Conjectures and results on modular representations of $\mathrm{GL}_n(K)$ for a p-adic field K

Christophe Breuil[*] Florian Herzig[†] Yongquan Hu[‡]

Stefano Morra[§] Benjamin Schraen[¶]

Abstract

Let p be a prime number and K a finite extension of $\mathbb{Q}_p$. We state conjectures on the smooth representations of $\mathrm{GL}_n(K)$ that occur in spaces of mod p automorphic forms (for compact unitary groups). In particular, when K is unramified, we conjecture that they are of finite length and predict their internal structure (extensions, form of subquotients) from the structure of a certain algebraic representation of GL_n. When $n = 2$ and K is unramified, we prove several cases of our conjectures, including new finite length results.

Contents

[*]CNRS, Bâtiment 307, Faculté d'Orsay, Université Paris-Saclay, 91405 Orsay Cedex, France
[†]Dept. of Math., Univ. of Toronto, 40 St. George St., BA6290, Toronto, ON M5S 2E4, Canada
[‡]Morningside Center of Math., No. 55, Zhongguancun East Road, Beijing, 100190, China
[§]Lab. d'Analyse, Géométrie, Algèbre, 99 Av. Jean Baptiste Clément, 93430 Villetaneuse, France
[¶]Bâtiment 307, Faculté d'Orsay, Université Paris-Saclay, 91405 Orsay Cedex, France

1 Introduction

1.1 Preamble

Let p be a prime number and K a local field of residue characteristic p. In the early nineties, Barthel and Livné had the fancy idea to start classifying irreducible (admissible) smooth representations of $\mathrm{GL}_2(K)$ over an algebraically closed field of characteristic p ([BL94], [BL95]). They found four nonempty distinct classes of such representations: 1-dimensional ones, irreducible principal series, special series, and those which are not an irreducible constituent of a principal series that they called supersingular. In 2001, one of us classified supersingular representations of $\mathrm{GL}_2(\mathbb{Q}_p)$ with a central character ([Bre03a]) and showed that they are in "natural" bijection with 2-dimensional irreducible representations of $\mathrm{Gal}(\overline{\mathbb{Q}}_p/\mathbb{Q}_p)$ in characteristic p. This was one of the starting points of the mod p and p-adic Langlands programmes for $\mathrm{GL}_2(\mathbb{Q}_p)$, which was developed essentially during the decade 2000-2010 (see for instance [Bre03b], [Bre10], [Eme10b], [Kis10], [Col10], [Ber10], [Paš13], [Eme], [CDP14], [CEG$^+$18], ...).

There are two main novel features of the mod p local Langlands correspondence for $\mathrm{GL}_2(\mathbb{Q}_p)$ (compared to previous Langlands correspondences). The first one is that it involves *reducible* representations of $\mathrm{GL}_2(\mathbb{Q}_p)$. More precisely, the representation of $\mathrm{GL}_2(\mathbb{Q}_p)$ is irreducible (resp. semisimple, resp. indecomposable) if and only if its corresponding 2-dimensional representation of $\mathrm{Gal}(\overline{\mathbb{Q}}_p/\mathbb{Q}_p)$ is, and, in the reducible case, is given (at least generically) by an extension between two specific principal series. The second one, found by Colmez in [Col10], is that the correspondence can be made *functorial* by an exact functor from finite length representations of $\mathrm{GL}_2(\mathbb{Q}_p)$ to étale (φ, Γ)-modules, i.e. to finite length representations of $\mathrm{Gal}(\overline{\mathbb{Q}}_p/\mathbb{Q}_p)$ by Fontaine's equivalence. Thanks to this exact functor, one can extend the correspondence first to extensions of representations, and then to deformations on both sides.

When K is not $\mathbb{Q}_p$, trouble comes from supersingular representations. Contrary to the case $K = \mathbb{Q}_p$, they can be more numerous than 2-dimensional irreducible representations of $\mathrm{Gal}(\overline{K}/K)$ ([BP12]) and they cannot be described as quotients of a compact induction by a finite number of equations ([Hu12, Cor.5.5], [Sch15, Thm.0.1], [Wu21, Thm.1.1]), justifying *a posteriori* the terminology "very strange" that was used to describe them in the introduction of [BL95]. As a consequence, no classification of supersingular representations of $\mathrm{GL}_2(K)$ is known so far, which has hitherto made impossible to find a definition of a hypothetical local mod p correspondence for $\mathrm{GL}_2(K)$ by purely local (either representation theoretic or geometric) means.

Fortunately, the global theory comes to the rescue. If a local correspondence exists, there is a place where it should be realized: the mod p cohomology of Shimura varieties. Let us assume now that K is a finite unramified extension of $\mathbb{Q}_p$ with residue field $\mathbb{F}_{p^f}$ and let $K_1 \stackrel{\text{def}}{=} 1 + pM_2(\mathcal{O}_K) \subseteq \mathrm{GL}_2(\mathcal{O}_K)$. Following the pioneering work of [BDJ10] on Serre weight conjectures, a series of articles ([BP12], [EGS15], [HW18], [LMS22], [Le19]) led to a complete description of the K_1-invariants of the $\mathrm{GL}_2(K)$-representations carried by Hecke isotypic subspaces in such mod p cohomology groups. Although these invariants are only a tiny piece of the representations of $\mathrm{GL}_2(K)$, combined with weight cycling this turned out to give a strong hint on the form of these representations, as well as being a useful technical result. Indeed, very recently, building on this description and on results of [BHH$^+$23], Hu and Wang could prove that, at least when K is quadratic unramified and the representation of $\mathrm{Gal}(\overline{\mathbb{Q}}_p/K)$ is a nonsplit extension between two (sufficiently generic) characters, these $\mathrm{GL}_2(K)$-representations are indecomposable of length 3 (in particular are of finite length), with similar principal series as in the case $K = \mathbb{Q}_p$ in socle and cosocle, and a supersingular representation "in the middle" ([HW22, Thm.1.7]).

These recent results maintain the hope of a local Langlands correspondence for $\mathrm{GL}_2(K)$. They also prompted us to make public some conjectures we had in mind for many years on the form of the $\mathrm{GL}_n(K)$-representations carried by Hecke isotypic subspaces, and on a functorial link to representations of $\mathrm{Gal}(\overline{\mathbb{Q}}_p/\mathbb{Q}_p)$ via (φ, Γ)-modules. We state such conjectures in the present work (Conjecture 2.1.3.1, Conjecture 2.1.4.5, Conjecture 2.5.1) and we prove some special cases in the case $n = 2$ and K unramified, including some new finite length results (Theorem 3.4.4.3, Theorem 3.4.4.6, Corollary 3.4.4.7). Moreover, when $n = 2$ and K is unramified, we also define (and use in the proofs!) an abelian category $\mathcal{C}$ of smooth admissible representations of $\mathrm{GL}_2(K)$ in characteristic p (containing the representations coming from the global theory) together with an exact functor from $\mathcal{C}$ to a new category of multivariable (φ, Γ)-modules.

1.2 Conjectures

Let us first describe our conjectures with some details. As usual, we mostly work in the setting of compact unitary groups (except in §2.1.4), so that we do not (yet) mix delicate representation theoretic issues with difficult geometric problems (ultimately, we think that the representations of $\mathrm{GL}_n(K)$ should not change from one global setting to another). We fix F a CM-field, i.e. a totally imaginary quadratic extension of a totally real number field F^+, and we assume *for simplicity in this introduction* that p is inert in F^+. We also assume (not for simplicity) that the unique p-adic place v of F^+ splits in F. We fix a continuous absolutely irreducible representation

$$\overline{r} : \mathrm{Gal}(\overline{F}/F) \longrightarrow \mathrm{GL}_n(\mathbb{F}),$$

where $\mathbb{F}$ is a (sufficiently large) extension of $\mathbb{F}_p$ and we assume that $\overline{r}$ is automorphic for a unitary group H over F^+ that is compact at all infinite places and becomes GL_n

over F. Equivalently there exists a compact open subgroup $U^v \subseteq H(\mathbb{A}_{F^+}^{\infty,v})$ such that

$$S(U^v, \mathbb{F})[\mathfrak{m}] \stackrel{\text{def}}{=} \{f : H(F^+)\backslash H(\mathbb{A}_{F^+}^{\infty})/U^v \to \mathbb{F} \text{ locally constant}\}[\mathfrak{m}] \neq 0,$$

where $[\mathfrak{m}]$ means the Hecke-isotypic subspace associated to $\overline{r}$ (one has to choose a finite set of bad places Σ in the definition of $\mathfrak{m}$, but we forget this issue here, see §2.1.3 below).

Let $\tilde{v}|v$ in F, $K \stackrel{\text{def}}{=} F_{\tilde{v}}$ the corresponding completion and $\overline{r}_{\tilde{v}}$ the restriction of $\overline{r}$ to a decomposition subgroup at $\tilde{v}$. Then $S(U^v, \mathbb{F})[\mathfrak{m}]$ is an admissible smooth representation of $\mathrm{GL}_n(K)$ over $\mathbb{F}$ by the usual right translation action on functions. Our main conjecture gives the form of this $\mathrm{GL}_n(K)$-representation (assuming it is of finite length) as well as a functorial link to $\overline{r}_{\tilde{v}}$. But to state it we need a few preliminaries on certain algebraic representations of GL_n over $\mathbb{F}$.

Let us first assume *for simplicity* that $K = \mathbb{Q}_p$. We let Std be the standard n-dimensional algebraic representation of GL_n over $\mathbb{F}$ and define the following algebraic representation of GL_n over $\mathbb{F}$:

$$\overline{L}^{\otimes} \stackrel{\text{def}}{=} \bigotimes_{i=1}^{n-1} \bigwedge\nolimits_{\mathbb{F}}^{i} \mathrm{Std}.$$

We fix $P \subseteq \mathrm{GL}_n$ a parabolic subgroup containing the Borel B of upper-triangular matrices, and let M_P be its Levi subgroup containing the torus T of diagonal matrices. We fix $\widetilde{P} \subseteq P$ a Zariski closed algebraic subgroup containing M_P and we consider the algebraic representation $\overline{L}^{\otimes}|_{\widetilde{P}}$ of $\widetilde{P}$ over $\mathbb{F}$.

Definition 1.2.1 (Definition 2.2.1.3)**.** A subquotient of $\overline{L}^{\otimes}|_{\widetilde{P}}$ is a *good* subquotient if its restriction to the center Z_{M_P} of M_P is a (direct) sum of isotypic components of $\overline{L}^{\otimes}|_{Z_{M_P}}$.

Note that an isotypic component of $\overline{L}^{\otimes}|_{Z_{M_P}}$ carries an action of M_P (Lemma 2.2.1.2). Hence, viewing an isotypic component of $\overline{L}^{\otimes}|_{Z_{M_P}}$ as a representation of $\widetilde{P}$ via the surjection $\widetilde{P} \twoheadrightarrow M_P$, one can see $\overline{L}^{\otimes}|_{\widetilde{P}}$ as a successive extension of such isotypic components (Lemma 2.2.1.5). On the $\mathrm{GL}_n(\mathbb{Q}_p)$-side, the isotypic components of $\overline{L}^{\otimes}|_{Z_{M_P}}$ will play the role of irreducible constituents. Note that the isotypic components of $\overline{L}^{\otimes}|_{Z_{M_P}}$ are by definition all distinct.

To an isotypic component C of $\overline{L}^{\otimes}|_{Z_{M_P}}$, we associate a parabolic subgroup $P(C)$ of GL_n containing B as follows. Let $\lambda \in X(T) = \mathrm{Hom}_{\mathrm{Gr}}(T, \mathbb{G}_{\mathrm{m}})$ be any weight such that C is the isotypic component of $\lambda|_{Z_{M_P}}$ and define (see (37))

$$\lambda' \stackrel{\text{def}}{=} \frac{1}{|W(P)|} \sum_{w' \in W(P)} w'(\lambda) \ \in \ X(T) \otimes_{\mathbb{Z}} \mathbb{Q},$$

where $W(P)$ is the Weyl group of M_P. Let θ be the highest weight of $\overline{L}^{\otimes}|_T$ and w in the Weyl group of GL_n such that $w(\lambda')$ is dominant with respect to B. Then one can check that (see Proposition 2.2.2.6)

$$\theta - w(\lambda') = \sum_{\alpha \in S} n_\alpha \alpha,$$

where S is the set of simple roots of GL_n (with respect to B) and the n_α are in $\mathbb{Q}_{\geq 0}$. Then $P(C)$ is by definition the parabolic subgroup of GL_n corresponding to the subset $\{\alpha \in S : n_\alpha \neq 0\}$ of S. We denote by $P(C)^-$ its opposite parabolic subgroup.

We now go back to the above global setting. Assuming a weak genericity condition on $\overline{r}_{\tilde{v}}$, one can replace $\overline{r}_{\tilde{v}}$ by a suitable conjugate so that the image of $\overline{r}_{\tilde{v}}$ is contained in the $\mathbb{F}$-points of a Zariski closed algebraic subgroup $\widetilde{P}_{\overline{r}_{\tilde{v}}}$ of a parabolic $P_{\overline{r}_{\tilde{v}}}$ as above which is "as small as possible" (see Definition 2.3.2.3 and Theorem 2.3.2.5). The following conjecture is part of Conjecture 2.5.1 (see Definition 2.4.2.7 and Definition 2.4.1.5).

Conjecture 1.2.2. *Assume that $\overline{r}_{\tilde{v}}$ has distinct irreducible constituents and that the ratio of any two 1-dimensional constituents is not in $\{\omega, \omega^{-1}\}$, where ω is the mod p cyclotomic character. Then we have a $\mathrm{GL}_n(\mathbb{Q}_p)$-equivariant isomorphism for some integer $d \geq 1$:*

$$S(U^v, \mathbb{F})[\mathfrak{m}] \cong \left(\Pi_{\tilde{v}} \otimes (\omega^{n-1} \circ \det)\right)^{\oplus d},$$

where $\Pi_{\tilde{v}}$ is an admissible smooth representation of $\mathrm{GL}_n(\mathbb{Q}_p)$ over $\mathbb{F}$ of finite length with distinct irreducible constituents such that there exists a bijection Φ between the (finite) set of subquotients of $\Pi_{\tilde{v}}$ and the (finite) set of good subquotients of $\overline{L}^{\otimes}|_{\widetilde{P}_{\overline{r}_{\tilde{v}}}}$ satisfying the following properties:

(i) *Φ respects inclusions, and thus extends to a bijection between the sets of all subquotients on both sides;*

(ii) *Φ^{-1} sends an isotypic component C of $\overline{L}^{\otimes}|_{Z_{M_{P_{\overline{r}_{\tilde{v}}}}}}$ to an irreducible constituent of $\Pi_{\tilde{v}}$ of the form $\mathrm{Ind}_{P(C)^-(\mathbb{Q}_p)}^{\mathrm{GL}_n(\mathbb{Q}_p)} \pi(C)$, where $\pi(C)$ is a supersingular representation of $M_{P(C)}(\mathbb{Q}_p)$ over $\mathbb{F}$.*

When K is not necessarily $\mathbb{Q}_p$, the conjecture is completely analogous, defining $\overline{L}^{\otimes}$ by

$$\overline{L}^{\otimes} \stackrel{\text{def}}{=} \bigotimes_{\mathrm{Gal}(K/\mathbb{Q}_p)} \Big(\bigotimes_{i=1}^{n-1} \bigwedge\nolimits_{\mathbb{F}}^{i} \mathrm{Std}\Big),$$

replacing $\widetilde{P}$ by $\widetilde{P}^{\mathrm{Gal}(K/\mathbb{Q}_p)} \stackrel{\text{def}}{=} \underbrace{\widetilde{P} \times \cdots \times \widetilde{P}}_{\mathrm{Gal}(K/\mathbb{Q}_p)}$ and taking isotypic components of $\overline{L}^{\otimes}|_{Z_{M_P}}$ for the *diagonal embedding* $Z_{M_P} \hookrightarrow Z_{M_P}^{\mathrm{Gal}(K/\mathbb{Q}_p)}$ in the definition of good subquotients of $\overline{L}^{\otimes}|_{\widetilde{P}^{\mathrm{Gal}(K/\mathbb{Q}_p)}}$.

Example 1.2.3. (i) If $\overline{r}_{\tilde{v}}$ is irreducible, then $\widetilde{P}_{\overline{r}_{\tilde{v}}} = \mathrm{GL}_n = M_{P_{\overline{r}_{\tilde{v}}}}$ and there is only one isotypic component C in $\overline{L}^{\otimes}|_{Z_{\mathrm{GL}_n}}$. It is such that $P(C) = \mathrm{GL}_n$: the representation $\Pi_{\tilde{v}}$ in Conjecture 1.2.2 is irreducible and supersingular.
(ii) If $\overline{r}_{\tilde{v}}$ is semisimple, then $\widetilde{P}_{\overline{r}_{\tilde{v}}} = M_{P_{\overline{r}_{\tilde{v}}}}$, and since the direct sum decomposition of $\overline{L}^{\otimes}|_{Z_{M_{P_{\overline{r}_{\tilde{v}}}}}}$ into isotypic components for the (diagonal) $Z_{M_{P_{\overline{r}_{\tilde{v}}}}}$-action is a direct sum decomposition as a $\widetilde{P}_{\overline{r}_{\tilde{v}}} = M_{P_{\overline{r}_{\tilde{v}}}}$-representation, we see that the representation $\Pi_{\tilde{v}}$ in Conjecture 1.2.2 is also semisimple.
(iii) If $K = \mathbb{Q}_p$ and $n = 2$, we have $\overline{L}^{\otimes} = \mathrm{Std}$. When $\overline{r}_{\tilde{v}}$ is irreducible, by (i) the representation $\Pi_{\tilde{v}}$ of $\mathrm{GL}_2(\mathbb{Q}_p)$ in Conjecture 1.2.2 is supersingular. When $\overline{r}_{\tilde{v}}$ is reducible split, then $\widetilde{P}_{\overline{r}_{\tilde{v}}} = T = M_{P_{\overline{r}_{\tilde{v}}}}$, and $\overline{L}^{\otimes}|_T = \mathbb{F}\lambda_1 \oplus \mathbb{F}\lambda_2$, where $\lambda_i : \mathrm{diag}(x_1, x_2) \mapsto x_i$, $i \in \{1, 2\}$. There are two isotypic components $C = \mathbb{F}\lambda_1$ or $C = \mathbb{F}\lambda_2$, both with $P(C) = B$: the representation $\Pi_{\tilde{v}}$ in Conjecture 1.2.2 is a direct sum of two irreducible principal series. Finally, when $\overline{r}_{\tilde{v}}$ is reducible nonsplit, then $\widetilde{P}_{\overline{r}_{\tilde{v}}} = B$, $\overline{L}^{\otimes}|_B$ is a nonsplit extension of $\mathbb{F}\lambda_2$ by $\mathbb{F}\lambda_1$ and $\Pi_{\tilde{v}}$ is a nonsplit extension between two irreducible principal series. Note that Conjecture 1.2.2 is known in that case ([CS17b], [CS17a] for $\overline{r}_{\tilde{v}}$ irreducible, [BD20, Cor.7.40] for arbitrary $\overline{r}_{\tilde{v}}$, all generalizing methods of [Eme]).
(iv) For K arbitrary (unramified) and $n = 2$, see Example 2.2.2.9 and Example 1 of §2.4.3.

Conjecture 1.2.2 only gives part of the picture. For instance there should be reducible subquotients of $\Pi_{\tilde{v}}$ which are also parabolic inductions $\mathrm{Ind}_{P(C)^-(\mathbb{Q}_p)}^{\mathrm{GL}_n(\mathbb{Q}_p)} \pi(C)$ with $\pi(C)$ of the form $\pi(C) \cong \pi_1(C) \otimes \cdots \otimes \pi_d(C)$, where the (reducible) $\pi_i(C)$ have themselves the same form as $\Pi_{\tilde{v}}$ but for the smaller $\mathrm{GL}_{n_i}(K)$ appearing in the Levi $M_{P(C)}(K)$ (which gives a "fractal" flavour to the whole picture!). In fact, it is possible that, in the end, this "fractal" picture will automatically follow from property (ii) in Conjecture 1.2.2 (i.e. from the statement for *irreducible* subquotients only), as one can already see in many of the examples of §2.4.3 using the work of Hauseux ([Hau18], [Hau19]), see Remark 2.4.1.6(iv). Also some parabolic (possibly reducible) inductions as above should be deduced from others by a permutation on the factors $\pi_i(C)$. Tracking down all these internal symmetries (with the various twists by characters that occur) and all the implications between them is not really difficult but a bit tedious, as the reader will see from the technical lemmas in §2.4.1 (see e.g. Proposition 2.4.1.8). The interested reader should maybe first have a look at the various examples in §2.4.3 before going into the full combinatorics.

Finally, the full picture has to take into account the Galois action. There is a simple way to extend Colmez's functor from representations of $\mathrm{GL}_2(\mathbb{Q}_p)$ to representations of $\mathrm{GL}_n(K)$ that we recall now (see [Bre15] or §2.1.1). Let $\xi : \mathbb{G}_{\mathrm{m}} \to T$ be the cocharacter $x \mapsto \mathrm{diag}(x^{n-1}, x^{n-2}, \ldots, 1)$ and $N_1 \overset{\mathrm{def}}{=} \mathrm{Ker}(N_0 \xrightarrow{\ell} \mathcal{O}_K \xrightarrow{\mathrm{trace}} \mathbb{Z}_p)$, where N_0 is the unipotent radical of $B(\mathcal{O}_K)$ and the map ℓ is the sum of the entries on the first diagonal (following the notation of [SV11]). Let π be a smooth representation of $\mathrm{GL}_n(K)$ over $\mathbb{F}$ and endow the algebraic dual $(\pi^{N_1})^\vee$ of π^{N_1} with the residual $\mathbb{F}[\![N_0/N_1]\!] \cong \mathbb{F}[\![\mathbb{Z}_p]\!] \cong \mathbb{F}[\![X]\!]$-module structure (where $X \stackrel{\text{def}}{=} [1]-1$), an action of $\mathbb{Z}_p^\times$ and an endomorphism ψ which commutes with the $\mathbb{Z}_p^\times$-action by

$$\begin{cases} (xf)(v) & \stackrel{\text{def}}{=} & f(\xi(x^{-1})v), \ x \in \mathbb{Z}_p^\times, \ f \in (\pi^{N_1})^\vee, \ v \in \pi^{N_1} \\ \psi(f)(v) & \stackrel{\text{def}}{=} & f\Big(\sum_{N_1/\xi(p)N_1\xi(p)^{-1}} n_1\xi(p)v\Big), \ f \in (\pi^{N_1})^\vee, \ v \in \pi^{N_1}. \end{cases}$$

Then one defines a covariant left exact functor V from the category of smooth representations of $\mathrm{GL}_n(K)$ over $\mathbb{F}$ to the category of (filtered) direct limits of continuous finite-dimensional representations of $\mathrm{Gal}(\overline{\mathbb{Q}}_p/\mathbb{Q}_p)$ over $\mathbb{F}$ by

$$V(\pi) \stackrel{\text{def}}{=} \Big(\varinjlim_{D} \mathbf{V}^\vee(D)\Big) \otimes \delta, \tag{1}$$

where the inductive limit is taken over the continuous morphisms of $\mathbb{F}[\![X]\!]$-modules $h : (\pi^{N_1})^\vee \to D$, where D is an étale (φ,Γ)-module of finite rank over $\mathbb{F}(\!(X)\!)$ and h intertwines the actions of $\mathbb{Z}_p^\times$ (recall $\Gamma \cong \mathbb{Z}_p^\times$), commutes with ψ and is surjective when tensored by $\mathbb{F}(\!(X)\!)$. (Here $\mathbf{V}^\vee$ is Fontaine's contravariant functor associating a representation of $\mathrm{Gal}(\overline{\mathbb{Q}}_p/\mathbb{Q}_p)$ to D and recall that any étale (φ,Γ)-module is endowed with an endomorphism ψ which is left inverse to the Frobenius φ.) In (1), δ is a certain power of ω which is here for normalization issues (see Example 2.1.1.3, see also the end of §2.1.4). In general, one doesn't know when $V(\pi)$ is nonzero or if it is finite-dimensional.

Using (1), one can strengthen Conjecture 1.2.2 (when $K=\mathbb{Q}_p$) so that it takes into account the action of $\mathrm{Gal}(\overline{\mathbb{Q}}_p/\mathbb{Q}_p)$ as follows.

Conjecture 1.2.4 (see Definition 2.4.1.5 and Conjecture 2.5.1)**.** *There is a bijection Φ as in Conjecture 1.2.2 that moreover commutes with the action of* $\mathrm{Gal}(\overline{\mathbb{Q}}_p/\mathbb{Q}_p)$ *in the following sense: for each subquotient $\Pi'_{\tilde v}$ of $\Pi_{\tilde v}$ one has $V(\Pi'_{\tilde v}) = \Phi(\Pi'_{\tilde v})\circ\overline{r}_{\tilde v}$. (Recall that $\Phi(\Pi'_{\tilde v})$ is an algebraic representation of $\widetilde{P}_{\overline{r}_{\tilde v}}$ over $\mathbb{F}$ and that $\overline{r}_{\tilde v}$ takes values in $\widetilde{P}_{\overline{r}_{\tilde v}}(\mathbb{F})$.)*

If K is not necessarily $\mathbb{Q}_p$, then by definition $\Phi(\Pi'_{\tilde v})$ is an algebraic representation of $\widetilde{P}^{\mathrm{Gal}(K/\mathbb{Q}_p)}_{\overline{r}_{\tilde v}}$ and there is a completely analogous conjecture replacing $\Phi(\Pi'_{\tilde v})\circ\overline{r}_{\tilde v}$ by $\Phi(\Pi'_{\tilde v})\circ(\overline{r}^{\sigma}_{\tilde v})_{\sigma\in\mathrm{Gal}(K/\mathbb{Q}_p)}$, which is again a representation of $\mathrm{Gal}(\overline{\mathbb{Q}}_p/\mathbb{Q}_p)$.

