SMOOTHING EFFECT FOR SCHRÖDINGER BOUNDARY VALUE PROBLEMS

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Abstract

We show the necessity of the nontrapping condition for the plain smoothing effect ($H^{1/2}$) for the Schrödinger equation with Dirichlet boundary conditions in exterior problems. We also give a class of trapped obstacles (Ikawa's example) for which we can prove a weak ($H^{1/2-\varepsilon}$) smoothing effect.

Résumé

On démontre que l'hypothèse de non capture est nécessaire pour l'effet régularisant ($H^{1/2}$) pour l'équation de Schrödinger avec conditions aux limites de Dirichlet à l'extérieur d'un domaine de $\mathbb{R}^d$. On donne aussi une classe d'obstacles captifs (l'exemple d'Ikawa) pour lesquels on démontre un effet régularisant affaibli ($H^{1/2-\varepsilon}$).

1. Introduction

Consider $u = e^{it\Delta}u_0$ a solution of the Schrödinger equation

$$\begin{cases} (i\partial_t + \Delta)u = 0 \quad \text{in } \mathbb{R} \times \mathbb{R}^d, \\ u|_{t=0} = u_0 \in L^2(\mathbb{R}^d). \end{cases} \tag{1.1}$$

It is well known that $u \in L^\infty(\mathbb{R}_t; L^2(\mathbb{R}^d))$ satisfies the following smoothing effect (for any $s > 1/2$ if $d \geq 3$):

$$\|u\|_{L^2(\mathbb{R}_t; \dot{H}_s^{1/2}(\mathbb{R}^d))} \leq C\|u_0\|_{L^2}, \tag{1.2}$$

where

$$\dot{H}_s^{1/2} = \big\{u \in \mathscr{D}'(\mathbb{R}^d);\, \langle x\rangle^{-s}\Delta^{1/4}u \in L^2(\mathbb{R}^d)\big\}. \tag{1.3}$$

This result, which can be proved by explicit calculations, has been extended to more complicated operators, satisfying a nontrapping assumption (see the results of Constantin and Saut [9], Ben-Artzi and Devinatz [1], Ben-Artzi and Klainerman [2],

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Doi [11], [10], and Kato and Yajima [19]). It has been recently extended to the case of boundary value problems by Gérard, Tzvetkov, and the author [7].

On the other hand, in [12] Doi has proved that, for Schrödinger operators in $\mathbb{R}^d$, the nontrapping assumption is necessary for the $H^{1/2}$ smoothing effect.

In this paper we extend this latter result to the case of boundary value problems. Our first result reads as follows.

THEOREM 1.1

Consider an arbitrary smooth domain with boundary $\Omega \subset \mathbb{R}^d$, with no infinite order contact with its boundary (see the precise definition in Section 3), and consider P a second-order self-adjoint operator on $L^2(\Omega)$, with domain $D \subset H^1_0(\Omega)$ and such that the boundary is noncharacteristic. Denote by $\varphi_s : {}^bT^*\Omega \setminus \{0\} \to {}^bT^*\Omega \setminus \{0\}$ the bicharacteristic flow of the operator P (given by the integral curves of the Hamiltonian vector field of the principal symbol of P reflecting on the boundary according to the law of geometric optics; see Section 3) defined on the boundary cotangent bundle. Let $A \in \Psi^{(1/2)}$ be a classical tangential pseudodifferential operator of order $1/2$. Suppose that $(z_0, \zeta_0) \in {}^bT^*\Omega \setminus \{0\}$ satisfy the trapping assumption

$$\int_{-\infty}^{0} \bigl|\sigma_{1/2}(A)(\varphi_s(x_0, \zeta_0)\bigr|^2 \, ds = +\infty, \tag{1.4}$$

where $\sigma_{1/2}(A)$ is the principal symbol of the operator A. Then for any $t_0 > 0$ the map

$$u_0 \in C^\infty_0 \subset L^2(\Omega) \mapsto Ae^{itP}u_0 \in L^2\bigl([0, t_0]; L^2(M)\bigr) \tag{1.5}$$

is not bounded (even for data with fixed compact support).

Remark 1.2

The assumption (1.4) can be essentially fulfilled in two distinct cases.

(1) If A is compactly supported (in the z variable), then (1.4) means that the bicharacteristic starting from (z_0, ζ_0) spends an infinite time in the support of A, which corresponds to a "trapped trajectory."

(2) If A is not compactly supported—a typical example is $\bigl($in the case $P = -\Delta\bigr)$ $A(z, D_z) = a(|z|)|D_z|^{1/2}$—then (1.4) might correspond to a lack of decay of $a(x)$ at infinity. Suppose that the trajectory starting from (z_0, ζ_0) is not trapped; hence it leaves any compact set and for $\pm s \to +\infty$, $(z(s), \zeta(s)) \sim (s\zeta_\pm, \zeta_\pm)$ and (1.4) is equivalent to $|a|^2 \notin L^1(\mathbb{R})$ (and we recover the usual assumption required for proving the smoothing effect; see [10]).

Remark 1.3

We could have added lower order terms to P and supposed that the Cauchy problem is well posed in L^2 (in the case of first-order terms). The condition (1.4) in this case has to be modified.

Remark 1.4
In [10], [12], Doi proved this result in the case of a manifold without boundary and gave some variants of this result for operators of higher order, and with weights in times. The proof we present below is essentially self-contained in this case, and it can also handle these variants modulo slight modifications. The proof in presence of a boundary is much more technical.

Remark 1.5
For $P = -\Delta_g$, the x-projection of the integral curves of H_p are the geodesics for the metric g.

Remark 1.6
The smoothness assumption can be relaxed to C^2-coefficients and C^3-domains (and even to C^1-coefficients, but the assumption (1.4) is then more complicated since the Hamiltonian flow is no more well defined). We also can prove Theorem 1.1 for systems (see Remark 3.1).

Having Theorem 1.1 in mind, one can see that a natural question is whether a weakened version of (1.2) might hold for some trapping geometries. In the case of a stable (elliptic) trapped trajectory, the existence of quasimodes well localized along this trajectory shows that no such result may hold (see Remark 4.3). However, in the case of hyperbolic trapped trajectories, we do obtain such a weak smoothing effect.

THEOREM 1.7
Consider $\Theta = \bigcup_{i=1}^N \Theta_i \subset \mathbb{R}^d$ a finite union of strictly convex obstacles satisfying the assumptions of Section 4. Denote by $\Omega = \Theta^c$ its complement. Then, for any $\varepsilon > 0$ and $\chi \in C_0^\infty(\mathbb{R}^d)$, there exists $C > 0$ such that for any $u_0 \in L^2(\Omega)$,

$$\|\chi e^{it\Delta_D} u_0\|_{L^2(\mathbb{R}_t; H^{1/2-\varepsilon}(\Omega))} \leq C\|u_0\|_{L^2(\Omega)}. \tag{1.6}$$

Remark 1.8
This result was proved in [7] with no ε loss under the nontrapping assumption that "any geodesic of the metric g reflecting on the boundary according to the laws of geometric optics goes to the infinity," which is clearly not fulfilled here.

To prove Theorem 1.1, we follow the same kind of strategy as Doi in his papers [10], [12]. However, we replace the use of Egorov's theorem in his argument with the use of the theorem of propagation of Wigner measures, which has three advantages: first, it simplifies the rest of the proof; second, it allows us to relax assumptions (on the regularity of the coefficients); and finally, the proof holds also for a (system of) boundary value problem (whereas Egorov's theorem is not true in these cases).

To prove Theorem 1.7, we reduce, following [7], the estimate (1.6) to obtaining estimates for the outgoing resolvent of Δ_D, $(-\Delta - (z \pm i0))^{-1}$. Then we show that these estimates can be deduced from a combination of other estimates proved by M. Ikawa [16], [17], [18] and some form of the maximum principle.

The article is written as follows: in Section 2 we recall the definition of Wigner measures which is used in the sequel, and we prove Theorem 1.1 in the simpler case where $\Theta = \emptyset$. In Section 3 we give the necessary modifications required to handle the general case $\Theta \neq \emptyset$. In Section 4 we prove Theorem 1.7. Finally, we state at the end of Section 4 an application of our smoothing result to the global existence of nonlinear Schrödinger (NLS) equations.

2. Proof of Theorem 1.1: The case of empty boundary

2.1. Wigner measures

In this section we recall the definition of Wigner measures (or semiclassical measures) introduced by Gérard and Leichtnam [14] and Lions and Paul [22] (see also the survey by Gérard, Markowich, Mauser, and Poupaud [15]). We work in the context of functions of $1+d$ variables $((t,z))$ in $L^2_{\text{loc}}(\mathbb{R}_t; L^2(\mathbb{R}^d_z)) = \mathscr{L}^2$, and we have adapted the definitions in [14], [22] to fit our purpose.

Definition 2.1

We say that a sequence of functions $(f_n) \in \mathscr{L}^2$ is bounded in $\mathscr{L}^2$ if, for any $\varphi \in C_0^\infty(\mathbb{R}_t)$, the sequence (φf_n) is bounded in L^2.

