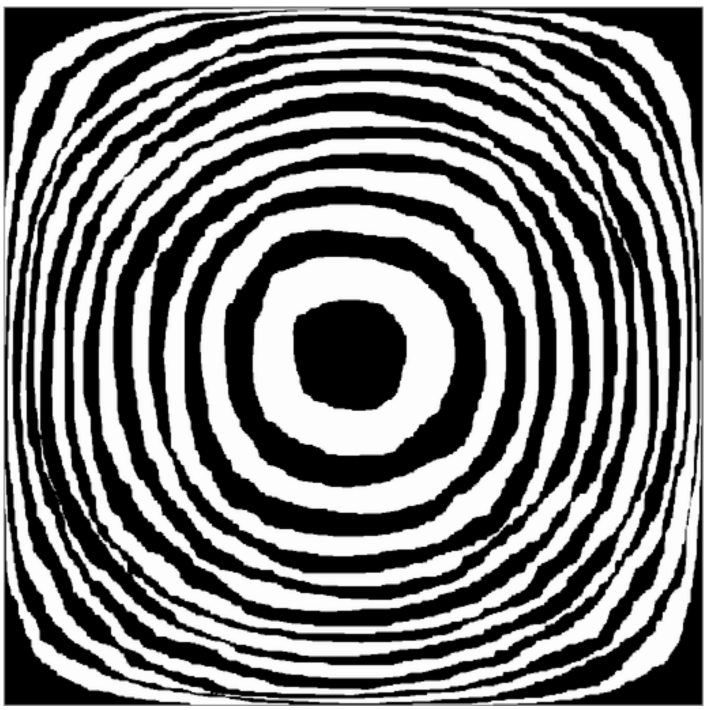
The initial drift of a 2D droplet at zero temperature

Raphaël Cerf * and Sana Louhichi †

Abstract

We consider the 2D stochastic Ising model evolving according to the Glauber dynamics at zero temperature. We compute the initial drift for droplets which are discretizations of smooth domains. A specific spatial average of the derivative at time 0 of the volume variation of a droplet close to a boundary point is equal to its curvature multiplied by a direction dependent coefficient. For a boundary point having a tangent with angle θ, this coefficient is equal to $-\frac{1}{2}|\cos 2\theta|$.



Key words. 2D Ising model, Glauber dynamics, zero temperature, Markov process, mean curvature, velocity.

Mathematics subject Classification 2000. 60K35, 82C22

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Introduction

The phenomenological theory asserts that the evolution of the shape of a droplet of one phase immersed in another phase is governed by the motion by mean curvature. We are still far from being able to verify this assertion starting from a genuine microscopic dynamics. Very interesting results have been obtained in a series of works in the context of the Ising model with Kać potentials [3, 4, 5, 6]. However, motion by mean curvature is recovered in some scaling limit where the range of the interactions diverges to infinity: the model becomes somehow close to a mean–field model and the ensuing motion is isotropic. For the true Ising model with only nearest–neighbour interactions, it is expected that an interface between the minus and the plus phase evolves according to an anisotropic motion by mean curvature, that is, each point x of the interface has velocity

$$v(x) = -c(\nu(x))\,\xi\,\nu(x)$$

where $\nu(x)$ is the vector normal to the interface at x, ξ is the curvature of the interface at x and $c(\nu)$ is a coefficient depending on the direction of ν. This anisotropy stems from the anisotropy of the cubic lattice.

In this paper, we consider the zero temperature Glauber dynamics for the 2D Ising model. Although we do not succeed in deriving the full motion by mean curvature, we manage to compute the initial drift for droplets which are discretizations of smooth domains and we believe this is a crucial step. Four works are directly relevant. In [8], Spohn claims to establish rigorously the mean curvature motion in the context of the 2D Ising model at zero temperature for interfaces which can be represented as the graph of a function. Although his results do not apply directly to the case of a droplet, we note that his formula for the direction dependent coefficient does not seem to agree with ours. We have not been able so far to explain the reason of this disagreement. The computation we present here can be considered to be a refinement of the observation of [1]. Chayes, Schonmann and Swindle proved a Lifshitz law for the volume of a two-dimensional droplet at zero temperature. Instead of looking at the total volume of the droplet, we shall concentrate here on the volume variation of the droplet in a small ball attached to its boundary. In [2], by interpreting the interface as a one dimensional exclusion process, Chayes and Swindle manage to prove that, starting from a square droplet, the evolution of the shape of one corner is described in the hydrodynamical limit by an appropriate Stefan problem. Finally, Sowers develops in [7] a framework of geometric measure theory to obtain the hydrodynamical limit. His convergence theorem is conditional on the verification of several assumptions, some of them concerning the structure of the interface. It might be that these estimates are the missing pieces to complete the picture.

Let us turn now to the description of our result. We work with the stochastic Ising model evolving according to the Glauber dynamics at zero temperature. We consider the diffusive limit where space is rescaled by a factor N and time is speeded up by a factor N^2. We start with a plus droplet whose boundary is a $\mathcal{C}_1$ simple Jordan curve γ: the initial configuration at step N is the discretization of the smooth droplet, consisting of the squares of the lattice $\mathbb{Z}^2/N$ which intersect the interior of γ. The droplet is immersed in the minus phase, hence all the squares of the discretization are initially set to plus, while the other squares of the lattice are set to minus. We then look at the process $(\sigma_{N^2t}, t \geq 0)$ and we denote by $\mathcal{A}^N(t)$ the plus droplet at time N^2t. Let x be a point of γ. We study the variation of the magnetization inside the ball $B(x,r)$ centered at x with radius r, for r small. Equivalently, we look at the volume $\mathrm{vol}(\mathrm{B}(\mathrm{x},\mathrm{r}) \cap \mathcal{A}^{\mathrm{N}}(\mathrm{t}))$ of the plus droplet in this ball and we aim at computing its derivative

$$\lim_{t\to 0} \frac{1}{t}\Big(\mathrm{vol}(\mathrm{B}(\mathrm{x},\mathrm{r}) \cap \mathcal{A}^{\mathrm{N}}(\mathrm{t})) - \mathrm{vol}(\mathrm{B}(\mathrm{x},\mathrm{r}) \cap \mathcal{A}^{\mathrm{N}}(0)\Big).$$

Several problems arise. Since the dynamics proceeds by jumps, we have to take the expectation to get a differentiable quantity. Next we wish to link the infinitesimal volume variation with the curvature of the droplet's boundary at x. To achieve this, we need to recover approximately the slope of the continuous curve from its discretized version. We perform a spatial averaging. Letting x_0, x_1 be the two points of γ which belong to the sphere $\partial B(x,r)$, we consider the domain

$$\mathcal{S}(x,r,\alpha_1,\alpha_2) = B(x,r) \cup B(x_0,\alpha_1) \cup B(x_1,\alpha_2),$$

and we denote by $\mathcal{S}_N$ its discretization at step N. The quantity of primary interest to link the volume variation and the curvature is

$$\mathbf{A}_N^\gamma(x,r,\delta) = \frac{1}{\delta^2}\int_0^\delta\int_0^\delta \lim_{t\to 0}\frac{1}{t}\Big(\mathbb{E}(\mathrm{vol}(\mathcal{A}^N(t)\cap\mathcal{S}_N)) - \mathrm{vol}(\mathcal{A}^N(0)\cap\mathcal{S}_N)\Big)\, d\alpha_1\, d\alpha_2.$$

Our main result states that

$$\lim_{r\to 0}\lim_{\delta\to 0}\liminf_{N\to\infty}\frac{1}{2r}\mathbf{A}_N^\gamma(x,r,\delta) = \lim_{r\to 0}\lim_{\delta\to 0}\limsup_{N\to\infty}\frac{1}{2r}\mathbf{A}_N^\gamma(x,r,\delta) = -\frac{1}{2}|\cos(2\theta)|\xi_\gamma(x)$$

where θ is the angle of the tangent to γ at x and $\xi_\gamma(x)$ is the curvature of γ at x. This indicates that the limit $(\mathcal{A}(t), t\geq 0)$ of any decently converging subsequence of the stochastic motion $(\mathcal{A}^N(t), t \geq 0)$ should satisfy the equation, for any $s>0$ and for any $x \in \partial\mathcal{A}(s)$

$$\lim_{r\to 0}\lim_{\delta\to 0}\frac{1}{2r\delta^2}\int_0^\delta\int_0^\delta \lim_{t\to s}\frac{1}{t-s}\left(\mathbb{E}(\mathrm{vol}(\mathcal{A}(t)\cap\mathcal{S})) - \mathrm{vol}(\mathcal{A}(s)\cap\mathcal{S})\right) d\alpha_1\, d\alpha_2 = -\frac{1}{2}|\cos(2\theta)|\,\xi_\gamma(x)$$

or at least a weaker variant of it. Here $(\mathcal{A}(t), t\geq 0)$ is a random process describing the evolution of the shape of the droplet. A standard computation shows that the deterministic motion by mean curvature satisfies this equation. However we do not know whether it is the only solution to this equation; we have not investigated the corresponding theory so far. For instance, can one get rid of the expectation? Anyway, we are still far from establishing that the hydrodynamical limit of the droplet process satisfies the above equation. An important further step would be to deal with more relaxed initial configurations, namely, to start with a droplet which is close to a continuous droplet either in the volume sense or in the sense of the Hausdorff metric, instead of being exactly its discretization. The main obstacle is to control dynamically the proportion of the corners in a microscopic random interface when its average slope is known. This would probably require some additional probabilistic input.

1 The model

We consider a zero-temperature 2D-stochastic Ising model. More precisely it is a continuous time Markov process $(\sigma_t)_{t\geq 0}$ taking values in $\{-1,+1\}^{\mathbb{Z}^2}$ with generator L which acts on each local function $f:\{-1,+1\}^{\mathbb{Z}^2}\to\mathbb{R}$ as

$$(Lf)(\sigma)=\sum_{x\in\mathbb{Z}^2}c(x,\sigma)(f(\sigma^x)-f(\sigma)).$$

Here

$$\sigma^x(y)=\begin{cases}\sigma(y) & \text{if } y\neq x,\\ -\sigma(y) & \text{if } y=x,\end{cases}$$

and $c(x,\sigma)$ is the rate with which the spin at site x flips when the configuration is σ. The rates $c(x,\sigma)$ define the dynamics. For the zero-temperature Ising model, the rates $c(x,\sigma)$ are given by

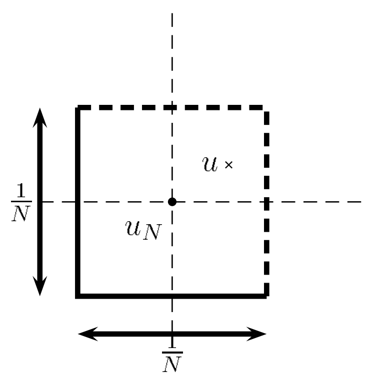
$$c(x,\sigma)=\begin{cases}1 & \text{if more than 2 neighbors of } x \text{ have a spin opposite to } x;\\ \alpha & \text{if exactly 2 neighbors of } x \text{ have a spin opposite to } x;\\ 0 & \text{otherwise,}\end{cases}$$

where $0<\alpha\leq 1$ is a fixed parameter. For technical reasons, we will take $\alpha=\frac{1}{2}$ in the sequel.

Initial condition. Let γ be a Jordan curve of $\mathbb{R}^2$. Suppose that γ encloses a connected, compact and bounded set Ω of $\mathbb{R}^2$ i.e. $\gamma=\partial\Omega$. Let N be a fixed positive integer. We define the spin configuration, $\sigma_0:=\sigma_{0,N}$, at time 0 as :

$$\forall\, x\in\mathbb{Z}^2,\quad \sigma_0(x)=\begin{cases}+1 & \text{if } \Lambda_{x/N}\cap\Omega\neq\emptyset\\ -1 & \text{otherwise,}\end{cases}$$

where, for all $x=(x_1,x_2)\in\mathbb{Z}^2$ and $N\in\mathbb{N}$, $\Lambda_{x/N}$ is the box defined as

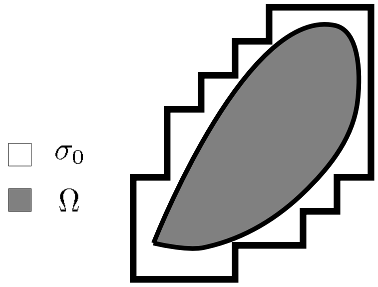


A point u and the box Λ_{u_N}.

$$\Lambda_{x/N}=\left\{(u_1,u_2)\in\mathbb{R}^2,\quad -\frac{1}{2N}\leq u_1-\frac{x_1}{N}<\frac{1}{2N};\ -\frac{1}{2N}\leq u_2-\frac{x_2}{N}<\frac{1}{2N}\right\}. \qquad (1)$$

We will say that σ_0 is the spin configuration associated to the curve γ at step N.

Having both the initial condition and the generator, the Markov process $(\sigma_t)_{t\geq 0}$ at step N is well defined.



The curve $\gamma = \partial\Omega$ and the configuration σ_0.

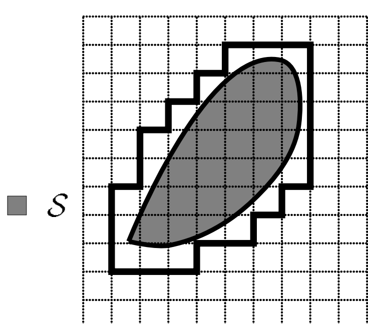
2 Notation

Let N be a fixed positive integer. We denote by $\mathbb{Z}_N^2$ the grid $\frac{\mathbb{Z}^2}{N}$. The family of boxes $(\Lambda_x,\, x \in \mathbb{Z}_N^2)$, as defined by (1), forms a partition of $\mathbb{R}^2$:

$$\mathbb{R}^2 = \bigcup_{x\in\mathbb{Z}_N^2} \Lambda_x, \qquad \forall\, x, y \in \mathbb{Z}_N^2\,, x \neq y\,, \Lambda_x \cap \Lambda_y = \emptyset.$$

Hence, for each $u = (u_1, u_2) \in \mathbb{R}^2$ there exists a unique $u_N \in \mathbb{Z}_N^2$ such that $u \in \Lambda_{u_N}$. Moreover $\|u - u_N\|_\infty \leq \frac{1}{2N}$, where $\|u\|_\infty = \max(|u_1|, |u_2|)$.
To each bounded set $\mathcal{S}$ of $\mathbb{R}^2$, we associate the set $\mathcal{S}_N$ defined by

$$\mathcal{S}_N = \bigcup_{x\in\mathbb{Z}_N^2 : \Lambda_x \cap \mathcal{S} \neq \emptyset} \Lambda_x.$$



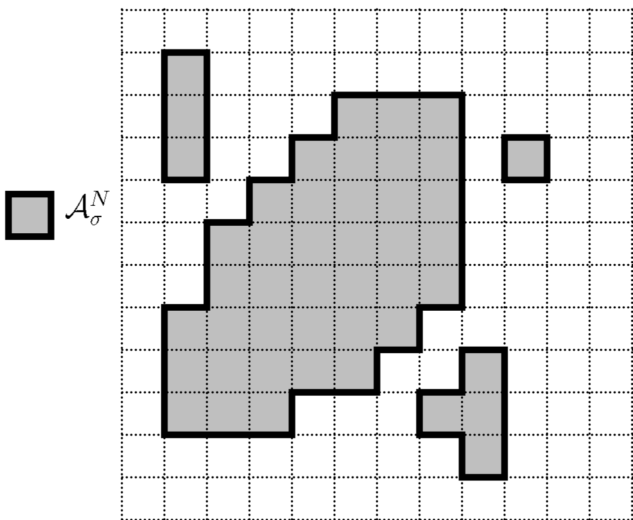
The set $\mathcal{S}$ is included in the set $\mathcal{S}_N$ with polygonal boundary.

For $\sigma \in \{-1,+1\}^{\mathbb{Z}^2}$ and for $x \in \mathbb{Z}^2$, we denote by $s(\sigma, x)$, the number of the neighbors of x having a spin opposite to x in the configuration σ:

$$s(\sigma, x) = \frac{1}{2} \sum_{y \in \mathbb{Z}^2 : |x-y|=1} |\sigma(x) - \sigma(y)|,$$

where $|x| = \sqrt{x_1^2 + x_2^2}$ for $x = (x_1, x_2)$.
Let N be a positive fixed integer, we define the set

$$\mathcal{A}_\sigma^N = \bigcup_{x \in \mathbb{Z}^2,\ \sigma(x)=+1} \Lambda_{x/N}.$$



For all $x \in \mathbb{Z}^2$, $\sigma(x) = +1$ if and only if $x \in N\mathcal{A}_\sigma^N$.

Let us note that, when σ is the spin configuration associated to a continuous Jordan curve γ at step N, then the corresponding set $\mathcal{A}_\sigma^N$ is connected; that means that $\mathbb{I}_{s(\sigma,x)=4} = 0$.

