}(\lambda) = \int_{\mathbb{R}^d} z_1^{\alpha_1} \dots z_d^{\alpha_d} \exp\langle \lambda, z\rangle \, d\nu(z)$$

We refer to [7] and [9] for the proofs of these results.

We define the Cramér transform of ν by

$$J : x \in \mathbb{R}^d \longmapsto \sup_{\lambda\in\mathbb{R}^d} \big(\langle \lambda, x\rangle - L(\lambda) \big)$$

It is the Fenchel-Legendre transform of L. We write

$$D_J = \{\, x \in \mathbb{R}^d : J(x) < +\infty \,\}$$

Proposition 11. ***(a)*** *J is a non-negative convex and lower semi-continuous function.*

(b) *If L is finite in a neighbourhood of the origin then the level sets of J, $\{\, x \in \mathbb{R}^d : J(x) \leq a \,\}$, $a \in \mathbb{R}$, are compact.*

(c) *If $t \in \overset{o}{D}_L$ and $u = \nabla L(t)$ then $J(u) = \langle t, u\rangle - L(t)$.*

(d) *If ν has a finite first moment then $J(m) = 0$ where*

$$m = \int_{\mathbb{R}^d} x \, d\nu(x)$$

Moreover, if $0 \in \overset{o}{D}_L$, then J has a unique minimum at m.

Proof. We refer to [7] for the proof of the points (a), (b) and (c). We prove the point (d) (see for instance chapter V. of [15]). Let $\lambda \in \mathbb{R}^d$. Jensen's inequality implies that

$$\int_{\mathbb{R}^d} e^{\langle \lambda, x\rangle}\, d\nu(x) \geq \exp \int_{\mathbb{R}^d} \langle \lambda, x\rangle\, d\nu(x) = e^{\langle \lambda, m\rangle}$$

Therefore $L(\lambda) \geq \langle \lambda, m\rangle$ and thus $J(m) \leq 0$. Since J is a non-negative function, it follows that $J(m) = 0$ hence m is a minimum of J. We show that it is the only one : suppose that x_0 is a minimum of J. Then $J(x_0) = 0$ and thus

$$\forall \lambda \in \mathbb{R}^d \qquad \langle \lambda, x_0\rangle - L(\lambda) \leq 0$$

Hence for all $t > 0$ and $\lambda \in \mathbb{R}^d$,

$$\frac{L(t\lambda) - L(0)}{t} = \frac{L(t\lambda)}{t} \geq \frac{\langle t\lambda, x_0\rangle}{t} = \langle \lambda, x_0\rangle$$

Since L is differentiable at $0 \in \overset{o}{D}_L$, letting t go to 0, we get

$$\forall \lambda \in \mathbb{R}^d \qquad \langle \nabla L(0), \lambda\rangle \geq \langle x_0, \lambda\rangle$$

It follows that $x_0 = \nabla L(0) = m$. □

We notice that Cramér's theorem (see [5]) links J and the large deviations of $(X_1 + \cdots + X_n)/n$ where $(X_n)_{n\in\mathbb{N}}$ is a sequence of real-valued independent and identically distributed random variables in $\mathbb{R}^d$. This is why J is called the Cramér transform.

A probability measure ν on $\mathbb{R}$ is said to be degenerate if it is a Dirac point mass. We will generalize this definition for measures on $\mathbb{R}^d$. We refer to [4] and [11].

Definition 12. *A probability measure ν on $\mathbb{R}^d$, $d \geq 2$, is said to be degenerate if its support is included in a hyperplane of $\mathbb{R}^d$, i.e., there exists a hyperplane $\mathcal{H}$ of $\mathbb{R}^d$ such that $\nu(\mathcal{H}) = 1$.*

The following lemma illustrates the interest of this concept.

Lemma 13. ***(a)*** *If ν is degenerate then its Cramér transform J vanishes outside of a hyperplane containing its support.*

(b) *If Z is a random variable whose distribution is ν, which is non-degenerate, then its covariance matrix G_Z is invertible.*

Proof. (a) We assume that $\mathcal{H}$ is the hyperplane given by $\langle a_0, z\rangle = t$, with $t \in \mathbb{R}$ and $a_0 \in \mathbb{R}^d \backslash \{0\}$. We set $t_0 = a_0 t/\|a_0\|$. We notice that $z \in \mathcal{H}$ if and only if $z - t_0$ belongs to the orthogonal of a_0. Thus for any $x \notin \mathcal{H}$,

$$\begin{aligned} J(x) &= \sup_{\lambda \in \mathbb{R}^d} \left(\langle \lambda, x - t_0\rangle - \ln \int_{\mathcal{H}} e^{\langle \lambda, z - t_0\rangle}\, d\nu(z) \right) \\ &\geq \sup_{\lambda \in \mathbb{R}a_0} \left(\langle \lambda, x - t_0\rangle - \ln \int_{\mathcal{H}} e^{0}\, d\nu(z) \right) \\ &= \sup_{k \in \mathbb{R}} \left(k\langle a_0, x - t_0\rangle \right) = +\infty \end{aligned}$$

(b) We have $\Gamma_Z = \mathbb{E}(Y\,{}^tY)$ with $Y = Z - \mathbb{E}(Z)$. The matrix Γ_Z is symmetric and thus it is diagonalizable. To conclude that Γ_Z is invertible, it remains to prove that 0 is not an eigenvalue . Suppose that it is the case : there exists a vector $x \neq 0$ such that $\Gamma_Z x =^t (0, \dots, 0)$. Then

$$\mathbb{E}(\|{}^tYx\|^2) = \mathbb{E}({}^txY\,{}^tYx) =^t x\mathbb{E}(Y\,{}^tY)x =^t x\Gamma_Z x = 0$$

Therefore $\|{}^tYx\|^2 = 0$ almost surely and thus

$$\sum_{i=1}^{d} x_i Y_i = 0 \qquad \text{a.s}$$

That is, with probability 1,

$$Z \in \{\, z \in \mathbb{R}^d : \langle x, z\rangle = \mathbb{E}(\langle x, Z\rangle)\,\}$$

This is absurd since ν is non-degenerate. Hence Γ_Z is invertible. □

From now onwards, we assume that ν is a non-degenerate probability measure in $\mathbb{R}^d$. We are interested in the points λ realizing the supremum defining $J(x)$, for $x \in D_J$. We denote by $\mathcal{C}$ the closed convex hull of the support of ν.

Lemma 14. *Let ν be a non-degenerate probability measure in $\mathbb{R}^d$. The interior of $\mathcal{C}$ is not empty and $\overset{o}{\mathcal{C}} \subset D_J \subset \mathcal{C}$. Moreover for any $x \in \overset{o}{\mathcal{C}}$, the supremum defining $J(x)$ is realized for some value $\lambda(x) \in D_L$.*

Proof. The non-degeneracy of ν means that its support is not included in a hyperplane of $\mathbb{R}^d$. Therefore the support of ν contains d linearly independent vectors and the interior of the convex hull of these vectors is non-empty. Thus $\overset{o}{\mathcal{C}}$ is non-empty.

We prove next the second assertion. We first show that $D_J \subset \mathcal{C}$ (see corollary 12.8 of [5]). Suppose that $\mathcal{C} \neq \mathbb{R}^d$ (otherwise the result is immediate). Let $x \notin \mathcal{C}$. By the Hahn-Banach theorem, there exists $\lambda \in \mathbb{R}^d$ and $a \in \mathbb{R}$ such that

$$\forall y \in \mathcal{C} \qquad \langle \lambda, y\rangle \leq a < \langle \lambda, x\rangle$$

Since $\nu(\mathcal{C}) = 1$, for any $t > 0$,

$$\begin{aligned} J(x) &\geq \langle t\lambda, x\rangle - \ln \int_{\mathbb{R}^d} \exp(\langle t\lambda, y\rangle)\, d\nu(y) \\ &= -\ln \int_{\mathcal{C}} \exp(t\langle \lambda, y\rangle - t\langle \lambda, x\rangle)\, d\nu(y) \geq t(\langle \lambda, x\rangle - a) \end{aligned}$$

Sending t to $+\infty$, we conclude that $J(x) = +\infty$. Thus $D_J \subset \mathcal{C}$. Let $x \in \overset{o}{\mathcal{C}}$ and let $(\lambda_n)_{n\in\mathbb{N}}$ be a sequence in $\mathbb{R}^d$ such that

$$\begin{aligned} J(x) &= \lim_{n\to+\infty} \Big(\langle \lambda_n, x\rangle - \ln \int_{\mathbb{R}^d} \exp(\langle \lambda_n, z\rangle)\, d\nu(z) \Big) \\ &= -\ln \lim_{n\to+\infty} \int_{\mathbb{R}^d} \exp(\langle \lambda_n, z - x\rangle)\, d\nu(z) \end{aligned}$$

We suppose that $|\lambda_n| \to +\infty$ and we show that it leads to a contradiction. For all $n \in \mathbb{N}$, we set $u_n = \lambda_n |\lambda_n|^{-1}$. Then $(u_n)_{n\in\mathbb{N}}$ is a bounded sequence. Thus, up to the extraction of a subsequence, we might assume that it converges to some vector $u \in \mathbb{R}^d$ whose norm is 1. Let v belong to the support of ν and let U be an open subset of $\mathbb{R}^d$ containing v. We have then $\nu(U) > 0$. Suppose that for any $z \in U$, $\langle u, z - x\rangle > 0$. Then, by Fatou's lemma,

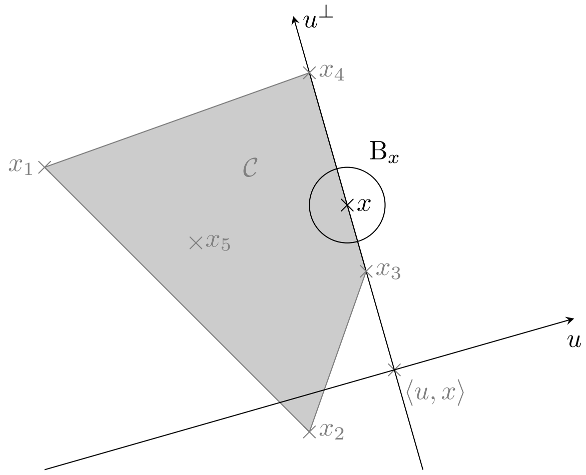
$$\begin{aligned} +\infty &= \int_{\mathrm{B}} \liminf_{n\to+\infty} \exp(|\lambda_n|\langle u_n, z - x\rangle)\, d\nu(z) \\ &\leq \liminf_{n\to+\infty} \int_U \exp(|\lambda_n|\langle u_n, z - x\rangle)\, d\nu(z) \end{aligned}$$

Hence

$$\exp(-J(x)) = \lim_{n\to+\infty} \int_{\mathbb{R}^d} \exp(|\lambda_n|\langle u_n, z - x\rangle)\, d\nu(z) = +\infty$$

Thus $J(x) = -\infty$, which is absurd since J is a non-negative function. We conclude that for all v in the support of ν and for any open subset U of $\mathbb{R}^d$ containing v, there exists $z \in U$ such that $\langle u, z - x\rangle \leq 0$. It follows that, for any v in the support of ν, $\langle u, v\rangle \leq \langle u, x\rangle$. This inequality is stable by convex combinations, thus

$$\forall y \in \mathcal{C} \qquad \langle u, y\rangle \leq \langle u, x\rangle$$



Case where ν is discrete and charges five points of $\mathbb{R}^2$

Since $x \in \overset{o}{\mathcal{C}}$, there exists a ball B_x centered at x and contained in $\mathcal{C}$. Thus there exists $y_0 \in \mathrm{B}_x$ such that $\langle u, y_0\rangle > \langle u, x\rangle$, which is absurd. Therefore $(\lambda_n)_{n\in\mathbb{N}}$ is a bounded sequence. Hence there exists a subsequence $(\lambda_{\varphi(n)})_{n\in\mathbb{N}}$ and $\lambda(x) \in \mathbb{R}^d$ such that $\lambda_{\varphi(n)} \to \lambda(x)$. By Fatou's lemma,

$$\begin{aligned} J(x) &= \langle \lambda(x), x\rangle - \ln \lim_{n\to+\infty} \int_{\mathbb{R}^d} \exp(\langle \lambda_n, z\rangle)\, d\nu(z) \\ &\leq \langle \lambda(x), x\rangle - \ln \liminf_{n\to+\infty} \int_{\mathbb{R}^d} \exp(\langle \lambda_n, z\rangle)\, d\nu(z) \\ &\leq \langle \lambda(x), x\rangle - \ln \int_{\mathbb{R}^d} \liminf_{n\to+\infty} \exp(\langle \lambda_n, z\rangle)\, d\nu(z) \\ &= \langle \lambda(x), x\rangle - \ln \int_{\mathbb{R}^d} \exp(\langle \lambda(x), z\rangle)\, d\nu(z) \\ &\leq J(x) \end{aligned}$$

Thus $J(x) = \langle \lambda(x), x \rangle - L(\lambda(x))$. Since $L(\lambda(x)) \neq -\infty$, this formula implies that $J(x) < +\infty$ and thus that $\overset{o}{\mathcal{C}} \subset D_J$. Moreover if $L(\lambda(x)) = +\infty$ then $J(x) = -\infty$, which is absurd. Therefore $L(\lambda(x)) < \infty$. This shows that the supremum defining $J(x)$ is realized at a point $\lambda(x)$ such that $\Lambda(\lambda(x)) < +\infty$. □

If D_L is an open subset of $\mathbb{R}^d$ then for all $(x, y) \in \overset{o}{D}_J = \overset{o}{\mathcal{C}}$, the supremum defining $J(x)$ is realized at some $\lambda(x) \in \overset{o}{D}_L$. This is the case when the support of ν is bounded, and also for the distribution ν_ρ when ρ is the Gaussian $\mathcal{N}(0, \sigma^2)$, where we have then $D_L = \mathbb{R} \times]-\infty, 1/(2\sigma^2)[$.

Now we study the smoothness of J.

Notation. *If f is a differentiable function on an open subset U of $\mathbb{R}^d$, we denote by $\mathrm{D}_x f$ the differential of f at $x \in U$. If f is real-valued, we denote :*
⋆ $\mathrm{D}^2_x f$ its second differential at $x \in U$ (considered as a matrix of size $d \times d$)
⋆ ∇f the function $U \longrightarrow \mathbb{R}^d$ such that

$$\forall x \in U \quad \forall y \in \mathbb{R}^d \qquad \langle \nabla f(x), y \rangle = \mathrm{D}_x f(y)$$

We define the admissible domain of J :

Definition 15. *Let ν be a non-degenerate probability measure on $\mathbb{R}^d$ such that the interior of D_L is non-empty. The admissible domain of J is the set $A_J = \nabla L\,(\overset{o}{D}_L)$.*

The following proposition states that A_J, the admissible domain of J, is an open subset of $\mathbb{R}^d$, and that J is $\mathcal{C}^\infty$ on A_J.

Proposition 16. *Let ν be a non-degenerate probability measure on $\mathbb{R}^d$ such that the interior of D_L is non-empty. Let A_J be the admissible domain of J.*
***(a)** The function ∇L is a $\mathcal{C}^\infty$-diffeomorphism from $\overset{o}{D}_L$ to A_J. Moreover*

$$A_J \subset D_J = \{\, x \in \mathbb{R}^d : J(x) < +\infty \,\}$$

***(b)** Denote by λ the inverse $\mathcal{C}^\infty$-diffeomorphism of ∇L. Then the map J is $\mathcal{C}^\infty$ on A_J and for any $x \in A_J$,*

$$J(x) = \langle x, \lambda(x) \rangle - L(\lambda(x))$$

$$\nabla J(x) = (\nabla L)^{-1}(x) = \lambda(x) \qquad \text{and} \qquad \mathrm{D}^2_x J = \left(\mathrm{D}^2_{\lambda(x)} L\right)^{-1}$$

***(c)** If D_L is an open subset of $\mathbb{R}^d$ then $A_J = \overset{o}{D}_J = \overset{o}{\mathcal{C}}$ where $\mathcal{C}$ denotes the convex hull of the support of ν.*

The points (a) and (b) are proved in [1] and [4]. For the sake of completeness, we reproduce the proof below.