Definition 2.2

We say that an operator A is bounded on $\mathscr{L}^2$ if there exists $\varphi \in C_0^\infty(\mathbb{R}_t)$ such that for any $f \in \mathscr{L}^2$,

$$\|Af\|_{L^2_{t,z}} \leq C\|\varphi f\|_{L^2_{t,z}}.$$

Denote by $(x,\xi) = (t,z,\tau,\zeta)$ a point in $T^*\mathbb{R}^{d+1}$, and consider, for $a(x,\xi) \in C_0^\infty(\mathbb{R}^{2d+2})$ and $\varphi \in C_0^\infty(\mathbb{R}_t)$ equal to 1 near the support of a, the operator $\mathrm{Op}_\varphi(a)(x,hD_x)$ defined on $\mathscr{L}^2$ by

$$\begin{aligned}\mathrm{Op}_\varphi(a)(x,hD_x)f &= \mathrm{Op}_\varphi(a)(t,z,hD_t,hD_z)f \\ &= \frac{1}{(2\pi)^{d+1}}\int e^{i(t\cdot\tau+z\cdot\zeta)}a(t,z,h\tau,h\zeta)\widehat{\varphi(t)f}(\tau,\zeta)\,d\tau\,d\zeta. \quad (2.1)\end{aligned}$$

The operator $\mathrm{Op}(a)_\varphi(t, z, hD_t, hD_z)$ is (uniformly with respect to $0 < h < 1$) bounded on $\mathscr{L}^2$, and we have the following weak form of the Gårding inequality.

PROPOSITION 2.3
For any $a \in C_0^\infty(\mathbb{R}^{2d+2})$ and any sequence (f_n) bounded in $\mathscr{L}^2$ and $(h_n) \in]0,1]$, $\lim_{n\to+\infty} h_n = 0$,

$$a(x,\xi) \geq 0 \Rightarrow \liminf_{n\to+\infty} \mathrm{Re}\left(\mathrm{Op}(a)_\varphi(x, h_n D_x) f_n, f_n\right)_{L^2(\mathbb{R}^{d+1})} \geq 0. \tag{2.2}$$

To prove this result, consider, for $\varepsilon > 0$, $\psi \in C_0^\infty(\mathbb{R}^{2d+2})$ equal to 1 near the (t, z, τ, ζ) projection of the support of a and $b = \varphi(t)\sqrt{\varepsilon + a}\,\psi(t, z, \tau, \zeta) \in C_0^\infty(\mathbb{R}^{2d+2})$. Then the symbolic calculus shows

$$\begin{aligned} 0 &\leq \left(\mathrm{Op}(b)_\varphi^* \mathrm{Op}(b)_\varphi f_n, f_n\right) \\ &= \left(\mathrm{Op}(a)_\varphi(x, h_n D_x) f_n, f_n\right)_{L^2} + \varepsilon\left(\varphi(t)\psi^2(x, h_n D_x)\varphi(t) f_n, f_n\right)_{L^2} + \mathscr{O}(h_n); \end{aligned} \tag{2.3}$$

hence, taking the lim inf and using the fact that $\liminf(\alpha_n + \beta_n) \leq \liminf(\alpha_n) + \limsup(\beta_n)$, we get

$$\liminf_{n\to+\infty} \mathrm{Re}\left(\mathrm{Op}(a)(x, h_n D_x) f_n, f_n\right)_{L^2(\mathbb{R}^d)} + \varepsilon \limsup_{n\to+\infty} \|\psi(x, h_n D_x)\varphi(t) f_n\|^2 \geq 0. \tag{2.4}$$

When $\varepsilon > 0$ tends to zero, we obtain Proposition 2.3.

By the symbolic calculus, the operator $\mathrm{Op}(a)_\varphi$ is modulo operators bounded on $\mathscr{L}$ by $\mathscr{O}(h^\infty)$, independent of the choice of the function φ. For conciseness, in the sequel we drop the index φ. As in [14] (see also [3]), we can prove the following.

PROPOSITION 2.4
Consider a sequence (f_n) bounded in $\mathscr{L}^2$. There exist a subsequence (n_k) and a positive Radon measure on $\mathbb{R}^{2d+2}$, μ, such that for any $a \in C_0^\infty(\mathbb{R}^{2d+2})$

$$\lim_{k\to+\infty} \left(\mathrm{Op}(a)(x, h_{n_k} D_x) f_{n_k}, f_{n_k}\right)_{L^2} = \langle \mu, a(x,\xi)\rangle. \tag{2.5}$$

The idea for extracting such a sequence is to fix a and consider the bounded sequence $(L(a)_n) = (\mathrm{Op}(a)(x, h_n D_x) f_n, f_n)_{L^2}$. By compactness, we can extract a subsequence that converges. Iterating this process for a sequence (a_j) dense in C_0^∞, we obtain, by diagonal extraction, a sequence (f_{n_k}) such that the limit exists for any a_j. By (2.2), the limit defines a positive functional on a dense subset of C_0^∞ (hence this limit is continuous for the C^0-topology). It is consequently a Radon measure, and the limit (2.5) exists for any $a \in C_0^\infty$. For the sake of conciseness, we denote again by (f_n) the extracted subsequences.

The measure μ represents at points (x_0, ξ_0) the oscillations of the sequence (f_n) at point x_0 and scale ξ_0/h_n. The oscillations at frequencies smaller than h_n^{-1} are concentrated in $\{\xi_0 = 0\}$, whereas the oscillations at higher ($\gg h_n^{-1}$) frequencies are lost.

2.2. Invariance of the Wigner measure
2.2.1. Elliptic regularity
Suppose that the sequence (f_n) is a solution of the equation

$$(ih_n\partial_t + h_n^2 P)f_n = \mathscr{O}(h_n)_{\mathscr{L}^2}. \tag{2.6}$$

Take $a \in C_0^\infty$, and consider first

$$\big(\operatorname{Op}(a)(x, h_n D_x)(ih_n\partial_t + h_n^2 P)f_n, f_n\big)_{L^2} = o(1). \tag{2.7}$$

Taking into account that the operator $\operatorname{Op}(a)(x, h_n D_x)(ih_n\partial_t + h_n^2 P)$ is equal to $\operatorname{Op}(a \times (-\tau + p(z,\zeta))(x, hD_x)$ modulo an operator bounded by $\mathscr{O}(h_n)$ on $\mathscr{L}^2$ and passing to the limit in (2.7), we obtain

$$\big\langle \mu, a(x,\xi)(-\tau + p(z,\zeta))\big\rangle = 0, \tag{2.8}$$

from which we deduce the following.

PROPOSITION 2.5
The measure μ is supported in the semiclassical characteristic set of the operator

$$\operatorname{Char}(ih_n\partial_t + h_n^2 P) = \big\{(x,\xi) = (t, z, \tau, \zeta); \tau = p(z,\zeta)\big\}. \tag{2.9}$$

Remark 2.6
Suppose that the sequence (f_n) is a solution of the equation (2.6). Then for any $a \in C_0^\infty(\mathbb{R}^{2d})$, the function

$$t \mapsto \big(\operatorname{Op}(a)(z, h_n D_z)f_n\,|_t, f_n\,|_t\,\big)_{L^2(\mathbb{R}_z^d)}(t) \tag{2.10}$$

is, according to (2.6), locally uniformly equicontinuous. Hence, using Ascoli's theorem, it is possible to extract a subsequence (f_{n_k}) (independent of t) such that there exists a family of positive measures μ_t continuous with respect to t and such that, for any t and any $a \in C_0^\infty(\mathbb{R}^{2d})$, we have

$$\lim_{n\to+\infty} \big(\operatorname{Op}(a)(z, h_n D_z)f_n, f_n\big)_{L^2(\mathbb{R}_z^d)}(t) = \langle \mu_t, a\rangle. \tag{2.11}$$

Of course, from μ_t one can recover the measure μ (assuming that the extracted sequences are the same):

$$\mu = dt \otimes \delta_{\tau = p(z,\zeta)} \otimes \mu_t. \tag{2.12}$$

2.2.2. Propagation of the Wigner measure

Suppose now that

$$(ih_n\partial_t + h_n^2 P)f_n = o(h_n)_{\mathscr{L}^2}. \tag{2.13}$$

Consider the bracket ($P^* = P$):

$$\begin{aligned} & h_n^{-1}\big([\mathrm{Op}(a)(x, h_n D_x), ih_n\partial_t + h_n^2 P]f_n, f_n\big)_{L^2} \\ & \quad = h_n^{-1}\big((ih_n\partial_t + h_n^2 P)\,\mathrm{Op}(a)(x, h_n D_x)f_n, f_n\big)_{L^2} + o(1), \\ & \quad = o(1). \end{aligned} \tag{2.14}$$

Taking into account that the operator

$$h_n^{-1}\big[\,\mathrm{Op}(a)(x, D_x), ih_n\partial_t + h_n^2 P\big] \tag{2.15}$$

is equal to

$$\frac{1}{i}\big\{a, -\tau + p(z,\zeta)\big\}(x, h_n D_x) + \mathscr{O}(h_n)_{\mathscr{L}(\mathscr{L}^2)}, \tag{2.16}$$

where the Poisson bracket of a and q, $\{a, q\}$, is defined by

$$\{a, q\} = \nabla_{\tau,\zeta} a \cdot \nabla_{t,z} q - \nabla_{t,z} a \nabla_{\tau,\zeta} q, \tag{2.17}$$

we can pass to the limit in (2.14) and obtain

$$\big\langle \mu, \big\{a, -\tau + p(z,\zeta)\big\}\big\rangle = 0 \tag{2.18}$$

or, equivalently (with $H_{\tau - p(z,\zeta)}$ the Hamiltonian vector field of $\tau - p$),

$$H_{\tau - p(z,\zeta)}(\mu) = (\partial_t - H_p)\mu = 0. \tag{2.19}$$

Gathering Proposition 2.5 and (2.19), we prove the following.

