THE 2D-ISING MODEL NEAR CRITICALITY: A FK-PERCOLATION ANALYSIS

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26 November 2008

Abstract. We study the 2d-Ising model defined on finite boxes at temperatures that are below but very close from the critical point. When the temperature approaches the critical point and the size of the box grows fast enough, we establish large deviations estimates on FK-percolation events that concern the phenomenon of phase coexistence.

1. Introduction

The present paper is a study of the influence of criticality on surface order large deviations. Surface order large deviations occur in supercritical FK-percolation and hence, by the FK-Potts coupling, in the Potts models at sub-critical temperatures. Originally, the study of such atypical large deviations and their corresponding Wulff construction has started for two dimensional models: the Ising model [18, 27, 28, 29, 34, 35], independent Bernoulli percolation [5, 3] and the random cluster model [4]. The just cited papers rely on a direct study of the contours. This leads to results that go beyond large deviations and give an extensive understanding of phase coexistence in two dimensions and at fixed temperatures. In higher dimensions, other techniques had to be used to achieve the Wulff construction, [7, 11, 14, 15]. There, the probabilistic estimates rely on block coarse graining techniques [36]. These coarse graining techniques also found applications in other problems not related to the Wulff construction, for example in the study of the random walk on the infinite percolation cluster [6]. A two-dimensional version of block coarse graining of Pisztora has been given in [17], using weak mixing results of Alexander [2].

In all the cited works, the percolation parameter (or the temperature) is kept fixed. The subject of our work is to understand how surface order large deviations and in particular block coarse graining techniques are influenced by criticality. In other words, our goal is to apply these coarse graining techniques in a joint limit where not only the blocks size increases but also the temperature approaches the critical point from below. It turns out that the study of block coarse graining in such a joint limit gives rise to several new problems. Indeed, ideas that are most natural and understood in the fixed temperature case become tricky when

1991 *Mathematics Subject Classification.* 60F10.

Key words and phrases. Large deviations, criticality, phase coexistence.

we approach criticality. This gives rise to questions like: how does the empirical density of the infinite cluster converge when we approach the critical point ? or how does the boundary condition influence the configuration inside the box when exponential decay starts to degenerate ? We address and to a certain extend solve these questions in the special case of the 2d-Ising model.

One may wonder why we limit our self to the particular case of the 2d-Ising model. Indeed, at fixed temperature, block coarse graining techniques are known to be adequate for the study of all FK-percolation models in all dimensions not smaller than two. But even in the fixed temperature case in dimensions higher than three, block coarse graining techniques are known to work up to the critical point only in the percolation model [25] and for the Ising model [8]. Unfortunately very little is known concerning the critical behavior of these models in dimension greater than two. When the dimension is greater than a certain threshold, many of the critical exponents take their so called mean-field values [38]. Despite these results, to our knowledge, no information is available on the critical behavior of the surface tension, i.e, the exponential price per unit area for the probability of a large interface of co-dimension one. Therefore we are limited to the two dimensional case, where two potential candidates are possible: site percolation on the triangular lattice, where a lot of progress has been made in the rigorous justification of critical exponents [37, 39, 10] and the 2d-Ising model where even more accurate information is available thanks to explicit computations, see [32] and the references therein. Site percolation model would have been an easier model to tackle and the techniques we use could handle this case with straightforward modifications. But the analysis of the corresponding Wulff construction is still out of reach. The reason for that is related to the open question number 3 at the end of [39]. Therefore, we chose to treat the 2d-Ising case and proof enough block estimates which permit the use of the techniques of [14] to establish the existence of the Wulff shape near criticality [13] under certain constrains on the simultaneous limit (thermodynamical and going to the critical point).

1.1. Statement of the main results. Our results concern the FK-measures of parameter $q = 2$ on finite boxes $\Lambda(n) = (-n/2, n/2]^2 \cap \mathbb{Z}^2$, where n is a positive integer. We denote by $\mathcal{FK}(p, \widetilde{\Lambda}(n))$ the set of the partially wired FK-measures on boxes $\widetilde{\Lambda}(n) = (-6n/10, 6n/10]^2 \cap \mathbb{Z}^2$ at percolation parameter p. The use of slightly enlarged boxes $\widetilde{\Lambda}$ is merely technical. When $p > p_c = \sqrt{2}/(1+\sqrt{2})$, we denote by $\theta(p)$ the density of the infinite cluster. In what follows we will say that a cluster C of a box Λ is crossing, if C intersects all the faces of the boundary of Λ. When $p > p_c$, it is known [17] that up to large deviations of the order of the linear size of the box $\Lambda(n)$, there exists a crossing cluster. It is also known that with overwhelming probability this crossing cluster has a density close to θ and that the crossing cluster intersect all the sub-boxes of at least logarithmic size. Our main results essentially state that this qualitative picture still holds when we approach the critical point and let the boxes grow fast enough. To formulate our results, we define for every box Λ the following events:

$$U(\Lambda) = \{\exists \text{ an open crossing cluster } C^* \text{ in } \Lambda\}.$$

Moreover, for $M > 0$, we define

$$R(\Lambda, M) = U(\Lambda) \cap \{ \text{ every open path } \gamma \subset \Lambda \text{ with } \operatorname{diam}(\gamma) \geq M \text{ is in } C^*\}$$
$$\cap \{C^* \text{ crosses every sub-box of } \Lambda \text{ with diameter } \geq M\},$$

where $\mathrm{diam}(\gamma) = \max_{x,y\in\gamma} |x-y|$ with $|\cdot|$ denoting the Euclidean norm.

Theorem 1. *Let $n > 1$ and $a > 5$. There exist two positive constants $\lambda, c = c(a)$ such that if $p > p_c$ and $n > c(p-p_c)^{-a}$ then*

$$\forall \Phi \in \mathcal{FK}(\widetilde{\Lambda}(n), p) \qquad \log \Phi[U(\Lambda(n))^c] \leq -\lambda(p-p_c)n.$$

Moreover, if M is such that

$$\frac{\log n}{\kappa(p-p_c)} < M \leq n, \tag{1}$$

with $\kappa > 0$ small enough, then

$$\forall \Phi \in \mathcal{FK}(\widetilde{\Lambda}(n), p) \qquad \log \Phi[R(\Lambda(n), M)^c] \leq -\lambda(p-p_c)M.$$

Note that the speed of the large deviations slows down by a factor $(p_c - p)$ when $p \downarrow p_c$. This is directly related to the critical exponent $\nu = 1$ of the inverse correlation length of the 2d-Ising model. The exponent $a > 5$ restrict our result to be valid only for boxes of width much larger than the inverse correlation length.
Next, we consider deviations for empirical densities of the infinite cluster when $p \downarrow p_c$. For $n > 0$, we consider the number of boundary connected sites

$$M_{\Lambda(n)} = |\{x \in \Lambda(n) : \; x \leftrightarrow \partial\Lambda(n)\},$$

where we have used the notation $|E|$ to denote the cardinality of a set $E \subset \mathbb{Z}^2$ and where $\partial\Lambda$ denotes the site boundary of Λ. It is known that for all $p > p_c$,

$$\lim_{n\to\infty} \frac{1}{|\Lambda(n)|} \Phi^{w,p}_{\Lambda(n)}[M_{\Lambda(n)}] = \theta(p). \tag{2}$$

On the other hand, from the solution of Onsager [33] we know that $\theta(p) \sim (p - p_c)^{1/8}$ when $p \downarrow p_c$. This degeneracy requires us to control the speed at which the convergence (2) occurs. To this end, for each $\delta > 0$, we define

$$m_{\sup}(\delta, p) = \inf\left\{ m \geq 1 : \; \forall n \geq m \quad \Phi^{w,p}_{\Lambda(n)}[M_{\Lambda(n)}] \leq |\Lambda(n)|(1+\delta/2)\theta \right\},$$

which represents the minimal size of the box required to approximate the density of the infinite cluster within an error of $\delta\theta/2$. The subadditivity of the map $\Lambda \mapsto M_\Lambda$, makes it handy to consider large deviations from above. To do so, we define the event

$$W(\Lambda, \delta) = \{M_\Lambda \leq (1+\delta)\theta|\Lambda|\}.$$

and obtain

Theorem 2. *Let $p > p_c$ and $\delta > 0$. If $n > 8m_{\sup}(\delta,p)/\delta$ then*

$$\log \Phi^{w,p}_{\Lambda(n)}[W(\Lambda(n),\delta)^c] \le -\left(\frac{\delta\theta n}{4m_{\sup}(\delta,p)}\right)^2 . \tag{3}$$

In particular, for every $a > 5/4$, there exists a positive constant $c = c(a,\delta)$ such that whenever $n \uparrow \infty$ and $p \downarrow p_c$ in such a way that $n > c(p-p_c)^{-a}$ then

$$\lim_{n,p} \frac{1}{(p-p_c)^{2a+1/4}n^2} \log \Phi^{w,p}_{\Lambda(n)}[W(\Lambda(n),\delta)^c] < 0 .$$

It is natural to take the density of the crossing cluster as an empirical density of the infinite cluster. Next we consider the deviations from below of this quantity. For any $\delta > 0$, we define the event

$$V(\Lambda,\delta) = U(\Lambda) \cap \{|C^*| \ge (1-\delta)\theta|\Lambda|\} .$$

When $p > p_c$ is kept fixed, an upper bound of the correct exponential speed can be obtained using coarse graining techniques of Pisztora. We proof that similar ideas can be used to obtain a priori estimates in the joint limit.

