A lower bound on the two–arms exponent for critical percolation on the lattice

Raphaël Cerf
Université Paris Sud and IUF

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Abstract

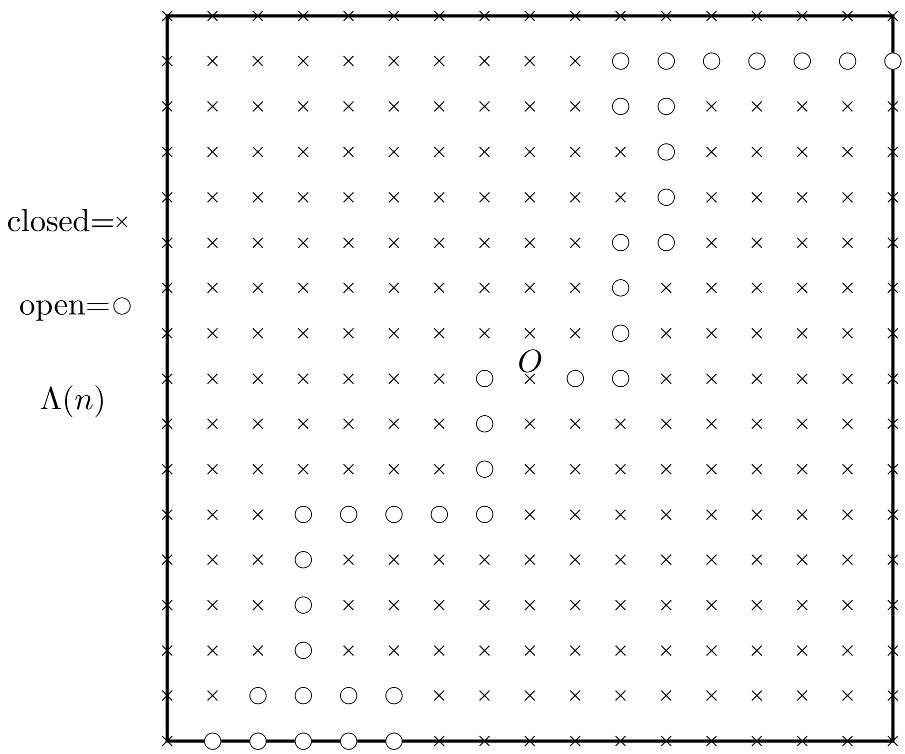
We consider the standard site percolation model on the d dimensional lattice. A direct consequence of the proof of the uniqueness of the infinite cluster of Aizenman, Kesten and Newman [1] is that the two–arms exponent is larger than or equal to 1/2. We improve slightly this lower bound in any dimension $d \geq 2$. Next, starting only with the hypothesis that $\theta(p) > 0$, without using the slab technology, we derive a quantitative estimate establishing long–range order in a finite box.

1 Introduction

We consider the site percolation model on $\mathbb{Z}^d$. Each site is declared open with probability p and closed with probability $1-p$, and the sites are independent. Little is rigorously known on the percolation model at the critical point p_c in three dimensions. There exists one remarkable result, a rigorous lower bound on the two–arms exponent, which says that, for any $d \geq 2$,

$$\exists\, \kappa > 0 \quad \forall n \geq 1 \qquad P_{p_c}\big(\text{two–arms}(0,n)\big) \;\leq\; \frac{\kappa \ln n}{\sqrt{n}}\,.$$

The event "two–arms$(0,n)$" is the event that two neighbours of 0 are connected to the boundary of the box $\Lambda(n) = [-n,n]^d$ by two disjoint open clusters. Although some percolationists are aware of this estimate (for instance, it is explicitly used by Zhang in [11]), it does not seem to be fully written in the literature. This estimate can be obtained as a byproduct of the proof of the uniqueness of the infinite cluster of Aizenman, Kesten and Newman [1]. This deep proof was originally written for a quite general percolation model. A simplified and illuminating version has been worked out by Gandolfi, Grimmett and Russo [2]. The two–arms estimate is obtained by taking $\varepsilon = \kappa \ln n/\sqrt{n}$ in the proof of [2]. Nowadays the uniqueness of the infinite cluster in percolation is proved with the help of the more robust Burton–Keane argument: see for instance [3] or [5]. Yet the Burton–Keane argument relies on translation invariance, and it does not yield any quantitative estimate, contrary to the argument



two–arms$(0,n)$

of Aizenman, Kesten and Newman. The first main result of this paper is a slightly improved lower bound on the two–arms exponent.

Theorem 1.1. *Let $d \geq 2$ and let p_c be the critical probability of the site percolation model in d dimensions. We have*

$$\limsup_{n\to\infty} \frac{1}{\ln n} \ln P_{p_c}\big(\textit{two–arms}(0,n)\big) \;\leq\; \frac{2d^2+3d-3}{4d^2+5d-5}\,.$$

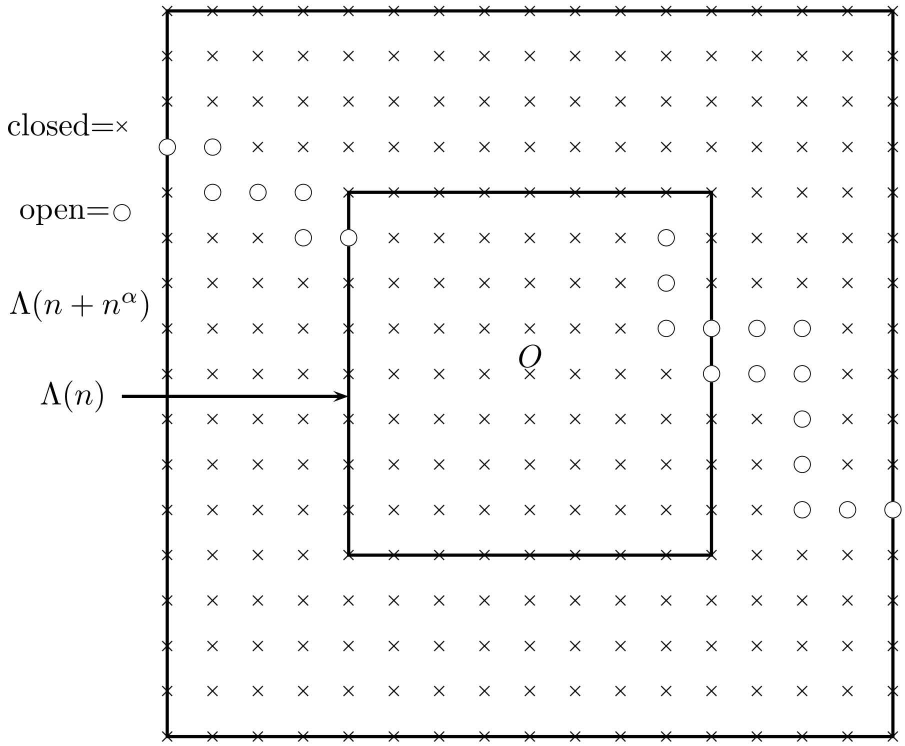
In two dimensions, our two arms event correspond to a four arms event with alternating colors. The corresponding exponent is rigorously known to be equal to 5/4 for site percolation on the triangular lattice (see [10]), and our lower bound is 11/21. In three dimensions, we obtain the following estimate:

$$\forall \gamma < \frac{12}{23} \quad \exists\, c>0 \quad \forall n\geq 1 \qquad P_{p_c}\big(\text{two–arms}(0,n)\big) \;\leq\; \frac{c}{n^\gamma}\,.$$

To prove theorem 1.1, we rework the proof of [2] in order to obtain an inequality of the form

$$P_{p_c}\big(\text{two–arms}(0,n)\big) \;\leq\; \frac{2d\ln n}{\sqrt{\big|\Lambda(n)\big|}} E\Big(\sqrt{|\mathcal{C}|}\Big) + \text{negligible term}\,,$$

where $\mathcal{C}$ is the collection of the clusters joining $\Lambda(n)$ to the boundary of $\Lambda(2n)$. From this inequality, we obtain the previously known estimate on the two–arms



two–arms$(\Lambda(n), n^\alpha)$

event by bounding the number of clusters in $\mathcal{C}$ by $2d(2n+1)^{d-1}$. We next try to enhance the control on the number of clusters. It turns out that the expectation of this number can be bounded with the help of the probability of the two–arms event. Our strategy consists in controlling the two–arms event associated to a box. This is the purpose of our second main result. The event "two–arms$(\Lambda(n), n^\alpha)$" is the event that two sites of the box $\Lambda(n) = [-n, n]^d$ are connected to the boundary of the box $\Lambda(n+n^\alpha)$ by two disjoint open clusters.