PROPOSITION 2.7

The measure μ is invariant along the integral curves of the vector field $H_{\tau-p}$ drawn on the surface $\{\tau = p(z,\zeta)\}$. Equivalently, if we denote by φ_s the Hamiltonian flow of the function $p(z,\zeta)$ on $T^*\mathbb{R}^2$, and if μ_t is as in (2.12)*, we have the equality for any $s \in \mathbb{R}$,*

$$\mu_s = \varphi_s^*(\mu_0) \Leftrightarrow \langle \mu_s, a \circ \varphi_s\rangle = \langle \mu_0, a\rangle. \tag{2.20}$$

2.3. Proof of Theorem 1.1 in the case $\Omega = \mathbb{R}^d$

Take (z_0, ζ_0) satisfying the assumption (1.4), and consider $\varphi \in C_0^\infty(\mathbb{R}^d)$ such that $\int |\varphi|^2 = 1$ and

$$u_{0,n} = n^{d/4}\varphi\big(n^{1/2}(z - z_0)\big)e^{in(z-z_0)\cdot\zeta_0}. \tag{2.21}$$

Denote by $v_n = e^{itP}u_{0,n}$ the corresponding solution of the Schrödinger equation. To prove Theorem 1.1, we show that

$$\forall \varepsilon > 0, \quad \lim_{n\to+\infty} \|A(z, D_z)v_n\|_{L^2([0,\varepsilon]\times\mathbb{R}^d)} = +\infty \tag{2.22}$$

if $A \in S^{1/2}(\mathbb{R}^{2d})$ satisfies the assumptions of Theorem 1.1.

For this we compute, with $h_n = 1/n$ and $\Psi \in C_0^\infty(\mathbb{R}^{2d})$, $0 \leq \Psi \leq 1$ equal to 1 near 0 and $\alpha > 0$ fixed,

$$\begin{aligned} \|A(z, D_z)v_n\|^2_{L^2([0,\varepsilon]\times\mathbb{R}^d)} &= \int_0^\varepsilon \big(A^*(z, D_z)A(z, D_z)v_n, v_n\big)_{L^2(\mathbb{R}^d_z)}\,dt \\ &\geq \int_0^\varepsilon \big(\Psi(\alpha z, \alpha h_n D_z)A^*(z, D_z)A(z, D_z)\Psi(\alpha z, \alpha h_n D_z)v_n, v_n\big)_{L^2(\mathbb{R}^d_z)}\,dt \\ &\quad - C \\ &\geq \int_0^\varepsilon \big(h_n^{-1}b^*(z, h_n D_z)b(z, h_n D_z)v_n, v_n\big)_{L^2(\mathbb{R}^d_z)}\,dt - C \end{aligned} \tag{2.23}$$

with $b(z, \zeta) = \sigma_{1/2}(A)(z, \zeta)\Psi(\alpha z, \alpha\zeta)$.

But, for any T, if n is large enough

$$\begin{aligned} &\int_0^\varepsilon \big(h_n^{-1}b^*(z, h_n D_z)b(z, h_n D_z)v_n, v_n\big)_{L^2(\mathbb{R}^d_z)}\,dt \\ &\qquad \geq \int_0^{h_n T} \big(h_n^{-1}b^*(z, h_n D_z)b(z, h_n D_z)v_n, v_n\big)_{L^2(\mathbb{R}^d_z)}\,dt. \end{aligned} \tag{2.24}$$

Denoting by $u_n(s, z) = v(h_n s, z)$ the solution of the semiclassical Schrödinger equation

$$(ih_n\partial_s + h_n^2 P)u_n = 0, \tag{2.25}$$

we obtain for any $T > 0$,

$$\begin{aligned} &\int_0^\varepsilon \big(h_n^{-1}b^*(z, h_n D_z)b(z, h_n D_z)v_n, v_n\big)_{L^2(\mathbb{R}^d_z)}\,dt \\ &\qquad \geq \int_0^{T} \big(b^*(z, h_n D_z)b(z, h_n D_z)u_n, u_n\big)_{L^2(\mathbb{R}^d_z)}\,ds. \end{aligned} \tag{2.26}$$

According to (2.21), the Wigner measure μ_0 of the sequence $(u_n \mid_{t=0})$ is equal to

$$\delta_{(z,\zeta)=(z_0,\zeta_0)}.$$

From Proposition 2.7 and (2.12), we deduce that the Wigner measure μ_s of $(u_n \mid_{t=s})$ is equal to $\delta_{(z,\zeta)=\varphi_{-s}(z_0,\zeta_0)}$, where φ_s is the flow of H_p.

Hence (for T fixed)

$$\begin{aligned}\lim_{n\to+\infty}\int_0^T \big(b^*(z,h_nD_z)b(z,h_nD_z)u_n,u_n\big)_{L^2(\mathbb{R}^d)}\,dt &= \int_0^T \langle\mu_s,b\rangle\,ds\\ &= \int_0^T |b|^2\big(\varphi_{-s}(z_0,\zeta_0)\big)\,ds\\ &= \int_0^T \big|\sigma_{1/2}(A)(\varphi_{-s}(z_0,\zeta_0))\big|^2\big|\Psi(\alpha\varphi_{-s}(z_0,\zeta_0))\big|^2\,ds. \qquad (2.27)\end{aligned}$$

From (2.23), (2.24), (2.26), and (2.27), we deduce (if α is chosen small enough) that, for any $T > 0$ and with a fixed constant C independent of T,

$$\liminf_{n\to+\infty}\|A(z,D_z)v_n\|^2_{L^2([0,\varepsilon]\times\mathbb{R}^d)} \geq \int_0^T \big|\sigma_{1/2}(A)(\varphi_{-s}(z_0,\zeta_0))\big|^2\,ds - C. \qquad (2.28)$$

Letting T tend to the infinity (and using the assumption (1.4)), we obtain (2.22).

3. Proof of Theorem 1.1 for a Dirichlet problem

In this section we give the outline of the proof of Theorem 1.1 in the general case. In fact, the proof is essentially the same as in the previous section. The differences are that we have to define Wigner measures for sequences bounded in $L^2_{\mathrm{loc}}(\mathbb{R}_t;L^2(\Omega))$ and prove the elliptic (Proposition 2.5) and propagation (Proposition 2.7) results for these measures. Then we construct a sequence of initial data whose Wigner measure is $\delta_{(z_0,\zeta_0)}$, where (z_0,ζ_0) satisfy the assumption (1.4) and the sequence of solutions of the Schrödinger equation with these initial data prove the result. Fortunately, all these constructions have already been done (see the works by Gérard and Leichtnam [14], Miller [25], [26], Burq and Lebeau [8], and Burq [5]) in some slightly different settings. All that we have to do is to adapt these constructions to our framework and glue the pieces together.

For the sake of completeness, we give an outline of the constructions. However, we insist on the fact that in this section most of the material is taken from the works cited above.

Remark 3.1

For simplicity, we restrict the study to the case of a scalar equation; however, following [8], it would not be much more difficult to prove the result for systems.

3.1. Geometry

Denote by $M=\mathbb{R}_t\times\Omega$, $x=(t,z)\in M$, by ${}^bTM$ the bundle of rank $d+1$ whose sections are the vector fields tangent to ∂M, by ${}^bT^*M$ the dual bundle (Melrose's compressed cotangent bundle), and by $j : T^*M \to {}^bT^*M$ the canonical map. In any coordinate system where $M = \{x = (x_n > 0, x')\})$, the bundle ${}^bTM$ is generated by the fields $\frac{\partial}{\partial x'}$, $x_n \frac{\partial}{\partial x_n}$ and j is defined by

$$j(x_n, x', \xi_n, \xi') = (x_n, x', v = x_n \xi_n, \xi'). \tag{3.1}$$

Denote by Char $\widetilde{P}$ the semiclassical characteristic manifold of $\widetilde{P} = ih\partial_t + h^2 P$, and denote by Z its projection:

$$\operatorname{Char} \widetilde{P} = \Big\{(x, \xi) = (t, z, \tau, \zeta) \in T^*\mathbb{R}^d \mid_{\overline{M}}; \, p(x, \xi) = \tau\Big\}, \qquad Z = j(\operatorname{Char} \widetilde{P}). \tag{3.2}$$

The set Z is a locally compact metric space.

Consider, near a point $x_0 \in \partial M$, a geodesic system of coordinates for which $x_0 = (0, 0)$, $M = \{(x_n, x') \in \mathbb{R}^+ \times \mathbb{R}^d\}$, and the operator $\widetilde{P}$ has the form (near x_0)

$$\widetilde{P} = -h^2 D_{x_n}^2 + R(x_n, x', hD_{x'}) + hQ(x, hD_x), \tag{3.3}$$

with R a second-order tangential operator and Q a first-order operator.