In particular the functor V, when applied to $\Pi_{\tilde v}$ and its subquotients $\Pi'_{\tilde v}$, should behave like an exact functor. Note that Conjecture 1.2.4 is known when $K=\mathbb{Q}_p$ and $n=2$ by the same references as in Example 1.2.3(iii). In the special case $\Pi'_{\tilde v}=\Pi_{\tilde v}$, Conjecture 1.2.4 implies in particular

Conjecture 1.2.5 (Conjecture 2.1.3.1)**.** *The functor V induces an isomorphism*

$$V\Big(S(U^v,\mathbb{F})[\mathfrak{m}]\otimes(\omega^{-(n-1)}\circ\det)\Big) \cong \Big(\mathrm{ind}_K^{\otimes\mathbb{Q}_p}\Big(\bigotimes_{i=1}^{n-1}\bigwedge\nolimits^{i}_{\mathbb{F}}\overline{r}_{\tilde v}\Big)\Big)^{\oplus d},$$

where $\mathrm{ind}_K^{\otimes\mathbb{Q}_p}$ *is the tensor induction from* $\mathrm{Gal}(\overline{\mathbb{Q}}_p/K)$ *to* $\mathrm{Gal}(\overline{\mathbb{Q}}_p/\mathbb{Q}_p)$*.*

The statement in Conjecture 1.2.5 makes sense even if K is ramified, and we conjecture it for an arbitrary finite extension K of $\mathbb{Q}_p$ and an arbitrary representation $\overline{r}_{\tilde{v}}$ (see Conjecture 2.1.3.1). In fact, using C-parameters ([BG14]), it can even be formulated in a more intrinsic way and in a more general global setting, see Conjecture 2.1.4.5.

Remark 1.2.6. Assuming $K = \mathbb{Q}_p$, the first appearance of the $\mathrm{Gal}(\overline{\mathbb{Q}}_p/\mathbb{Q}_p)$-representation on the right-hand side of the isomorphism in Conjecture 1.2.5 is in [BH15], where its "ordinary part" was related to the "ordinary part" of $S(U^v, \mathbb{F})[\mathfrak{m}]$ (see Theorem 2.5.9 for an improvement). Note that the algebraic representation $\overline{L}^{\otimes}$ of GL_n is *not* irreducible for $n > 2$. One could have thought about using the irreducible algebraic representation of GL_n of highest weight θ instead of the reducible $\overline{L}^{\otimes}$ to make predictions (at least for p big enough the latter strictly contains the former as a direct factor). However, we chose the representation $\overline{L}^{\otimes}$. One reason is that it can also be seen as a representation of $\mathrm{GL}_n \times \cdots \times \mathrm{GL}_n$ ($n-1$ times) in an obvious way – in which case a better notation is $\overline{L}^{\boxtimes} \stackrel{\text{def}}{=} \boxtimes_{i=1}^{n-1} \bigwedge_{\mathbb{F}}^i \mathrm{Std}$ – and one can hope to state a stronger variant of Conjecture 1.2.4 replacing $\overline{L}^{\otimes}$ by $\overline{L}^{\boxtimes}$ and $\Phi(\Pi'_{\tilde{v}}) \circ \overline{r}_{\tilde{v}}$ by $\Phi(\Pi'_{\tilde{v}}) \circ (\overline{r}_{\tilde{v}}, \overline{r}_{\tilde{v}}, \dots, \overline{r}_{\tilde{v}})$ (see [Záb18b], [Záb18a] where such a possibility is mentioned). However one has to be careful with defining a "multivariable" functor V in that context (there is a tentative definition in [Záb18b] when $K = \mathbb{Q}_p$ generalizing (1), but see Remark 3.1.2.12 when $n = 2$ and $K \neq \mathbb{Q}_p$).

If a representation $\Pi_{\tilde{v}}$ as in Conjecture 1.2.4 exists, we do hope that it will realize a mod p local Langlands correspondence for $\mathrm{GL}_n(K)$.

1.3 Results

Let us now describe our main results when $n = 2$ and $K = \mathbb{Q}_{p^f}$ is unramified. For a finite place $\tilde{w}$ of F we denote by $R^{\square}_{\overline{r}_{\tilde{w}}}$ the (unrestricted) framed deformation ring of $\overline{r}_{\tilde{w}} \stackrel{\text{def}}{=} \overline{r}|_{\mathrm{Gal}(\overline{F}_{\tilde{w}}/F_{\tilde{w}})}$ over $W(\mathbb{F})$. We let $I_K \subseteq \mathrm{Gal}(\overline{\mathbb{Q}}_p/K)$ be the inertia subgroup and $\omega_{f'}$ for $f' \in \{f, 2f\}$ be Serre's fundamental character of level f'. We make the following extra assumptions on F, H, $\overline{r}$ and $U^v = \prod_{w \neq v} U_w$ (recall we assumed p inert in F^+ for simplicity):

(i) F/F^+ is unramified at all finite places of F^+;

(ii) H is quasi-split at all finite places of F^+;

(iii) $\overline{r}|_{\mathrm{Gal}(\overline{F}/F(\sqrt[p]{1}))}$ is adequate ([Tho17, Def.2.20]);

(iv) $\overline{r}_{\tilde{w}}$ is unramified if $\tilde{w}|_{F^+}$ is inert in F;

(v) $R^{\square}_{\overline{r}_{\tilde{w}}}$ is formally smooth over $W(\mathbb{F})$ if $\overline{r}_{\tilde{w}}$ is ramified and $\tilde{w}|_{F^+} \neq v$;

(vi) $\overline{r}_{\tilde{v}}|_{I_K}$ is, up to twist, of one of the following forms:

$$\begin{cases} \overline{r}_{\tilde{v}}|_{I_K} \cong \begin{pmatrix} \omega_f^{(r_0+1)+\cdots+p^{f-1}(r_{f-1}+1)} & 0 \\ 0 & 1 \end{pmatrix}, \\ \overline{r}_{\tilde{v}}|_{I_K} \cong \begin{pmatrix} \omega_{2f}^{(r_0+1)+\cdots+p^{f-1}(r_{f-1}+1)} & 0 \\ 0 & \omega_{2f}^{p^f(\text{same})} \end{pmatrix}, \end{cases}$$

where the r_i satisfy the following bounds:

$$\begin{cases} \max\{12, 2f-1\} \leq r_j \leq p - \max\{15, 2f+2\} & \text{if } j > 0 \text{ or } \overline{r}_{\tilde{v}} \text{ is reducible,} \\ \max\{13, 2f\} \qquad \leq r_0 \leq p - \max\{14, 2f+1\} & \text{if } \overline{r}_{\tilde{v}} \text{ is irreducible;} \end{cases} \tag{2}$$

(vii) U_w is maximal hyperspecial in $H(F_w^+)$ if w is inert in F.

(We also need to fix a place v_1 which splits in F, where nothing ramifies and U_{v_1} is contained in the Iwahori subgroup at v_1, we forget that here along with the set Σ of bad places and the definition of the ideal $\mathfrak{m}$.)

Theorem 1.3.1 (Theorem 3.4.4.3)**.** *Assume $n = 2$, $K/\mathbb{Q}_p$ unramified, and the above conditions (i)–(vii). Then Conjecture 1.2.5 holds.*

We sketch the proof of Theorem 1.3.1. We denote by I_1 the pro-p Iwahori subgroup in $\mathrm{GL}_2(\mathcal{O}_K)$ and set

$$\overline{\rho} \stackrel{\text{def}}{=} \overline{r}_{\tilde{v}}(1) \qquad \Pi \stackrel{\text{def}}{=} S(U^v, \mathbb{F})[\mathfrak{m}].$$

Note that the central character of Π is $\det(\overline{\rho})\omega^{-1}$ (Lemma 2.1.3.3). There are two main steps in the proof which involve quite different arguments:

(i) one proves a $\mathrm{Gal}(\overline{\mathbb{Q}}_p/\mathbb{Q}_p)$-equivariant injection $(\mathrm{ind}_K^{\otimes \mathbb{Q}_p} \overline{\rho})^{\oplus d} \hookrightarrow V(\Pi)$;

(ii) one proves $\dim_{\mathbb{F}} V(\Pi) \leq 2^f d$ $(= \dim_{\mathbb{F}}(\mathrm{ind}_K^{\otimes \mathbb{Q}_p} \overline{\rho})^{\oplus d})$.

We first sketch the proof of (i). Arguing as in the proof of [BHH+23, Prop.8.2.6], there is an integer $d \geq 1$ and a $\mathrm{GL}_2(\mathcal{O}_K)K^\times$-equivariant isomorphism $\Pi^{K_1} \cong D_0(\overline{\rho})^{\oplus d}$, where $D_0(\overline{\rho})$ is defined as in [BP12, §13] (see Corollary 3.4.2.2). Taking into account the action of $\left(\begin{smallmatrix} 0 & 1 \\ p & 0 \end{smallmatrix}\right)$ on $\Pi^{I_1} \subseteq \Pi^{K_1}$, one can promote this isomorphism to an isomorphism of *diagrams*:

Theorem 1.3.2 ([DL21, Thm.1.3] when $d = 1$, Theorem 3.4.1.1 when $d > 1$)**.** *There is a diagram $D(\overline{\rho}) = (D_1(\overline{\rho}) \hookrightarrow D_0(\overline{\rho}))$ only depending on $\overline{\rho}$ such that one has an isomorphism of diagrams:*

$$D(\overline{\rho})^{\oplus d} \cong (\Pi^{I_1} \hookrightarrow \Pi^{K_1}).$$

Theorem 1.3.2 can actually be made stronger, i.e. one can show that certain constants $\nu_i \in \mathbb{F}^\times$ associated to the weight cycling on $D_1(\overline{\rho}) \cong D_0(\overline{\rho})^{I_1}$ as in [Bre11, §6] (up to suitable normalization) are as predicted in [Bre11, Thm.6.4]. When $d = 1$, Theorem 1.3.2 (and its strengthening) is entirely due to Dotto and Le ([DL21, Thm.1.3]). When $d > 1$, we check from their proof that the action of $\left(\begin{smallmatrix} 0 & 1 \\ p & 0 \end{smallmatrix}\right)$ on $\Pi^{I_1} \cong (D_0(\overline{\rho})^{I_1})^{\oplus d}$ "respects" each copy of $D_0(\overline{\rho})^{I_1}$. Note that Theorem 1.3.2 holds under much weaker bounds on the r_i than the bounds (2), see §3.4.1.

Then item (i) above follows from the following purely local result.

Theorem 1.3.3 (Theorem 3.2.1.1)**.** *Let π be an (admissible) smooth representation of $\mathrm{GL}_2(K)$ over $\mathbb{F}$ such that one has an isomorphism of diagrams $D(\overline{\rho})^{\oplus d} \cong (\pi^{I_1} \hookrightarrow \pi^{K_1})$. Then one has a $\mathrm{Gal}(\overline{\mathbb{Q}}_p/\mathbb{Q}_p)$-equivariant injection $(\mathrm{ind}_K^{\otimes \mathbb{Q}_p} \overline{\rho})^{\oplus d} \hookrightarrow V(\pi)$.*

The proof of Theorem 1.3.3 is a long and technical computation of (φ, Γ)-modules that is given in §3.2. It uses the previous computations in [Bre11] and the bounds (2) (though one can slightly weaken them, see (126)).

We now sketch the (longer) proof of (ii). We let Z_1 be the center of I_1 (or of K_1) and $\mathfrak{m}_{I_1/Z_1}$ the maximal ideal of the Iwasawa algebra $\mathbb{F}[\![I_1/Z_1]\!]$. The main idea is to focus on the structure of the (algebraic) dual $\pi^\vee$ as an $\mathbb{F}[\![I_1/Z_1]\!]$-module and to use the results of [BHH+23]. Recall that the graded ring $\mathrm{gr}(\mathbb{F}[\![I_1/Z_1]\!])$ for the $\mathfrak{m}_{I_1/Z_1}$-adic filtration (we use the normalization of [LvO96, §I.2.3]) is not commutative, but contains a regular sequence of central elements $(h_0, \dots, h_{f-1})$ such that $R \stackrel{\text{def}}{=} \mathrm{gr}(\mathbb{F}[\![I_1/Z_1]\!])/(h_0, \dots, h_{f-1})$ is a commutative polynomial algebra in $2f$ variables $\mathbb{F}[y_i, z_i, 0 \leq i \leq f-1]$ (see [BHH+23, §5.3] and (101), (117)). We let $J \stackrel{\text{def}}{=} (y_i z_i, h_i, 0 \leq i \leq f-1)$ (an ideal of $\mathrm{gr}(\mathbb{F}[\![I_1/Z_1]\!])$) and define

$$\overline{R} \stackrel{\text{def}}{=} \mathrm{gr}(\mathbb{F}[\![I_1/Z_1]\!])/J \cong \mathbb{F}[y_i, z_i, 0 \leq i \leq f-1]/(y_i z_i, 0 \leq i \leq f-1). \tag{3}$$

Then $\mathfrak{p}_0 \stackrel{\text{def}}{=} (z_i, 0 \leq i \leq f-1)$ is one of the 2^f minimal prime ideals of $\overline{R}$. If N is any finite type $\mathrm{gr}(\mathbb{F}[\![I_1/Z_1]\!])$-module killed by a power of J, one can define its multiplicity $m_{\mathfrak{p}_0}(N) \in \mathbb{Z}_{\geq 0}$ at $\mathfrak{p}_0$, see (123).

For π a smooth representation of $\mathrm{GL}_2(K)$ over $\mathbb{F}$ with a central character, we endow $\pi^\vee$ with the $\mathfrak{m}_{I_1/Z_1}$-adic filtration and we let $\mathrm{gr}(\pi^\vee)$ be the associated graded $\mathrm{gr}(\mathbb{F}[\![I_1/Z_1]\!])$-module.

Theorem 1.3.4 (Theorem 3.3.2.3)**.** *Let* π *be an* (*admissible*) *smooth representation of* $\mathrm{GL}_2(K)$ *over* $\mathbb{F}$ *satisfying the following two properties:*

(i) *there is a* $\mathrm{GL}_2(\mathcal{O}_K)K^\times$*-equivariant isomorphism* $D_0(\overline{\rho})^{\oplus d} \cong \pi^{K_1}$*;*

(ii) *for any character* $\chi : I \to \mathbb{F}^\times$ *appearing in* $\pi[\mathfrak{m}_{I_1/Z_1}]$ *there is an equality of multiplicities*

$$[\pi[\mathfrak{m}^3_{I_1/Z_1}] : \chi] = [\pi[\mathfrak{m}_{I_1/Z_1}] : \chi].$$

Then $\mathrm{gr}(\pi^\vee)$ *is killed by* J *and one has* $m_{\mathfrak{p}_0}(\mathrm{gr}(\pi^\vee)) \leq 2^f d$*.*

By the proof of [BHH$^+$23, Cor.5.3.5], property (ii) in Theorem 1.3.4 implies that $\mathrm{gr}(\pi^\vee)$ is killed by J. By an explicit computation (using both properties (i) and (ii)), one proves in Theorem 3.3.2.1 that there is a surjection of $\overline{R}$-modules

$$(\oplus_{\lambda\in\mathscr{P}} R/\mathfrak{a}(\lambda))^{\oplus d} \twoheadrightarrow \mathrm{gr}(\pi^\vee),$$

where $\mathscr{P}$ is a combinatorial finite set associated to $\overline{\rho}$ (in bijection with the set of χ appearing in $\pi[\mathfrak{m}_{I_1/Z_1}]$, see §3.3.1) and the $\mathfrak{a}(\lambda)$ are explicit ideals of R containing the image of J (see Definition 3.3.1.1). Then Theorem 1.3.4 follows from the equality $m_{\mathfrak{p}_0}(\oplus_{\lambda\in\mathscr{P}} R/\mathfrak{a}(\lambda)) = 2^f$ which is an easy computation.

Arguing as in [BHH$^+$23], the representation Π satisfies all assumptions of Theorem 1.3.4, see Corollary 3.4.2.2 and Theorem 3.4.4.1. Hence the upper bound in item (ii) below Theorem 1.3.1 follows from Theorem 1.3.4 combined with the next result:

Theorem 1.3.5 (Corollary 3.1.4.5)**.** *Let* π *be an admissible smooth representation of* $\mathrm{GL}_2(K)$ *over* $\mathbb{F}$ *with a central character such that* $\mathrm{gr}(\pi^\vee)$ *is killed by some power of* J*. Then one has* $\dim_{\mathbb{F}} V(\pi) \leq m_{\mathfrak{p}_0}(\mathrm{gr}(\pi^\vee))$*.*

We prove Theorem 1.3.5 by first associating to π an "étale $(\varphi, \mathcal{O}_K^\times)$-module over A" (Definition 3.1.3.1). This is the "multivariable (φ, Γ)-module" mentioned at the end of §1.1. Though one could probably give a more direct proof without explicitly introducing them, these étale $(\varphi, \mathcal{O}_K^\times)$-modules are important for our finite length results below and are likely to play a role later, so we describe them now.

We start with the ring A. Let $\mathbb{F}[\![N_0]\!] \cong \mathbb{F}[\![\mathcal{O}_K]\!]$ be the Iwasawa algebra of the unipotent radical N_0 of $B(\mathcal{O}_K)$. Then $\mathbb{F}[\![N_0]\!] \cong \mathbb{F}[\![Y_0, \dots, Y_{f-1}]\!]$, where the Y_i are eigenvectors for the action of the finite torus on $\mathbb{F}[\![N_0]\!]$ (see (101)). Let S be the *multiplicative system* in $\mathbb{F}[\![N_0]\!]$ generated by the Y_i. The filtration on $\mathbb{F}[\![N_0]\!]$ by powers of its maximal ideal $\mathfrak{m}_{N_0}$ naturally extends to a filtration on the localized ring $\mathbb{F}[\![N_0]\!]_S$ and we define A to be the completion of $\mathbb{F}[\![N_0]\!]_S$ (where $\mathbb{F}[\![N_0]\!]_S$ denotes the localization of $\mathbb{F}[\![N_0]\!]$ at S) for this filtration ([LvO96, §I.3.4]). The ring A is *not* local, but it is a regular noetherian domain (Corollary 3.1.1.2) and a complete filtered ring in the sense of [LvO96, §I.3.3] with associated graded ring $\mathrm{gr}(A) \cong \mathrm{gr}(\mathbb{F}[\![N_0]\!]_S)$ (see Remark 3.1.1.3(iii) for a concrete description of A). Most importantly, the natural action of $\mathcal{O}_K^\times$ on $\mathbb{F}[\![N_0]\!] \cong \mathbb{F}[\![\mathcal{O}_K]\!]$ by multiplication on $\mathcal{O}_K$ extends by continuity to A (Lemma 3.1.1.4) and any ideal of A preserved by $\mathcal{O}_K^\times$ is either 0 or A (Corollary 3.1.1.7).

Let π be an admissible smooth representation of $\mathrm{GL}_2(K)$ over $\mathbb{F}$ with a central character and recall that $\pi^\vee$ is endowed with the $\mathfrak{m}_{I_1/Z_1}$-adic filtration (which, in general, *strictly* contains the $\mathfrak{m}_{N_0}$-adic filtration). We endow $(\pi^\vee)_S \stackrel{\mathrm{def}}{=} \mathbb{F}[\![N_0]\!]_S \otimes_{\mathbb{F}[\![N_0]\!]} \pi^\vee$ with the tensor product filtration and define $D_A(\pi)$ as the completion of $(\pi^\vee)_S$. Then $D_A(\pi)$ is a complete filtered A-module such that $\mathrm{gr}(D_A(\pi)) \cong \mathrm{gr}((\pi^\vee)_S)$ (Lemma 3.1.1.1). The action of $\mathcal{O}_K^\times$ on $\pi^\vee$ extends by continuity to $D_A(\pi)$, as well as the map

$$\psi : \pi^\vee \longrightarrow \pi^\vee, \quad f \longmapsto \psi(f) \stackrel{\mathrm{def}}{=} \Big(v \in \pi \mapsto f(\xi(p)v) = f\Big(\big(\begin{smallmatrix} p & 0 \\ 0 & 1 \end{smallmatrix}\big)v\Big)\Big)$$

(Lemma 3.1.2.5). The latter can be linearized into an A-linear morphism

$$\beta : D_A(\pi) \longrightarrow A \otimes_{\phi, A} D_A(\pi),$$

where ϕ is a Frobenius endomorphism on the characteristic p ring A (see (116) for the definition of β, and §3.1.1 for the definition of ϕ on A).

We let $\mathcal{C}$ be the abelian category of admissible smooth representations π with a central character such that $\mathrm{gr}((\pi^\vee)_S)$ is a finite type $\mathrm{gr}(\mathbb{F}[\![N_0]\!]_S)$-module. It follows from (3) that

$$\Big(\mathrm{gr}(\mathbb{F}[\![I_1/Z_1]\!])/J\Big)[(y_0 \cdots y_{f-1})^{-1}] \cong \mathbb{F}[y_0, \dots, y_{f-1}][(y_0 \cdots y_{f-1})^{-1}] \cong \mathrm{gr}(\mathbb{F}[\![N_0]\!]_S)$$

which easily implies that, if $\mathrm{gr}(\pi^\vee)$ is killed by a power of J, then π is in $\mathcal{C}$ (Proposition 3.1.2.11). In particular the representation Π is in $\mathcal{C}$. Note that *any* finite length admissible smooth representation π of $\mathrm{GL}_2(\mathbb{Q}_p)$ over $\mathbb{F}$ with a central character is such that $\mathrm{gr}(\pi^\vee)$ is killed by a power of J (Corollary 3.3.3.5), hence is in $\mathcal{C}$.

For π in $\mathcal{C}$, by general results of [Lyu97], there exists a largest quotient $D_A(\pi)^{\text{ét}}$ of $D_A(\pi)$ such that the map β induces an isomorphism $\beta^{\text{ét}} : D_A(\pi)^{\text{ét}} \xrightarrow{\sim} A \otimes_{\phi, A} D_A(\pi)^{\text{ét}}$ (see the beginning of §3.1.2). We let $\varphi : D_A(\pi)^{\text{ét}} \to D_A(\pi)^{\text{ét}}$ such that $\mathrm{Id} \otimes \varphi = (\beta^{\text{ét}})^{-1}$. Then $D_A(\pi)^{\text{ét}}$ equipped with φ and the induced action of $\mathcal{O}_K^\times$ is our *étale* $(\varphi, \mathcal{O}_K^\times)$*-module over* A associated to π in $\mathcal{C}$.

Theorem 1.3.6 (Proposition 3.1.2.3, Corollary 3.1.2.9, Theorem 3.1.3.3 and Corollary 3.1.4.5)**.**

(i) *If π is in $\mathcal{C}$, then $D_A(\pi)$ and $D_A(\pi)^{\text{ét}}$ are finite projective A-modules and* $\mathrm{rk}_A(D_A(\pi)^{\text{ét}}) \leq m_{\mathfrak{p}_0}(\mathrm{gr}(\pi^\vee))$.

(ii) *The (contravariant) functors $\pi \to D_A(\pi)$ and $\pi \to D_A(\pi)^{\text{ét}}$ are exact on the abelian category $\mathcal{C}$.*

One key ingredient in the proof of Theorem 1.3.6 (cf. the proof of Proposition 3.1.1.8) is that if the annihilator of an A-module endowed with an A-semilinear $\mathcal{O}_K^\times$-action is nonzero, then this annihilator is A (since there are no proper nonzero ideals of A which are preserved by $\mathcal{O}_K^\times$, see above) and hence the A-module must be 0.

For a smooth representation π of $\mathrm{GL}_2(K)$ over $\mathbb{F}$ such that $\dim_{\mathbb{F}} V(\pi) < +\infty$, we denote by $D_\xi^\vee(\pi)$ the unique étale (φ, Γ)-module over $\mathbb{F}(\!(X)\!)$ such that $V(\pi) = \mathbf{V}^\vee(D_\xi^\vee(\pi)) \otimes \delta$ (see (1)). We denote by $\mathrm{tr} : A \to \mathbb{F}(\!(X)\!)$ the ring morphism induced by the trace $\mathrm{tr} : \mathbb{F}[\![N_0]\!] \to \mathbb{F}[\![\mathbb{Z}_p]\!] \cong \mathbb{F}[\![X]\!]$.

Theorem 1.3.7 (Theorem 3.1.3.7)**.** *If π is in $\mathcal{C}$, then we have an isomorphism of étale (φ, Γ)-modules over $\mathbb{F}(\!(X)\!)$:*

$$D_A(\pi)^{\text{ét}} \otimes_A \mathbb{F}(\!(X)\!) \xrightarrow{\sim} D_\xi^\vee(\pi).$$

In particular, $\dim_{\mathbb{F}} V(\pi) = \mathrm{rk}_A(D_A(\pi)^{\text{ét}}) < +\infty$ *and the functor* $\pi \longmapsto V(\pi)$ *in* (1) *is exact on the category $\mathcal{C}$.*

The proof essentially follows by a careful unravelling of all the definitions and constructions involved. The last statement follows from the first and from Theorem 1.3.6.

Theorem 1.3.7 and Theorem 1.3.6(i) imply in particular the bound on $V(\pi)$ in Theorem 1.3.5, which finally proves Theorem 1.3.1.

We see that the multivariable $(\varphi, \mathcal{O}_K^\times)$-module $D_A(\pi)^{\text{ét}}$ plays an important role in the proof of Theorem 1.3.5. One natural question therefore is to understand more the internal structure of $D_A(\Pi)^{\text{ét}}$ (at least conjecturally): does $D_A(\Pi)^{\text{ét}}$ only depend on $\overline{\rho}$? Does it determine $\overline{\rho}$? We plan to come back to these questions, as well as generalizations in higher dimension, in future work.

The modules $D_A(\Pi)^{\text{ét}}$ and $D_\xi^\vee(\Pi)$ are also crucial tools in the proof of our finite length results on the representation Π which provide evidence to Conjecture 1.2.2 and Conjecture 1.2.4 and that we describe now.