Theorem 1.2. *Let $d \geq 2$ and let p_c be the critical probability of the site percolation model in d dimensions. Let α be such that*

$$\alpha \,>\, \frac{2d^2+2d-2}{2d^2+3d-3}(4d^2+5d-5)\,.$$

We have

$$\lim_{n\to\infty} P_{p_c}\big(\textit{two–arms}(\Lambda(n), n^\alpha)\big) \,=\, 0\,.$$

For $d=3$, this gives

$$\lim_{n\to\infty} P_{p_c}\big(\text{two–arms}(\Lambda(n), n^{43})\big) \,=\, 0\,.$$

Next, we cover the boundary of the box $\Lambda(n) = [-n, n]^d$ by a collection of boxes of side length n^β, with β small. Theorem 1.2 yields an estimate on the number of small boxes joined to the boundary of $\Lambda(2n)$ by at most one cluster, from which we obtain an upper bound on the mean number of open clusters joining $\Lambda(n)$ to the boundary of $\Lambda(2n)$. This gives an upper bound on $E(|\mathcal{C}|)$ in terms of the two–arms event. This way we obtain an inequality of the form

$$P(\text{two–arms}(0,3n)) \leq \frac{c' \ln n}{\sqrt{n}} \left(\frac{1}{k^{d-1}} + k^{2d^2+2d-2} P\big(\text{two–arms}(0,n)\big) \right)^{1/2} .$$

Iterating this inequality with an adequate choice of $k \leq n$, we progressively improve the exponent 1/2. We obtain a sequence of exponents converging geometrically towards the limiting value presented in theorem 1.1. The final improvement is quite disappointing and the value is probably quite far from the correct one.

Our third main result is a little minor step for the establishment of long–range order in a finite box. This is a central question, which, if correctly answered, should lead to a proof that $\theta(p_c) = 0$. For Λ a box and x, y in Λ, we denote by $\{\, x \longleftrightarrow y \text{ in } \Lambda \,\}$ the event that x, y are connected by an open path inside Λ.

Theorem 1.3. *Let $d \geq 2$ and let p be such that $\theta(p) > 0$. Let α be such that*

$$\alpha > \frac{4d^2+5d-5}{2d^2+3d-3}(3d-1) .$$

We have

$$\inf_{n\geq 1} \inf \left\{ P_p\big(x \longleftrightarrow y \textit{ in } \Lambda(n^\alpha)\big) : x, y \in \Lambda(n) \right\} > 0 .$$

For $d = 3$, this gives the following estimate:

$$\exists\, \rho > 0 \quad \forall n \geq 1 \quad \forall x, y \in \Lambda(n) \qquad P_p\big(x \longleftrightarrow y \text{ in } \Lambda(n^{16})\big) \geq \rho .$$

One of the most important problems in percolation is to prove that, in three dimensions, there is no infinite cluster at the critical point. The most promising strategy so far seems to perform a renormalization argument [3, 9]. The missing ingredient is a suitable construction helping to define a good block, starting solely with the hypothesis that $\theta(p) > 0$. For instance, it would be enough to have the above estimate within a box of side length proportional to n. Moreover, if the famous conjecture $\theta(p_c) = 0$ was true, such an estimate would indeed hold. Here again, we are still far from the desired result. Our technique to prove theorem 1.3 is to inject the hypothesis $\theta(p) > 0$ inside the proof of the two–arms estimate for a box. This allows to obtain a much better control on the probability of a long connection, which is unfortunately still far from optimal.

2 Basic notation

Two sites x, y of the lattice $\mathbb{Z}^d$ are said to be connected if they are nearest neighbours, i.e., if $|x - y| = 1$. Let A be a subset of $\mathbb{Z}^d$. We define its internal boundary $\partial^{\,in} A$ and its external boundary $\partial^{\,out} A$ by

$$\partial^{\,in} A = \left\{ x \in A : \exists y \in A^c \quad |x - y| = 1 \right\},$$

$$\partial^{out} A \,=\, \big\{\, x \in A^c : \exists y \in A \quad |x-y| = 1 \,\big\}\,.$$

For $x \in \mathbb{Z}^d$, we denote by $C(x)$ the open cluster containing x, i.e., the connected component of the set of the open sites containing x. If x is closed, then $C(x)$ is empty. For $n \in \mathbb{N}$, we denote by $\Lambda(n)$ the cubic box

$$\Lambda(n) \,=\, [-n,n]^d\,.$$

Let n,ℓ be two integers. We consider the open clusters of the percolation configuration restricted to $\Lambda(n{+}\ell)$. These open clusters are the connected components of the graph having for vertices the sites of $\Lambda(n+\ell)$ which are open, endowed with edges between nearest neighbours. We denote by $\mathcal{C}$ the collection of the open clusters in $\Lambda(n+\ell)$ which intersect both $\Lambda(n)$ and $\partial^{in}\Lambda(n+\ell)$, i.e.,

$$\mathcal{C} \,=\, \big\{\, C \text{ open cluster in } \Lambda(n+\ell) : C \cap \Lambda(n) \neq \varnothing,\, C \cap \partial^{in}\Lambda(n+\ell) \neq \varnothing \,\big\}\,.$$

3 The proof of Gandolfi, Grimmett and Russo

We reproduce here the initial step of the argument of Gandolfi, Grimmett and Russo to prove the uniqueness of the infinite cluster [2]. This argument was obtained from the more complex work of Aizenman, Kesten and Newman [1]. The only difference is that we introduce an additional parameter ℓ. We will use specific values for ℓ later on. We define the following three subsets of $\Lambda(n)$:

$$F \,=\, \bigcup_{C \in \mathcal{C}} C \cap \Lambda(n)\,, \qquad G \,=\, \bigcup_{C \in \mathcal{C}} \partial^{out} C \cap \Lambda(n)\,,$$
$$H \,=\, \bigcup_{C_1, C_2 \in \mathcal{C}} \big(\partial^{out} C_1 \cap \partial^{out} C_2 \cap \Lambda(n)\big)\,.$$

A site of $\Lambda(n)$ belongs to F if it is connected to $\partial^{in}\Lambda(n{+}\ell)$ by an open path. A site of $\Lambda(n)$ belongs to G if it is closed and it has a neighbour which is connected to $\partial^{in}\Lambda(n+\ell)$ by an open path. A site of $\Lambda(n)$ belongs to $F \cup G$ if it has a neighbour which is connected to $\partial^{in}\Lambda(n+\ell)$ by an open path. Yet, for any $x \in \Lambda(n)$, the event

$$\big\{\, \text{a neighbour of } x \text{ is connected to } \partial^{in}\Lambda(n+\ell) \text{ by an open path} \,\big\}$$

is independent of the status of the site x itself, therefore

$$P\big(\, x \in F \,|\, x \in F \cup G \,\big) \,=\, P(x \text{ is open}) \,=\, p\,,$$
$$P\big(\, x \in G \,|\, x \in F \cup G \,\big) \,=\, P(x \text{ is closed}) \,=\, 1-p\,.$$

Summing over $x \in \Lambda(n)$, we obtain

$$\begin{aligned} E\big(|F|\big) \,=\, & E\Big(\sum_{x\in\Lambda(n)} 1_{x\in F}\Big) \\ &= \sum_{x\in\Lambda(n)} P(x\in F) \,=\, \sum_{x\in\Lambda(n)} P\big(\,x\in F\,|\,x\in F\cup G\,\big)P\big(\,x\in F\cup G\,\big) \\ &\qquad = \sum_{x\in\Lambda(n)} p\,P\big(\,x\in F\cup G\,\big) \,=\, p\,E\big(|F\cup G|\big)\,. \end{aligned}$$

Similarly, we have

$$E\big(|G|\big) \,=\, (1-p)\,E\big(|F\cup G|\big)\,.$$

We wish to estimate the cardinality of H. To this end, we write

$$\begin{aligned} |H| \,=\, & \Big|\bigcup_{C_1,C_2\in\mathcal{C}} \big(\partial^{\,out}C_1\cap\partial^{\,out}C_2\cap\Lambda(n)\big)\Big| \\ &\leq \sum_{C\in\mathcal{C}} \Big|\partial^{\,out}C\cap\Lambda(n)\Big| - \Big|\bigcup_{C\in\mathcal{C}} \partial^{\,out}C\cap\Lambda(n)\Big| \\ &\qquad \leq \sum_{C\in\mathcal{C}} \Big|\partial^{\,out}C\cap\Lambda(n)\Big| - \big|G\big|\,. \end{aligned}$$

Taking the expectation in this inequality, we obtain

$$\begin{aligned} E\big(|H|\big) \,\leq\, & E\bigg(\sum_{C\in\mathcal{C}} \Big|\partial^{\,out}C\cap\Lambda(n)\Big|\bigg) - E\Big(\big|G\big|\Big) \\ &= E\bigg(\sum_{C\in\mathcal{C}} \Big|\partial^{\,out}C\cap\Lambda(n)\Big|\bigg) - \frac{1-p}{p}E\Big(\big|F\big|\Big) \\ &= (1-p)E\bigg(\sum_{C\in\mathcal{C}} \Big(\frac{1}{1-p}\big|\partial^{\,out}C\cap\Lambda(n)\big| - \frac{1}{p}\big|C\cap\Lambda(n)\big|\Big)\bigg)\,. \end{aligned}$$

For A a subset of $\mathbb{Z}^d$, we define

$$h(A) \,=\, \frac{1}{1-p}\Big|\big\{\,x\in A : x \text{ is closed}\,\big\}\Big| - \frac{1}{p}\Big|\big\{\,x\in A : x \text{ is open}\,\big\}\Big|\,.$$

For C an open cluster, we define

$$\overline{C} \,=\, C\cup\partial^{\,out}C\,.$$

With these definitions, we can rewrite the previous inequality as

$$E\big(|H|\big) \,\leq\, (1-p)E\bigg(\sum_{C\in\mathcal{C}} h\big(\overline{C}\cap\Lambda(n)\big)\bigg)\,.$$

Our next goal is to control the expectation on the right hand side. We first notice that, for x in the box $\Lambda(n)$, the expected value of $h(\overline{C}(x)\cap\Lambda(n))$ is zero.