Theorem 3. *Let $a > 5$ and $\alpha \in]0,(1+\frac{1}{8a})^{-1}[$. There exists a positive constant $c = c(a,\alpha)$ such that, if $n \uparrow \infty$ and $p \downarrow p_c$ in such a way that $n^\alpha (p-p_c)^a > c$ then*

$$\sup_{\Phi \in \mathcal{FK}(\Lambda(n),p)} \Phi[V(\Lambda(n),\delta)^c] \le \exp(-\lambda\delta(p-p_c)n^\alpha) + \exp(-\frac{\delta^2}{4}(p-p_c)^{1/4}n^{2-2\alpha}), \tag{4}$$

where λ is a positive constant. In particular

$$\lim_{n,p} \inf_{\Phi \in \mathcal{FK}(\Lambda(n),p)} \Phi[V(\Lambda(n),\delta)] = 1.$$

When $p > p_c$ is kept fixed, the right hand side of (4) can be replaced by an expression of the form $\exp(-cn)$ where c is a positive constant. The appearance of two terms in the joint limit $n \to \infty$ and $p \downarrow p_c$ comes from the fact that the size of the blocks in the coarse graining cannot be taken constant anymore, they have to diverge like n^α. Note that the two terms on the right hand side of (4) are competing, indeed when α increases the first term decreases and the second one increases.

1.2. Organisation of the paper. In section 2, we start by introducing the basic definitions and notations used in the rest of the paper. In this section, we also provide preliminary results on the critical behavior of the 2d-Ising model. Then, in section 3, we establish weak mixing results in a situation where $p \to p_c$. These results will enable us to control adequately the influence of the boundary conditions. Finally, the proofs of the main theorem are given in section 4. In the appendix, we prove a technical result concerning the speed of convergence of the empirical magnetization near criticality.

2.1. The FK-representation. There exists a useful and well known coupling between the Ising model at inverse temperature β and the random cluster model with parameter $q=2$ and $p=1-\exp(-2\beta)$, see [19, 21]. The coupling is a probability measure $\mathbb{P}_n^+$ on the edge-spin configuration space $\{0,1\}^{\mathbb{E}(\Lambda(n))}\times\{-1,+1\}^{\Lambda(n)}$.

To construct $\mathbb{P}_n^+$ we first consider Bernoulli percolation of parameter p on the edge space $\{0,1\}^{\mathbb{E}(\Lambda(n))}$, then we choose the spins of the sites in $\Lambda(n)$ independently with the uniform distribution on $\{-1,+1\}$ and finally we condition the edge-spin configuration on the event that there is no open edge in $\Lambda(n)$ between two sites with different spin values. The construction can be summed up with a formula, we have

$$\forall(\sigma,\omega)\in\{0,1\}^{\mathbb{E}(\Lambda(n))}\times\{-1,+1\}^{\Lambda(n)}$$
$$\mathbb{P}_n^+(\sigma,\omega)=\frac{1}{Z}\prod_{e\in\mathbb{E}(\Lambda(n))}p^{\omega(e)}(1-p)^{1-\omega(e)}1_{(\sigma(x)-\sigma(y))\omega(e)=0},$$

where Z is the appropriate normalization factor. It can be verified that the marginal of $\mathbb{P}_n^+$ on the spin configurations is the Ising model at inverse temperature β given by the formula $p=1-\exp(-2\beta)$ and the marginal on the edge configurations is the random cluster measure with parameters p, $q=2$ and subject to wired boundary conditions, i.e., the probability measure on $\Omega_{\Lambda(n)}=\{0,1\}^{\mathbb{E}(\Lambda(n))}$ defined by

$$\forall\omega\in\Omega_{\Lambda(n)}\qquad\Phi^{p,w}_{\Lambda(n)}[\omega]=\frac{1}{Z}q^{\mathrm{cl}^w(\omega)}\prod_{e\in\mathbb{E}(\Lambda(n))}p^{\omega(e)}(1-p)^{1-\omega(e)},\tag{5}$$

where $\mathrm{cl}^w(\omega)$ is the number of connected components with the convention that two clusters that touch the boundary $\partial\Lambda(n)$ are identified. This coupling says that one may obtain an Ising configuration by first drawing a FK-percolation configuration with the measure $\Phi^{w,p}_{\Lambda(n)}$, then coloring all the sites in the clusters that touch the boundary $\partial\Lambda(n)$ in $+1$ and finally coloring the remaining clusters independently in $+1$ and -1 with probability $1/2$ each. Also, the coupling permits to obtain a $\Phi^{w,p}_{\Lambda(n)}$ percolation configuration by first drawing a spin configuration with $\mu^{+,\beta}_{\Lambda(n)}$, then declaring that all the edges between two sites with different spins are closed, while the other edges are independently declared open with probability p and closed with probability $1-p$.

Let $\Lambda\subset\mathbb{Z}^2$ and $0\le p\le 1$. In addition to the wired boundary conditions we will also work with *partially wired boundary conditions*. In order to define them, we consider a partition π of $\partial\Lambda=\{x\in\Lambda:\exists y\in\mathbb{Z}^2\setminus\Lambda,\,|x-y|_1=1\}$. Let us say that π consists of $\{B_1,\cdots,B_k\}$, where the B_i are non-empty disjoint subsets of $\partial\Lambda$ and such that $\cup_i B_i=\partial\Lambda$. For every configuration $\omega\in\Omega_\Lambda$, we define $\mathrm{cl}^\pi(\omega)$ as the number of open connected clusters in Λ computed by identifying two clusters that are connected to the same set B_i. The π-wired FK-measure $\Phi^{p,\pi}_\Lambda$ is defined by substituting $\mathrm{cl}^w(\omega)$ for $\mathrm{cl}^\pi(\omega)$ in (5). We will denote the set of all partially wired FK-measures in Λ by $\mathcal{FK}(p,\Lambda)$. Note that $\Phi^{p,w}_\Lambda$ corresponds to $\pi=\{\partial\Lambda\}$. We define the FK-measure with free boundary conditions $\Phi^{p,f}_\Lambda$ as the partially wired measure corresponding to $\pi=\emptyset$.

Let $U\subseteq V\subseteq\mathbb{Z}^2$. For every configuration $\omega\in\{0,1\}^{\mathbb{E}(\mathbb{Z}^2)}$, we denote by ω_V the restriction of ω to $\Omega_V=\{0,1\}^{\mathbb{E}(V)}$. More generally we will denote by ω_V^U the restriction of ω to $\Omega_V^U = \{0,1\}^{\mathbb{E}(V)\setminus\mathbb{E}(U)}$. If $V = \mathbb{Z}^2$ or $U = \emptyset$ then we drop them from the notation. We will denote by $\mathcal{F}_V^U$ the σ-algebra generated by the finite dimensional cylinders of Ω_V^U.

Note that every configuration $\eta \in \Omega_V$ induces a partially wired boundary condition $\pi(\eta)$ on the set U. The partition $\pi(\eta)$ is obtained by identifying the sites of ∂U that are connected through an open path of η^U. We will denote by $\Phi_U^{p,\pi(\eta)}$ the corresponding FK measure.

2.2. Planar Duality. The duality of the FK-measures in dimension two is well known. In this paper we will use the notation of [17] that we summarize next. Let $0 \le p \le 1$ and Λ be a box of $\mathbb{Z}^2$. To construct the dual model we associate to a box Λ the set $\widehat{\Lambda} \subset \mathbb{Z}^2 + (1/2, 1/2)$, which is defined as the smallest box of $\mathbb{Z}^2 + (1/2,1/2)$ containing Λ. To each edge $e \in \mathbb{E}(\Lambda)$ we associate the edge $\widehat{e} \in \mathbb{E}(\widehat{\Lambda})$ that crosses the edge e. Note that $\{e' \in \mathbb{E}(\widehat{\Lambda}) : \exists e \in \mathbb{E}(\Lambda), \widehat{e} = e'\} = \mathbb{E}(\widehat{\Lambda}) \setminus \mathbb{E}(\partial\widehat{\Lambda})$.

This allows us to build a bijective application from Ω_Λ to $\Omega_{\widehat{\Lambda}}^{\partial\widehat{\Lambda}}$ that maps each original configuration $\omega \in \Omega_\Lambda$ into its dual configuration $\widehat{\omega} \in \Omega_{\widehat{\Lambda}}^{\partial\widehat{\Lambda}}$ such that

$$\forall e \in \mathbb{E}(\Lambda) : \widehat{\omega}(\widehat{e}) = 1 - \omega(e).$$

The duality property states that for any $0 < p < 1$ and any $\mathcal{F}_\Lambda$-measurable event A we have

$$\Phi_\Lambda^{f,p}[A] = \Phi_{\widehat{\Lambda}}^{w,\widehat{p}}[\widehat{A}],$$

where $\widehat{A} = \{\eta \in \Omega_{\widehat{\Lambda}} : \exists \omega \in A, \widehat{\omega} = \eta^{\partial\widehat{\Lambda}}\} \subset \Omega_{\widehat{\Lambda}}^{\partial\widehat{\Lambda}}$ is the dual event of A and where $\widehat{p} = 2(1-p)/(2-p)$. It is useful to remark that when we translate an $\mathcal{F}_\Lambda$-measurable event A into it's dual $\widehat{A}$, we obtain an event which is in $\mathcal{F}_{\widehat{\Lambda}}^{\partial\widehat{\Lambda}}$. and that $\Phi_{\widehat{\Lambda}}^{w,\widehat{p},q}[\widehat{A}]$ does note depend on the states of the edges in $\mathbb{E}(\partial\widehat{\Lambda})$. Note also that under the measure $\Phi_\Lambda^{w,p}$ the law of $\omega_{\partial\Lambda}$ is an independent percolation of parameter p and $\omega_{\partial\Lambda}$ is also independent from $\omega^{\partial\Lambda}$.