Theorem 1.3.8 (Theorem 3.4.4.5)**.** *Assume moreover $d = 1$, i.e. $\Pi^{K_1} \cong D_0(\overline{\rho})$ (the so-called minimal case). Then the* $\mathrm{GL}_2(K)$*-representation* Π *is generated by its* $\mathrm{GL}_2(\mathcal{O}_K)$*-socle, in particular is of finite type.*

Note that the last finiteness assertion in Theorem 1.3.8 (with Π^{K_1} instead of the $\mathrm{GL}_2(\mathcal{O}_K)$-socle) was known for $\overline{\rho}$ *non*-semisimple (and sufficiently generic) by [HW22, Thm.1.6], but the proof there doesn't extend to the semisimple case.

We sketch the proof of Theorem 1.3.8. Let $\Pi' \subseteq \Pi$ be a nonzero subrepresentation and $\Pi'' \stackrel{\text{def}}{=} \Pi/\Pi'$. As $\mathrm{gr}(\Pi^\vee)$ and hence its quotient $\mathrm{gr}(\Pi'^\vee)$ are killed by J, the representations Π, Π', Π'' are all in $\mathcal{C}$, thus Theorem 1.3.6(i) and Theorem 1.3.7

imply $\dim_{\mathbb{F}} V(\Pi') \leq m_{\mathfrak{p}_0}(\mathrm{gr}(\Pi'^\vee))$ and $\dim_{\mathbb{F}} V(\Pi'') \leq m_{\mathfrak{p}_0}(\mathrm{gr}(\Pi''^\vee))$. Since $V(\Pi'') \cong V(\Pi)/V(\Pi')$ by the last statement in Theorem 1.3.7, and since $m_{\mathfrak{p}_0}$ is an additive function by Lemma 3.3.4.4 (and Definition 3.3.4.1), we deduce $\dim_{\mathbb{F}} V(\Pi') = m_{\mathfrak{p}_0}(\mathrm{gr}(\Pi'^\vee))$ and $\dim_{\mathbb{F}} V(\Pi'') = m_{\mathfrak{p}_0}(\mathrm{gr}(\Pi''^\vee))$ as we have seen that $\dim_{\mathbb{F}} V(\Pi) = m_{\mathfrak{p}_0}(\mathrm{gr}(\Pi^\vee))$ $(= 2^f)$. On the other hand, by computations analogous to the ones used in the proofs of Theorem 1.3.3 and Theorem 1.3.4, we also have inequalities

$$m_{\mathfrak{p}_0}(\mathrm{gr}(\Pi'^\vee)) \leq \lg(\mathrm{soc}_{\mathrm{GL}_2(\mathcal{O}_K)}(\Pi')) \leq \dim_{\mathbb{F}} V(\Pi')$$

and thus we deduce

$$m_{\mathfrak{p}_0}(\mathrm{gr}(\Pi'^\vee)) = \lg(\mathrm{soc}_{\mathrm{GL}_2(\mathcal{O}_K)}(\Pi')) = \dim_{\mathbb{F}} V(\Pi') \neq 0. \tag{4}$$

Now take Π' to be the nonzero subrepresentation generated over $\mathrm{GL}_2(K)$ by the $\mathrm{GL}_2(\mathcal{O}_K)$-socle of Π. We wish to prove $\Pi'' = 0$. As

$$\lg(\mathrm{soc}_{\mathrm{GL}_2(\mathcal{O}_K)}(\Pi')) = \lg(\mathrm{soc}_{\mathrm{GL}_2(\mathcal{O}_K)}(\Pi)) = 2^f = \dim_{\mathbb{F}} V(\Pi)$$

we already have by (4) and the exactness of V that

$$m_{\mathfrak{p}_0}(\mathrm{gr}(\Pi''^\vee)) = \dim_{\mathbb{F}} V(\Pi'') = 0. \tag{5}$$

To deduce $\Pi'' = 0$ from (5), we need the following key new ingredient: Π is *essentially self-dual of grade (or codimension)* $2f$, i.e. $\mathrm{Ext}^j_{\mathbb{F}[\![I_1/Z_1]\!]}\big(\Pi^\vee, \mathbb{F}[\![I_1/Z_1]\!]\big) = 0$ if $j < 2f$ and there is a $\mathrm{GL}_2(K)$-equivariant isomorphism

$$\mathrm{Ext}^{2f}_{\mathbb{F}[\![I_1/Z_1]\!]}\Big(\Pi^\vee, \mathbb{F}[\![I_1/Z_1]\!]\Big) \cong \Pi^\vee \otimes (\det(\overline{\rho})\omega^{-1}), \tag{6}$$

where $\mathrm{Ext}^{2f}_{\mathbb{F}[\![I_1/Z_1]\!]}(\Pi^\vee, \mathbb{F}[\![I_1/Z_1]\!])$ is endowed with the action of $\mathrm{GL}_2(K)$ defined by Kohlhaase in [Koh17, Prop.3.2]. This follows by the same argument as in [HW22, Thm.8.2] (using Remark 3.4.4.2). We then define $\widetilde{\Pi}$ as the admissible smooth representation of $\mathrm{GL}_2(K)$ over $\mathbb{F}$ such that

$$\widetilde{\Pi}^\vee \otimes (\det(\overline{\rho})\omega^{-1}) \cong \mathrm{Im}\Big(\mathrm{Ext}^{2f}_{\mathbb{F}[\![I_1/Z_1]\!]}\Big(\Pi^\vee, \mathbb{F}[\![I_1/Z_1]\!]\Big) \to \mathrm{Ext}^{2f}_{\mathbb{F}[\![I_1/Z_1]\!]}\Big(\Pi''^\vee, \mathbb{F}[\![I_1/Z_1]\!]\Big)\Big),$$

and by (6) $\widetilde{\Pi}$ is a subrepresentation of Π. By (6) and general results on $\mathrm{Ext}^j_\Lambda(-,\Lambda)$ for Auslander regular rings Λ, $\Pi''^\vee \subseteq \Pi^\vee$ is also of grade $2f$ if it is nonzero, and hence $\mathrm{Ext}^{2f}_{\mathbb{F}[\![I_1/Z_1]\!]}(\Pi''^\vee, \mathbb{F}[\![I_1/Z_1]\!])$ is nonzero if and only if $\Pi'' \neq 0$. From the short exact sequence

$$0 \to \widetilde{\Pi}^\vee \otimes (\det(\overline{\rho})\omega^{-1}) \to \mathrm{Ext}^{2f}_{\mathbb{F}[\![I_1/Z_1]\!]}\Big(\Pi''^\vee, \mathbb{F}[\![I_1/Z_1]\!]\Big) \to \mathrm{Ext}^{2f+1}_{\mathbb{F}[\![I_1/Z_1]\!]}\Big(\Pi'^\vee, \mathbb{F}[\![I_1/Z_1]\!]\Big) \tag{7}$$

and the fact that the last Ext^{2f+1} has grade $\geq 2f+1$, we finally obtain:

$$\widetilde{\Pi} \text{ is nonzero if and only if } \Pi'' \text{ is nonzero.} \tag{8}$$

We now use the following general theorem.

Theorem 1.3.9 (Theorem 3.3.4.5)**.** *Let π be an admissible smooth representation of $\mathrm{GL}_2(K)$ over $\mathbb{F}$ with a central character such that $\mathrm{gr}(\pi^\vee)$ is killed by a power of J. Then the $\mathrm{gr}(\mathbb{F}[\![I_1/Z_1]\!])$-module (for the $\mathfrak{m}_{I_1/Z_1}$-adic filtration on $\mathrm{Ext}^{2f}_{\mathbb{F}[\![I_1/Z_1]\!]}(\pi^\vee, \mathbb{F}[\![I_1/Z_1]\!])$):*

$$\mathrm{gr}\left(\mathrm{Ext}^{2f}_{\mathbb{F}[\![I_1/Z_1]\!]}\left(\pi^\vee, \mathbb{F}[\![I_1/Z_1]\!]\right)\right)$$

is also finitely generated and annihilated by a power of J, and we have

$$m_{\mathfrak{p}_0}(\mathrm{gr}(\pi^\vee)) = m_{\mathfrak{p}_0}\left(\mathrm{gr}\left(\mathrm{Ext}^{2f}_{\mathbb{F}[\![I_1/Z_1]\!]}\left(\pi^\vee, \mathbb{F}[\![I_1/Z_1]\!]\right)\right)\right).$$

From the injection in (7) and from Theorem 1.3.9 applied to $\pi = \Pi''$ we have $m_{\mathfrak{p}_0}(\mathrm{gr}(\widetilde{\Pi}^\vee)) \leq m_{\mathfrak{p}_0}(\mathrm{gr}(\Pi''^\vee))$, hence we obtain

$$m_{\mathfrak{p}_0}(\mathrm{gr}(\widetilde{\Pi}^\vee)) = m_{\mathfrak{p}_0}(\mathrm{gr}(\Pi''^\vee)) \overset{(5)}{=} 0.$$

This implies $\widetilde{\Pi} = 0$ by (4) (applied to the subrepresentation $\Pi' = \widetilde{\Pi}$) and thus $\Pi'' = 0$ by (8), finishing the proof of Theorem 1.3.8.

The following corollary immediately follows from Theorem 1.3.8 and from [BP12, Thm.19.10(i)].

Corollary 1.3.10 (Theorem 3.4.4.5)**.** *Assume moreover $d = 1$ and $\overline{\rho}$ irreducible. Then the $\mathrm{GL}_2(K)$-representation Π is irreducible and is a supersingular representation.*

When $\overline{\rho}$ is reducible (split), we can prove the following result.

Theorem 1.3.11 (Theorem 3.4.4.6)**.** *Assume moreover $d = 1$ and $\overline{\rho}$ reducible, i.e. $\overline{\rho} = \begin{pmatrix} \chi_1 & 0 \\ 0 & \chi_2 \end{pmatrix}$. Then one has*

$$\Pi = \mathrm{Ind}_{B(K)}^{\mathrm{GL}_2(K)}(\chi_1 \otimes \chi_2\omega^{-1}) \oplus \Pi' \oplus \mathrm{Ind}_{B(K)}^{\mathrm{GL}_2(K)}(\chi_2 \otimes \chi_1\omega^{-1}),$$

where Π' is generated by its $\mathrm{GL}_2(\mathcal{O}_K)$-socle and $\Pi'^\vee$ is essentially self-dual of grade $2f$, i.e. satisfies (6)*. Moreover, when $f = 2$, Π' is irreducible and supersingular* (*and hence Π is semisimple*)*.*

The fact that the two principal series in Theorem 1.3.11 occur as subobjects of Π was already known (and is not difficult). To prove that they also occur as quotients (and that the obvious composition is the identity), we again crucially use the essential self-duality (6). The rest of the statement follows from Theorem 1.3.8 and [BP12, Thm.19.10(ii)].

The following last corollary sums up the above results.

Corollary 1.3.12 (Theorem 3.4.4.7)**.** *Assume (i) to (vii) as at the beginning of §1.3 and assume $d = 1$ as in Theorem 1.3.8. Then Conjecture 1.2.4 holds for $n = 2$ and $\overline{\rho}$ irreducible, or for $n = 2$, K quadratic and $\overline{\rho}$ semisimple.*

Note finally that when $f = 2$, $\overline{\rho}$ is *non*-semisimple (sufficiently generic) and $d = 1$, Conjecture 1.2.2 at least is known and follows from [HW22, Thm.1.7].

1.4 Notation

We finish this introduction with some very general notation (many more will be defined in the text).

Throughout the text, we fix $\overline{\mathbb{Q}}_p$ an algebraic closure of $\mathbb{Q}_p$ and K an arbitrary finite extension of $\mathbb{Q}_p$ in $\overline{\mathbb{Q}}_p$ with residue field $\mathbb{F}_q$, $q = p^f$ ($f \in \mathbb{Z}_{\geq 1}$). The field K is unramified from §2.2 on. We also fix a finite extension E of $\mathbb{Q}_p$, with ring of integers $\mathcal{O}_E$, uniformizer ϖ_E and residue field $\mathbb{F}$, and we assume that $\mathbb{F}$ contains $\mathbb{F}_q$. The finite field $\mathbb{F}$ is the main coefficient field in this work. We denote by ε the p-adic cyclotomic character of $\mathrm{Gal}(\overline{\mathbb{Q}}_p/\mathbb{Q}_p)$ and by ω its reduction mod p. We normalize Hodge–Tate weights so that ε has Hodge–Tate weight 1 at each embedding $K \hookrightarrow E$. We normalize local class field theory so that uniformizers correspond to geometric Frobeniuses.

If H is any split connected reductive algebraic group, we denote by Z_H the center of H and by T_H a split maximal torus. If P_H is a parabolic subgroup of H containing T_H, we denote by M_{P_H} its Levi subgroup containing T_H, N_{P_H} its unipotent radical and P_H^- its opposite parabolic subgroup with respect to T_H (so $P_H \cap P_H^- = M_H$).

We let $n \geq 2$ be an integer and denote by G the algebraic group GL_n over $\mathbb{Z}$. The integer n is arbitrary in §2 and is 2 in §3.

Irreducible for a representation always means absolutely irreducible.

Finally, though we mainly work with the group GL_n, several proofs in §2 can be extended more or less *verbatim* to a split connected reductive algebraic group over $\mathbb{Z}$ with connected center, and §2.1.4 deals with possibly nonsplit reductive groups.

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2 Local-global compatibility conjectures

We state local-global compatibility conjectures (Conjecture 2.1.3.1, Conjecture 2.1.4.5 and Conjecture 2.5.1) which "functorially" relate Hecke-isotypic components with their action of $\mathrm{GL}_n(K)$ in spaces of mod p automorphic forms to representations of $\mathrm{Gal}(\overline{\mathbb{Q}}_p/\mathbb{Q}_p)$. Conjecture 2.5.1 assumes K is unramified but is much stronger and more precise than Conjecture 2.1.3.1 and Conjecture 2.1.4.5 as it predicts the number, position and form of the irreducible constituents of these Hecke-isotypic components, as well as their contribution on the Galois side.

Throughout this section, we let $T \subseteq G = \mathrm{GL}_n$ the diagonal torus over $\mathbb{Z}$ and $X(T)$ the $\mathbb{Z}$-module $\mathrm{Hom}_{\mathrm{Gr}}(T, \mathbb{G}_{\mathrm{m}})$. As usual, we identify $X(T)$ with $\oplus_{i=1}^n \mathbb{Z}e_i$ via $e_i \mapsto \big(\mathrm{diag}(x_1, \dots, x_n) \mapsto x_i\big)$ and define $\langle\ ,\ \rangle : X(T) \times X(T) \to \mathbb{Z}$, $\langle e_i, e_j\rangle \overset{\mathrm{def}}{=} \delta_{i,j}$, which we extend by $\mathbb{Q}$-bilinearity to $X(T) \otimes_{\mathbb{Z}} \mathbb{Q}$. This provides an isomorphism of $\mathbb{Z}$-modules $X(T) \xrightarrow{\sim} \mathrm{Hom}_{\mathbb{Z}}(X(T), \mathbb{Z}) \cong \mathrm{Hom}_{\mathrm{Gr}}(\mathbb{G}_{\mathrm{m}}, T)$ given by

$$e_i \longmapsto e_i^* \overset{\mathrm{def}}{=} \Big(x \mapsto \mathrm{diag}(\underbrace{1, \dots, 1}_{i-1}, x, 1, \dots, 1)\Big), \quad i \in \{1, \dots, n\}. \tag{9}$$

We denote by $R = \{e_i - e_j : 1 \leq i \neq j \leq n\} \subseteq X(T)$ the roots of (G, T), by $B \subseteq G$ the Borel subgroup (over $\mathbb{Z}$) of upper-triangular matrices and by N the unipotent radical of B, so that the positive roots are $R^+ = \{e_i - e_j : 1 \leq i < j \leq n\} \subseteq R$ and the simple roots are $S = \{e_i - e_{i+1} : 1 \leq i \leq n-1\} \subseteq R^+$. An element of $X(T) \otimes_{\mathbb{Z}} \mathbb{Q}$ is dominant if $\langle \lambda, e_i - e_{i+1}\rangle \geq 0$ for all $i \in \{1, \dots, n-1\}$. If $\lambda, \mu \in X(T) \otimes_{\mathbb{Z}} \mathbb{Q}$, we write $\lambda \leq \mu$ if $\mu - \lambda \in \sum_{i=1}^{n-1} \mathbb{Q}_{\geq 0}(e_i - e_{i+1})$. If $\lambda = \sum_{i=1}^{n-1} n_i(e_i - e_{i+1})$ for some $n_i \in \mathbb{Q}$, its support is by definition the set of simple roots $e_i - e_{i+1}$ such that $n_i \neq 0$. Finally, we denote by $W \cong \mathcal{S}_n$ the Weyl group of (G, T), which acts on the left on $X(T)$ by $w(\lambda)(t) \overset{\mathrm{def}}{=} \lambda(w^{-1}tw)$ for $\lambda \in X(T)$ and $t \in T$.

If P is a standard parabolic subgroup of G (that is, containing B), we denote by $S(P) \subseteq S$ the subset of simple roots of M_P, $R(P)^+ \subseteq R^+$ the positive roots of M_P (generated by $S(P)$) and $W(P) \subseteq W$ its Weyl group.

2.1 Weak local-global compatibility conjecture

We state our first local-global compatibility conjecture (see Conjecture 2.1.3.1 and its generalization Conjecture 2.1.4.5) which relate Hecke-isotypic components with their action of $\mathrm{GL}_n(K)$ to representations of $\mathrm{Gal}(\overline{\mathbb{Q}}_p/\mathbb{Q}_p)$ without taking care of their irreducible constituents.

2.1.1 The functors $D^\vee_{\xi_H}$ and V_H

We review the simple generalization of Colmez's functor defined in [Bre15].

Throughout this section, we fix a connected reductive algebraic group H which is split over K with a connected center, $B_H \subseteq H$ a Borel subgroup and $T_H \subseteq B_H$ a split maximal torus in B_H. We let $\big(X(T_H), R_H, X^\vee(T_H), R_H^\vee\big)$ be the associated root datum, $R_H^+ \subseteq X(T_H)$ the (positive) roots of B_H, $S_H \subseteq R_H^+$ the simple roots and $S_H^\vee$ the associated simple coroots.

We need to recall some notation of [Bre15] (to which we refer the reader for any further details). For $\alpha \in R_H^+$, we let $N_\alpha \subseteq N_H$ be the associated (commutative) root subgroup, where $N_H \overset{\text{def}}{=} N_{B_H}$ is the unipotent radical of B_H. For $\alpha \in S_H$, we fix an isomorphism $\iota_\alpha : N_\alpha \overset{\sim}{\to} \mathbb{G}_{\mathrm{a}}$ of algebraic groups over K such that

$$\iota_\alpha(tn_\alpha t^{-1}) = \alpha(t)\iota_\alpha(n_\alpha) \ \ \forall\, t \in T_H, \ \ \forall\, n_\alpha \in N_\alpha. \tag{10}$$

We fix an open compact subgroup $N_0 \subseteq N_H(K)$ such that $\prod_{\alpha \in R_H^+} N_\alpha \overset{\sim}{\to} N_H$ induces a bijection $\prod_{\alpha \in R_H^+} N_\alpha(K) \cap N_0 \overset{\sim}{\to} N_0$ for any order on the $\alpha \in R_H^+$ and such that ι_α induces isomorphisms for $\alpha \in S_H$:

$$N_\alpha(K) \cap N_0 \overset{\sim}{\longrightarrow} \mathcal{O}_K \subseteq K = \mathbb{G}_{\mathrm{a}}(K).$$

We denote by ℓ the composite $N_H \twoheadrightarrow \prod_{\alpha \in S_H} N_\alpha \xrightarrow{\sum_{\alpha \in S_H} \iota_\alpha} \mathbb{G}_{\mathrm{a}}$ (a morphism of algebraic groups over K). The morphism ℓ thus induces a group morphism still denoted $\ell : N_0 \to \mathcal{O}_K$ and we define

$$N_1 \overset{\text{def}}{=} \mathrm{Ker}\Big(N_0 \overset{\ell}{\to} \mathcal{O}_K \overset{\mathrm{Tr}_{K/\mathbb{Q}_p}}{\longrightarrow} \mathbb{Q}_p\Big) \tag{11}$$

which is a normal open compact subgroup of N_0. We fix an isomorphism of $\mathbb{Z}_p$-modules $\psi : \mathrm{Tr}_{K/\mathbb{Q}_p}(\mathcal{O}_K) \overset{\sim}{\to} \mathbb{Z}_p$. When $N_H \neq 0$, i.e. when $H \neq T_H$, this fixes an isomorphism

$$N_0/N_1 \overset{\mathrm{Tr}_{K/\mathbb{Q}_p} \circ \ell}{\underset{\sim}{\longrightarrow}} \mathrm{Tr}_{K/\mathbb{Q}_p}(\mathcal{O}_K) \overset{\psi}{\underset{\sim}{\longrightarrow}} \mathbb{Z}_p. \tag{12}$$

We fix fundamental coweights $(\lambda_{\alpha^\vee})_{\alpha \in S_H}$ (which exist since H has a connected center) and set

$$\xi_H \overset{\text{def}}{=} \sum_{\alpha^\vee \in S_H^\vee} \lambda_{\alpha^\vee} \in \mathrm{Hom}_{\mathrm{Gr}}(\mathbb{G}_{\mathrm{m}}, T_H) = X^\vee(T_H). \tag{13}$$

Note that $\xi_H(x) N_1 \xi_H(x^{-1}) \subseteq N_1$ for any $x \in \mathbb{Z}_p \backslash \{0\}$. Let $\mathbb{F}[\![X]\!][F]$ be the noncommutative polynomial ring in F over the ring of formal power series $\mathbb{F}[\![X]\!]$ such that $FS(X) = S(X^p)F$.

For π a smooth representation of $B_H(K)$ over $\mathbb{F}$, we endow the invariant subspace $\pi^{N_1} \subseteq \pi$ with a structure of an $\mathbb{F}[\![X]\!][F]$-module as follows:

(i) $\mathbb{F}[\![X]\!] \cong \mathbb{F}[\![\mathbb{Z}_p]\!]$ acts via $\mathbb{F}[\![N_0/N_1]\!] \overset{(12)}{\cong} \mathbb{F}[\![\mathbb{Z}_p]\!]$ (here $X \overset{\text{def}}{=} [1]-1$);

(ii) F acts via the "Hecke" action $F(v) \overset{\text{def}}{=} \sum_{n_1\in N_1/\xi_H(p)N_1\xi_H(p^{-1})} n_1\xi_H(p)v \ \in \pi^{N_1}$ for $v\in\pi^{N_1}$.

Note that π^{N_1} is a torsion $\mathbb{F}[\![X]\!]$-module (but not a torsion $\mathbb{F}[F]$-module in general). We also endow π^{N_1} with an action of $\mathbb{Z}_p^\times$ by making $x\in\mathbb{Z}_p^\times$ act by $\xi_H(x)$. This action commutes with F and satisfies $\xi_H(x)\circ(1+X)=(1+X)^x\circ\xi_H(x)$.

As in [Bre15], we denote by $\Phi\Gamma_{\mathbb{F}}^{\text{ét}}$ the category of finite-dimensional étale (φ,Γ)-modules over $\mathbb{F}[\![X]\!][X^{-1}]=\mathbb{F}(\!(X)\!)$ and by $\widehat{\Phi\Gamma}_{\mathbb{F}}^{\text{ét}}$ the corresponding category of (pseudocompact) pro-objects, see [Bre15, §2] for more details. Both $\Phi\Gamma_{\mathbb{F}}^{\text{ét}}$ and $\widehat{\Phi\Gamma}_{\mathbb{F}}^{\text{ét}}$ are abelian categories. Let $M\subseteq\pi^{N_1}$ be a finite type $\mathbb{F}[\![X]\!][F]$-submodule which is $\mathbb{Z}_p^\times$-stable and assume that M is admissible as an $\mathbb{F}[\![X]\!]$-module, that is, $M[X]\overset{\text{def}}{=}\{m\in M: Xm=0\}$ is finite-dimensional over $\mathbb{F}$. Let $M^\vee\overset{\text{def}}{=}\mathrm{Hom}_{\mathbb{F}}(M,\mathbb{F})$ (algebraic $\mathbb{F}$-linear dual) which is also an $\mathbb{F}[\![X]\!]$-module (but not a torsion $\mathbb{F}[\![X]\!]$-module in general). Then by a key result of Colmez $M^\vee[X^{-1}]$ can be endowed with the structure of an object of $\Phi\Gamma_{\mathbb{F}}^{\text{ét}}$ ([Col10], see also [Bre15, Lemma 2.6]). More precisely X acts on $f\in M^\vee$ by $(Xf)(m)\overset{\text{def}}{=}f(Xm)$ $(m\in M)$, $x\in\mathbb{Z}_p^\times$ acts by $(xf)(m)\overset{\text{def}}{=}f(x^{-1}m)$, and the operator φ is defined as follows. Take the $\mathbb{F}$-linear dual of $\mathrm{Id}\otimes F:\mathbb{F}[\![X]\!]\otimes_{\varphi,\mathbb{F}[\![X]\!]}M\longrightarrow M$, compose with[1]

$$\begin{aligned}(\mathbb{F}[\![X]\!]\otimes_{\varphi,\mathbb{F}[\![X]\!]}M)^\vee &\overset{\sim}{\longrightarrow} \mathbb{F}[\![X]\!]\otimes_{\varphi,\mathbb{F}[\![X]\!]}M^\vee\\ f &\longmapsto \sum_{i=0}^{p-1}(1+X)^i\otimes f\Big(\frac{1}{(1+X)^i}\otimes\cdot\Big)\end{aligned}\tag{14}$$

and invert X: the resulting morphism $M^\vee[X^{-1}]\to\mathbb{F}[\![X]\!]\otimes_{\varphi,\mathbb{F}[\![X]\!]}M^\vee[X^{-1}]$ turns out to be an $\mathbb{F}(\!(X)\!)$-linear isomorphism whose inverse is by definition $\mathrm{Id}\otimes\varphi$.