We recall now the usual decomposition of $T^*\partial M$ (in this coordinate system). Denote by $r(x_n, x', \xi')$ the semiclassical principal symbol of R, and let $r_0 = r \mid_{x_n=0}$. Then $T^*\partial M$ is the disjoint union of $\mathscr{E} \cup \mathscr{G} \cup \mathscr{H}$ with

$$\mathscr{E} = \{r_0 < 0\}, \qquad \mathscr{G} = \{r_0 = 0\}, \qquad \mathscr{H} = \{r_0 > 0\}. \tag{3.4}$$

Note that j gives a natural identification between $Z \mid_{\partial M}$ and $\mathscr{H} \cup \mathscr{G} \subset T^*\partial M$. In $\mathscr{G}$ we distinguish between the *diffractive* points $\mathscr{G}^{2,+} = \{r_0 = 0, r_1 = \partial_{x_n} r \mid_{x_n=0} > 0\}$ and the *gliding* points $\mathscr{G}^- = \{r_0 = 0, r_1 = \partial_{x_n} r \mid_{x_n=0} \leq 0\}$. We make the assumption (Ω has no infinite order contact with its tangents) that, for any $\varrho_0 \in T^*\partial M$, there exists $N \in \mathbb{N}$ such that

$$H_{r_0}^N(r_1) \neq 0.$$

The definition of the generalized bicharacteristic flow φ_s associated to the operator P is essentially the definition given in [24].

Definition 3.2

A generalized bicharacteristic curve $\gamma(s)$ is a continuous curve from an interval $I \subset \mathbb{R}$ to Z such that,

(1) if $s_0 \in I$ and $\gamma(s_0) \in T^*M$, then close to s_0, γ is an integral curve of the Hamiltonian vector field $H_{\widetilde{p}}$;

(2) if $s_0 \in I$ and $\gamma(s_0) \in \mathscr{H} \cup \mathscr{G}^{2,+}$, then there exists $\varepsilon > 0$ such that for $0 < |s - s_0| < \varepsilon$, $x_n(\gamma(s)) > 0$;

(3) if $s_0 \in I$ and $\gamma(s_0) \in \mathscr{G}^-$, then for any function $f \in C^\infty(T^*\mathbb{R}^{d+1}|_{\overline{M}})$ satisfying the symmetry condition

$$\forall \varrho_0 \in Z,\ \forall \widehat{\varrho_0}, \widetilde{\varrho_0} \in j^{-1}(\varrho_0) \cap \mathrm{Char}(\widetilde{P}), \quad f(\widehat{\varrho_0}) = f(\widetilde{\varrho_0}), \tag{3.5}$$

we have

$$\frac{d}{ds} f\big(j(\gamma(s))\big)\big|_{s=s_0} = H_{\widetilde{p}}\,|_{j^{-1}(\gamma(s_0))}\, f\big(j^{-1}(\gamma(s_0))\big).$$

It is proved in [24] that under the assumption of no infinite order contact, through every point $\varrho_o \in {}^bT^*M \setminus \{0\}$ there exists a unique generalized bicharacteristic (which is furthermore a limit of bicharacteristics having only hyperbolic contacts with the boundary). This defines the flow Φ. Finally, note that, since $\widetilde{p} = p - \tau$, we have a natural flow φ on Char $P \subset {}^bT^*\Omega$ (the generalized flow of $p(z,\zeta)$) given by

$$\Phi_s(t, \tau, z, \zeta) = \big(t - s, \tau, \varphi_s(z, \zeta)\big). \tag{3.6}$$

3.2. Wigner measures

Consider functions $a = a_i + a_\partial$ with $a_i \in C_0^\infty(T^*M)$, and $a_\partial \in C_0^\infty(\mathbb{R}^{2d-1})$. Such symbols are quantized in the following way: take $\varphi_i \in C_0^\infty(M)$ (resp., $\varphi_\partial \in C_0^\infty(\mathbb{R}^d)$) equal to 1 near the x-projection of $\mathrm{supp}(a_i)$ (resp., the x-projection of $\mathrm{supp}(a_\partial)$), and define

$$\begin{aligned}\mathrm{Op}_{\varphi_i,\varphi_\partial}(a)(x, hD_x)f &= \frac{1}{(2\pi h)^d}\int e^{i(x-y)\cdot\xi/h} a_i(x,\xi)\varphi_i(y) f(y)\,dy\,d\xi \\ &+ \frac{1}{(2\pi h)^{d-1}}\int e^{i(x'-y')\cdot\xi'/h} a_\delta(x_n, x', \xi)\varphi_\delta(x_n, y') f(x_n, y')\,dy'\,d\xi'. \end{aligned}\tag{3.7}$$

Note that, according to the symbolic semiclassical calculus, the operator $\mathrm{Op}_{\varphi_i,\varphi_\partial}(a)$ does not depend on the choice of functions $\varphi_i, \varphi_\partial$, modulo operators on $\mathscr{L}^2$ of norms bounded by $O(h^\infty)$. As in the previous section, in the sequel we drop the index $\varphi_i, \varphi_\partial$.

Denote by $\mathscr{A}$ the space of the operators that are a finite sum of operators obtained as above in suitable coordinate systems near the boundary, and for $A \in \mathscr{A}$, denote by $a = \sigma(A)$ the semiclassical symbol of the operator A. For such functions a, we can define $\kappa(a) \in C^0(Z)$ by

$$\kappa(a)(\rho) = a\big(j^{-1}(\rho)\big). \tag{3.8}$$

(The value is independent of the choice of $j^{-1}(\rho)$ since the operator is tangential.)

The set

$$\big\{\kappa(a), a = \sigma(A), A \in \mathscr{A}\big\} \tag{3.9}$$

is a locally dense subset of $C_c^0(Z)$.

3.3. Elliptic regularity

Consider a sequence (f_k) bounded in $\mathscr{L}^2 = L^2_{\mathrm{loc}}(\mathbb{R}_t; L^2(\Omega))$ a solution of the equation (with $\lim_{k\to+\infty} h_k = 0$)

$$\begin{cases} (ih_k\partial_t + h_k^2 P)f_k = o(h_n)_{L^2_{\mathrm{loc}}(\mathbb{R}_t; L^2(\Omega))}, \\ u \mid_{\partial\Omega} = 0. \end{cases} \tag{3.10}$$

The same argument as in Section 2.2.1 shows the following.

PROPOSITION 3.3

If a_i is equal to zero near Char $\widetilde{P}$, *then*

$$\lim_{k\to+\infty} \big(\mathrm{Op}(a_i)(x, h_k D_x) f_k, f_k\big)_{L^2} = 0, \tag{3.11}$$

and the analysis of the boundary value problem shows the following.

PROPOSITION 3.4

If a_∂ is equal to zero near Z (i.e., a_i is supported in the elliptic region), then

$$\lim_{k\to+\infty} \big(\mathrm{Op}(a_\partial)(x_n, x', h_k D_{x'}) f_k, f_k\big)_{L^2} = 0. \tag{3.12}$$

3.4. Definition of the measure

The analog of Proposition 2.4 is the following.

PROPOSITION 3.5

There exists a subsequence (k_p) and a Radon positive measure μ on Z such that

$$\forall Q \in \mathscr{A}, \quad \lim_{p\to\infty} (Qf_{k_p}, f_{k_p})_{L^2} = \big\langle \mu, \kappa(\sigma(Q))\big\rangle. \tag{3.13}$$

The proof of this result in the interior of Ω is the same as in Section 2, and near a boundary point it relies on the Gårding inequality for tangential operators (see G. Lebeau [21] for a proof in the classical context and [14], [3] for the semiclassical construction). As before, we denote again by (f_k) the extracted sequence.

PROPOSITION 3.6 (First properties of the measure μ)

The measure μ satisfies the following:

$$\mu(\mathscr{H}) = 0, \tag{3.14}$$

$$\limsup_{k\to+\infty} \big|(\mathrm{Op}(a) h_k D_{x_n} f_k, f_k)_{L^2}\big| \le C \sup_{\varrho\in\mathrm{supp}(a)} |r|^{1/2} |a|. \tag{3.15}$$

The relation (3.14) is a simple consequence of the micro-local analysis of the boundary problem near a point $\varrho_0 \in \mathscr{H}$, for which a parametrix for the solution can be written in terms of a semiclassical Fourier integral operator by geometric optics methods. To prove (3.15), compute (with $\varphi \in C_0^\infty$ equal to 1 near the t-projection of the support of a)

$$\begin{aligned}
\left|(\mathrm{Op}(a)h_k D_{x_n} f_k, f_k)_{L^2}\right| &\leq \|\mathrm{Op}(a)h_k D_{x_n} f_k\|_{L^2}\|\varphi(t)f_k\|_{L^2}\\
&\leq \left(h_k D_{x_n}\mathrm{Op}(a)^*\mathrm{Op}(a)h_k D_{x_n} f_k, f_k\right)_{L^2}^{1/2}\|\varphi(t)f_k\|_{L^2}\\
&\leq \left(\mathrm{Op}(a)^*\mathrm{Op}(a)h_k^2 D_{x_n}^2 f_k, f_k\right)_{L^2}^{1/2}\|\varphi(t)f_k\|_{L^2} + o(1)\\
&\leq \left(\mathrm{Op}(a)^*\mathrm{Op}(a)(R-\widetilde{P})f_k, f_k\right)_{L^2}^{1/2}\|\varphi(t)f_k\|_{L^2} + o(1)\\
&\leq \left(\mathrm{Op}(a)^*\mathrm{Op}(a)Rf_k, f_k\right)_{L^2}^{1/2}\|\varphi(t)f_k\|_{L^2} + o(1), \quad (3.16)
\end{aligned}$$

and we obtain

$$\limsup_{k\to+\infty}\left|(\mathrm{Op}(a)h_k D_{x_n} f_k, f_k)_{L^2}\right| \leq C|\langle\mu, a^2 r\rangle|^{1/2} \leq C \sup_{\varrho\in\mathrm{supp}(a)}|a||r|^{1/2}. \quad (3.17)$$

3.5. Invariance of the measure

Consider now a sequence (f_k) bounded in $L^2_{\mathrm{loc}}(\mathbb{R}_t; L^2(\Omega))$ a solution of the equation (with $\lim_{k\to+\infty} h_k = 0$)

$$\begin{cases}(ih_k\partial_t + h_k^2 P)f_k = o(h_k)_{L^2_{\mathrm{loc}}(\mathbb{R}_t;L^2(\Omega))},\\ u\ |_{\partial\Omega} = 0.\end{cases} \quad (3.18)$$

PROPOSITION 3.7

Consider $q \in C^\infty(T^*\mathbb{R}^{d+1}\ |_{\overline{M}})$ satisfying the symmetry condition (3.5). In general, $\{\widetilde{p}, q\} = -2\xi_n\partial_{x_n}q + \{r, q\}$ is not a function defined on Z (because of the ξ_n dependence). To obtain a function on Z, we take the convention

$$\{\widetilde{p}, q\} \stackrel{\mathrm{def}}{=} -2\xi_n\partial_{x_n}q\,1_{\varrho\notin\mathscr{H}} + \{r, q\}. \quad (3.19)$$

This function is μ-integrable and, thanks to (3.14), μ-a.e. continuous.