Lemma 3.1. *For any $x \in \Lambda(n)$, we have $E\big(h(\overline{C}(x) \cap \Lambda(n))\big) = 0$.*

Proof. Let $x \in \Lambda(n)$. For any lattice animal A containing x and included in $\Lambda(n)$, we have

$$P\big(C(x) = A\big) \;=\; p^{|A|}(1-p)^{|\partial^{\,out} A \cap \Lambda(n)|}\,.$$

Summing over all such lattice animals A, we get

$$1 \;=\; \sum_A p^{|A|}(1-p)^{|\partial^{\,out} A \cap \Lambda(n)|}\,.$$

Differentiating with respect to p, we obtain

$$0 \;=\; \sum_A \Big(\frac{|A|}{p} - \frac{|\partial^{\,out} A \cap \Lambda(n)|}{1-p}\Big) p^{|A|}(1-p)^{|\partial^{\,out} A \cap \Lambda(n)|}$$

and we notice that this last sum is equal to $E\big(h(\overline{C}(x) \cap \Lambda(n))\big)$. ☐

It turns out that, for large clusters, the value $h(\overline{C} \cap \Lambda(n))$ is close to 0 with high probability. This is quantified by the next proposition.

4 The large deviation estimate

The basic inequality leading to the control of the two–arms event relies on the following large deviation estimate. This estimate is a variant of the one stated in [1, 2]. We have introduced an additional parameter ℓ and we use Hoeffding's inequality.

Proposition 4.1. *For any p in $]0,1[$, any $n \geq 1$, $\ell \geq 0$, we have*

$\forall x \in \Lambda(n+\ell) \quad \forall k \geq 1 \quad \forall t \geq 0$

$$P\Big(\big|h\big(\overline{C}(x) \cap \Lambda(n)\big)\big| \geq t,\, \big|\overline{C}(x) \cap \Lambda(n)\big| = k\Big) \;\leq\; \exp\Big(-2p^2(1-p)^2 \frac{t^2}{k}\Big)\,.$$

Proof. Let $x \in \Lambda(n+\ell)$. In order to estimate the above probability, we build $\overline{C}(x) \cap \Lambda(n)$ in two steps. First we explore all the sites of $\Lambda(n+\ell) \setminus \Lambda(n)$. Second we use a standard growth algorithm in $\Lambda(n)$ to find the sites belonging to $\overline{C}(x) \cap \Lambda(n)$. This algorithm is driven by a sequence of i.i.d. Bernoulli random variables $(X_m)_{m \geq 1}$ with parameter p. Let us describe precisely this strategy. The first step amounts to condition on the percolation configuration in $\Lambda(n+\ell) \setminus \Lambda(n)$. We denote this configuration by $\omega|_{\Lambda(n+\ell)\setminus\Lambda(n)}$ and we write

$$\begin{aligned}
&P\Big(\big|h\big(\overline{C}(x) \cap \Lambda(n)\big)\big| \geq t,\, \big|\overline{C}(x) \cap \Lambda(n)\big| = k\Big) \\
&\quad = \sum_\eta P\Big(\big|h\big(\overline{C}(x) \cap \Lambda(n)\big)\big| \geq t,\, \big|\overline{C}(x) \cap \Lambda(n)\big| = k,\, \omega|_{\Lambda(n+\ell)\setminus\Lambda(n)} = \eta\Big) \\
&\quad = \sum_\eta P\Big(\big|h\big(\overline{C}(x) \cap \Lambda(n)\big)\big| \geq t,\, \big|\overline{C}(x) \cap \Lambda(n)\big| = k \,\Big|\, \omega|_{\Lambda(n+\ell)\setminus\Lambda(n)} = \eta\Big) \\
&\qquad\qquad\qquad\qquad\qquad\qquad \times P\Big(\omega|_{\Lambda(n+\ell)\setminus\Lambda(n)} = \eta\Big)\,.
\end{aligned}$$

The summation runs over all the percolation configurations η in $\Lambda(n+\ell)\setminus\Lambda(n)$. Let us fix one such configuration η. The second step corresponds to the growth algorithm. At each iteration, the algorithm updates three sets of sites:

• The set A_k: these are the active sites, which are to be explored.
• The set O_k: these are open sites, which belong to $\overline{C}(x)\cap\Lambda(n)$.
• The set C_k: these are closed sites, which have been visited by the algorithm.

All the sites of the sets A_k, O_k, C_k are in $\Lambda(n)$. Initially, we set $O_0 = C_0 = \varnothing$ and A_0 is the set of the sites of $\Lambda(n)$ which are connected to x by an open path in η. Recall that a path is a sequence of sites such that each site is a neighbour of its predecessor. Thus a site y belongs to A_0 if and only if

$$\begin{gathered}\exists\, z_0,\dots,z_r\in\Lambda(n+\ell)\setminus\Lambda(n)\qquad z_0,\dots,z_r \text{ are open in } \eta\,,\\ \qquad\qquad z_0=x,\quad z_0,\dots,z_r,y \text{ is a path}\,.\end{gathered}$$

Suppose that the sets A_k, O_k, C_k are built and let us explain how to build the sets $A_{k+1}, O_{k+1}, C_{k+1}$. If $A_k=\varnothing$, the algorithm terminates and

$$\overline{C}(x)\cap\Lambda(n)\,=\,O_k\cup C_k\,.$$

If A_k is not empty, we pick an element x_k of A_k. The site x_k has not been explored previously, and its status will be decided by the random variable X_k. We consider two cases, according to the value of X_k.

• $X_k=0$. The site x_k is declared closed, and we set

$$A_{k+1}\,=\,A_k\setminus\{\,x_k\,\}\,,\quad O_{k+1}\,=\,O_k\,,\quad C_{k+1}\,=\,C_k\cup\{\,x_k\,\}\,.$$

• $X_k=1$. The site x_k is declared open, and we set

$$\begin{aligned}O_{k+1}\,&=\,O_k\cup\{\,x_k\,\}\,,\quad C_{k+1}\,=\,C_k\,,\\ A_{k+1}\,&=\,A_k\cup V_k\setminus\Big(\{\,x_k\,\}\cup O_k\cup C_k\Big)\,,\end{aligned}$$

where V_k is the set of the sites of $\Lambda(n)$ which are neighbours of x_k or which are connected to x_k by an open path in $\Lambda(n+\ell)\setminus\Lambda(n)$. More precisely, a site y of $\Lambda(n)$ belongs to V_k if and only if it is a neighbour of x_k or

$$\begin{gathered}\exists\, z_1,\dots,z_r\in\Lambda(n+\ell)\setminus\Lambda(n)\qquad z_1,\dots,z_r \text{ are open in } \eta\,,\\ \qquad\qquad x_k,z_1,\dots,z_r,y \text{ is a path}\,.\end{gathered}$$

Since $O_k\cup C_k\cup A_k$ is included in $\Lambda(n)$ and the sequence of sets $O_k\cup C_k$, $k\geq 0$, is increasing, necessarily A_k is empty after at most $|\Lambda(n)|$ steps and the algorithm terminates. Suppose $\big|\overline{C}(x)\cap\Lambda(n)\big|=k$. This means that the growth algorithm stops after having explored k sites in $\Lambda(n)$. The status of these k sites is given by the first k variables of the sequence $(X_m)_{m\geq 1}$, so that

$$\begin{aligned}\big|C(x)\cap\Lambda(n)\big|\,&=\,X_1+\cdots+X_k\,,\\ \big|\partial^{\,out}C(x)\cap\Lambda(n)\big|\,&=\,k-\big(X_1+\cdots+X_k\big)\end{aligned}$$

and

$$\begin{aligned}
h\big(\overline{C}(x)\cap\Lambda(n)\big) \;&=\; \frac{1}{1-p}\big|\partial^{out}C(x)\cap\Lambda(n)\big| - \frac{1}{p}\big|C(x)\cap\Lambda(n)\big| \\
&=\; \frac{1}{1-p}\big(k-(X_1+\cdots+X_k)\big) - \frac{1}{p}\big(X_1+\cdots+X_k\big) \\
&=\; \frac{pk-(X_1+\cdots+X_k)}{p(1-p)}\,.
\end{aligned}$$

Therefore we can write

$$\begin{aligned}
&P\Big(\big|h\big(\overline{C}(x)\cap\Lambda(n)\big)\big|\geq t,\ \big|\overline{C}(x)\cap\Lambda(n)\big|=k\,\Big|\,\omega|_{\Lambda(n+\ell)\setminus\Lambda(n)}=\eta\Big) \\
&\quad= P\Big(\Big|\frac{pk-(X_1+\cdots+X_k)}{p(1-p)}\Big|\geq t,\ \big|\overline{C}(x)\cap\Lambda(n)\big|=k\,\Big|\,\omega|_{\Lambda(n+\ell)\setminus\Lambda(n)}=\eta\Big) \\
&\qquad\leq P\Big(\Big|\frac{pk-(X_1+\cdots+X_k)}{p(1-p)}\Big|\geq t\,\Big|\,\omega|_{\Lambda(n+\ell)\setminus\Lambda(n)}=\eta\Big) \\
&\qquad\quad= P\Big(\big|X_1+\cdots+X_k-pk\big|\geq tp(1-p)\Big) \\
&\qquad\qquad\leq\; 2\exp\Big(-\frac{2}{k}t^2p^2(1-p)^2\Big)\,.
\end{aligned}$$