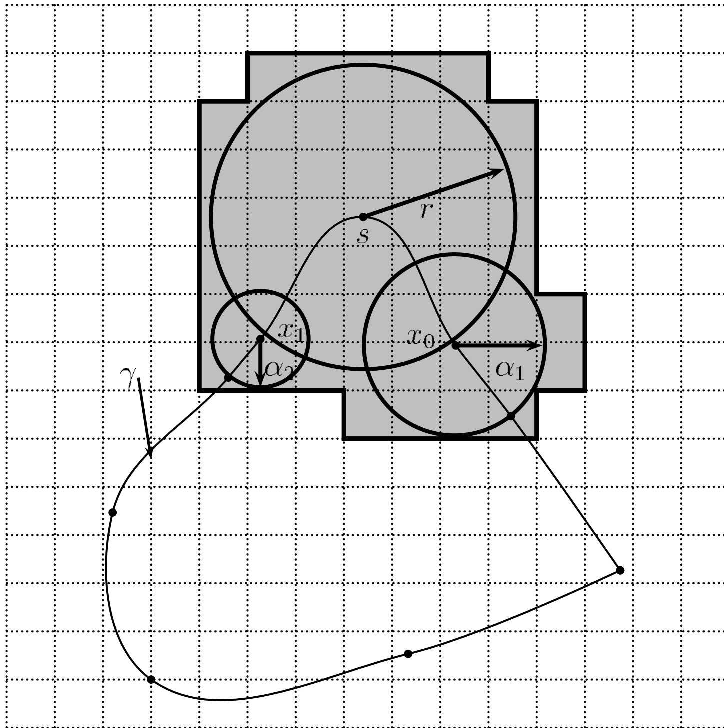
Let r, α_1, α_2 be positive real numbers. We define for $s \in \gamma$, the set

$$\mathcal{S}(s, r, \alpha_1, \alpha_2) = B(s, r) \cup B(x_0, \alpha_1) \cup B(x_1, \alpha_2),$$

where $B(s, r)$ is the closed ball centered at s with radius r chosen sufficiently small, so that $\partial B(s, r) \cap \gamma$ contains exactly 2 points x_0 and x_1.
Let

$$L_N^\gamma(s, r, \alpha_1, \alpha_2) = \lim_{t \to 0} \frac{1}{t} \left(\mathbb{E}_{\sigma_0}(\text{vol}(\mathcal{A}_{\sigma_{tN^2}}^N \cap \mathcal{S}_N)) - \text{vol}(\mathcal{A}_{\sigma_0}^N \cap \mathcal{S}_N) \right),$$

where $\mathcal{S}_N = (\mathcal{S}(s, r, \alpha_1, \alpha_2))_N = (B(s, r) \cup B(x_0, \alpha_1) \cup B(x_1, \alpha_2))_N$ and vol denotes the planar Lebesgue measure.



The set $(B(s,r) \cup B(x_0,\alpha_1) \cup B(x_1,\alpha_2))_N$.

Finally, we define the average

$$\mathbf{A}_N^\gamma(s,r,\delta) = \frac{1}{\delta^2}\int_0^\delta \int_0^\delta L_N^\gamma(s,r,\alpha_1,\alpha_2)\, d\alpha_1\, d\alpha_2.$$

3 Main result

Theorem 1 *Let γ be a Jordan curve of $\mathbb{R}^2$ of class $\mathcal{C}_2$. Suppose that γ encloses a connected, compact and bounded set Ω of $\mathbb{R}^2$ i.e. $\gamma = \partial\Omega$. Let $s \in \gamma$ be fixed. Then,*

$$\lim_{r\to 0}\lim_{\delta\to 0}\liminf_{N\to+\infty}\frac{1}{2r}\mathbf{A}_N^\gamma(s,r,\delta) = \lim_{r\to 0}\lim_{\delta\to 0}\limsup_{N\to+\infty}\frac{1}{2r}\mathbf{A}_N^\gamma(s,r,\delta) = -\frac{1}{2}\left|\cos 2\theta\right|\xi_\gamma(s),$$

where $\xi_\gamma(s)$ is the curvature of γ at the point s and θ is the angle between the horizontal axis and the tangent to the curve γ at the point s.

4 Proofs

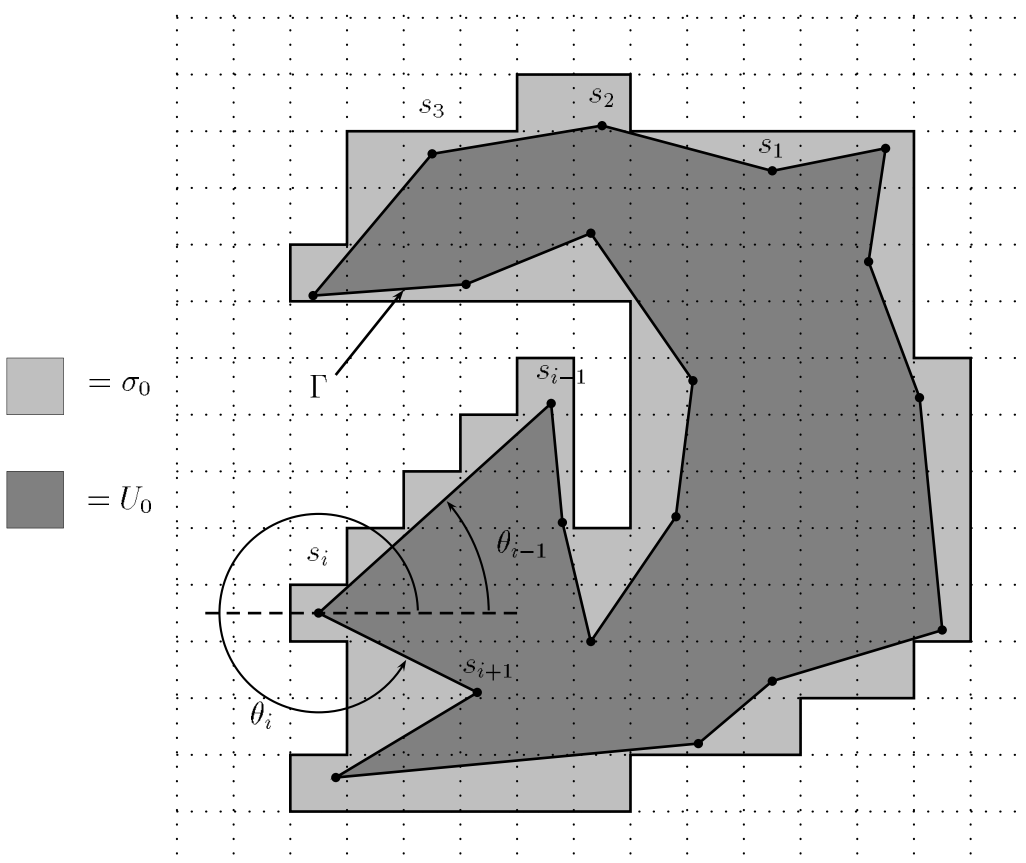
We will first evaluate the average $\mathbf{A}_N^\gamma(s,r,\delta)$, when γ is a suitable polygon in $\mathbb{R}^2$ and s is any corner point of γ. Next we will check that the proofs for polygons can be extended with little efforts to any Jordan curve γ of class $\mathcal{C}_1$. The end of the proof of theorem 1 is given in the section 4.3.

4.1 Proofs for polygons

We introduce a class of regular polygons as follows.

m-smooth polygons. Let $s_1, \dots, s_m$ be m points of $\mathbb{R}^2$. We denote by $\Gamma(s_1, \dots, s_m)$ or by Γ, if there is no ambiguities, the polygon in $\mathbb{R}^2$ linking the points $[s_1, s_2, \dots, s_m, s_1]$; the points $s_1, s_2, \dots, s_m$ are then the corner points of Γ. We suppose that the points $s_1, s_2, \dots, s_m$ are arranged counterclockwise. By convention, we set $s_0 = s_m$. To each site s_i, we associate two oriented angles $\theta_i(s_i)$ and $\theta_{i-1}(s_i)$ such that $\theta_{i-1}(s_i)$ (respectively $\theta_i(s_i)$) is the oriented angle between the half horizontal axis $[0, +\infty[$ and the segment $[s_i, s_{i-1}[$ (respectively $[s_i, s_{i+1}[$). Finally, we suppose that Γ encloses a connected, compact, bounded set U of $\mathbb{R}^2$ i.e. $\Gamma = \partial U$ and that $\Gamma \cap \mathbb{Z}^2/N = \emptyset$ for all $N \geq 1$.

Initial condition. We will consider σ_0 the spin configuration associated to the polygon Γ at step N.



A polygon Γ and the configuration σ_0

The purpose of the following proposition is the evaluation of the average $\mathbf{A}_N^\Gamma(s_i, r, \delta)$.

Proposition 1 *Let Γ be an m-smooth polygon in $\mathbb{R}^2$ associated to the m points $s_1, \ldots, s_m$ and let $\sigma_0 := \sigma_{0,N}$ be the associated initial configuration at step N. Let $\theta_i \in [0, 2\pi]$ (respectively $\theta_{i-1} \in [0, 2\pi]$) be the oriented angle between the half horizontal axis $[0, +\infty[$ and the segment $[s_i, s_{i+1}[$ (respectively $[s_i, s_{i-1}[$) with the convention that $s_0 = s_m$. Then, for each $i = 1, \ldots, m$, and for any positive real numbers r, δ small enough, one has*

$$\lim_{N\to+\infty} \mathbf{A}_N^\Gamma(s_i, r, \delta) = \frac{1}{4}\sin 2\theta_{i-1}\left(2\mathbb{1}_{|\sin\theta_{i-1}|<|\cos\theta_{i-1}|} - 1\right) - \frac{1}{4}\sin 2\theta_i\left(2\mathbb{1}_{|\sin\theta_i|<|\cos\theta_i|} - 1\right)$$

$$+\frac{1}{2}\left(\mathrm{sgn}(\tan\theta_{\mathrm{i}})\mathbb{1}_{|\sin\theta_{\mathrm{i}}|<|\cos\theta_{\mathrm{i}}|} - \mathrm{sgn}(\tan\theta_{\mathrm{i-1}})\mathbb{1}_{|\sin\theta_{\mathrm{i-1}}|<|\cos\theta_{\mathrm{i-1}}|}\right) + \mathrm{sgn}(\theta_{\mathrm{i}} - \theta_{\mathrm{i-1}})\mathbb{1}_{\sin\theta_{\mathrm{i-1}}\sin\theta_{\mathrm{i}}>0}.$$

Hence,

- *if* $(\theta_{i-1}, \theta_i) \in [(2k+1)\frac{\pi}{4}, (2k+3)\frac{\pi}{4}] \times [(2k+5)\frac{\pi}{4}, (2k+7)\frac{\pi}{4}]$, *with* $k \in \{0, 2\}$, *then*

$$\lim_{N\to+\infty} \mathbf{A}_N^\Gamma(s_i, r, \delta) = \frac{1}{4}\left(\sin 2\theta_i - \sin 2\theta_{i-1}\right).$$

- *if* $(\theta_{i-1}, \theta_i) \in [(2k+1)\frac{\pi}{4}, (2k+3)\frac{\pi}{4}] \times [(2k+5)\frac{\pi}{4}, (2k+7)\frac{\pi}{4}]$, *with* $k \in \{1, 3\}$, *then*

$$\lim_{N\to+\infty} \mathbf{A}_N^\Gamma(s_i, r, \delta) = \frac{1}{4}\left(\sin 2\theta_{i-1} - \sin 2\theta_i\right).$$

- *if* $(\theta_{i-1}, \theta_i) \in [(2k+1)\frac{\pi}{4}, (2k+3)\frac{\pi}{4}]^2$, *with* $k \in \{0, 2\}$, *then*

$$\lim_{N\to+\infty} \mathbf{A}_N^\Gamma(s_i, r, \delta) = \frac{1}{4}\left(4\,\mathrm{sgn}(\theta_{\mathrm{i}} - \theta_{\mathrm{i-1}}) + \sin 2\theta_{\mathrm{i}} - \sin 2\theta_{\mathrm{i-1}}\right).$$

- *if* $(\theta_{i-1}, \theta_i) \in [(2k+1)\frac{\pi}{4}, (2k+3)\frac{\pi}{4}]^2$, *with* $k \in \{1, 3\}$, *then*

$$\lim_{N\to+\infty} \mathbf{A}_N^\Gamma(s_i, r, \delta) = \frac{1}{4}\left(4\,\mathrm{sgn}(\sin(\theta_{\mathrm{i}} - \theta_{\mathrm{i-1}})) + \sin 2\theta_{\mathrm{i-1}} - \sin 2\theta_{\mathrm{i}}\right).$$

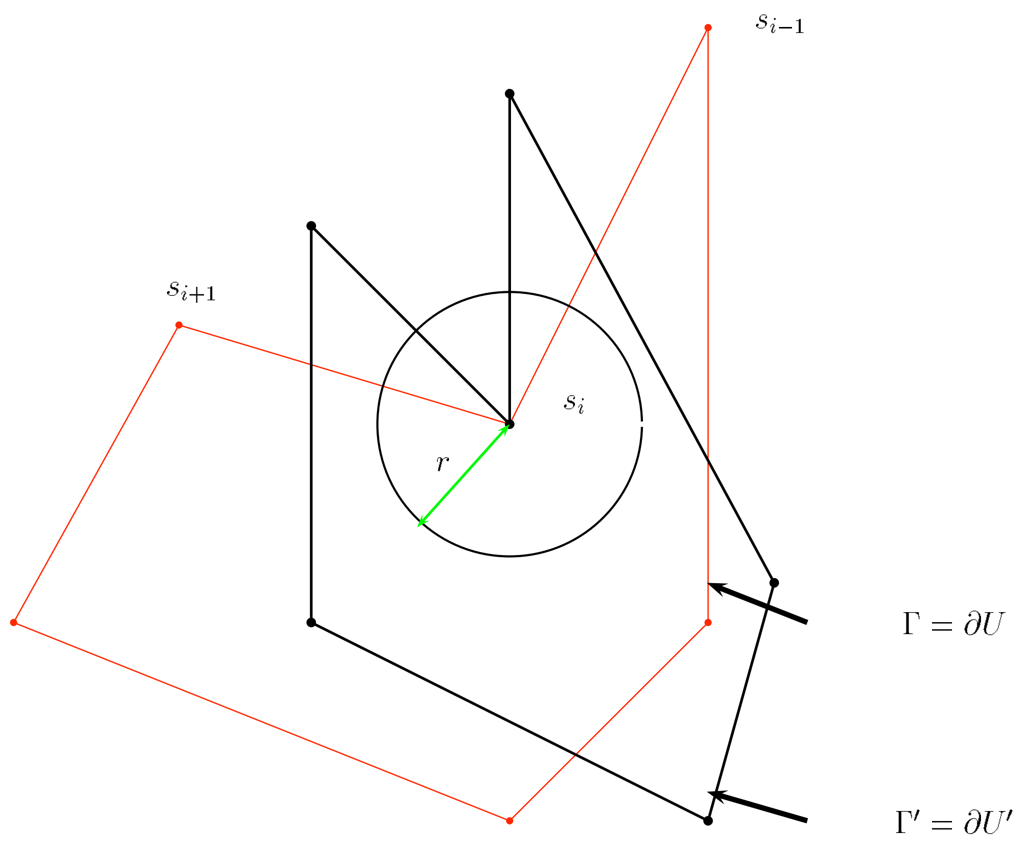
- *if* $(\theta_{i-1}, \theta_i) \in [(2k+1)\frac{\pi}{4}, (2k+3)\frac{\pi}{4}] \times [(2k+3)\frac{\pi}{4}, (2k+5)\frac{\pi}{4}] \cup [(2k+3)\frac{\pi}{4}, (2k+5)\frac{\pi}{4}] \times [(2k+1)\frac{\pi}{4}, (2k+3)\frac{\pi}{4}]$, *with* $k \in \{0, 2\}$, *then*

$$\lim_{N\to+\infty} \mathbf{A}_N^\Gamma(s_i, r, \delta) = \begin{cases} \frac{1}{4}\left(2 - \sin 2\theta_i - \sin 2\theta_{i-1}\right) & if\ |\tan\theta_i| \le 1, \quad |\tan\theta_{i-1}| \ge 1 \\ \frac{1}{4}\left(-2 + \sin 2\theta_{i-1} + \sin 2\theta_i\right) & if\ |\tan\theta_i| \ge 1, \quad |\tan\theta_{i-1}| \le 1. \end{cases}$$

- *if* $(\theta_{i-1},\theta_i) \in [(2k+1)\frac{\pi}{4},(2k+3)\frac{\pi}{4}] \times [(2k+3)\frac{\pi}{4},(2k+5)\frac{\pi}{4}] \cup [(2k+3)\frac{\pi}{4},(2k+5)\frac{\pi}{4}] \times [(2k+1)\frac{\pi}{4},(2k+3)\frac{\pi}{4}]$, *with* $k \in \{1,3\}$, *then*

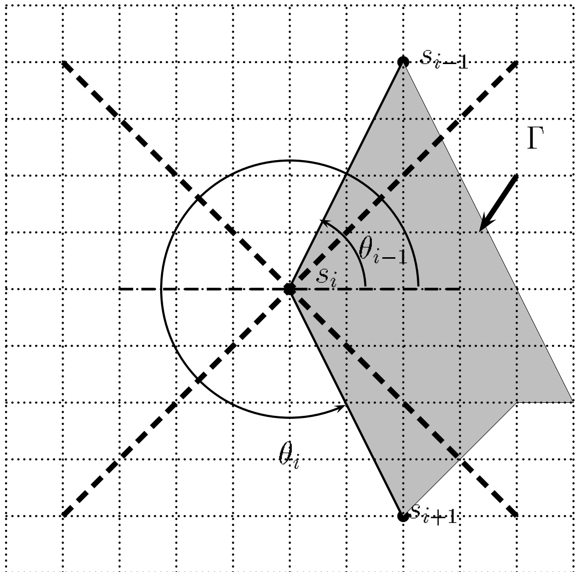
$$\lim_{N\to+\infty} \mathbf{A}_N^\Gamma(s_i,r,\delta) = \begin{cases} \frac{1}{4}\left(-2-\sin 2\theta_i - \sin 2\theta_{i-1}\right) & if \ |\tan\theta_i| \leq 1, \quad |\tan\theta_{i-1}| \geq 1 \\ \\ \frac{1}{4}\left(2+\sin 2\theta_{i-1} + \sin 2\theta_i\right) & if \ |\tan\theta_i| \geq 1, \quad |\tan\theta_{i-1}| \leq 1. \end{cases}$$

Remark. We denote by $L_\Gamma(s_i) = \lim_{N\to+\infty} \mathbf{A}_N^\Gamma(s_i,r,\delta)$, where Γ is a polygon as described by proposition 1. Then we can check the following comparison criterion.