Then, with this convention, the measure μ satisfies

$$\left\langle\mu, \{\widetilde{p}, q\}\right\rangle = 0. \quad (3.20)$$

The proof of Proposition 3.7 is simply integration by parts (and some careful study of the terms arising). We give it below.

Since in M the equation (3.20) is simply (2.18), we restrict the study to the case where q is supported near a point $\varrho_0 \in T^*\partial M$. Suppose first only that $q \in$

$C^\infty(T^*\mathbb{R}^{d+1}\ |_{\overline{M}})$. From the Malgrange preparation theorem, there exist functions $q_0(x_n, x', \xi'), q_1(x_n, x', \xi') \in C^\infty$ such that

$$q\ |_{\mathrm{Char}\ \widetilde{P}} = q_0\ |_{\mathrm{Char}\ \widetilde{P}} + \xi_n q_1\ |_{\mathrm{Char}\ \widetilde{P}}\ . \tag{3.21}$$

Let $Q = \mathrm{Op}(q_0) + \mathrm{Op}(q_1) h D_{x_n}$, and compute ($P^* = P$)

$$h_k^{-1}\big((\widetilde{P}Q - Q\widetilde{P}) f_k, f_k\big)_{L^2}. \tag{3.22}$$

Two integrations by part, (3.3) and the boundary condition $f_k\ |_{x_n=0} = 0$, show that

$$\begin{aligned} h_k^{-1}([\widetilde{P}, Q] f_k, f_k)_{L^2} &= h_k^{-1}(\widetilde{P} Q f_k, f_k)_{L^2} + o(1) \\ &= -i\ \big(Q_1\ |_{x_n=0}\ h_k D_{x_n} f_k\ |_{x_n=0}, h_k D_{x_n} f_k\ |_{x_n=0}\big)_{L^2(\mathbb{R}^{d-1}_{x'})}\ . \end{aligned} \tag{3.23}$$

On the other hand, $[\widetilde{P}, Q]$ can be written under the form

$$\frac{i}{h_k}[\widetilde{P}, Q] = A_0 + A_1 h_k D_{x_n} + A_2 \widetilde{P} + h A_3,$$

where A_0, A_1, and A_2 are tangential operators, A_3 is differential of order at most 1 in D_{x_n}, and on Char $\widetilde{P}$ we have

$$a_0 + a_1 \xi_n = \{\widetilde{p}, q\}. \tag{3.24}$$

From (3.14), we deduce that for μ-a.e.,

$$a_0 + a_1 \xi_n 1_{x_n > 0} = \{\widetilde{p}, q\}. \tag{3.25}$$

Consequently,

$$\begin{aligned} h_k^{-1}\big([\widetilde{P}, Q] f_k, f_k\big)_{L^2} &= h_k^{-1}(\widetilde{P} Q f_k, f_k)_{L^2} + o(1). \\ &= \big(- i(A_0 + A_1 h_k D_{x_n} + A_2 \widetilde{P} + o(1)) f_k, f_k\big)_{L^2} + o(1). \end{aligned} \tag{3.26}$$

Passing to the limit in (3.26), we obtain

$$\lim_{k\to+\infty} \big((A_0 + A_1 h_k D_{x_n} + A_2(\widetilde{P})) f_k, f_k\big)_{L^2} = \langle \mu, a_0\rangle + \lim_{k\to+\infty} \big(A_1 h_k D_{x_n} f_k, f_k\big)_{L^2}. \tag{3.27}$$

Take $\varepsilon > 0$ and $\varphi \in C_0^\infty(]-1, 1[)$ equal to 1 near 0. Decompose

$$\begin{aligned} A_1 = \left(1 - \varphi\Big(\frac{x_n}{\varepsilon}\Big)\right) A_1 + \mathrm{Op}\left(\varphi\Big(\frac{x_n}{\varepsilon}\Big)\varphi\Big(\frac{r(x_n, x', \xi')}{\varepsilon}\Big)\right) A_1 \\ + \mathrm{Op}\left(\varphi\Big(\frac{x_n}{\varepsilon}\Big)\left(1 - \varphi\Big(\frac{r(x_n, x', \xi')}{\varepsilon}\Big)\right)\right) A_1. \end{aligned} \tag{3.28}$$

The first term in the right-hand side of (3.28) is supported in the interior of Ω; its contribution to the limit in (3.27) is equal to

$$\Big\langle \mu, \Big(1-\varphi\Big(\frac{x_n}{\varepsilon}\Big)\Big) a_1 \xi_n \Big\rangle. \tag{3.29}$$

The contribution of the second term is, according to (3.15), smaller than

$$C \sup_{\varrho \in \operatorname{supp}(\varphi(x_n/\varepsilon)(\varphi(r(x_n,x',\xi')/\varepsilon)))} |r|^{1/2} |a_1| \leq C\varepsilon^{1/2}, \tag{3.30}$$

and the contribution of the last term is smaller than

$$\begin{aligned} \|h_k D_{x_n} f_k\| \Big\| A_1^* \operatorname{Op}\Big(\varphi\Big(\frac{x_n}{\varepsilon}\Big)\Big(1-\varphi\Big(\frac{r(x_n,x',\xi')}{\varepsilon}\Big)\Big)\Big)^* f_k \Big\| \\ \leq C\Big\langle \mu, |a_1|^2 \varphi^2\Big(\frac{x_n}{\varepsilon}\Big)\Big(1-\varphi\Big(\frac{r(x_n,x',\xi')}{\varepsilon}\Big)\Big)^2\Big\rangle^{1/2} + o(1). \end{aligned} \tag{3.31}$$

Passing to the limit $\varepsilon \to 0$, we obtain that the contribution of the first term is equal to

$$\langle \mu, a_1 \xi_n 1_{x_n>0} \rangle, \tag{3.32}$$

the contribution of the second term is (according to (3.30)) equal to zero, and the contribution of the last term is, according to (3.14), smaller than

$$\langle \mu, a_1^2 1_{x_n=0} 1_{r\neq 0} \rangle = \langle \mu, a_1^2 1_{\varrho \in \mathscr{H}} \rangle = 0. \tag{3.33}$$

Finally, we prove

$$\lim_{n\to+\infty} -i(Q_1 \mid_{x_n=0} h_k D_{x_n} f_k \mid_{x_n=0}, h_k D_{x_n} f_k \mid_{x_n=0})_{L^2(\mathbb{R}^{d-1}_{x'})} = -i\langle \mu, \{\widetilde{p}, q\}\rangle. \tag{3.34}$$

But, if q satisfies the symmetry condition (3.5), the function $q \mid_{x_n=0}$ is independent of ξ_n on Char P. Hence $q_1 \mid_{x_n=0} = 0$ on $\mathscr{H}$ and consequently on $\overline{H} = \mathscr{H} \cup \mathscr{G}$; and the left-hand side in (3.34) tends to zero, which proves Proposition 3.7.