We end this section by setting the following convention concerning the use of the word *dual* in the rest of the paper: we always consider that the original model is the super-critical one, i.e., $p > p_c$, which is defined on the edges of $\mathbb{Z}^2$. The *dual* model is always the dual of the super-critical model. That is, it is a sub-critical model defined on the edges of $\mathbb{Z}^2 + (1/2,1/2)$ and at percolation parameter $\widehat{p} = 2(1-p)/(2-p) \le p_c$. A dual path, circuit or site will always denote a path, circuit or site in $\mathbb{Z}^2 + (1/2,1/2)$. The term *open dual* will always designate edges $\widehat{e}$ of $\mathbb{Z}^2 + (1/2,1/2)$ that are open with respect to the dual configuration, i.e., $\widehat{\omega}(\widehat{e}) = 1$. The law of the dual edges $\widehat{e}$ will always be the dual measure $\Phi^{\widehat{p}}$ which is sub-critical, i.e., $\widehat{p} < p_c$.

2.3. Preliminary results on criticality in the 2d-Ising model. In this section we review some known results about the nature of the phase transition of the 2d-Ising model. These properties are important for our analysis and their proofs uses the specificities of the 2d-Ising model: explicit computations and correlation inequalities. Even though similar results are believed to hold for all the two dimensional FK-measures with parameter $1 \le q \le 4$, the FK-percolation with parameter $q = 2$ is the only model where such results can be established via-explicit computations. That is why our results are restricted to the 2d-Ising model. Let us also mention that if the analogues of the results stated in the section where available for other two dimensional FK-measures then the techniques used in this paper can be generalized to treat such cases. The extension to higher dimensional models is potentially also possible along the ideas of [36] but, to our knowledge, information about the critical behavior of the surface tension near criticality is nowadays unavailable even in the form of conjectures.

2.3.1 The critical point. It is known that the critical point of the Ising model on $\mathbb{Z}^2$ is given by the fixed point of a duality relation (see [23]). For the random cluster model with $q = 2$, the dual point $\widehat{p}$ is related to p through the relation

$$\frac{p}{1-p}\,\frac{\widehat{p}}{1-\widehat{p}} = 2, \text{ and the fixed point is } p_c = \frac{\sqrt{2}}{1+\sqrt{2}}. \tag{6}$$

For the general q-Potts model, the identification of the critical point and the self-dual point, i.e., $p_c = \sqrt{q}/(1+\sqrt{q})$, is still an open problem for the values $2 < q < 25$. When $q > 25.72$, this identity has been established and in this situation the Potts model exhibits a first order phase transition [22,30]. Thus the 2d-Ising model is the only two dimensional Potts model exhibiting a second order phase transition for which the critical point has been rigorously identified to be the self-dual point.

2.3.2 The surface tension. In the two dimensional supercritical FK-percolation model, large interfaces are best studied via duality. Indeed, a large interface implies a long connection in the sub-critical dual model. This is why the surface tension at $p > p_c$ is given by the exponential decay of connectivities in the sub-critical dual model:

$$\forall x \in \mathbb{Z}^2 \quad \tau_p(x) = -\lim_{n\to\infty} \frac{1}{n} \log \Phi^{\widehat{p}}_{\infty}[0 \leftrightarrow nx],$$

where $\Phi^{\widehat{p}}_{\infty}$ denotes the unique infinite FK-measure for $\widehat{p} < p_c$ [24]. In this paper, we are interested in the situation where the spatial scale n goes to infinity and simultaneously p goes to p_c. Using sub-additivity and the formula for τ_p, it is possible to show that

Proposition 4. *When $n \uparrow \infty$ and $p \downarrow p_c$ we have uniformly in $x \in Z^2$ that*

$$\frac{1}{(p-p_c)n|x|} \log \Phi^{\widehat{p}}_{\infty}[0 \leftrightarrow nx] \leq -\tau_c, \tag{7}$$

where τ_c is a positive constant.

The proof of the last proposition and even stronger results is the subject of [32].

2.3.3 The magnetization. The magnetization of the Ising model corresponds to the density $\theta(p)$ of the infinite cluster in the FK-representation. When $q = 2$, it is known that $\theta(p)$ approaches zero when $p \downarrow p_c$. Thanks to the Onsager's exact solution, it is also known at which speed this occurs:

$$\theta(p) \sim (p-p_c)^{1/8} \quad \text{when } p \downarrow p_c. \tag{8}$$

To apply our techniques, we will also need to know at which speed the empirical magnetization converges to $\theta(p)$ when approaching p_c. More precisely, we need to control

$$\frac{1}{n^2}\Phi^{p,w}\left[|\{x \in \Lambda(n) : \, x \leftrightarrow \partial\Lambda(n)\}|\right] - \theta(p) \tag{9}$$

in the joint limit $n \to \infty$ and $p \to p_c$. In turns out that the control of (9) is delicate. Indeed, we where unable to control the speed of convergence of (9) in the joint limit using only Proposition 4, (8) and robust FK-percolation techniques. We found a solution to this problem using further specificities of the 2d-Ising model, namely correlation inequalities. Using the ideas of [9], we get the following result

Proposition 5. *Let $\xi > 0$ and $a > \xi + 1$. There exist two positive constants $c = c(\xi, a)$ and ρ such that*

$$\forall p \neq p_c,\ n > c|p - p_c|^{-a} \qquad \frac{1}{n^2}\Phi^{p,w}\left[\sum_{x \in \Lambda(n)} 1_{x \leftrightarrow \partial\Lambda(n)}\right] - \theta(p) \leq \rho|p - p_c|^{\xi}.$$

We defer the proof of the last proposition to the end of the paper in Appendix A.

3. Weak mixing near criticality

In this part we establish weak mixing properties in the situation where $p \downarrow p_c$. These results are crucial in order to bound the influence of the boundary conditions. As it appears from [17], in order to implement a useful coarse graining in dimension two, it is necessary to have a control of the boundary conditions. When p is fixed, this control can be obtained by using the weak mixing properties proved in [1, 2]. To handle the situation where $p \downarrow p_c$, we give an alternative way to establish weak mixing and generalize the results of [1, 2] to a situation where the exponential decay of connectivities becomes degenerate.

3.1. Control of the number of boundary connected sites. Let $p < p_c, n \geq 1$. In this paragraph, we are interested in the control of the number of boundary connected sites

$$M_{\Lambda(n)} = |\{x \in \Lambda(n) : x \leftrightarrow \partial\Lambda(n)\}|. \tag{10}$$

The coming results depend on the speed of convergence of the mean of M_n near the critical point. We characterize this speed by introducing the following quantity:
(11)

$$\forall p < p_c,\ \delta > 0 \quad m_{\rm sub}(\delta, p) = \inf\left\{m \geq 1 : \forall n > m \quad \frac{1}{|\Lambda(n)|}\Phi^{w,p}_{\Lambda(n)}[M_{\Lambda(n)}] \leq \delta\right\}.$$

The main tool used in this section is subadditivity which permit us to reduce the problem to a family of bounded i.i.d random variables. Which are then well under control thanks to the following concentration bound:

Lemma 6. *(Theorem 1 of [26]) If $(X_i)_{1 \leq i \leq n}$ are independent random variables with values in $[0, 1]$ and with mean m, then*

$$\forall t \in]0, 1 - m[\qquad P\Big[\sum_{i=1}^{n}(X_i - m) \geq n\ t\Big] \leq \exp(-nt^2).$$

Lemma 7. *Let $\delta > 0$, $p \le p_c$. If $n \ge 16 m_{\rm sub}(\delta/2,p)/\delta$, then*

$$\log \Phi^{w,p}_{\Lambda(n)}\left[\frac{M_{\Lambda(n)}}{|\Lambda(n)|} \ge \delta\right] \le -\left(\frac{\delta n}{6 m_{\rm sub}(\delta/2,p)}\right)^2.$$

Proof. First we partition $\Lambda(n)$ into translates of the square $\Lambda(m)$ where

$$m = m_{\rm sub}(\delta/2,p). \tag{12}$$

Next, we take

$$n > 16m/\delta, \tag{13}$$

and consider the set

$$\Lambda'(n) = \bigcup_{\underline{x}\in\mathbb{Z}^2 : B(\underline{x})\subset\Lambda(n)} B(\underline{x}),$$

where $B(\underline{x}) = m\underline{x} + \Lambda(m)$. Note that $|\Lambda(n) \setminus \Lambda'(n)| \le 4mn$. The number of partitioning blocks satisfies

$$\frac{n^2}{2m^2} \le |\underline{\Lambda}'(n)| \le \frac{n^2}{m^2}. \tag{14}$$

Since M_Λ is subadditive, by (14) and (13), we obtain

$$\begin{aligned}\frac{M_{\Lambda(n)}}{|\Lambda(n)|} &\le \frac{1}{n^2}\sum_{\underline{x}\in\underline{\Lambda}'(n)} |\{v\in B(\underline{x}) : v \leftrightarrow \partial\Lambda(n)\}| + \frac{4m}{n}\\ &\le \frac{1}{|\underline{\Lambda}'(n)|}\sum_{\underline{x}\in\underline{\Lambda}'(n)} \frac{M_{B(\underline{x})}}{|B(\underline{x})|} + \frac{\delta}{4}.\end{aligned}$$