Proof. (a) We know that the function L is $\mathcal{C}^\infty$ on $\overset{o}{D}_L$ and that for any $\lambda \in \overset{o}{D}_L$ and $i, j \in \{1, \dots, d\}$,

$$\frac{\partial L}{\partial \lambda_i}(\lambda) = \frac{1}{\exp L(\lambda)} \int_{\mathbb{R}^d} z_i e^{\langle \lambda, z \rangle}\, d\nu(z)$$

$$\frac{\partial^2 L}{\partial\lambda_i\lambda_j}(\lambda) = \frac{\int_{\mathbb{R}^d} z_i z_j e^{\langle\lambda,z\rangle}\, d\nu(z)}{\exp L(\lambda)} - \frac{\left(\int_{\mathbb{R}^d} z_i e^{\langle\lambda,z\rangle}\, d\nu(z)\right)\left(\int_{\mathbb{R}^d} z_j e^{\langle\lambda,z\rangle}\, d\nu(z)\right)}{(\exp L(\lambda))^2}$$
$$= \mathbb{E}(Z_{\lambda,i}Z_{\lambda,j}) - \mathbb{E}(Z_{\lambda,i})\mathbb{E}(Z_{\lambda,j})$$

where $Z_\lambda = {}^t(Z_{\lambda,1},\dots,Z_{\lambda,d})$ is a random vector in $\mathbb{R}^d$ with distribution μ_ν whose density is

$$z \longmapsto \exp(\,\langle\lambda,z\rangle - L(\lambda)\,)$$

with respect to ν. Thus $\mathrm{D}^2_\lambda L$ is the covariance matrix of μ_ν. Moreover μ_ν has the same support as ν and thus it is non-degenerate. Therefore lemma 13 implies that $\mathrm{D}^2_\lambda L$ is invertible. Hence, for all $\lambda \in \overset{o}{D}_L$, $\mathrm{D}^2_\lambda L$ is a symmetric positive definite matrix. It follows that, for any $x \in \mathbb{R}^d$, the equation

$$\nabla L(\lambda) = x$$

has at most one solution $\lambda(x) \in \overset{o}{D}_L$. Indeed, if there exist two different vectors λ_1 and λ_2 in $\overset{o}{D}_L$ such that $\nabla L(\lambda_1) = \nabla L(\lambda_2)$, then the function

$$\psi : t \longmapsto \langle \nabla L(t\lambda_1 + (1-t)\lambda_2), \lambda_1 - \lambda_2\rangle$$

is $\mathcal{C}^1$ and real-valued on $[0,1]$ and verifies $\psi(0) = \psi(1) = 0$. Rolle's theorem implies that there exists $t_0 \in]0,1[$ such that $\psi'(t_0) = 0$, i.e.,

$$\left\langle \frac{d}{dt}\left(\nabla L(t\lambda_1 + (1-t)\lambda_2)\right)\Big|_{t=t_0}, \lambda_1 - \lambda_2 \right\rangle = 0$$

Setting $\lambda_0 = t_0\lambda_1 + (1-t_0)\lambda_2 \in \overset{o}{D}_L$ and $v = \lambda_1 - \lambda_2 \in \mathbb{R}^d\backslash\{0\}$, we have

$$\langle \mathrm{D}_{\lambda_0} L(v), v\rangle = 0$$

This contradicts the fact that $\mathrm{D}^2_\lambda L$ is positive definite. Hence ∇L is a bijection from $\overset{o}{D}_L$ to A_J with inverse function

$$\lambda : x \in A_J \longmapsto \lambda(x)$$

Moreover L is $\mathcal{C}^\infty$ on $\overset{o}{D}_L$ and for any $\lambda \in \overset{o}{D}_L$, $\mathrm{D}_\lambda(\nabla L) = \mathrm{D}^2_\lambda L$ is an isomorphism. Thus the inverse function theorem implies that ∇L is a $\mathcal{C}^\infty$-diffeomorphism from $\overset{o}{D}_L$ to A_J.

(b) For $x \in A_J$, we define

$$f_x : \lambda \in \mathbb{R}^d \longmapsto \langle x, \lambda\rangle - L(\lambda)$$

The map f_x is differentiable on $\overset{o}{D}_L$ and

$$\forall \lambda \in \overset{o}{D}_L \qquad \nabla f_x(\lambda) = x - \nabla L(\lambda)$$

We have shown in (a) that, for all $x \in A_J$, $\nabla f_x(\lambda) = 0$ if and only if $\lambda = \lambda(x)$. Since f_x is concave, its supremum is realized at $\lambda(x)$, that is

$$J(x) = f_x(\lambda(x)) = \langle x, \lambda(x)\rangle - L(\lambda(x))$$

It follows that $A_J \subset D_J$ and that J is $\mathcal{C}^\infty$ on A_J. Finally for any $x \in A_J$, $u \in \mathbb{R}^d$,

$$\langle \nabla J(x), u\rangle = \langle u, \lambda(x)\rangle + \langle \mathrm{D}_x\lambda(u), x\rangle - \langle \nabla L(\lambda(x)), \mathrm{D}_x\lambda(u)\rangle = \langle u, \lambda(x)\rangle$$

since $\nabla L(\lambda(x)) = x$. Hence

$$\forall x \in A_J \qquad \nabla J(x) = \lambda(x) = (\nabla L)^{-1}(x)$$

Differentiating $\nabla L(\lambda(x)) = x$, we get

$$\mathrm{D}^2_{\lambda(x)} L \circ \mathrm{D}^2_x J = \mathrm{D}^2_{\lambda(x)} L \circ \mathrm{D}_x \lambda = \mathrm{Id}$$

whence the expression of $\mathrm{D}^2_x J$ since $\mathrm{D}^2_{\lambda(x)} L$ is an isomorphism.

(c) If D_L is an open subset of $\mathbb{R}^d$ then lemma 14 implies that for $x \in \overset{o}{\mathcal{C}} = \overset{o}{D}_J$, the supremum defining $J(x)$ is realized at some point $\lambda(x) \in D_L = \overset{o}{D}_L$. The function L is differentiable at $\lambda(x)$ and the point (b) yields that

$$x = \nabla L(\lambda(x)) \in \Lambda(\overset{o}{D}_L) = A_J$$

Thus $\overset{o}{D}_J \subset A_J$. Finally, since $A_J \subset D_J$ and A_J is open, we have $A_J = \overset{o}{D}_J = \overset{o}{\mathcal{C}}$. This proves (c). □

Let ν be a probability distribution on $\mathbb{R}^d$ having a density with respect to the Lebesgue measure and let $(X_n)_{n\geq 1}$ be a sequence of independent and identically distributed random variables with distribution ν. The following theorem states that, under some hypothesis allowing the Fourier inversion, the density of the distribution of $(X_1 + \cdots + X_n)/n$ is asymptotically a function of

$$J : x \in \mathbb{R}^d \longmapsto \sup_{t \in \mathbb{R}^d} \left(\langle t, x \rangle - \ln \int_{\mathbb{R}^d} e^{\langle t, z \rangle}\, d\nu(z) \right)$$

We propose a proof, extracted from the article of C. Andriani and P. Baldi [1]. It relies on proposition 16.

Theorem 17. *Let ν be a non-degenerate probability measure on $\mathbb{R}^d$. We denote by L its Log-Laplace and by J its Cramér transform. Suppose that $\overset{o}{D}_L \neq \varnothing$ and that there exists $n_0 \geq 1$ such that*

$$\widehat{\nu^{*n_0}} \in \mathrm{L}^1(\mathbb{R}^d)$$

We denote by A_J the admissible domain of J. For any $x \in A_J$, we set μ_x the probability measure on $\mathbb{R}^d$ such that

$$d\mu_x(y) = \frac{\exp\langle y + x, \lambda(x) \rangle}{\exp L(\lambda(x))}\, d\nu(y + x)$$

(where λ is the inverse function of ∇L). For n large enough, the Fourier transform of μ_x belongs to $\mathrm{L}^n(\mathbb{R}^d)$. Let $(X_n)_{n\geq 1}$ be a sequence of independent and identically distributed random variables with distribution ν. For any $n \geq n_0$, the random variable $\overline{X}_n = (X_1 + \cdots + X_n)/n$ has a density g_n with respect to the Lebesgue measure on $\mathbb{R}^d$ satisfying :

(a) *For $x \in A_J$ and for n large enough,*

$$g_n(x) = \left(\frac{n}{2\pi}\right)^d e^{-nJ(x)} \int_{\mathbb{R}^d} \left(\widehat{\mu}_x(t)\right)^n\, dt$$

(b) *If K_J is a compact subset of A_J then, uniformly over $x \in K_J$, when n goes to $+\infty$,*

$$g_n(x) \sim \Big(\frac{n}{2\pi}\Big)^{d/2} \Big(\det \mathrm{D}_x^2 J\Big)^{1/2} e^{-nJ(x)}$$

The proof requires some preliminary results, which are presented next.

Lemma 18 (Uniform dominated convergence theorem)**.** *Let $\mathcal{X}$ be a separable space and let $(\Omega, \mathcal{F}, \mu)$ be a measurable space. Let f and f_n, $n \geq 1$, be real or complex-valued measurable functions defined on $\mathcal{X} \times \Omega$. Suppose that, for any $\omega \in \Omega$, the functions $x \longmapsto f(x, \omega)$ and $x \longmapsto f_n(x, \omega)$, $n \in \mathbb{N}$, are continuous on $\mathcal{X}$ and that*

$$\sup_{x \in \mathcal{X}} |f_n(x, \omega) - f(x, \omega)| \underset{n \to \infty}{\longrightarrow} 0$$

Suppose also that there exists a non-negative and integrable function g on Ω such that

$$\forall n \in \mathbb{N} \quad \forall x \in \mathcal{X} \quad \forall \omega \in \Omega \qquad |f_n(x, \omega)| \leq g(\omega)$$

Then for any $x \in \mathcal{X}$, the function $\omega \longmapsto f(x, \omega)$ is integrable and

$$\sup_{x \in \mathcal{X}} \left| \int_\Omega f_n(x, \omega)\, d\mu(\omega) - \int_\Omega f(x, \omega)\, d\mu(\omega) \right| \underset{n \to \infty}{\longrightarrow} 0$$

Proof. We adapt the proof of the classical dominated convergence theorem in [16]. Sending n to $+\infty$ in the domination inequality, we get

$$\forall (x, \omega) \in \mathcal{X} \times \Omega \qquad |f(x, \omega)| \leq g(\omega)$$

This shows that $\omega \longmapsto f(x, \omega)$ is integrable. For any $n \in \mathbb{N}$, we set

$$h_n : \omega \longmapsto \sup_{x \in \mathcal{X}} |f_n(x, \omega) - f(x, \omega)|$$

For all $n \in \mathbb{N}$ and $\omega \in \Omega$, the function $x \in \mathcal{X} \longmapsto |f_n(x, \omega) - f(x, \omega)|$ is continuous and, since $\mathcal{X}$ is separable, its supremum is equal to its supremum on a countable dense subset of $\mathcal{X}$. Therefore h_n is a measurable function. We have that $(2g - h_n)_{n \in \mathbb{N}}$ is a sequence of non-negative functions whose limit is the function $2g$. Fatou's lemma implies that

$$\begin{aligned} \int_\Omega 2g\, d\mu = \int_\Omega \liminf_{n \to +\infty} (2g - h_n)\, d\mu &\leq \liminf_{n \to +\infty} \int_\Omega (2g - h_n)\, d\mu \\ &= \int_\Omega 2g\, d\mu - \limsup_{n \to +\infty} \int_\Omega h_n\, d\mu \end{aligned}$$

Since g is integrable, we get that

$$\limsup_{n \to +\infty} \int_\Omega h_n\, d\mu \leq 0$$

Hence $\int_\Omega h_n\, d\mu \to 0$ since for any $n \in \mathbb{N}$, h_n is a non-negative function. Finally

$$\begin{aligned} \sup_{x \in \mathcal{X}} \left| \int_\Omega f_n(x, \omega)\, d\mu(\omega) - \int_\Omega f(x, \omega)\, d\mu(\omega) \right| &\leq \sup_{x \in \mathcal{X}} \int_\Omega |f_n(x, \omega) - f(x, \omega)|\, d\mu(\omega) \\ &\leq \int_\Omega h_n\, d\mu \underset{n \to \infty}{\longrightarrow} 0 \end{aligned}$$

and the lemma is proved. □

Lemma 19. *Let ν be a probability measure on $\mathbb{R}^d$. We denote by L its Log-Laplace. Let K be a compact subset of $\overset{o}{D}_L$. Then the function*

$$(s,t) \longmapsto M(s+it) = \int_{\mathbb{R}^d} e^{\langle s+it,x\rangle}\,d\nu(x)$$

is uniformly continuous on $K \times \mathbb{R}^d$.

Proof. For $n \geq 1$, we denote by B_n the open ball of radius n centered at the origin and we set

$$f_n : y \longmapsto \int_{\mathrm{B}_n^c} e^{\langle y,x\rangle}\,d\nu(x)$$

The sequence $(f_n)_{n\geq 1}$ is a non-increasing sequence of continuous functions on $K \subset \overset{o}{D}_L$, which converges to the null function. By Dini's theorem, the sequence $(f_n)_{n\geq 1}$ converges uniformly to the null function on K. Let $\varepsilon > 0$. There exists $n_0 \geq 1$ such that

$$\forall y \in K \qquad \int_{\mathrm{B}_{n_0}^c} e^{\langle y,x\rangle}\,d\nu(x) \leq \frac{\varepsilon}{4}$$

We define next

$$\forall x,s,t \in \mathbb{R}^d \qquad g(x,s,t) = \exp(\langle s+it,x\rangle)$$

The function g is uniformly continuous on $\mathrm{B}_{n_0} \times K \times \mathbb{R}^d$ (its differential is bounded on this set, hence g is lipschitz). Thus there exists $\delta > 0$ such that for $s,u \in K$ and $t,v \in \mathbb{R}^d$,

$$\|(s,t)-(u,v)\| \leq \delta \quad \Longrightarrow \quad \forall x \in \mathrm{B}_{n_0} \qquad |g(x,s,t)-g(x,u,v)| \leq \frac{\varepsilon}{2}$$

Therefore

$$\begin{aligned}
&|M(s+it)-M(u+iv)| \\
&\qquad \leq \int_{\mathrm{B}_{n_0}} |g(x,s,t)-g(x,u,v)|\,d\nu(x) + 2\sup_{y\in K}\int_{\mathrm{B}_{n_0}^c} e^{\langle y,x\rangle}\,d\nu(x) \\
&\qquad \leq \int_{\mathrm{B}_{n_0}} \frac{\varepsilon}{2}\,d\nu(x) + \frac{\varepsilon}{2} \leq \varepsilon
\end{aligned}$$

This proves the uniform continuity of $(s,t) \longmapsto M(s+it)$ on $K \times \mathbb{R}^d$. □

We will use the Riesz-Thorin theorem to prove our last lemma. Recall that the norm of a continuous linear operator T from $\mathrm{L}^p(\mathbb{R}^d)$ to $\mathrm{L}^q(\mathbb{R}^d)$, with $(p,q) \in [1,+\infty]^2$, is defined by

$$\|T\|_{p,q} = \sup\left\{ \frac{\|T(f)\|_q}{\|f\|_p} : f \in \mathrm{L}^p,\ f \neq 0 \right\}$$

Theorem 20 (Riesz-Thorin)**.** *Let p_0, p_1, q_0 and q_1 in $[1,+\infty]$ such that $p_0 \neq p_1$ and $q_0 \neq q_1$. For any $t \in [0,1]$, we put*

$$\frac{1}{p_t} = \frac{1-t}{p_0} + \frac{t}{p_1} \qquad \textit{and} \qquad \frac{1}{q_t} = \frac{1-t}{q_0} + \frac{t}{q_1}$$

Let T be a continuous linear operator from $\mathrm{L}^{p_0}(\mathbb{R}^d)$ to $\mathrm{L}^{q_0}(\mathbb{R}^d)$ with operator norm M_0. Suppose that T is also continuous from $\mathrm{L}^{p_1}(\mathbb{R}^d)$ to $\mathrm{L}^{q_1}(\mathbb{R}^d)$ with operator norm M_1. Then, for any $t \in [0,1]$, T is a continuous linear operator from $\mathrm{L}^{p_t}(\mathbb{R}^d)$ to $\mathrm{L}^{q_t}(\mathbb{R}^d)$ and

$$\forall f \in \mathrm{L}^{p_t}(\mathbb{R}^d) \qquad \|T(f)\|_{q_t} \leq M_0^{1-t} M_1^t \|f\|_{p_t}$$

We refer to chapter 1 of [3] for the proof of this theorem. We apply next the Riesz-Thorin theorem to the Fourier transform :

The map which associates each integrable function to its Fourier transform is a continuous linear operator from $\mathrm{L}^1(\mathbb{R}^d)$ to $\mathrm{L}^\infty(\mathbb{R}^d)$ with operator norm 1. Moreover Plancherel theorem guarantees that

$$\forall f \in \mathrm{L}^2(\mathbb{R}^d) \qquad \|\widehat{f}\|_2 = (2\pi)^{d/2}\|f\|_2$$

For any $t \in [0,1]$, we put $p_t = 2/(2-t)$ and $q_t = 2/t$. We easily check that p_t and q_t are conjugate and that

$$\frac{1}{p_t} = \frac{1-t}{p_0} + \frac{t}{p_1} \qquad \text{and} \qquad \frac{1}{q_t} = \frac{1-t}{q_0} + \frac{t}{q_1}$$

with $p_0 = 1$, $p_1 = 2$, $q_0 = +\infty$ and $q_1 = 2$. The Riesz-Thorin theorem implies that

$$\forall f \in \mathrm{L}^{p_t}(\mathbb{R}^d) \qquad \|\widehat{f}\|_{q_t} \leq 1^{1-t}(2\pi)^{td/2}\|f\|_{p_t} = (2\pi)^{d/q_t}\|f\|_{p_t}$$