When $H\neq T_H$ we then define

$$D_{\xi_H}^\vee(\pi)\overset{\text{def}}{=}\varprojlim_M M^\vee[X^{-1}],\tag{15}$$

where the projective limit is taken over the finite type $\mathbb{F}[\![X]\!][F]$-submodules M of π^{N_1} (for the preorder defined by inclusion) which are admissible as $\mathbb{F}[\![X]\!]$-modules and invariant under the action of $\mathbb{Z}_p^\times$. When $H=T_H$, one has to replace $M^\vee[X^{-1}]$ by $\mathbb{F}(\!(X)\!)\otimes_{\mathbb{F}}M^\vee$, we refer the reader to [Bre15, §3]. The functor $D_{\xi_H}^\vee$ is right exact contravariant from the category of smooth representations of $B_H(K)$ over $\mathbb{F}$ to the category $\widehat{\Phi\Gamma}_{\mathbb{F}}^{\text{ét}}$ and, up to isomorphism, only depends on the choice of the cocharacter

[1]The formula for this isomorphism given in the proof of [Bre15, Lemma 2.6] is actually wrong, the present formula is the correct one. Note that it is also the same as $f\mapsto\sum_{i=0}^{p-1}\frac{1}{(1+X)^i}\otimes f((1+X)^i\otimes\cdot)$.

ξ_H. Moreover, if $D^\vee_{\xi_H}(\pi)$ turns out to be in $\Phi\Gamma_{\mathbb{F}}^{\text{ét}}$ (and not just $\widehat{\Phi\Gamma_{\mathbb{F}}^{\text{ét}}}$), then $D^\vee_{\xi_H}(\pi)$ is exactly the maximal étale (φ,Γ)-module which occurs as a quotient of $(\pi^{N_1})^\vee[X^{-1}]$, see [Bre15, Rem.5.6(iii)].

Remark 2.1.1.1. If $H = \mathbb{G}_{\mathrm{m}} = T_H$, then by definition $\xi_H = 1$. It follows, for $\dim_{\mathbb{F}} \pi = 1$, that $D^\vee_{\xi_H}(\pi)$ is always the trivial (rank one) (φ,Γ)-module (even if π is a nontrivial character).

Let us now assume that the dual group $\widehat{H}$ of H also has a connected center, and let us fix $\theta_H \in X(T_H)$ such that $\theta_H \circ \alpha^\vee = \mathrm{Id}_{\mathbb{G}_{\mathrm{m}}}$ for all $\alpha \in S_H$ ([BH15, Prop.2.1.1], such an element is called a *twisting element*). In §2.1.4 below, it is possible to avoid this assumption using C-parameters, but since our main aim is $G = \mathrm{GL}_n$ in the rest of the paper, there is no harm in making this assumption.

Consider the smooth character

$$K^\times \longrightarrow \mathbb{F}^\times,\ x \longmapsto \omega\Big(\theta_H(\xi_H(x))\Big)$$

and denote by δ_H the restriction of this character to $\mathbb{Q}_p^\times \subseteq K^\times$. Seeing $\omega \circ \theta_H \circ \xi_H$ as a character of $\mathrm{Gal}(\overline{\mathbb{Q}}_p/K)$ via local class field theory for K (as normalized in §1), and remembering that the restriction from $K^\times$ to $\mathbb{Q}_p^\times$ corresponds via local class field theory to the composition with the transfer $\mathrm{Gal}(\overline{\mathbb{Q}}_p/\mathbb{Q}_p)^{\mathrm{ab}} \to \mathrm{Gal}(\overline{\mathbb{Q}}_p/K)^{\mathrm{ab}}$, we see that

$$\delta_H \cong \mathrm{ind}_K^{\otimes \mathbb{Q}_p}(\omega \circ \theta_H \circ \xi_H),$$

where $\mathrm{ind}_K^{\otimes \mathbb{Q}_p}$ is the tensor induction from $\mathrm{Gal}(\overline{\mathbb{Q}}_p/K)$ to $\mathrm{Gal}(\overline{\mathbb{Q}}_p/\mathbb{Q}_p)$ (see the end of §2.1.2 below).

Denote by $\mathrm{Rep}_{\mathbb{F}}$ the abelian category of continuous linear representations of $\mathrm{Gal}(\overline{\mathbb{Q}}_p/\mathbb{Q}_p)$ on finite-dimensional $\mathbb{F}$-vector spaces (equipped with the discrete topology) and $\mathrm{IndRep}_{\mathbb{F}}$ the corresponding category of ind-objects, i.e. the category of filtered direct limits of objects of $\mathrm{Rep}_{\mathbb{F}}$. Recall that there is a covariant equivalence of categories $\mathbf{V} : \Phi\Gamma_{\mathbb{F}}^{\text{ét}} \xrightarrow{\sim} \mathrm{Rep}_{\mathbb{F}}$ (see [Fon90, Thm.A.3.4.3] where this functor is denoted $\mathbf{V}_{\mathcal{E}}$) compatible with tensor products and duals on both sides. We denote by $\mathbf{V}^\vee$ the dual of $\mathbf{V}$ (i.e. the dual Galois representation). When $H \neq T_H$, we then define the covariant functor V_H from the category of smooth representations of $B_H(K)$ over $\mathbb{F}$ to the category $\mathrm{IndRep}_{\mathbb{F}}$ by

$$V_H(\pi) \stackrel{\text{def}}{=} \varinjlim_{M} \Big(\mathbf{V}^\vee(M^\vee[X^{-1}])\Big) \otimes \delta_H, \tag{16}$$

where the inductive limit is taken over the finite type $\mathbb{F}[\![X]\!][F]$-submodules of π^{N_1} which are admissible as $\mathbb{F}[\![X]\!]$-modules and preserved by $\mathbb{Z}_p^\times$. Likewise, when $H = T_H$, with $\mathbb{F}(\!(X)\!) \otimes_{\mathbb{F}} M^\vee$ instead of $M^\vee[X^{-1}]$ (note that δ_H is then 1).

Lemma 2.1.1.2. *The functor V_H is left exact.*

Proof. We give the proof for $H \neq T_H$, leaving the case $H = T_H$ to the reader. Let $0 \to \pi' \to \pi \xrightarrow{s} \pi'' \to 0$ be an exact sequence of smooth $B_H(K)$-representations over $\mathbb{F}$, which gives a short exact sequence $0 \to \pi'^{N_1} \to \pi^{N_1} \xrightarrow{s} \pi''^{N_1}$. If M is a finite type $\mathbb{F}[\![X]\!][F]$-submodule of π^{N_1} which is admissible as an $\mathbb{F}[\![X]\!]$-module and stable under the action of $\mathbb{Z}_p^\times$, then so are $M \cap \pi'^{N_1}$ and $s(M)$ (see e.g. [Bre15, Lemma 2.1(i)]). The functor $M \to \mathbf{V}^\vee(M^\vee[X^{-1}])$ being covariant exact (since both $M \mapsto M^\vee[X^{-1}]$ and $\mathbf{V}^\vee$ are contravariant exact), each such $M \subseteq \pi^{N_1}$ gives rise to a short exact sequence in $\mathrm{Rep}_{\mathbb{F}}$:

$$0 \to \mathbf{V}^\vee\Big((M \cap \pi'^{N_1})^\vee[X^{-1}]\Big) \to \mathbf{V}^\vee\Big(M^\vee[X^{-1}]\Big) \to \mathbf{V}^\vee\Big(s(M)^\vee[X^{-1}]\Big) \to 0.$$

Twisting by δ_H and taking the inductive limit over such M, we obtain a short exact sequence $0 \to V_H(\pi') \to V_H(\pi) \to \varinjlim_M \mathbf{V}^\vee\Big(s(M)^\vee[X^{-1}]\Big) \otimes \delta_H \to 0$ in $\mathrm{IndRep}_{\mathbb{F}}$. But we have an injection

$$\varinjlim_M \mathbf{V}^\vee\Big(s(M)^\vee[X^{-1}]\Big) \otimes \delta_H \hookrightarrow V_H(\pi'')$$

in $\mathrm{IndRep}_{\mathbb{F}}$ since all transitions maps in the inductive limits are injective, therefore we end up with an exact sequence $0 \to V_H(\pi') \to V_H(\pi) \to V_H(\pi'')$. $\square$

Example 2.1.1.3. For $H = G \times_{\mathbb{Z}} K = \mathrm{GL}_{n/K}$ (so $H \cong \widehat{H}$), we take in the sequel (writing just G as a subscript instead of $G \times_{\mathbb{Z}} K$)

$$\xi_G(x) \stackrel{\text{def}}{=} \mathrm{diag}(x^{n-1}, \dots, x, 1) \quad \text{and} \quad \theta_G\Big(\mathrm{diag}(x_1, \dots, x_n)\Big) = x_1^{n-1} x_2^{n-2} \cdots x_{n-1},$$

so that $\delta_G = \mathrm{ind}_K^{\otimes \mathbb{Q}_p}(\omega^{(n-1)^2+(n-2)^2+\cdots+4+1})$. (In fact, since the tensor induction of a character is given by composition with the transfer map [Col89], by local class field theory we see that $\delta_G = \omega^{[K:\mathbb{Q}_p]((n-1)^2+(n-2)^2+\cdots+4+1)}$.)

Remark 2.1.1.4. (i) The covariant functor V_H depends on the choices of ξ_H and δ_H (though we don't include it in the notation). The reader may wonder why we need to assume the existence of θ_H and normalize V_H using the strange twist δ_H above. This comes from the local-global compatibility: it turns out that this normalization is essentially what is going on in spaces of mod p automorphic forms (see [BH15, §4], [Bre15, Cor.9.8], Example 2.1.1.6 and §§2.1.3, 2.5 below). This normalization is also natural if one uses C-parameters, see §2.1.4.
(ii) For H as in Example 2.1.1.3, π a smooth representation of $B(K)$ over $\mathbb{F}$ and $\chi : K^\times \to \mathbb{F}^\times$ a smooth character, one checks that $V_G(\pi \otimes (\chi \circ \det)) \cong V_G(\pi) \otimes \delta$, where δ is the continuous character of $\mathrm{Gal}(\overline{\mathbb{Q}}_p/\mathbb{Q}_p)$ associated via local class field theory to $x \mapsto \chi\Big(\det(\xi_G(x))\Big)$ for $x \in \mathbb{Q}_p^\times$. An explicit computation gives $\delta = (\chi|_{\mathbb{Q}_p^\times})^{\frac{n(n-1)}{2}} \cong \mathrm{ind}_K^{\otimes \mathbb{Q}_p}(\chi^{\frac{n(n-1)}{2}})$.

When restricted to the abelian category of finite length admissible smooth representations of $H(K)$ over $\mathbb{F}$ with all irreducible constituents isomorphic to irreducible constituents of principal series, it is proven in [Bre15, §9] that the functors $D^{\vee}_{\xi_H}$ and V_H are exact. It seems reasonable to us, and also consistent with the conjectural formalism developed in the sequel (see e.g. Remark 2.4.2.8(iii)), to hope that there exists a suitable abelian category of admissible smooth representations of $H(K)$ over $\mathbb{F}$ containing the previous abelian category and the representations "coming from the global theory" on which the functors $D^{\vee}_{\xi_H}$ and V_H are still exact. See for instance the category $\mathcal{C}$ in §3.1.2 when $H = \mathrm{GL}_{2/K}$ and K is unramified.

We now recall the behaviour of the functor V_H with respect to parabolic induction.

We assume for simplicity $H = G \times_{\mathbb{Z}} K = \mathrm{GL}_{n/K}$ and let ξ_G, θ_G as in Example 2.1.1.3. We let P be a standard parabolic subgroup of $G\times_{\mathbb{Z}}K$ and write $M_P = \prod_{i=1}^d M_i$ with $M_i \cong \mathrm{GL}_{n_i/K}$. We define V_{M_P} as in (16) using $\xi_{M_P} \overset{\text{def}}{=} \xi_G$ and $\theta_{M_P} \overset{\text{def}}{=} \theta_G$ (to define $D^{\vee}_{\xi_{M_P}}$ and δ_{M_P}). We write $\xi_{M_P} = \oplus_{i=1}^d \xi_{M_P,i}$ in $X^{\vee}(T) = \oplus_{i=1}^d X^{\vee}(T_i)$ and $\theta_{M_P} = \oplus_{i=1}^d \theta_{M_P,i}$ in $X(T) = \oplus_{i=1}^d X(T_i)$, where T_i is the diagonal torus in M_i, and let $V_{M_P,i} \overset{\text{def}}{=} V_{\mathrm{GL}_{n_i}}$ but defined with $\xi_{M_P,i}$ and $\theta_{M_P,i}$. Finally we define $V_{M_i} \overset{\text{def}}{=} V_{\mathrm{GL}_{n_i}}$ with ξ_{M_i} and θ_{M_i} as in Example 2.1.1.3 replacing n by n_i, and we recall that ξ_{M_i}, θ_{M_i} and δ_{M_i} are trivial characters if $n_i = 1$.

If π_P is a smooth representation of $M_P(K)$ over $\mathbb{F}$, that we see as a representation of $P^-(K)$ via $P^-(K) \twoheadrightarrow M_P(K)$, we define the usual smooth parabolic induction

$$\mathrm{Ind}_{P^-(K)}^{G(K)} \pi_P \overset{\text{def}}{=} \{f : G(K) \to \pi_P \text{ loc. const.},\ f(px) = p(f(x)),\ p \in P^-(K),\ x \in \pi_P\},$$

with $G(K)$ acting (smoothly) on the left by $(gf)(g') \overset{\text{def}}{=} f(g'g)$.

Lemma 2.1.1.5. *Let π_P be a smooth representation of $M_P(K)$ over $\mathbb{F}$ of the form $\pi_P = \pi_1 \otimes \cdots \otimes \pi_d$, where the π_i are smooth representations of $M_i(K)$ over $\mathbb{F}$. Assume that the π_i have central characters $Z(\pi_i) : K^{\times} \to \mathbb{F}^{\times}$ and that $V_{M_P}(\pi_P) \cong \bigotimes_{i=1}^d V_{M_P,i}(\pi_i)$. Then we have an isomorphism in* $\mathrm{IndRep}_{\mathbb{F}}$ *(using implicitly local class field theory for* $\mathrm{Gal}(\overline{\mathbb{Q}}_p/\mathbb{Q}_p)$*):*

$$V_G\Big(\mathrm{Ind}_{P^-(K)}^{G(K)} \pi_P\Big) \otimes \delta_G^{-1} \cong \bigotimes_{i=1}^d \Big(V_{M_i}(\pi_i) \otimes \Big(Z(\pi_i)^{n-\sum_{j=1}^i n_j}\Big)|_{\mathbb{Q}_p^{\times}} \delta_{M_i}^{-1}\Big).$$

Proof. By [Bre15, Thm.6.1] we have $V_G\Big(\mathrm{Ind}_{P^-(K)}^{G(K)} \pi_P\Big) \cong V_{M_P}(\pi_P)$ so that from the assumption (all isomorphisms are in $\mathrm{IndRep}_{\mathbb{F}}$):

$$V_G\Big(\mathrm{Ind}_{P^-(K)}^{G(K)} \pi_P\Big) \cong \bigotimes_{i=1}^d V_{M_P,i}(\pi_i). \tag{17}$$

An easy computation yields in $M_i(K)$ for $x \in K^\times$:

$$\xi_{M_P,i}(x) = \mathrm{diag}(\underbrace{x^{n-\sum_{j=1}^i n_j}, \dots, x^{n-\sum_{j=1}^i n_j}}_{n_i})\xi_{M_i}(x)$$

which implies by [Bre15, Rem.4.3] that

$$V_{M_P,i}(\pi_i) \otimes \delta_{M_P,i}^{-1} \cong V_{M_i}(\pi_i) \otimes \Big(Z(\pi_i)^{n-\sum_{j=1}^i n_j}\Big)|_{\mathbb{Q}_p^\times}\delta_{M_i}^{-1}, \qquad (18)$$

where $\delta_{M_P,i} \overset{\mathrm{def}}{=} \mathrm{ind}_K^{\otimes \mathbb{Q}_p}(\omega \circ \theta_{M_P,i} \circ \xi_{M_P,i})$ (and recall $V_{M_i}(\pi_i) = 1$ if $n_i = \dim_{\mathbb{F}} \pi_i = 1$, see Remark 2.1.1.1). Since $\delta_G = \prod_{i=1}^d \delta_{M_P,i}$, twisting (17) by δ_G^{-1} gives the result by (18). □

Example 2.1.1.6. An enlightening and important example is the case of principal series $\mathrm{Ind}_{B^-(K)}^{G(K)}(\chi_1 \otimes \cdots \otimes \chi_n)$, where the $\chi_i : K^\times \to \mathbb{F}^\times$ are smooth characters. The assumptions of Lemma 2.1.1.5 are then trivially satisfied and thus we have

$$V_G\Big(\mathrm{Ind}_{B^-(K)}^{G(K)}(\chi_1 \otimes \cdots \otimes \chi_n)\Big) \otimes \delta_G^{-1} \cong (\chi_1^{n-1}\chi_2^{n-2}\cdots\chi_{n-1})|_{\mathbb{Q}_p^\times}.$$

In particular we deduce (using Example 2.1.1.3 for δ_G) that

$$\begin{aligned} V_G\Big(\mathrm{Ind}_{B^-(K)}^{G(K)}(\chi_1\omega^{-(n-1)} \otimes \chi_2\omega^{-(n-2)} \otimes \cdots \otimes \chi_n)\Big) &\cong (\chi_1^{n-1}\chi_2^{n-2}\cdots\chi_{n-1})|_{\mathbb{Q}_p^\times} \\ &\cong \mathrm{ind}_K^{\otimes \mathbb{Q}_p}(\chi_1^{n-1}\chi_2^{n-2}\cdots\chi_{n-1}), \end{aligned}$$

where $\chi_1^{n-1}\chi_2^{n-2}\cdots\chi_{n-1}$ on the last line is seen as a character of $\mathrm{Gal}(\overline{\mathbb{Q}}_p/K)$ via local class field theory for K.

Remark 2.1.1.7. Using [Bre15, Prop.5.5] the assumptions of Lemma 2.1.1.5 are satisfied when all finite type $\mathbb{F}[\![X]\!][F]$-submodules of $\pi_i^{N_1}$ for $i \in \{1, \dots, d\}$ are automatically admissible as $\mathbb{F}[\![X]\!]$-modules. This happens for instance if the π_i are principal series or (when $K = \mathbb{Q}_p$) are finite length representations of $\mathrm{GL}_2(\mathbb{Q}_p)$ with a central character, but is not known otherwise. Contrary to what is stated in [Bre15, Rem.5.6(ii)], we currently do not have a proof of an isomorphism $V_{M_P}(\pi_P) \cong \bigotimes_{i=1}^d V_{M_P,i}(\pi_i)$ for any smooth representations π_i, though we expect that it will indeed be satisfied for representations "coming from" the global theory. Note that, in [Záb18b, Prop.3.2], Zábrádi does prove a compatibility of his functor with the tensor product which looks close to the isomorphism above. However, *loc.cit.* deals with an *external* tensor product, whereas we have an *internal* tensor product. In particular he has *two* operators F, one for each factor in the external tensor product (whereas we consider the resulting diagonal operator), and his argument doesn't extend.

2.1.2 Global setting

We recall our global setting (see e.g. [EGH13, §7.1] or [Tho12, §6] or [BH15, §4.1] or many other references) and define the $\mathrm{Gal}(\overline{\mathbb{Q}}_p/\mathbb{Q}_p)$-representation $\overline{L}^{\otimes}(\overline{\rho})$ for $\overline{\rho} : \mathrm{Gal}(\overline{\mathbb{Q}}_p/K) \longrightarrow G(\mathbb{F})$.

We let F^+ be a totally real finite extension of $\mathbb{Q}$ with ring of integers $\mathcal{O}_{F^+}$, F/F^+ a totally imaginary quadratic extension with ring of integers $\mathcal{O}_F$ (do not confuse F with the operator F of §2.1.1!) and c the nontrivial element of $\mathrm{Gal}(F/F^+)$. If v (resp. $\tilde{v}$) is a finite place of F^+ (resp. F), we let F_v^+ (resp. $F_{\tilde{v}}$) be the completion of F^+ (resp. F) at v (resp. $\tilde{v}$) and $\mathcal{O}_{F_v^+}$ (resp. $\mathcal{O}_{F_{\tilde{v}}}$) the ring of integers of F_v^+ (resp. $F_{\tilde{v}}$). If v splits in F and $\tilde{v}, \tilde{v}^c$ are the two places of F above v, we have $\mathcal{O}_{F_v^+} = \mathcal{O}_{F_{\tilde{v}}} \overset{c}{\cong} \mathcal{O}_{F_{\tilde{v}^c}}$, where the last isomorphism is induced by c. We let $\mathbb{A}_{F^+}^\infty$ (resp. $\mathbb{A}_{F^+}^{\infty,v}$) denote the finite adèles of F^+ (resp. the finite adèles of F^+ outside v). Finally we always assume that all places of F^+ above p split in F.

We let $n \in \mathbb{Z}_{>1}$, N a positive integer prime to p and H a connected reductive algebraic group over $\mathcal{O}_{F^+}[1/N]$ satisfying the following conditions:

(i) there is an isomorphism $\iota : H \times_{\mathcal{O}_{F^+}[1/N]} \mathcal{O}_F[1/N] \xrightarrow{\sim} G \times_{\mathbb{Z}} \mathcal{O}_F[1/N]$;

(ii) $H \times_{\mathcal{O}_{F^+}[1/N]} F^+$ is an outer form of $G \times_{\mathbb{Z}} F^+ = \mathrm{GL}_{n/F^+}$;

(iii) $H \times_{\mathcal{O}_{F^+}[1/N]} F^+$ is isomorphic to $\mathrm{U}_n(\mathbb{R})$ at all infinite places of F^+.

One can prove that such groups exist (cf. e.g. [EGH13, §7.1.1]). Condition (i) implies that if v is any finite place of F^+ that splits in F and if $\tilde{v}|v$ in F the isomorphism ι induces $\iota_{\tilde{v}} : H(F_v^+) \xrightarrow{\sim} \mathrm{GL}_n(F_{\tilde{v}}) = G(F_{\tilde{v}})$ which restricts to an isomorphism still denoted by $\iota_{\tilde{v}} : H(\mathcal{O}_{F_v^+}) \xrightarrow{\sim} \mathrm{GL}_n(\mathcal{O}_{F_{\tilde{v}}})$ if v doesn't divide N. Condition (ii) implies that $c \circ \iota_{\tilde{v}} : H(F_v^+) \xrightarrow{\sim} \mathrm{GL}_n(F_{\tilde{v}^c})$ (resp. $c \circ \iota_{\tilde{v}} : H(\mathcal{O}_{F_v^+}) \xrightarrow{\sim} \mathrm{GL}_n(\mathcal{O}_{F_{\tilde{v}^c}})$ if v doesn't divide N) is conjugate in $\mathrm{GL}_n(F_{\tilde{v}^c})$ (resp. in $\mathrm{GL}_n(\mathcal{O}_{F_{\tilde{v}^c}})$) to $\tau^{-1} \circ \iota_{\tilde{v}^c}$, where τ is the transpose in $\mathrm{GL}_n(F_{\tilde{v}^c})$ (resp. in $\mathrm{GL}_n(\mathcal{O}_{F_{\tilde{v}^c}})$).

If U is any compact open subgroup of $H(\mathbb{A}_{F^+}^\infty)$ then

$$S(U, \mathbb{F}) \overset{\mathrm{def}}{=} \{f : H(F^+)\backslash H(\mathbb{A}_{F^+}^\infty)/U \to \mathbb{F}\}$$

is a finite-dimensional $\mathbb{F}$-vector space since $H(F^+)\backslash H(\mathbb{A}_{F^+}^\infty)/U$ is a finite set. Fix $v|p$ in F^+ and a compact open subgroup U^v of $H(\mathbb{A}_{F^+}^{\infty,v})$, we define

$$S(U^v, \mathbb{F}) \overset{\mathrm{def}}{=} \varinjlim_{U_v} S(U^v U_v, \mathbb{F}),$$

where U_v runs among compact open subgroups of $H(\mathcal{O}_{F_v^+})$. We endow $S(U^v, \mathbb{F})$ with a linear left action of $H(F_v^+)$ by $(h_v f)(h) \overset{\mathrm{def}}{=} f(hh_v)$ ($h_v \in H(F_v^+)$, $h \in H(\mathbb{A}_{F^+}^\infty)$). Thus, for $\tilde{v}$ dividing v in F, the isomorphism $\iota_{\tilde{v}}$ gives an admissible smooth action of $G(F_v^+) = \mathrm{GL}_n(F_{\tilde{v}})$ on $S(U^v, \mathbb{F})$. By what is above, the action of $G(F_v^+)$ induced by $\iota_{\tilde{v}}$ is the inverse transpose of the one induced by $\iota_{\tilde{v}^c}$.