By the FKG inequality, we get

$$\Phi^{w,p}_{\Lambda(n)}\left[\frac{M_{\Lambda(n)}}{|\Lambda(n)|} \ge \delta\right] \le \Phi^{w,p}_{\Lambda(n)}\left[\frac{1}{|\underline{\Lambda}'(n)|}\sum_{\underline{x}\in\underline{\Lambda}'(n)}\frac{M_{B(\underline{x})}}{|B(\underline{x})|} \ge \frac{3\delta}{4}\,\middle|\, E\right] \tag{15}$$

where E is the increasing event $\big\{\forall \underline{x}\in\underline{\Lambda}'(n),$ all the edges of $\partial B(\underline{x})$ are open$\big\}$. The random variables $M_{B(\underline{x})}/|B(\underline{x})|$, $\underline{x}\in\underline{\Lambda}'(n)$, take their values in $[0,1]$ and they are independent under $\Phi^{w,p}_{\Lambda(n)}[\cdot\,|E]$. By (12), their mean satisfies

$$\forall\, \underline{x}\in\underline{\Lambda}'(n) \qquad \Phi^{w,p}_{\Lambda(n)}\left[\frac{M_{B(\underline{x})}}{|B(\underline{x})|}\,\middle|\, E\right] = \Phi^{w,p}_{B(\underline{x})}\left[\frac{M_{B(\underline{x})}}{|B(\underline{x})|}\right] \le \frac{\delta}{2}. \tag{16}$$

Finally, by lemma 6 and by the inequalities (14), (15) and (16) we get

$$\Phi^{w,p}_{\Lambda(n)}\left[\frac{M_{\Lambda(n)}}{|\Lambda(n)|} \ge \delta\right] \le \exp\left(-\frac{\delta^2 n^2}{32 m^2}\right).$$

□

3.2. Control of the boundary conditions. In this section, we determine a regime where we can still control the influence of the boundary conditions when $p \to p_c$. The regime will be characterized by the speed by which the quantity m_{sub} defined in (11) diverges near the critical point. We thus need to give an upper bound for the speed of this divergence.

Lemma 8. *Let $\kappa > 0, \xi > 0$. For every $a > \xi + 1$ there exists a positive constant $c = c(a, \kappa)$ such that*

$$\forall p < p_c \qquad m_{\text{sub}}(\kappa(p_c - p)^\xi, p) \le c(p_c - p)^{-a}.$$

Proof. Let $a > 1$ and $\xi \in (0, a-1)$. From proposition 5 we know that for every $\eta \in (\xi, \xi+1)$ there exist two positive constants ρ and c_1 such that

$$\forall p < p_c \quad \forall n > c_1(p_c - p)^{-a} \qquad \frac{1}{|\Lambda(n)|} \Phi^{w,p}_{\Lambda(n)}[M_{\Lambda(n)}] \le \rho(p_c - p)^\eta.$$

Furthermore, since $\eta > \xi$, there exists a positive constant $\varepsilon = \varepsilon(\rho, \xi, \kappa, \eta)$ such that

$$\forall p \in (p_c - \varepsilon, p_c) \qquad \rho(p_c - p)^\eta \le \kappa(p_c - p)^\xi.$$

Note also that if $p \le p_c - \varepsilon$ then $\kappa(p - p_c)^\xi \ge \kappa\varepsilon^\xi$ and there exists $n_0(\varepsilon^\xi)$ such that $n > n_0$ implies

$$\forall p < p_c - \varepsilon \qquad m_{\text{sub}}(\kappa(p - p_c)^\xi, p) \le n_0.$$

Hence the result follows by choosing $c = \max(c_1, \varepsilon^a n_0)$. $\square$

Proposition 9. *Let $p < p_c$ and $a > 5$. There exist two positive constants $c = c(a)$ and λ such that if $n > c(p_c - p)^{-a}$ then*

$$\log \Phi^{w,p}_{\Lambda(n)}[0 \leftrightarrow \partial\Lambda(n)] \le -\lambda(p_c - p)n\,.$$

Proof. Let $A = \{0 \leftrightarrow \partial\Lambda(n/2)\}$. In order to control the influence of the boundary conditions imposed on $\Lambda(n)$ we first write

$$\begin{aligned} \Phi^{w,p}_{\Lambda(n)}[A] \le &\Phi^{w,p}_{\Lambda(n)}[A \cap \{M_{\Lambda(n)} \le |\Lambda(n)|\delta\}] \\ &+ \Phi^{w,p}_{\Lambda(n)}[M_{\Lambda(n)} > |\Lambda(n)|\delta\ ], \end{aligned} \tag{17}$$

where $M_{\Lambda(n)}$ is defined in (10). On the event $A' = A \cap \{M_{\Lambda(n)} \le |\Lambda(n)|\delta\}$ of the first term we can bound the influence of the boundary conditions in an adequate way by using a judicious trick due to David Barbato [6], while the second term will be made negligible thanks to lemma 7.
Barbato's trick: This trick has initially been introduced in order to simplify the proof of the so called interface lemma in the case of dimensions higher or equal to three. Here we will use this trick in a different context. From the definition of the FK-measures it is clear that the influence of the boundary conditions comes from the connected components that connect $\partial\Lambda(n/2)$ to $\partial\Lambda(n)$. Thus if one can cut all these connections without altering too much the probability of the event A then

one gets a control over the influence of the boundary conditions. To do this we first define $M'_{\Lambda(n)}$ as

$$M'_{\Lambda(n)} \;=\; \Big|\{x \in \Lambda(n) : x \leftrightarrow \partial\Lambda(n) \text{ in } \Lambda(n) \setminus \Lambda(2|x|_\infty)\}\Big|.$$

This is the same quantity as $M_{\Lambda(n)}$ with the difference that we count only the sites x that are connected to the boundary with a *direct* path that does not use the edges in $\mathbb{E}(\Lambda(2|x|_\infty))$. Now suppose that $A' = A \cap \{M_{\Lambda(n)} \le |\Lambda(n)|\delta\}$ occurs. Since $M'_{\Lambda(n)} \le M_{\Lambda(n)}$ we also have $M'_{\Lambda(n)} \le \delta|\Lambda(n)|$. Next, for $0 < h < 1/4$, we define the set

$$\mathfrak{b}(h) = \partial[-n(1-h)/2, n(1-h)/2]^2.$$

Note that for $0 < h < 1/4$, we always have

$$\mathfrak{b}(h) \cap \Lambda(n/2) = \emptyset.$$

Next, we concentrate on the finite set of values $0 < h_1 < \cdots < h_K$ that satisfy

$$\mathfrak{b}(h_k) \cap \Lambda(n) \neq \emptyset.$$

We notice that the number K of such values h_k satisfies

$$\frac{n}{8} - 1 < K < \frac{n}{8} + 1.$$

Until here, the construction does not depend on the configuration. Next, we scan the configuration in $\Lambda(n)$ from outside inwards and define for each h_k the set of *bad* sites intersected by $\mathfrak{b}(h_k)$:

$$V(h_k) \;=\; M'_{\Lambda(n)} \cap \mathfrak{b}(h_k).$$

On A' we have that $\sum_{k=1}^{K} |V(h_k)| \le M'_{\Lambda(n)} \le \delta|\Lambda(n)|$ whence, for n large enough,

$$\min_k |V(h_k)| \le \frac{\delta|\Lambda(n)|}{K} \le \frac{\delta|\Lambda(n)|}{\frac{n}{8}-1} \;\le\; 16\delta n\,.$$

Thus there exists at least one $k \in \{1, \ldots, K\}$ such that

$$|V(h_k)| \le 16\delta n. \tag{18}$$

We define h^* as the first (smallest) value h_k that satisfies (18). Notice that h^* is a sort of stopping time, in the sense that

$$\forall 0 < h < 1/4 \qquad \{h^* = h\} \in \mathcal{F}_{\Lambda(n)\setminus\Lambda((1-h)n)}. \tag{19}$$

Then we define the set of *bad* edges as the set of edges that have one extremity in $\Lambda((1-h^*)n)$ and the other in $V(h^*)$:

$$I_n \;=\; \big\{\, e = \{v, u\} \in \mathbb{E}^2 \,:\; v \in \Lambda((1-h^*)n),\; u \in V(h^*)\,\big\}.$$

Even though

$$I_n \cap \mathbb{E}(\Lambda(n) \setminus \Lambda((1-h^*)n)) = \emptyset, \tag{20}$$

we obtain from (19) and from the definition of $V(h^*)$ that

$$\forall I \subseteq \mathbb{E}(\Lambda(n)) \quad \{I_n = I\} \in \mathcal{F}_{\Lambda(n)\setminus\Lambda((1-h^*)n)}. \tag{21}$$

It is also important to notice that

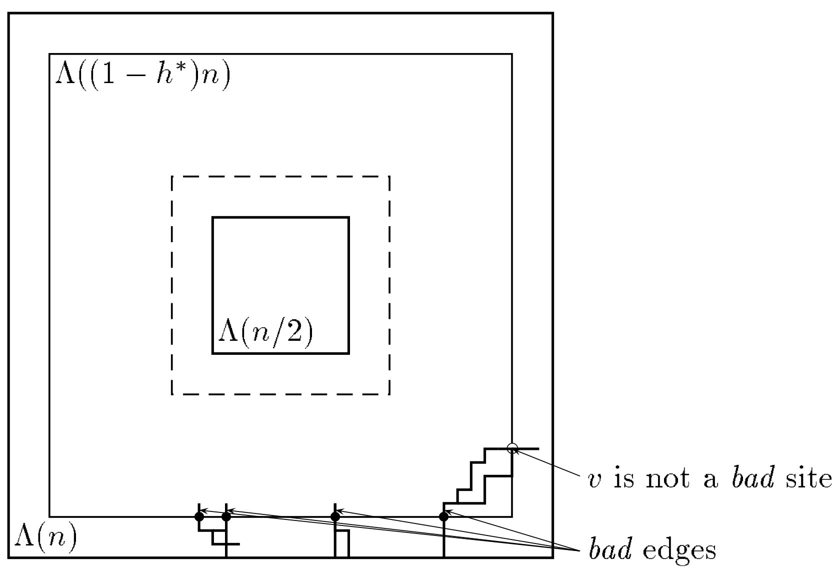
$$I_n \cap \mathbb{E}(\Lambda(n/2)) = \emptyset. \tag{22}$$