Since $p_t \in [1,2]$, we get the following inequality :

Lemma 21 (Hausdorff-Young inequality)**.** *Let $d \geq 1$ and $p \in]1,2]$. We denote $q = p/(p-1) \in [2,+\infty[$. Then*

$$\forall f \in \mathrm{L}^p(\mathbb{R}^d) \qquad \|\widehat{f}\|_q \leq (2\pi)^{d/q}\|f\|_p$$

Proof of theorem 17. We denote by φ the Fourier transform of ν. We have

$$\forall n \geq 1 \qquad \widehat{\nu^{*n}} = \widehat{\nu}^n = \varphi^n$$

By hypothesis, $\varphi^{n_0} \in \mathrm{L}^1(\mathbb{R}^d)$ and

$$\forall n \geq n_0 \qquad |\widehat{\nu^{*n}}| = |\varphi|^{n_0}|\varphi|^{n-n_0} \leq |\varphi|^{n_0}$$

Thus $\widehat{\nu^{*n}} \in \mathrm{L}^1(\mathbb{R}^d)$ and the Fourier inversion formula implies that ν^{*n} has a density f_n given by

$$\forall x \in \mathbb{R} \qquad f_n(x) = \frac{1}{(2\pi)^d}\int_{\mathbb{R}^d} e^{-i\langle t,x\rangle}\varphi^n(t)\,dt$$

We also get that $f_n \in \mathrm{L}^\infty(\mathbb{R}^d)$. Let $s \in \overset{o}{D}_L$ and $n \geq 1$. The function $x \longmapsto e^{\langle s,x\rangle}f_n(x)$ is non-negative and its integral over $\mathbb{R}^d$ is $M(s)^n < +\infty$. Let us denote by $\varphi_{s,n}$ its Fourier transform. Let $p \in]1,2]$ be such that $ps \in \overset{o}{D}_L$. We have

$$\int_{\mathbb{R}^d} \left|e^{\langle s,x\rangle}f_{n_0}(x)\right|^p\,dx \leq \int_{\mathbb{R}^d} e^{\langle sp,x\rangle}f_{n_0}(x)\|f_{n_0}\|_\infty^{p-1}\,dx = M(ps)^{n_0}\|f_{n_0}\|_\infty^{p-1}$$

which is finite. The Hausdorff-Young inequality implies that $\varphi_{s,n_0} \in \mathrm{L}^q(\mathbb{R}^d)$ where $q = p/(p-1) \in [2, +\infty[$. However

$$\forall t \in \mathbb{R}^d \qquad \varphi_{s,n}(t) = \int_{\mathbb{R}^d} e^{i\langle t,x\rangle} e^{\langle s,x\rangle} f_n(x)\, dx = M(s+it)^n$$

Thus for any $n \geq qn_0$,

$$\begin{aligned}
\int_{\mathbb{R}^d} |\varphi_{s,n}(t)|\, dt &= \int_{\mathbb{R}^d} |M(s+it)|^{n_0 q} |M(s+it)|^{n-n_0 q}\, dt \\
&\leq M(s)^{n-n_0 q} \int_{\mathbb{R}^d} \left(|M(s+it)|^{n_0}\right)^q\, dt \\
&= M(s)^{n-n_0 q} \int_{\mathbb{R}^d} |\varphi_{s,n_0}|^q\, dt < +\infty
\end{aligned}$$

Thus $\varphi_{s,n} \in \mathrm{L}^1(\mathbb{R}^d)$ and the Fourier inversion theorem yields that for all $x \in \mathbb{R}$,

$$e^{\langle s,x\rangle} f_n(x) = \frac{1}{(2\pi)^d} \int_{\mathbb{R}^d} e^{-i\langle t,x\rangle} \varphi_{s,n}(t)\, dt = \frac{1}{(2\pi)^d} \int_{\mathbb{R}^d} e^{-i\langle t,x\rangle} M(s+it)^n\, dt$$

Moreover for $x \in A_J$ and $t \in \mathbb{R}^d$,

$$\widehat{\mu}_x(t) = \int_{\mathbb{R}^d} e^{i\langle t,y\rangle + \langle y+x,\lambda(x)\rangle - L(\lambda(x))}\, d\nu(y+x) = e^{-i\langle t,x\rangle} \frac{M(\lambda(x)+it)}{M(\lambda(x))}$$

where we made the change of variables $z = y + x$. It follows that $\widehat{\mu}_x \in \mathrm{L}^n(\mathbb{R}^d)$ for $n \geq n_0 q$. Notice that

$$\forall x \in \mathbb{R}^d \qquad g_n(x) = n^d f_n(nx)$$

Therefore

$$g_n(x) = \left(\frac{n}{2\pi}\right)^d \int_{\mathbb{R}^d} \left(e^{-\langle it+s,x\rangle + L(s+it)}\right)^n\, dt$$

If $x \in A_J$ then proposition 16 implies

$$J(x) = \langle \lambda(x), x\rangle - L(\lambda(x))$$

thus, applying the above inequality to $s = \lambda(x)$, we get

$$\begin{aligned}
g_n(x) &= \left(\frac{n}{2\pi}\right)^d \int_{\mathbb{R}^d} \left(e^{-\langle \lambda(x)+it,x\rangle + L(\lambda(x)+it)}\right)^n\, dt \\
&= \left(\frac{n}{2\pi}\right)^d e^{-nJ(x)} \int_{\mathbb{R}^d} \left(e^{J(x) - \langle \lambda(x)+it,x\rangle + L(\lambda(x)+it)}\right)^n\, dt \\
&= \left(\frac{n}{2\pi}\right)^d e^{-nJ(x)} \int_{\mathbb{R}^d} \left(e^{-i\langle t,x\rangle - L(\lambda(x)) + L(\lambda(x)+it)}\right)^n\, dt \\
&= \left(\frac{n}{2\pi}\right)^d e^{-nJ(x)} \int_{\mathbb{R}^d} \left(\widehat{\mu}_x(t)\right)^n\, dt
\end{aligned}$$

This equality is valid when $n \geq n_0 q$. This proves the point (a).

Now let us prove the point (b). Let K_J be a compact subset of A_J. We notice that q depends on $x \in A_J$, but, by compactness of K_J, we can choose q uniformly over $x \in K_J$. For any $x \in K_J$, the mean of μ_x is

$$\int_{\mathbb{R}^d} y \frac{e^{\langle x+y,\lambda(x)\rangle}}{\exp M(\lambda(x))}\, d\nu(y+x) = \int_{\mathbb{R}^d} (z-x) \frac{e^{\langle z,\lambda(x)\rangle}}{M(\lambda(x))}\, d\nu(z) = \nabla L(\lambda(x)) - x = 0$$

and its covariance matrix is $\Gamma_x = \mathrm{D}^2_{\lambda(x)} L$ since for $1 \le i, j \le d$ and $s \in D_L$,

$$\begin{aligned}(\Gamma_x)_{i,j} &= \frac{\int_{\mathbb{R}^d} y_i y_j e^{\langle \lambda(x), y+x\rangle}\, d\nu(y+x)}{M(\lambda(x))} = \frac{\int_{\mathbb{R}^d} (z_i - x_i)(z_j - x_j) e^{\langle \lambda(x), z\rangle}\, d\nu(z)}{M(\lambda(x))} \\ &= \frac{\int_{\mathbb{R}^d} z_i z_j e^{\langle \lambda(x), z\rangle}\, d\nu(z)}{M(\lambda(x))} - x_i x_j = \frac{\partial^2 L}{\partial s_i s_j}(\lambda(x))\end{aligned}$$

When $t \to 0$, uniformly over $x \in K_J$,

$$\widehat{\mu_x}(t) = 1 - \frac{1}{2}\langle \Gamma_x t, t\rangle + o(\|t\|^2)$$

Indeed

$$(x, t) \longmapsto \widehat{\mu_x}(t) = e^{-i\langle t, x\rangle} \frac{M(\lambda(x) + it)}{M(\lambda(x))}$$

is $\mathcal{C}^\infty$ on $A_J \times \mathbb{R}^d$ (by proposition 16), thus the Taylor-Lagrange formula guarantees that the remainder term is uniformly controlled over $x \in K_J$. Therefore, for any $t \in \mathbb{R}^d$, uniformly over $x \in K_J$,

$$\widehat{\mu_x}\left(\frac{t}{\sqrt{n}}\right)^n \underset{n\to\infty}{\longrightarrow} \exp\left(-\frac{1}{2}\langle \Gamma_x t, t\rangle\right)$$

The functions $x \longmapsto \widehat{\mu}_x$ and $x \longmapsto \exp\left(-\langle \Gamma_x t, t\rangle/2\right)$, $t \in \mathbb{R}^d$, are continuous on K_J. In order to apply the dominated convergence theorem (the uniform variant), we need to get a uniform domination of the sequence of functions. For $x \in A_J$, Γ_x is a positive definite symmetric matrix thus ε_x, its smallest eigenvalue, is positive. The largest eigenvalue of the inverse of Γ_x is ε_x^{-1}. Therefore, for any $x \in A_J$,

$$\varepsilon_x = \Big(\max\{\, \alpha : \alpha \text{ eigenvalue of } \Gamma_x^{-1}\,\}\Big)^{-1} = \left(\sup_{y\neq 0} \frac{\langle \Gamma_x^{-1} y, \Gamma_x^{-1} y\rangle}{\langle y, y\rangle}\right)^{-1/2}$$

The term on the right is the inverse of the operator norm of the linear application associated to the matrix Γ_x^{-1}. Moreover $x \longmapsto \Gamma_x = \mathrm{D}^2_{\lambda(x)} L$ is continuous on A_J thus the function $x \longmapsto \varepsilon_x$ is continuous. Let us denote by ε_0 its minimum on K_J. The compactness of K_J ensures that $\varepsilon_0 > 0$. The previous expansion implies that there exists $\delta > 0$ such that

$$\forall (t, x) \in \mathrm{B}(0, \delta) \times K_J \qquad |\widehat{\mu_x}(t)| \le 1 - \frac{1}{2}\left\langle \left(\Gamma_x - \frac{\varepsilon_0}{2}\mathrm{I}_d\right) t, t\right\rangle$$

The spectral theorem for real symmetric matrices yields that, for any $x \in K_J$, the matrix $\Gamma_x - \varepsilon_0 \mathrm{I}_d$ is positive symmetric. Thus

$$\forall t \in \mathbb{R}^d \qquad \left\langle \left(\Gamma_x - \frac{\varepsilon_0}{2}\mathrm{I}_d\right) t, t\right\rangle - \frac{\varepsilon_0}{2}\|t\|^2 = \langle (\Gamma_x - \varepsilon_0 \mathrm{I}_d) t, t\rangle \ge 0$$

It follows that

$$\forall (t, x) \in \mathrm{B}(0, \delta) \times K_J \qquad |\widehat{\mu_x}(t)| \le 1 - \frac{\varepsilon_0}{4}\|t\|^2$$

Since $1 - y \le e^{-y}$ for all $y \ge 0$, we get

$$\forall n \ge 1 \quad \forall (t, x) \in \mathrm{B}(0, \delta\sqrt{n}) \times K_J \qquad \left|\widehat{\mu_x}\left(\frac{t}{\sqrt{n}}\right)\right|^n \le \exp\left(-\frac{\varepsilon_0}{4}\|t\|^2\right)$$

The right term is integrable and does not depend on $x \in K_J$ and n. The uniform dominated convergence theorem (lemma 18) implies that, uniformly in K_J,

$$\int_{\|t\|<\delta\sqrt{n}} \widehat{\mu_x}\left(\frac{t}{\sqrt{n}}\right)^n dt \underset{n\to\infty}{\longrightarrow} \int_{\mathbb{R}^d} \exp\left(-\frac{1}{2}\langle \Gamma_x t, t\rangle\right) dt$$

Moreover this second integral is equal to $(2\pi)^{d/2}\left(\det \Gamma_x\right)^{-1/2}$ and proposition 16 guarantees that for $x \in A_J$, $\mathrm{D}^2_{\lambda(x)}L$ is the inverse matrix of $\mathrm{D}^2_x J$. Hence, when $n \to \infty$, uniformly over $x \in K_J$,

$$\int_{\|t\|<\delta} \widehat{\mu_x}(t)^n\, dt = n^{-d/2}\int_{\|t\|<\delta\sqrt{n}} \widehat{\mu_x}\left(\frac{t}{\sqrt{n}}\right)^n dt \sim \left(\frac{2\pi}{n}\right)^{d/2}\left(\det \mathrm{D}^2_x J\right)^{1/2}$$

Let us focus on the remainder of the integral. We set

$$h : x \in K_J \longmapsto \sup_{\|t\|\geq\delta} |\widehat{\mu_x}(t)|$$

The function λ is continuous thus $\lambda(K_J)$ is compact and lemma 19 states that the function $(s,t) \longmapsto M(s+it)$ is uniformly continuous on $\lambda(K_J)\times\mathbb{R}^d$. Therefore the function

$$x \longmapsto \sup_{\|t\|\geq\delta} |M(\lambda(x)+it)|$$

is continuous on K_J. However

$$\forall x \in K_J \qquad h(x) = \sup_{\|t\|\geq\delta} |\widehat{\mu_x}(t)| = \frac{1}{M(\lambda(x))}\sup_{\|t\|\geq\delta} |M(\lambda(x)+it)|$$

Hence h is continuous on K_J. By compactness of K_J, there exists $x_0 \in K_J$ such that

$$\sup_{x\in K_J}\sup_{\|t\|\geq\delta} |\widehat{\mu_x}(t)| = \sup_{x\in K_J} h(x) = h(x_0) = \sup_{\|t\|\geq\delta} |\widehat{\mu_{x_0}}(t)|$$

Finally, just like ν, the law $\mu_{x_0}^{*n_0}$ has a density and the Riemann-Lebesgue lemma implies that

$$\widehat{\mu_{x_0}^{*n_0}}(t) \underset{\|t\|\to+\infty}{\longrightarrow} 0$$

Moreover lemma 4 of chapter XV.1 of [11] guarantees that for any $t \neq 0$, $|\widehat{\mu_{x_0}^{*n_0}}(t)| < 1$. Therefore there exists $\kappa \in]0,1[$ such that

$$\sup_{\|t\|\geq\delta}\left|\widehat{\mu_{x_0}^{*n_0}}(t)\right| \leq \kappa^{n_0}$$

We get

$$\sup_{x\in K_J}\sup_{\|t\|\geq\delta} |\widehat{\mu_x}(t)| \leq \kappa < 1$$

It follows that for any $x \in K_J$ and $n \geq n_0 q$, uniformly over $x \in K_J$,

$$\left|\int_{\|t\|\geq\delta} \widehat{\mu_x}(t)^n\, dt\right| \leq \int_{\|t\|\geq\delta} |\widehat{\mu_x}(t)|^n\, dt \leq \kappa^{n-n_0 q}\int_{\|t\|\geq\delta} |\widehat{\mu_{x_0}}(t)|^{n_0 q}\, dt = o(1)$$

since $\kappa < 1$ and $\widehat{\mu_{x_0}} \in \mathrm{L}^{n_0 q}(\mathbb{R}^d)$. Finally

$$\begin{aligned} g_n(x) &= \left(\frac{n}{2\pi}\right)^d e^{-nJ(x)} \left(\int_{\|t\|\geq\delta} \widehat{\mu_x}(t)^n \, dt + \int_{\|t\|<\delta} \widehat{\mu_x}(t)^n \, dt \right) \\ &= \left(\frac{n}{2\pi}\right)^d e^{-nJ(x)} \left(o(1) + \left(\frac{2\pi}{n}\right)^{d/2} \left(\det \mathrm{D}_x^2 J\right)^{1/2} (1+o(1)) \right) \\ &= \left(\frac{n}{2\pi}\right)^{d/2} e^{-nJ(x)} \left(\det \mathrm{D}_x^2 J\right)^{1/2} (1+o(1)) \end{aligned}$$

The expansion is uniform over $x \in K_J$, by the previous results and the boundedness of $x \longmapsto \left(\det \mathrm{D}_x^2 J\right)^{1/2}$ on K_J, since it is continuous. This ends the proof of theorem 17. □

Proposition 22. *Let ν be a non-degenerate probability measure on $\mathbb{R}^d$ such that $\overset{o}{D}_L \neq \varnothing$. If there exists $m \in \mathbb{N}$ and $p \in]1,2]$ such that ν^{*m} has a density $f_m \in \mathrm{L}^p(\mathbb{R}^d)$ then the hypothesis of theorem 17 are verified.*

Proof. It follows from the Hausdorff-Young inequality that $\widehat{f_m} \in \mathrm{L}^r(\mathbb{R}^d)$, with $r = p/(p-1)$. Moreover $\widehat{f_m}$ is bounded thus $\widehat{f_m} \in \mathrm{L}^q(\mathbb{R}^d)$, where q is a positive integer larger than r. Therefore

$$\widehat{\nu^{*mq}} = \left(\widehat{\nu^{*m}}\right)^q = \left(\widehat{f_m}\right)^q \in \mathrm{L}^1(\mathbb{R}^d)$$

Hence the hypothesis of the theorem are verified with $n_0 = mq$. □

7 Minima of I-F

Let ρ be a probability measure on $\mathbb{R}$. We define

$$\Lambda : (u,v) \in \mathbb{R}^2 \longmapsto \ln \int_{\mathbb{R}} e^{uz+vz^2} \, d\rho(z)$$

We denote by I the Fenchel-Legendre transform of Λ.