For the last step, we have applied Hoeffding's inequality [7]. The above inequality is uniform with respect to the configuration η. Plugging this bound in the initial summation, we obtain the desired estimate. □

5 The central inequality

We will now put together the previous estimates in order to obtain an inequality between the probability of the two–arms event and the number of clusters in the collection $\mathcal{C}$. Our goal is to bound the expectation

$$E\Big(\sum_{C\in\mathcal{C}} h\big(\overline{C}\cap\Lambda(n)\big)\Big)\,.$$

Let $\mathcal{E}$ be the event

$$\mathcal{E} \;=\; \Big\{\,\forall C\in\mathcal{C}\quad \big|h\big(\overline{C}\cap\Lambda(n)\big)\big| \;<\; (\ln n)\,\big|\overline{C}\cap\Lambda(n)\big|^{1/2}\,\Big\}\,.$$

On the event $\mathcal{E}$, we bound the sum as follows:

$$\begin{aligned}
\sum_{C\in\mathcal{C}}\big|h\big(\overline{C}\cap\Lambda(n)\big)\big| \;&\leq\; \sum_{C\in\mathcal{C}}(\ln n)\,\big|\overline{C}\cap\Lambda(n)\big|^{1/2} \\
&\leq\; (\ln n)\sqrt{|\mathcal{C}|}\Big(\sum_{C\in\mathcal{C}}\big|\overline{C}\cap\Lambda(n)\big|\Big)^{1/2}\,.
\end{aligned}$$

A site x belongs to at most $2d$ sets of the collection $\{\,\overline{C}:C\in\mathcal{C}\,\}$, therefore

$$\sum_{C\in\mathcal{C}}\big|\overline{C}\cap\Lambda(n)\big|\;\leq\;2d\big|\Lambda(n)\big|\,.$$

If $\mathcal{E}$ does not occur, then we use the inequality

$$\forall C\in\mathcal{C}\qquad\big|h\big(\overline{C}\cap\Lambda(n)\big)\big|\;\leq\;\frac{1}{p(1-p)}\big|\overline{C}\cap\Lambda(n)\big|$$

and we bound the sum as follows:

$$\sum_{C\in\mathcal{C}}\big|h\big(\overline{C}\cap\Lambda(n)\big)\big|\;\leq\;\frac{1}{p(1-p)}\sum_{C\in\mathcal{C}}\big|\overline{C}\cap\Lambda(n)\big|\;\leq\;\frac{2d}{p(1-p)}\big|\Lambda(n)\big|\,.$$

We bound the probability of the complement of $\mathcal{E}$ with the help of proposition 4.1:

$$\begin{aligned}
P(\mathcal{E}^c)\;&=\;P\big(\,\exists C\in\mathcal{C}\quad\big|h\big(\overline{C}\cap\Lambda(n)\big)\big|\;\geq\;(\ln n)\,\big|\overline{C}\cap\Lambda(n)\big|^{1/2}\,\big)\\
&\leq\;P\big(\,\exists x\in\Lambda(n)\quad\big|h\big(\overline{C}(x)\cap\Lambda(n)\big)\big|\;\geq\;(\ln n)\,\big|\overline{C}(x)\cap\Lambda(n)\big|^{1/2}\;\geq\;1\,\big)\\
&\leq\;\sum_{x\in\Lambda(n)}\sum_{k=1}^{|\Lambda(n)|}P\Big(\,\big|\overline{C}(x)\cap\Lambda(n)\big|=k,\\
&\qquad\qquad\big|h\big(\overline{C}(x)\cap\Lambda(n)\big)\big|\;\geq\;(\ln n)\,\big|\overline{C}(x)\cap\Lambda(n)\big|^{1/2}\Big)\\
&\leq\;\big|\Lambda(n)\big|^2\,2\exp\Big(-2(\ln n)^2p^2(1-p)^2\Big)\,.
\end{aligned}$$

Putting together the previous inequalities, we obtain

$$\begin{aligned}
E\big(|H|\big)\;&\leq\;2d(\ln n)\sqrt{|\Lambda(n)|}\,E\Big(\sqrt{|\mathcal{C}|}\Big)\\
&\qquad+\frac{4d}{p(1-p)}\big|\Lambda(n)\big|^3\exp\Big(-2(\ln n)^2p^2(1-p)^2\Big)\,.
\end{aligned}$$

For $x\in\mathbb{Z}^d$ and $n\geq 1$, we define the event two–arms(x,n) as follows:

$$\text{two–arms}(x,n)\;=\;\left\{\begin{matrix}\text{in the configuration restricted to } x+\Lambda(n)\\ \text{two neighbours of } x \text{ are connected to the boundary}\\ \text{of the box } x+\Lambda(n) \text{ by two disjoint open clusters}\end{matrix}\right\}.$$

If x belongs to $\Lambda(n)$ and the event two–arms$(x,2n+\ell)$ occurs, then x belongs to H as well. Thus

$$|H|\;\geq\;\sum_{x\in\Lambda(n)}1_{\text{two–arms}(x,2n+\ell)}$$

and taking expectation, we obtain the following central inequality.

Lemma 5.1. *For any p in $]0,1[$, any $n\geq 1$, $\ell\geq 0$, we have the inequality*

$$P(\textit{two–arms}(0,2n+\ell))\ \leq\ \frac{2d\ln n}{\sqrt{\big|\Lambda(n)\big|}}E\Big(\sqrt{|\mathcal{C}|}\Big)$$
$$+\ \frac{4d}{p(1-p)}\big|\Lambda(n)\big|^2\,\exp\Big(-2(\ln n)^2p^2(1-p)^2\Big)\,.$$

In order to obtain the initial estimate on the two–arms event stated in the introduction, we remark that the cardinality of $\mathcal{C}$ is bounded by the cardinality of $\partial^{\,in}\Lambda(n)$, because different clusters of $\mathcal{C}$ intersect $\partial^{\,in}\Lambda(n)$ at different sites. Taking $\ell=0$ in the inequality, we have

$$P(\text{two–arms}(0,2n))\ \leq\ 2d(\ln n)\bigg(\frac{\big|\partial^{\,in}\Lambda(n)\big|}{\big|\Lambda(n)\big|}\bigg)^{1/2}$$
$$+\ \frac{4d}{p(1-p)}\big|\Lambda(n)\big|^2\,\exp\Big(-2(\ln n)^2p^2(1-p)^2\Big)\,.$$

This inequality readily implies the initial estimate stated in the introduction.

Proposition 5.2. *Let $d\geq 2$ and let $p\in]0,1[$. There exists a constant κ depending on d and p only such that*

$$\forall n\geq 1\qquad P_p\big(\textit{two–arms}(0,n)\big)\ \leq\ \frac{\kappa\ln n}{\sqrt{n}}\,.$$

In order to improve this estimate on the two–arms exponent, we will try to improve the estimate on the cardinality of $\mathcal{C}$.

6 Lower bound for the connection probability

For x,y two sites belonging to a box Λ, we define the event

$$\big\{\,x\longleftrightarrow y\text{ in }\Lambda\,\big\}\ =\ \left\{\begin{matrix}\text{the sites }x\text{ and }y\text{ are joined by}\\ \text{an open path of sites inside }\Lambda\end{matrix}\right\}.$$

The next lemma gives a polynomial lower bound for the probability of connection of two sites of $\Lambda(n)$ if one allows the path to be in $\Lambda(2n)$. At criticality, the expected behavior is indeed a power of n, but with a different exponent. In Lemma 1.1 of [8], Kozma and Nachmias derive a smaller lower bound, however only paths staying inside $\Lambda(n)$ are allowed.