Now, for each site $v \in V(h^*)$ there is at most one edge e in I_n with extremity v thus we get from (18) that

$$|I_n| \leq 16\delta n. \tag{23}$$

Let $\Psi : A' \to \Omega$ be the map defined by:

$$\forall \omega \in A' \quad \forall e \in \Lambda(n) \quad \Psi(\omega)(e) = \begin{cases} 0 & \text{if } e \in I_n(\omega) \\ \omega(e) & \text{otherwise} \end{cases}$$



The configurations in $\Psi(A')$ have the following three crucial properties:

i) We claim that

$$\max_{\omega' \in \Psi(A')} |\Psi^{-1}(\omega')| \leq 2^{16\delta n}. \tag{24}$$

To prove (24), we first write for each $\widetilde{\omega} \in \Psi(A')$

$$|\Psi^{-1}(\widetilde{\omega})| \leq \sum_{I \subset \mathbb{E}(\Lambda(n))} \left|\{\omega \in \Omega_{\Lambda(n)} : I_n(\omega) = I, \, \omega^I = \widetilde{\omega}^I\}\right|.$$

By (20) and (21), the above sum contains only one term corresponding to $I = I(\widetilde{\omega})$. Hence

$$|\Psi^{-1}(\widetilde{\omega})| \le \left|\{\omega \in \Omega_{\Lambda(n)} : I_n(\omega) = I(\widetilde{\omega}),\, \omega^I = \widetilde{\omega}^I\}\right| \le 2^{|I_n(\widetilde{\omega})|},$$

and the claim follows from (23). Finally, using the finite energy property and (24) we get

$$\text{(25)}\qquad \begin{aligned}\Phi^{w,p}_{\Lambda(n)}[A'] &\le \max_{\omega' \in \Psi(A')} \left|\Psi^{-1}(\omega')\right| \left(1 \vee \frac{p}{1-p}\right)^{16\delta n} \Phi^{w,p}_{\Lambda(n)}[\Psi(A')]\\ &\le \exp(c_1 \delta n) \Phi^{w,p}_{\Lambda(n)}[\Psi(A')],\end{aligned}$$

where $0 < c_1 < \infty$ is a constant.

ii) By (22), the map Ψ does not modify the configuration inside $\Lambda(n/2)$, thus

$$\Psi(A') \subset A.$$

iii) By our cutting procedure we disconnect $\Lambda((1-h^*)n)$ from $\partial\Lambda(n)$ hence

$$\Psi(A') \subset \left\{ \Lambda(3n/4) \nleftrightarrow \partial\Lambda(n) \right\}.$$

By the property *iii)* and by duality, if the event $\Psi(A')$ occurs, there exists an outermost open dual circuit Γ in $\Lambda(n)$ that surrounds $\Lambda(3n/4)$. Let Ξ be the set of such dual circuits surrounding $\Lambda(3n/4)$. For every $\widehat{\gamma} \in \Xi$, we define $\mathrm{Int}(\widehat{\gamma})$ as the set of all the sites of $\Lambda(n)$ that are surrounded by $\widehat{\gamma}$ and $\mathrm{Ext}(\widehat{\gamma})$, the set of the sites of $\Lambda(n)$ that are not surrounded by $\widehat{\gamma}$. Note that

$$\text{(26)}\qquad \{\Gamma = \widehat{\gamma}\} = \mathrm{Open}(\widehat{\gamma}) \cap G_{\widehat{\gamma}},$$

where $\mathrm{Open}(\widehat{\gamma}) = \{\forall \widehat{e} \in \widehat{\gamma} : \widehat{\omega}(\widehat{e}) = 1\}$ and where $G_{\widehat{\gamma}}$ is a $\mathcal{F}_{\mathrm{Ext}(\widehat{\gamma})}$-measurable event. By using properties ii) and iii) and by (26) we can write

$$\text{(27)}\qquad \begin{aligned}\Phi^{w,p}_{\Lambda(n)}[\Psi(A')] &\le \Phi^{w,p}_{\Lambda(n)}[A \cap \bigcup_{\widehat{\gamma}\in\Xi} \{\Gamma = \widehat{\gamma}\}]\\ &= \sum_{\widehat{\gamma}\in\Xi} \Phi^{w,p}_{\Lambda(n)}[A \cap G_{\widehat{\gamma}} | \mathrm{Open}(\widehat{\gamma})]\; \Phi^{w,p}_{\Lambda(n)}[\mathrm{Open}(\widehat{\gamma})].\end{aligned}$$

Since A is $\mathcal{F}_{\mathrm{Int}\widehat{\gamma}}$-measurable, $G_{\widehat{\gamma}}$ is $\mathcal{F}_{\mathrm{Ext}\widehat{\gamma}}$-measurable, we can use the independence of the σ-algebras $\mathcal{F}_{\mathrm{Int}\widehat{\gamma}}$ and $\mathcal{F}_{\mathrm{Ext}\widehat{\gamma}}$ under $\Phi^{w,p}_{\Lambda(n)}[\,\cdot\,|\mathrm{Open}(\widehat{\gamma})]$ and the spatial Markov property to get

$$\text{(28)}\qquad \begin{aligned}\Phi^{w,p}_{\Lambda(n)}[A \cap G_{\widehat{\gamma}} | \mathrm{Open}(\widehat{\gamma})] &= \Phi^{w,p}_{\Lambda(n)}[A | \mathrm{Open}(\widehat{\gamma})]\; \Phi^{w,p}_{\Lambda(n)}[G_{\widehat{\gamma}} | \mathrm{Open}(\widehat{\gamma})]\\ &= \Phi^{f,p}_{\mathrm{Int}(\widehat{\gamma})}[A]\; \Phi^{w,p}_{\Lambda(n)}[G_{\widehat{\gamma}} | \mathrm{Open}(\widehat{\gamma})].\end{aligned}$$

Also A is an increasing event, so using (28), we get

$$\text{(29)}\qquad \forall \widehat{\gamma} \in \Xi \quad \Phi^{w,p}_{\Lambda(n)}[A \cap G_{\widehat{\gamma}} | \mathrm{Open}(\widehat{\gamma})] \le \Phi^{f,p}_{\Lambda(n)}[A]\; \Phi^{w,p}_{\Lambda(n)}[G_{\widehat{\gamma}} | \mathrm{Open}(\widehat{\gamma})].$$

Using (27) and (29) we obtain

$$\begin{aligned}\Phi^{w,p}_{\Lambda(n)}[\Psi(A')] \le & \Phi^{f,p}_{\Lambda(n)}[A] \sum_{\widehat{\gamma}\in\Xi} \Phi^{w,p}_{\Lambda(n)}[G_{\widehat{\gamma}}|\mathrm{Open}(\widehat{\gamma})]\ \Phi^{w,p}_{\Lambda(n)}[\mathrm{Open}(\widehat{\gamma})] \\ = & \Phi^{f,p}_{\Lambda(n)}[A]\ \Phi^{w,p}_{\Lambda(n)}[\exists \widehat{\gamma}\in\Xi:\ \Gamma=\widehat{\gamma}] \le \Phi^{p}_{\infty}[A].\end{aligned} \tag{30}$$

Combining (30) with (25) gives us

$$\Phi^{w,p}_{\Lambda(n)}[A'] \le \exp(c_1\delta n)\Phi^{p}_{\infty}[A]. \tag{31}$$

Now we turn to the second term of (17), namely $\Phi^{w,p}_{\Lambda(n)}[M_{\Lambda(n)} > |\Lambda(n)|\delta\ ]$. Assuming that n is bigger than $16m_{\mathrm{sub}}(\delta/2,p)/\delta$, we can apply lemma 7 to get

$$\Phi^{w,p}_{\Lambda(n)}[M_{\Lambda(n)} > |\Lambda(n)|\delta\ ] \le \exp\left[-\left(\frac{\delta n}{6m_{\mathrm{sub}}(\delta/2,p)}\right)^2\right]. \tag{32}$$

Substituting (31) and (32) into (17) one has

$$\Phi^{w,p}_{\Lambda(n)}[A] \le \exp(c_1\delta n)\ \Phi^{p}_{\infty}[A] + \exp\left[-\left(\frac{\delta n}{6m_{\mathrm{sub}}(\delta/2,p)}\right)^2\right]. \tag{33}$$

It follows from the comments after proposition 4 that there exists a positive τ_c such that for all $p < p_c$ and $n > 1$,

$$\Phi^{p}_{\infty}[A]\ \le\ |\partial\Lambda(n/2)| \sup_{x\in\partial\Lambda(n/2)} \Phi^{p}_{\infty}[0 \leftrightarrow x]\ \le\ 2n\exp(-\tau_c(p_c-p)n/4).$$

So that (33) becomes

$$\Phi^{w,p}_{\Lambda(n)}[A] \le 2n\exp(-\tau_c(p_c-p)n/4 + c_1\delta n) + \exp\left[-\left(\frac{\delta n}{6m_{\mathrm{sub}}(\delta/2,p)}\right)^2\right]. \tag{34}$$

From (34), it is clear that the only way not to destroy our estimates is to take δ at most of order $(p_c - p)$. So let us choose $\delta = \frac{\tau_c}{8c_1}(p_c-p)$. Let $a > 2$. By lemma 8 we know that there exists a positive constant c_2 such that $m_{\mathrm{sub}}(\tau_c(p_c-p)/(16c_1), p) < c_2(p_c-p)^{-a}$. Thus there exists a positive c_3 such that for all $n > c_3(p_c-p)^{-1-a}$, (34) becomes

$$\Phi^{w,p}_{\Lambda(n)}[A] \le \exp(-(\tau_c/16)(p_c-p)n) + \exp(-c_4(p_c-p)^{2+2a}n^2), \tag{35}$$

where $c_4 > 0$. Furthermore, we require that the first term is the main contribution, we do this by imposing that $n > \tau_c(p_c-p)^{-1-2a}/(16c_4)$. We conclude the proof by choosing $c = (c_3p_c^a) \vee (\tau_c/(16c_4))$ and $\lambda = \tau_c/16$. $\square$

The last proposition permits us to control adequately the influence of boundary conditions near criticality.