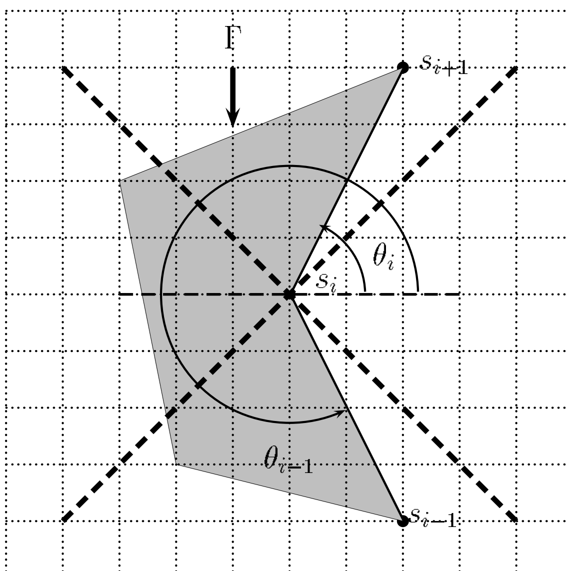
If $U \cap B(s_i,r) \subset U' \cap B(s_i,r)$ for some $r>0$ and $s_i \in \Gamma\cap\Gamma'$, then $L_\Gamma(s_i) \leq L_{\Gamma'}(s_i)$.

We illustrate the results of proposition 1 with the help of the following pictures.

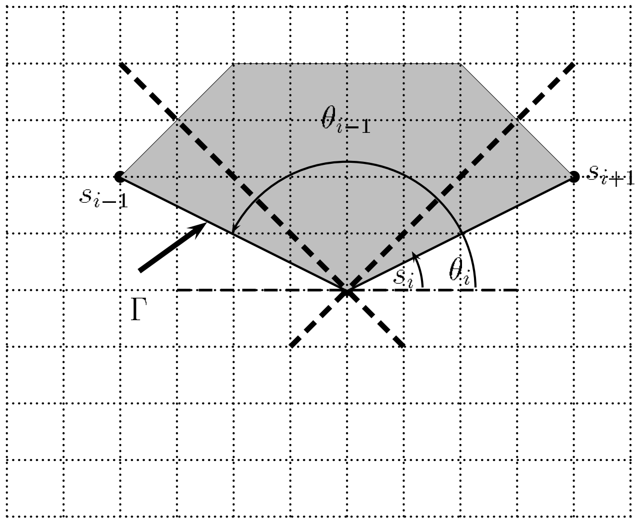


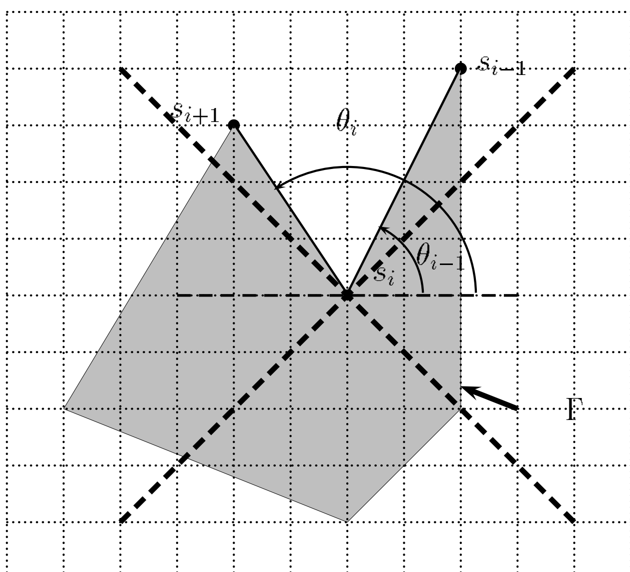
Here $\lim_{N\to+\infty} \mathbf{A}_N^\Gamma(s_i, r, \delta) = \frac{1}{4}\left(\sin 2\theta_i - \sin 2\theta_{i-1}\right).$

In the first picture this limit is negative, while for the second one it is positive.



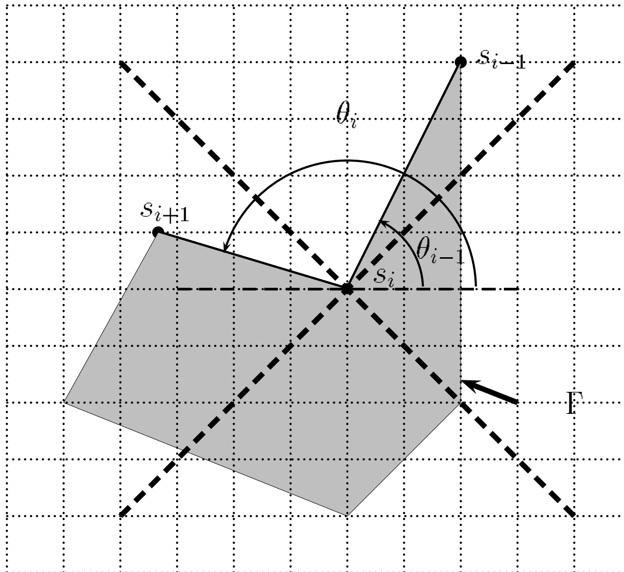
In the following picture, we have $\lim_{N\to+\infty}\mathbf{A}_N^\Gamma(s_i,r,\delta)=\frac{1}{4}\left(\sin 2\theta_{i-1}-\sin 2\theta_i\right).$
This limit is negative.

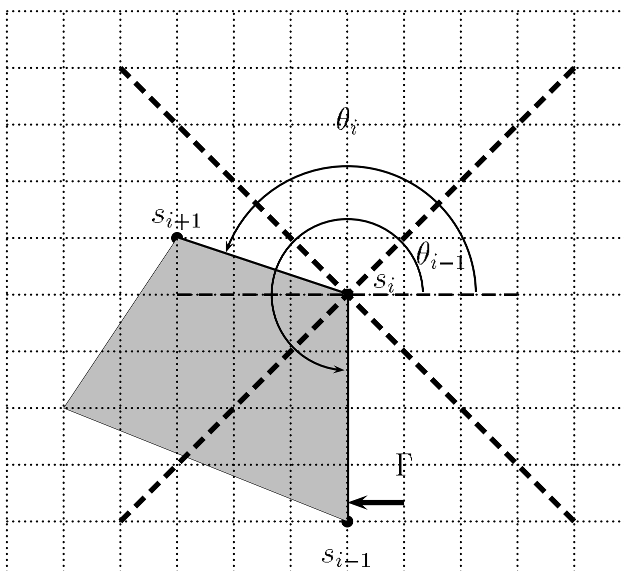




Here $\lim_{N\to+\infty}\mathbf{A}_N^\Gamma(s_i,r,\delta)=\frac{1}{4}\left(4+\sin 2\theta_i-\sin 2\theta_{i-1}\right).$
This limit is positive.

In the following picture, we have $\lim_{N\to+\infty} \mathbf{A}_N^\Gamma(s_i, r, \delta) = \frac{1}{4}\left(2 - \sin 2\theta_{i-1} - \sin 2\theta_i\right).$
This limit is positive.





Here $\lim_{N\to+\infty} \mathbf{A}_N^\Gamma(s_i, r, \delta) = \frac{1}{4}\left(-2 - \sin 2\theta_i - \sin 2\theta_{i-1}\right).$
This limit is negative.

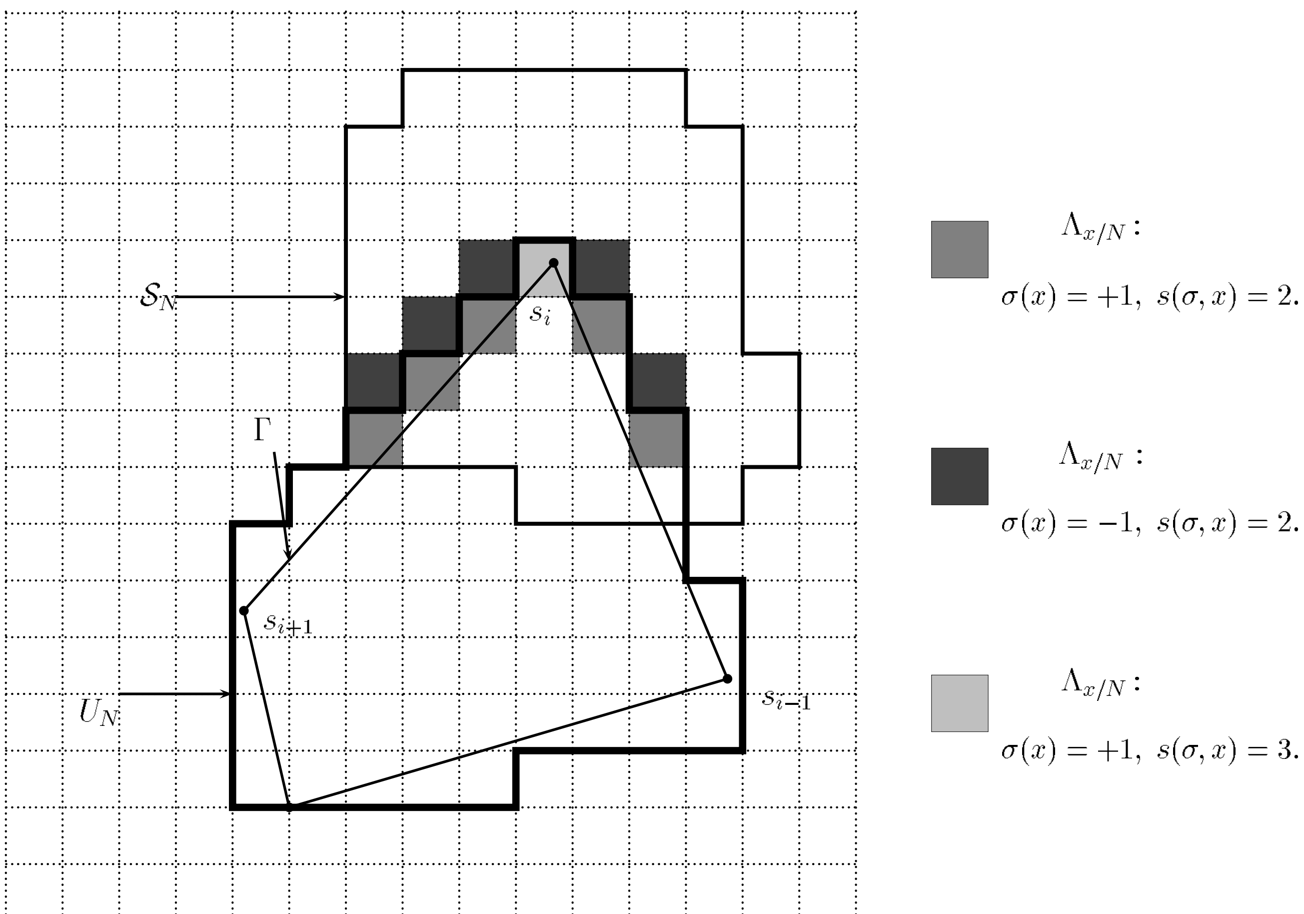
4.1.1 Proof of proposition 1.

We need the following preliminary lemma.

Lemma 1 *Let $\mathcal{S}$ be a compact set of $\mathbb{R}^2$. Let $\sigma \in \{-1,+1\}^{\mathbb{Z}^2}$ be fixed. Then*

$$\lim_{t\to 0} \frac{1}{t}\left(\mathbb{E}_\sigma(\mathrm{vol}(\mathcal{A}^N_{\sigma_{tN^2}} \cap \mathcal{S}_N)) - \mathrm{vol}(\mathcal{A}^N_\sigma \cap \mathcal{S}_N)\right) =$$

$$\sum_{x\in\mathbb{Z}^2:\Lambda_{\frac{x}{N}}\subset\mathcal{S}_N}\left(\mathbb{1}_{\sigma(x)=-1,\ s(\sigma,x)\geq 3} - \mathbb{1}_{\sigma(x)=+1,\ s(\sigma,x)\geq 3}\right) + \alpha \sum_{x\in\mathbb{Z}^2:\Lambda_{\frac{x}{N}}\subset\mathcal{S}_N}\left(\mathbb{1}_{\sigma(x)=-1,\ s(\sigma,x)=2} - \mathbb{1}_{\sigma(x)=+1,\ s(\sigma,x)=2}\right).$$



In this picture, we suppose that $\mathcal{A}^N_\sigma = U_N$, and $\Gamma(s_1,\ldots,s_m) = \partial U$.

Proof of lemma 1. Let $f_N(\sigma) = \mathrm{vol}(\mathcal{A}_\sigma^N \cap \mathcal{S}_N)$ and $S(t)f_N(\sigma) = \mathbb{E}_\sigma(\mathrm{vol}(\mathcal{A}_{\sigma_t}^N \cap \mathcal{S}_N))$. We deduce from

$$\lim_{t\to 0} \frac{1}{t}\left(S(t)f_N - f_N\right) = Lf_N,$$

that

$$\lim_{t\to 0} \frac{1}{t}\left(S(t)f_N(\sigma) - f_N(\sigma)\right) = \sum_{x\in\mathbb{Z}^2} c(x,\sigma)(f_N(\sigma^x) - f_N(\sigma)). \tag{2}$$

Now,

$$f_N(\sigma^x) - f_N(\sigma) = \frac{1}{N^2}\mathbb{1}_{\Lambda_{\frac{x}{N}}\subset\mathcal{S}_N}\left(\mathbb{1}_{\sigma(x)=-1} - \mathbb{1}_{\sigma(x)=1}\right),$$

this fact together with (2) gives

$$\lim_{t\to 0} \frac{1}{t}\left(S(tN^2)f_N(\sigma) - f_N(\sigma)\right) = \sum_{x\in\mathbb{Z}^2:\,\Lambda_{\frac{x}{N}}\subset\mathcal{S}_N} c(x,\sigma)\left(\mathbb{1}_{\sigma(x)=-1} - \mathbb{1}_{\sigma(x)=1}\right),$$

which proves lemma 1 since $c(x,\sigma) = \mathbb{1}_{s(\sigma,x)\geq 3} + \alpha\mathbb{1}_{s(\sigma,x)=2}$. □

4.1.2 Evaluation of $L_N(s_i, r, \alpha_1, \alpha_2)$

Without loss of generality, we prove the statements of proposition 1 for the site s_1 instead of s_i. Throughout this step, we consider the set

$$\mathcal{S}_N = \left(\mathcal{S}(s_1, r, \alpha_1, \alpha_2)\right)_N = \left(B(s_1, r) \cup B(x_0, \alpha_1) \cup B(x_1, \alpha_2)\right)_N, \tag{3}$$

where α_1, α_2 are positive real numbers less than δ, the positive real numbers r and δ are such that $r + \delta$ fulfills

$$r + \delta < \min_{j,k\neq i} \inf_{x\in[s_j,s_k]} |s_i - x| \tag{4}$$

so that $\partial B(s_1, r) \cap \Gamma$ contains exactly 2 points x_0 and x_1, where x_j (for $j = 0$ or $j = 1$) is the point of the side $[s_j, s_{j+1}]$ belonging to the boundary of $B(s_1, r)$.