PROPOSITION 3.8 (see [6] and [25])
We have

$$\mu(\mathscr{G}^{2,+}) = 0. \tag{3.35}$$

Consider a point $\varrho_0 \in \mathscr{G}^{2,+}$. Apply (3.34) to a family of functions $q = \xi_n \times q_\varepsilon$ with

$$q_\varepsilon = \varphi\Big(\frac{x_n}{\varepsilon^{1/3}}\Big)\varphi\Big(\frac{(r(x_n,x',\xi'))}{\varepsilon}\Big) a(x_n,x',\xi'). \tag{3.36}$$

Then we get

$$\lim_{n\to+\infty}\Big(\varphi\Big(\frac{(r(0,x',h_kD_{x'}))}{\varepsilon}\Big)$$
$$\cdot\, a(0,x',h_kD_{x'})h_kD_{x_n}f_k\ |_{x_n=0}, h_kD_{x_n}f_k\ |_{x_n=0}\Big)_{L^2(\mathbb{R}^{d-1}_{x'})}$$
$$=\Big\langle\mu, -2\xi_n^2\partial_{x_n}\Big(\varphi\Big(\frac{x_n}{\varepsilon^{1/3}}\Big)\varphi\Big(\frac{(r(x_n,x',\xi'))}{\varepsilon}\Big)a(x_n,x',\xi')\Big)\Big\rangle$$
$$-\Big\langle\mu, \partial_{x_n}r\varphi\Big(\frac{x_n}{\varepsilon^{1/3}}\Big)\varphi\Big(\frac{(r(x_n,x',\xi'))}{\varepsilon}\Big)a(x_n,x',\xi')+\xi_n\{r,q_\varepsilon\}'\Big\rangle, \quad (3.37)$$

where $\{r, q_\varepsilon\}'$ is the Poisson bracket with respect to the x', ξ' variables. On the support of the measure μ, $\xi_n^2 = r(x_n, x', \xi')$. Hence we can apply the dominated convergence theorem and obtain that the right-hand side in (3.37) tends to

$$\langle\mu, -\partial_{x_n}r(0,x',\xi')a(x_n,x',\xi')1_{x_n=0}1_{r=0}\rangle = \langle\mu, -\partial_{x_n}r(0,x',\xi')a(x_n,x',\xi')1_{\rho\in\mathscr{G}}\rangle. \quad (3.38)$$

According to the assumption $\varrho_0 \in \mathscr{G}^{2,+}$, $\partial_{x_n}r > 0$ at the point ϱ_0 . If the support of a is chosen small enough so that $\partial_{x_n}r > 0$ on this support, then the right-hand side in (3.38) is nonpositive. On the other hand, by Gårding inequality, the limit on the left-hand side is nonnegative. Both sides are then equal to zero. This implies Proposition 3.8.

It is now possible to prove, as in [8, Théorème 1] (see also [5]), by measure theory methods, that the invariance of the measure μ along the generalized bicharacteristic flow is equivalent to Propositions 3.7 and 3.8 (in fact, the proof of this result is presented in [8, Section 3.3] for classical measures in the more general context of systems, but the proof for semiclassical measures is the same word by word).

3.6. Proof of Theorem 1.1

All that remains to do to complete the proof of Theorem 1.1 in the case of a Dirichlet boundary value problem is to construct a sequence of initial data $(u_{0,n})$ and a sequence h_n, $\lim_{n\to+\infty} h_n = 0$, such that the sequence of solutions of the semiclassical Schrödinger equations admits

$$dt \otimes \delta_{\tau=p(z_0,\zeta_0)} \otimes \delta_{(z,\zeta)=(\varphi_{-t}(z_0,\zeta_0))} \quad (3.39)$$

as the Wigner measure.

In the case where the bicharacteristic starting from $(t_0 = 0, \tau_0 = p(z_0, \zeta_0), z_0, \zeta_0)$ has an interior point $(t_1, \tau = \tau_0, z_1 \in \Omega, \zeta_1)$, we perform the construction as in the previous section; since, by finite speed of propagation (modulo

$\mathcal{O}(h^\infty)$), the boundary is not seen, (3.39) is satisfied close to $(t_1, \tau = \tau_0, z_1, \zeta_1)$. Using the propagation result, we deduce that (3.39) is satisfied everywhere.

In the case where the bicharacteristic starting from $(t_0 = 0, \tau = \tau_0, z_0, \zeta_0)$ has no interior point, we know that it can be approximated by bicharacteristics γ_k which have an interior point (see [23], [24]). For these bicharacteristics, we can construct sequences of initial data $(u_{n,k})$ associated to $(h_{n,k})$, $\lim_{n\to+\infty} h_{n,k} = 0$. Taking $(u_{n_k,k})$ with n_k large enough as initial data matches our aim.

The rest of the proof of the estimate (2.22) in the case of a boundary value problem is now the same as in Section 2.

4. Smoothing effect

In this section we prove a weaker smoothing effect for a class of trapping obstacles.

Assumptions

Consider $\Theta \subset \mathbb{R}^d$ a compact smooth obstacle whose complement, $\Omega = \Theta^c$, is connected. Let Δ_D be the Laplace operator acting on $L^2(\Omega)$, with domain $D = H^2(\Omega) \cap H^1_0(\Omega)$. For $u_0 \in L^2(\Omega)$, denote by $e^{-it\Delta_D}u_0 = u$ the solution of the Schrödinger equation with Dirichlet boundary conditions:

$$\begin{cases} (i\partial_t - \Delta)u = 0 \quad \text{on } \mathbb{R}_t \times \Omega, \\ u \mid_{\partial\Omega} = 0, \\ u \mid_{t=0} = u_0. \end{cases} \tag{4.1}$$

We suppose that $\Theta = \bigcup_{i=1}^N \Theta_i \subset \mathbb{R}^d$ is the union of a finite number of strictly convex obstacles, Θ_i satisfying the following.

- For any $1 \leq i, j, k \leq N$, $i \neq j$, $j \neq k$, $k \neq i$, one has

$$\text{Convex Hull}(\Theta_i \cup \Theta_j) \cap \Theta_k = \emptyset. \tag{4.2}$$

- Denote by κ the infimum of the principal curvatures of the boundaries of the obstacles Θ_i, and denote by L the infimum of the distances between two obstacles. Then, if $N > 2$, we assume that $\kappa L > N$ (no assumption if $N = 2$).

Remark 4.1

If there are only two obstacles, then the assumptions are automatically fulfilled. The first assumption is essentially technical, whereas the second one is an assumption about the strong hyperbolicity of the dynamical system given by the billiard flow.

In this case, since there are trapped trajectories (e.g., any line minimizing the distance between two obstacles is trapped), we show in Section 3 that the plain smoothing effect $H^{1/2}$ does not hold. However, the result below (a more precise version of Theorem 1.7) shows that the smoothing effect with a logarithmic loss still holds.

THEOREM 4.2
Under the assumptions above, for any $\chi \in C_0^\infty(\mathbb{R}^d)$, there exists $C > 0$ such that the solution of

$$\begin{cases} (i\partial_t - \Delta)u = 0 \quad \text{on } \mathbb{R}_t \times \Omega, \\ u\mid_{t=0}= u_0, \\ u\mid_{\partial\Omega}= 0 \end{cases} \tag{4.3}$$

and the solution of

$$\begin{cases} (i\partial_t - \Delta)v = \chi f, \quad \chi f \text{ compactly supported in time}, \\ v\mid_{t\ll 0}= 0, \\ v\mid_{\partial\Omega}= 0 \end{cases} \tag{4.4}$$

satisfy

$$\begin{aligned} \|\chi u\|_{L^2(\mathbb{R}_t; H_D^{1/2,-}(\Omega))} &\leq C\|u_0\|_{L^2(\Omega)}, \\ \|\chi v\|_{L^2(\mathbb{R}_t; H_D^{1/2,-}(\Omega))} &\leq C\|\chi f\|_{L^2(\mathbb{R}_t; H_D^{-1/2,+}(\Omega))}, \end{aligned} \tag{4.5}$$

where $H_D^{1/2,-} = D\big((\mathrm{Id} - \Delta_D)^{1/4}\log^{-1/2}(2\,\mathrm{Id} - \Delta_D)\big)$ and $H^{-1/2,+} = (H_D^{1/2,-})'$. In particular,

$$\begin{gathered} \forall \varepsilon > 0, \quad H^{-1/2+\varepsilon}(\Omega) \subset H_D^{-1/2,+} \subset H^{-1/2}(\Omega), \\ H^{1/2}(\Omega) \subset H_D^{1/2,-} \subset H^{1/2-\varepsilon}(\Omega) \end{gathered}$$

with continuous injections.

Remark 4.3
In the case where there exists an elliptic (stable) periodic trajectory, it is possible to construct quasimodes with compact support, that is, functions $(e_n)_{n\in\mathbb{N}}$ with compact supports associated to a particular sequence $(\lambda_n) \to +\infty$ and satisfying

$$\begin{aligned} -\Delta e_n &= \lambda_n e_n + r_n, \\ \|r_n\|_{H^N} &\leq C_{N,M}\lambda_n^{-M}, \quad \forall N, M \in \mathbb{N}. \end{aligned} \tag{4.6}$$

From this we deduce easily that the sequence of solutions of the Schrödinger equation with initial data (e_n) is, for any $\varepsilon > 0$, $s > 0$, not bounded in $L^1([0,\varepsilon[; H^s_{\mathrm{loc}})$, which implies that no smoothing effect at all is true any more. Under the assumptions of Theorem 4.2, the periodic trajectories are hyperbolic (unstable), which forbids the construction of such well-localized quasimodes.

Theorem 4.2 is deduced from the following estimate of the cut-off resolvent.

PROPOSITION 4.4
Suppose that the obstacle Θ satisfies the assumptions in Theorem 4.2. Then the resolvent of the operator Δ_D, $(-\Delta_D - \lambda)^{-1}$ (which is analytic in $\mathbb{C} \setminus \mathbb{R}^+$) satisfies

$$\begin{gathered}\forall \chi \in C_0^\infty(\mathbb{R}^2), \quad \exists C > 0; \qquad \forall \lambda \in \mathbb{R}, \quad 0 < \varepsilon \ll 1, \\ \left\| \chi(-\Delta_D - (\lambda \pm i\varepsilon))^{-1} \chi \right\|_{L^2 \to L^2} \le \frac{C \log(2 + |\lambda|)}{1 + \sqrt{|\lambda|}}.\end{gathered} \tag{4.7}$$

We prove this estimate for $\lambda \gg 1$. The proof for $|\lambda| \ll 1$ can be found in [4, Annexe B.2], whereas the result for $c \le |\lambda| \le C$ follows from the Rellich uniqueness theorem (see [20] or [4, Annexe B.1]), and the result for $\lambda < -\varepsilon$ is clear because in this case the operator is semiclassically elliptic.