Corollary 10. *Let $p \neq p_c$, $a > 5$ and $\delta > 0$. There exist two positive constants $c = c(a,\delta)$ and λ such that uniformly over the events $A \in \mathcal{F}_{\Lambda(n)}$ and uniformly over two measures Φ_1, Φ_2 in $\mathcal{FK}(\Lambda(n(1+\delta)), p)$ we have*

$$n > c|p-p_c|^{-a} \Rightarrow (1 - e^{-\delta\lambda|p-p_c|n/2})^2 \Phi_1[A] \leq \Phi_2[A] \leq (1 + e^{-\delta\lambda|p-p_c|n/2})^2 \Phi_1[A].$$

Proof. Consider $A \in \mathcal{F}_{\Lambda(n)}$ and two partially wired boundary conditions π_1 and π_2 on the boundary $\partial\Lambda((1+\delta)n)$. It is sufficient to prove the statement for the measures $\Phi_1 = \Phi^{\pi_1,p}_{\Lambda((1+\delta)n)}$ and $\Phi_2 = \Phi^{\pi_2,p}_{\Lambda((1+\delta)n)}$. Let $m > (1+2\delta)n$ and define the following $\mathcal{F}^{\Lambda((1+\delta)n)}_{\Lambda(m)}$-measurable events, for $i = 1, 2$:

$$W_i = \left\{ \omega \in \Omega_{\Lambda(m)} : \begin{array}{l} \text{with wired boundary conditions on } \Lambda(m) \\ \text{and the configuration } \omega \text{ on } \Lambda(m) \setminus \Lambda((1+\delta)n), \\ \text{the boundary conditions induced on } \Lambda((1+\delta)n) \text{ are } \pi_i \end{array} \right\}$$

Since π_1 and π_2 are partially wired boundary conditions, it is possible to find a large enough finite m such that $\Phi^{w,p}_{\Lambda(m)}[W_i] > 0, i = 1, 2$. We fix such an m and write $\Phi_i[A] = \Phi^{w,p}_{\Lambda(m)}[A|W_i], i = 1, 2$. We note that $d(\Lambda(m) \setminus \Lambda((1+\delta)n), \Lambda(n)) > \delta n/2$. Therefore, Proposition 9 and an adaptation of the arguments of lemma 3.2 in [4] ensures the existence of a positive $c = c(a,\delta)$ such that

$$n > c|p-p_c|^{-a} \quad \Rightarrow \quad |\Phi^{w,p}_{\Lambda(m)}[A|W_i] - \Phi^{w,p}_{\Lambda(m)}[A]| \leq e^{-\delta\lambda|p-p_c|n/2} \Phi^{w,p}_{\Lambda(m)}[A] \quad i = 1, 2.$$

Using the last inequality, we finally get

$$\Phi_2[A] \geq (1 - e^{-\delta\lambda|p-p_c|n/2}) \Phi^{w,p}_{\Lambda(m)}[A] \geq (1 - e^{-\delta\lambda|p-p_c|n/2})^2 \Phi_1[A],$$

and

$$\Phi_2[A] \leq (1 + e^{-\delta\lambda|p-p_c|n/2}) \Phi^{w,p}_{\Lambda(m)}[A] \leq (1 - e^{-\delta\lambda|p-p_c|n/2})^2 \Phi_1[A].$$

□

4. Proof of the theorems

Proof of Theorem 1. Since $U(\Lambda(n))$ is increasing, we have that

$$\forall \Phi \in \mathcal{FK}(\widetilde{\Lambda}(n), p) \quad \Phi[U(\Lambda(n))^c] \leq \Phi^{f,p}_{\Lambda(n)}[U(\Lambda(n))^c].$$

By duality we get that

$$\Phi[U(\Lambda(n))^c] \leq 2\ \Phi^{f,p}_{\Lambda(n)}[\exists \text{ an open dual path in } \widehat{\Lambda}(n) \text{ of diameter } \geq n],$$

Let $a > 5$. By Corollary 10 and proposition 4 there exist two positive constants $c = c(a)$ and λ_1 such that for all $p > p_c$ and for all $n > c(p-p_c)^{-a}$ we have

$$\begin{aligned} &\Phi^{f,p}_{\Lambda(n)}[\exists \text{ an open dual path in } \widehat{\Lambda}(n) \text{ of diameter} \geq n] \\ &\qquad \leq \Phi^{f,p}_{\Lambda(n)}[\exists \text{ an open dual path in } \widehat{\Lambda}(n) \text{ of diameter} \geq n] \\ &\qquad \leq \Phi^{p}_{\infty}[\exists \text{ an open dual path in } \widehat{\Lambda}(n) \text{ of diameter} \geq n] \\ &\qquad \leq 2n^4 \exp(-\lambda_1(p-p_c)n) \\ &\qquad \leq 2\exp((p_c-p)n(\lambda_1 - 4\frac{\log n}{n(p-p_c)})), \end{aligned}$$

Note that there exists n_0 independent of everything such that

$$\forall n > \max(n_0, c(p-p_c)^{-a}) \qquad \frac{\log n}{n(p-p_c)} \le \frac{n^{-1/2}}{p-p_c} \le \frac{1}{c}(p-p_c)^{3/2}.$$

Thus, the result follows by choosing $\lambda = \lambda_1/2$ and c big enough. To estimate the event R, notice that

$$\Phi[R(\Lambda(n),M)^c] \le \Phi[U(\Lambda(n))^c] + \Phi^{f,p}_{\Lambda(n)}[\exists \text{ an open dual path of diameter } \ge M].$$

Then, as before, we use Corollary 10 and proposition 4 to get

$$\begin{aligned}\Phi[R(\Lambda(n),M)^c] &\le \exp(-\lambda(p-p_c)n) + n^4\exp(-\lambda(p-p_c)M)\\ &\le (1+n^4)\exp(-\lambda(p-p_c)M).\end{aligned}$$

Finally, condition (1) ensures that the prefactor does not destroy our estimates and this concludes the proof. □

Now we turn to the estimation of the crossing cluster's size:

Proof of Theorem 2.
To get (3), one proceeds as in lemma 7. For the second statement, one proceeds as in lemma 8 to prove that for every $a > 9/8$, there exists a positive constant $c = c(a,\delta)$ such that $m_{\sup}(\delta,p) \le C(p-p_c)^{-a}$. The desired result follows then from (3). □

Proof of Theorem 3. Let $\Phi \in \mathcal{FK}(\widetilde{\Lambda}(n),p)$. We renormalize $\Lambda(n)$ into $\underline{\Lambda}(n)$ by partitioning it into blocks $B(\underline{x})$ of size $N \le n$ to get the renormalized box

$$\underline{\Lambda}(n) = \{\underline{x} \in \mathbb{Z}^2 : \ (-N/2,N/2]^2 + N\underline{x} \subset (-n/2,n/2]^2\}.$$

Next, we define the following events:

- For $\{\underline{x},\underline{y}\} \in \mathbb{E}(\underline{\Lambda}(n))$, we denote by $m(\underline{x},\underline{y})$ the middle point of the face between $B(\underline{x})$ and $B(\underline{y})$. We also introduce the box $D_{\underline{x},\underline{y}} = m(\underline{x},\underline{y}) + \Lambda(\lfloor N/4 \rfloor)$ of width $\lfloor N/4 \rfloor$ and centered at $m(\underline{x},\underline{y})$. Then, we define

$$K_{\underline{x},\underline{y}} = \{\exists \text{ crossing in } D_{\underline{x},\underline{y}}\}, \qquad K_{\underline{x}} = \bigcap_{\underline{z}\in\underline{\Lambda}(n) :\, |\underline{x}-\underline{z}|=1} K_{\underline{x},\underline{z}}.$$

- For $\underline{x} \in \underline{\Lambda}(n)$ and $M > 0$, we define

$$\begin{aligned}R(\underline{x}) = \{&\exists! \text{ crossing cluster } C^*_{\underline{x}} \text{ in } B(\underline{x})\}\cap\\ &\{\text{every open path } \gamma \subset B(\underline{x}) \text{ with } \mathrm{diam}(\gamma) \ge M \text{ is included in } C^*_{\underline{x}}\}.\end{aligned} \tag{36}$$

On $\underline{\Lambda}(n)$, we define the $0-1$ renormalized process $(X(\underline{x}), \underline{x} \in \underline{\Lambda}(n))$ as the indicator of the occurrence of the above mentioned events:

$$\forall \underline{x} \in \underline{\Lambda}(n) \quad X(\underline{x}) = \begin{cases} 1 \text{ on } R(\underline{x}) \cap K(\underline{x}) \\ 0 \text{ otherwise}\end{cases}$$