In this section, we consider the minima of the function $I - F$ when ρ is symmetric.

a) Admissible domain of I

We begin by giving some properties of I, which are consequences of the results stated in the previous section. Let ν_ρ be the distribution of (Z, Z^2) when Z is a random variable with law ρ. We suppose that the support of ρ contains at least three points so that ν_ρ is a non-degenerate measure on $\mathbb{R}^2$. The function Λ is the Log-Laplace of ν_ρ and its domain of definition D_Λ contains $\mathbb{R}\times]-\infty,0[$, thus its interior is non-empty. The function I is the Cramér transform of ν_ρ and we denote by $A_I = \nabla\Lambda(\overset{o}{D}_\Lambda)$ the admissible domain of I. Proposition 16 implies that :

(a) The function $\nabla\Lambda$ is a $\mathcal{C}^\infty$-diffeomorphism from $\overset{o}{D}_\Lambda$ to A_I. Moreover

$$A_I \subset D_I = \{\, x \in \mathbb{R}^d : I(x) < +\infty \,\}$$

(b) The function I is $\mathcal{C}^\infty$ on A_I. If $(x,y) \longmapsto (u(x,y),v(x,y))$ is the inverse function of $\nabla\Lambda$ then, for any $(x,y)\in A_I$,

$$I(x,y) = xu(x,y)+yv(x,y)-\Lambda(u(x,y),v(x,y))$$
$$\nabla I(x,y) = (\nabla\Lambda)^{-1}(x,y) = (u(x,y),v(x,y))$$
$$\mathrm{D}^2_{(x,y)} I = \left(\mathrm{D}^2_{(u(x,y),v(x,y))}\Lambda\right)^{-1}$$

(c) If D_Λ is an open subset of $\mathbb{R}^2$ then $A_I = \overset{o}{D}_I = \overset{o}{\mathcal{C}}$ where $\mathcal{C}$ is the convex hull of the set $\{\,(x,x^2) : x \text{ is in the support of } \rho\,\}$.

b) Minimum of $I-F$ on Δ^*

Let ρ be a symmetric and non-degenerate probability measure on $\mathbb{R}$. Jensen's inequality gives us

$$\forall (u,v)\in\mathbb{R}^2 \qquad \ln\int_{\mathbb{R}} e^{uz+vz^2}\,d\rho(z) \geq \int_{\mathbb{R}} (uz+vz^2)\,d\rho(z) = v\sigma^2$$

thus $I(0,\sigma^2)\leq 0$. Since I is non-negative, then $I(0,\sigma^2)=0$ and

$$\inf_{\Delta^*}(I-F)\in[-\frac{1}{2},0]$$

The function I is even in the first variable. Indeed, if $(x,y)\in\mathbb{R}^2$, then

$$\begin{aligned} I(-x,y) &= \sup_{(u,v)\in\mathbb{R}^2}\left(-xu+yv-\ln\int_{\mathbb{R}} e^{uz+vz^2}\,d\rho(z)\right) \\ &= \sup_{(u,v)\in\mathbb{R}^2}\left(xu+yv-\ln\int_{\mathbb{R}} e^{-uz+vz^2}\,d\rho(z)\right) = I(x,y) \end{aligned}$$

Assume that $I-F$ has a unique minimum (x_0,y_0) on Δ^*. Then $(-x_0,y_0)$ is also a minimum of $I-F$ since

$$I(-x_0,y_0)-F(-x_0,y_0) = I(x_0,y_0)-F(x_0,y_0)$$

The uniqueness of the minimum implies that $x_0=0$ so that

$$\inf_{\Delta^*}(I-F) = I(0,y_0)-F(0,y_0) = I(0,y_0)\geq 0$$

Since $I(0,\sigma^2)=0$ we have $y_0=\sigma^2$.

In this section, we will show that, if ρ is symmetric, then $I-F$ has a unique minimum on Δ^*, which is at $(0,\sigma^2)$.

Consider first the case of a Bernoulli distribution. Let $c>0$. Suppose that $\rho=(\delta_{-c}+\delta_c)/2$. The law ρ is centered and its variance is c^2. For all $(u,v)\in\mathbb{R}^2$,

$$\Lambda(u,v) = \ln\int_{\mathbb{R}} e^{ux+vx^2}\,d\rho(x) = vc^2+\ln\cosh(uc)$$

For any $(x,y)\in\mathbb{R}^2$, by studying the function $(u,v)\longmapsto xu+yv-\Lambda(u,v)$, we can determinate its supremum. We get that I is finite on $D_I=[-c,c]\times\{c^2\}$, $I(-c,c^2)=I(c,c^2)=\ln 2$ and for any $x\in\,]-c,c[$,

$$I(x,c^2) = \frac{1}{2c}\left((c+x)\ln(c+x)+(c-x)\ln(c-x)\right)-\ln c$$

The function $g : x \longmapsto I(x, c^2) - x^2/(2c^2)$ is $\mathcal{C}^2$ on $]-c, c[$ and

$$\forall x \in]-c,c[\qquad g'(x) = \frac{1}{c}\left(\operatorname{arctanh}\left(\frac{x}{c}\right) - \frac{x}{c}\right)$$
$$g''(x) = \frac{1}{c^2 - x^2} - \frac{1}{c^2} \geq 0$$

Thus g' is non-decreasing on $[-c, c]$ and, since $g'(0) = 0$, it follows that g has a unique minimum at 0. Therefore $I - F$ has a unique minimum in Δ^* at $(0, c^2) = (0, \sigma^2)$.

The previous results yield the following lemma :

Lemma 23. *Let* $c > 0$*. We define*

$$\varphi_c : x \in \mathbb{R} \longmapsto \sup_{u \in \mathbb{R}} (\, ux - \ln\cosh(uc)\,)$$

The function

$$x \in [-c, c] \longmapsto \varphi_c(x) - \frac{x^2}{2c^2}$$

has a unique minimum at 0 *and* $\varphi_c(0) = 0$*.*

Notice that the Bernoulli case is special since, if X is a random variable with distribution $\rho = (\delta_{-c} + \delta_c)/2$, then $X^2 = c^2$ almost surely. Thus

$$\frac{1}{Z_n} \exp\left(\frac{1}{2}\frac{(x_1 + \cdots + x_n)^2}{x_1^2 + \cdots + x_n^2}\right) \mathbb{1}_{\{x_1^2 + \cdots + x_n^2 > 0\}} \prod_{i=1}^n d\rho(x_i)$$
$$= \frac{1}{Z_n} \exp\left(\frac{(x_1 + \cdots + x_n)^2}{2nc^2}\right) \prod_{i=1}^n d\rho(x_i)$$

This is exactly the classical Curie-Weiss model.

In the following, we suppose that the support of ν contains at least three distinct points. We first show that, if D_Λ is an open subset of $\mathbb{R}^2$, then $I - F$ has a unique minimum at $(0, \sigma^2)$.

In the subsection a), we saw that, if the support of ν contains at least three distinct points and D_Λ is an open subset of $\mathbb{R}^2$, then I is differentiable on the interior of its domain of definition D_I and, if $(x, y) \longmapsto (u(x, y), v(x, y))$ is the inverse function of $\nabla\Lambda$, then

$$\forall (x,y) \in \overset{o}{D}_I \qquad \frac{\partial I}{\partial x}(x, y) = u(x, y)$$

If we show that $u(x, y) \geq x/y$ for $x \geq 0$ and $y > 0$, then, by integrating this inequality and using the fact that I is even in its first variable, we get that

$$\forall (x,y) \in \overset{o}{D}_I \qquad I(x, y) - I(0, y) \geq \frac{x^2}{2y} = F(x, y)$$

To obtain that $I - F$ has a unique minimum at $(0, \sigma^2)$, it is enough to extend this inequality to the boundary points of D_I (if they exist) and to show that $I(0, .)$ has a unique minimum at σ^2.

The following lemma is the key result to establish the uniqueness of the minimum of $I - F$, when ρ is symmetric.

Lemma 24. *Let ρ be a symmetric probability measure whose support contains at least three points. We have $u(x,y)=0$ if $x=0$ and*

$$u(x,y)\geq\frac{x}{y}\quad\text{if}\quad x>0$$

$$u(x,y)\leq\frac{x}{y}\quad\text{if}\quad x<0$$

Proof. The vector $(u,v)=(u(x,y),v(x,y))$ verifies

$$(x,y)=\nabla\Lambda(u,v)=\left(\frac{\int_{\mathbb{R}}ze^{uz+vz^2}\,d\rho(z)}{\int_{\mathbb{R}}e^{uz+vz^2}\,d\rho(z)},\frac{\int_{\mathbb{R}}z^2e^{uz+vz^2}\,d\rho(z)}{\int_{\mathbb{R}}e^{uz+vz^2}\,d\rho(z)}\right)$$

The distribution ρ is symmetric, thus

$$\int_{\mathbb{R}}ze^{uz+vz^2}\,d\rho(z)=\int_0^{+\infty}2z\sinh(uz)e^{vz^2}\,d\rho(z)$$

This formula shows that u and x have the same sign. Moreover for any $z\geq 0$, $\tanh(z)\leq z$ thus, if $x>0$ then $\sinh(uz)\leq uz\cosh(uz)$. Therefore, using the symmetry of ρ,

$$x\leq u\,\frac{\int_0^{+\infty}2z^2\cosh(uz)e^{vz^2}\,d\rho(z)}{\int_{\mathbb{R}}e^{uz+vz^2}\,d\rho(z)}=u\,\frac{\int_{\mathbb{R}}z^2e^{uz+vz^2}\,d\rho(z)}{\int_{\mathbb{R}}e^{uz+vz^2}\,d\rho(z)}=uy$$

Since $x>0$, $u>0$ and $y>0$, we conclude that $u\geq x/y$. Similarly, we show that if $x<0$ then $u\leq x/y$. □

We can now prove the following inequality :

Proposition 25. *If ρ is a symmetric probability measure on $\mathbb{R}$ with variance $\sigma^2>0$ and such that D_Λ is an open subset of $\mathbb{R}^2$ then*

$$\forall(x,y)\in\mathbb{R}\times\mathbb{R}\backslash\{0\}\qquad I(x,y)-\frac{x^2}{2y}\geq I(0,y)$$

Notice that this result encompasses the case of a symmetric measure with bounded support, because in this case $D_\Lambda=\mathbb{R}^2$. In proposition 31, we shall extend the inequality to any symmetric distribution on $\mathbb{R}$.

Proof. We already treated the Bernoulli case (see lemma 23). We assume next that the support of ρ contains at least three points. We denote by A_I the admissible domain of I and by $\mathcal{C}$ the convex hull of the set

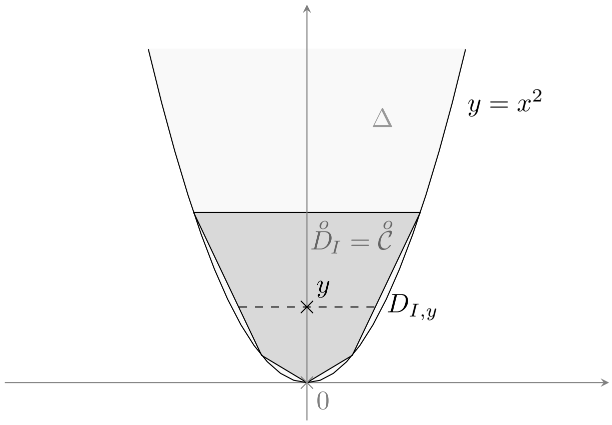
$$\{\,(x,x^2): x\text{ is in the support of }\rho\,\}$$

In the subsection a), we saw that, if D_Λ is an open subset of $\mathbb{R}^2$, then $A_I=\overset{o}{D}_I=\overset{o}{\mathcal{C}}\subset\Delta^*$. Moreover I is $\mathcal{C}^\infty$ on $\overset{o}{D}_I$ and if $(x,y)\longmapsto(u(x,y),v(x,y))$ is the inverse function of $\nabla\Lambda$ then

$$\forall(x,y)\in\overset{o}{D}_I\qquad\frac{\partial I}{\partial x}(x,y)=u(x,y)$$

Let us examine the structure of the set D_I. We put

$$\forall y>0\qquad D_{I,y}=D_I\cap(\mathbb{R}\times\{y\})$$



Case where ρ is symmetric discrete and charges 5 points

Let $y > 0$ be such that $(x, y) \in \overset{o}{D}_I$ for some $x \in \mathbb{R}$. The set $D_{I,y}$ is a convex subset of $\mathbb{R}$. Moreover $x \longmapsto I(x, y)$ is even, therefore $\overset{o}{D}_{I,y}$ is an open interval $]-a(y), a(y)[$ with $a(y) \in [0, \sqrt{y}]$. We have

$$\forall x \in \overset{o}{D}_{I,y} \qquad I(x, y) - I(0, y) = \int_0^x u(t, y)\, dt$$

Lemma 24 implies that for any $t \geq 0$, $u(t, y) \geq t/y$. By integrating and using the fact that I is even, we get that

$$\forall x \in \overset{o}{D}_{I,y} \qquad I(x, y) - I(0, y) \geq \frac{x^2}{2y}$$

and there is no problem of definition at $y = 0$ since $\overset{o}{D}_I \subset \Delta^*$ does not contain $\mathbb{R} \times \{0\}$ and $\overset{o}{D}_{I,0} = \varnothing$. Moreover

$$x \longmapsto \frac{I(x, y) - I(0, y)}{x}$$

is non-decreasing on $D_{I,y} \backslash \{0\}$ since I is convex. Therefore, if $-a(y)$ and $a(y)$ belong to $D_{I,y}$, then the previous inequality extends to $x = -a(y)$ and $x = a(y)$. We have shown that

$$\forall (x, y) \in D_I \quad \forall y > 0 \qquad I(x, y) - I(0, y) \geq \frac{x^2}{2y}$$

except for the points (x, y) of the superior and inferior borders of D_I, if they exist. More precisely, we set

$$K^2 = \inf \{\, x^2 : x \text{ is in the support of } \rho\} \geq 0$$

and

$$L^2 = \sup \{\, x^2 : x \text{ is in the support of } \rho\} \leq +\infty$$

If $K = 0$ and $L = +\infty$ then the inequality is already proved on $D_I \backslash \{(0, 0)\}$. Suppose that $K^2 > 0$. Let $y = K^2$ and $x \in \mathbb{R}$. We define

$$f : (u, v) \in \mathbb{R}^2 \longmapsto ux + vK^2 - \Lambda(u, v)$$

Denoting $c_K = \rho(\{K\})$, we have for all $(u, v) \in \mathbb{R}^2$,

$$f(u,v) = ux - \ln(2c_K \cosh(uK)) - \ln \int_{\mathbb{R}\backslash[-K,K]} e^{uz+v(z^2-K^2)}\, d\rho(z)$$

For any $z \in \mathbb{R}\backslash]-K,K[$, the function $v \longmapsto \exp(v(z^2-K^2))$ is non-decreasing. Therefore

$$\begin{aligned}\sup_{v\in\mathbb{R}} f(u,v) &= ux - \ln(2c_K \cosh(uK)) - \ln\left(\lim_{v\to-\infty} \int_{\mathbb{R}\backslash[-K,K]} e^{uz+v(z^2-K^2)}\, d\rho(z)\right)\\ &= ux - \ln(2c_K \cosh(uK))\end{aligned}$$

by the dominated convergence theorem. Indeed

$$\forall z \in \mathbb{R}\backslash[-K,K] \quad \forall v < -1 \qquad \left|e^{uz+v(z^2-K^2)}\right| \leq e^{uz-(z^2-K^2)}$$

and the map $z \in \mathbb{R}\backslash[-K,K] \longmapsto e^{uz-(z^2-K^2)}$ is integrable with respect to ρ since it is bounded (it is continuous and goes to 0 when $|z|$ goes to $+\infty$). Hence

$$I(x, K^2) = \sup_{u,v\in\mathbb{R}} f(u,v) = \sup_{u\in\mathbb{R}} \{\, ux - \ln(2c_K \cosh(uK))\,\}$$

In fact, we come back to the Bernoulli case. The reason is that, if we condition on $T_n = K^2$ in our model, then for any i, $X_n^i = -K$ or K.