Lemma 6.1. *There exists a positive constant c which depends only on the dimension d such that, for $n\geq 1$,*

$$\forall x,y\in\Lambda(n)\qquad P_{p_c}\big(x\longleftrightarrow y\ \textit{in}\ \Lambda(2n)\big)\ \geq\ \frac{c}{n^{2(d-1)d}}\,.$$

Proof. The basic ingredient to prove lemma 6.1 is the following lower bound. For any box Λ centered at 0, we have

$$\sum_{x\in\partial^{\,in}\Lambda} P_{p_c}\big(0\longleftrightarrow x \text{ in } \Lambda\big)\,\geq\,1\,.$$

This lower bound is proved in Lemma 3.1 of [8], or in the proof of theorem 5.3 of [4]. The reason is that, by an argument due to Hammersley [6], if the converse inequality holds, then this implies that the probability of long connections decays exponentially fast with the distance, and the system would be in the subcritical regime. Applying the above inequality to the box $\Lambda(n)$, we conclude that there exists x^* in $\partial^{\,in}\Lambda(n)$ such that

$$P_{p_c}\big(0\longleftrightarrow x^* \text{ in } \Lambda(n)\big)\,\geq\,\frac{1}{|\partial^{\,in}\Lambda(n)|}\,\geq\,\frac{1}{(2d)(2n+1)^{d-1}}\,.$$

Suppose for instance that x^* belongs to $\{\,n\,\}\times\mathbb{Z}^{d-1}$. Let us set $e_1=(1,0,\dots,0)$. By the FKG inequality and the symmetry of the model, we have

$$\begin{aligned}
&P_{p_c}\big(0\longleftrightarrow 2ne_1 \text{ in } \Lambda(n)\cup\big(2ne_1+\Lambda(n)\big)\big)\,\geq\\
&P_{p_c}\Big(0\longleftrightarrow x^* \text{ in } \Lambda(n)\cup\big(2ne_1+\Lambda(n)\big),\, x^*\longleftrightarrow 2ne_1 \text{ in } \Lambda(n)\cup\big(2ne_1+\Lambda(n)\big)\Big)\\
&\geq\,P_{p_c}\Big(0\longleftrightarrow x^* \text{ in } \Lambda(n)\cup\big(2ne_1+\Lambda(n)\big)\Big)\\
&\qquad\qquad\times P_{p_c}\Big(x^*\longleftrightarrow 2ne_1 \text{ in } \Lambda(n)\cup\big(2ne_1+\Lambda(n)\big)\Big)\\
&\geq\,P_{p_c}\big(0\longleftrightarrow x^* \text{ in } \Lambda(n)\big)\,P_{p_c}\big(x^*\longleftrightarrow 2ne_1 \text{ in } 2ne_1+\Lambda(n)\big)\\
&\qquad\qquad\geq\,\left(\frac{1}{(2d)(2n+1)^{d-1}}\right)^2\,.
\end{aligned}$$

By symmetry, the same inequality holds for the other axis directions. Let now x,y be two sites in $\Lambda(n)$ with coordinates

$$x=(x_1,\dots,x_d)\,,\quad y=(y_1,\dots,y_d)\,.$$

We suppose first that y_i-x_i is even, for $1\leq i\leq d$, and we set

$$z_0=x\,,\quad z_1=(y_1,x_2,\dots,x_d)\,,\dots,\; z_{d-1}=(y_1,\dots,y_{d-1},x_d)\,,\quad z_d=y\,.$$

Again by the FKG inequality, we have

$$\begin{aligned}
P_{p_c}\big(x\longleftrightarrow y \text{ in } \Lambda(2n)\big)\,&\geq\,P_{p_c}\big(\forall i\in\{\,0,\dots,d-1\,\}\quad z_i\longleftrightarrow z_{i+1} \text{ in } \Lambda(2n)\big)\\
&\geq\,\prod_{0\leq i\leq d-1} P_{p_c}\big(z_i\longleftrightarrow z_{i+1} \text{ in } \Lambda(2n)\big)\,.
\end{aligned}$$

Let $i\in\{\,0,\dots,d-1\,\}$ and let $n_i=(y_i-x_i)/2$. We have $n_i\leq n$ and

$$\big(z_i+\Lambda(n_i)\big)\cup\big(z_{i+1}+\Lambda(n_i)\big)\,\subset\,\Lambda(2n)\,,$$

whence

$$P_{p_c}\big(z_i \longleftrightarrow z_{i+1} \text{ in } \Lambda(2n)\big) \,\geq\, P_{p_c}\big(z_i \longleftrightarrow z_{i+1} \text{ in } \big(z_i{+}\Lambda(n_i)\big){\cup}\big(z_{i+1}{+}\Lambda(n_i)\big)\big)$$
$$\geq \left(\frac{1}{(2d)(2n_i+1)^{d-1}}\right)^2 .$$

Coming back to the previous inequality, we obtain

$$P_{p_c}\big(x \longleftrightarrow y \text{ in } \Lambda(2n)\big) \,\geq\, \prod_{0\leq i\leq d-1} \left(\frac{1}{(2d)(2n_i+1)^{d-1}}\right)^2 \,\geq\, \frac{c}{n^{2(d-1)d}}\,,$$

where the last inequality holds for some positive constant c. In the general case, if $x \neq y$, we can find z in $\Lambda(n)$ such that $|z-x| \leq |y-x|$ and

$$\forall i \in \{\,1,\dots,d\,\} \qquad |z_i - y_i| \leq 1\,, \quad z_i - x_i \text{ is even}\,.$$

We then use the FKG inequality to write

$$P_{p_c}\big(x \longleftrightarrow y \text{ in } \Lambda(2n)\big) \,\geq\, P_{p_c}\big(x \longleftrightarrow z \text{ in } \Lambda(2n)\big) P_{p_c}\big(z \longleftrightarrow y \text{ in } \Lambda(2n)\big)\,.$$

The probability of connection between x and z is controlled with the help of the previous case, while the probability of connection between z and y is larger than $(p_c)^d$. ☐

7 Two–arms for distant sites

We derive here an estimate for the two–arms event associated to two distant sites. For $n, \ell \geq 1$ and two sites a, b belonging to $\Lambda(n)$, we define the event two–arms$\big(\Lambda(n), a, b, \ell\big)$ as follows:

$$\text{two–arms}\big(\Lambda(n), a, b, \ell\big) \,=\, \left\{ \begin{matrix} \text{the open clusters of } a \text{ and } b \text{ in } \Lambda(n+\ell) \\ \text{are disjoint and they intersect } \partial^{\,in}\Lambda(n+\ell) \end{matrix} \right\}.$$

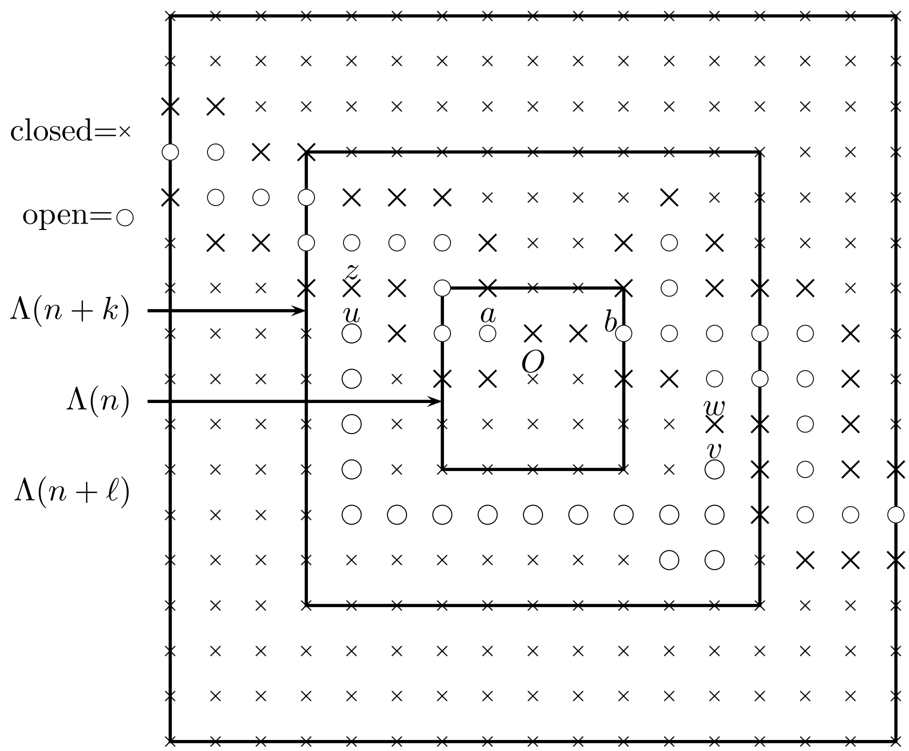
We will establish an inequality linking the two–arms event for distant sites to the two–arms event for neighbouring sites.

Lemma 7.1. *Let $p \in]0,1[$. For any $n, \ell \geq 1$ and any $a, b \in \Lambda(n)$, we have*

$$\forall k \leq \ell \qquad P\big(\textit{two–arms}\big(\Lambda(n), a, b, \ell\big)\big) \,\leq\, \frac{3^{4d}}{p}(n{+}k)^{2d} \frac{P\big(\textit{two–arms}(0, \ell-k)\big)}{P\big(\, a \longleftrightarrow b \textit{ in } \Lambda(n+k)\,\big)}\,.$$

Proof. Let $n, \ell \geq 1$, let $k \leq \ell$ and let $a, b \in \Lambda(n)$. We denote by $C(a)$ and $C(b)$ the open clusters of a and b in $\Lambda(n+\ell)$. We write

$$P\big(\text{two–arms}\big(\Lambda(n), a, b, \ell\big)\big) \,=\, \sum_{A,B} P\big(\, C(a) = A,\, C(b) = B\,\big)$$



where the sum runs over the pairs A, B of connected subsets of $\Lambda(n+\ell)$ such that

$$A \cap B = \varnothing\,, \quad a \in A\,, A \cap \partial^{in}\Lambda(n+\ell) \neq \varnothing\,, \quad b \in B\,, B \cap \partial^{in}\Lambda(n+\ell) \neq \varnothing\,.$$

For E a finite subset of $\mathbb{Z}^d$, we define

$$\overline{E} = E \cup \partial^{out}E\,, \qquad \Delta E = \partial^{in}\Big(\big(\overline{E}\big)^c\Big)\,.$$