Define

$$\gamma_N = \partial\left(\mathcal{S}_N \cap (U)_N\right).$$

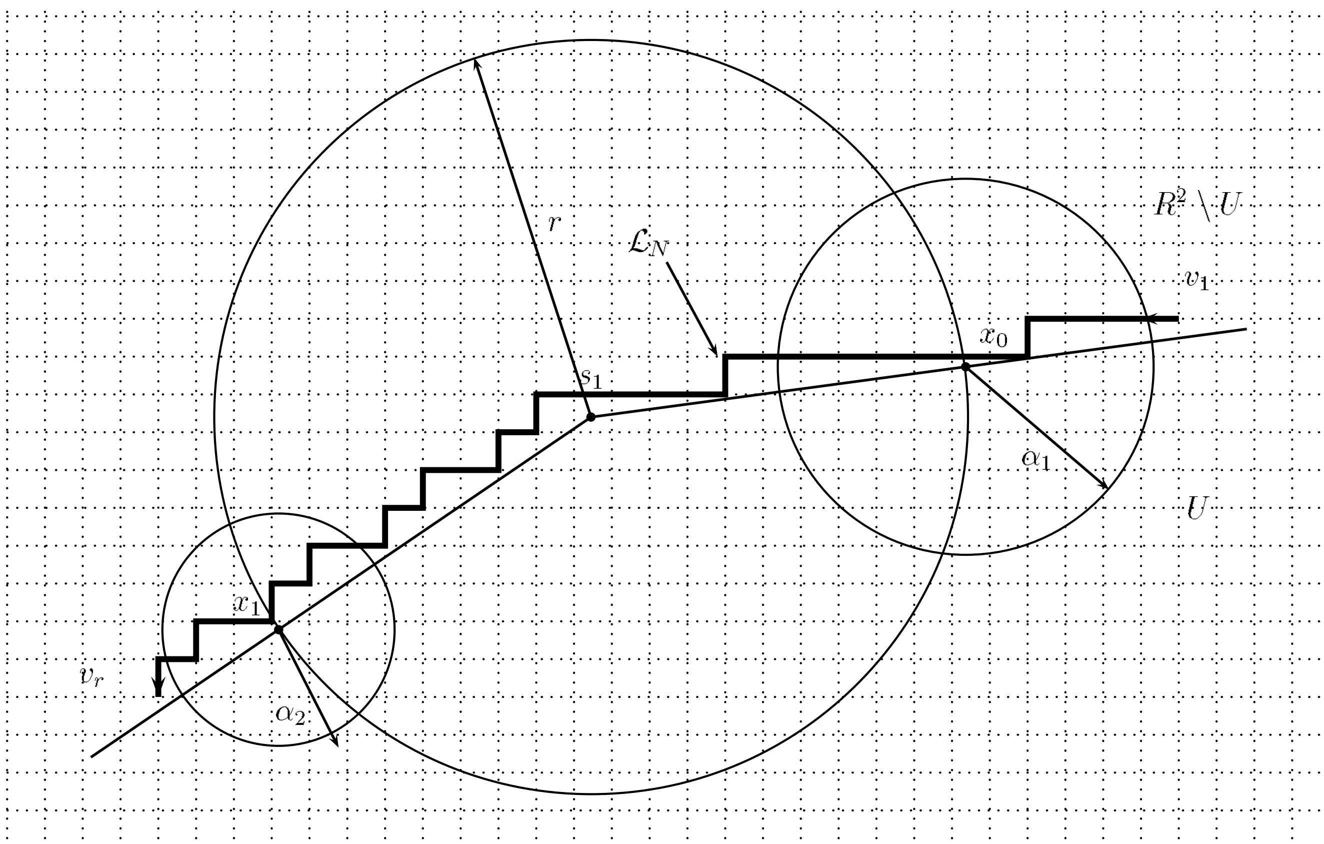
For N large enough, γ_N is a closed Jordan curve. We orient γ_N counterclockwise. This boundary can be described as a sequence $v_0, v_1, \ldots$ of horizontal or vertical vectors of norm $\frac{1}{N}$. The order in the sequence corresponds to the order of appearance along γ_N.

By construction and thanks to the hypothesis on Γ, there is exactly one vector v (resp. w) of the boundary γ_N going from U to $\mathbb{R}^2 \setminus U$ (resp. from $\mathbb{R}^2 \setminus U$ to U). We index the sequence of vectors describing γ_N in such a way that $v_0 = v$ and $v_{r+1} = w$. We denote by $e_N^1(\alpha_1)$, $e_N^2(\alpha_2)$ the two unit vectors defined by

$$e_N^1(\alpha_1) = Nv_1, \qquad e_N^2(\alpha_2) = Nv_r, \tag{5}$$

and by $\mathcal{L}_N$ the maximal subgraph of γ_N strictly included in $\mathbb{R}^2 \setminus U$:

$$\mathcal{L}_N = (v_1, \ldots, v_r). \tag{6}$$

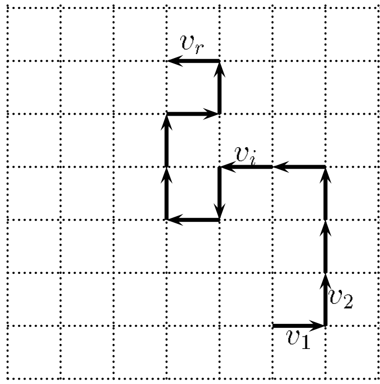


The polygonal line $\mathcal{L}_N = \partial(\mathcal{S}_N \cap (U)_N) \cap (\mathbb{R}^2 \setminus U)$.
Here $\mathcal{S}_N = (B(s_1, r) \cup B(x_0, \alpha_1) \cup B(x_1, \alpha_2))_N$.

We also need the following definition and notation.

Definition 1 *We say that $\mathcal{L}_N$ is a path on $\mathbb{Z}_N^2$ if $\mathcal{L}_N$ is a finite sequence of consecutive vectors $(v_i)_{1 \le i \le r}$ (this means that the endpoint of v_i is the starting point of v_{i+1} for $1 \le i < r$), of norm $1/N$, drawn on the grid $\mathbb{Z}_N^2$, and such that the endpoints of these vectors (resp. the starting points) are distinct.*

The following family of vectors $(v_1, \ldots, v_r)$ is a path on the grid $\mathbb{Z}_N^2$.



Notation Let $\mathcal{L}_N = (v_1, v_2, \ldots, v_r)$ be a path on $\mathbb{Z}_N^2$. We denote by

$$N_+(\mathcal{L}_N) = \text{card}\left\{\mathrm{i}:\ (\widehat{\mathrm{v_i}, \mathrm{v_{i+1}}}) = -\frac{\pi}{2}\right\},\quad \mathrm{N}_-(\mathcal{L}_\mathrm{N}) = \text{card}\left\{\mathrm{i}:\ (\widehat{\mathrm{v_i}, \mathrm{v_{i+1}}}) = +\frac{\pi}{2}\right\}, \tag{7}$$

where $(\widehat{v_i, v_{i+1}})$ denotes the oriented angle between v_i and v_{i+1}.
The purpose of the following proposition is to establish the relation between $N_-(\mathcal{L}_N) - N_+(\mathcal{L}_N)$ and $L_N^\Gamma(s_1, r, \alpha_1, \alpha_2)$, for the path $\mathcal{L}_N$ as defined by (6).

Proposition 2 *Let N be a fixed positive integer. Let $\Gamma = \Gamma(s_1, \ldots, s_m)$ be an m-smooth polygon enclosing U and let σ be the associated configuration at step N. Let $\mathcal{L}_N$ be the path as defined by (6). Then*

$$L_N^\Gamma(s_1, r, \alpha_1, \alpha_2) = \frac{1}{2}N_-(\mathcal{L}_N) - \frac{1}{2}N_+(\mathcal{L}_N). \tag{8}$$

Proof of proposition 2. Let $N \in \mathbb{N}$ be fixed and $\mathcal{S}_N = (B(s_1, r) \cup B(x_0, \alpha_1) \cup B(x_1, \alpha_2))_N$. We denote by f the function defined from $\{0, 1, \ldots, 4\}$ to $\{0, 1, 2\}$ by

$$f(s(\sigma, x)) = \begin{cases} 1 & \text{if } s(\sigma, x) = 2 \\ 2 & \text{if } s(\sigma, x) = 3 \\ 0 & \textit{otherwise.} \end{cases}$$

On the one hand, by definition of $N_-(\mathcal{L}_N)$ and $N_+(\mathcal{L}_N)$, we have

$$\sum_{x \in \mathbb{Z}^2 \cap N\mathcal{S}_N} \sigma(x) f(s(\sigma, x)) = N_+(\mathcal{L}_N) - N_-(\mathcal{L}_N), \tag{9}$$

on the other hand, we deduce from the definition of the function f,

$$\sum_{x \in \mathbb{Z}^2 \cap N\mathcal{S}_N} \sigma(x) f(s(\sigma, x)) = -\sum_{x \in \mathbb{Z}^2,\, \Lambda_{x/N} \subset \mathcal{S}_N} \left(\mathbb{1}_{\sigma(x)=-1\,,\, s(\sigma,x)=2} - \mathbb{1}_{\sigma(x)=+1,\, s(\sigma,x)=2}\right)$$
$$-2 \sum_{x \in \mathbb{Z}^2,\, \Lambda_{x/N} \subset \mathcal{S}_N} \left(\mathbb{1}_{\sigma(x)=-1,\, s(\sigma,x)=3} - \mathbb{1}_{\sigma(x)=+1,\, s(\sigma,x)=3}\right).$$

We combine the last formula, lemma 1 (with $\alpha = \frac{1}{2}$) together with the fact that $\mathbb{I}_{s(\sigma,x)=4} = 0$, and we obtain

$$L_N^\Gamma(s_1, r, \alpha_1, \alpha_2) = -\frac{1}{2} \sum_{x \in \mathbb{Z}^2 \cap \mathcal{S}_N} \sigma(x) f(s(\sigma, x)). \tag{10}$$

The proof of proposition 2 is deduced from (9) and (10). □

In order to evaluate $N_+(\mathcal{L}_N) - N_-(\mathcal{L}_N)$, we need the following lemma.

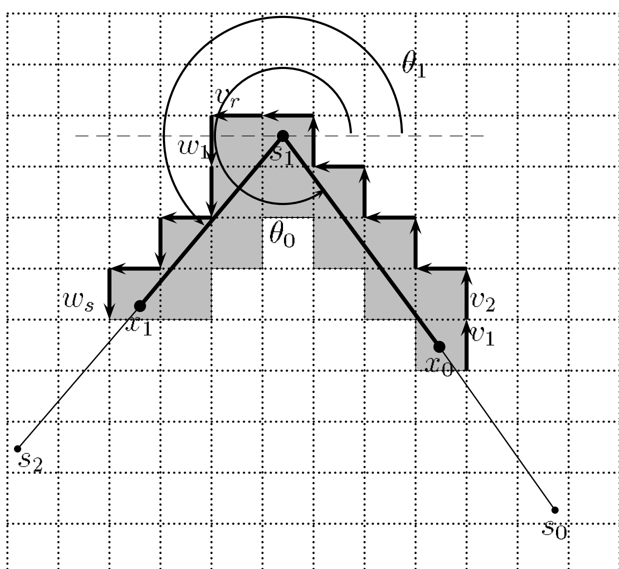
Lemma 2 *Let s_0, s_1 and s_2 be three points in $\mathbb{R}^2$ and x_0, x_1 be two points belonging to the segments $[s_0, s_1]$ and $[s_1, s_2]$ respectively. Let θ_0 (resp. θ_1) be the oriented angle between the half horizontal axis $[0, +\infty[$ and the segment $[s_1, s_0[$ (respectively $[s_1, s_2[$). Let $\mathcal{S}_N = ([x_0, s_1] \cup [s_1, x_1])_N$, then $\partial \mathcal{S}_N$ is a Jordan curve, which we orient counterclockwise. Let $\mathcal{L}_N$ be one of the two maximal subpaths of $\partial \mathcal{S}_N$ not intersecting $[x_0, s_1] \cup [s_1, x_1]$. Let $N^{-1}u_e$ (respectively, $N^{-1}u_s$) be the first (respectively, the last) vector in $\mathcal{L}_N$. Then, for any positive integer N large enough,*

$$N_-(\mathcal{L}_N) - N_+(\mathcal{L}_N) = \mathrm{sgn}(\sin\theta_0)\mathrm{u_e} \cdot \mathrm{i} + \mathrm{sgn}(\sin\theta_1)\mathrm{u_s} \cdot \mathrm{i} + \mathrm{f}(\theta_1, \theta_0), \tag{11}$$

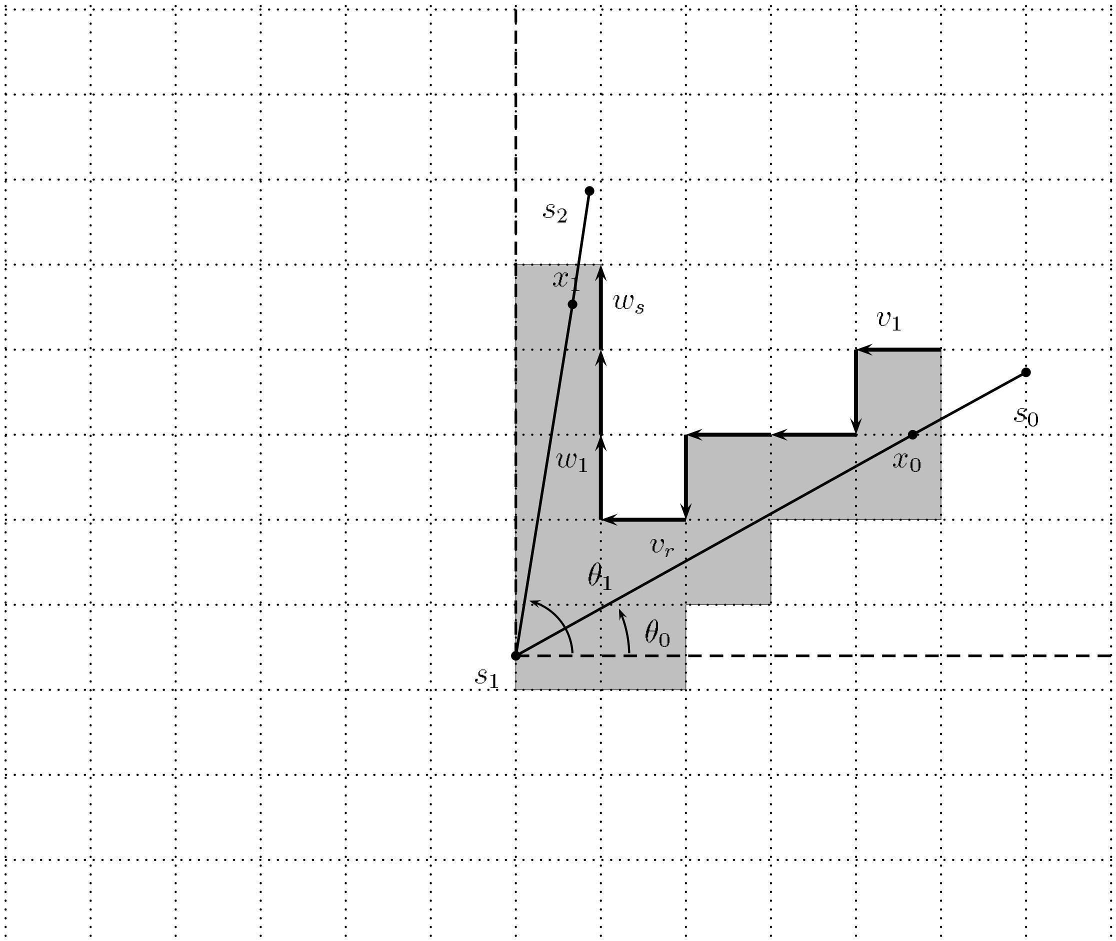
where i is the unit vector $(1,0)$, $\cdot$ is the usual scalar product in $\mathbb{R}^2$ and

$$f(\theta_1, \theta_0) = \begin{cases} 2\mathrm{sgn}(\theta_1 - \theta_0) & if \quad \sin\theta_0 \sin\theta_1 > 0 \\ 0 & otherwise. \end{cases}$$

Remark. The smallest value of N for which (11) holds depends only on θ_0, θ_1 and $|s_1 - s_0|$, $|s_2 - s_1|$.



$\mathcal{L}_N$ is the circuit $(v_1, \ldots, v_r, w_1, \ldots, w_s)$.
Here, $u_e = Nv_1$, $u_s = Nw_s$ and $f(\theta_1, \theta_0) = 2\mathrm{sgn}(\theta_1 - \theta_0)$.