If $c_K > 0$, then lemma 23 implies that for all $x \in [-K, K]$,

$$I(x, K^2) - I(0, K^2) = \varphi_K(x) \geq \frac{x^2}{2K^2}$$

If $c_K = 0$ then for any $x \neq 0$, $I(x, K^2) = +\infty$ so that the inequality is verified for $y = K^2$.

If $L < +\infty$ then we show similarly that for all $x \in [-L, L]$,

$$I(x, L^2) - I(0, L^2) \geq \frac{x^2}{2L^2}$$

Therefore for any $(x, y) \in D_I\backslash\{(0,0)\}$,

$$I(0, y) \leq I(x, y) - \frac{x^2}{2y}$$

Notice that for any $y \in \mathbb{R}$, by the convexity and the symmetry of $x \longmapsto I(x, y)$, if $I(0, y) = +\infty$ then for all $x \neq 0$, $I(x, y) = +\infty$. Therefore the inequality extends to each subset of $\mathbb{R}^2$ which does not contain $\mathbb{R} \times \{0\}$. $\square$

In the previous proof, if we take $x = y = 0$, then for any $u \in \mathbb{R}$, the function $v \longmapsto \Lambda(u, v)$ is non-decreasing on $\mathbb{R}$. Therefore

$$\begin{aligned}\inf_{v\in\mathbb{R}} \Lambda(u,v) = \lim_{v\to-\infty} \Lambda(u,v) &= \lim_{v\to-\infty} \left(\ln \rho(\{0\}) + \ln \int_{\mathbb{R}\backslash\{0\}} e^{uz+vz^2}\, d\rho(z)\right)\\ &= \ln \rho(\{0\})\end{aligned}$$

by the dominated convergence theorem. Hence

$$\inf_{u,v\in\mathbb{R}^2} \Lambda(u,v) = \inf_{u\in\mathbb{R}} (\ln \rho(\{0\})) = \ln \rho(\{0\})$$

This is valid for any probability measure ρ in $\mathbb{R}$. This yields the following lemma :

Lemma 26. *If ρ is a probability measure on $\mathbb{R}$ then $I(0,0) = -\ln \rho(\{0\})$.*

If D_Λ is an open subset of $\mathbb{R}^2$ then $(0,0) \in D_\Lambda = \overset{o}{D}_\Lambda$. It follows from the point (d) of proposition 11 that $I(0,.)$ has a unique minimum at σ^2. Therefore, the inequality of proposition 25 implies the following corollary :

Corollary 27. *If ρ is a symmetric probability measure on $\mathbb{R}$ with variance $\sigma^2 > 0$ and such that D_Λ is an open subset of $\mathbb{R}^2$ then the function*

$$(x, y) \in \Delta^* \longmapsto I(x, y) - x^2/(2y)$$

has a unique minimum at $(0, \sigma^2)$.

Now we will extend this result to any symmetric probability measure such that $(0,0) \in \overset{o}{D}_\Lambda$. For this we need Mosco's theorem, which we restate next.

Definition 28. *Let f and f_n, $n \in \mathbb{N}$, be convex functions from $\mathbb{R}^d$ to $[-\infty, +\infty]$. The sequence $(f_n)_{n\in\mathbb{N}}$ is said to Mosco converge to f if for any $x \in \mathbb{R}^d$,*
$\star$ for each sequence $(x_n)_{n\in\mathbb{N}}$ in $\mathbb{R}^d$ converging to x,

$$\liminf_{n\to+\infty} f_n(x_n) \geq f(x)$$

$\star$ there exists a sequence $(x_n)_{n\in\mathbb{N}}$ in $\mathbb{R}^d$ converging to x and such that

$$\limsup_{n\to+\infty} f_n(x_n) \leq f(x)$$

We write then

$$f_n \underset{n\to\infty}{\overset{\mathcal{M}}{\longrightarrow}} f$$

If f is a convex function from $\mathbb{R}^d$ to $[-\infty, +\infty]$, we define its Fenchel-Legendre transform f^* by

$$\forall x \in \mathbb{R}^d \qquad f^*(x) = \sup_{t\in\mathbb{R}^d} \left(\langle t, x\rangle - f(t)\right)$$

Theorem 29 (Mosco)**.** *Let f and f_n, $n \in \mathbb{N}$, be convex functions from $\mathbb{R}^d$ to $[-\infty, +\infty]$ which are convex and lower semi-continuous. We have the equivalence*

$$f_n \underset{n\to\infty}{\overset{\mathcal{M}}{\longrightarrow}} f \quad \Longleftrightarrow \quad f_n^* \underset{n\to\infty}{\overset{\mathcal{M}}{\longrightarrow}} f^*$$

We refer to [13] for a proof.

Proposition 30. *Let ν be a probability measure on $\mathbb{R}^d$. We denote by L its Log-Laplace. Let $(K_n)_{n\in\mathbb{N}}$ be a non-decreasing sequence of compact sets whose union is $\mathbb{R}^d$. For all $n \in \mathbb{N}$, we set $\nu_n = \nu(.|K_n)$ the probability ν conditioned by K_n and we denote by L_n its Log-Laplace. Then*

$$L_n \underset{n\to\infty}{\overset{\mathcal{M}}{\longrightarrow}} L$$

Proof. For n large enough, the compact set K_n meets the support of ν, and we have for any borel set A,

$$\nu_n(A) = \frac{\nu(A \cap K_n)}{\nu(K_n)}$$

Thus, for n large enough and $\lambda \in \mathbb{R}^d$, we have

$$L_n(\lambda) = \ln \int_{\mathbb{R}^d} e^{\langle \lambda, z \rangle} \, d\nu_{K_n}(z) = \ln \int_{K_n} e^{\langle \lambda, z \rangle} \, d\nu(z) - \ln \nu(K_n)$$

By the monotone convergence theorem,

$$\lim_{n \to +\infty} L_n(\lambda) = \ln \int_{\mathbb{R}^d} \lim_{n \to +\infty} \left(\mathbb{1}_{K_n}(z) e^{\langle \lambda, z \rangle} \right) \, d\nu(z) - \lim_{n \to +\infty} \ln \nu(K_n) = L(\lambda)$$

Hence the second condition of Mosco convergence (with the limsup) is satisfied with the sequence $(\lambda_n)_{n \in \mathbb{N}}$ constant equal to λ.

Let $\lambda \in \mathbb{R}^d$ and $(\lambda_n)_{n \in \mathbb{N}}$ be any sequence converging to λ. Fatou's lemma implies that

$$\exp L(\lambda) = \int_{\mathbb{R}^d} \liminf_{n \to +\infty} \mathbb{1}_{K_n}(z) e^{\langle \lambda_n, z \rangle} \, d\nu(z) \leq \liminf_{n \to +\infty} \int_{\mathbb{R}^d} \mathbb{1}_{K_n}(z) e^{\langle \lambda_n, z \rangle} \, d\nu(z)$$

Therefore

$$L(\lambda) \leq \liminf_{n \to +\infty} \left(L_n(\lambda_n) + \ln \nu(K_n) \right) = \liminf_{n \to +\infty} L_n(\lambda_n)$$

Thus the first condition of Mosco convergence (with the liminf) is verified. □

Proposition 31. *Let ρ be a symmetric probability measure on $\mathbb{R}$ with variance $\sigma^2 > 0$. We have*

$$\forall (x, y) \in \Delta^* \qquad I(x, y) - \frac{x^2}{2y} \geq I(0, y)$$

Moreover, if Λ is finite in a neighbourhood of $(0, 0)$, then the function

$$(x, y) \in \Delta^* \longmapsto I(x, y) - \frac{x^2}{2y}$$

has a unique minimum at $(0, \sigma^2)$ where it is equal to 0.

Proof. For any $n \in \mathbb{N}$, we put $K_n = [-n, n]^2$. For n large enough so that K_n meets the support of ν_ρ, we define $\nu_n = \nu_\rho(.|K_n)$, Λ_n its Log-Laplace and I_n its Fenchel-Legendre transform. For all $(u, v) \in \mathbb{R}^2$,

$$\Lambda_n(u, v) = \ln \int_{K_n} e^{us + vt} \, d\nu_\rho(s, t) - \ln \nu_\rho(K_n) \leq \Lambda(u, v) - \ln \nu_\rho(K_n)$$

Applying the Fenchel-Legendre transformation, we get

$$\forall y \in \mathbb{R} \qquad I(0, y) \leq I_n(0, y) - \ln \nu_\rho(K_n)$$

Moreover the measure ν_n has a bounded support thus proposition 25 and the previous inequality imply that

$$\forall x \in \mathbb{R} \quad \forall y > 0 \qquad I(0,y) + \frac{x^2}{2y} \le I_n(x,y) - \ln \nu_\rho(K_n)$$

It follows from proposition 30 that $(\Lambda_n)_{n\in\mathbb{N}}$ Mosco converges to Λ. Hence, by Mosco's theorem, $(I_n)_{n\in\mathbb{N}}$ Mosco converges to I. In particular, for $(x,y) \in \mathbb{R}^2$ such that $y > 0$, there exists a sequence $(x_n, y_n) \in \mathbb{R}^2$ converging to (x,y) and such that

$$\limsup_{n\to+\infty} I_n(x_n,y_n) \le I(x,y)$$

Since $y > 0$, there exists $n_0 \ge 1$ such that $y_n > 0$ for all $n \ge n_0$. Therefore

$$\forall n \ge n_0 \qquad I(0,y_n) + \frac{x_n^2}{2y_n} \le I_n(x_n,y_n) - \ln \nu_\rho(K_n)$$

Moreover $\nu_\rho(K_n) \to 1$ when $n \to \infty$. Hence

$$\limsup_{n\to+\infty} I(0,y_n) + \frac{x^2}{2y} \le I(x,y)$$

Finally I is lower semi-continuous, thus

$$\liminf_{n\to+\infty} I(0,y_n) \ge I(0,y)$$

It follows that for $(x,y) \in \mathbb{R}^2$ such that $y \ne 0$,

$$I(0,y) \le I(x,y) - \frac{x^2}{2y}$$

Suppose in addition that Λ is finite in a neighbourhood of $(0,0)$. Point (d) of proposition 11 implies then that $I(0,.)$ has a unique minimum at σ^2. Therefore $(x,y) \in \Delta^* \longmapsto I(x,y) - x^2/(2y)$ has a unique minimum at $(0,\sigma^2)$ where its value is 0. □

c) Expansion of $I - F$ around its minimum

If ρ is a symmetric probability measure whose support contains at least three points and if $(0,0) \in \overset{o}{D}_L$ then $(0,\sigma^2) = \nabla\Lambda(0,0) \in \nabla\Lambda(\overset{o}{D}_\Lambda) = A_I$, the admissible domain of I. We saw in subsection a) that I is $\mathcal{C}^\infty$ in the neighbourhood of $(0,\sigma^2)$ and that

$$\nabla I(0,\sigma^2) = (u(0,\sigma^2), v(0,\sigma^2)) = (\nabla\Lambda)^{-1}(0,\sigma^2) = (0,0)$$

$$\mathrm{D}^2_{(0,\sigma^2)} I = \left(\mathrm{D}^2_{(0,0)}\Lambda\right)^{-1} = \begin{pmatrix} \sigma^2 & 0 \\ 0 & \mu_4 - \sigma^4 \end{pmatrix}^{-1} = \begin{pmatrix} 1/\sigma^2 & 0 \\ 0 & 1/(\mu_4 - \sigma^4) \end{pmatrix}$$

since $\mathrm{D}^2_{(0,0)}\Lambda$ is the covariance matrix of ν_ρ (see the proof of proposition 16). Up to the second order, the expansion of I in the neighbourhood of $(0,\sigma^2)$ is

$$I(x,y) = \frac{x^2}{2\sigma^2} + \frac{(y-\sigma^2)^2}{2(\mu_4-\sigma^4)} + o(\|x, y-\sigma^2\|^2)$$

The expansion of F up to the second order in the neighbourhood of $(0,\sigma^2)$ is

$$F(x,y)=\frac{x^2}{2y}=\frac{x^2}{2\sigma^2}\frac{1}{1+(y-\sigma^2)/\sigma^2}=\frac{x^2}{2\sigma^2}+o(\|x,y-\sigma^2\|^2)$$

Therefore, in the neighbourhood of $(0,\sigma^2)$,

$$I(x,y)-F(x,y)=\frac{(y-\sigma^2)^2}{2(\mu_4-\sigma^4)}+o(\|x,y-\sigma^2\|^2)$$

We need to push further the expansion of $I-F$.

Consider the case of the centered Gaussian distribution with variance σ^2. We can compute explicitly I :

$$\forall (x,y)\in\Delta^* \qquad I(x,y)=\frac{1}{2}\left(\frac{y}{\sigma^2}-1-\ln\left(\frac{y-x^2}{\sigma^2}\right)\right)$$

If ψ is the function defined in proposition 8, then

$$I(x,y)-F(x,y)=\psi\left(\frac{x}{\sigma},\frac{y}{\sigma^2}\right)-\frac{1}{2}$$

Lemma 9 implies then that, in the neighbourhood of $(0,\sigma^2)$,

$$I(x,y)-F(x,y)\sim\frac{1}{4}\left(\frac{x^4}{\sigma^4}+\left(\frac{y}{\sigma^2}-1\right)^2\right)=\frac{x^4}{4\sigma^4}+\frac{(y-\sigma^2)^2}{4\sigma^2}$$

In fact, we have a similar expansion in a more general case :

Proposition 32. *If ρ is a symmetric probability measure on $\mathbb{R}$ whose support contains at least three points and such that $(0,0)\in\overset{o}{D}_\Lambda$ then I is $\mathcal{C}^\infty$ in the neighbourhood of $(0,\sigma^2)$. If μ_4 denotes the fourth moment of ρ then, when (x,y) goes to $(0,\sigma^2)$,*

$$I(x,y)-\frac{x^2}{2y}\sim\frac{(y-\sigma^2)^2}{2(\mu_4-\sigma^4)}+\frac{\mu_4x^4}{12\sigma^8}$$

Whenever $\rho=\mathcal{N}(0,\sigma^2)$, we have

$$2(\mu_4-\sigma^4)=2(3\sigma^4-\sigma^2)=4\sigma^4$$

and

$$\mu_4/(12\sigma^8)=3\sigma^4/(12\sigma^8)=(4\sigma^4)^{-1}$$

This is what we obtained before the proposition in the Gaussian case.

Proof. If $(0,0)\in\overset{o}{D}_\Lambda$ then

$$(0,\sigma^2)=\nabla\Lambda(0,0)\in\nabla\Lambda(\overset{o}{D}_\Lambda)=A_I$$

The function I is $\mathcal{C}^\infty$ on A_I and, if $(x,y)\longmapsto(u(x,y),v(x,y))$ is the inverse function of $\nabla\Lambda$ then, for all $(x,y)\in A_I$,

$$I(x,y)=xu(x,y)+yv(x,y)-\Lambda(u(x,y),v(x,y))$$

$$\nabla I(x,y) = (\nabla\Lambda)^{-1}(x,y) = (u(x,y),v(x,y))$$

$$\mathrm{D}^2_{(x,y)} I = \left(\mathrm{D}^2_{(u(x,y),v(x,y))}\Lambda\right)^{-1}$$

Moreover the hypothesis $(0,0) \in \overset{o}{D}_\Lambda$ implies that ρ has finite moments of all order. The expansion of F to the fourth order in the neighbourhood of $(0,\sigma^2)$ is

$$F(x,y) = \frac{x^2}{2\sigma^2} - \frac{x^2(y-\sigma^2)}{2\sigma^4} + \frac{x^2(y-\sigma^2)^2}{2\sigma^6} + o(\|x,y-\sigma^2\|^4)$$

Therefore, in the neighbourhood of $(0,0)$,

$$\begin{aligned} I(x,h+\sigma^2) - F(x,h+\sigma^2) = \frac{h^2}{2(\mu_4-\sigma^4)} + a_{3,0}x^3 + a_{2,1}x^2h + a_{1,2}xh^2 + a_{0,3}h^3 \\ + a_{4,0}x^4 + a_{3,1}x^3h + a_{2,2}x^2h^2 + a_{1,3}xh^3 + a_{0,4}h^4 + o(\|x,h\|^4) \end{aligned}$$

with, for any $(i,j) \in \mathbb{N}$ such that $i+j \in \{3,4\}$,

$$a_{i,j} = \frac{1}{i!j!}\frac{\partial^{i+j} I}{\partial x^i \partial y^j}(0,\sigma^2)$$

except for

$$a_{2,1} = \frac{1}{2}\left(\frac{\partial^3 I}{\partial x^2\partial y}(0,\sigma^2) + \frac{1}{\sigma^4}\right) \qquad \text{and} \qquad a_{2,2} = \frac{1}{4}\frac{\partial^4 I}{\partial x^2 \partial y^2}(0,\sigma^2) - \frac{1}{2\sigma^6}$$

If we prove that $a_{4,0} > 0$ then the terms xh^2, h^3, x^3h, x^2h^2, xh^3 and h^4 are negligible compared to $a_{4,0}x^4 + a_{0,2}h^2$ when (x,h) goes to $(0,0)$. Next, the symmetry of $I - F$ in the first variable implies that $a_{3,0} = 0$. If we show that $a_{2,1} = 0$ then we get

$$I(x,y) - F(x,y) = \left(\frac{(y-\sigma^2)^2}{2(\mu_4-\sigma^4)} + a_{4,0}x^4\right)(1+o(1))$$

when $(x,y) \to (0,\sigma^2)$, so we have the desired expansion.