If U is a compact open subgroup of $H(\mathbb{A}_{F^+}^\infty)$, following [EGH13, §7.1.2] we say that U is *unramified* at a finite place v of F^+ which splits in F and doesn't divide N if we have $U = U^v \times H(\mathcal{O}_{F_v^+})$, where U^v is a compact open subgroup of $H(\mathbb{A}_{F^+}^{\infty,v})$. Note that a compact open subgroup of $H(\mathbb{A}_{F^+}^{\infty})$ is unramified at all but a finite number of finite places of F^+ which split in F. If U is a compact open subgroup of $H(\mathbb{A}_{F^+}^{\infty})$ and Σ a finite set of finite places of F^+ containing the set of places of F^+ that split in F and divide pN and the set of places of F^+ that split in F at which U is *not* unramified, we denote by $\mathcal{T}^\Sigma \overset{\text{def}}{=} \mathcal{O}_E[T_{\tilde{w}}^{(j)}]$ the commutative polynomial $\mathcal{O}_E$-algebra generated by formal variables $T_{\tilde{w}}^{(j)}$ for $j \in \{1, \dots, n\}$ and $\tilde{w}$ a place of F lying above a finite place w of F^+ that splits in F and *doesn't* belong to Σ. The algebra $\mathcal{T}^\Sigma$ acts on $S(U, \mathbb{F})$ by making $T_{\tilde{w}}^{(j)}$ act by the double coset

$$\iota_{\tilde{w}}^{-1}\left[\mathrm{GL}_n(\mathcal{O}_{F_{\tilde{w}}})\begin{pmatrix}\mathbf{1}_{n-j} & \\ & \varpi_{\tilde{w}}\mathbf{1}_j\end{pmatrix}\mathrm{GL}_n(\mathcal{O}_{F_{\tilde{w}}})\right],$$

where $\varpi_{\tilde{w}}$ is a uniformizer in $\mathcal{O}_{F_{\tilde{w}}}$. Explicitly, if we write

$$\mathrm{GL}_n(\mathcal{O}_{F_{\tilde{w}}})\left(\begin{smallmatrix}\mathbf{1}_{n-j} & \\ & \varpi_{\tilde{w}}\mathbf{1}_j\end{smallmatrix}\right)\mathrm{GL}_n(\mathcal{O}_{F_{\tilde{w}}}) = \coprod_i g_i\left(\begin{smallmatrix}\mathbf{1}_{n-j} & \\ & \varpi_{\tilde{w}}\mathbf{1}_j\end{smallmatrix}\right)\mathrm{GL}_n(\mathcal{O}_{F_{\tilde{w}}}),$$

we have for $f \in S(U, \mathbb{F})$ and $g \in H(\mathbb{A}_{F^+}^{\infty})$:

$$(T_{\tilde{w}}^{(j)} f)(g) \overset{\text{def}}{=} \sum_i f\left(g\iota_{\tilde{w}}^{-1}\left(g_i\left(\begin{smallmatrix}\mathbf{1}_{n-j} & \\ & \varpi_{\tilde{w}}\mathbf{1}_j\end{smallmatrix}\right)\right)\right).$$

One checks that $T_{\tilde{w}^c}^{(j)} = (T_{\tilde{w}}^{(n)})^{-1}T_{\tilde{w}}^{(n-j)}$ on $S(U, \mathbb{F})$. We let $\mathcal{T}^\Sigma(U, \mathbb{F})$ be the image of $\mathcal{T}^\Sigma$ in $\mathrm{End}_{\mathcal{O}_E}(S(U, \mathbb{F}))$ (if $U' \subseteq U$, we thus have $S(U, \mathbb{F}) \subseteq S(U', \mathbb{F})$ and $\mathcal{T}^\Sigma(U', \mathbb{F}) \twoheadrightarrow \mathcal{T}^\Sigma(U, \mathbb{F})$). If S is any $\mathcal{T}^\Sigma$-module and I any ideal of $\mathcal{T}^\Sigma$, we set in the sequel $S[I] \overset{\text{def}}{=} \{x \in S : Ix = 0\}$.

We now fix $v|p$ and a compact open subgroup U^v of $H(\mathbb{A}_{F^+}^{\infty,v})$. If Σ is a finite set of finite places of F^+ containing the set of places of F^+ that split in F and divide pN and the set of places of F^+ prime to p that split in F and at which U^vU_v (for any U_v) is not unramified, the algebra $\mathcal{T}^\Sigma$ acts on $S(U^vU_v, \mathbb{F})$ (via its quotient $\mathcal{T}^\Sigma(U^vU_v, \mathbb{F})$) for any U_v and thus also on $S(U^v, \mathbb{F})$. This action commutes with that of $H(F_v^+)$. If $\mathfrak{m}^\Sigma$ is a maximal ideal of $\mathcal{T}^\Sigma$ with residue field $\mathbb{F}$, we can define the localized subspaces $S(U^vU_v, \mathbb{F})_{\mathfrak{m}^\Sigma}$ and their inductive limit

$$\varinjlim_{U_v} S(U^vU_v, \mathbb{F})_{\mathfrak{m}^\Sigma} = S(U^v, \mathbb{F})_{\mathfrak{m}^\Sigma},$$

which inherits an induced (admissible smooth) action of $H(F_v^+)$ together with a commuting action of $\varprojlim_{U_v} \mathcal{T}^\Sigma(U^vU_v, \mathbb{F})_{\mathfrak{m}^\Sigma}$. We have

$$S(U^vU_v, \mathbb{F})[\mathfrak{m}^\Sigma] \subseteq S(U^vU_v, \mathbb{F})_{\mathfrak{m}^\Sigma} \subseteq S(U^vU_v, \mathbb{F})$$

and thus inclusions of admissible smooth $H(F_v^+)$-representations over $\mathbb{F}$:

$$S(U^v, \mathbb{F})[\mathfrak{m}^\Sigma] \subseteq S(U^v, \mathbb{F})_{\mathfrak{m}^\Sigma} \subseteq S(U^v, \mathbb{F}).$$

Moreover, as representations of $H(F_v^+)$, $S(U^v,\mathbb{F})_{\mathfrak{m}^\Sigma}$ is a direct summand of $S(U^v,\mathbb{F})$ (= the maximal vector subspace on which the elements of $\mathfrak{m}^\Sigma$ act nilpotently).

We now go back to the notation of §2.1.1. For $\lambda \in X(T)$ a dominant weight with respect to B, we consider the following algebraic representation of $G \times_{\mathbb{Z}} \mathbb{F}$ over $\mathbb{F}$:

$$\overline{L}(\lambda) \stackrel{\text{def}}{=} \Big(\mathrm{ind}_{B^-}^{G}\lambda\Big)_{/\mathbb{Z}} \otimes_{\mathbb{Z}} \mathbb{F} = \Big(\mathrm{ind}_{B^-\times_{\mathbb{Z}}\mathbb{F}}^{G\times_{\mathbb{Z}}\mathbb{F}}\lambda\Big)_{/\mathbb{F}}, \tag{19}$$

where ind means the algebraic induction functor of [Jan03, §I.3.3] and the last equality follows from [Jan03, II.8.8(1)]. For $\alpha = e_i - e_{i+1} \in S$, we set

$$\lambda_\alpha \stackrel{\text{def}}{=} e_1 + \cdots + e_i \in X(T), \tag{20}$$

so that the λ_α for $\alpha \in S$ are fundamental weights of G (see e.g. [BH15, §2.1]). Let $\overline{\rho} : \mathrm{Gal}(\overline{\mathbb{Q}}_p/K) \longrightarrow G(\mathbb{F})$ be a continuous homomorphism, viewing $\overline{L}(\lambda_\alpha)$ as a continuous homomorphism

$$G(\mathbb{F}) \longrightarrow \mathrm{Aut}\Big(\overline{L}(\lambda_\alpha)(\mathbb{F})\Big)$$

(where $\overline{L}(\lambda_\alpha)(\mathbb{F})$ is the underlying $\mathbb{F}$-vector space of the algebraic representation $\overline{L}(\lambda_\alpha)$), we define the Galois representations for $\alpha \in S$:

$$\overline{L}(\lambda_\alpha)(\overline{\rho}) : \mathrm{Gal}(\overline{\mathbb{Q}}_p/K) \stackrel{\overline{\rho}}{\longrightarrow} G(\mathbb{F}) \stackrel{\overline{L}(\lambda_\alpha)}{\longrightarrow} \mathrm{Aut}\Big(\overline{L}(\lambda_\alpha)(\mathbb{F})\Big).$$

Recall that $\overline{L}(\lambda_\alpha)(\overline{\rho}) = \bigwedge_{\mathbb{F}}^i \overline{\rho}$ if $\alpha = e_i - e_{i+1}$ ([BH15, Ex.2.1.3]). We let

$$\bigotimes_{\alpha\in S}\Big(\overline{L}(\lambda_\alpha)(\overline{\rho})\Big) \cong \bigotimes_{i=1}^{n-1} \bigwedge\nolimits_{\mathbb{F}}^{i} \overline{\rho}$$

be the tensor product of the representations $\overline{L}(\lambda_\alpha)(\overline{\rho})$ (over $\mathbb{F}$) and define the following finite-dimensional continuous representation of $\mathrm{Gal}(\overline{\mathbb{Q}}_p/\mathbb{Q}_p)$ over $\mathbb{F}$:

$$\overline{L}^{\otimes}(\overline{\rho}) \stackrel{\text{def}}{=} \mathrm{ind}_K^{\otimes\mathbb{Q}_p}\Big(\bigotimes_{\alpha\in S}\Big(\overline{L}(\lambda_\alpha)(\overline{\rho})\Big)\Big), \tag{21}$$

where $\mathrm{ind}_K^{\otimes\mathbb{Q}_p}$ means the *tensor induction from* $\mathrm{Gal}(\overline{\mathbb{Q}}_p/K)$ *to* $\mathrm{Gal}(\overline{\mathbb{Q}}_p/\mathbb{Q}_p)$ ([Col89], [CR81, §13], see also the end of the proof of Lemma 2.4.2.3). Note that there are $\mathrm{Gal}(\overline{\mathbb{Q}}_p/\mathbb{Q}_p)$-equivariant isomorphisms

$$\overline{L}^{\otimes}(\overline{\rho}^\vee) \cong \overline{L}^{\otimes}(\overline{\rho})^\vee \cong \overline{L}^{\otimes}(\overline{\rho}) \otimes \mathrm{ind}_K^{\otimes\mathbb{Q}_p}\Big(\det(\overline{\rho})^{-(n-1)}\Big) \tag{22}$$

(recall $\mathrm{ind}_K^{\otimes\mathbb{Q}_p}\Big(\det(\overline{\rho})^{-(n-1)}\Big)$ is still one dimensional).

Example 2.1.2.1. For $n = 2$, we thus just have $\overline{L}^{\otimes}(\overline{\rho}) = \mathrm{ind}_K^{\otimes\mathbb{Q}_p}(\overline{\rho})$.

2.1.3 Weak local-global compatibility conjecture

We state our weak local-global compatibility conjecture (Conjecture 2.1.3.1).

Let $\overline{r} : \mathrm{Gal}(\overline{F}/F) \to \mathrm{GL}_n(\mathbb{F})$ be a continuous representation and $\overline{r}^\vee$ its dual. We assume:

(i) $\overline{r}^c \cong \overline{r}^\vee \otimes \omega^{1-n}$ (where $\overline{r}^c(g) \overset{\text{def}}{=} \overline{r}(cgc)$ for $g \in \mathrm{Gal}(\overline{F}/F)$);

(ii) $\overline{r}$ is an absolutely irreducible representation of $\mathrm{Gal}(\overline{F}/F)$.

Fix $v|p$ in F^+, $V^v \subseteq U^v \subseteq H(\mathbb{A}_{F^+}^{\infty,v})$ compact open subgroups and Σ a finite set of finite places of F^+ containing

(a) the set of places of F^+ that split in F and divide pN;

(b) the set of places of F^+ that split in F at which V^v is not unramified;

(c) the set of places of F^+ that split in F at which $\overline{r}$ is ramified.

We associate to $\overline{r}$ and Σ the maximal ideal $\mathfrak{m}^\Sigma$ in $\mathcal{T}^\Sigma$ with residue field $\mathbb{F}$ generated by ϖ_E and all elements

$$\left((-1)^j \mathrm{Norm}(\tilde{w})^{j(j-1)/2} T_{\tilde{w}}^{(j)} - a_{\tilde{w}}^{(j)}\right)_{j,\tilde{w}},$$

where $j \in \{1, \ldots, n\}$, $\tilde{w}$ is a place of F lying above a finite place w of F^+ that splits in F and doesn't belong to Σ, $X^n + \overline{a}_{\tilde{w}}^{(1)} X^{n-1} + \cdots + \overline{a}_{\tilde{w}}^{(n-1)} X + \overline{a}_{\tilde{w}}^{(n)}$ is the characteristic polynomial of $\overline{r}(\mathrm{Frob}_{\tilde{w}})$ (an element of $\mathbb{F}[X]$, $\mathrm{Frob}_{\tilde{w}}$ is a geometric Frobenius at $\tilde{w}$) and where $a_{\tilde{w}}^{(j)}$ is any element in $\mathcal{O}_E$ lifting $\overline{a}_{\tilde{w}}^{(j)}$. Note that $S(V^v, \mathbb{F})[\mathfrak{m}^\Sigma] \neq 0$ in fact implies assumption (i) above on $\overline{r}$ (though strictly speaking we need (i) to define $\mathfrak{m}^\Sigma$ in $\mathcal{T}^\Sigma$). Note also that if U is any subgroup of $H(\mathbb{A}_{F^+}^\infty)$ containing V^v as a normal subgroup, then U naturally acts on $S(V^v, \mathbb{F})$ and $S(V^v, \mathbb{F})[\mathfrak{m}^\Sigma]$.

For $\tilde{v}|v$ in F, we denote by $V_{G,\tilde{v}}$ the functor defined in (16) applied to smooth representations of $H(F_v^+)$ over $\mathbb{F}$, where we identify $H(F_v^+)$ with $\mathrm{GL}_n(F_{\tilde{v}}) = G(F_{\tilde{v}})$ via $\iota_{\tilde{v}}$. For any finite place $\tilde{w}$ of F, we denote by $\overline{r}_{\tilde{w}}$ the restriction of $\overline{r}$ to a decomposition subgroup at $\tilde{w}$.

Conjecture 2.1.3.1. *Let $\overline{r} : \mathrm{Gal}(\overline{F}/F) \to \mathrm{GL}_n(\mathbb{F})$ be a continuous representation that satisfies conditions (i) and (ii) above and fix a place v of F^+ which divides p. Assume that there exist compact open subgroups $V^v \subseteq U^v \subseteq H(\mathbb{A}_{F^+}^{\infty,v})$ with V^v normal in U^v, a finite-dimensional representation σ^v of U^v/V^v over $\mathbb{F}$ and a finite set Σ of finite places of F^+ as above such that* $\mathrm{Hom}_{U^v}(\sigma^v, S(V^v, \mathbb{F})[\mathfrak{m}^\Sigma]) \neq 0$. *Let $\tilde{v}|v$ in F.*

Then there is an integer $d \in \mathbb{Z}_{>0}$ depending only on v, U^v, V^v, σ^v and $\overline{r}$ such that there is an isomorphism of representations of $\mathrm{Gal}(\overline{\mathbb{Q}}_p/\mathbb{Q}_p)$ *on* $\mathbb{F}$*:*

$$V_{G,\tilde{v}}\Big(\mathrm{Hom}_{U^v}(\sigma^v, S(V^v,\mathbb{F})[\mathfrak{m}^\Sigma]) \otimes (\omega^{-(n-1)} \circ \det)\Big) \cong \overline{L}^{\otimes}(\overline{r}_{\tilde{v}})^{\oplus d}. \quad (23)$$

Remark 2.1.3.2. (i) In the special case $\sigma^v = 1$, Conjecture 2.1.3.1 boils down to $V_{G,\tilde{v}}(S(U^v,\mathbb{F})[\mathfrak{m}^\Sigma] \otimes (\omega^{-(n-1)} \circ \det)) \cong \overline{L}^{\otimes}(\overline{r}_{\tilde{v}})^{\oplus d}$.
(ii) Conjecture 2.1.3.1 implies that the $G(F_{\tilde{v}})$-representation $\mathrm{Hom}_{U^v}(\sigma^v, S(V^v,\mathbb{F})[\mathfrak{m}^\Sigma])$ determines the $\mathrm{Gal}(\overline{\mathbb{Q}}_p/\mathbb{Q}_p)$-representation $\overline{L}^{\otimes}(\overline{r}_{\tilde{v}})$. Note that this doesn't imply in general that $\mathrm{Hom}_{U^v}(\sigma^v, S(V^v,\mathbb{F})[\mathfrak{m}^\Sigma])$ determines the $\mathrm{Gal}(\overline{F_{\tilde{v}}}/F_{\tilde{v}})$-representation $\overline{r}_{\tilde{v}}$ itself (though this is also expected, see [PQ22] and the references therein).
(iii) See §§3.2, 3.4 below for nontrivial evidence on Conjecture 2.1.3.1 when K is unramified and $n = 2$.

We now check that, at least when p is odd, F/F^+ is unramified at finite places and $H \times_{\mathcal{O}_{F^+}[1/N]} F^+$ is quasi-split at finite places, Conjecture 2.1.3.1 holds for $\tilde{v}$ if and only if it holds for $\tilde{v}^c$ (these extra assumptions come from the use of [Tho12, §6] in the next lemma).

Lemma 2.1.3.3. *Assume $p > 2$, F/F^+ unramified at finite places and $H \times_{\mathcal{O}_{F^+}[1/N]} F^+$ quasi-split at finite places of F^+. Let $\tilde{v}|v$ in F. Then the action of the center $(F_v^+)^\times \subseteq \mathrm{GL}_n(F_v^+)$ on $S(V^v,\mathbb{F})[\mathfrak{m}^\Sigma]$ via $\iota_{\tilde{v}}$ is given by $\det(\overline{r}_{\tilde{v}})\omega^{\frac{n(n-1)}{2}}$ (via local class field theory for F_v^+).*

Proof. We can assume $S(V^v,\mathbb{F})[\mathfrak{m}^\Sigma] \neq 0$. The map $S(V^vU_v,\mathcal{O}_E) \longrightarrow S(V^vU_v,\mathbb{F})$ being surjective for U_v small enough (see e.g. [BH15, Lemma 4.4.1] or [EGH13, §7.1.2]), we have a surjection of smooth $H(F_v^+)$-representations:

$$S(V^v,\mathcal{O}_E)_{\mathfrak{m}^\Sigma} \twoheadrightarrow S(V^v,\mathbb{F})_{\mathfrak{m}^\Sigma} \quad (24)$$

(where $S(V^vU_v,\mathcal{O}_E)$, $S(V^v,\mathcal{O}_E)_{\mathfrak{m}^\Sigma}$ are defined as $S(V^vU_v,\mathbb{F})$, $S(V^v,\mathbb{F})_{\mathfrak{m}^\Sigma}$ replacing $\mathbb{F}$ by $\mathcal{O}_E$). By classical local-global compatibility applied to $\big(\varinjlim_U S(U,\mathcal{O}_E)\big) \otimes_{\mathcal{O}_E} E$, see e.g. [EGH13, Thm.7.2.1], we easily deduce with (24) that *if* $(F_v^+)^\times$ acts via $\iota_{\tilde{v}}$ on the whole $S(V^v,\mathbb{F})[\mathfrak{m}^\Sigma]$ (inside $S(V^v,\mathbb{F})_{\mathfrak{m}^\Sigma}$) by a single character, then this character must be $\det(\overline{r}_{\tilde{v}})\omega^{\frac{n(n-1)}{2}}$.
Let us prove that $(F_v^+)^\times$ indeed acts by a character. The functor associating to any local artinian $\mathcal{O}_E$-algebra A with residue field $\mathbb{F}$ the set of isomorphism classes of deformations r_A of $\overline{r}$ to A such that $r_A^c \cong r_A^\vee \otimes \varepsilon^{1-n}$ is pro-representable by a local complete noetherian $\mathcal{O}_E$-algebra $R_{\overline{r},\Sigma}$ of residue field $\mathbb{F}$. When $p > 2$, F/F^+ is unramified at finite places and $H \times_{\mathcal{O}_{F^+}[1/N]} F^+$ is quasi-split at finite places of F^+, it follows from [Tho12, Prop.6.7] that there is a natural such deformation with values in $\mathcal{T}^\Sigma(V^vU_v,\mathcal{O}_E)_{\mathfrak{m}^\Sigma}$ for any U_v (where $\mathcal{T}^\Sigma(V^vU_v,\mathcal{O}_E)_{\mathfrak{m}^\Sigma}$ is defined as $\mathcal{T}^\Sigma(V^vU_v,\mathbb{F})_{\mathfrak{m}^\Sigma}$

in §2.1.2 replacing $\mathbb{F}$ by $\mathcal{O}_E$), and hence by universality a continuous morphism of local $\mathcal{O}_E$-algebras:

$$R_{\overline{r},\Sigma} \longrightarrow \mathcal{T}^{\Sigma}(V^v U_v, \mathcal{O}_E)_{\mathfrak{m}^\Sigma}. \tag{25}$$

Likewise, the functor associating to any A as above the set of isomorphism classes of $\mathrm{Gal}(\overline{F_{\tilde v}}/F_{\tilde v})^{\mathrm{ab}}$-deformations of $\det(\overline{r}_{\tilde v})$ over A is pro-representable by the Iwasawa algebra $\mathcal{O}_E[\![\mathrm{Gal}(\overline{F_{\tilde v}}/F_{\tilde v})^{\mathrm{ab}}]\!]$, and considering $\det_A(r_A|_{\mathrm{Gal}(\overline{F_{\tilde v}}/F_{\tilde v})})$ for $A = R_{\overline{r},\Sigma}$ provides by the universal property again a continuous morphism of local $\mathcal{O}_E$-algebras:

$$\mathcal{O}_E[\![\mathrm{Gal}(\overline{F_{\tilde v}}/F_{\tilde v})^{\mathrm{ab}}]\!] \longrightarrow R_{\overline{r},\Sigma}. \tag{26}$$

Since $\mathcal{T}^{\Sigma}(V^v U_v, \mathcal{O}_E)_{\mathfrak{m}^\Sigma}$ acts by a character on $S(V^v U_v, \mathbb{F})[\mathfrak{m}^\Sigma]$ for any U_v, so is the case of $R_{\overline{r},\Sigma}$ on $S(V^v, \mathbb{F})[\mathfrak{m}^\Sigma]$ by (25). Using (24), we see that it is enough to prove that the induced morphism

$$\mathcal{O}_E[\![\mathrm{Gal}(\overline{F_{\tilde v}}/F_{\tilde v})^{\mathrm{ab}}]\!] \overset{(26)}{\longrightarrow} R_{\overline{r},\Sigma} \overset{(25)}{\longrightarrow} \varprojlim_{U_v} \mathcal{T}^{\Sigma}(V^v U_v, \mathcal{O}_E)_{\mathfrak{m}^\Sigma}$$

gives an action of $\mathrm{Gal}(\overline{F_{\tilde v}}/F_{\tilde v})^{\mathrm{ab}}$ on $S(V^v, \mathcal{O}_E)_{\mathfrak{m}^\Sigma}$ which, when restricted to $F_{\tilde v}^\times \hookrightarrow \mathrm{Gal}(\overline{F_{\tilde v}}/F_{\tilde v})^{\mathrm{ab}}$ (via the local reciprocity map), coincides with the action of $F_{\tilde v}^\times$ on $S(V^v, \mathcal{O}_E)_{\mathfrak{m}^\Sigma}$ as center of $H(F_v^+) \overset{\iota_{\tilde v}}{\cong} G(F_{\tilde v})$. We can work in $S(V^v, \mathcal{O}_E)_{\mathfrak{m}^\Sigma} \otimes_{\mathcal{O}_E} E$, in which case this follows from local-global compatibility (as in [EGH13, Thm.7.2.1]) and from the fact that, by construction of the map (25) (see [Tho12, §6]) and by (26), $\mathrm{Gal}(\overline{F_{\tilde v}}/F_{\tilde v})^{\mathrm{ab}}$ acts on $\pi^{V^v} \subseteq S(V^v, \mathcal{O}_E)_{\mathfrak{m}^\Sigma} \otimes_{\mathcal{O}_E} E$ by multiplication by the character $\det(r_\pi)|_{\mathrm{Gal}(\overline{F_{\tilde v}}/F_{\tilde v})}$, where π is an irreducible $H(\mathbb{A}_{F^+}^\infty)$-subrepresentation of $\left(\varinjlim_{U} S(U, \mathcal{O}_E)\right) \otimes_{\mathcal{O}_E} E$ such that π^{V^v} occurs in $S(V^v, \mathcal{O}_E)_{\mathfrak{m}^\Sigma} \otimes_{\mathcal{O}_E} E$ and where r_π is its associated (irreducible) p-adic representation of $\mathrm{Gal}(\overline{F}/F)$ ([EGH13, Thm.7.2.1] again). ☐

Let π be a smooth representation of $G(K) = \mathrm{GL}_n(K)$ over $\mathbb{F}$ with central character $Z(\pi)$ and denote by $\pi^\star$ the smooth representation of $G(K)$ with the same underlying vector space as π but where $g \in G(K)$ acts by $\tau(g)^{-1}$.