$\mathcal{L}_N$ is the circuit $(v_1, \ldots, v_r, w_1, \ldots, w_s)$.
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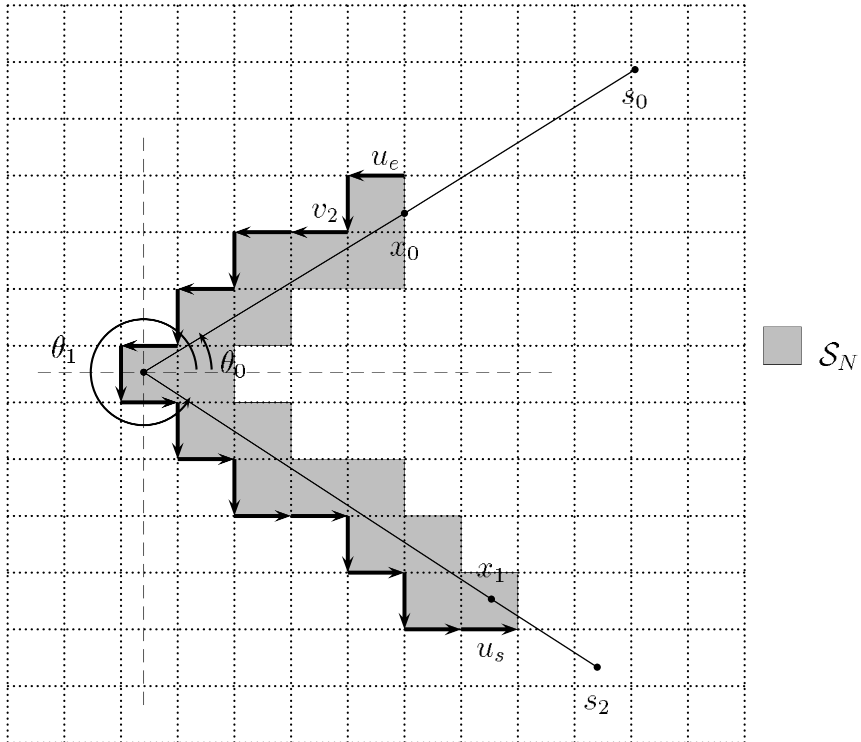
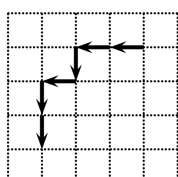


In this picture, $f(\theta_1, \theta_0) = 0$.

In order to prove lemma 2, we introduce the following definition.

Definition 2 *A path on $\mathbb{Z}_N^2$ is said monotone if all its horizontal as well as all its vertical vectors are oriented in the same sense.*



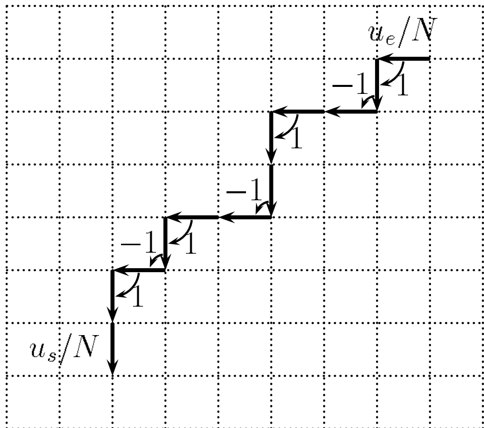
A monotone path on the grid $\mathbb{Z}_N^2$.

The following lemma evaluates $N_+(\mathcal{L}_N) - N_-(\mathcal{L}_N)$, whenever $\mathcal{L}_N$ is a monotone path on $\mathbb{Z}_N^2$.

Lemma 3 *Let $(v_i)_{1 \le i \le r}$ be a sequence of r consecutive vectors drawn on the grid $\mathbb{Z}_N^2$. Those vectors are enumerated beginning from $N^{-1}u_e := v_1$ until $N^{-1}u_s := v_r$. We suppose that they form a* **monotone** *path on $\mathbb{Z}_N^2$, say $\mathcal{L}_N$. Then*

$$N_+(\mathcal{L}_N) - N_-(\mathcal{L}_N) = [u_e \wedge u_s],$$

where $[u_e \wedge u_s] = (u_e \cdot i)(u_s \cdot j) - (u_e \cdot j)(u_s \cdot i)$.



For this monotone path $\mathcal{L}_N$, we have $u_e = (-1, 0)$ and $u_s = (0, -1)$, hence $[u_e \wedge u_s] = 1$. On the other hand $N_+(\mathcal{L}_N) - N_-(\mathcal{L}_N) = 1 - 1 + 1 - 1 + 1 - 1 + 1 = 1$.

Remark. Let us note that for any path $\mathcal{L}_N = (v_1, \dots, v_r)$, we have

$$(\widehat{u_e, u_s}) = \frac{\pi}{2}\left(N_-\left(\mathcal{L}_N\right) - N_+\left(\mathcal{L}_N\right)\right),$$

where $u_e = Nv_1$ and $u_s = Nv_r$.

Proof of lemma 3. We denote by $\mathcal{L}_N(r) = (v_1, \dots, v_r)$ a monotone path on $\mathbb{Z}_N^2$. The proof of lemma 3 is done by induction on r.

For $r = 1$, we have $N_-\left(\mathcal{L}_N(1)\right) - N_+\left(\mathcal{L}_N(1)\right) = 0$ which corresponds to $[u_e \wedge u_s]$, since in this case $N^{-1}u_e = N^{-1}u_s = v_1$.

We suppose now that the property is true at step $r \geq 1$ and we prove it at step $r + 1$. We consider the path $\mathcal{L}_N(r+1)$. Since $\mathcal{L}_N(r+1)$ is monotone, we can suppose without loss of generality that

$$(\mathcal{H}) \qquad \forall\, l \in \{1, \dots, r+1\}, \ \ (Nv_l) \cdot i \in \{0, -1\}, \quad (Nv_l) \cdot j \in \{0, -1\}.$$

Once the hypothesis $(\mathcal{H})$ is assumed, we have only three cases to discuss on the expression of (v_r, v_{r+1}),

- If $v_r = v_{r+1}$, then $N_+\left(\mathcal{L}_N(r+1)\right) - N_-\left(\mathcal{L}_N(r+1)\right) = N_+\left(\mathcal{L}_N(r)\right) - N_-\left(\mathcal{L}_N(r)\right)$, and the inductive assumption gives

$$N_+\left(\mathcal{L}_N(r+1)\right) - N_-\left(\mathcal{L}_N(r+1)\right) = [Nv_1 \wedge Nv_{r+1}].$$

- If $(Nv_r) \cdot j = -1 = (Nv_{r+1}) \cdot i$, then $(\widehat{v_r, v_{r+1}}) = \frac{\pi}{2}$ and $N_+\left(\mathcal{L}_N(r+1)\right) - N_-\left(\mathcal{L}_N(r+1)\right) = N_+\left(\mathcal{L}_N(r)\right) - N_-\left(\mathcal{L}_N(r)\right) - 1$. Together with the inductive assumption, this gives

$$N_+\left(\mathcal{L}_N(r+1)\right) - N_-\left(\mathcal{L}_N(r+1)\right) = -(Nv_1) \cdot i - 1 = (Nv_1) \cdot j = [Nv_1 \wedge Nv_{r+1}].$$

• If $(Nv_r)\cdot i = -1 = (Nv_{r+1})\cdot j$, then $(\widehat{v_r, v_{r+1}}) = -\frac{\pi}{2}$, $N_+\left(\mathcal{L}_N(r+1)\right) - N_-\left(\mathcal{L}_N(r+1)\right) = N_+\left(\mathcal{L}_N(r)\right) - N_-\left(\mathcal{L}_N(r)\right) + 1$ and

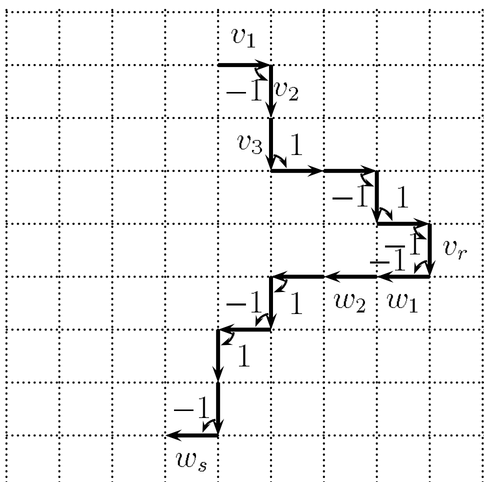
$$N_+\left(\mathcal{L}_N(r+1)\right) - N_-\left(\mathcal{L}_N(r+1)\right) = (Nv_1)\cdot j + 1 = -(Nv_1)\cdot i = [Nv_1 \wedge Nv_{r+1}].$$

The equality $N_+\left(\mathcal{L}_N(r+1)\right) - N_-\left(\mathcal{L}_N(r+1)\right) = [Nv_1 \wedge Nv_{r+1}]$ is then always valid and lemma 3 is proved. □

Remark Let $\mathcal{L}_N = (v_1, \ldots, v_r, w_1, \ldots, w_s)$ be a path on $\mathbb{Z}_N^2$. We suppose that the family $(v_1, \ldots, v_r)$ (respectively $(w_1, \ldots, w_s)$) forms a monotone path on $\mathbb{Z}_N^2$, say $\mathcal{L}_N^1$ (respectively $\mathcal{L}_N^2$). We suppose also that $v_r \cdot w_1 = 0$. We deduce, applying lemma 3 to the monotone paths $\mathcal{L}_N^1$, $\mathcal{L}_N^2$ and (v_r, w_1), that,

$$N_+\left(\mathcal{L}_N\right) - N_-\left(\mathcal{L}_N\right) = [Nv_1 \wedge Nv_r] + [Nv_r \wedge Nw_1] + [Nw_1 \wedge Nw_s]. \tag{12}$$

In the following picture, we have have $Nv_1 = (1,0)$, $Nv_r = (0,-1)$, $Nw_1 = (-1,0)$, $Nw_s = (-1,0)$. Hence $[Nv_1 \wedge Nv_r] + [Nw_1 \wedge Nw_s] + [Nv_r \wedge Nw_1] = -2$. On the other hand, we have $N_+(\mathcal{L}_N) - N_-(\mathcal{L}_N) = -2$.



Now, we have all the ingredients in order to prove lemma 2.

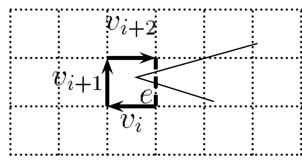
Proof of lemma 2. For $i = 1, 2$, let $\mathcal{S}_N^i = ([x_{i-1}, s_1])_N$. We orient the Jordan curve $\partial\mathcal{S}_N^i$ counterclockwise. Let $\mathcal{L}_N^i$ be one of the two maximal subpaths of $\partial\mathcal{S}_N^i$ not intersecting $[x_{i-1}, s_1]$. The union of the two half lines $[s_1, s_0[\cup[s_1, s_2[$ divides the plane in two connected regions. We suppose that the paths $\mathcal{L}_N^1$ and $\mathcal{L}_N^2$ are situated in the same region. The path $\mathcal{L}_N^1$ is constructed from a finite sequence of consecutive vectors on $\mathbb{Z}_N^2$, say $(v_1, \ldots, v_r)$.
In the same way, we suppose that the path $\mathcal{L}_N^2$ is constructed from the consecutive vectors $(w_1, \ldots, w_s)$. By construction

$$\mathcal{L}_N = \mathcal{L}_N^1 \cup \mathcal{L}_N^2 = (v_1, \ldots, v_r, w_1, \ldots, w_s). \tag{13}$$

We first claim that the paths $\mathcal{L}_N^1$ and $\mathcal{L}_N^2$ are monotone. In fact, if $\mathcal{L}_N^1$ is not a monotone path, then we can suppose without loss of generality the existence of three consecutive vectors v_i, v_{i+1}, v_{i+2} such that

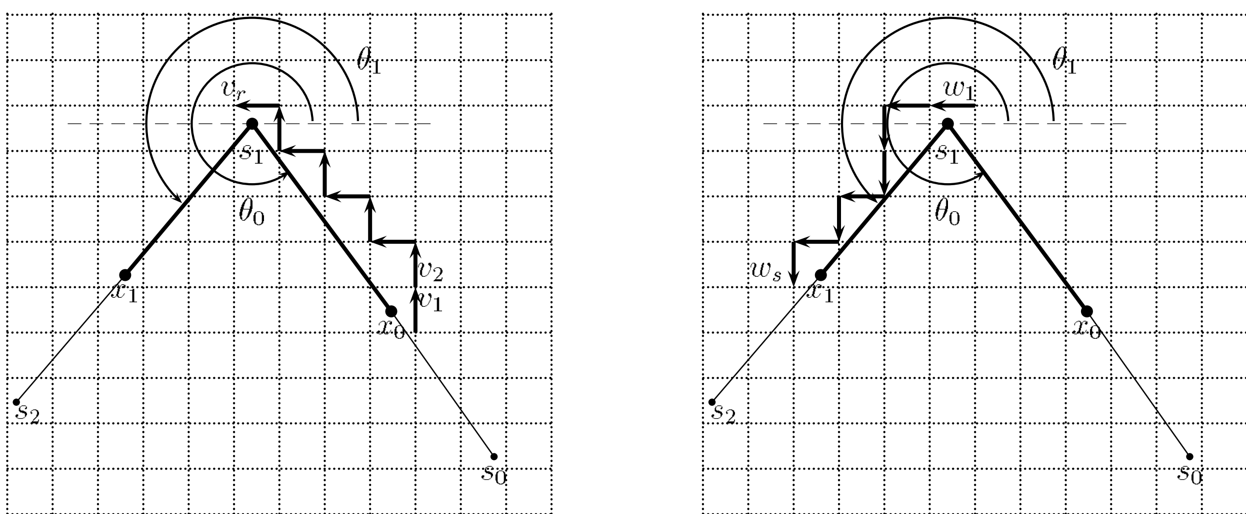
$$(Nv_i)\cdot(Nv_{i+2}) = -1, \quad v_i\cdot v_{i+1} = 0, \text{ and } v_{i+1}\cdot v_{i+2} = 0.$$

Let e denote the vector on the grid $\mathbb{Z}_N^2$ such that $(v_i, v_{i+1}, v_{i+2}, e)$ is a box drawn on $\mathbb{Z}_N^2$.



By construction, the circuit (v_i, v_{i+1}, v_{i+2})
is included in the path $\mathcal{L}_N^1$ that covers the segment $[x_0, s_1]$.

By definition of $\mathcal{L}_N^1$, the segment $[x_0, s_1]$ would cross e in two distinct points, this is a contradiction. Hence the path $\mathcal{L}_N^1$ is monotone. The same arguments apply for the path $\mathcal{L}_N^2$.



The paths $\mathcal{L}_N^1 = (v_1, \ldots, v_r)$ and $\mathcal{L}_N^2 = (w_1, \ldots, w_s)$ are monotone.

We check also that

$$\text{for any } 1 \leq l \leq r, \quad (Nv_l)\cdot i \in \{0, -\text{sgn}(\cos\theta_0)\}, \quad \text{and} \quad (\mathrm{Nv_l})\cdot \mathrm{j} \in \{0, -\text{sgn}(\sin\theta_0)\}.$$

Hence, for any $1 \leq l \leq r$,

$$\text{sgn}(\cos\theta_0)(\mathrm{Nv_l})\cdot \mathrm{i} + \text{sgn}(\sin\theta_0)(\mathrm{Nv_l})\cdot \mathrm{j} = -1. \tag{14}$$

In the same way, we deduce that for any $1 \leq l \leq s$,

$$\mathrm{sgn}(\cos\theta_1)(\mathrm{Nw_l})\cdot \mathrm{i} + \mathrm{sgn}(\sin\theta_1)(\mathrm{Nw_l})\cdot \mathrm{j} = 1. \tag{15}$$

We can suppose that $v_r \cdot w_1 = 0$ (since the points s_1, x_0, x_1 are not on the same line). In order to prove (11), we combine equalities (12), (14), (15) and we use the fact that $v_r \cdot w_1 = 0$:

$$\begin{aligned} N_-(\mathcal{L}_N) - N_+(\mathcal{L}_N) &= -[Nv_1 \wedge Nv_r] - [Nv_r \wedge Nw_1] - [Nw_1 \wedge Nw_s] \\ &= \mathrm{sgn}(\sin\theta_0)(\mathrm{Nv_1})\cdot \mathrm{i} + \mathrm{sgn}(\sin\theta_1)(\mathrm{Nw_s})\cdot \mathrm{i} \\ &+ (\mathrm{sgn}(\sin\theta_0) + \mathrm{sgn}(\sin\theta_1))\left[\mathbb{1}_{w_1\cdot i=0}\left(\mathrm{sgn}(\cos\theta_0)+\mathrm{sgn}(\cos\theta_1)\right) - \mathrm{sgn}(\cos\theta_1)\right]. \end{aligned} \tag{16}$$

If $\mathrm{sgn}(\sin\theta_0) + \mathrm{sgn}(\sin\theta_1) = 0$, then (11) is immediately deduced from (16). We suppose now that $\mathrm{sgn}(\sin\theta_0) = \mathrm{sgn}(\sin\theta_1)$ and $\mathrm{sgn}(\cos\theta_0) = -\mathrm{sgn}(\cos\theta_1)$, then (11) becomes

$$N_-(\mathcal{L}_N) - N_+(\mathcal{L}_N) = \mathrm{sgn}(\sin\theta_0)(\mathrm{Nv_1})\cdot \mathrm{i} + \mathrm{sgn}(\sin\theta_1)(\mathrm{Nw_s})\cdot \mathrm{i} + 2\mathrm{sgn}(\tan\theta_0).$$

The last formula is exactly equality (11), since $\mathrm{sgn}(\tan\theta_0) = \mathrm{sgn}(\theta_1 - \theta_0)$ whenever $\mathrm{sgn}(\sin\theta_0) = \mathrm{sgn}(\sin\theta_1)$ and $\mathrm{sgn}(\cos\theta_0) = -\mathrm{sgn}(\cos\theta_1)$.