Equivalently, we have

$$\Delta E = \big\{\, z \notin E \cup \partial^{out}E : z \text{ is the neighbour of a point in } \partial^{out}E \big\}\,.$$

Let $a, b \in \Lambda(n)$ and let A, B be two connected subsets of $\Lambda(n+\ell)$ as above. Suppose that the open clusters of a and b in $\Lambda(n+\ell)$ are exactly A and B, i.e., we have $C(a) = A$ and $C(b) = B$. Suppose that $\partial^{out}A \cap \partial^{out}B \cap \Lambda(n) \neq \varnothing$. Then the event two–arms(z, ℓ) occurs, where z is any point in the previous intersection. Suppose next that

$$\partial^{out}A \cap \partial^{out}B \cap \Lambda(n) = \varnothing\,.$$

We will transform the configuration in $\Lambda(n)$ in order to create a two–arms event. The idea is that, for $k \leq \ell$, the sets ΔA and ΔB are rather likely to be connected by an open path inside

$$\Lambda(n+k) \setminus \big(A \cup \partial^{out}A \cup B \cup \partial^{out}B\big)\,.$$

By modifying the status of one site in $\partial^{out} B$, we can then create a connection between ΔA and $\partial^{in}\Lambda(n+\ell)$, which does not use the sites of A. Let us make this strategy more precise. Any open path joining a to b in $\Lambda(n+k)$ has to go through both ΔA and ΔB, thus

$$\begin{aligned}
&P\big(a \longleftrightarrow b \text{ in } \Lambda(n+k) \big) \,\leq \\
&\qquad\qquad P\big(\Delta A \cap \Lambda(n+k) \longleftrightarrow \Delta B \cap \Lambda(n+k) \text{ in } \Lambda(n+k) \setminus (\overline{A} \cup \overline{B}) \big)\,.
\end{aligned}$$

The event $\big\{ C(a)=A,\, C(b)=B \big\}$ depends only on the sites in $\overline{A} \cup \overline{B}$, hence it is independent from the event above, therefore

$$\begin{aligned}
P\big(C(a)=A,\, C(b)=B,\, &\Delta A \cap \Lambda(n+k) \longleftrightarrow \Delta B \cap \Lambda(n+k) \text{ in } \Lambda(n+k) \setminus (\overline{A} \cup \overline{B}) \big) \\
&\geq P\big(C(a)=A,\, C(b)=B \big) \times P\big(a \longleftrightarrow b \text{ in } \Lambda(n+k) \big)\,.
\end{aligned}$$

Plugging this inequality in the initial sum, we get

$$\begin{aligned}
&\sum_{A,B} P\big(C(a)=A,\, C(b)=B \big) \,\leq \\
&\sum_{A,B} \frac{P\begin{pmatrix} C(a)=A,\, C(b)=B \\ \Delta A \cap \Lambda(n+k) \longleftrightarrow \Delta B \cap \Lambda(n+k) \text{ in } \Lambda(n+k) \setminus (\overline{A} \cup \overline{B}) \end{pmatrix}}{P\big(a \longleftrightarrow b \text{ in } \Lambda(n+k) \big)} \,\leq \\
&\frac{P\begin{pmatrix} C(a) \cap \partial^{in}\Lambda(n+\ell) \,\neq\, \varnothing,\, C(b) \cap \partial^{in}\Lambda(n+\ell) \,\neq\, \varnothing,\, C(a) \cap C(b) = \varnothing \\ \Delta C(a) \cap \Lambda(n+k) \longleftrightarrow \Delta C(b) \cap \Lambda(n+k) \text{ in } \Lambda(n+k) \setminus (\overline{C(a)} \cup \overline{C(b)}) \end{pmatrix}}{P\big(a \longleftrightarrow b \text{ in } \Lambda(n+k) \big)} \\
&\qquad \leq \frac{P\begin{pmatrix} C(a) \cap \partial^{in}\Lambda(n+\ell) \,\neq\, \varnothing,\, C(b) \cap \partial^{in}\Lambda(n+\ell) \,\neq\, \varnothing \\ C(a) \cap C(b) = \varnothing \\ \exists\, u \in \Delta C(a) \cap \Lambda(n+k) \quad \exists\, v \in \Delta C(b) \cap \Lambda(n+k) \\ u \longleftrightarrow v \text{ in } \Lambda(n+k) \setminus (\overline{C(a)} \cup \overline{C(b)}) \end{pmatrix}}{P\big(a \longleftrightarrow b \text{ in } \Lambda(n+k) \big)} \\
&\qquad\qquad \leq \sum_{u,v \in \Lambda(n+k)} \frac{P\begin{pmatrix} C(a) \cap \partial^{in}\Lambda(n+\ell) \,\neq\, \varnothing \\ C(b) \cap \partial^{in}\Lambda(n+\ell) \,\neq\, \varnothing \\ C(a) \cap C(b) = \varnothing \\ u \in \Delta C(a),\, v \in \Delta C(b) \\ u \longleftrightarrow v \text{ in } \Lambda(n+k) \setminus (\overline{C(a)} \cup \overline{C(b)}) \end{pmatrix}}{P\big(a \longleftrightarrow b \text{ in } \Lambda(n+k) \big)}\,.
\end{aligned}$$

Let us consider the event inside the probability appearing on the numerator. Let z (respectively w) be a neighbour of u (respectively v) belonging to $\partial^{out} C(a)$ (respectively $\partial^{out} C(b)$). Suppose that we change the status of w to open. The site u is connected to v by an open path, and v is now connected to w and $C(b)$, hence to $\partial^{in}\Lambda(n+\ell)$, and this connection does not use any site of $C(a)$.

Thus the site z, which is closed, will admit two neighbours which are connected to $\partial^{in}\Lambda(n+\ell)$: the site u and another one belonging to $C(a)$, and these two neighbours do not belong to the same cluster in $\Lambda(n+\ell)$. Therefore the event two–arms$(z,\ell-k)$ occurs, and we conclude that

$$P\left(\begin{array}{c} C(a)\cap\partial^{in}\Lambda(n+\ell)\neq\varnothing,\, C(b)\cap\partial^{in}\Lambda(n+\ell)\neq\varnothing,\, C(a)\cap C(b)=\varnothing \\ u\in\Delta C(a),\, v\in\Delta C(b),\, u\longleftrightarrow v \text{ in } \Lambda(n+k)\setminus(\overline{C(a)}\cup\overline{C(b)}) \end{array}\right)$$
$$\leq \frac{4d^2}{p}P\big(\text{two–arms}(0,\ell-k)\big)\,.$$

Plugging this inequality in the previous sum, we obtain

$$\begin{aligned} P\big(\text{two–arms}(\Lambda(n),a,b,\ell)\big) &\leq \sum_{u,v\in\Lambda(n+k)}\frac{4d^2}{p}\frac{P\big(\text{two–arms}(0,\ell-k)\big)}{P\big(\,a\longleftrightarrow b \text{ in } \Lambda(n+k)\,\big)} \\ &\leq \big|\Lambda(n+k)\big|^2\frac{4d^2}{p}\frac{P\big(\text{two–arms}(0,\ell-k)\big)}{P\big(\,a\longleftrightarrow b \text{ in } \Lambda(n+k)\,\big)} \\ &\leq \frac{3^{4d}}{p}(n+k)^{2d}\frac{P\big(\text{two–arms}(0,\ell-k)\big)}{P\big(\,a\longleftrightarrow b \text{ in } \Lambda(n+k)\,\big)}\,. \end{aligned}$$

This is the inequality we wanted to prove. □

We derive next an estimate for the two–arms event associated to a box. For $n,\ell\geq 1$, we define the event two–arms$\big(\Lambda(n),\ell\big)$ as follows:

$$\text{two–arms}\big(\Lambda(n),\ell\big) = \left\{\begin{array}{c}\text{there exist two distinct open clusters}\\ \text{in } \Lambda(n+\ell) \text{ joining } \Lambda(n) \text{ to } \partial^{in}\Lambda(n+\ell)\end{array}\right\}.$$

Corollary 7.2. *For any $n\geq 1$, $\ell\geq n$, we have*

$$P\big(\textit{two–arms}(\Lambda(n),\ell)\big) \leq \frac{3^{9d}}{p}\frac{n^{4d-2}P\big(\textit{two–arms}(0,\ell-n)\big)}{\inf\big\{\,P\big(\,a\longleftrightarrow b \textit{ in } \Lambda(2n)\,\big):a,b\in\partial^{in}\Lambda(n)\,\big\}}$$

Proof. From the definition of the two–arms event, we have

$$\text{two–arms}\big(\Lambda(n),\ell\big) = \bigcup_{a,b\in\partial^{in}\Lambda(n)}\text{two–arms}\big(\Lambda(n),a,b,\ell\big)\,.$$