Lemma 2.1.3.4. *There is a* $\mathrm{Gal}(\overline{\mathbb{Q}}_p/\mathbb{Q}_p)$*-equivariant isomorphism*

$$V_G(\pi^\star) \cong V_G(\pi) \otimes Z(\pi)^{-(n-1)}|_{\mathbb{Q}_p^\times},$$

where $Z(\pi)|_{\mathbb{Q}_p^\times}$ *is seen as a character of* $\mathrm{Gal}(\overline{\mathbb{Q}}_p/\mathbb{Q}_p)$ *via local class field theory.*

Proof. We use the notation of §2.1.1. Let $w_0 \in W$ be the element of maximal length, the isomorphism $\pi^{N_1} \xrightarrow{\sim} \pi^{w_0 N_1 w_0}$, $v \mapsto w_0 v$ shows that one can compute $V_G(\pi)$ using $w_0 N_1 w_0$ instead of N_1 and conjugating everything by w_0 (e.g. $x \in \mathbb{Z}_p^\times$ acts by $w_0 \xi_G(x) w_0$, etc.). Now, it is easy to check that the $\mathbb{F}$-linear isomorphism

$(\pi^\star)^{N_1} \xrightarrow{\sim} \pi^{w_0 N_1 w_0}$, $v \mapsto w_0 v$ is compatible with the $\mathbb{F}[\![X]\!][F]$-module structure on both sides but where we twist the $\mathbb{F}[\![X]\!][F]$-action as follows on the right-hand side: X acts by $(1+X)^{-1}-1$ and F acts by $p^{-(n-1)}F$, $p^{-(n-1)}$ being here in the center of $G(K)$. Likewise, it is compatible with the action of $\mathbb{Z}_p^\times$ but where $x \in \mathbb{Z}_p^\times$ acts by $x^{-(n-1)}\xi_G(x)$ on the right-hand side (with $x^{-(n-1)}$ in the center of $G(K)$). All this easily implies the lemma. □

Lemma 2.1.3.5. *Assume $p>2$, F/F^+ unramified at finite places and $H \times_{\mathcal{O}_{F^+}[1/N]} F^+$ quasi-split at finite places of F^+. We have a $\mathrm{Gal}(\overline{\mathbb{Q}}_p/\mathbb{Q}_p)$-equivariant isomorphism*

$$V_{G,\tilde{v}^c}\Big(\mathrm{Hom}_{U^v}(\sigma^v, S(V^v,\mathbb{F})[\mathfrak{m}^\Sigma])\Big)$$
$$\cong V_{G,\tilde{v}}\Big(\mathrm{Hom}_{U^v}(\sigma^v, S(V^v,\mathbb{F})[\mathfrak{m}^\Sigma])\Big) \otimes \mathrm{ind}_{F_{\tilde{v}}}^{\otimes \mathbb{Q}_p}\Big(\det(\overline{r}_{\tilde{v}})^{-(n-1)}\omega^{\frac{-n(n-1)^2}{2}}\Big).$$

Proof. This follows from Lemma 2.1.3.4 applied to $\pi = \mathrm{Hom}_{U^v}(\sigma^v, S(V^v,\mathbb{F})[\mathfrak{m}^\Sigma])$ together with Lemma 2.1.3.3, recalling that $Z(\pi)|_{\mathbb{Q}_p^\times}$, seen as a character of $\mathrm{Gal}(\overline{\mathbb{Q}}_p/\mathbb{Q}_p)$ via local class field theory, is $\mathrm{ind}_{F_{\tilde{v}}}^{\otimes \mathbb{Q}_p}(Z(\pi))$ (where $Z(\pi)$ is here seen as a character of $\mathrm{Gal}(\overline{F_{\tilde{v}}}/F_{\tilde{v}})$). □

Proposition 2.1.3.6. *Assume $p>2$, F/F^+ unramified at finite places and $H \times_{\mathcal{O}_{F^+}[1/N]} F^+$ quasi-split at finite places of F^+. Conjecture 2.1.3.1 holds for $\tilde{v}$ if and only if it holds for $\tilde{v}^c$.*

Proof. This follows from Lemma 2.1.3.5 together with $\overline{r}_{\tilde{v}^c} \cong \overline{r}_{\tilde{v}}^\vee \otimes \omega^{1-n}$, (22) and an easy computation. □

2.1.4 A reformulation using C-groups

We show that one can give a more general and more natural formulation of Conjecture 2.1.3.1 (in the special case of Remark 2.1.3.2(i)) using C-parameters (Conjecture 2.1.4.5).

We start by some reminders about L-groups and C-groups.

Let k be a field and k^{sep} a separable closure of k. We note $\Gamma_k \stackrel{\mathrm{def}}{=} \mathrm{Gal}(k^{\mathrm{sep}}/k)$. Let H be a connected reductive group defined over k, let $\widehat{H}$ be its dual group, ${}^L H$ its L-group and ${}^C H$ its C-group. We refer to [Bor79, §2], [BG14, §§2,5], [GHS18, §9] and [Zhu, §1.1] for more details concerning these L-groups and C-groups. Note that these two groups can be defined over $\mathbb{Z}$. Their construction depends on the choice of a pinning $(B_H, T_H, \{x_\alpha\}_{\alpha\in S_H})$ of $H_{k^{\mathrm{sep}}}$. The dual group $\widehat{H}$ has a natural pinned structure $(B_{\widehat{H}}, T_{\widehat{H}}, \{x_{\widehat{\alpha}}\}_{\alpha\in S_H})$ with $B_{\widehat{H}}$ a Borel subgroup of $\widehat{H}$, $T_{\widehat{H}} \subseteq B_{\widehat{H}}$ a maximal split torus and $\{x_{\widehat{\alpha}}\}_{\alpha\in S_H}$ a pinning of $(B_{\widehat{H}}, T_{\widehat{H}})$ (see [Con14, §§5,6] for the fact that everything can be defined over $\mathbb{Z}$) on which the group Γ_k is acting. Let $1 \to \mathbb{G}_m \to \widetilde{H} \to H \to 1$ be the central $\mathbb{G}_m$-extension of H (over k) whose existence is proved in [BG14, Prop.5.3.1(a)]. The inverse images $T_{\widetilde{H}}$ and $B_{\widetilde{H}}$ of T_H and B_H in $\widetilde{H}_{k^{\mathrm{sep}}}$ are respectively a maximal torus and a Borel subgroup of $\widetilde{H}_{k^{\mathrm{sep}}}$. Moreover, since the above extension is central, there is a unique pinning $\{\widetilde{x}_\alpha\}_{\alpha\in S_H}$ of $(B_{\widetilde{H}}, T_{\widetilde{H}})$ inducing $\{x_\alpha\}_{\alpha\in S_H}$ on (B,T) via the map $\widetilde{H}_{k^{\mathrm{sep}}} \to H_{k^{\mathrm{sep}}}$. This gives rise to a pinned dual data $(\widehat{\widetilde{H}}, B_{\widehat{\widetilde{B}}}, T_{\widehat{\widetilde{H}}}, \{\widetilde{x}_{\widehat{\alpha}}\}_{\alpha\in S_H})$ with an action of Γ_k (trivial on some open subgroup) and a Γ_k-equivariant injection $(\widehat{H}, B_{\widehat{H}}, T_{\widehat{H}}) \hookrightarrow (\widehat{\widetilde{H}}, B_{\widehat{\widetilde{H}}}, T_{\widehat{\widetilde{H}}})$ such that $\{x_{\widehat{\alpha}}\}_{\alpha\in S_H}$ induces $\{\widetilde{x}_{\widehat{\alpha}}\}_{\alpha\in S_H}$.

The *L-groups and C-groups* are then defined as the group schemes

$$ {}^L H \stackrel{\mathrm{def}}{=} \widehat{H} \rtimes \Gamma_k \qquad {}^C H \stackrel{\mathrm{def}}{=} \widehat{\widetilde{H}} \rtimes \Gamma_k. \tag{27}$$

We have the following simple description of $\widehat{\widetilde{H}}$ given in [Zhu, §1.1]. Let $\widehat{H}^{\mathrm{ad}}$ and $T^{\mathrm{ad}}_{\widehat{H}}$ be the quotients of $\widehat{H}$ and $T_{\widehat{H}}$ by the center of $\widehat{H}$ and let δ_{ad} be the cocharacter of $T^{\mathrm{ad}}_{\widehat{H}} \subseteq \widehat{H}^{\mathrm{ad}}$ defined as the half sum of positive roots of $\widehat{H}$ with respect to $(B_{\widehat{H}}, T_{\widehat{H}})$. The group $\widehat{H}^{\mathrm{ad}}$ acts on $\widehat{H}$ by the adjoint action and, after precomposition with δ_{ad}, this defines an action, in the category of $\mathbb{Z}$-group schemes, of $\mathbb{G}_m$ on $\widehat{H}$. There is an isomorphism of $\mathbb{Z}$-group schemes $\widehat{\widetilde{H}} \cong \widehat{H} \rtimes \mathbb{G}_m$ identifying $B_{\widehat{\widetilde{H}}}$ with $B_{\widehat{H}} \rtimes \mathbb{G}_m$ and $T_{\widehat{\widetilde{H}}}$ with $T_{\widehat{H}} \rtimes \mathbb{G}_m = T_{\widehat{H}} \times \mathbb{G}_m$. We note that, since δ_{ad} is fixed by the Galois action, this isomorphism is Galois equivariant. Using this isomorphism, we identify $X(T_{\widehat{\widetilde{H}}})$ with $X(T_{\widehat{H}}) \times \mathbb{Z} \cong X^\vee(T_H) \times \mathbb{Z}$. This shows that we have an exact sequence of $\mathbb{Z}$-group schemes:

$$1 \longrightarrow {}^L H \longrightarrow {}^C H \xrightarrow{d} \mathbb{G}_m \longrightarrow 1.$$

Let A be a topological $\mathbb{Z}_p$-algebra and assume from now on that k is either a number field or a finite extension of $\mathbb{Q}_p$, so that we have an A-valued p-adic cyclotomic character. We recall that a morphism $\rho : \Gamma_k \to {}^L H(A)$ is called *admissible* if its composition with the second projection ${}^L H(A) \to \Gamma_k$ is the identity (see [Bor79, §3]).

Definition 2.1.4.1. An *L-parameter* (resp. *C-parameter*) of H over A is an admissible continuous morphism $\rho : \Gamma_k \longrightarrow {}^L H(A)$ (resp. $\rho : \Gamma_k \longrightarrow {}^C H(A)$ such that $d \circ \rho$ is the p-adic cyclotomic character). When A is moreover an algebraically closed field, we say that two L-parameters (resp. C-parameters) of H over A are *equivalent* if they are conjugate by an element of $\widehat{H}(A)$ (resp. $\widehat{\widetilde{H}}(A)$).

Remark 2.1.4.2. Assume A is an algebraically closed field. Each element of $\widehat{\widetilde{H}}(A)$ is the product of an element of $\widehat{H}(A)$ and an element of the center of $\widehat{\widetilde{H}}(A)$. This can be deduced from [BG14, Prop.5.3.3] or [Zhu, (1.2)]. This implies that two C-parameters of H over A are equivalent if and only if they are conjugate by an element of $\widehat{H}(A)$.

For simplicity, we assume from now on that A is moreover an algebraically closed field. We also assume (not for simplicity) that H has a connected center. We generalize now the representation $\overline{L}^{\otimes}(\overline{\rho}) \otimes_{\mathbb{F}} \overline{\mathbb{F}}_p$ (see (21) for $\overline{L}^{\otimes}(\overline{\rho})$).

Let $(\lambda_{\alpha^\vee})_{\alpha \in S_H}$ be a family of fundamental coweights of H such that

$$\xi_H \overset{\text{def}}{=} \sum_{\alpha \in S_H} \lambda_{\alpha^\vee} \in X(T_{\widehat{H}}) \cong X^\vee(T_H) \tag{28}$$

is fixed under the action of Γ_k (compare with (13) and note that the cocharacters $\lambda_{\alpha^\vee}$ exist since H has a connected center but each of them doesn't have to be fixed by Γ_k). Let $(r_{\lambda_{\alpha^\vee}}, V_{\lambda_{\alpha^\vee}})$ be the irreducible algebraic representation of $\widehat{H}$ of highest weight $\lambda_{\alpha^\vee}$ over A and let $(r^{\otimes}_{\xi_H}, V^{\otimes}_{\xi_H})$ be the irreducible algebraic representation of $\widehat{H}^{S_H}$ over A of highest weight $(\lambda_{\alpha^\vee})_{\alpha \in S_H}$ = the character of $T^{S_H}_{\widehat{H}}$ defined by $(x_\alpha)_{\alpha \in S_H} \mapsto \sum_\alpha \lambda_{\alpha^\vee}(x_\alpha)$. Note that we have an isomorphism of algebraic representations of $\widehat{H}^{S_H}$:

$$(r^{\otimes}_{\xi_H}, V^{\otimes}_{\xi_H}) \cong \bigotimes_{\alpha \in S_H} (r_{\lambda_{\alpha^\vee}}, V_{\lambda_{\alpha^\vee}}). \tag{29}$$

Let $\gamma \in \Gamma_k$ and $\chi_{\alpha,\gamma}$ be the character of $\widehat{H}$ corresponding to the cocharacter $\gamma(\lambda_{\alpha^\vee}) - \lambda_{\gamma\alpha^\vee} \in X^\vee(Z_H) \subseteq X^\vee(T_H)$. Comparing the highest weights, for $\gamma \in \Gamma_k$ there is an isomorphism of algebraic irreducible representations of $\widehat{H}^{S_H}$:

$$\left(r^{\otimes}_{\xi_H}(\gamma^{-1}\cdot), V^{\otimes}_{\xi_H}\right) \cong \left(\otimes_{\alpha \in S_H}(r_{\lambda_{\alpha^\vee}} \otimes \chi_{\gamma^{-1}\alpha,\gamma}) \circ c_\gamma, V^{\otimes}_{\xi_H}\right),$$

where c_γ is the automorphism of $\widehat{H}^{S_H}$ defined by $(x_\alpha)_{\alpha \in S_H} \mapsto (x_{\gamma^{-1}\alpha})_{\alpha \in S_H}$. Therefore there exists an A-linear automorphism M_γ of $V^{\otimes}_{\xi_H}$, well defined up to a nonzero scalar, such that, for $(x_\alpha)_{\alpha \in S_H} \in \widehat{H}(A)^{S_H}$:

$$M_\gamma \left(r^{\otimes}_{\xi_H}((\gamma^{-1}x_\alpha)_{\alpha \in S_H})\right) M_\gamma^{-1} = \left(\otimes_{\alpha \in S_H} r_{\lambda_{\alpha^\vee}}(x_{\gamma^{-1}\alpha})\right) \prod_{\alpha \in S_H} \chi_{\alpha,\gamma}(x_\alpha). \tag{30}$$

Moreover the subspaces of highest weight of these two representations over $V^{\otimes}_{\xi_H}$ being the same, we can choose M_γ such that it induces the identity on this line. With this choice, the map $\gamma \mapsto M_\gamma$ is a representation of Γ_k over $V^{\otimes}_{\xi_H}$. Since $\xi_H \in X^\vee(T_H)^{\Gamma_k}$, we have $\prod_{\alpha \in S_H} \chi_{\alpha,\gamma} = 1$ for all $\gamma \in \Gamma_k$ so that, for $x \in \widehat{H}(A)$, we have from (30) and (29) (replacing $\gamma^{-1}x_\alpha$ by x for all $\alpha \in S_H$):

$$M_\gamma \left(\otimes_{\alpha \in S_H} r_{\lambda_{\alpha^\vee}}(x)\right) M_\gamma^{-1} = \left(\otimes_{\alpha \in S_H} r_{\lambda_{\alpha^\vee}}(\gamma x)\right).$$

All this proves that there is an algebraic representation $(L^{\otimes}_{\xi_H}, V^{\otimes}_{\xi_H})$ of ${}^L H$ on $V^{\otimes}_{\xi_H}$ defined by

$$L^{\otimes}_{\xi_H}(x, \gamma) \overset{\text{def}}{=} \left(\otimes_{\alpha \in S_H} r_{\lambda_{\alpha^\vee}}(x)\right) M_\gamma$$

for $x \in \widehat{H}(A)$ and $\gamma \in \Gamma_k$. The isomorphism class of this representation does not depend on the choice of the $\lambda_{\alpha^\vee}$ such that $\xi_H = \sum \lambda_{\alpha^\vee}$. Namely any other choice will twist each $r_{\lambda_{\alpha^\vee}}$ by a character whose product over all α is trivial.

If ρ is an L-parameter of H over A we define the Γ_k-representation $L^{\otimes}_{\xi_H}(\rho)$ as the composition $L^{\otimes}_{\xi_H} \circ \rho$. Moreover if two L-parameters ρ_1 and ρ_2 are equivalent, the representations $L^{\otimes}_{\xi_H}(\rho_1)$ and $L^{\otimes}_{\xi_H}(\rho_2)$ are clearly isomorphic. If ρ is a C-parameter of H over A, ρ is in particular an L-parameter of $\widetilde{H}$ over A by (27), and we define the Γ_k-representation $L^{\otimes,C}_{\xi_H}(\rho) \stackrel{\text{def}}{=} L^{\otimes}_{\xi_{\widetilde{H}}}(\rho)$, where

$$\xi_{\widetilde{H}} \stackrel{\text{def}}{=} (\xi_H, 0) \in X(T_{\widehat{\widetilde{H}}}) \cong X(T_{\widehat{H}}) \times \mathbb{Z}. \tag{31}$$

We now compare $L^{\otimes}_{\xi_H}(\rho)$, $L^{\otimes,C}_{\xi_H}(\rho)$ between k and finite extensions k' of k.

We fix $k' \subseteq k^{\text{sep}}$ a finite extension of k, H' a connected reductive group over k' and we let $H \stackrel{\text{def}}{=} \operatorname{Res}_{k'/k}(H')$. We let $\Sigma_{k'}$ be the set of embeddings $k' \hookrightarrow k^{\text{sep}}$ inducing the identity on k and $\tau_0 \in \Sigma_{k'}$ the inclusion $k' \subseteq k^{\text{sep}}$. For $\tau \in \Sigma_{k'}$ we choose $g_\tau \in \Gamma_k$ such that $\tau = g_\tau \circ \tau_0$, and we have $\Gamma_k = \coprod_{\tau \in \Sigma_{k'}} g_\tau \Gamma_{k'}$. The dual group $\widehat{H}$ of H is isomorphic to $\operatorname{ind}_{\Gamma_{k'}}^{\Gamma_k} \widehat{H}'$, i.e. the group scheme of functions $f : \Gamma_k \to \widehat{H}'$ such that $f(gh) = h^{-1}f(g)$ for all $g \in \Gamma_k$ and $h \in \Gamma_{k'}$ (see [Bor79, §5.1(4)]). More explicitly, the map $f \mapsto (f(g_\tau))_{\tau \in \Sigma_{k'}}$ induces an isomorphism $\operatorname{ind}_{\Gamma_{k'}}^{\Gamma_k} \widehat{H}' \cong \widehat{H}'^{\Sigma_{k'}}$ and the action of Γ_k on $\widehat{H}'^{\Sigma_{k'}}$ is given by

$$g \cdot (x_\tau)_{\tau \in \Sigma_{k'}} = \Big((g_\tau^{-1} g g_{g^{-1}\circ\tau}) x_{g^{-1}\circ\tau} \Big)_{\tau \in \Sigma_{k'}}.$$

The map $(x_\tau)_{\tau \in \Sigma_{k'}} \mapsto x_{\tau_0}$ is a $\Gamma_{k'}$-equivariant map $\widehat{H} \to \widehat{H}'$. It extends to a morphism of group schemes $\widehat{H} \rtimes \Gamma_{k'} \to {}^L H'$ (resp. $\widehat{\widetilde{H}} \rtimes \Gamma_{k'} \to {}^C H'$) inducing the identity on the $\Gamma_{k'}$ factor (resp. the $\mathbb{G}_m$ and $\Gamma_{k'}$ factors). If ρ is an L-parameter (resp. a C-parameter) of H over A, we can define an L-parameter (resp. a C-parameter) ρ' of H' by restriction of ρ to $\Gamma_{k'}$ and composition with the above morphism.

Lemma 2.1.4.3. *The map $\rho \mapsto \rho'$ induces a bijection between equivalence classes of L-parameters (resp. of C-parameters) of H over A and equivalence classes of L-parameters (resp. C-parameters) of H' over A.*

Proof. A map ρ from Γ_k to ${}^L H(A)$ of the form (c_ρ, Id) is admissible if and only if c_ρ is a 1-cocycle of Γ_k in $\widehat{H}(A)$ and is continuous if and only if c_ρ is continuous. Moreover two admissible ρ are equivalent if and only if they are conjugate by an element of $\widehat{H}(A)$. Therefore the map associating to ρ the class $[c_\rho]$ of c_ρ induces a bijection between the set of equivalence classes of L-parameters and the set of classes $[c] \in H^1_{\text{cont}}(\Gamma_k, \widehat{H}(A))$. The fact that the above map $\rho \mapsto \rho'$ induces an isomorphism $H^1_{\mathrm{cont}}(\Gamma_k, \widehat{H}(A)) \xrightarrow{\sim} H^1_{\mathrm{cont}}(\Gamma_{k'}, \widehat{H'}(A))$ is a consequence of a nonabelian version of Shapiro's Lemma (see for example [Sti10, Prop.8] noting that everything can be made continuous there or [GHS18, Lemma 9.4.1] in a more restricted context).

Therefore the map associated to a C-parameter ρ the class $[c_\rho]$ of c_ρ induces a bijection between the set of equivalence classes of C-parameters and the set of classes $c \in H^1_{\mathrm{cont}}(\Gamma_k, \widehat{\widetilde{H}}(A))$ such that $d(c) \in H^1_{\mathrm{cont}}(\Gamma_k, A^\times) \cong \mathrm{Hom}^{\mathrm{cont}}_{\mathrm{gp}}(\Gamma_k, A^\times)$ coincides with the p-adic cyclotomic character. Let $\widetilde{H}_1 \stackrel{\mathrm{def}}{=} \mathrm{Res}_{k'/k} \widetilde{H}'$, so that $\widetilde{H}$ can be identified to a quotient of $\widetilde{H}_1$. It follows from Remark 2.1.4.2 that $H^1(\Gamma_k, \widehat{\widetilde{H}_1}(A))$ is the set of classes of 1-cocycles of Γ_k with values in $\widehat{\widetilde{H}_1}(A)$ up to $\widehat{H}(A)$-conjugation. It follows again from Remark 2.1.4.2 that the set of equivalence classes of C-parameters of H over A is in bijection with the subset of $H^1_{\mathrm{cont}}(\Gamma_k, \widehat{\widetilde{H}_1}(A))$ of classes whose image in $H^1_{\mathrm{cont}}(\Gamma_k, (A^\times)^{[k':k]}) \cong \mathrm{Hom}^{\mathrm{cont}}_{\mathrm{gp}}(\Gamma_k, (A^\times)^{[k':k]})$ is the image of the p-adic cyclotomic character via the diagonal embedding $A^\times \hookrightarrow (A^\times)^{[k':k]}$. The conclusion follows from the commutativity of the following diagram

$$\begin{array}{ccc} H^1_{\mathrm{cont}}\Big(\Gamma_k, \widehat{\widetilde{H}}(A)\Big) & \longrightarrow & H^1_{\mathrm{cont}}\Big(\Gamma_k, (\widehat{\mathrm{Res}_{k'/k}\mathbb{G}_m})(A)\Big) \\ \downarrow \wr & & \downarrow \wr \\ H^1_{\mathrm{cont}}\Big(\Gamma_{k'}, \widehat{\widetilde{H}'}(A)\Big) & \longrightarrow & H^1_{\mathrm{cont}}\Big(\Gamma_{k'}, \widehat{\mathbb{G}_m}(A)\Big) \end{array}$$

and from the fact that the classes corresponding to the cyclotomic characters correspond under the right vertical arrow. □

Lemma 2.1.4.4. *Let ρ be an L-parameter, resp. a C-parameter, of H over A and ρ' the L-parameter, resp. C-parameter, of H' over A corresponding to ρ by Lemma 2.1.4.3. Let $\xi_{H'} \in X(T_{\widehat{H'}})$ be as in* (28) (*with H' instead of H*) *and let $\xi_H \in X(T_{\widehat{H}}) \cong X(T_{\widehat{H'}})^{\Sigma_{k'}}$ be the character $(\xi_{H'})_{\tau\in\Sigma_{k'}}$ (which is fixed by Γ_k). Then we have an isomorphism of representations of Γ_k over A:*

$$L^{\otimes}_{\xi_H}(\rho) \cong \mathrm{ind}^{\otimes k}_{k'}\Big(L^{\otimes}_{\xi_{H'}}(\rho')\Big) \quad \textit{resp.}\ L^{\otimes,C}_{\xi_H}(\rho) \cong \mathrm{ind}^{\otimes k}_{k'}\Big(L^{\otimes,C}_{\xi_{H'}}(\rho')\Big).$$

Proof. Let ρ' be an L-parameter of H' over A. If $g \in \Gamma_k$ and $\tau \in \Sigma_{k'}$, let $gg_\tau = g_{g\circ\tau}h(g,\tau)$ with $h(g,\tau) \in \Gamma_{k'}$. For $g \in \Gamma_k$, we can check that the above automorphism M_g of $V^{\otimes}_{\xi_H} = (V^{\otimes}_{\xi_{H'}})^{\otimes[k':k]}$ is defined by $M_g(\otimes_{\tau\in\Sigma_{k'}} v_\tau) = \otimes_{\tau\in\Sigma_{k'}}(M_{h(g,g^{-1}\circ\tau)} v_{g^{-1}\circ\tau})$. Moreover, setting for $g \in \Gamma_k$:

$$\rho(g) \stackrel{\mathrm{def}}{=} \Big((\rho'(h(g, g^{-1}\circ\tau))_{\tau\in\Sigma_{k'}}, g\Big) \in \widehat{H'}(A)^{\Sigma_{k'}} \rtimes \Gamma_k$$

it is easy to check that ρ is an admissible morphism and that the equivalence class of ρ corresponds to ρ' via Lemma 2.1.4.3. The result follows from an explicit computation together with the definition of the tensor induction ([CR81, §13], see also the end of the proof of Lemma 2.4.2.3 below). The case of C-parameters can be deduced from the case of L-parameters as in the proof of Lemma 2.1.4.3. $\square$

We will later need to "untwist" a C-parameter into an L-parameter. This can be done when the group H has a twisting element (as we assumed in §2.1.1), i.e. a character $\theta_H \in X(T_H)^{\Gamma_k} \cong X^\vee(T_{\widehat{H}})^{\Gamma_k}$ such that $\langle \theta_H, \alpha^\vee \rangle = 1$ for all $\alpha \in S_H$. By [Zhu, (1.3)], there exists a Galois equivariant isomorphism $\widehat{\widetilde{H}} \cong \widehat{H} \times \mathbb{G}_m$ given explicitly by

$$t_{\theta_H} : \begin{array}{ccc} \widehat{H} \rtimes \mathbb{G}_m & \cong & \widehat{H} \times \mathbb{G}_m \\ (h,t) & \mapsto & (h\theta_H(t), t). \end{array}$$

This induces an isomorphism of group schemes ${}^C H \cong {}^L H \times \mathbb{G}_m$. The choice of θ_H gives a bijection between the equivalence classes of C-parameters and of L-parameters of H over A given by $\rho^C \mapsto \rho$, so that $t_{\theta_H} \circ \rho^C \cong (\rho, \varepsilon)$, where ε is (the image in $A^\times$) of the p-adic cyclotomic character.