Now, we have to prove (11) when $\mathrm{sgn}(\sin\theta_0) = \mathrm{sgn}(\sin\theta_1)$ and $\mathrm{sgn}(\cos\theta_0) = \mathrm{sgn}(\cos\theta_1)$. In this later case, equality (16) becomes,

$$\begin{aligned} &N_-(\mathcal{L}_N) - N_+(\mathcal{L}_N) = \\ &\quad \mathrm{sgn}(\sin\theta_0)(\mathrm{Nv_1})\cdot \mathrm{i} + \mathrm{sgn}(\sin\theta_1)(\mathrm{Nw_s})\cdot \mathrm{i} + 2\mathrm{sgn}(\tan\theta_0)\left[2\mathbb{1}_{\mathrm{w_1\cdot i=0}} - 1\right]. \end{aligned} \tag{17}$$

We claim that, if $\sin\theta_0 \sin\theta_1 \geq 0$ and $\cos\theta_0\cos\theta_1 \geq 0$, then

$$\mathrm{sgn}(\tan\theta_0)\left(2\mathbb{1}_{\mathrm{w_1\cdot i=0}} - 1\right) = \mathrm{sgn}(\theta_1 - \theta_0). \tag{18}$$

This formula together with (17) proves (11) in the special case $\mathrm{sgn}(\sin\theta_0) = \mathrm{sgn}(\sin\theta_1)$ and $\mathrm{sgn}(\cos\theta_0) = \mathrm{sgn}(\cos\theta_1)$.

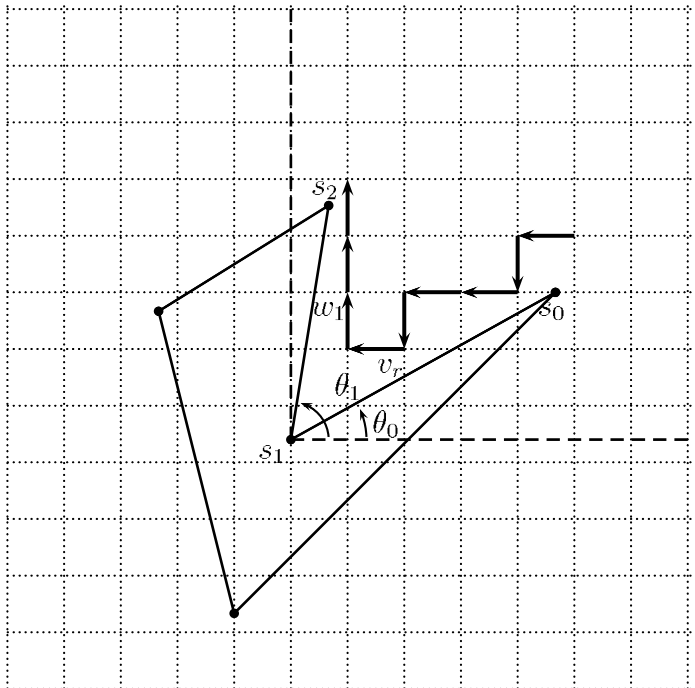
Let us now prove (18). We suppose without loss of generality that θ_0 and θ_1 belong to $[0, \frac{\pi}{2}]$, hence $\mathrm{sgn}(\tan\theta_0) = +1$, and we discuss only the case when $\mathrm{sgn}(\theta_1 - \theta_0) = +1$ (the other case is similar). The monotone path $\mathcal{L}_N^1$ (respectively $\mathcal{L}_N^2$) is constructed from the vectors $(v_i)_{1\leq i\leq r}$ (respectively $(w_i)_{1\leq i\leq s}$) which are either $N^{-1}(-1,0)$ or $N^{-1}(0,-1)$ (respectively $N^{-1}(1,0)$ or $N^{-1}(0,1)$). Those paths are L^1-connected and $v_r \cdot w_1 = 0$, so we have only two possibilities :

$$\text{either } \left(v_r = N^{-1}(-1,0) \text{ and } w_1 = N^{-1}(0,1)\right), \text{ or } \left(v_r = N^{-1}(0,-1) \text{ and } w_1 = N^{-1}(1,0)\right).$$

Since $\theta_1 > \theta_0$, the segment $[s_1, x_1]$ belongs to the upper plane (separated by (x_0, s_1)). Since the path $\mathcal{L}_N^2$ (respectively $\mathcal{L}_N^1$) approximates $[s_1, x_1]$ (respectively $[x_0, s_1]$) for $N > (2\tan(\theta_1 - \theta_0))^{-1}$, we deduce that

$$v_r = N^{-1}(-1,0) \text{ and } w_1 = N^{-1}(0,1)$$

hence $w_1 \cdot i = 0$ as soon as $\theta_1 > \theta_0$. $\square$



If $\cos\theta_0 \geq 0$, $\cos\theta_1 \geq 0$, $\sin\theta_0 \geq 0$, $\sin\theta_1 \geq 0$ and $\theta_1 > \theta_0$, then $w_1 \cdot i = 0$.

4.1.3 Evaluation of $\int_0^\delta \int_0^\delta L_N^\Gamma(s_1, r, \alpha_1, \alpha_2)\, d\alpha_1\, d\alpha_2$

We deduce from proposition 2 combined with lemma 2 that there exists N_0 depending only on Γ such that, for any $N \geq N_0$,

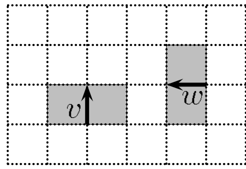
$$L_N^\Gamma(s_1, r, \alpha_1, \alpha_2) = \frac{1}{2}\mathrm{sgn}(\sin\theta_0)\, \mathrm{e}_\mathrm{N}^1(\alpha_1) \cdot \mathrm{i} + \frac{1}{2}\mathrm{sgn}(\sin\theta_1)\, \mathrm{e}_\mathrm{N}^2(\alpha_2) \cdot \mathrm{i} + \mathrm{sgn}(\theta_1 - \theta_0) \mathbb{1}_{\sin\theta_0 \sin\theta_1 > 0},$$

where $e_N^1(\alpha_1)$, $e_N^2(\alpha_2)$ are the two unit vectors as defined by (5). (Recall that i is the unit vector $(1,0)$ and $\cdot$ is the usual scalar product in $\mathbb{R}^2$).
Hence, in order to evaluate the quantity

$$\int_0^\delta \int_0^\delta L_N^\Gamma(s_1, r, \alpha_1, \alpha_2)\, d\alpha_1\, d\alpha_2,$$

for δ and r small enough, it suffices to evaluate the terms $\int_0^\delta e_N^1(\alpha) \cdot i\, d\alpha$, $\int_0^\delta e_N^2(\alpha) \cdot i\, d\alpha$. We begin by the first quantity, for this we need some further notations.

Notation. For a vector v drawn on the grid $\mathbb{Z}_N^2$, we denote by $R(v)$ the union of the two boxes of the family $(\Lambda_{x/N})_{x \in \mathbb{Z}^2}$ having v as an edge vector.



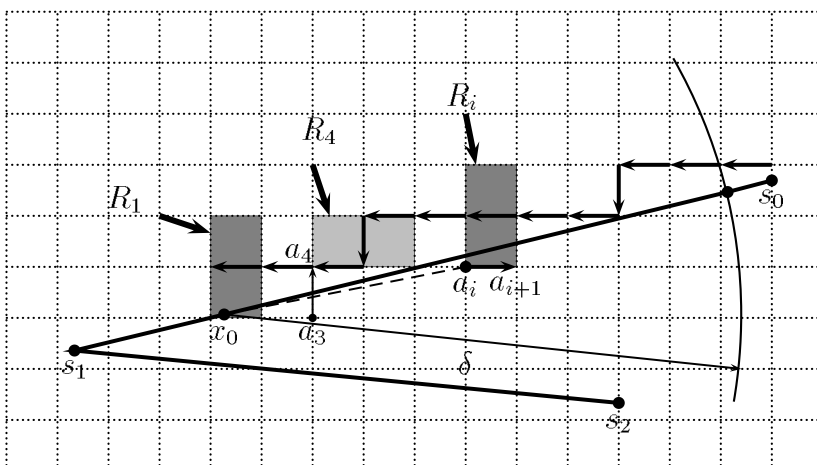
The two blocks $R(v)$ and $R(w)$.

Let $\mathcal{L}_N = (v_1, \ldots, v_r)$ be the oriented path as defined by (6). Let $\mathcal{L}_{1,N} = (v_1, \ldots, v_s)$ be the minimal subgraph of $\mathcal{L}_N$ covering the segment $[s_0, x_0]$; the vector v_s is the entering vector in the box containing x_0.

To each vector v_l ($1 \leq l \leq s$), we associate the block $R_{s-l+1} := R(v_l)$. These blocks $(R_l)_{1 \leq l \leq s}$ are enumerated according to their distances to x_0, R_1 being the block containing x_0. Let $(a_l)_{1 \leq l \leq s}$ be the sequence of vertices such that

$$a_l \in R_l, \quad \text{and} \quad d_l := d(x_0, R_l) = |a_l - x_0|,$$

then this sequence of vertices $(a_l)_{1 \leq l \leq s}$ is L^1 connected and the vector $a_l a_{l+1}$ is either vertical or horizontal. Finally, let $\mathcal{H}_N$ be the set of indices $l \in \{1, \ldots, s\}$ for which v_l is horizontal.



$$d_l = d(x_0, R_l) = |x_0 - a_l|.$$

For N large enough, the vector $a_l a_{l+1}$ is either horizontal or vertical, and $|a_l - a_{l+1}| = \frac{1}{N}$.

By construction, $e_N^1(\alpha) \cdot i = -\text{sgn}(\cos\theta_0)$ if and only if there exists $l \in \mathcal{H}_N$ such that $\alpha \in]d_l, d_{l+1}]$ (such an index is necessarily unique). Then,

$$\left| \int_0^\delta e_N^1(\alpha) \cdot i \, d\alpha + \text{sgn}(\cos\theta_0) \sum_{\mathrm{l} \in \mathcal{H}_\mathrm{N}} (\mathrm{d}_{\mathrm{l}+1} - \mathrm{d}_\mathrm{l}) \right| \leq \frac{2}{N}. \tag{19}$$

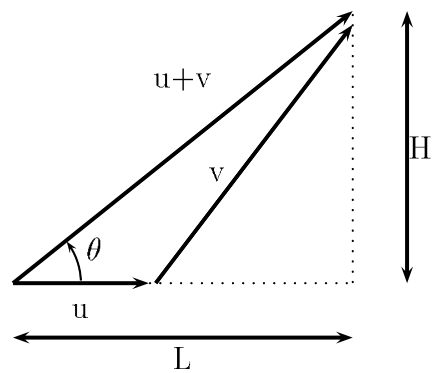
In order to evaluate $d_{l+1} - d_l$, we need the following lemma.

Lemma 4 *Let u and v be two vectors such that $\|u\| \le \|v\|$. Then*

$$\|u+v\| - \|v\| = \frac{(u+v)\cdot u}{\|u+v\|} - \frac{\|u\|^2}{\|v\|}\frac{\sin^2\theta}{1+\sqrt{1-\frac{\|u\|^2}{\|v\|^2}\sin^2\theta}},$$

where θ is the angle between u and $u+v$.

Proof of lemma 4. Let u, v and θ be as defined in lemma 4.



We have

$$\begin{aligned}
\|u+v\|^2 &= L^2 + H^2 \\
&= \cos^2\theta\|u+v\|^2 + \|v\|^2 - (\cos\theta\|u+v\| - \|u\|)^2 \\
&= \|v\|^2 + 2\cos\theta\|u\| \times \|u+v\| - \|u\|^2.
\end{aligned}$$

The quantity $\|u+v\|$ is then a positive solution of an algebraic equation of degree two. We deduce from $\|u\| \le \|v\|$, that

$$\|u+v\| = \|u\|\cos\theta + \sqrt{\|v\|^2 - \sin^2\theta\|u\|^2}.$$

Hence

$$\begin{aligned}
\|u+v\| - \|v\| &= \|u\|\cos\theta + \|v\|\left(\sqrt{1-\sin^2\theta\frac{\|u\|^2}{\|v\|^2}} - 1\right) \\
&= \|u\|\cos\theta - \frac{\|u\|^2}{\|v\|}\frac{\sin^2\theta}{1+\sqrt{1-\frac{\|u\|^2}{\|v\|^2}\sin^2\theta}}.
\end{aligned}$$

The last equality together with the fact that $\|u\|\cos\theta = \dfrac{(u+v)\cdot u}{\|u+v\|}$ proves lemma 4. □

We continue the proof of proposition 1. We apply lemma 4 with

$$u = a_l a_{l+1}, \qquad v = x_0 a_l,$$

with this choice

$$\frac{(u+v)\cdot u}{\|u+v\|} = \frac{(x_0 a_{l+1})\cdot(a_l a_{l+1})}{|x_0 - a_{l+1}|}.$$

Moreover, we deduce from lemma 4,

$$\left| d_{l+1} - d_l - \frac{(x_0 a_{l+1})\cdot(a_l a_{l+1})}{|x_0 - a_{l+1}|} \right| \leq \frac{1}{N^2 |x_0 - a_l|}. \tag{20}$$

We first evaluate the term $\sum_{l\in\mathcal{H}_N} \frac{(x_0 a_{l+1})\cdot(a_l a_{l+1})}{|x_0 - a_{l+1}|}$, with the help of the following lemma.

Lemma 5 *For $l \in \mathcal{H}_N$, let x_l be the point of the segment $[x_0, s_0]$ such that $|x_0 - a_{l+1}| = |x_0 - x_l|$. Then*

$$\left| \frac{(x_0 a_{l+1})\cdot(a_l a_{l+1})}{|x_0 - a_{l+1}|} - \frac{(x_0 x_l)\cdot(a_l a_{l+1})}{|x_0 - x_l|} \right| \leq \frac{1}{N(\phi(l)-1)},$$

where $\phi(l)$ is the cardinality of the set $\mathcal{H}_N \cap \{1, \ldots, l\}$.

Proof of lemma 5. We have

$$|(x_0 a_{l+1})\cdot(a_l a_{l+1}) - (x_0 x_l)\cdot(a_l a_{l+1})| \leq \frac{1}{N^2},$$

since $|a_l - a_{l+1}| \leq 1/N$ and $|x_l - a_{l+1}| \leq 1/N$. We deduce from the definition of the edge $a_l a_{l+1}$, that $|(x_0 a_{l+1})\cdot i| \geq \frac{\phi(l)-1}{N}$, hence

$$|x_0 - x_l| = |x_0 - a_{l+1}| \geq |(x_0 a_{l+1})\cdot i| \geq \frac{\phi(l)-1}{N}.$$

These two inequalities prove lemma 5. □

We deduce, applying lemma 5, and noting that $\mathrm{sgn}(\cos\theta_0)(\mathrm{a_l a_{l+1}})\cdot \mathrm{i} + \mathrm{sgn}(\sin\theta_0)(\mathrm{a_l a_{l+1}})\cdot \mathrm{j} = 1$,

$$\begin{aligned} \sum_{l\in\mathcal{H}_N} \frac{(x_0 a_{l+1})\cdot(a_l a_{l+1})}{|x_0 - a_{l+1}|} &= \sum_{l\in\mathcal{H}_N} \frac{(x_0 x_l)\cdot(a_l a_{l+1})}{|x_0 - x_l|} + \mathcal{O}\left(\frac{\ln(|\mathcal{H}_N|)}{N}\right) \qquad (21) \\ &= \frac{\mathrm{sgn}(\cos\theta_0)}{N} \sum_{l\in\mathcal{H}_{1,N}} \frac{(x_0 x_l)\cdot i}{|x_0 - x_l|} + \frac{\mathrm{sgn}(\sin\theta_0)}{N} \sum_{l\in\mathcal{H}_{2,N}} \frac{(x_0 x_l)\cdot j}{|x_0 - x_l|} + \mathcal{O}\left(\frac{\ln(|\mathcal{H}_N|)}{N}\right), \end{aligned}$$

where

$$\mathcal{H}_{1,N} = \{\, l \in \mathcal{H}_N : (a_l a_{l+1})\cdot j = 0 \,\} \quad \text{and} \quad \mathcal{H}_{2,N} = \{\, l \in \mathcal{H}_N : (a_l a_{l+1})\cdot i = 0 \,\}. \tag{22}$$

We have

$$\frac{(x_0 x_l)\cdot i}{|x_0 - x_l|} = \cos\theta_0 \quad \text{and} \quad \frac{(x_0 x_l)\cdot j}{|x_0 - x_l|} = \sin\theta_0.$$

Theses identities ensure

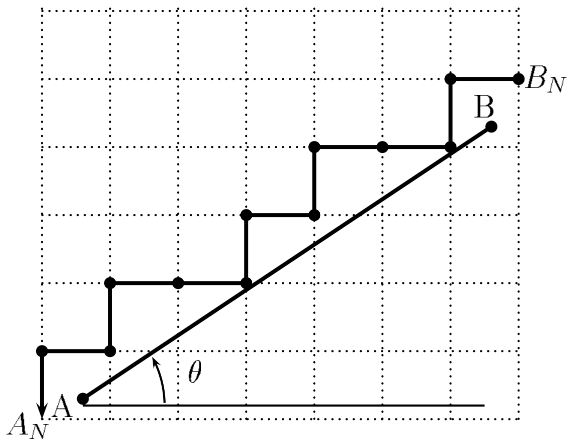
$$\sum_{l\in\mathcal{H}_N} \frac{(x_0 a_{l+1})\cdot(a_l a_{l+1})}{|x_0 - a_{l+1}|} = \frac{|\mathcal{H}_{1,N}|}{N}|\cos\theta_0| + \frac{|\mathcal{H}_{2,N}|}{N}|\sin\theta_0| + \mathcal{O}\left(\frac{\ln(|\mathcal{H}_N|)}{N}\right). \tag{23}$$

In order to calculate the limits as N goes to infinity of the right hand side of the last equality, we need the following lemma.