To conclude it is enough to show that $a_{2,1} = 0$ and $a_{4,0} = \mu_4/(12\sigma^8)$, that is

$$\frac{\partial^3 I}{\partial x^2 \partial y}(0,\sigma^2) = -\frac{1}{\sigma^4} \qquad \text{and} \qquad \frac{\partial^4 I}{\partial x^4}(0,\sigma^2) = \frac{2\mu_4}{\sigma^2}$$

For any $j \in \mathbb{N}$, we introduce the function f_j defined on $\overset{o}{D}_\Lambda$ by

$$\forall (u,v) \in \overset{o}{D}_\Lambda \qquad f_j(u,v) = \frac{\displaystyle\int_{\mathbb{R}} x^j e^{ux+vx^2}\, d\rho(x)}{\displaystyle\int_{\mathbb{R}} e^{ux+vx^2}\, d\rho(x)}$$

These functions are $\mathcal{C}^\infty$ on $\overset{o}{D}_\Lambda$ and they verify the following properties :
⋆ f_0 is the identity function on $\mathbb{R}^2$
⋆ For all $j \in \mathbb{N}$, $f_j(0,0) = \mu_j$ is the j-th moment of ρ. It is null if j is odd, since ρ is symmetric. Moreover

$$f_1 = \frac{\partial\Lambda}{\partial u} \qquad \text{and} \qquad f_2 = \frac{\partial\Lambda}{\partial v}$$

⋆ For any $j \in \mathbb{N}$,

$$\frac{\partial f_j}{\partial u} = f_{j+1} - f_j f_1 \qquad \text{and} \qquad \frac{\partial f_j}{\partial v} = f_{j+2} - f_j f_2$$

Therefore, for all $(x,y) \in A_I$,

$$\mathrm{D}^2_{(x,y)} I = \left(\mathrm{D}^2_{(u(x,y),v(x,y))}\Lambda\right)^{-1} = \begin{pmatrix} f_2 - f_1^2 & f_3 - f_1 f_2 \\ f_3 - f_1 f_2 & f_4 - f_2^2 \end{pmatrix}^{-1} (u(x,y), v(x,y))$$

Denoting by $g = (f_2 - f_1^2)(f_4 - f_2^2) - (f_3 - f_1 f_2)^2$, the determinant of the positive definite symmetric matrix $\mathrm{D}^2\Lambda$, we get that for any $(x,y) \in A_I$,

$$\mathrm{D}^2_{(x,y)} I = \frac{1}{g(u(x,y),v(x,y))} \begin{pmatrix} f_4 - f_2^2 & f_1 f_2 - f_3 \\ f_1 f_2 - f_3 & f_2 - f_1^2 \end{pmatrix} (u(x,y), v(x,y))$$

Moreover $(u(0,\sigma^2), v(0,\sigma^2)) = (0,0)$ thus

$$\frac{\partial u}{\partial x}(0,\sigma^2) = \frac{\partial^2 I}{\partial x^2}(0,\sigma^2) = \frac{f_4 - f_2^2}{g}(0,0) = \frac{\mu_4 - \sigma^4}{\sigma^2(\mu_4 - \sigma^4)} = \frac{1}{\sigma^2}$$

$$\frac{\partial v}{\partial y}(0,\sigma^2) = \frac{\partial^2 I}{\partial y^2}(0,\sigma^2) = \frac{f_2 - f_1^2}{g}(0,0) = \frac{\sigma^2}{\sigma^2(\mu_4 - \sigma^4)} = \frac{1}{\mu_4 - \sigma^4}$$

$$\frac{\partial u}{\partial y}(0,\sigma^2) = \frac{\partial v}{\partial x}(0,\sigma^2) = \frac{\partial^2 I}{\partial x \partial y}(0,\sigma^2) = \frac{f_1 f_2 - f_3}{g}(0,0) = 0$$

Differentiating with respect to y, we get

$$\frac{\partial^3 I}{\partial y \partial x^2} = \frac{\partial u}{\partial y} \times \frac{\partial}{\partial u}\left(\frac{f_4 - f_2^2}{g}\right)(u,v) + \frac{\partial v}{\partial y} \times \frac{\partial}{\partial v}\left(\frac{f_4 - f_2^2}{g}\right)(u,v)$$

The first term of the addition, taken at $(0,\sigma^2)$, is null. For the second term, we need to compute the partial derivative of $(f_4 - f_2^2)/g$ with respect to v :

$$\begin{aligned} \frac{\partial}{\partial v}\left(\frac{f_4 - f_2^2}{g}\right) &= \frac{1}{g} \times \frac{\partial}{\partial v}\left(f_4 - f_2^2\right) - \frac{f_4 - f_2^2}{g^2} \times \frac{\partial g}{\partial v} \\ &= \frac{f_6 - 3f_2 f_4 + 2f_2^3}{g} - \frac{f_4 - f_2^2}{g^2} \times \frac{\partial g}{\partial v} \end{aligned}$$

Developing the expression of g, we get

$$g = f_2 f_4 - f_1^2 f_4 - f_2^3 - f_3^2 + 2 f_1 f_2 f_3$$

Let us differentiate with respect to v :

$$\begin{aligned} \frac{\partial g}{\partial v} &= f_2(f_6 - f_4 f_2) + f_4(f_4 - f_2^2) - f_1^2(f_6 - f_4 f_2) - 2f_4 f_1(f_3 - f_1 f_2) - 3f_2^2(f_4 - f_2^2) \\ &\quad - 2f_3(f_5 - f_3 f_2) + 2f_1 f_2(f_5 - f_3 f_2) + 2f_2 f_3(f_3 - f_1 f_2) + 2f_1 f_3(f_4 - f_2^2) \end{aligned}$$

Taken at $(0,0)$, each term with even subscript vanishes and we have

$$\begin{aligned} \frac{\partial g}{\partial v}(0,0) &= \sigma^2(\mu_6 - \mu_4\sigma^2) + \mu_4(\mu_4 - \sigma^4) - 3\sigma^4(\mu_4 - \sigma^4) \\ &= \sigma^2\mu_6 - 3\mu_4\sigma^4 + 2\sigma^8 + (\mu_4 - \sigma^4)^2 \end{aligned}$$

Finally

$$\frac{\partial}{\partial v}\left(\frac{f_4-f_2^2}{g}\right)(0,0) = \frac{\mu_6-3\sigma^2\mu_4+2\sigma^6}{\sigma^2(\mu_4-\sigma^4)} - \frac{\sigma^2\mu_6-3\mu_4\sigma^4+2\sigma^8+(\mu_4-\sigma^4)^2}{\sigma^4(\mu_4-\sigma^4)}$$
$$= \frac{\sigma^4-\mu_4}{\sigma^4}$$

Therefore

$$\frac{\partial^3 I}{\partial y\partial x^2}(0,\sigma^2) = 0+\frac{\partial v}{\partial x}(0,\sigma^2)\frac{\partial}{\partial v}\left(\frac{f_4-f_2^2}{g}\right)(0,0) = \frac{1}{\mu_4-\sigma^4}\times\frac{\sigma^4-\mu_4}{\sigma^4} = -\frac{1}{\sigma^4}$$

This is what we wanted to prove. Let us compute now the fourth partial derivative of I with respect to x. We have to obtain first an expression of the third partial derivative of I with respect to x :

$$\frac{\partial^3 I}{\partial x^3} = \frac{\partial u}{\partial x}\times\frac{\partial}{\partial u}\left(\frac{f_4-f_2^2}{g}\right)(u,v)+\frac{\partial v}{\partial x}\times\frac{\partial}{\partial v}\left(\frac{f_4-f_2^2}{g}\right)(u,v)$$

The only term we do not know is the partial derivative with respect to u of $(f_4-f_2^2)/g$. We have

$$\frac{\partial}{\partial u}\left(\frac{f_4-f_2^2}{g}\right) = \frac{1}{g}\times\frac{\partial}{\partial u}\left(f_4-f_2^2\right) - \frac{f_4-f_2^2}{g^2}\times\frac{\partial g}{\partial u}$$
$$= \frac{f_5-f_4f_1-2f_2f_3+2f_2^2f_1}{g} - \frac{f_4-f_2^2}{g^2}\times\frac{\partial g}{\partial u}$$

with

$$\frac{\partial g}{\partial u} = f_2(f_5-f_4f_1)+f_4(f_3-f_2f_1)-f_1^2(f_5-f_4f_1)-2f_4f_1(f_2-f_1^2)$$
$$-3f_2^2(f_3-f_2f_1)-2f_3(f_4-f_3f_1)+2f_1f_2(f_4-f_3f_1)$$
$$+2f_2f_3(f_2-f_1^2)+2f_1f_3(f_3-f_2f_1)$$

Notice that this quantity vanishes at $(0,0)$. Therefore the partial derivative of $(f_4-f_2^2)/g$ with respect to u, taken at $(0,0)$, is null as well and we get back

$$\frac{\partial^3 I}{\partial x^3}(0,\sigma^2) = 0$$

Differentiating once more, we obtain

$$\frac{\partial^4 I}{\partial x^4} = \frac{\partial u}{\partial x}\times\left(\frac{\partial u}{\partial x}\times\frac{\partial^2}{\partial u^2}\left(\frac{f_4-f_2^2}{g}\right)(u,v)+\frac{\partial v}{\partial x}\times\frac{\partial^2}{\partial v\partial u}\left(\frac{f_4-f_2^2}{g}\right)(u,v)\right)$$
$$+\frac{\partial^2 u}{\partial x^2}\times\frac{\partial}{\partial u}\left(\frac{f_4-f_2^2}{g}\right)(u,v)+\frac{\partial^2 v}{\partial x^2}\times\frac{\partial}{\partial v}\left(\frac{f_4-f_2^2}{g}\right)(u,v)$$
$$+\frac{\partial v}{\partial x}\times\left(\frac{\partial u}{\partial x}\times\frac{\partial^2}{\partial u\partial v}\left(\frac{f_4-f_2^2}{g}\right)(u,v)+\frac{\partial v}{\partial x}\times\frac{\partial^2}{\partial v^2}\left(\frac{f_4-f_2^2}{g}\right)(u,v)\right)$$

Let us compute it at $(0,\sigma^2)$:

$$\frac{\partial^4 I}{\partial x^4}(0,\sigma^2) = \frac{1}{\sigma^2}\left(\frac{1}{\sigma^2}\frac{\partial^2}{\partial u^2}\left(\frac{f_4-f_2^2}{g}\right)(0,0)+0\right)+0+\frac{\sigma^4-\mu_4}{\sigma^4}\frac{\partial^2 v}{\partial x^2}(0,\sigma^2)+0$$

with

$$\frac{\partial^2 v}{\partial x^2}(0,\sigma^2) = \frac{\partial}{\partial x}\left(\frac{\partial^2 I}{\partial x \partial y}\right)(0,\sigma^2) = \frac{\partial^3 I}{\partial x^2 \partial y}(0,\sigma^2) = -\frac{1}{\sigma^4}$$

and

$$\frac{\partial^2}{\partial u^2}\left(\frac{f_4 - f_2^2}{g}\right) = \frac{1}{g}\frac{\partial^2}{\partial u^2}(f_4 - f_2^2) + \frac{1}{g^2}\frac{\partial g}{\partial u}\frac{\partial}{\partial u}\left(\frac{f_4 - f_2^2}{g}\right) - \frac{1}{g^2}\frac{\partial g}{\partial u}\frac{\partial^2}{\partial u^2}(f_4 - f_2^2)$$
$$- \frac{f_4 - f_2^2}{g^2}\frac{\partial^2 g}{\partial u^2} + \frac{2}{g^3}\left(\frac{\partial g}{\partial u}\right)^2 (f_4 - f_2^2)$$

Hence

$$\frac{\partial^2}{\partial u^2}\left(\frac{f_4 - f_2^2}{g}\right)(0,0) = \frac{1}{\sigma^4(\mu_4 - \sigma^4)}\left(\sigma^2 \frac{\partial^2}{\partial u^2}(f_4 - f_2^2)(0,0) - \frac{\partial^2 g}{\partial u^2}(0,0)\right)$$

The two remaining terms are the derivatives of quantities which we have already computed. We evaluate them directly at $(0,0)$, which is straightforward since $f_j(0,0) = 0$ when j is odd :

$$\frac{\partial^2}{\partial u^2}(f_4 - f_2^2)(0,0) = \frac{\partial}{\partial u}(f_5 - f_4 f_1 - 2 f_2 f_3 + 2 f_2^2 f_1)(0,0) = \mu_6 - 3\sigma^2\mu_4 + 2\sigma^6$$

and

$$\frac{\partial^2 g}{\partial u^2}(0,0) = \frac{\partial}{\partial u}\left(\frac{\partial g}{\partial u}\right)(0,0) = \sigma^2(\mu_6 - \mu_4\sigma^2) + \mu_4(\mu_4 - \sigma^4) - 0 - 2\mu_4\sigma^4$$
$$- 3\sigma^4(\mu_4 - \sigma^4) - 2\mu_4^2 + 2\sigma^4\mu_4 + 2\sigma^4\mu_4 + 0$$

This is equal to $\sigma^2\mu_6 - \mu_4^2 + 3\sigma^8 - 3\mu_4\sigma^4$ after simplification. Thus we have

$$\frac{\partial^2}{\partial u^2}\left(\frac{f_4 - f_2^2}{g}\right)(0,0) = \frac{\sigma^2\mu_6 - 3\sigma^4\mu_4 + 2\sigma^8 - \sigma^2\mu_6 + \mu_4^2 - 3\sigma^8 + 3\mu_4\sigma^4}{\sigma^4(\mu_4 - \sigma^4)}$$
$$= \frac{\mu_4^2 - \sigma^8)}{\sigma^4(\mu_4 - \sigma^4} = \frac{\mu_4 + \sigma^4}{\sigma^4}$$

Finally

$$\frac{\partial^2 I}{\partial x^4}(0,\sigma^2) = \frac{\mu_4 + \sigma^4}{\sigma^8} - \frac{\sigma^4 - \mu_4}{\sigma^8} = \frac{2\mu_4}{\sigma^8}$$

We obtain the announced term and the proof is completed. □

8 Around Varadhan's lemma

We denote by $\widetilde{\nu}_{n,\rho}$ the distribution of $(S_n/n, T_n/n)$ under $\rho^{\otimes n}$ and by $\theta_{n,\rho}$ the distribution of $(S_n/n, T_n/n)$ under $\widetilde{\mu}_{n,\rho}$. We saw in section 3 that, if Λ is finite in the neighbourhood of $(0,0)$, then the sequence $(\widetilde{\nu}_{n,\rho})_{n\geq 1}$ satisfies a large deviation principle with speed n, governed by the good rate function I. Moreover, for any $A \subset \mathbb{R}^2$,

$$\theta_{n,\rho}(A) = \frac{\int_{A\cap\Delta^*} \exp\left(\frac{nx^2}{2y}\right)\, d\widetilde{\nu}_{n,\rho}(x,y)}{\int_{\Delta^*} \exp\left(\frac{nx^2}{2y}\right)\, d\widetilde{\nu}_{n,\rho}(x,y)}$$

Yet we cannot apply Varadhan's lemma directly since Δ^* is not a closed set and $F : (x,y) \longmapsto x^2/(2y)$ is not continuous on Δ. However we have the following proposition :

Proposition 33. *Suppose that ρ is a non-degenerate symmetric probability measure on $\mathbb{R}$ such that $(0,0) \in \overset{o}{D}_\Lambda$. Then*

$$\liminf_{n\to+\infty} \frac{1}{n} \ln \int_{\Delta^*} \exp\left(\frac{nx^2}{2y}\right) \, d\widetilde{\nu}_{n,\rho}(x,y) \geq 0$$

We assume that there exists $r > 0$ such that $M_r + \ln \rho(\{0\}) < 0$ with

$$M_r = \sup\left\{ \frac{x^2}{2y} : (x,y) \in \mathcal{C} \cap \mathrm{B}_r \backslash \{(0,0)\} \right\}$$

where B_r is the open ball of radius r centered at $(0,0)$ and $\mathcal{C}$ is the closed convex hull of $\{ (x,x^2) : x$ is in the support of $\rho \}$. If A is a closed subset of $\mathbb{R}^2$ which does not contain $(0,\sigma^2)$ then

$$\limsup_{n\to+\infty} \frac{1}{n} \ln \int_{\Delta^* \cap A} \exp\left(\frac{nx^2}{2y}\right) \, d\widetilde{\nu}_{n,\rho}(x,y) < 0$$

Let us give first some sufficient conditions to fulfill the hypothesis of the proposition.