Let $\xi_H \in X^\vee(T_H)^{\Gamma_k} \cong X(T_{\widehat{H}})^{\Gamma_k}$ be a dominant character of $\widehat{H}$ fixed by Γ_k as above. The algebraic representation $r_{\xi_{\widetilde{H}}} \circ t_{\theta_H}^{-1}$ of $\widehat{H} \times \mathbb{G}_m$ (see (31) for $\xi_{\widetilde{H}}$) is the representation of highest weight $(\xi_H, -\langle \xi_H, \theta_H \rangle)$ and similarly $L^{\otimes}_{\xi_{\widetilde{H}}} \circ t_{\theta_H}^{-1} = L^{\otimes}_{\xi_H} \otimes x^{-\langle \xi_H, \theta_H \rangle}$ (where we note x^h the character $x \mapsto x^h$ of $\mathbb{G}_m$). This proves that we have

$$L^{\otimes, C}_{\xi_H}(\rho^C) \cong L^{\otimes}_{\xi_H}(\rho) \otimes \varepsilon^{-\langle \xi_H, \theta_H \rangle}. \tag{32}$$

On order to state the reformulation/generalization Conjecture 2.1.3.1 (more precisely of its variant in Remark 2.1.3.2(i) and extending scalars from $\mathbb{F}$ to $\overline{\mathbb{F}}_p$), we broaden the global setting of §2.1.2 following [DPS].

We now let H be a connected reductive group defined over $\mathbb{Q}$. We fix some compact open subgroup $U^p \subseteq H(\mathbb{A}_{\mathbb{Q}}^{\infty,p})$ satisfying the hypotheses of [DPS, §9.2] (U^p there is denoted K_f^p). For $i \geq 0$ an integer, let $\widetilde{H}^i(\mathbb{F}_p)$ be the completed cohomology of the tower of locally symmetric spaces associated to H of tame level U^p defined in [Eme06] (see [DPS, §9.2]). Let Σ be a set of finite places of $\mathbb{Q}$ containing p and the places of $\mathbb{Q}$ where H is not unramified or U^p is not hyperspecial. Let $\mathbb{T}^\Sigma$ be the abstract Hecke algebra defined as the tensor product of the spherical $\mathbb{Z}[p^{-1}]$-Hecke algebras $\mathcal{H}_\ell$ of $H(\mathbb{Q}_\ell)$ with respect to U^p_ℓ. We recall that a maximal open ideal $\mathfrak{m} \subseteq \mathbb{T}^\Sigma$ is *weakly non-Eisenstein* [DPS, Def.9.13] if the equivalent assumptions of [DPS, Lemma 9.10] are satisfied. In this case there is a unique $q_0 \geq 0$ such that $\widetilde{H}^{q_0}(\mathbb{F}_p)_{\mathfrak{m}} \neq 0$. Then the $H(\mathbb{Q}_p)$-representation $\widetilde{H}^{q_0}(\mathbb{F}_p)[\mathfrak{m}]$ is smooth and admissible and the residue field of $\mathfrak{m}$ is finite. We choose an embedding $\mathbb{T}/\mathfrak{m} \hookrightarrow \overline{\mathbb{F}}_p$.

Considering [DPS, Conj.9.3.1], the following construction is natural. Let $\overline{r}^C$: $\mathrm{Gal}(\overline{\mathbb{Q}}/\mathbb{Q}) \to {}^C H(\overline{\mathbb{F}}_p)$ be a C-parameter unramified outside a finite number of primes and choose Σ big enough to contain all the primes of ramification of $\overline{r}^C$. For each

$\ell \notin \Sigma$, let $x_\ell : \mathcal{H}_\ell \to \overline{\mathbb{F}}_p$ be the character such that the semisimplification of $\overline{r}^C(\mathrm{Frob}_\ell)$ is contained in the $\widehat{H}(\overline{\mathbb{F}}_p)$-conjugacy class $CC(x_\ell)\zeta(\ell^{-1})$ of ${}^C H(\overline{\mathbb{F}}_p)$ defined by the version of Satake isomorphism for C-groups in [Zhu] and ζ is the cocharacter $t \mapsto (2\delta_{\mathrm{ad}}(t^{-1}), t^2)$ of $\widehat{\widehat{H}}$ (recall δ_{ad} is defined at the beginning of this section). We define $\mathfrak{m}^\Sigma$ as the maximal ideal of $\mathbb{T}^\Sigma$ generated by the kernels of all the x_ℓ with $\ell \notin \Sigma$. Note that this gives us a natural embedding $\mathbb{T}^\Sigma/\mathfrak{m}^\Sigma \hookrightarrow \overline{\mathbb{F}}_p$.

Assume that $H_{\mathbb{Q}_p} \stackrel{\mathrm{def}}{=} H \times_{\mathbb{Q}} \mathbb{Q}_p$ is isomorphic to $\mathrm{Res}_{K/\mathbb{Q}_p}(H')$ for a finite extension K of $\mathbb{Q}_p$ and some split connected reductive group H' over K (in particular $H_{\mathbb{Q}_p}$ is quasi-split) and that H' has a connected center. Then we can fix a cocharacter $\xi_{H'}$ of H' such that $\langle \xi_{H'}, \alpha\rangle = 1$ for all $\alpha \in S_{H'}$ and define $\xi_{H_{\mathbb{Q}_p}} \stackrel{\mathrm{def}}{=} \mathrm{Res}_{K/\mathbb{Q}_p}(\xi_{H'})|_{\mathbb{G}_m}$ (restriction to the diagonal embedding $\mathbb{G}_m \hookrightarrow \mathrm{Res}_{K/\mathbb{Q}_p}(\mathbb{G}_m) = \mathbb{G}_m^{[K:\mathbb{Q}_p]}$), which is a cocharacter of $H_{\mathbb{Q}_p}$ satisfying $\langle \xi_{H_{\mathbb{Q}_p}}, \alpha\rangle = 1$ for all $\alpha \in S_{H_{\mathbb{Q}_p}}$. We can finally conjecture:

Conjecture 2.1.4.5. *Assume that the $H(\mathbb{Q}_p)$-representation $\pi \stackrel{\mathrm{def}}{=} \widetilde{H}^{q_0}(\mathbb{F})[\mathfrak{m}^\Sigma]$ is nonzero. Then $D^\vee_{\xi_{H'}}(\pi)$ (defined similarly to (15)) is finite-dimensional over $\overline{\mathbb{F}}_p(\!(X)\!)$ and there is an integer $d \in \mathbb{Z}_{>0}$ such that we have an isomorphism of representations of* $\mathrm{Gal}(\overline{\mathbb{Q}}_p/\mathbb{Q}_p)$ *over $\overline{\mathbb{F}}_p$:*

$$\mathbf{V}^\vee\Big(D^\vee_{\xi_{H'}}(\pi)\Big) \otimes_{\mathbb{T}^\Sigma/\mathfrak{m}^\Sigma} \overline{\mathbb{F}}_p \cong \Big(L^{\otimes, C}_{\xi_{H_{\mathbb{Q}_p}}}\Big(\overline{r}^C|_{\mathrm{Gal}(\overline{\mathbb{Q}}_p/\mathbb{Q}_p)}\Big)\Big)^{\oplus d}.$$

We now check that, when H is the restriction of scalars of a compact unitary group as in §2.1.2, Conjecture 2.1.4.5 is equivalent to the special case of Conjecture 2.1.3.1 in Remark 2.1.3.2(i) where the coefficient field is $\overline{\mathbb{F}}_p$ instead of $\mathbb{F}$.

We go back to the notation of §§2.1.2, 2.1.3 and we fix an embedding $\mathbb{F} \hookrightarrow \overline{\mathbb{F}}_p$. For simplification we assume that there is a unique place v of F^+ over p and we fix $\tilde{v}$ in F above v, so that we have an isomorphism $(\mathrm{Res}_{F^+/\mathbb{Q}} H) \times_{\mathbb{Q}} \mathbb{Q}_p \cong \mathrm{Res}_{F_{\tilde{v}}/\mathbb{Q}_p} \mathrm{GL}_n$. The base field k at the beginning is now F^+, the connected reductive group H over k is the compact unitary group H of §2.1.2 (so that $\widehat{H} \cong G = \mathrm{GL}_n$), ξ_H is the cocharacter ξ_G of Example 2.1.1.3, the twisting element θ_H is the character θ_G of Example 2.1.1.3 and the algebraically closed field A is $\overline{\mathbb{F}}_p$.

Let $\overline{r}$ be a continuous irreducible representation $\mathrm{Gal}(\overline{\mathbb{Q}}/F) \to \mathrm{GL}_n(\overline{\mathbb{F}}_p)$ as in §2.1.3 (composed with our embedding $\mathbb{F} \hookrightarrow \overline{\mathbb{F}}_p$). Let $\overline{r}' : \mathrm{Gal}(\overline{\mathbb{Q}}/F^+) \to \mathcal{G}_n(\overline{\mathbb{F}}_p)$ be the continuous morphism associated to $\overline{r}$ using [CHT08, Lemma 2.1.4] and denote by $(\overline{r}')^C : \mathrm{Gal}(\overline{\mathbb{Q}}/F^+) \to {}^C H(\overline{\mathbb{F}}_p)$ the C-parameter of H over $\overline{\mathbb{F}}_p$ obtained by the construction of [BG14, §8.3]. A simple computation shows that $(\overline{r}')^C$ (or more precisely its composition with $\widehat{\widehat{H}} \rtimes \mathrm{Gal}(\overline{\mathbb{Q}}/F^+) \twoheadrightarrow \widehat{\widehat{H}} \rtimes \mathrm{Gal}(F/F^+)$) is the composition of $(\overline{r}', \omega)$ with

$$\begin{array}{ccc} \mathcal{G}_n \times \mathbb{G}_m & \longrightarrow & \widehat{H} \rtimes (\mathbb{G}_m \times \mathrm{Gal}(F/F^+)) \\ (g, \mu, \gamma, \lambda) & \longmapsto & (g\theta'_H(\lambda)^{-1}, \lambda, \gamma) \end{array} \tag{33}$$

where $g \in \mathrm{GL}_n(\overline{\mathbb{F}}_p)$, $\mu, \lambda \in \overline{\mathbb{F}}_p^\times$, $\gamma \in \mathrm{Gal}(F/F^+)$ and $\theta'_H \in X(T)$ is the character $\theta'_H(\mathrm{diag}(x_1, \ldots, x_n)) = x_2^{-1}x_3^{-2} \cdots x_n^{1-n}$. Finally we define $\overline{r}^C$ as the C-parameter of $\mathrm{Res}_{F^+/\mathbb{Q}}(H)$ over $\overline{\mathbb{F}}_p$ obtained from the application of Lemma 2.1.4.3 to $(\overline{r}')^C$. We can check that the maximal ideal $\mathfrak{m}^\Sigma$ of $\mathbb{T}^\Sigma$ defined by $\overline{r}^C$ coincides with the ideal $\mathfrak{m}^\Sigma$ defined in §2.1.3. This can be checked using the formulas relating the Satake isomorphism for C-groups with the usual Satake isomorphism ([Zhu, §1.4]) and the explicit formulas [Gro98, (3.13)], [Gro98, (3.14)].

Note that, seeing now θ_H and θ'_H as *cocharacters* of T (recall $\widehat{\mathrm{GL}_n} \cong \mathrm{GL}_n$), we have $\theta_H \circ \omega = (\theta'_H \circ \omega)\omega^{n-1}$, so that we have, using (33):

$$(\overline{r}')^C = t_{\theta_H}^{-1} \circ ((\overline{r} \otimes \omega^{n-1}), \omega).$$

Let $\xi_v \stackrel{\mathrm{def}}{=} \xi_H \times_{F^+} F_v^+$ and $\xi_p \stackrel{\mathrm{def}}{=} \mathrm{Res}_{F_v^+/\mathbb{Q}_p}(\xi_v)|_{\mathbb{G}_m}$. Then (32) and Lemma 2.1.4.4 imply (note that ξ_v is fixed by $\mathrm{Gal}(\overline{\mathbb{Q}}_p/F_v^+)$ since $H \times_{F^+} F_v^+$ is split):

$$L_{\xi_p}^{\otimes, C}(\overline{r}^C|_{\mathrm{Gal}(\overline{\mathbb{Q}}_p/\mathbb{Q}_p)}) \cong \mathrm{ind}_{F_v^+}^{\otimes \mathbb{Q}_p}\left(r_{\xi_v}^{\otimes}(\overline{r}_{\tilde{v}} \otimes \omega^{n-1})\omega^{-\langle \xi_H, \theta_H \rangle}\right) = \overline{L}^{\otimes}(\overline{r}_{\tilde{v}}) \otimes \delta_G^{-1}.$$

This shows that Conjecture 2.1.4.5 is equivalent to the special case of Conjecture 2.1.3.1 in Remark 2.1.3.2(i) (with $\overline{\mathbb{F}}_p$ instead of $\mathbb{F}$).

2.2 Good subquotients of $\overline{L}^{\otimes}$

From now on we assume that K is unramified (i.e. $K = \mathbb{Q}_{p^f}$), and we remind the reader that $G = \mathrm{GL}_n /\mathbb{Z}$. We define the algebraic representation $\overline{L}^{\otimes}$ of $\prod_{\sigma \in \mathrm{Gal}(K/\mathbb{Q}_p)} G$ together with "good subquotients" of $\overline{L}^{\otimes}$, and prove various properties of these good subquotients. This section is entirely on the "Galois side" (though no Galois representation appears yet). All the results, except Remark 2.2.3.12, in fact hold for any split reductive connected algebraic group $G/\mathbb{Z}$ with connected center.

2.2.1 Definition and first properties

We define good subquotients of $\overline{L}^{\otimes}$.

If H is an algebraic group over $\mathbb{Z}$, we now write H instead of $H \times_{\mathbb{Z}} \mathbb{F}$ (in order not to burden the notation) and $H^{\mathrm{Gal}(K/\mathbb{Q}_p)}$ for the group product $\prod_{\sigma \in \mathrm{Gal}(K/\mathbb{Q}_p)} H$ (it is not a subgroup of H!).

We define the following algebraic representation of $G^{\mathrm{Gal}(K/\mathbb{Q}_p)}$ over $\mathbb{F}$:

$$\overline{L}^{\otimes} \stackrel{\mathrm{def}}{=} \bigotimes_{\mathrm{Gal}(K/\mathbb{Q}_p)} \Big(\bigotimes_{\alpha \in S} \overline{L}(\lambda_\alpha) \Big) \tag{34}$$

(recall that $\overline{L}(\lambda_\alpha)$ is defined in (19) and (20)). Note that $\overline{L}^{\otimes}$ is also the tensor product of all fundamental representations of the product group $G^{\mathrm{Gal}(K/\mathbb{Q}_p)}$. In particular the center $Z_G^{\mathrm{Gal}(K/\mathbb{Q}_p)}$ acts on $\overline{L}^{\otimes}$ by the character $\underbrace{\theta_G|_{Z_G} \otimes \cdots \otimes \theta_G|_{Z_G}}_{\mathrm{Gal}(K/\mathbb{Q}_p)}$, where θ_G is as in Example 2.1.1.3, i.e.

$$\theta_G = \sum_{\alpha \in S} \lambda_\alpha \in X(T). \tag{35}$$

Remark 2.2.1.1. (i) With the notation of §2.1.4, the representation $\overline{L}^{\otimes}$ is the restriction to $\widehat{H}$ of the representation $(L^{\otimes}_{\xi_H}, V^{\otimes}_{\xi_H})$ of ${}^L H$, where $k = \mathbb{F}$, $H = \mathrm{Res}_{K/\mathbb{Q}_p}(G)$ and $\xi_H = (\xi_G, \ldots, \xi_G) \in X(T_{\widehat{H}})$ (ξ_G as in Example 2.1.1.3).
(ii) Since $\lambda_\alpha \in \oplus_{i=1}^n \mathbb{Z}_{\geq 0} e_i$ by (20), all the weights of $X(T)$ appearing in each $\overline{L}(\lambda_\alpha)|_T$ are also in $\oplus_{i=1}^n \mathbb{Z}_{\geq 0} e_i$, and thus the same holds for the weights of $\overline{L}^{\otimes}|_T$ (where T is diagonally embedded into $G^{\mathrm{Gal}(K/\mathbb{Q}_p)}$). This follows from the classical fact that the weights appearing in $\overline{L}(\lambda)|_T$ for any dominant $\lambda \in X(T)$ are the points in $\oplus_{i=1}^n \mathbb{Z} e_i \cong X(T)$ of the convex hull in $\oplus_{i=1}^n \mathbb{R} e_i$ of the weights $w(\lambda)$, $w \in W$.

Fix P a standard parabolic subgroup of G, if R is a finite-dimensional algebraic representation of $P^{\mathrm{Gal}(K/\mathbb{Q}_p)}$ over $\mathbb{F}$, we write $R|_{Z_{M_P}}$ for the restriction of R to Z_{M_P} acting via the diagonal embedding

$$Z_{M_P} \hookrightarrow Z_{M_P}^{\mathrm{Gal}(K/\mathbb{Q}_p)} \subseteq G^{\mathrm{Gal}(K/\mathbb{Q}_p)}. \tag{36}$$

Since Z_{M_P} is a torus, it follows from [Jan03, §I.2.11] that $R|_{Z_{M_P}}$ is the *direct sum* of its isotypic components. For instance, if $P = G$ and $R = \overline{L}^{\otimes}$, there is only one isotypic component as $Z_{M_G} = Z_G$ acts on $\overline{L}^{\otimes}$ via the character $f\theta_G|_{Z_G}$.

Lemma 2.2.1.2. *Any isotypic component of $R|_{Z_{M_P}}$ carries an action of $M_P^{\mathrm{Gal}(K/\mathbb{Q}_p)}$ when viewed inside $R|_{M_P^{\mathrm{Gal}(K/\mathbb{Q}_p)}}$.*

Proof. This just comes from the fact that the action of Z_{M_P} commutes with that of $M_P^{\mathrm{Gal}(K/\mathbb{Q}_p)}$. □

Definition 2.2.1.3. Let $\widetilde{P} \subseteq P$ be a Zariski closed algebraic subgroup containing M_P and R an algebraic representation of $P^{\mathrm{Gal}(K/\mathbb{Q}_p)}$ over $\mathbb{F}$, a subquotient (resp. subrepresentation, resp. quotient) of $R|_{\widetilde{P}^{\mathrm{Gal}(K/\mathbb{Q}_p)}}$ is a *good* subquotient (resp. subrepresentation, resp. quotient) if its restriction to Z_{M_P} is a (direct) sum of isotypic components of $R|_{Z_{M_P}}$.

Remark 2.2.1.4. A Zariski closed subgroup $\widetilde{P}$ as in Definition 2.2.1.3 actually determines the standard parabolic subgroup P that contains it. Indeed, assume there is another standard parabolic subgroup P' such that $M_{P'} \subseteq \widetilde{P} \subseteq P'$. Then we have $M_{P'} \subseteq P$ which implies $P' \subseteq P$. Symmetrically, we also have $P \subseteq P'$, hence $P = P'$.

Since isotypic components of $R|_{Z_{M_P}}$ tautologically occur with multiplicity 1, we see in particular that there is only a finite number of good subquotients of $R|_{\widetilde{P}^{\mathrm{Gal}(K/\mathbb{Q}_p)}}$. For instance the entire $\overline{L}^{\otimes}$ is the only good subquotient of $\overline{L}^{\otimes}|_{G^{\mathrm{Gal}(K/\mathbb{Q}_p)}}$. If $\widetilde{\widetilde{P}} \subseteq \widetilde{P}$ is another Zariski closed algebraic subgroup as in Definition 2.2.1.3, any good subquotient (resp. subrepresentation, resp. quotient) of $R|_{\widetilde{P}^{\mathrm{Gal}(K/\mathbb{Q}_p)}}$ is a good subquotient (resp. subrepresentation, resp. quotient) of $R|_{\widetilde{\widetilde{P}}^{\mathrm{Gal}(K/\mathbb{Q}_p)}}$ (but the converse is wrong).

Lemma 2.2.1.5. *There exists a filtration on $\overline{L}^{\otimes}|_{\widetilde{P}^{\mathrm{Gal}(K/\mathbb{Q}_p)}}$ by good subrepresentations such that the graded pieces exhaust the isotypic components of $\overline{L}^{\otimes}|_{Z_{M_P}}$ seen as representations of $\widetilde{P}^{\mathrm{Gal}(K/\mathbb{Q}_p)}$ via the surjection $\widetilde{P}^{\mathrm{Gal}(K/\mathbb{Q}_p)} \twoheadrightarrow M_P^{\mathrm{Gal}(K/\mathbb{Q}_p)}$ and Lemma 2.2.1.2.*

Proof. It is enough to prove the lemma for $\widetilde{P} = P$. We prove the following statement (which implies the lemma): let H be a split connected reductive algebraic group over $\mathbb{Z}$ with connected center, $T_H \subseteq H$ a split maximal torus in H, $B_H \subseteq H$ a Borel subgroup containing T_H with set of (positive) roots R_H^+, V a finite-dimensional H-module over $\mathbb{F}$, $Q_H \subseteq H$ a parabolic subgroup containing B_H with Levi decomposition $M_{Q_H}N_{Q_H}$ and center $Z_{M_{Q_H}} \subseteq T_H$, $Z'_{M_{Q_H}}$ a subtorus of $Z_{M_{Q_H}}$ and $\lambda'_{Q_H} \in X(Z'_{M_{Q_H}}) \stackrel{\mathrm{def}}{=} \mathrm{Hom}_{\mathrm{Gr}}(Z'_{M_{Q_H}}, \mathbb{G}_{\mathrm{m}})$. Then the $Z'_{M_{Q_H}}$-isotypic component $V_{\lambda'_{Q_H}}$ of V is a quotient of two subrepresentations in $V|_{Q_H}$ which are both direct sums of isotypic components of $V|_{Z'_{M_{Q_H}}}$ (one applies this result to $H = G^{\mathrm{Gal}(K/\mathbb{Q}_p)}$, $V = \overline{L}^{\otimes}$, $Q_H = P^{\mathrm{Gal}(K/\mathbb{Q}_p)}$ and $Z'_{M_{Q_H}} = Z_{M_P}$). Note that as above $V = \oplus_{\lambda'_{Q_H}} V_{\lambda'_{Q_H}}$ and that $V_{\lambda'_{Q_H}}$ carries from $V|_{M_{Q_H}}$ an action of M_{Q_H} by the same proof as for Lemma 2.2.1.2. Let $R(Q_H)^+ \subseteq R_H^+$ be the positive roots of M_{Q_H}, if $\alpha \in R_H^+ \backslash R(Q_H)^+$, denote by $\overline{\alpha}$ its image via the quotient map $X(T_H) \twoheadrightarrow X(Z'_{M_{Q_H}})$ and $N_\alpha \subseteq N_{Q_H}$ the root subgroup. If $n_\alpha \in N_\alpha$ and $\lambda'_{Q_H} \in X(Z'_{M_{Q_H}})$, then we have $n_\alpha(V_{\lambda'_{Q_H}}) \subseteq \sum_{i=0}^{+\infty} V_{\lambda'_{Q_H}+i\overline{\alpha}}$ by [Jan03, §II.1.19] (the sum being finite inside V). Fix $\lambda'_{Q_H} \in X(Z'_{M_{Q_H}})$ that occurs in $V|_{Z'_{M_{Q_H}}}$ and let $\mathcal{W}(\lambda'_{Q_H})$ be the set of $\lambda''_{Q_H} \in X(Z'_{M_{Q_H}})$ of the form $\lambda'_{Q_H} + \left(\sum_{\alpha \in R_H^+ \backslash R(Q_H)^+} \mathbb{Z}_{\geq 0}\overline{\alpha}\right)$ that occur in $V|_{Z'_{M_{Q_H}}}$, we deduce that both subspaces

$$\sum_{\lambda''_{Q_H} \in \mathcal{W}(\lambda'_{Q_H}) \backslash \{\lambda'_{Q_H}\}} V_{\lambda''_{Q_H}} \subsetneq \sum_{\lambda''_{Q_H} \in \mathcal{W}(\lambda'_{Q_H})} V_{\lambda''_{Q_H}}$$

are preserved by N_{Q_H}, hence by Q_H, inside V. Since their cokernel is exactly $V_{\lambda'_{Q_H}}$, this proves the statement. $\square$

We will use the following lemma extensively.

Lemma 2.2.1.6. *If Q is a (standard) parabolic subgroup of G containing P, any isotypic component of $R|_{Z_{M_Q}}$ is a good subquotient of $R|_{P^{\mathrm{Gal}(K/\mathbb{Q}_p)}}$ (hence of $R|_{\widetilde{P}^{\mathrm{Gal}(K/\mathbb{Q}_p)}}$).*

Proof. By Lemma 2.2.1.5 (applied in the case $\widetilde{P} = P$ and with P there being Q), such an isotypic component is a good subquotient of $R|_{Q^{\mathrm{Gal}(K/\mathbb{Q}_p)}}$, and thus is a subquotient of $R|_{P^{\mathrm{Gal}(K/\mathbb{Q}_p)}}$ since $P \subseteq Q$. It is also obviously a direct sum of isotypic components of $R|_{Z_{M_P}}$ since $Z_{M_Q} \subseteq Z_{M_P}$. This proves the lemma. $\square$

Remark 2.2.1.7. Let $\widetilde{P}$, P and R as in Definition 2.2.1.3 and define a good subquotient of $R|_{\widetilde{P}}$ (for the diagonal embedding $\widetilde{P} \hookrightarrow \widetilde{P}^{\mathrm{Gal}(K/\mathbb{Q}_p)}$ similar to (36)) as a subquotient of $R|_{\widetilde{P}}$ such that its restriction to Z_{M_P} is a sum of isotypic components of $R|_{Z_{M_P}}$. Then, using the same kind of argument as for the proof of Lemma 2.2.1.5, one can prove that a good subquotient of $R|_{\widetilde{P}}$ is also a good subquotient of $R|_{\widetilde{P}^{\mathrm{Gal}(K/\mathbb{Q}_p)}}$, so that good subquotients of $R|_{\widetilde{P}}$ and of $R|_{\widetilde{P}^{\mathrm{Gal}(K/\mathbb{Q}_p)}}$ are actually the same.