Let us perform a change of variables $\lambda = \tau^2$ and consider $\chi(-\Delta_D - (\tau^2))^{-1}\chi$, which is holomorphic in $\{\operatorname{Im} \tau < 0\}$ and satisfies there (according to the standard estimate for self-adjoint operators)

$$\left\| (-\Delta_D - (\tau^2))^{-1} \right\|_{L^2(\Omega) \to L^2(\Omega)} \le \frac{1}{|\tau| |\operatorname{Im} \tau|}. \tag{4.8}$$

M. Ikawa proved in [16], [17] and more precisely in [18, Theorem 2.1] (see also the work by C. Gérard [13], where such an estimate is implicit) that, under the assumptions above, the following estimate on the cut-off resolvent holds.

THEOREM 4.5 (Ikawa, [18, Theorem 2.1])
The cut-off resolvent $\chi(-\Delta_D - (\lambda \pm i\varepsilon))^{-1}\chi$ admits a holomorphic continuation in a strip of the upper half-plane

$$\left\{ \tau \in \mathbb{C}; |\tau| > 1, \operatorname{Im} \tau \le \alpha \right\}, \quad \alpha > 0, \tag{4.9}$$

and satisfies there (for a large N)

$$\left\| \chi(-\Delta_D - (\tau^2))^{-1} \chi \right\|_{L^2(\Omega) \to L^2(\Omega)} \le C |\tau|^N. \tag{4.10}$$

Remark 4.6
In [18, Theorem 2.1] the proof is done with the additional assumption that the dimension of space is equal to 3 (which is the relevant dimension the author had in mind for applications to the wave equation). However, the proof could be equally performed in any space dimension $d \ge 2$ (see [13] in the case $N = 2$, $d \ge 2$).

Using (4.8) and (4.10) (and writing $\tau = h^{-1}z$, $z \sim 1$, $h \to 0$), one easily sees, with

$$f(h, z) = \left(\chi(-h^2 \Delta_D - z^2)^{-1} \chi u, v \right)_{L^2(\Omega)}, \quad u, v \in L^2(\Omega), \tag{4.11}$$

that (4.7), for large $|\lambda|$, follows from Theorem 4.5 and from the following semiclassical maximum principle (a variant of Phrägmen Lindelöf principle) adapted from the work by Tang and Zworski [28].

LEMMA 4.7
Suppose that $f(h,z)$ is a family of holomorphic functions defined for $0 < h < 1$ in a neighbourhood of

$$\Omega(h) = \Big[\frac{1}{2}, \frac{3}{2}\Big] \times i[h\alpha, -h\alpha] \tag{4.12}$$

such that

$$\begin{aligned} |f(h,z)| &\le Ch^{-M} \quad \textit{on } \Omega(h), \\ |f(h,z)| &\le \frac{1}{|\operatorname{Im} z|} \quad \textit{on } \Omega(h) \cap \{\operatorname{Im} z < 0\}. \end{aligned} \tag{4.13}$$

Then there exist $h_0 > 0$, $C > 0$, such that, for any $0 < h < h_0$,

$$|f(h,z)| \le C\frac{\log(h^{-1})}{h} \quad \textit{on } \Big[\frac{4}{5}, \frac{6}{5}\Big]. \tag{4.14}$$

To prove this lemma, first consider the function

$$\varphi(z,h) = (\pi h)^{-1/2} \int e^{-(x-z)^2/h} \Psi(x)\, dx, \tag{4.15}$$

where $\Psi \in C_0^\infty(]2/3, 4/3[)$ is nonnegative and equal to 1 in $[3/4, 5/4]$. Then the function $\varphi(z,h)$ satisfies the following:
(1) $\varphi(z,h)$ is holomorphic in $\Omega(h)$,
(2) $|\varphi(z,h)| \le C$ in $\Omega(h)$,
(3) $|\varphi(z,h)| \ge c > 0$ in $[4/5, 6/5]$,
(4) $|\varphi(z,h)| \le Ce^{-c/h}$ on $\Omega(h) \cap \{|\operatorname{Re} z - 1| \ge 1/2\}$.
Then apply the classical maximum principle to the function $g(z,h) = e^{-iN\log(h)z/h}\varphi(z,h) f(z,h)$ on the domain

$$\widetilde{\Omega}(h) = \Big[\frac{1}{2}, \frac{3}{2}\Big] \times i\Big[h\alpha, \frac{-h}{\log(h^{-1})}\Big]. \tag{4.16}$$

Using the bounds (4.7) on f and the properties of φ above, we can estimate g by

$$\begin{aligned} |g(z,h)| &\le Ch^{N\alpha - M} \quad \text{on } \partial\widetilde{\Omega}(h) \cap \{\operatorname{Im} z = h\alpha\}, \\ |g(z,h)| &\le C_N e^{-c/h} \quad \text{on } \partial\widetilde{\Omega}(h) \cap \Big\{\operatorname{Re} x \in \Big\{\frac{1}{2}, \frac{3}{2}\Big\}\Big\}, \\ |g(z,h)| &\le C_N \frac{\log(h^{-1})}{h} \quad \text{on } \partial\widetilde{\Omega}(h) \cap \Big\{\operatorname{Im} z = \frac{-h}{\log(h^{-1})}\Big\}. \end{aligned} \tag{4.17}$$

Taking N large enough and applying the maximum principle, we get

$$|g(z,h)| \leq C' \frac{\log(h^{-1})}{h} \quad \text{on } \widetilde{\Omega}(h), \tag{4.18}$$

which implies

$$|f(z,h)| \leq C' \frac{\log(h^{-1})}{h} \quad \text{on } \Big[\frac{4}{5}, \frac{6}{5}\Big] \tag{4.19}$$

and ends the proof of Lemma 4.7.

We deduce from (4.7)

$$\big\| \chi(-\Delta_D - (\lambda \pm i\varepsilon))^{-1} \chi \big\|_{H^{-1/2,+} \to H^{1/2,-}} \leq C. \tag{4.20}$$

Indeed, for bounded λ, integrations by parts show that we can in fact replace $H^{1/2,-}$ by $H_0^1(\Omega)$ and $H^{-1/2,+}$ by $H^{-1}(\Omega)$, and for large λ, we decompose, with $\Psi \in C_0^\infty(]1/2, 2[)$ equal to 1 close to 1,

$$u = (P - \lambda)^{-1} \chi f = \Psi\Big(\frac{-\Delta_D}{\lambda}\Big) u + \Big(1 - \Psi\Big(\frac{-\Delta_D}{\lambda}\Big)\Big) u. \tag{4.21}$$

We get by the functional calculus of self-adjoint operators

$$\Big\| \Big(1 - \Psi\Big(\frac{-\Delta_D}{\lambda}\Big)\Big) u \Big\|_{H_0^1(\Omega)} \leq C \Big\| \Big(1 - \Psi\Big(\frac{-\Delta_D}{\lambda}\Big)\Big) \chi f \Big\|_{H^{-1}(\Omega)}. \tag{4.22}$$

On the other hand, the function $v = \Psi(-\Delta_D/\lambda) u$ satisfies

$$(P - \lambda) v = \Psi\Big(\frac{-\Delta_D}{\lambda}\Big) \chi f. \tag{4.23}$$

If $\widetilde{\chi} \in C_0^\infty(\mathbb{R}^d)$ is equal to 1 on the support of χ, we have, modulo negligible terms,

$$\widetilde{\chi} \Psi\Big(\frac{-\Delta_D}{\lambda}\Big) \chi = \Big(\frac{-\Delta_D}{\lambda}\Big) \chi \tag{4.24}$$

because

$$\widetilde{\chi} \Psi\Big(\frac{-\Delta_D}{\lambda}\Big) \chi - \Big(\frac{-\Delta_D}{\lambda}\Big) \chi = \Big[\widetilde{\chi}, \Psi\Big(\frac{-\Delta_D}{\lambda}\Big)\Big] \chi$$

and, on the support of $\nabla \widetilde{\chi}$, the operator P is a differential operator and consequently $\Psi(-\Delta_D/\lambda)$ is a pseudodifferential operator on this set (see, e.g., [27, Section 4]).

According to (4.23), (4.24), and Lemma 4.7, we get

$$\Big(\frac{\sqrt{|\lambda|}}{\log(2 + |\lambda|)}\Big)^{1/2} \Big\| \chi \Psi\Big(\frac{-\Delta_D}{\lambda}\Big) u \Big\|_{L^2} \leq C \Big(\frac{\log(2 + |\lambda|)}{\sqrt{|\lambda|}}\Big)^{1/2} \Big\| \Psi\Big(\frac{-\Delta_D}{\lambda}\Big) \chi f \Big\|_{L^2} \tag{4.25}$$

to replace the weights in λ above by the $H^{\pm 1/2,\mp}$-norms (i.e., to replace the weights in λ by weights in $-\Delta_D$); it is enough to check that modulo negligible terms, if $\widetilde{\Psi}$ is a function equal to 1 on the support of Ψ,

$$\widetilde{\Psi}\Big(\frac{-\Delta_D}{\lambda}\Big)\chi\Psi\Big(\frac{-\Delta_D}{\lambda}\Big) = \chi\Psi\Big(\frac{-\Delta_D}{\lambda}\Big), \tag{4.26}$$

which follows from the same arguments as above.