Lemma 6 *Let A and B be two points of $\mathbb{R}^2$. For each fixed integer N, let $\mathcal{L}_N$ denote one of the two maximal subpaths of $\partial([AB]_N)$ not crossing the line (AB). Let $N_+(\mathcal{L}_N)$ be as defined in (7). Then*

$$\lim_{N\to+\infty} \frac{N_+(\mathcal{L}_N)}{N} = |(AB)\cdot i| \wedge |(AB)\cdot j|,$$

where $a \wedge b = \min(a,b)$.



Proof of lemma 6. We suppose without loss of generality that $AB\cdot i$ and $AB\cdot j$ are positive. Let θ denote the angle between AB and i. We consider only the case $0 \le \tan\theta < 1$, since the proofs for the cases $\tan\theta > 1$ and $\tan\theta = 1$ are similar. We denote by A_N and B_N the extremes points of $\mathcal{L}_N$. Our task is to prove that

$$(A_N B_N)\cdot j = \frac{N_+(\mathcal{L}_N)}{N}. \tag{24}$$

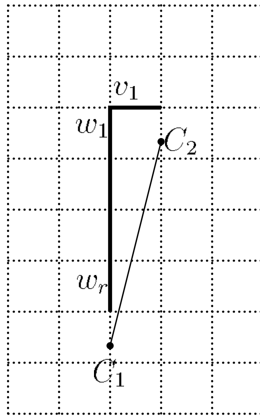
The identity (24) will prove lemma 6 since $\lim_{N\to+\infty}(A_N B_N)\cdot j = (AB)\cdot j$ and $0 \le \tan\theta < 1$. We first prove the equality (24) for $N_+(\mathcal{L}_N) = 1$. When $N_+(\mathcal{L}_N) = 1$, the path $\mathcal{L}_N$ contains a unique monotone path $\mathcal{L}'_N = (v_1, w_1, \ldots, w_r)$ such that $v_1\cdot j = 0$, $v_1 \cdot w_1 = 0$ and $w_1\cdot i = \ldots = w_r\cdot i = 0$. These vectors are drawn on the lattice $\mathbb{Z}^2_N$ and arranged according to the direct sense.

Let C_1, C_2 be the two points of (AB) such that $(C_1C_2)\cdot i = 1/N$ and that the path $\mathcal{L}'_N$ cover the segment $[C_1, C_2]$. By construction

$$(C_1C_2)\cdot j > \frac{r-1}{N},$$

hence

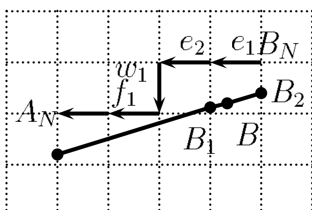
$$\tan\theta = \frac{(C_1C_2)\cdot j}{(C_1C_2)\cdot i} > r - 1.$$



Since $0 \leq \tan\theta < 1$, we deduce that $r = 1$. Now let B_1, B_2 be the two points of (AB) belonging to the boundary of the box of $[AB]_N$ that contains the point B. Since $0 \leq \tan\theta < 1$, we have

$$|(B_1B_2) \cdot j| < \frac{1}{N}.$$

This fact together with $r = 1$ proves that the path $\mathcal{L}_N$ is equal to $(e_1, \dots, e_m, w_1, f_1, \dots, f_n)$, where the vectors (e_i) and (f_i) are copies of the vector v_1, so that they are all horizontal. Hence $(A_NB_N) \cdot j = 1/N$.



The general case when $N_+(\mathcal{L}_N) > 1$ is proved by induction on $N_+(\mathcal{L}_N)$. □

Let $\mathcal{L}_N$ be the monotone path $(v_1, \dots, v_s)$ as defined by (6). We obtain using the definition of $\mathcal{H}_{2,N}$, that $|\mathcal{H}_{2,N}|$ is either $N_+(\mathcal{L}_N)$ or $N_-(\mathcal{L}_N)$. This fact together with the constatation that $|N_+(\mathcal{L}_N) - N_-(\mathcal{L}_N)| \leq 1$, gives

$$\left| \frac{|\mathcal{H}_{2,N}|}{N} - \frac{N_+(\mathcal{L}_N)}{N} \right| \leq \frac{1}{N},$$

Lemma 6 applied with $[AB] = [x_0s_0]$ and $\mathcal{L}_N = (v_1, \dots, v_s)$, together with the last inequality ensures

$$\lim_{N \to +\infty} \frac{|\mathcal{H}_{2,N}|}{N\delta} = |\cos\theta_0| \wedge |\sin\theta_0| .$$

Now the definition of $\mathcal{H}_N$ gives

$$\lim_{N \to +\infty} \frac{|\mathcal{H}_N|}{N\delta} = |\cos\theta_0| .$$

The two sets of indices $\mathcal{H}_{1,N}$ and $\mathcal{H}_{2,N}$ form a partition of $\mathcal{H}_N$, hence

$$\lim_{N\to+\infty} \frac{|\mathcal{H}_{1,N}|}{N\delta} = |\cos\theta_0| - |\cos\theta_0| \wedge |\sin\theta_0| \, .$$

All these limits together with (23) yield

$$\lim_{N\to+\infty} \sum_{l\in\mathcal{H}_N} \frac{(x_0 a_{l+1})\cdot(a_l a_{l+1})}{|x_0 - a_{l+1}|} \;=\; \delta\cos^2\theta_0 + \delta\,|\cos\theta_0| \wedge |\sin\theta_0|\,(|\sin\theta_0| - |\cos\theta_0|)\, . \quad (25)$$

We have

$$\begin{aligned} \sum_{l\in\mathcal{H}_N} \frac{|a_l - a_{l+1}|^2}{|x_0 - a_l|} &= \frac{1}{N^2}\sum_{l\in\mathcal{H}_N} \frac{1}{|x_0 - a_l|} \\ &\le \frac{1}{N}\sum_{l\in\mathcal{H}_N}\frac{1}{\phi(l)} \qquad \text{since} \qquad |(x_0 a_l)\cdot i| \ge N^{-1}(\phi(l) - 1) \\ &\le \frac{\ln|\mathcal{H}_N|}{N}. \end{aligned} \quad (26)$$

Finally we obtain collecting (19), (20), (25), (26) :

$$\begin{aligned} &\lim_{N\to+\infty} \frac{1}{\delta}\int_0^\delta e_N^1(\alpha)\cdot i\, d\alpha \\ &\quad = -\mathrm{sgn}(\cos\theta_0)\cos^2\theta_0 - \mathrm{sgn}(\cos\theta_0)\,|\cos\theta_0| \wedge |\sin\theta_0|\,(|\sin\theta_0| - |\cos\theta_0|)\, . \end{aligned} \quad (27)$$

Using the same method, we prove that

$$\begin{aligned} &\lim_{N\to+\infty} \frac{1}{\delta}\int_0^\delta e_N^2(\alpha)\cdot i\, d\,\alpha \\ &\quad = \mathrm{sgn}(\cos\theta_1)\cos^2\theta_1 + \mathrm{sgn}(\cos\theta_1)\,|\cos\theta_1| \wedge |\sin\theta_1|\,(|\sin\theta_1| - |\cos\theta_1|)\, . \end{aligned} \quad (28)$$

We finish the proof of proposition 1 by combining proposition 2 and the limits (27) and (28).
□

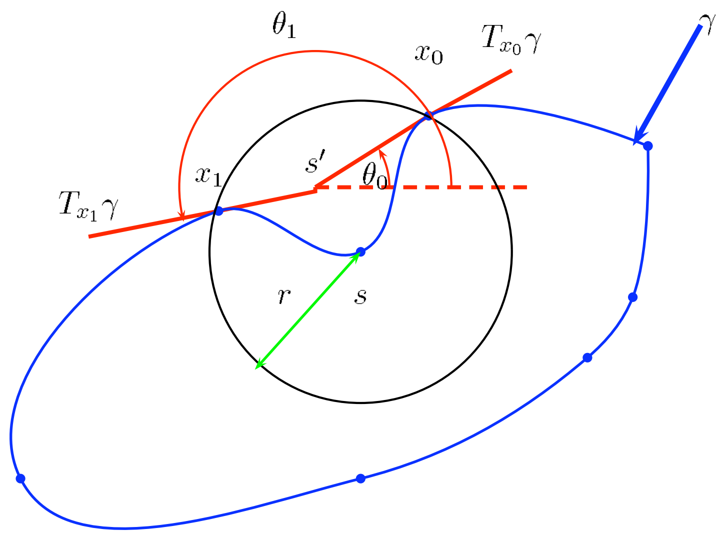
4.2 Extension to an arbitrary Jordan curve of class $\mathcal{C}_1$.

The purpose of the following proposition is to evaluate $\mathbf{A}_N^\gamma(s, r, \delta)$ for any Jordan curve γ of class $\mathcal{C}_1$. For any $x \in \gamma$, we denote by $T_x\gamma$ the tangent to the curve γ at the point x.

Proposition 3 *Let γ be a Jordan curve of $\mathbb{R}^2$ of class $\mathcal{C}_1$. Suppose that γ encloses a connected, compact and bounded set Ω of $\mathbb{R}^2$ i.e. $\gamma = \partial\Omega$. Let $s \in \gamma$ be fixed. Let r be a positive real number such that $\partial B(s, r) \cap \gamma$ contains exactly two points x_0 and x_1. Suppose that x_0, s and x_1 are arranged counterclockwise. Let s' be the common point to $T_{x_0}\gamma$ and $T_{x_1}\gamma$. Let $\theta_1 := \theta_1(r)$*

(respectively $\theta_0 := \theta_0(r)$) be the oriented angle in $[0, 2\pi[$ between the half horizontal axis $[0, +\infty[$ and the segment $[s', x_1[$ (respectively $[s', x_0[$). Then, for r small enough,

$$\begin{aligned}
&\lim_{\delta\to 0}\liminf_{N\to+\infty} \mathbf{A}_N^\gamma(s,r,\delta) = \lim_{\delta\to 0}\limsup_{N\to+\infty} \mathbf{A}_N^\gamma(s,r,\delta) \\
&\quad = \frac{1}{4}\sin 2\theta_0 \left(2\mathbb{1}_{|\sin\theta_0|<|\cos\theta_0|} - 1\right) - \frac{1}{4}\sin 2\theta_1 \left(2\mathbb{1}_{|\sin\theta_1|<|\cos\theta_1|} - 1\right) \\
&\quad + \frac{1}{2}\left(\operatorname{sgn}(\tan\theta_1)\mathbb{1}_{|\sin\theta_1|<|\cos\theta_1|} - \operatorname{sgn}(\tan\theta_0)\mathbb{1}_{|\sin\theta_0|<|\cos\theta_0|}\right) + \operatorname{sgn}(\theta_1-\theta_0)\mathbb{1}_{\sin\theta_0\sin\theta_1>0}.
\end{aligned}$$



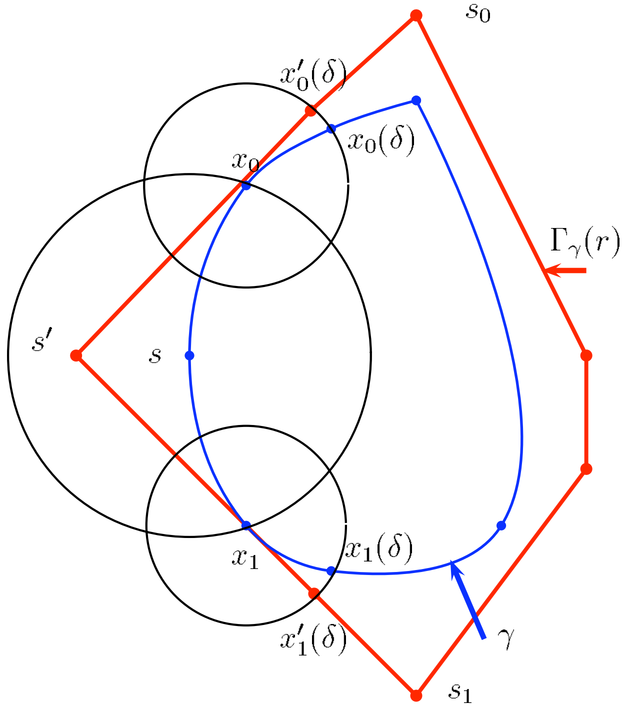
Proof of Proposition 3. We deal only with the lim sup, the lim inf can be handled similarly. Let $\Gamma_\gamma(r)$ be a polygon having three consecutive corners s_0, s' and s_1 arranged counterclockwise such that, for $l \in \{0,1\}$, x_l is a point of the segment $[s', s_l]$ and $|s' - s_l| > r + \delta$. Suppose that $\Gamma_\gamma(r) = \partial U_r$. For $l = 0, 1$, we denote by $x_l(\delta)$ (resp. by $x'_l(\delta)$) the point of $\gamma \cap B(x_l,\delta) \cap \left(\mathbb{R}^2 \setminus B(s,r)\right)$ $\left(\text{resp. of } \Gamma_\gamma(r) \cap B(x_l,\delta) \cap \left(\mathbb{R}^2 \setminus B(s,r)\right)\right)$. We choose r and δ sufficiently small, so that for any point x of γ varying between $x_l(\delta)$ and s, for $l \in \{0,1\}$, one has,

$$\operatorname{sgn}(\mathrm{sx}\cdot \mathrm{i}) = \operatorname{sgn}(\mathrm{sx_l}(\delta)\cdot \mathrm{i}) = \operatorname{sgn}(\cos\theta_\mathrm{l}),$$

and

$$\operatorname{sgn}\left(\mathrm{sx}\cdot \mathrm{j}\right) = \operatorname{sgn}\left(\mathrm{sx_l}(\delta)\cdot \mathrm{j}\right) = \operatorname{sgn}(\sin\theta_\mathrm{l}).$$

This restriction allows to deduce that the parts of the curves γ and $\Gamma_\gamma(r)$ respectively between $x_l(\delta)$, s and $x'_l(\delta)$, s' are both nonincreasing or both nondecreasing.



For a Jordan curve ϕ which is the boundary of a domain Φ, we denote by $\mathcal{L}_N^\phi$ the polygonal line

$$\mathcal{L}_N^\phi = \partial(\mathcal{S}_N \cap \Phi_N) \cap (\mathbb{R}^2 \setminus \Phi).$$

From what precedes, we see that $\mathcal{L}_N^\gamma$ and $\mathcal{L}_N^{\Gamma_\gamma(r)}$ have the same monotony.