By Theorem 1, we get the following estimate on the probability that a specific box is *bad*. There exist $\kappa, \lambda > 0$ such that if

$$n > N > 4M > \frac{\log N}{\kappa(p - p_c)} \tag{37}$$

then

$$\forall \underline{x} \in \underline{\Lambda}(n) \qquad \Phi[X(\underline{x}) = 0] \leq \exp\left(-\lambda(p - p_c)M\right). \tag{38}$$

As M will grow, we can restrict ourselves to the case where there is no bad block at all and where the event $R(\Lambda(n), N)$ is satisfied, namely for all $\Phi \in \mathcal{FK}(\widetilde{\Lambda}(n), p)$, we write

$$\begin{aligned} \Phi[V(\Lambda(n),\delta)^c] \leq & \Phi[\exists \text{ a bad block }] + \Phi[R(\Lambda(n),N)^c] \\ & + \Phi[\nexists \text{ a bad block } \cap R(\Lambda(n),N) \cap V(\Lambda(n),\delta)^c]. \end{aligned} \tag{39}$$

By (38), we get

$$\Phi[\exists \text{ a bad block }] \leq \frac{n^2}{N^2} \exp(-\lambda_1(p - p_c)M). \tag{40}$$

For the second term of (39),we apply Theorem 1 to get

$$\Phi[R(\Lambda(n), N)^c] \leq \exp(-\lambda_2(p - p_c)N), \tag{41}$$

For the third term of (39), we observe that if there is no bad block then there is one single cluster in the renormalized process that consists of all the blocks of $\underline{\Lambda}(n)$. By the definition of the events associated to $(X(\underline{x}),\ \underline{x} \in \underline{\Lambda}(n))$, this induces one crossing cluster $\widetilde{C}^*$ of $\cup_{\underline{x}\in\underline{\Lambda}(n)} B(\underline{x})$ that contains all the crossing clusters $C^*_{\underline{x}}$, $\underline{x} \in \underline{\Lambda}(n)$. On the other hand, since $R(\Lambda(n), N)$ is satisfied, we have that $\widetilde{C}^* \subset C^*$, where C^* is the crossing cluster of $\Lambda(n)$, which is guaranteed to exists thanks to the event $U(\Lambda(n))$. Now, we define for every $\underline{x} \in \underline{\Lambda}(n)$ the random variables

$$Y(\underline{x}) = N^{-2} |\{v \in B(\underline{x}) : \text{diam}(C_v) \geq M\}|$$

and observe that

$$|C^*| < (1 - \delta)\theta n^2 \quad \Rightarrow \quad \sum_{\underline{x}\in\underline{\Lambda}(n)} |C^*_{\underline{x}}| < (1 - \delta)\theta n^2 \,. \tag{42}$$

Yet if $B(\underline{x})$ is a good box then every cluster of $B(\underline{x})$ that is of diameter larger than M is included in $C^*_{\underline{x}}$, thus using (39), (40),(41), (42) and by the FKG inequality we get

(43)

$$\Phi[V(\Lambda(n),\delta)^c] \leq 2\frac{n^2}{N^2} \exp(-\lambda_3(p - p_c)M) + \Phi\Big[\left(\frac{N}{n}\right)^2 \sum_{\underline{x}\in\underline{\Lambda}(n)} Y(\underline{x}) \leq (1 - \delta)\theta |E\Big],$$

where E is the event that all the edges that touch the boundary of the boxes $B(\underline{x})$ are *closed* and $\lambda_3 = \min(\lambda_1, \lambda_2)$. Now we choose N and M such that the mean of the random variables $Y(\underline{x})$ is big enough: by using Corollary 10 we have for $\underline{x} \in \underline{\Lambda}(n)$

$$\begin{aligned}
\Phi^{f,p}_{\Lambda(N)}[Y(\underline{x})] &\geq N^{-2}\Phi^{f,p}_{\Lambda(N)}[|\{x \in \Lambda(N-4M) : x \leftrightarrow \partial\Lambda(2M) + x\}|] \\
&\geq N^{-2} \sum_{x \in \Lambda(N-4M)} \Phi^{f,p}_{\Lambda(N)}[x \leftrightarrow \partial\Lambda(2M) + x] \\
&\geq N^{-2} \sum_{x \in \Lambda(N-4M)} \Phi^{f,p}_{x+\Lambda(4M)}[x \leftrightarrow \partial\Lambda(2M) + x] \\
&\geq (1 - e^{-(p-p_c)M/2})^2 \frac{(N-4M)^2}{N^2} \Phi^{w,p}_{\Lambda(4M)}[0 \leftrightarrow \partial\Lambda(2M)] \\
&\geq (1 - e^{-(p-p_c)M/2})^2 \frac{(N-4M)^2}{N^2} \Phi^{p}_{\infty}[0 \leftrightarrow \partial\Lambda(2M)] \\
&\geq (1 - 2e^{-(p-p_c)M/2})(1 - \frac{8M}{N})\theta\,.
\end{aligned}$$

By Onsager's formula, we have

$$\theta = (p-p_c)^{1/8} + o((p-p_c)^{1/8})\,, \qquad p \downarrow p_c\,.$$

Thus if we choose

$$M = \frac{\delta}{32}N \quad \text{and} \quad M(p-p_c) \geq c, \tag{44}$$

where $c > 0$ is a large enough constant we get

$$\forall \underline{x} \in \underline{\Lambda}(n) \qquad \Phi^{f,p}_{\Lambda(N)}[Y(\underline{x})] \geq \theta(1 - \frac{\delta}{2})\,. \tag{45}$$

The random variables $Y(\underline{x})$, $\underline{x} \in \underline{\Lambda}(n)$, take their values in $[0,1]$ and they are independent under $\Phi[\cdot\, |E]$, thus we can use lemma 6 with (45) to bound (43) by

$$\Phi[V(\Lambda(n),\delta)^c] \leq 2\frac{n^2}{N^2}\exp(-\lambda(p-p_c)M) + \exp\left(-\frac{\delta^2\theta^2(p)}{4}\frac{n^2}{N^2}\right)\,. \tag{46}$$

Let $a > 3$ and $0 < \alpha < 1$. If $n^\alpha > (p-p_c)^{-a}$ and letting $N = n^\alpha$, one gets

$$\Phi[V(\Lambda(n),\delta)^c] \leq 2\exp(-\frac{\lambda}{32}\delta(p-p_c)n^\alpha + 2(1-\alpha)\log n) + \exp\left(-\frac{\delta^2\theta^2(p)}{4}n^{2-2\alpha}\right)\,.$$

Also, under the above regime we have that $(p-p_c)n^\alpha/\log n \to \infty$. Thus, by choosing n, N, M such that (37) and (44) are satisfied, we obtain the desired result. $\square$

Appendix A

As promised, we give a proof of Proposition 5

Proof of Proposition 5. From the Ising-FK coupling it follows that

$$\frac{1}{|\Lambda(n)|}\Phi^{w,p}_{\Lambda(n)}[M_{\Lambda(n)}] = \frac{1}{|\Lambda(n)|}\sum_{x\in\Lambda(n)}\mu^{+,\beta}_{\Lambda(n)}[\sigma(x)],$$

where $\mu^{+,\beta}_{\Lambda(n)}$ is the plus boundary condition Ising measure on $\{-1,+1\}^{\Lambda(n)}$ taken at inverse temperature β. The proof we present here is an adaptation of arguments included in [9]. An alternative way to derive the result is to use the ideas of [16]. Let n,k,l be three integers larger than one. For $h>0$, we note $\mu^{+,\beta,h}_{n+k+l}$ the Ising measure on the box $\Lambda(n+k+l)$ with boundary conditions $+$, at inverse temperature β and where every spin in $\Lambda(n+k+l)\setminus\Lambda(n+k)$ is submitted to a positive field h/β. Let $x\in\Lambda(n)$. The measure $\mu^{+,\beta,h}_{n+k+l}$ has the property that

$$\lim_{h\uparrow\infty}\mu^{+,\beta,h}_{n+k+l}[\sigma(x)] = \mu^{+,\beta}_{\Lambda(n+k)}[\sigma(x)].$$

It is thus a sort of interpolation between the measures $\mu^{+,\beta}_{\Lambda(n+k+l)}$ and $\mu^{+,\beta,h}_{\Lambda(n+k)}$. Furthermore, it is easy to check that

$$\frac{\partial\mu^{+,\beta,h}_{n+k+l}[\sigma(x)]}{\partial h} = \sum_{y\in\Lambda(n+k+l)\setminus\Lambda(n+k)}\mu^{+,\beta,h}_{n+k+l}[\sigma(x)\sigma(y)] - \mu^{+,\beta,h}_{n+k+l}[\sigma(x)]\mu^{+,\beta,h}_{n+k+l}[\sigma(y)].$$

Therefore, we have

$$\begin{aligned}0 \le \mu^{+,\beta}_{\Lambda(n+k)}[\sigma(x)] - \mu^{+,\beta}_{\Lambda(n+k+l)}[\sigma(x)] =\\ \sum_{y\in\Lambda(n+k+l)\setminus\Lambda(n+k)}\int_0^\infty \mu^{+,\beta,h}_{n+k+l}[\sigma(x)\sigma(y)] - \mu^{+,\beta,h}_{n+k+l}[\sigma(x)]\mu^{+,\beta,h}_{n+k+l}[\sigma(y)]\,dh,\end{aligned}$$