To ensure that there exists $r > 0$ such that $M_r + \ln \rho(\{0\}) < 0$, it is enough that one of the following conditions is satisfied :

(a) ρ has a density

(b) $\rho(\{0\}) < 1/\sqrt{e}$

(c) There exists $c > 0$ such that $\rho(]0,c[) = 0$

(d) ρ is the sum of a finite number of Dirac masses

Indeed, the function F is bounded by $1/2$ on $\mathcal{C}\backslash\{(0,0)\} \subset \Delta^*$, thus for any $r > 0$, $M_r \leq 1/2$. Therefore, if ρ has a density, or more generally if $\rho(\{0\}) < e^{-1/2}$, then for all $r > 0$, $M_r + \ln \rho(\{0\}) < 0$.

On the other hand, if there exists $c > 0$ such that $]0,c[$ does not intersect the support of ρ (especially if ρ is the sum of a finite number of Dirac masses) then

$$\mathcal{C} \subset \{ (x,y) \in \mathbb{R}^2 : c|x| \leq y \}$$

Therefore

$$\forall (x,y) \in \mathcal{C} \cap \mathrm{B}_r \backslash \{(0,0)\} \qquad \frac{x^2}{2y} = \frac{c|x|^2}{2cy} \leq \frac{|x|}{2c} \leq \frac{r}{2c}$$

Hence for any $r > 0$, $M_r < r/2c$. Since ρ is non-degenerate, $\rho(\{0\}) < 1$, thus there exists $r > 0$ such that $\ln \rho(\{0\}) + r/2c < 0$. Therefore the conditions (c) and (d) imply that $M_r + \ln \rho(\{0\}) < 0$.

Before we prove proposition 33, we need two preliminary lemmas, the following one being very useful for handling superior limits.

Lemma 34. *If $(u_1(n))_{n\geq 1}, \dots, (u_k(n))_{n\geq 1}$ are k sequences of non-negative real numbers then*

$$\limsup_{n\to+\infty} \frac{1}{n} \ln \left(\sum_{i=1}^{k} u_i(n) \right) = \max_{1\leq i\leq k} \limsup_{n\to+\infty} \frac{1}{n} \ln u_i(n)$$

We refer to [7] for a proof. The second lemma we need is a variant of the upper bound of Varadhan's lemma. Recall that a topological space $\mathcal{X}$ is Hausdorff if, for any $(x, y) \in \mathcal{X}^2$ such that $x \neq y$, there exist two disjoint neighbourhoods of x and y. The Hausdorff space $\mathcal{X}$ is regular if, for any closed subset F of $\mathcal{X}$ and any $x \notin F$, there exist two disjoint open subsets O_1 and O_2 such that $F \subset O_1$ and $x \in O_2$.

Lemma 35. *Let $\mathcal{X}$ be a regular topological Hausdorff space endowed with its Borel σ-field $\mathcal{B}$. Let $(\nu_n)_{n\in\mathbb{N}}$ be a sequence of probability measures defined on $(\mathcal{X}, \mathcal{B})$ which satisfies a large deviation principle with speed n, governed by the good rate function J. For any bounded continuous function $f : \mathcal{X} \longrightarrow \mathbb{R}$, we have for any closed subset A of $\mathcal{X}$,*

$$\limsup_{n\to+\infty} \frac{1}{n} \ln \int_A e^{nf(x)}\, d\nu_n(x) \leq \sup_{x\in A} (f(x) - J(x))$$

Proof. Let $\lambda, \alpha > 0$. We define

$$J^{-1}([0, \lambda]) = \{\, x \in \mathcal{X} : J(x) \leq \lambda \,\}$$

The set $J^{-1}([0, \lambda]) \cap A$ is compact since J is good and A is closed. Since f is continuous, J is lower semi-continuous and $\mathcal{X}$ is regular, for any $x \in A$, there exists an open neighbourhood V_x of x such that

$$\sup_{y\in V_x} f(y) \leq f(x) + \alpha \qquad \text{and} \qquad \inf_{y\in \overline{V}_x} J(y) \geq J(x) - \alpha$$

The collection V_x, $x \in J^{-1}([0, \lambda]) \cap A$, is an open cover of the compact set $J^{-1}([0, \lambda]) \cap A$. Let $(V_{x_i}, 1 \leq i \leq k)$ be a finite subcover extracted from this covering. Setting $U = \bigcup_{i=1}^{k} V_{x_i}$, we have

$$\begin{aligned}\int_A e^{nf(y)}\, d\nu_n(y) &\leq \sum_{i=1}^{k} \int_{V_{x_i}} e^{nf(y)}\, d\nu_n(y) + \int_{A\backslash U} e^{nf(y)}\, d\nu_n(y)\\ &\leq \sum_{i=1}^{k} e^{nf(x_i)+n\alpha} \nu_n(V_{x_i}) + e^{n\|f\|_\infty} \nu_n(A\backslash U)\end{aligned}$$

Moreover

$$\inf\{\, J(x) : x \in A\backslash U \} \geq \inf\{\, I(x) : x \notin J^{-1}([0, \lambda])\} \geq \lambda$$

Therefore, using the large deviation upper bound and lemma 34,

$$\begin{aligned}\limsup_{n\to+\infty} \frac{1}{n} \ln \int_A e^{nf(y)}\, d\nu_n(y) &\leq \max\left(\max_{1\leq i\leq k} (f(x_i) - J(x_i) + 2\alpha), \|f\|_\infty - \lambda \right)\\ &\leq \max\left(\sup_{x\in A} (f(x) - J(x)) + 2\alpha, \|f\|_\infty - \lambda \right)\end{aligned}$$

We conclude by letting successively α go to 0 and λ go to $+\infty$. □

Proof of proposition 33. If $\mathcal{V}$ is an open neighbourhood of $(0, \sigma^2)$ which is included in Δ^* then

$$\int_{\Delta^*} \exp\left(\frac{nx^2}{2y}\right) d\widetilde{\nu}_{n,\rho}(x, y) \geq \int_{\mathcal{V}} d\widetilde{\nu}_{n,\rho}(x, y) = \widetilde{\nu}_{n,\rho}\,(\mathcal{V})$$

The large deviation principle satisfied by $\widetilde{\nu}_{n,\rho}$ implies that

$$\liminf_{n\to+\infty}\frac{1}{n}\ln\int_{\Delta^*}\exp\left(\frac{nx^2}{2y}\right)\,d\widetilde{\nu}_{n,\rho}(x,y)\geq-\inf_{\mathcal{V}}I\geq-I(0,\sigma^2)=0$$

We prove now the second inequality. Let $\alpha>0$. The function I is lower semi-continuous on $\mathbb{R}^2$, thus there exists an open neighbourhood $\mathcal{U}$ of $(0,0)$ such that

$$\forall(x,y)\in\overline{\mathcal{U}}\qquad I(x,y)\geq I(0,0)-\alpha=-\ln\rho(\{0\})-\alpha$$

The above equality follows from lemma 26. By hypothesis, there exists $r>0$ such that $M_r+\ln\rho(\{0\})<0$ thus, by choosing α sufficiently small, we can assume that

$$M_r+\ln\rho(\{0\})+\alpha<0$$

Since M_r decreases with r, we can take r small enough so that $\mathrm{B}_r\subset\mathcal{U}$. Notice next that

$$\left(\frac{S_n}{n},\frac{T_n}{n}\right)=\frac{1}{n}\sum_{k=1}^{n}(X_k,X_k^2)\in\mathcal{C}\qquad\text{a.s}$$

therefore, setting $\mathcal{C}^*=\mathcal{C}\backslash\{(0,0)\}$,

$$\int_{\Delta^*\cap A}\exp\left(\frac{nx^2}{2y}\right)\,d\widetilde{\nu}_{n,\rho}(x,y)=\int_{\mathcal{C}^*\cap A}\exp\left(\frac{nx^2}{2y}\right)\,d\widetilde{\nu}_{n,\rho}(x,y)$$

Let us decompose

$$\mathcal{C}^*\cap A\subset(\mathcal{C}^*\cap\mathrm{B}_r)\cup(\mathcal{C}\cap\mathrm{B}_r^c\cap A)$$

We have

$$\int_{\mathcal{C}^*\cap\mathrm{B}_r}\exp\left(\frac{nx^2}{2y}\right)\,d\widetilde{\nu}_{n,\rho}(x,y)\leq\exp(nM_r)\,\widetilde{\nu}_{n,\rho}\,(\mathcal{U})$$

The large deviation principle satisfied by $\widetilde{\nu}_{n,\rho}$ implies that

$$\limsup_{n\to+\infty}\frac{1}{n}\ln\int_{\mathcal{C}^*\cap\mathrm{B}_r}\exp\left(\frac{nx^2}{2y}\right)\,d\widetilde{\nu}_{n,\rho}(x,y)\leq M_r-\inf_{\overline{\mathcal{U}}}I\leq M_r+\ln\rho(\{0\})+\alpha$$

Next, the set $\mathcal{C}\cap\mathrm{B}_r^c\cap A$ is closed and does not contain $(0,0)$ thus the function F is continuous on this set. Moreover F is bounded on $\mathcal{C}^*$. Hence, by the previous lemma,

$$\limsup_{n\to+\infty}\frac{1}{n}\ln\int_{\mathcal{C}\cap\mathrm{B}_r^c\cap A}\exp\left(\frac{nx^2}{2y}\right)\,d\widetilde{\nu}_{n,\rho}(x,y)\leq\sup_{\mathcal{C}\cap\mathrm{B}_r^c\cap A}(F-I)$$

Lemma 34 implies that

$$\limsup_{n\to+\infty}\frac{1}{n}\ln\int_{\mathcal{C}^*\cap A}\exp\left(\frac{nx^2}{2y}\right)\,d\widetilde{\nu}_{n,\rho}(x,y)$$
$$\leq\max\left(M_r+\ln\rho(\{0\})+\alpha,\sup_{\mathcal{C}\cap\mathrm{B}_r^c\cap A}(F-I)\right)$$

Since ρ is symmetric and $(0,0)\in\overset{o}{D}_\Lambda$, proposition 31 implies that $G=I-F$ has a unique minimum at $(0,\sigma^2)$ on Δ^*. Suppose that

$$\inf_{\mathcal{C}\cap\mathrm{B}_r^c\cap A}G=0$$

Then there exists a sequence $(x_k,y_k)_{k\in\mathbb{N}}$ in $\mathcal{C}\cap \mathrm{B}_r^c\cap A\subset\Delta^*$ such that

$$\lim_{k\to+\infty} G(x_k,y_k)=\inf_{\mathcal{C}\cap\mathrm{B}_r^c\cap A} G=0$$

For k large enough, $G(x_k,y_k)\leq 1/2$ thus $I(x_k,y_k)\leq 1$, i.e., (x_k,y_k) belongs to the compact set $\{\,(u,v)\in\mathbb{R}^2 : I(u,v)\leq 1\,\}$. Up to the extraction of a subsequence, we suppose that $(x_k,y_k)_{k\in\mathbb{N}}$ converges to some (x_0,y_0), which belongs to the closed subset $\mathcal{C}\cap\mathrm{B}_r^c\cap A$. Moreover G is lower semi-continuous, hence

$$0=\liminf_{k\to+\infty} G(x_k,y_k)\geq G(x_0,y_0)\geq 0$$

Therefore $G(x_0,y_0)=0$ and thus $(x_0,y_0)=(0,\sigma^2)\in\mathcal{C}\cap\mathrm{B}_r^c\cap A$, which is absurd since A does not contain $(0,\sigma^2)$. Thus

$$\inf_{\mathcal{C}\cap\mathrm{B}_r^c\cap A} G>0$$

and

$$\max\left(M_r+\ln\rho(\{0\})+\alpha\,,\ \sup_{\mathcal{C}\cap\mathrm{B}_r^c\cap A}(F-I)\right)<0$$

This proves the second inequality. □

9 Proof of theorem 1

The proof of theorem 1 relies on the variant of Varadhan's lemma exposed in the previous section. Suppose that ρ is a symmetric probability measure on $\mathbb{R}$ with positive variance σ^2 and such that $(0,0)\in\overset{o}{D}_\Lambda$. We assume that one of the four conditions given in the paragraph below proposition 33 is satisfied.

We denote by $\theta_{n,\rho}$ the distribution of $(S_n/n,T_n/n)$ under $\widetilde{\mu}_{n,\rho}$. We saw in section 3 that for any $A\subset\mathbb{R}^2$,

$$\theta_{n,\rho}(A)=\frac{\int_{A\cap\Delta^*}\exp\left(\frac{nx^2}{2y}\right)\,d\widetilde{\nu}_{n,\rho}(x,y)}{\int_{\Delta^*}\exp\left(\frac{nx^2}{2y}\right)\,d\widetilde{\nu}_{n,\rho}(x,y)}$$

Let U be an open neighbourhood of $(0,\sigma^2)$ in $\mathbb{R}^2$. Proposition 33 implies that

$$\begin{aligned}\limsup_{n\to+\infty}\frac{1}{n}\ln\theta_{n,\rho}(U^c)=\limsup_{n\to+\infty}\frac{1}{n}\ln\int_{\Delta^*\cap U^c}\exp\left(\frac{nx^2}{2y}\right)\,d\widetilde{\nu}_{n,\rho}(x,y)\\ -\liminf_{n\to+\infty}\frac{1}{n}\ln\int_{\Delta^*}\exp\left(\frac{nx^2}{2y}\right)\,d\widetilde{\nu}_{n,\rho}(x,y)<0\end{aligned}$$

Hence there exist $\varepsilon>0$ and $n_0\in\mathbb{N}$ such that for any $n>n_0$,

$$\theta_{n,\rho}(U^c)\leq e^{-n\varepsilon}\underset{n\to\infty}{\longrightarrow}0$$

Thus, for each open neighbourhood U of $(0,\sigma^2)$,

$$\lim_{n\to+\infty}\widetilde{\mu}_{n,\rho}\left(\left(\frac{S_n}{n},\frac{T_n}{n}\right)\in U^c\right)=0$$

This means that, under $\widetilde{\mu}_{n,\rho}$, $(S_n/n,T_n/n)$ converges in probability to $(0,\sigma^2)$.

10 Proof of theorem 2

In this section, we first give conditions on ρ in order to apply theorem 17 to the distribution ν_ρ. Recall that this theorem states that, for n large enough, $\widetilde{\nu}_{n,\rho}$ has a density g_n with respect to the Lebesgue measure on $\mathbb{R}^2$ such that, for any compact subset K_I of A_I, the admissible domain of I, when $n \to +\infty$, uniformly in $(x,y) \in K_I$,

$$g_n(x,y) \sim \frac{n}{2\pi} \left(\det \mathrm{D}^2_{(x,y)} I\right)^{1/2} e^{-nI(x,y)}$$

We use then the Laplace method, as we announced in the heuristics of section 3, to obtain the fluctuations theorem 2. The proof relies on the variant of Varadhan's lemma and the expansion of $I-F$ in $(0,\sigma^2)$ given in proposition 32.

We first notice that $\overset{o}{D}_\Lambda \neq \varnothing$ since it contains $\mathbb{R}\times]-\infty,0[$. Next, we have the following lemma :

Lemma 36. *If ρ has a probability density f with respect to the Lebesgue measure on $\mathbb{R}$, then ν_ρ^{*2} has the density*

$$f_2 : (x,y) \in \mathbb{R}^2 \longmapsto \frac{1}{\sqrt{2y-x^2}}\, f\left(\frac{x+\sqrt{2y-x^2}}{2}\right) f\left(\frac{x-\sqrt{2y-x^2}}{2}\right) \mathbb{1}_{x^2<2y}$$

with respect to the Lebesgue measure on $\mathbb{R}^2$.

Proof. Let h be a bounded continuous function from $\mathbb{R}^2$ to $\mathbb{R}$. We have

$$\begin{aligned}
&\int_{\mathbb{R}^2} h(x,y)\, d\nu_\rho^{*2}(x,y) = \int_{\mathbb{R}^2} h((z,z^2)+(t,t^2))\, d\rho(z)\, d\rho(t) \\
&\quad = \int_{\mathbb{R}^2} h((z,z^2)+(t,t^2)) f(z) f(t)\, dz\, dt \\
&\quad = \int_{D^+} h(z+t, z^2+t^2) f(z) f(t)\, dz\, dt + \int_{D^-} h(z+t, z^2+t^2) f(z) f(t)\, dz\, dt \\
&\quad = I_+ + I_-
\end{aligned}$$

with $D^+ = \{\,(z,t) \in \mathbb{R}^2 : z > t\,\}$ and $D^- = \{\,(z,t) \in \mathbb{R}^2 : z < t\,\}$. Indeed, the Lebesgue measure of the set $\{\,(z,t) \in \mathbb{R}^2 : z = t\,\}$ is null.