All the calculations done with polygons as initial condition can be carried out to this case, that means that

$$\mathbf{A}_N^\gamma(s,r,\delta) = \frac{1}{2}\mathrm{sgn}(\sin\theta_0)\,\frac{1}{\delta}\int_0^\delta \mathrm{e}_\mathrm{N}^1(\alpha)\cdot\mathrm{i}\,\mathrm{d}\alpha + \frac{1}{2}\mathrm{sgn}(\sin\theta_1)\,\frac{1}{\delta}\int_0^\delta \mathrm{e}_\mathrm{N}^2(\alpha)\cdot\mathrm{i}\,\mathrm{d}\alpha + \mathrm{sgn}(\theta_1-\theta_0)\mathbb{1}_{\sin\theta_0\sin\theta_1>0}, \tag{29}$$

where the unit vectors $e_N^1(\alpha)$ and $e_N^2(\alpha)$ are as defined by (5) with

$$\mathcal{L}_N = \mathcal{L}_N^\gamma = \partial(\mathcal{S}_N \cap \Omega_N) \cap (\mathbb{R}^2 \setminus \Omega).$$

Our task now is to evaluate $\int_0^\delta e_N^l(\alpha)\cdot i\,d\alpha$, for $l=1,2$. We have, collecting (19), (20), (21), (26),

$$\left|\int_0^\delta e_N^1(\alpha)\cdot i\,d\alpha+\frac{1}{N}\sum_{l\in\mathcal{H}_{1,N}}\frac{(x_0a_{l+1})\cdot i}{|x_0-a_{l+1}|}+\frac{\operatorname{sgn}(\tan\theta_0)}{N}\sum_{l\in\mathcal{H}_{2,N}}\frac{(x_0a_{l+1})\cdot j}{|x_0-a_{l+1}|}\right|=\mathcal{O}\left(\frac{\ln(|\mathcal{H}_N|)}{N}\right),\tag{30}$$

where $\mathcal{H}_{1,N}$ and $\mathcal{H}_{2,N}$ are defined by (22).
Let x_l be the point of the curve γ situated between x_0 and $x_0(\delta)$ for which $|x_0-a_{l+1}|=|x_0-x_l|$. An analogue of lemma 5, together with (30), ensures

$$\left|\int_0^\delta e_N^1(\alpha)\cdot i\,d\alpha+\frac{1}{N}\sum_{l\in\mathcal{H}_{1,N}}\frac{(x_0x_l)\cdot i}{|x_0-x_l|}+\frac{\operatorname{sgn}(\tan\theta_0)}{N}\sum_{l\in\mathcal{H}_{2,N}}\frac{(x_0x_l)\cdot j}{|x_0-x_l|}\right|=\mathcal{O}\left(\frac{\ln(|\mathcal{H}_N|)}{N}\right).$$

For any $x\in\gamma$, we denote by θ_x the angle between the horizontal axis and the segment $[x_0,x]$ and we set $\theta(\delta)=\theta_{x_0(\delta)}$. Now

$$\frac{1}{N\delta}\sum_{l\in\mathcal{H}_{1,N}}\left|\frac{(x_0x_l)\cdot i}{|x_0-x_l|}-\cos\theta(\delta)\right|\leq\frac{|\mathcal{H}_N|}{N\delta}\sup_x|\cos\theta_x-\cos\theta(\delta)|,$$

where the supremum is taken over all the points x of γ situated between x_0 and $x_0(\delta)$. Since

$$\lim_{\delta\to0}\lim_{N\to+\infty}\frac{|\mathcal{H}_N|}{N\delta}=|\cos\theta_0|,\tag{31}$$

we deduce from the last bound that

$$\lim_{\delta\to0}\limsup_{N\to+\infty}\frac{1}{N\delta}\sum_{l\in\mathcal{H}_{1,N}}\left|\frac{(x_0x_l)\cdot i}{|x_0-x_l|}-\cos\theta(\delta)\right|=0.\tag{32}$$

Using the same arguments, we prove that

$$\lim_{\delta\to0}\limsup_{N\to+\infty}\frac{1}{N\delta}\sum_{l\in\mathcal{H}_{2,N}}\left|\frac{(x_0x_l)\cdot j}{|x_0-x_l|}-\sin\theta(\delta)\right|=0.\tag{33}$$

Now

$$\begin{aligned}&\frac{|\mathcal{H}_{1,N}|}{N\delta}\cos\theta(\delta)+\operatorname{sgn}(\tan\theta_0)\frac{|\mathcal{H}_{2,\mathrm{N}}|}{\mathrm{N}\delta}\sin\theta(\delta)\\&\quad=\frac{|\mathcal{H}_N|}{N\delta}\cos\theta(\delta)+\frac{|\mathcal{H}_{2,N}|}{N\delta}\left(\operatorname{sgn}(\tan\theta_0)\sin\theta(\delta)-\cos\theta(\delta)\right).\end{aligned}\tag{34}$$

In order to evaluate the quantity $\dfrac{|\mathcal{H}_{2,N}|}{N\delta}$, we need the following generalization of lemma 6.

Lemma 7 *Let ϕ be a monotone function of class $\mathcal{C}_1$ defined on $[a,b]$. Let $\mathcal{L}_N$ denote one of the two maximal subpaths of $\mathbb{Z}_N^2$ covering ϕ. Let $N_+(\mathcal{L}_N)$ be as defined in (7). Then*

$$\lim_{N\to+\infty}\frac{N_+(\mathcal{L}_N)}{N}=\int_a^b(|\phi'(x)|\wedge 1)\,dx.$$

Proof of lemma 7. We suppose without loss of generality that the function ϕ is nondecreasing on $[a,b]$. Let $(I_i)_{i\in I}$ be the collection of the open intervals where $\phi'-1$ is nonzero. Setting $I_i=]x_{i-1},x_i[$ for $i\in I$, we have

$$\left(\frac{\phi(x_i)-\phi(x_{i-1})}{x_i-x_{i-1}}\right)\wedge 1=\frac{1}{x_i-x_{i-1}}\int_{x_{i-1}}^{x_i}(\phi'(x)\wedge 1)\,dx. \tag{35}$$

We denote by ϕ_i the restriction of ϕ to $[x_{i-1},x_i[$ and by $\mathcal{L}_N^{(i)}$ the associated polygonal line. We deduce from the suitable construction of the intervals $(I_i)_{i\in I}$ and arguing as in the proof of lemma 6, that

$$\lim_{N\to+\infty}\frac{N_+(\mathcal{L}_N^{(i)})}{N}=(x_i-x_{i-1})\wedge(\phi(x_i)-\phi(x_{i-1})).$$

Hence

$$\lim_{N\to+\infty}\frac{N_+(\mathcal{L}_N)}{N}=\sum_{i\in I}(x_i-x_{i-1})\wedge(\phi(x_i)-\phi(x_{i-1})).$$

Lemma 7 is proved by collecting the last bound together with (35). □

We now continue the proof of proposition 3. We define a monotone function ϕ, such that the part of γ limited by x_0 and $x_0(\delta)$ is equal to the graph $\{(x,y):\,y=\phi(x)\}$ and we apply lemma 7 to the monotone path $\mathcal{L}_N$ covering the part of γ limited by x_0 and $x_0(\delta)$. We deduce, since $|N_+(\mathcal{L}_N)-|\mathcal{H}_{2,N}||\leq 1$, that

$$\begin{aligned}\lim_{N\to+\infty}\frac{|\mathcal{H}_{2,N}|}{N\delta} &= \frac{1}{\delta}\int_{I_\delta}(|\phi'(x)|\wedge 1)\,dx\\ &= |\cos\theta(\delta)|\frac{1}{I_\delta}\int_{I_\delta}(|\phi'(x)|\wedge 1)\,dx,\end{aligned}$$

where I_δ is the segment $[x_0\cdot i, x_0(\delta)\cdot i]$. We obtain, taking the limit over $\delta\to 0$ in the last equality,

$$\begin{aligned}\lim_{\delta\to 0}\lim_{N\to+\infty}\frac{|\mathcal{H}_{2,N}|}{N\delta} &= |\cos\theta_0|\,(|\phi'(x_0\cdot i)|\wedge 1)\\ &= |\cos\theta_0|\wedge|\sin\theta_0|.\end{aligned}$$

The last limit together with (34) and (31) ensures

$$\begin{aligned}&\lim_{\delta\to 0}\lim_{N\to+\infty}\left(\frac{|\mathcal{H}_{1,N}|}{N\delta}\cos\theta(\delta)+\operatorname{sgn}(\tan\theta_0)\frac{|\mathcal{H}_{2,\mathrm{N}}|}{\mathrm{N}\delta}\sin\theta(\delta)\right)\\ &=\operatorname{sgn}(\cos\theta_0)\cos^2\theta_0+\operatorname{sgn}(\cos\theta_0)\,|\cos\theta_0|\wedge|\sin\theta_0|\,(|\sin\theta_0|-|\cos\theta_0|)\,.\end{aligned} \tag{36}$$

We deduce, collecting (30), (32), (33), (36), that the successive limits over $N \to +\infty$ and $\delta \to 0$ of $\frac{1}{\delta}\int_0^\delta e_N^1(\alpha)\cdot i\,d\alpha$ are exactly the limit obtained in (27). In the same way, the successive limits over $N \to +\infty$ and $\delta \to 0$ of $\frac{1}{\delta}\int_0^\delta e_N^2(\alpha)\cdot i\,d\alpha$ are exactly the limit obtained in (28) (recall that the angles θ_0 and θ_1 are as defined in proposition 3). Those facts together with (29) prove proposition 3.

4.3 End of the proof of theorem 1

We deal only with the lim sup, the lim inf can be handled similarly. We suppose first that θ takes a value different from $(2k+1)\frac{\pi}{4}$, for $k \in \mathbb{N}$. Since the curve γ admits a tangent at the point s, then for r small enough $(\theta_0, \theta_1) \in [(2k+1)\frac{\pi}{4}, (2k+3)\frac{\pi}{4}] \times [(2k+5)\frac{\pi}{4}, (2k+7)\frac{\pi}{4}]$, for some $k \in \mathbb{N}$. We then deduce from proposition 3 that,

- if $(\theta_0, \theta_1) \in [(2k+1)\frac{\pi}{4}, (2k+3)\frac{\pi}{4}] \times [(2k+5)\frac{\pi}{4}, (2k+7)\frac{\pi}{4}]$, with $k \in \{0, 2\}$, then

$$\lim_{\delta\to 0}\limsup_{N\to+\infty} \mathbf{A}_N^\gamma(s, r, \delta) = \frac{1}{4}\left(\sin 2\theta_1 - \sin 2\theta_0\right).$$

- if $(\theta_0, \theta_1) \in [(2k+1)\frac{\pi}{4}, (2k+3)\frac{\pi}{4}] \times [(2k+5)\frac{\pi}{4}, (2k+7)\frac{\pi}{4}]$, with $k \in \{1, 3\}$, then

$$\lim_{\delta\to 0}\limsup_{N\to+\infty} \mathbf{A}_N^\gamma(s, r, \delta) = \frac{1}{4}\left(\sin 2\theta_0 - \sin 2\theta_1\right).$$

We now need the following lemma.

Lemma 8 *Let γ be a Jordan curve of $\mathbb{R}^2$ of class $\mathcal{C}_2$. Let s be a fixed point of γ. Let r be a positive real number sufficiently small such that $\partial B(s,r)\cap\gamma$ contains exactly two points x_0 and x_1. Suppose that x_0, s and x_1 are arranged counterclockwise. Let s' be the common point to $T_{x_0}\gamma$ and $T_{x_1}\gamma$. Let $\theta_1 \in [0, 2\pi]$ (respectively $\theta_0 \in [0, 2\pi]$) be the oriented angle between the half horizontal axis $[0, +\infty[$ and the segment $[s', x_1[$ (respectively $[s', x_0[$). Then*

$$\lim_{r\to 0} \frac{\sin(\theta_0 - \theta_1)}{2r} = \xi_\gamma(s),$$

and

$$\lim_{r\to 0} \cos(\theta_0 + \theta_1) = -\cos 2\theta,$$

where θ is the angle between the half horizontal axis $[0, +\infty[$ and $T_s\gamma$.

Lemma 8, together with the fact $\sin 2a - \sin 2b = 2\sin(a-b)\cos(a+b)$, gives

$$\lim_{r\to 0}\lim_{\delta\to 0}\limsup_{N\to+\infty} \frac{1}{2r}\mathbf{A}_N^\gamma(s, r, \delta) = \begin{cases} \frac{1}{2}(\cos 2\theta)\,\xi_\gamma(s) & \textit{if } \theta \in](1+4k)\frac{\pi}{4}, (3+4k)\frac{\pi}{4}[\\ -\frac{1}{2}(\cos 2\theta)\,\xi_\gamma(s) & \textit{if } \theta \in](3+4k)\frac{\pi}{4}, (5+4k)\frac{\pi}{4}[. \end{cases}$$

which proves theorem 1 when θ is different from $(2k+1)\frac{\pi}{4}$, for $k \in \mathbb{N}$. Now, suppose that $\theta = \frac{\pi}{4}$ and that for any r small enough $(\theta_0, \theta_1) \in [\frac{\pi}{4}, 3\frac{\pi}{4}] \times [3\frac{\pi}{4}, 5\frac{\pi}{4}]$ (the arguments for the proof for the other values of θ and the corresponding values of θ_1, θ_0 will be similar). Proposition 3 gives

$$\begin{aligned} \lim_{\delta\to 0} \limsup_{N\to+\infty} \mathbf{A}_N^\gamma(s,r,\delta) &= \frac{1}{4}(2 - \sin 2\theta_1 - \sin 2\theta_0) \\ &= \frac{1}{2}\left(\sin(\frac{\pi}{4} - \theta_1)\cos(\frac{\pi}{4} + \theta_1) + \sin(\frac{\pi}{4} - \theta_0)\cos(\frac{\pi}{4} + \theta_0)\right). \quad (37) \end{aligned}$$

Now the method of the proof of lemma 8 gives

$$\lim_{r\to 0} \frac{\sin(\theta - \theta_1)}{r} = \lim_{r\to 0} \frac{\sin(\theta - \theta_0)}{r} = -\xi_\gamma(s).$$

This fact, together with (37), leads to

$$\lim_{r\to 0} \lim_{\delta\to 0} \limsup_{N\to+\infty} \frac{1}{r} \mathbf{A}_N^\gamma(s,r,\delta) = 0,$$

which is the conclusion of theorem 1 for $\theta = \frac{\pi}{4}$.

Proof of lemma 8. We begin by giving the definition of the curvature of γ at any $s \in \gamma$.

Definition. Let γ be a smooth Jordan curve of $\mathbb{R}^2$. Suppose that $(\phi(t))_{t\in[-1,1]}$ is a parametrization of the curve γ. Let $s = \phi(t) = (x(t), y(t))$ be a fixed point of γ. The **curvature** of γ at the point s is defined by

$$\xi_\gamma(s) = \frac{x'(t)y''(t) - x''(t)y'(t)}{(x'^2(t) + y'^2(t))^{3/2}}.$$

Let s, x_0 and x_1 be as defined in lemma 8. Let t, t_0 and t_1 be three real numbers of $[-1,1]$ such that $s = \phi(t) = (x(t), y(t))$, and for $i \in \{0,1\}$, $x_i = \phi(t_i) = (x(t_i), y(t_i))$. We have $r^2 = (x(t_i) - x(t))^2 + (y(t_i) - y(t))^2$, for $i \in \{0,1\}$. Hence

$$\lim_{t_0\to t,\, t_0<t} \frac{r}{t - t_0} = \sqrt{x'^2(t) + y'^2(t)}, \qquad \lim_{t_1\to t,\, t<t_1} \frac{r}{t_1 - t} = \sqrt{x'^2(t) + y'^2(t)}.$$

For any $\tau \in [-1,1]$, define $f(\tau) = \dfrac{x'(\tau)}{\sqrt{x'^2(\tau) + y'^2(\tau)}}$. We have

$$f'(\tau) = \frac{x''(\tau)}{\sqrt{x'^2(\tau) + y'^2(\tau)}} - x'(\tau)\frac{x'(\tau)x''(\tau) + y'(\tau)y''(\tau)}{(x'^2(\tau) + y'^2(\tau))^{3/2}}.$$

Hence

$$\begin{aligned} \cos\theta_0 &= -\frac{x'(t_0)}{\sqrt{x'^2(t_0) + y'^2(t_0)}} \\ &= -\frac{x'(t)}{\sqrt{x'^2(t) + y'^2(t)}} + (t - t_0)f'(t) + o(|t - t_0|). \quad (38) \end{aligned}$$

$$\cos\theta_1 = \frac{x'(t)}{\sqrt{x'^2(t)+y'^2(t)}} + (t_1 - t)f'(t) + o\left(|t_1 - t|\right). \tag{39}$$

We obtain, combining the last two equalities

$$\lim_{t_1\to t,\, t_0\to t,\, t_0<t<t_1} \frac{\cos\theta_0 + \cos\theta_1}{r} = \frac{2x''(t)}{x'^2(t)+y'^2(t)} - 2x'(t)\frac{x'(t)x''(t)+y'(t)y''(t)}{(x'^2(t)+y'^2(t))^2}.$$

The last limit together with

$$\lim_{t_1\to t,\, t<t_1} \sin\theta_1 = \frac{y'(t)}{\sqrt{x'^2(t)+y'^2(t)}},$$

ensures

$$\lim_{t_1\to t,\, t_0\to t,\, t_0<t<t_1} \frac{1}{r}\sin\theta_1\left(\cos\theta_0+\cos\theta_1\right) = \frac{2x''(t)y'(t)}{(x'^2(t)+y'^2(t))^{3/2}} - 2x'(t)y'(t)\frac{x'(t)x''(t)+y'(t)y''(t)}{(x'^2(t)+y'^2(t))^{5/2}}.$$

In the same way, we prove that

$$\lim_{t_1\to t,\, t_0\to t,\, t_0<t<t_1} \frac{1}{r}\cos\theta_1\left(\sin\theta_0+\sin\theta_1\right) = \frac{2x'(t)y''(t)}{(x'^2(t)+y'^2(t))^{3/2}} - 2x'(t)y'(t)\frac{x'(t)x''(t)+y'(t)y''(t)}{(x'^2(t)+y'^2(t))^{5/2}}.$$

The last two limits together with

$$\sin\left(\theta_0-\theta_1\right) = \cos\theta_1\left(\sin\theta_1+\sin\theta_0\right) - \sin\theta_1\left(\cos\theta_0+\cos\theta_1\right),$$

prove that

$$\lim_{t_1\to t,\, t_0\to t,\, t_0<t<t_1} \frac{1}{2r}\sin\left(\theta_0-\theta_1\right) = \frac{x'(t)y''(t)-x''(t)y'(t)}{(x'^2(t)+y'^2(t))^{3/2}}.$$

Now the equality

$$\cos\left(\theta_0+\theta_1\right) = \cos\theta_0\cos\theta_1 - \sin\theta_0\sin\theta_1,$$

together with the limits (38), (39), yields

$$\lim_{t_1\to t,\, t_0\to t,\, t_0<t<t_1} \cos\left(\theta_0+\theta_1\right) = \frac{y'^2(t)-x'^2(t)}{x'^2(t)+y'^2(t)}.$$

The last limit is equal to $-\cos 2\theta$, where θ is the angle between the horizontal axis and $T_s\gamma$. □

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