Next, applying the Ising specific G.H.S inequality [20], we get that

$$\begin{aligned}&\mu^{+,\beta,h}_{n+k+l}[\sigma(x)\sigma(y)] - \mu^{+,\beta,h}_{n+k+l}[\sigma(x)]\mu^{+,\beta,h}_{n+k+l}[\sigma(y)]\\ &\qquad\le \mu^{+,\beta}_{\infty}[\sigma(x)\sigma(y)] - \mu^{+,\beta}_{\infty}[\sigma(x)]\mu^{+,\beta}_{\infty}[\sigma(y)].\end{aligned}$$

Note that the right hand side depends only on the infinite volume measure. On the other hand, by using Griffith's inequalities [20], we may estimate

$$\mu^{+,\beta,h}_{n+k+l}[\sigma(x)\sigma(y)] - \mu^{+,\beta,h}_{n+k+l}[\sigma(x)]\mu^{+,\beta,h}_{n+k+l}[\sigma(y)] \le \exp(-\lambda_1 h),$$

uniformly in $n+k+l$, x,y and in β, where λ_1 is a positive constant.
Combining the two last inequalities with the magnetic field representation of the boundary conditions, we finally obtain

$$\begin{aligned}&\mu^{+,\beta}_{\Lambda(n+k)}[\sigma(x)] - \mu^{+,\beta}_{\Lambda(n+k+l)}[\sigma(x)] \le\\ &\int_0^\infty dh \sum_{y\in\Lambda(n+k+l)\setminus\Lambda(n+k)}\left\{\left(\mu^{+,\beta}_{\infty}[\sigma(x)\sigma(y)] - \mu^{+,\beta}_{\infty}[\sigma(x)]\,\mu^{+,\beta}_{\infty}[\sigma(y)]\right)\wedge\exp(-\lambda_1 h)\right\}.\end{aligned}$$

First, let us consider the case where $0 < \beta < \beta_c$. In this situation, the explicit computation (see [32]) yields that the correlation is bounded above as follows:

$$\mu_\infty^{+,\beta}[\sigma(x)\sigma(y)] - \mu_\infty^{+,\beta}[\sigma(x)]\,\mu_\infty^{+,\beta}[\sigma(y)] \le \exp(-\lambda_2(\beta-\beta_c)|x-y|). \tag{47}$$

Thus

$$\begin{aligned}
&\mu_{\Lambda(n+k)}^{+,\beta}[\sigma(x)] - \mu_{\Lambda(n+k+l)}^{+,\beta}[\sigma(x)]\\
&\le \int_0^\infty dh \sum_{y\in\Lambda(n+k+l)\setminus\Lambda(n+k)} \exp(-(\lambda_1 h \vee \lambda_2(\beta_c-\beta)|x-y|))\\
&\le \int_0^\infty dh\, \exp(-\frac{\lambda_1}{2}h) \sum_{y\in\Lambda(n+k+l)\setminus\Lambda(n+k)} \exp(-\frac{\lambda_2}{2}(\beta_c-\beta)|x-y|)\\
&\le \frac{2}{\lambda_1} \sum_{y\in\Lambda(n+k+l)\setminus\Lambda(n+k)} \exp(-\frac{\lambda_2}{2}(\beta_c-\beta)|x-y|)\\
&\le \frac{8}{\lambda_1}\exp(-\frac{\lambda_2}{4}(\beta_c-\beta)k)\sum_{r=0}^{l}(n+k+r)\exp(-\frac{\lambda_2}{4}(\beta_c-\beta)r)\\
&\le c_1\frac{8}{\lambda_1}\frac{n+k}{\beta_c-\beta}\exp(-\frac{\lambda_2}{4}(\beta_c-\beta)k)\sum_{r=0}^{l}\exp(-\frac{\lambda_2}{8}(\beta_c-\beta)r).
\end{aligned}$$

The last inequality has been obtained by bounding $n+k+r$ by $(n+k)(r+1)$ and by choosing c_1 in such a way that

$$\forall r \ge 0 \qquad r+1 \le \frac{c_1}{\beta_c-\beta}\exp(\frac{\lambda_2}{8}(\beta_c-\beta)(r+1)).$$

Sending l to infinity yields

$$\mu_{\Lambda(n+k)}^{+,\beta}[\sigma(x)] \le \frac{8c_1}{\lambda_1\lambda_2}\frac{n+k}{(\beta_c-\beta)^2}(8+\lambda_2(\beta_c-\beta))\exp(-\frac{\lambda_2}{4}(\beta_c-\beta)k).$$

Thus, there exists a positive constant c_2 such that

$$\mu_{\Lambda(n+k)}^{+,\beta}[\sigma(x)] \le c_2\frac{n+k}{(\beta_c-\beta)^2}\exp(-\frac{\lambda_2}{4}(\beta_c-\beta)k).$$

Applying this inequality to the box $\Lambda(n)$ and the sites in $\Lambda(n-k)$, we deduce that for all $k < n$

$$\frac{1}{n^2}\sum_{x\in\Lambda(n)}\mu_{\Lambda(n)}^{+,\beta}[\sigma(x)] \le 2\frac{k}{n} + c_2\frac{n}{(\beta_c-\beta)^2}\exp(-\frac{\lambda_2}{4}(\beta_c-\beta)k).$$

We fix $\xi > 0$ and $a > \xi+1$. For all $0 < \beta < \beta_c$ we take $n > (\beta_c-\beta)^{-\xi}$ and choose $k = (\beta_c-\beta)^\xi n$ and obtain

$$\begin{aligned}
&\frac{1}{n^2}\sum_{x\in\Lambda(n)}\mu_{\Lambda(n)}^{+,\beta}[\sigma(x)] \le\\
&2(\beta_c-\beta)^\xi + \frac{c_2}{(n(\beta_c-\beta)^a)^{\frac{2}{a}}}\exp\left(\left(-\frac{\lambda_2}{4}(\beta_c-\beta)^{1+\xi}\frac{n}{\log n}+1+\frac{2}{a}\right)\log n\right).
\end{aligned} \tag{48}$$

The last expression suggests to impose a regime on $(\beta_c-\beta)^a n$. Indeed, there exists a positive $n_0 = n_0(\xi, a)$ such that $n > n_0$ implies that $n/\log n > n^{\frac{\xi+1}{a}}$, hence by imposing

$$(\beta_c-\beta)^a n > \Big(\frac{8}{\lambda_2}(1+\frac{2}{a})\Big)^{a/(\xi+1)} \vee 1 \vee \beta_c^a n_0$$

we obtain that

$$\begin{aligned}
&\frac{1}{n^2}\sum_{x\in\Lambda(n)} \mu^{+,\beta}_{\Lambda(n)}[\sigma(x)] \le 2(\beta_c-\beta)^\xi + c_2 \exp\left(-\frac{\lambda_2}{8}(\beta_c-\beta)^{\xi+1} n\right)\\
\le&(\beta_c-\beta)^\xi\left[2 + c_2\exp\left(\log\frac{1}{\beta_c-\beta}\Big(\xi - \frac{\lambda_2}{8}\frac{(\beta_c-\beta)^{\xi+1}}{\log\frac{1}{\beta_c-\beta}} n\Big)\right)\right],
\end{aligned}$$

Finally, note that there exists a positive $\varepsilon = \varepsilon(\xi, a)$ such that for all $\beta_c - \varepsilon \le \beta \le \beta_c$

$$\frac{(\beta_c-\beta)^{\xi+1}}{\log\frac{1}{\beta_c-\beta}} > (\beta_c-\beta)^a.$$

This implies that

$$\begin{aligned}
&\frac{1}{n^2}\sum_{x\in\Lambda(n)} \mu^{+,\beta}_{\Lambda(n)}[\sigma(x)]\\
\le&(\beta_c-\beta)^\xi\left[2 + c_2\exp\left(\log\frac{1}{\beta_c-\beta}\Big(\xi - \frac{\lambda_2}{8}\frac{(\beta_c-\beta)^{\xi+1}}{\log\frac{1}{\beta_c-\beta}} n\Big)\right)\right]\\
\le&(\beta_c-\beta)^\xi\left[2 + c_2(\beta_c-\beta)^{-\xi+\frac{\lambda_2}{8}(\beta_c-\beta)^a n}\right],
\end{aligned}$$

Thus, if we impose that

$$n > \frac{16\xi}{\lambda_2}(\beta_c-\beta)^{-a},$$

we get that

$$\frac{1}{n^2}\sum_{x\in\Lambda(n)} \mu^{+,\beta}_{\Lambda(n)}[\sigma(x)] \le (\beta_c-\beta)^\xi[2 + c_2(\beta_c-\beta)^\xi].$$

Thus, there exists a positive $c = c(\xi, a)$ such that for all $\beta_c - \varepsilon \le \beta \le \beta_c$ and for all $n > c(\beta_c-\beta)^{-a}$ we have that

$$\frac{1}{n^2}\sum_{x\in\Lambda(n)} \mu^{+,\beta}_{\Lambda(n)}[\sigma(x)] \le \rho'(\beta_c-\beta)^\xi, \tag{49}$$

where ρ' is a positive constant. When $\beta < \beta_c - \varepsilon$, (49) also holds, provided ρ' is replaced by $\rho = \rho' \vee \varepsilon^{-\xi}$.
In order to treat the case where $\beta > \beta_c$, one proceeds in the same way. In this situation (47) is replaced by the following bound that can be obtained from the results of [31]: there exist positive constants λ_3, c_3 and δ such that for all $x, y \in \mathbb{Z}^2$ satisfying $|x-y|(\beta-\beta_c) > 1/\delta$ we have that

$$\mu^{+,\beta}_\infty[\sigma(x)\sigma(y)] - m^*(\beta)^2 \le c_3\exp(-\lambda_3(\beta-\beta_c)|x-y|)\,.$$

□

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