Following [7], Theorem 4.2 is now a consequence of (4.20). For the sake of completeness and since the argument is short, we recall it: first, note that by the (TT^*)-argument it suffices to study the second (inhomogeneous) case. Indeed, denote $T = \chi e^{-it\Delta_D}$. The continuity of T from L^2 to $L^2(\mathbb{R}_t; H^{1/2,-})$ is equivalent to the continuity of the adjoint operator

$$T^* f = \int_{\mathbb{R}} e^{is\Delta_D}\chi f(s)\, ds \tag{4.27}$$

from $L^2(\mathbb{R}_t; H^{-1/2,+})$ to L^2, which in turn is equivalent to the continuity of the operator TT^* from $L^2(\mathbb{R}_t; H^{-1/2,+})$ to $L^2(\mathbb{R}_t; H^{1/2,-})$. But

$$\begin{aligned} TT^* f(t) &= \int_{\mathbb{R}} \chi e^{i(s-ts)\Delta_D}\chi f(s)\, ds \\ &= \int_{s<t} \chi e^{i(s-t)\Delta_D}\chi f(s)\, ds + \int_{t<s} \chi e^{i(s-t)\Delta_D}\chi f(s)\, ds, \end{aligned} \tag{4.28}$$

and (by time inversion) it clearly suffices to prove the continuity of any one of the terms in the right-hand side, which is the second (inhomogeneous) part of Theorem 4.2.

Consider now (v, f) a solution of (4.4). By translation invariance, we can suppose that f (and hence v) is supported in $\{t > 0\}$. The Fourier transforms of v and f are (according to the support property) holomorphic in the set $\{\operatorname{Im} z < 0\}$ and satisfy there, according to (4.4),

$$(-z + \Delta)\hat{u}(z, \cdot) = \chi\hat{f}(z, \cdot). \tag{4.29}$$

Taking $z = \tau - i\varepsilon$, $\tau \in \mathbb{R}$, and having ε tend to zero, using (4.20), we get

$$\|\chi\hat{u}\|_{L^2(\mathbb{R}_\tau; H^{1/2,-})} \le C\|\chi\hat{f}\|_{L^2(\mathbb{R}_\tau; H^{-1/2,+})}, \tag{4.30}$$

and since the Fourier transform is an isometry on $L^2(\mathbb{R}; H)$ if H is a Hilbert space, we get (4.5).

Finally, as in [7], we can deduce from Theorem 4.2 the following.

THEOREM 4.8 (Global existence for 2-dimensional defocusing NLS equations) *Consider $\Theta \subset \mathbb{R}^2$ an obstacle that is the union of N strictly convex obstacles satisfying the assumptions above. Denote by $\Omega = \Theta^c$ its complement, and let P be a polynomial with real coefficients. For every $u_0 \in H_0^1(\Omega)$, there exists a unique maximal solution $u \in C(I, H_0^1(\Omega))$ of the equation*

$$i\partial_t u + \Delta u = P'(|u|^2)u, \quad u(0,x) = u_0(x). \tag{4.31}$$

Moreover we have the following.

(i) *If $\|u_0\|_{H_0^1(\Omega)}$ is bounded from above, the length of $I \cap \mathbb{R}_\pm$ is bounded from below by a positive constant.*

(ii) *For any finite p, $u \in L^p_{\mathrm{loc}}(I, L^\infty(\Omega))$.*

(iii) *If $P(r) \longrightarrow +\infty$ as $r \longrightarrow +\infty$, $I = \mathbb{R}$.*

(iv) *If $u_0 \in H^s_D(\Omega)$ for some $s > 1$, $u \in C(I, H^s_D(\Omega))$. In particular, if $u_0 \in C_0^\infty(\Omega)$, $u \in C^\infty(I \times \Omega)$.*

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References

[1] M. BEN-ARTZI and A. DEVINATZ, *Regularity and decay of solutions to the Stark evolution equation*, J. Funct. Anal. **154** (1998), 501 – 512. MR 1612650

[2] M. BEN-ARTZI and S. KLAINERMAN, *Decay and regularity for the Schrödinger equation*, J. Anal. Math. **58** (1992), 25 – 37. MR 1226935

[3] N. BURQ, *Mesures semi-classiques et mesures de défaut*, Astérisque **245** (1997), 167 – 195, Séminaire Bourbaki 1996/97, no. 826. MR 1627111

[4] ———, *Décroissance de l'énergie locale de l'équation des ondes pour le problème extérieur et absence de résonance au voisinage du réel*, Acta Math. **180** (1998), 1 – 29. MR 1618254

[5] ———, *Semi-classical estimates for the resolvent in nontrapping geometries*, Int. Math. Res. Not. **2002**, no. 5, 221 – 241. MR 1876933

[6] N. BURQ and P. GÉRARD, *Condition nécessaire et suffisante pour la contrôlabilité exacte des ondes*, C. R. Acad. Sci. Paris Sér I Math. **325** (1997), 749 – 752. MR 1483711

[7] N. BURQ, P. GÉRARD, and N. TZVETKOV, *On nonlinear Schrödinger equations in exterior domains*, to appear in Ann. Inst. H. Poincaré Non Linéaire, preprint, 2002, http://www.math.u-psud.fr/~burq/

[8] N. BURQ and G. LEBEAU, *Mesures de défaut de compacité, application au système de Lamé*, Ann. Sci. École Norm. Sup. (4) **34** (2001), 817 – 870. MR 1872422

[9] P. CONSTANTIN and J.-C. SAUT, *Local smoothing properties of Schrödinger equations*, Indiana Univ. Math. J. **38** (1989), 791 – 810. MR 1017334

[10] S.-I. DOI, *Remarks on the Cauchy problem for Schrödinger-type equations*, Comm. Partial Differential Equations **21** (1996), 163 – 178. MR 1373768

[11] ———, *Smoothing effects of Schrödinger evolution groups on Riemannian manifolds*, Duke Math. J. **82** (1996), 679 – 706. MR 1387689

[12] ———, *Smoothing effects for Schrödinger evolution equation and global behaviour of geodesic flow*, Math. Ann. **318** (2000), 355 – 389. MR 1795567

[13] C. GÉRARD, *Asymptotique des pôles de la matrice de scattering pour deux obstacles strictement convexes*, Mém. Soc. Math. France (N.S.) (1988), no. 31. MR 0998698

[14] P. GÉRARD and E. LEICHTNAM, *Ergodic properties of eigenfunctions for the Dirichlet problem*, Duke Math. J. **71** (1993), 559 – 607. MR 1233448

[15] P. GÉRARD, P. MARKOWICH, N. J. MAUSER, and F. POUPAUD, *Homogenization limits and Wigner transforms*, Comm. Pure Appl. Math. **50** (1997), 323 – 379. MR 1438151

[16] M. IKAWA, *Decay of solutions of the wave equation in the exterior of two convex obstacles*, Osaka J. Math. **19** (1982), 459 – 509. MR 0676233

[17] ———, *On the poles of the scattering matrix for two strictly convex obstacles*, J. Math. Kyoto Univ. **23** (1983), 127 – 194. MR 0692733

[18] ———, *Decay of solutions of the wave equation in the exterior of several convex bodies*, Ann. Inst. Fourier (Grenoble) **38** (1988), 113 – 146. MR 0949013

[19] T. KATO and K. YAJIMA, *Some examples of smooth operators and the associated smoothing effect*, Rev. Math. Phys. **1** (1989), 481 – 496. MR 1061120

[20] P. D. LAX and R. S. PHILLIPS, *Scattering Theory*, 2nd ed., Pure Appl. Math. **26**, Academic Press, Boston, 1989. MR 1037774

[21] G. LEBEAU, "Équation des ondes amorties" in *Algebraic and Geometric Methods in Mathematical Physics (Kaciveli, Ukraine, 1993)*, Math. Phys. Stud. **19**, Kluwer, Dordrecht, Netherlands, 1996, 73 – 109. MR 1385677

[22] P.-L. LIONS and T. PAUL, *Sur les mesures de Wigner*, Rev. Mat. Iberoamericana **9** (1993), 553 – 618. MR 1251718

[23] R. B. MELROSE AND J. SJÖSTRAND, *Singularities of boundary value problems, I*, Comm. Pure Appl. Math. **31** (1978), 593 – 617. MR 0492794

[24] ———, *Singularities of boundary value problems, II*, Comm. Pure Appl. Math. **35**, (1982), 129 – 168. MR 0644020

[25] L. MILLER, *Réfraction d'ondes semi-classiques par des interfaces franches*, C. R. Acad. Sci. Paris Sér. I Math. **325** (1997), 371 – 376. MR 1467089

[26] ———, *Refraction of high-frequency waves density by sharp interfaces and semiclassical measures at the boundary*, J. Math. Pures Appl. (9) **79** (2000), 227 – 269. MR 1750924

[27] J. SJÖSTRAND, "A trace formula and review of some estimates for resonances" in *Microlocal Analysis and Spectral Theory (Lucca, Italy, 1996)*, NATO Adv. Sci. Inst. Ser. C Math. Phys. Sci. **490**, Kluwer, Dordrecht, Netherlands, 1997, 377 – 437. MR 1451399

[28] S.-H. TANG and M. ZWORSKI, *Resonance expansions of scattered waves*, Comm. Pure Appl. Math. **53** (2000), 1305 – 1334. MR 1768812

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