Therefore, applying the inequality of lemma 7.1 with $k=n$, we obtain

$$\begin{aligned} P\big(\text{two–arms}(\Lambda(n),\ell)\big) &\leq \sum_{a,b\in\partial^{in}\Lambda(n)}P\big(\text{two–arms}(\Lambda(n),a,b,\ell)\big) \\ &\leq \sum_{a,b\in\partial^{in}\Lambda(n)}\frac{3^{4d}}{p}\frac{(2n)^{2d}P\big(\text{two–arms}(0,\ell-n)\big)}{P\big(\,a\longleftrightarrow b \text{ in } \Lambda(2n)\,\big)} \\ &\leq \frac{3^{4d}}{p}\frac{4d^2(2n+1)^{2d-2}(2n)^{2d}P\big(\text{two–arms}(0,\ell-n)\big)}{\inf\big\{\,P\big(\,a\longleftrightarrow b \text{ in } \Lambda(2n)\,\big):a,b\in\partial^{in}\Lambda(n)\,\big\}}\,. \end{aligned}$$

This yields the desired inequality. □

Corollary 7.3. *We have*

$$\lim_{n\to\infty} P\big(\textit{two–arms}\big(\Lambda(n), n^{4d^2+4d-3}\big)\big) \;=\; 0\,.$$

Proof. We apply the inequality given in corollary 7.2. We use proposition 5.2 to control the probability of the two–arms event and lemma 6.1 to control from below the connection probability. We obtain

$$P\big(\text{two–arms}\big(\Lambda(n),\ell\big)\big) \;\leq\; \frac{3^{9d}}{pc} n^{2d^2+2d-2} \frac{\kappa \ln(\ell-n)}{\sqrt{\ell-n}}\,.$$

We take $\ell = n^{4d^2+4d-3}$ in this inequality and we send n to ∞. □

For $d=3$, this yields the exponent $4d^2+4d-3=45$.

8 Control on the number of arms

We try next to improve the previous estimates. The idea is the following. With the help of corollary 7.3, we will improve slightly the control on the number of clusters in the collection $\mathcal{C}$ (these are the clusters intersecting both $\Lambda(n)$ and $\partial^{\,in}\Lambda(n+\ell)$). Thanks to the central inequality stated in lemma 5.1, this will permit to improve the bound on the two–arms event for a site, and subsequently the bound on the two–arms event for a box. This leads to a better exponent in corollary 7.3. We can then iterate this scheme to improve further the exponents. Unfortunately, the sequence of exponents converges geometrically and the final result is still quite weak.

Let n,ℓ,k be three integers, with $k\leq n\leq \ell$. Let Λ_i, $i\in I$, be a collection of boxes which are translates of $\Lambda(k)=[-k,k]^d$, which are included in $\Lambda(n)$ and which covers the inner boundary $\partial^{\,in}\Lambda(n)$. Such a covering can be realized with disjoint boxes if $2n+1$ is a multiple of $2k+1$, otherwise we do not require that the boxes are disjoint. In any case, there exists such a covering Λ_i, $i\in I$, whose cardinality $|I|$ satisfies

$$\big|I\big| \;\leq\; 2d\Big(2\frac{n}{k}\Big)^{d-1}\,.$$

Let us fix such a covering. Given a percolation configuration in $\Lambda(n+\ell)$, a box Λ_i of the covering is said to be good if the event two–arms(Λ_i,ℓ) does not occur. Let us compute the expected number of bad boxes:

$$\begin{aligned} E\left(\begin{array}{c}\text{number of bad boxes in}\\ \text{the collection } \Lambda_i,\, i\in I\end{array}\right) \;&=\; E\Big(\sum_{i\in I} 1_{\text{the box } \Lambda_i \text{ is bad}}\Big)\\ &=\; \big|I\big| P\big(\text{two–arms}(\Lambda(k),\ell)\big)\,. \end{aligned}$$

The clusters of the collection $\mathcal{C}$ intersect $\partial^{\,in}\Lambda(n)$, hence they have to go into one box of the collection Λ_i, $i\in I$. If two clusters of $\mathcal{C}$ intersect the same box Λ_i, this box has to be bad, because these two clusters go all the way till $\partial^{\,in}\Lambda(n+\ell)$, hence they realize the event two–arms(Λ_i,ℓ). Thus a good box of the collection Λ_i, $i\in I$, meets at most one cluster of $\mathcal{C}$. Moreover, a bad box of the collection Λ_i, $i\in I$, meets at most $\big|\partial^{\,in}\Lambda(k)\big|$ clusters of $\mathcal{C}$. We conclude that

$$|\mathcal{C}|\,\leq\,\begin{pmatrix}\text{number of good boxes in}\\ \text{the collection } \Lambda_i,\ i\in I\end{pmatrix}+\big|\partial^{\,in}\Lambda(k)\big|\times\begin{pmatrix}\text{number of bad boxes in}\\ \text{the collection } \Lambda_i,\ i\in I\end{pmatrix}.$$

We bound the number of good boxes by $|I|$ and we take the expectation in this inequality. We obtain

$$\begin{aligned}E\big(|\mathcal{C}|\big)\,&\leq\,\big|I\big|+\big|\partial^{\,in}\Lambda(k)\big|\times\big|I\big|\times P\big(\text{two–arms}(\Lambda(k),\ell)\big)\\&\leq\,d2^d\Big(\frac{n}{k}\Big)^{d-1}\Big(1+2d(2k+1)^{d-1}P\big(\text{two–arms}(\Lambda(k),\ell)\big)\Big)\\&\leq\,c\Big(\frac{n}{k}\Big)^{d-1}+cn^{d-1}P\big(\text{two–arms}(\Lambda(k),\ell)\big)\,,\end{aligned}$$

where c is a constant depending on d and p. Plugging the inequality of corollary 7.2 in the previous inequality, we get, with some larger constant c,

$$E\big(|\mathcal{C}|\big)\,\leq\,c\Big(\frac{n}{k}\Big)^{d-1}+\frac{cn^{d-1}k^{4d-2}P\big(\text{two–arms}(0,\ell-k)\big)}{\inf\Big\{\,P\big(\,a\longleftrightarrow b\text{ in }\Lambda(2k)\,\big):a,b\in\partial^{\,in}\Lambda(k)\,\Big\}}\,.$$

Noticing that $E(\sqrt{|\mathcal{C}|})\leq E(|\mathcal{C}|)^{1/2}$, we deduce from the central inequality stated in lemma 5.1 and the previous inequality that

$$\begin{aligned}&P(\text{two–arms}(0,2n+\ell))\,\leq\\&\frac{2d\ln n}{\sqrt{\big|\Lambda(n)\big|}}\Bigg(c\Big(\frac{n}{k}\Big)^{d-1}+\frac{cn^{d-1}k^{4d-2}P\big(\text{two–arms}(0,\ell-k)\big)}{\inf\Big\{\,P\big(\,a\longleftrightarrow b\text{ in }\Lambda(2k)\,\big):a,b\in\partial^{\,in}\Lambda(k)\,\Big\}}\Bigg)^{1/2}\\&\qquad\qquad+\frac{4d}{p(1-p)}\big|\Lambda(n)\big|^2\,\exp\Big(-2(\ln n)^2p^2(1-p)^2\Big)\,.\end{aligned}$$

We choose $\ell=n$, and we conclude that, for some constant c, we have

$$\begin{aligned}&P(\text{two–arms}(0,3n))\,\leq\\&\qquad\frac{c\ln n}{\sqrt{n}}\Bigg(\frac{1}{k^{d-1}}+\frac{k^{4d-2}P\big(\text{two–arms}(0,n-k)\big)}{\inf\Big\{\,P\big(\,a\longleftrightarrow b\text{ in }\Lambda(2k)\,\big):a,b\in\partial^{\,in}\Lambda(k)\,\Big\}}\Bigg)^{1/2}.\end{aligned}$$

We shall next iterate this inequality in order to enhance the lower bound on the two–arms exponent.

9 Iterating at p_c

In this section, we work at $p=p_c$ and we complete the proofs of theorems 1.1 and 1.2. Lemma 6.1 yields that

$$\forall k\geq 1\qquad\inf\Big\{\,P\big(\,a\longleftrightarrow b\text{ in }\Lambda(2k)\,\big):a,b\in\partial^{\,in}\Lambda(k)\,\Big\}\,\geq\,\frac{c}{k^{2(d-1)d}}\,,$$

whence, for $1 \leq k \leq n$, for some $c > 0$,

$$P(\text{two–arms}(0,3n)) \leq \frac{c \ln n}{\sqrt{n}} \left(\frac{1}{k^{d-1}} + k^{2d^2+2d-2} P\big(\text{two–arms}(0,n-k)\big) \right)^{1/2}.$$

Suppose that for some positive constants c', β, γ, with $\gamma < 1$, we have

$$\forall n \geq 2 \qquad P\big(\text{two–arms}(0,n)\big) \leq \frac{c'(\ln n)^{\beta}}{n^{\gamma}}.$$

Choosing $k = n^{\delta}$ with

$$\delta = \frac{\gamma}{2d^2+3d-3},$$

we obtain that

$$\forall n \geq 2 \qquad P\big(\text{two–arms}(0,3n)\big) \leq \frac{2c\sqrt{c'}(\ln n)^{\beta/2+1}}{n^{\gamma'}},$$

where

$$\gamma' = \frac{1}{2} + \frac{d-1}{4d^2+6d-6}\gamma.$$

By monotonicity,

$$\forall n \geq 3 \qquad P\big(\text{two–arms}(0,n)\big) \leq P\big(\text{two–arms}(0,\lfloor n/3 \rfloor)\big),$$

therefore there exists also a constant c'' such that

$$\forall n \geq 2 \qquad P\big(\text{two–arms}(0,n)\big) \leq \frac{c''(\ln n)^{\beta+1}}{n^{\gamma'}}.$$