We define $\varphi : (z,t) \in \mathbb{R}^2 \longmapsto (u,v) = (z+t, z^2+t^2)$. If $(z,t) \in D^+$, then

$$\frac{u^2}{4} = \left(\frac{z+t}{2}\right)^2 < \frac{z^2+t^2}{2} = \frac{v}{2}$$

thus

$$\varphi(D^+) \subset \Delta_2 = \{\,(u,v) \in \mathbb{R}^2 : u^2 < 2v\,\}$$

Similarly $\varphi(D^-) \subset \Delta_2$. Conversely, if $(u,v) \in \Delta_2$ and $u = z+t$, $v = z^2+t^2$ then $t = u-z$ and $2z^2 - 2uz + (u^2-v) = 0$. This last quadratic equation in z has for discriminant $4(2v-u^2) > 0$, thus there are two distinct real-valued roots. Therefore

$$(z,t) = \frac{1}{2}(u+\sqrt{2v-u^2}, u-\sqrt{2v-u^2}) \in D^+$$

or

$$(z,t) = \frac{1}{2}(u - \sqrt{2v-u^2}, u + \sqrt{2v-u^2}) \in D^-$$

This proves that φ is a one to one map from D^+ (resp. from D^-) onto Δ_2. Moreover φ is $\mathcal{C}^1$ on $D^+ \cup D^-$ with Jacobian in (z,t)

$$\left|\det\begin{pmatrix} 1 & 1 \\ 2z & 2t \end{pmatrix}\right| = 2|z-t| = 2\sqrt{2v-u^2} \neq 0$$

The change of variables given by φ yields

$$I_+ = I_- = \int_{\Delta_2} h(u,v)\frac{1}{2\sqrt{2v-u^2}}\, f\left(\frac{u+\sqrt{2v-u^2}}{2}\right)\, f\left(\frac{u-\sqrt{2v-u^2}}{2}\right) du\, dv$$

By adding theses two terms, we get the lemma. □

Notice that, if $\rho = \mathcal{N}(0,1)$, then for any $(x,y) \in \mathbb{R}^2$,

$$\begin{aligned} f_2(x,y) &= \frac{1}{2\pi\sqrt{2y-x^2}}\, \exp\left(-\frac{y}{2}\right) \mathbb{1}_{x^2<2y} \\ &= \left(\sqrt{2^2\pi 2}\,\Gamma\left(\frac{2-1}{2}\right)\right)^{-1} \exp\left(-\frac{y}{2}\right)\left(y - \frac{x^2}{2}\right)^{(2-3)/2} \mathbb{1}_{x^2<2y} \end{aligned}$$

This is precisely the formula of proposition 7 for $n = 2$.

By theorem 17, the expansion of g_n holds as soon as there exists $q \in [1, +\infty[$ such that $\widehat{f_2} \in \mathrm{L}^q(\mathbb{R}^d)$. However the computation of $\widehat{f_2}$ is not feasible in general. Proposition 22 says that the previous condition is satisfied if there exists $p \in]1,2]$ such that $f_2 \in \mathrm{L}^p(\mathbb{R}^d)$ so that the expansion is true. Let us take a look at this :

$$\begin{aligned} &\int_{\mathbb{R}^2} |f_2(u,v)|^p\, du\, dv \\ &\qquad = \int_{\mathbb{R}^2} \frac{f^p\left((u+\sqrt{2v-u^2})/2\right)\, f^p\left((u-\sqrt{2v-u^2})/2\right)}{(2v-u^2)^{p/2}} \mathbb{1}_{u^2<2v}\, du\, dv \end{aligned}$$

Let us make the change of variables given by

$$(u,v) \longmapsto (x,y) = \frac{1}{2}(u + \sqrt{2v-u^2}, u + \sqrt{2v-u^2})$$

which is a $\mathcal{C}^1$-diffeomorphism from Δ_2 to D^+ (see the proof of the previous lemma) with Jacobian in (u,v), $2\sqrt{2v-u^2} = 2(y-x) > 0$:

$$\int_{\mathbb{R}^2} |f_2(u,v)|^p\, du\, dv = \int_{\mathbb{R}^2} \frac{f^p(x) f^p(y)}{(y-x)^p}\, 2(y-x)\mathbb{1}_{y>x}\, dx\, dy$$

By symmetry in x and y, we get

$$\int_{\mathbb{R}^2} |f_2(u,v)|^p\, du\, dv = \int_{\mathbb{R}^2} f^p(x) f^p(y)|y-x|^{1-p}\, dx\, dy$$

The next proposition follows from theorem 17, proposition 22 and the previous equality :

Proposition 37. *Suppose that ρ has a density f with respect to the Lebesgue measure on $\mathbb{R}$ such that, for some $p \in]1,2]$,*

$$(x,y) \longmapsto f^p(x+y)f^p(y)|x|^{1-p}$$

is integrable. Then, for n large enough, $\widetilde{\nu}_{n,\rho}$ has a density g_n with respect to the Lebesgue measure on $\mathbb{R}^2$ such that, for any compact subset K_I of A_I, when $n \to +\infty$, uniformly over $(x,y) \in K_I$.

$$g_n(x,y) \sim \frac{n}{2\pi}\left(\det \mathrm{D}^2_{(x,y)} I\right)^{1/2} e^{-nI(x,y)}$$

Let us prove now theorem 2. Suppose that ρ is a probability measure on $\mathbb{R}$ with an even density f such that there exist $v_0 > 0$ and $p \in]1,2]$ such that

$$\int_{\mathbb{R}} e^{v_0 z^2} f(z)\, dz < +\infty \qquad \text{and} \qquad \int_{\mathbb{R}^2} f^p(x+y)f^p(y)|x|^{1-p}\, dx\, dy < +\infty$$

The first inequality implies that

$$\forall v < v_0 \quad \forall u \in \mathbb{R} \qquad e^{-v_0 z^2} e^{uz+vz^2} = e^{uz-(v_0-v)z^2} \underset{|z|\to+\infty}{\longrightarrow} 0$$

Therefore $\mathbb{R}\times]-\infty, v_0[\subset D_\Lambda$ and thus $(0,0) \in \overset{o}{D}_\Lambda$. Moreover ρ is symmetric (since f is even) and its support contains at least three points (since ρ has a density). Proposition 32 implies that I is $\mathcal{C}^\infty$ in a neighbourhood of $(0,\sigma^2)$ and, when $(x,y) \to (0,\sigma^2)$,

$$I(x,y) - \frac{x^2}{2y} \sim \frac{(y-\sigma^2)^2}{2(\mu_4-\sigma^4)} + \frac{\mu_4 x^4}{12\sigma^8}$$

where μ_4 is the fourth moment of ρ. Denote by B_δ the open ball of radius δ centered at $(0,\sigma^2)$. It follows from the previous expansion that there exists $\delta > 0$ such that for any $(x,y) \in \mathrm{B}_\delta$,

$$G(x,y) = I(x,y) - \frac{x^2}{2y} \geq \frac{(y-\sigma^2)^2}{4(\mu_4-\sigma^4)} + \frac{\mu_4 x^4}{24\sigma^8} \tag{$*$}$$

We can reduce δ, in order to have $\mathrm{B}_\delta \subset K_I$ where K_I is a compact subset of A_I. Moreover $A_I \subset \overset{o}{D}_I \subset \Delta^*$ thus $\mathrm{B}_\delta \cap \Delta^* = \mathrm{B}_\delta$.

Let $n \in \mathbb{N}$ and let $f : \mathbb{R} \longrightarrow \mathbb{R}$ be a bounded continuous function. We have

$$\mathbb{E}_{\widetilde{\mu}_{n,\rho}}\left(f\left(\frac{S_n}{n^{3/4}}\right)\right) = \frac{1}{Z_n}\int_{\Delta^*} f(xn^{1/4}) \exp\left(\frac{nx^2}{2y}\right)\, d\widetilde{\nu}_{n,\rho}(x,y) = \frac{A_n + B_n}{Z_n}$$

with

$$A_n = \int_{\mathrm{B}_\delta} f(xn^{1/4}) \exp\left(\frac{nx^2}{2y}\right)\, d\widetilde{\nu}_{n,\rho}(x,y)$$

$$B_n = \int_{\Delta^*\cap \mathrm{B}_\delta^c} f(xn^{1/4}) \exp\left(\frac{nx^2}{2y}\right)\, d\widetilde{\nu}_{n,\rho}(x,y)$$

Proposition 37 implies that, for n large enough, $\widetilde{\nu}_{n,\rho}$ has a density g_n with respect to the Lebesgue measure on $\mathbb{R}^2$ which verifies, when $n \to +\infty$, uniformly over $(x,y) \in K_I$,

$$g_n(x,y) \sim \frac{n}{2\pi}\left(\det \mathrm{D}^2_{(x,y)} I\right)^{1/2} e^{-nI(x,y)}$$

Let us put $e^{-nI(x,y)}$ in the expression of A_n :

$$\begin{aligned}A_n &= \int_{\mathrm{B}_\delta} f(xn^{1/4}) e^{-n\left(I(x,y)-x^2/2y\right)} e^{nI(x,y)} g_n(x,y)\,dx\,dy \\ &= n\int_{\mathrm{B}_\delta} f(xn^{1/4}) e^{-nG(x,y)} H_n(x,y)\,dx\,dy\end{aligned}$$

where we set $H_n = e^{nI(x,y)} g_n(x,y)/n$. We define

$$\mathrm{B}_{\delta,n} = \{\,(x,y)\in\mathbb{R}^2 : x^2/\sqrt{n} + y^2/n \leq \delta^2\,\}$$

Let us make the change of variables given by $(x,y)\longmapsto(xn^{-1/4}, yn^{-1/2}+\sigma^2)$, with Jacobian $n^{-3/4}$:

$$A_n = n^{1/4}\int_{\mathrm{B}_{\delta,n}} f(x)\exp\left(-nG\left(\frac{x}{n^{1/4}},\frac{y}{\sqrt{n}}+\sigma^2\right)\right) H_n\left(\frac{x}{n^{1/4}},\frac{y}{\sqrt{n}}+\sigma^2\right)\,dx\,dy$$

We check now that we can apply the dominated convergence theorem to this integral. The uniform expansion of g_n means that for any $\alpha>0$, there exists $n_0\in\mathbb{N}$ such that

$$(x,y)\in K_I \quad\text{and}\quad n\geq n_0 \quad\Longrightarrow\quad \left|H_n(x,y)\,2\pi\left(\det \mathrm{D}^2_{(x,y)} I\right)^{-1/2} - 1\right| \leq \alpha$$

If $(x,y)\in\mathrm{B}_{\delta,n}$, then $(x_n,y_n)=(xn^{-1/4}, yn^{-1/2}+\sigma^2)\in\mathrm{B}_\delta\subset K_I$, thus for all $n\geq n_0$ and $(x,y)\in\mathrm{B}_{\delta,n}$,

$$\left|H_n\left(\frac{x}{n^{1/4}},\frac{y}{\sqrt{n}}+\sigma^2\right)2\pi\left(\det \mathrm{D}^2_{(x_n,y_n)} I\right)^{-1/2} - 1\right| \leq \alpha$$

Moreover $(x_n,y_n)\to(0,\sigma^2)$ thus, by continuity,

$$\left(\mathrm{D}^2_{(x_n,y_n)} I\right)^{-1/2} \underset{n\to+\infty}{\longrightarrow} \left(\mathrm{D}^2_{(0,\sigma^2)} I\right)^{-1/2} = \left(\mathrm{D}^2_{(0,0)}\Lambda\right)^{1/2} = \sqrt{\sigma^2(\mu_4-\sigma^4)}$$

Therefore

$$\mathbb{1}_{\mathrm{B}_{\delta,n}}(x,y) H_n\left(\frac{x}{n^{1/4}},\frac{y}{\sqrt{n}}+\sigma^2\right) \underset{n\to+\infty}{\longrightarrow} \left(4\pi^2\sigma^2(\mu_4-\sigma^4)\right)^{-1/2}$$

The expansion of G in the neighbourhood of $(0,\sigma^2)$ implies that

$$\exp\left(-nG\left(\frac{x}{n^{1/4}},\frac{y}{\sqrt{n}}+\sigma^2\right)\right) \underset{n\to+\infty}{\longrightarrow} \exp\left(-\frac{(y-\sigma^2)^2}{2(\mu_4-\sigma^4)} - \frac{\mu_4 x^4}{12\sigma^8}\right)$$

Let us check that the integrand is dominated by an integrable function, which is independent of n. The function

$$(x,y)\longmapsto\left(\mathrm{D}^2_{(x,y)} I\right)^{-1/2}$$

is bounded on B_δ by some $M_\delta>0$. The uniform expansion of g_n implies that for all $(x,y)\in\mathrm{B}_\delta$, $H_n(x,y)\leq C_\delta$ for some constant $C_\delta>0$. Finally, the inequality $(*)$ above yields that

$$\begin{aligned}\mathbb{1}_{\mathrm{B}_{\delta,n}}(x,y) f(x)\exp\left(-nG\left(\frac{x}{n^{1/4}},\frac{y}{\sqrt{n}}+\sigma^2\right)\right) H_n\left(\frac{x}{n^{1/4}},\frac{y}{\sqrt{n}}+\sigma^2\right)& \\ \leq \|f\|_\infty C_\delta \exp\left(-\frac{(y-\sigma^2)^2}{4(\mu_4-\sigma^4)} - \frac{\mu_4 x^4}{24\sigma^8}\right)&\end{aligned}$$

and the right term is an integrable function on $\mathbb{R}^2$. It follows from the dominated convergence theorem that

$$A_n \underset{+\infty}{\sim} n^{1/4} \int_{\mathbb{R}^2} f(x) \frac{1}{\sqrt{2\pi\sigma^2}\sqrt{2\pi(\mu_4 - \sigma^4)}} \exp\left(-\frac{(y-\sigma^2)^2}{2(\mu_4 - \sigma^4)} - \frac{\mu_4 x^4}{12\sigma^8}\right) dx\, dy$$

By Fubini's theorem, we get

$$A_n \underset{+\infty}{\sim} \frac{n^{1/4}}{\sqrt{2\pi\sigma^2}} \int_{\mathbb{R}} f(x) \exp\left(-\frac{\mu_4 x^4}{12\sigma^8}\right) dx$$

Let us focus now on B_n. The distribution ρ is symmetric, it has a density and $(0,0)$ belongs to the interior of D_Λ, thus proposition 33 implies that

$$\limsup_{n\to+\infty} \frac{1}{n} \ln \int_{\Delta^* \cap \mathrm{B}_\delta^c} \exp\left(\frac{nx^2}{2y}\right) d\widetilde{\nu}_{n,\rho}(x,y) < 0$$

Hence there exist $\varepsilon > 0$ and $n_0 \geq 1$ such that for any $n \geq n_0$,

$$\int_{\Delta^* \cap \mathrm{B}_\delta^c} \exp\left(\frac{nx^2}{2y}\right) d\widetilde{\nu}_{n,\rho}(x,y) \leq e^{-n\varepsilon}$$

and thus $B_n \leq \|f\|_\infty e^{-n\varepsilon}$ so that $B_n = o(n^{1/4})$. Therefore

$$A_n + B_n \underset{+\infty}{\sim} \frac{n^{1/4}}{\sqrt{2\pi\sigma^2}} \int_{\mathbb{R}} f(x) \exp\left(-\frac{\mu_4 x^4}{12\sigma^8}\right) dx$$

Applying this to $f = 1$, we get

$$Z_n \underset{+\infty}{\sim} \frac{2n^{1/4}}{\sqrt{2\pi\sigma^2}} \int_0^{+\infty} \exp\left(-\frac{\mu_4 x^4}{12\sigma^8}\right) dx = \frac{n^{1/4}}{\sqrt{2\pi\sigma^2}} \frac{1}{2} \left(\frac{12\sigma^8}{\mu_4}\right)^{1/4} \Gamma\left(\frac{1}{4}\right)$$

where we made the change of variables $y = \mu_4 x^4/(12\sigma^8)$. Finally

$$\mathbb{E}_{\tilde{\mu}_{n,\rho}}\left(f\left(\frac{S_n}{n^{3/4}}\right)\right) \underset{+\infty}{\sim} \left(\frac{4\mu_4}{3\sigma^8}\right)^{1/4} \Gamma\left(\frac{1}{4}\right)^{-1} \int_{\mathbb{R}} f(x) \exp\left(-\frac{\mu_4 x^4}{12\sigma^8}\right) dx$$

An ultimate change of variables $s = \mu_4^{1/4} x/\sigma^2$ gives us

$$\mathbb{E}_{\mu_{n,\rho}}\left(f\left(\frac{\mu_4^{1/4} S_n}{\sigma^2 n^{3/4}}\right)\right) \underset{n\to+\infty}{\longrightarrow} \left(\frac{4}{3}\right)^{1/4} \Gamma\left(\frac{1}{4}\right)^{-1} \int_{\mathbb{R}} f(s) \exp\left(-\frac{s^4}{12}\right) dx$$

This ends the proof of theorem 2.

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