The initial estimate stated in proposition 5.2 yields that

$$\forall n \geq 2 \qquad P\big(\text{two–arms}(0,n)\big) \leq \frac{\kappa \ln n}{\sqrt{n}}.$$

We define a sequence of exponents $(\gamma_i)_{i \geq 0}$ by setting $\gamma_0 = 1/2$ and

$$\forall i \geq 0 \qquad \gamma_{i+1} = \frac{1}{2} + \frac{d-1}{4d^2+6d-6}\gamma_i.$$

Iterating the previous argument, we conclude that, for any $i \geq 1$, there exists a constant α_i such that

$$\forall n \geq 2 \qquad P\big(\text{two–arms}(0,n)\big) \leq \frac{\alpha_i(\ln n)^{i+1}}{n^{\gamma_i}}.$$

It follows that

$$\forall i \geq 0 \qquad \limsup_{n \to \infty} \frac{1}{\ln n} \ln P\big(\text{two–arms}(0,n)\big) \leq \gamma_i.$$

The sequence $(\gamma_i)_{i\geq 0}$ converges geometrically towards

$$\gamma_\infty \,=\, \frac{2d^2+3d-3}{4d^2+5d-5}\,.$$

Letting i go to ∞ in the previous inequality, we obtain the result stated in theorem 1.1. Theorem 1.1 and the inequality of corollary 7.2 readily imply theorem 1.2. To prove theorem 1.2, we proceed as in the proof of corollary 7.3, but instead of the initial estimate of proposition 5.2, we use the enhanced estimate provided by theorem 1.1.

10 Proof of theorem 1.3

Throughout this section, we work with a parameter p such that $\theta(p)>0$. We will use the hypothesis $\theta(p)>0$ to improve the lower bound for the probability of a connection inside a finite box.

Lemma 10.1. *Let $n,\ell\geq 2$. For any $x,y\in\Lambda(n)$, we have*

$$P\big(x\longleftrightarrow y\ \textit{in}\ \Lambda(n+\ell)\big)\,\geq\,\theta(p)^2-P\big(\textit{two–arms}(\Lambda(n),x,y,\ell)\big)\,.$$

Proof. We write

$$P\big(x\longleftrightarrow y\text{ in }\Lambda(n+\ell)\big)\,\geq\,P\begin{pmatrix}x\longleftrightarrow\partial^{\,in}\Lambda(n+\ell)\\ y\longleftrightarrow\partial^{\,in}\Lambda(n+\ell)\\ x\longleftrightarrow y\text{ in }\Lambda(n+\ell)\end{pmatrix}$$
$$\geq\,P\begin{pmatrix}x\longleftrightarrow\partial^{\,in}\Lambda(n+\ell)\\ y\longleftrightarrow\partial^{\,in}\Lambda(n+\ell)\end{pmatrix}-P\begin{pmatrix}x\longleftrightarrow\partial^{\,in}\Lambda(n+\ell)\\ y\longleftrightarrow\partial^{\,in}\Lambda(n+\ell)\\ x\not\longleftrightarrow y\text{ in }\Lambda(n+\ell)\end{pmatrix}\,.$$

By the FKG inequality, we have

$$P\begin{pmatrix}x\longleftrightarrow\partial^{\,in}\Lambda(n+\ell)\\ y\longleftrightarrow\partial^{\,in}\Lambda(n+\ell)\end{pmatrix}\,\geq\,P\begin{pmatrix}x\longleftrightarrow\infty\\ y\longleftrightarrow\infty\end{pmatrix}\,\geq\,\theta(p)^2\,.$$

Moreover

$$P\begin{pmatrix}x\longleftrightarrow\partial^{\,in}\Lambda(n+\ell)\\ y\longleftrightarrow\partial^{\,in}\Lambda(n+\ell)\\ x\not\longleftrightarrow y\text{ in }\Lambda(n+\ell)\end{pmatrix}\,\leq\,P\big(\text{two–arms}(\Lambda(n),x,y,\ell)\big)\,.$$

The last two inequalities imply the inequality stated in the lemma. $\square$

Since

$$\theta(p)\,\leq\,P\big(0\longleftrightarrow\partial^{\,in}\Lambda(n)\big)\,\leq\,\sum_{x\in\partial^{\,in}\Lambda(n)}P\big(0\longleftrightarrow x\text{ in }\Lambda(n)\big)\,,$$

then there exists x_n in $\partial^{in}\Lambda(n)$ such that

$$P\big(0\longleftrightarrow x_n \text{ in } \Lambda(n)\big)\ \geq\ \frac{\theta(p)}{\big|\partial^{in}\Lambda(n)\big|}\ \geq\ \frac{\theta(p)}{2d(2n+1)^{d-1}}\,.$$

We apply the inequality of lemma 7.1 to 0 and x_n with $k=0$:

$$P\big(\text{two–arms}\big(\Lambda(n),0,x_n,\ell\big)\big)\ \leq\ \frac{3^{4d}}{p}n^{2d}\frac{P\big(\text{two–arms}(0,\ell)\big)}{P\big(\,0\longleftrightarrow x_n \text{ in } \Lambda(n)\,\big)}\,.$$

Combining the two previous inequalities, we conclude that

$$P\big(\text{two–arms}\big(\Lambda(n),0,x_n,\ell\big)\big)\ \leq\ \frac{3^{7d}}{p\theta(p)}n^{3d-1}P\big(\text{two–arms}(0,\ell)\big)\,.$$

We apply the inequality of lemma 10.1 to 0 and x_n, and, together with the previous inequality, we obtain

$$P\big(0\longleftrightarrow x_n \text{ in } \Lambda(n+\ell)\big)\ \geq\ \theta(p)^2-\frac{3^{7d}}{p\theta(p)}n^{3d-1}P\big(\text{two–arms}(0,\ell)\big)\,.$$

Let α be such that

$$\alpha\ >\ \frac{4d^2+5d-5}{2d^2+3d-3}(3d-1)\,.$$

We take $\ell=n^\alpha$. By theorem 1.1, for n large enough,

$$P\big(0\longleftrightarrow x_n \text{ in } \Lambda(n+n^\alpha)\big)\ \geq\ \frac{1}{2}\theta(p)^2\,.$$

Suppose for instance that x_n belongs to $\{\,n\,\}\times\mathbb{Z}^{d-1}$. Let $e_1=(1,0,\dots,0)$. By symmetry and the FKG inequality, for n large enough,

$$\begin{aligned}
&P\big(0\longleftrightarrow 2ne_1 \text{ in } \Lambda(4n+n^\alpha)\big)\ \geq\\
&\qquad P\begin{pmatrix} 0\longleftrightarrow x_n \text{ in } \Lambda(n+n^\alpha)\\ x_n\longleftrightarrow 2ne_1 \text{ in } 2ne_1+\Lambda(n+n^\alpha)\end{pmatrix}\ \geq\\
&\quad P\big(0\longleftrightarrow x_n \text{ in } \Lambda(n+n^\alpha)\big)P\big(x_n\longleftrightarrow 2ne_1 \text{ in } 2ne_1+\Lambda(n+n^\alpha)\big)\\
&\qquad\qquad\geq\ P\big(0\longleftrightarrow x_n \text{ in } \Lambda(n+n^\alpha)\big)^2\ \geq\ \frac{1}{4}\theta(p)^4\,.
\end{aligned}$$

Thus there exists $N\geq 1$ such that

$$\forall n\geq N\qquad P\big(0\longleftrightarrow 2ne_1 \text{ in } \Lambda(4n+n^\alpha)\big)\ \geq\ \frac{1}{4}\theta(p)^4\,.$$

Let $n\geq N$ and let $k\in\{\,N,\dots,n\,\}$. We have

$$P\big(0\longleftrightarrow 2ke_1 \text{ in } \Lambda(4n+n^\alpha)\big)\ \geq\ P\big(0\longleftrightarrow 2ke_1 \text{ in } \Lambda(4k+k^\alpha)\big)\ \geq\ \frac{1}{4}\theta(p)^4\,.$$

This implies further that

$$\forall k \in \{ 2N, \dots, 2n \} \qquad P\big(0 \longleftrightarrow ke_1 \text{ in } \Lambda(4n+n^\alpha)\big) \,\geq\, \frac{p}{4}\theta(p)^4 \,.$$

Since N is independent of n, we conclude that there exists $\rho > 0$ such that

$$\forall n \geq N \quad \forall k \in \{ 0, \dots, 2n \} \qquad P\big(0 \longleftrightarrow ke_1 \text{ in } \Lambda(4n+n^\alpha)\big) \,\geq\, \rho \,.$$

Since N is fixed, this lower bound can be extended to every $n \geq 1$ by taking a smaller value of ρ. By symmetry, we have the same lower bounds for the probabilities of connections along the other axis directions. Using the FKG inequality, we conclude that

$$\forall n \geq 1 \quad \forall x \in \Lambda(2n) \qquad P\big(0 \longleftrightarrow x \text{ in } \Lambda(6n+n^\alpha)\big) \,\geq\, \rho^d \,.$$

This completes the proof of theorem 1.3.

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