2.2.2 The parabolic group associated to an isotypic component

Fix $P \subseteq G$ a standard parabolic subgroup and C_P an isotypic component of $\overline{L}^{\otimes}|_{Z_{M_P}}$, we associate to C_P a subset of the set of simple roots S (see (38)), as well as the standard parabolic subgroup of G, denoted by $P(C_P)$, corresponding to this subset.

We will use the following two lemmas, the first being well-known.

Lemma 2.2.2.1. *Let $\lambda \in X(T) \otimes_{\mathbb{Z}} \mathbb{Q}$ be dominant. Then the Weyl group of the root subsystem of R generated by the simple roots $\alpha \in S$ such that s_α fixes λ is the subgroup of W of elements fixing λ.*

Lemma 2.2.2.2. *Let $\alpha \in S$. Then $\sum_{w \in W(P)} w(\alpha) \geq 0$, and we have $\sum_{w \in W(P)} w(\alpha) = 0$ if and only if $\alpha \in S(P)$. Moreover, if $\alpha \in S \backslash S(P)$, then α is in the support of $\sum_{w \in W(P)} w(\alpha)$.*

Proof. If $\alpha \in S(P)$, it is clear that $\sum_{w \in W(P)} w(\alpha) = 0$ since, for each $w \in W(P)$, we also have $ws_\alpha \in W(P)$. If $\alpha \in S \backslash S(P)$, then $-\alpha$ is dominant for M_P, that is, $-\langle \alpha, \beta \rangle \geq 0$ for $\beta \in S(P)$. This implies that $w(-\alpha) \leq -\alpha$ for $w \in W(P)$. Summing over $W(P)$ gives $-\sum_{w \in W(P)} w(\alpha) \leq -|W(P)|\alpha$ or equivalently $|W(P)|\alpha \leq \sum_{w \in W(P)} w(\alpha)$. This proves the lemma. $\square$

If $w \in W$ satisfies $w(S(P)) \subseteq S$, we denote by ${}^wP$ the standard parabolic subgroup of G whose associated set of simple roots is $w(S(P))$. It has Levi subgroup $M_{{}^wP} = wM_Pw^{-1}$ (so ${}^wP = (wM_Pw^{-1})N$) and Weyl group $W({}^wP) = wW(P)w^{-1}$ (caution: ${}^wP$ is not wPw^{-1} if $w \neq 1$!). If $\lambda \in X(T)$, we define

$$\lambda' \stackrel{\text{def}}{=} \frac{1}{|W(P)|} \sum_{w' \in W(P)} w'(\lambda) \ \in \ (X(T) \otimes_{\mathbb{Z}} \mathbb{Q})^{W(P)}. \tag{37}$$

Remark 2.2.2.3. (i) Note that λ' only depends on $\lambda|_{Z_{M_P}}$ since two distinct λ with the same restriction to Z_{M_P} differ by an element in $\sum_{\alpha\in S(P)}\mathbb{Z}\alpha$ and since $\sum_{w'\in W(P)}w(\alpha)=0$ for $\alpha\in S(P)$ by Lemma 2.2.2.2.
(ii) It easily follows from the definitions and Lemma 2.2.2.2 that if $w\in W$ satisfies $w(S(P))\subseteq S$ and $\lambda\in X(T)$ is any weight, then $w(\lambda')=(w(\lambda))'$, where $(w(\lambda))'$ is given by the same formula as in (37) applied to the parabolic ${}^wP$ and the character $w(\lambda)$.

Lemma 2.2.2.4. *Let P be a standard parabolic subgroup of G.*

(i) *Let $\lambda\in X(T)$, there exists $w\in W$ such that $w(S(P))\subseteq S$ and $w(\lambda)|_{Z_{M_{wP}}}$ coincides with the restriction to $Z_{M_{wP}}$ of a dominant weight of $X(T)\otimes_{\mathbb{Z}}\mathbb{Q}$.*

(ii) *Let $\lambda\in X(T)$ such that $\lambda|_{Z_{M_P}}$ occurs in $\overline{L}^{\otimes}|_{Z_{M_P}}$ and let w as in (i). Then we have $f\theta_G-w(\lambda)=\sum_{\alpha\in S}n_\alpha\alpha$ for some $n_\alpha\in\mathbb{Z}_{\geq 0}$ (see (35) for θ_G) and the subset*

$$w(S(P))\cup\{\alpha\in S:n_\alpha\neq 0\}\subseteq S \tag{38}$$

only depends on $\lambda|_{Z_{M_P}}$.

Proof. (i) We first claim that it is equivalent to find w such that $w(S(P))\subseteq S$ and $w(\lambda')$ is dominant with λ' as in (37). Assume we have such a w, since $w'(\lambda)|_{Z_{M_P}}=\lambda|_{Z_{M_P}}$ for all $w'\in W(P)$, we have $\lambda'|_{Z_{M_P}}=\lambda|_{Z_{M_P}}$ and thus $w(\lambda)|_{Z_{M_{wP}}}=w(\lambda')|_{Z_{M_{wP}}}$. Conversely, assume that there is w such that $w(S(P))\subseteq S$ and $w(\lambda)|_{Z_{M_{wP}}}=\mu|_{Z_{M_{wP}}}$ for some dominant μ in $X(T)\otimes_{\mathbb{Z}}\mathbb{Q}$, and set $\mu'\overset{\text{def}}{=}\frac{1}{|W(P)|}\sum_{w'\in W({}^wP)}w'(\mu)\in(X(T)\otimes_{\mathbb{Z}}\mathbb{Q})^{W({}^wP)}$. Then we have $\mu'=w(\lambda')$ by Remark 2.2.2.3(ii) and $\mu\geq\mu'$ (as $\mu\geq w'(\mu)$ for any $w'\in W$ since μ is dominant). Thus $\mu-w(\lambda')=\mu-\mu'=\sum_{\alpha\in S({}^wP)}n_\alpha\alpha$ for some $n_\alpha\in\mathbb{Q}_{\geq 0}$ (recall $\mu|_{Z_{M_{wP}}}=\mu'|_{Z_{M_{wP}}}$). This implies that

$$\langle w(\lambda'),\beta\rangle=\langle\mu,\beta\rangle-\sum_{\alpha\in S({}^wP)}n_\alpha\langle\alpha,\beta\rangle\geq 0$$

for any $\beta\in S\backslash S({}^wP)$ (as μ is dominant and $\langle\alpha,\beta\rangle\leq 0$ if $\alpha\neq\beta\in S$). Since $\langle w(\lambda'),\beta\rangle=\langle\mu',\beta\rangle=0$ for $\beta\in S({}^wP)$ (use again Lemma 2.2.2.2), we see that $w(\lambda')$ is dominant.
Now let us find such a w. First, pick $w'\in W$ such that $w'(\lambda')$ is dominant, by Lemma 2.2.2.1 applied to $w'(\lambda')$ the set of elements β in S such that s_β fixes $w'(\lambda')$ generate a root subsystem of R with corresponding Weyl group the subgroup of W of elements that fix $w'(\lambda')$. This root subsystem has two natural bases of simple roots: namely the elements β above and the elements $w'(\gamma)\in w'(S)$ such that s_γ fixes λ' (they are usually distinct as W doesn't preserve S). This second basis obviously contains $w'(S(P))$. Therefore, there is w'' in the Weyl group of this root subsystem,

i.e. $w'' \in W$ fixing $w'(\lambda')$, that maps the second basis to the first. In particular we have $w''w'(S(P)) \subseteq S$ and $w''w'(\lambda') = w'(\lambda')$ dominant, thus we can take $w \stackrel{\text{def}}{=} w''w'$.

(ii) The positivity of the n_α follows from the fact $f\theta_G$ is the highest weight of $\overline{L}^{\otimes}|_T$ (for the diagonal embedding of T as in (36)). Let w_1, w_2 as in (i) and λ' as in (37). Then $w_1(\lambda') = w_2(\lambda')$ as these two weights are dominant (by the first part of the proof of (i)) and in a single W-orbit. Since λ' only depends on $\lambda|_{Z_{M_P}}$ by Remark 2.2.2.3(i), it is therefore enough to prove that the support of $f\theta_G - w(\lambda')$ is exactly the set of simple roots (38) for one (any) w as in (i). Writing $f\theta_G - w(\lambda') = (f\theta_G - w(\lambda)) + (w(\lambda) - w(\lambda'))$ and recalling that $w(\lambda) - w(\lambda')$ is a sum of roots in $w(S(P)) \subseteq S$ (as $w(\lambda), w(\lambda')$ have same restriction to $Z_{M_{wP}}$ from the proof of (i)), we see that this support is contained in (38) and that it contains $\{\alpha \in S \backslash w(S(P)) : n_\alpha \neq 0\}$. It is thus enough to prove that this support also contains $w(S(P))$. Since $f\theta_G \geq w(\lambda')$ (use $f\theta_G \geq ww'(\lambda)$ for any $w' \in W$ and sum over $w' \in W(P)$) and $\langle \beta, \alpha \rangle \leq 0$ if $\alpha \neq \beta \in S$, it is enough to check that $\langle f\theta_G - w(\lambda'), \alpha \rangle > 0$ (in $\mathbb{Q}$) for any $\alpha \in w(S(P))$. But this follows from Lemma 2.2.2.2 and $\langle f\theta_G - w(\lambda'), \alpha \rangle = f\langle \theta_G, \alpha \rangle - \langle w(\lambda'), \alpha \rangle = f - 0 = f$. □

Remark 2.2.2.5. Note that it is not true in general that, for λ as in Lemma 2.2.2.4(ii), one can find $w \in W$ such that $w(S(P)) \subseteq S$ and $w(\lambda)|_{Z_{M_{wP}}}$ is the restriction to $Z_{M_{wP}}$ of a dominant weight of $X(T)$ (one really needs $X(T) \otimes_{\mathbb{Z}} \mathbb{Q}$).

The proof of Lemma 2.2.2.4 also gives the following equivalent proposition that we will use repeatedly in the sequel.

Proposition 2.2.2.6. *Let P be a standard parabolic subgroup of G.*

(i) *Let $\lambda \in X(T)$ and λ' as in (37), there exists $w \in W$ such that $w(S(P)) \subseteq S$ and $w(\lambda')$ is a dominant weight of $X(T) \otimes_{\mathbb{Z}} \mathbb{Q}$.*

(ii) *Let $\lambda \in X(T)$ such that $\lambda|_{Z_{M_P}}$ occurs in $\overline{L}^{\otimes}|_{Z_{M_P}}$ and let w as in (i). Then we have $f\theta_G - w(\lambda') = \sum_{\alpha \in S} n_\alpha \alpha$ for some $n_\alpha \in \mathbb{Q}_{\geq 0}$ and the support of $f\theta_G - w(\lambda')$ is $S(P(C_P))$.*

Let C_P be an isotypic component of $\overline{L}^{\otimes}|_{Z_{M_P}}$ associated to some $\lambda_P \in X(Z_{M_P}) = \mathrm{Hom}_{\mathrm{Gr}}(Z_{M_P}, \mathbb{G}_{\mathrm{m}})$. We denote by $P(C_P)$ the unique standard parabolic subgroup of G whose associated set of simple roots $S(P(C_P))$ is (38) for one (equivalently any) weight $\lambda \in X(T)$ such that $\lambda|_{Z_{M_P}} = \lambda_P$. We also define

$$W(C_P) \stackrel{\text{def}}{=} \{w \in W \text{ as in Proposition 2.2.2.6(i) for } \lambda \in X(T) : \lambda|_{Z_{M_P}} = \lambda_P\} \quad (39)$$

($W(C_P)$ doesn't depend on the choice of such λ by the claim in the proof of Lemma 2.2.2.4(i) and by Remark 2.2.2.3(i)). We see from (38) that for all $w \in W(C_P)$ we have the inclusion

$$^{w}P \subseteq P(C_P). \quad (40)$$

Note that the set $W(C_P)$ is not in general a group, in particular it is distinct in general from the Weyl group $W(P(C_P))$ (see Lemma 2.2.2.10 below for the relation between the two).

Remark 2.2.2.7. The inclusion ${}^wP \subseteq P(C_P)$ for some $w \in W$ (such that $w(S(P)) \subseteq S$) *doesn't* imply $w \in W(C_P)$ (take $P = B$). Also $P(C_P)$ doesn't necessarily contain P, see e.g. the end of Example 2.2.2.9(ii) below. The subgroup generated by all ${}^wP$ for $w \in W(C_P)$ may also be *strictly* contained in $P(C_P)$ (see e.g. Example 2.2.2.9(i) below).

The parabolic subgroups $P(C_P)$ respect inclusions.

Lemma 2.2.2.8. *Let $P' \subseteq P$ be two standard parabolic subgroups of G, C_P an isotypic component of $\overline{L}^{\otimes}|_{Z_{M_P}}$ and $C_{P'}$ an isotypic component of $\overline{L}^{\otimes}|_{Z_{M_{P'}}}$ such that $C_{P'} \subseteq C_P|_{Z_{M_{P'}}}$. Then $P(C_{P'}) \subseteq P(C_P)$.*

Proof. Let $\lambda \in X(T)$ such that $C_{P'}$ is the isotypic component of $\lambda|_{Z_{M_{P'}}}$. Then by assumption C_P is the isotypic component of $\lambda|_{Z_{M_P}}$. Define $\lambda'_P \in (X(T) \otimes_{\mathbb{Z}} \mathbb{Q})^{W(P)}$, $\lambda'_{P'} \in (X(T) \otimes_{\mathbb{Z}} \mathbb{Q})^{W(P')}$ as in (37) for respectively P and P' and let $(w_P, w_{P'}) \in W \times W$ such that $w_P(S(P)) \subseteq S$ and $w_P(\lambda'_P)$ dominant, $w_{P'}(S(P')) \subseteq S$ and $w_{P'}(\lambda'_{P'})$ dominant ($w_P, w_{P'}$ exist by Proposition 2.2.2.6(i)). Then we have

$$w_P(\lambda'_P) = \frac{1}{|W(P)|} \sum_{w' \in W({}^{w_P}P)} w' w_P(\lambda), \quad w_P(\lambda'_{P'}) = \frac{1}{|W(P')|} \sum_{w' \in W({}^{w_P}P')} w' w_P(\lambda)$$

and also

$$w_P(\lambda'_P) = \frac{|W(P')|}{|W(P)|} \sum_{\sigma \in W({}^{w_P}P)/W({}^{w_P}P')} \sigma w_P(\lambda'_{P'}). \tag{41}$$

Since $w_{P'}(\lambda'_{P'})$ is dominant, we have $w_{P'}(\lambda'_{P'}) \geq w w_{P'}(\lambda'_{P'})$ for any $w \in W$ and in particular $w_{P'}(\lambda'_{P'}) \geq \sigma w_P(\lambda'_{P'}) = (\sigma w_P w_{P'}^{-1}) w_{P'}(\lambda'_{P'})$. Summing up these inequalities over $\sigma \in W({}^{w_P}P)/W({}^{w_P}P')$ and multiplying by $\frac{|W(P')|}{|W(P)|}$, one gets with (41):

$$w_{P'}(\lambda'_{P'}) \geq w_P(\lambda'_P). \tag{42}$$

Now the result follows from

$$f\theta_G - w_P(\lambda'_P) = \Big(f\theta_G - w_{P'}(\lambda'_{P'})\Big) + \Big(w_{P'}(\lambda'_{P'}) - w_P(\lambda'_P)\Big)$$

together with Proposition 2.2.2.6(ii) and (42). □

Example 2.2.2.9. We give a few simple examples (beyond $\mathrm{GL}_2(\mathbb{Q}_p)$).

(i) Assume $n = 2$ and $P = B$. Then $\overline{L}^{\otimes}|_{Z_{M_B}} = \overline{L}^{\otimes}|_T$ has $f+1$ isotypic components $C(\lambda_i)$ given by the characters $\lambda_i : \mathrm{diag}(x_1, x_2) \mapsto x_1^{f-i} x_2^i$ for $0 \leq i \leq f$. For $i < f/2$,

λ_i is dominant, $W(C(\lambda_i)) = \{1\}$ and $f\theta_G - \lambda_i = i(e_1 - e_2)$. For $i = f/2$ (if f is even), $\lambda_i = s_{e_1-e_2}(\lambda_i)$ is dominant, $W(C(\lambda_i)) = \{1, s_{e_1-e_2}\}$ and $f\theta_G - w(\lambda_i) = f/2(e_1 - e_2)$ for $w \in W(C(\lambda_i))$. For $i > f/2$, $s_{e_1-e_2}(\lambda_i)$ is dominant, $W(C(\lambda_i)) = \{s_{e_1-e_2}\}$ and $f\theta_G - s_{e_1-e_2}(\lambda_i) = (f-i)(e_1-e_2)$. We see that ${}^wB = B \subsetneq P(C(\lambda_i)) = G$ if $i \notin \{0, f\}$ and ${}^wB = P(C(\lambda_i)) = B$ if $i \in \{0, f\}$.

(ii) Assume $n = 3$ and $K = \mathbb{Q}_p$.

If $P = B$, then $\overline{L}^{\otimes}|_T$ has 7 isotypic components given by the 6 characters λ_w : $\mathrm{diag}(x_1, x_2, x_3) \mapsto x_{w^{-1}(1)}^2 x_{w^{-1}(2)}$ for $w \in \mathcal{S}_3$ and the character $\det : \mathrm{diag}(x_1, x_2, x_3) \mapsto x_1x_2x_3$. If C_P corresponds to some λ_w, one gets that $W(C_P)$ is the singleton $\{w\}$ and $\theta_G - w(\lambda_w) = 0$, which implies ${}^wB = P(C_P) = B$. If C_P corresponds to det, one gets $W(C_P) = W$ and $\theta_G - w(\det) = (e_1 - e_2) + (e_2 - e_3)$ for $w \in W$, which implies ${}^wB = B \subsetneq P(C_P) = G$.

If P is the standard parabolic subgroup of Levi $\mathrm{diag}(\mathrm{GL}_2, \mathrm{GL}_1)$, then $\overline{L}^{\otimes}|_{Z_{M_P}}$ has 3 isotypic components C_P given by the characters

$$\lambda_0 : \mathrm{diag}(x_1, x_1, x_2) \mapsto x_1^3,\ \lambda_1 : \mathrm{diag}(x_1, x_1, x_2) \mapsto x_1^2x_2,\ \lambda_2 : \mathrm{diag}(x_1, x_1, x_2) \mapsto x_1x_2^2.$$

One has $\lambda'_0 = 3/2(e_1 + e_2)$, $\lambda'_1 = e_1 + e_2 + e_3$, $\lambda'_2 = 1/2(e_1 + e_2) + 2e_3$ from which one deduces for the three respective isotypic components C_P (where $w \in W(C_P)$):

$$\begin{array}{lllllll} W(C_P) & = & \{1\} & \theta_G - w(\lambda'_0) & = & 1/2(e_1 - e_2) \\ W(C_P) & = & \{1, s_{e_1-e_2}s_{e_2-e_3}\} & \theta_G - w(\lambda'_1) & = & (e_1 - e_2) + (e_2 - e_3) \\ W(C_P) & = & \{s_{e_1-e_2}s_{e_2-e_3}\} & \theta_G - w(\lambda'_2) & = & 1/2(e_2 - e_3). \end{array}$$

If C_P corresponds to λ_0 one gets ${}^wP = P(C_P) = P$, if C_P corresponds to λ_1 one gets ${}^wP \subsetneq P(C_P) = G$ (${}^wP$ being P if $w = \mathrm{Id}$ and the standard parabolic subgroup of Levi $\mathrm{diag}(\mathrm{GL}_1, \mathrm{GL}_2)$ if $w = s_{e_1-e_2}s_{e_2-e_3}$), and if C_P corresponds to λ_2 one gets ${}^wP = P(C_P) =$ the standard parabolic subgroup of Levi $\mathrm{diag}(\mathrm{GL}_1, \mathrm{GL}_2)$. In this last case we see that $P(C_P)$ doesn't contain P.

Finally, if $M_P = \mathrm{diag}(\mathrm{GL}_1, \mathrm{GL}_2)$, the situation is symmetric.

Lemma 2.2.2.10. *We have $W(C_P) \subseteq W(P(C_P))w$ for any fixed element $w \in W(C_P)$. Equivalently $w'w^{-1} \in W(P(C_P))$ for any $w, w' \in W(C_P)$.*

Proof. Let $\lambda_P \in X(Z_{M_P})$ corresponding to C_P, $w_{C_P} \in W(C_P)$, $\lambda \in X(T)$ such that $\lambda|_{Z_{M_P}} = \lambda_P$ and define λ' as in (37). Recall that an element $w \in W$ is in $W(C_P)$ if and only if $w(S(P)) \subseteq S$ and $w(\lambda')$ is dominant (see Proposition 2.2.2.6(i)), and that we have $w(\lambda') = w_{C_P}(\lambda')$ for all $w \in W(C_P)$ (see the beginning of the proof of Lemma 2.2.2.4(ii)). We rewrite this $ww_{C_P}^{-1}(w_{C_P}(\lambda')) = w_{C_P}(\lambda')\ \forall\, w \in W(C_P)$. By the definition of $P(C_P)$ and Proposition 2.2.2.6(ii), we know that $S(P(C_P))$ is the set of simple roots in the support of $f\theta_G - w_{C_P}(\lambda')$. Since $w_{C_P}(\lambda')$ is dominant, by Lemma

2.2.2.1 the subgroup of W fixing $w_{C_P}(\lambda')$ is generated by the simple reflections s_β fixing $w_{C_P}(\lambda')$, or equivalently such that $\langle w_{C_P}(\lambda'), \beta\rangle = 0$. Since $\langle f\theta_G - w_{C_P}(\lambda'), \beta\rangle = f - 0 = f$, we see that these simple roots β are all in the support of $f\theta_G - w_{C_P}(\lambda')$. Therefore $W(P(C_P))$ contains the subgroup of W fixing $w_{C_P}(\lambda')$. Since $ww_{C_P}^{-1}$ fixes $w_{C_P}(\lambda')$, it follows that $ww_{C_P}^{-1} \in W(P(C_P))$. $\square$

Remark 2.2.2.11. The inclusion in Lemma 2.2.2.10 is not an equality in general (take $P = G$).

2.2.3 The structure of isotypic components of $\overline{L}^{\otimes}$

We let P be a standard parabolic subgroup of G, we prove an important structure theorem on the isotypic components of $\overline{L}^{\otimes}|_{Z_{M_P}}$ (Theorem 2.2.3.9) as well as several useful technical results.

Recall that $W(C_P)$ is defined in (39) and $P(C_P)$ is defined just before.

Lemma 2.2.3.1. *If $P(C_P) = {}^wP$ for some $w \in W(C_P)$ then $W(C_P)$ has just one element.*

Proof. Let $w_{C_P} \in W(C_P)$ such that $P(C_P) = {}^{w_{C_P}}P$ and let $w'_{C_P} \in W(C_P)$. Since $P(C_P) = {}^{w_{C_P}}P$ we get $S(P(C_P)) = w_{C_P}(S(P))$ and $W(P(C_P)) = w_{C_P}W(P)w_{C_P}^{-1}$. By Lemma 2.2.2.10 applied to the element w_{C_P}, we deduce $W(C_P) \subseteq w_{C_P}W(P)$ and thus $w_{C_P}^{-1}w'_{C_P} \in W(P)$. But since $S(P(C_P))$ contains $w(S(P))$ for all $w \in W(C_P)$ by definition of $W(C_P)$ and (38), we have $w'_{C_P}(S(P)) \subseteq S(P(C_P)) = w_{C_P}(S(P))$ which implies $w'_{C_P}(S(P)) = w_{C_P}(S(P))$ since the cardinalities are the same on both sides, that is, $w_{C_P}^{-1}w'_{C_P}(S(P)) = S(P)$. Since $w_{C_P}^{-1}w'_{C_P} \in W(P)$, this forces $w'_{C_P} = w_{C_P}$. $\square$

Remark 2.2.3.2. (i) The converse to Lemma 2.2.3.1 is wrong in general (e.g. consider the $C(\lambda_i)$ with $i \notin \{0, f/2, f\}$ in Example 2.2.2.9(i)).
(ii) For a general isotypic component C_P, it is not true that one can find $w \in W(C_P)$ such that $w^{-1}M_{P(C_P)}w$ is the Levi subgroup of a standard parabolic subgroup of G.

Proposition 2.2.3.3. *The isotypic components C_P such that $P(C_P) = {}^wP$ for some (necessarily unique) $w \in W(C_P)$ are those isotypic components which are associated to $fw^{-1}(\theta_G)|_{Z_{M_P}}$ for the $w \in W$ such that $w(S(P)) \subseteq S$.*

Proof. Let $w \in W$ such that $w(S(P)) \subseteq S$ and $\lambda \overset{\text{def}}{=} fw^{-1}(\theta_G) \in X(T)$. Since $w(\lambda) = f\theta_G$ is dominant and $f\theta_G - w(\lambda) = 0$, the set (38) is $w(S(P))$. This implies $P(C_P) = {}^wP$.
Conversely, let C_P as in the statement, $\lambda \in X(T)$ such that C_P is the isotypic

component associated to the character $\lambda|_{Z_{M_P}}$ of Z_{M_P} and define λ' as in (37). Since $S(P(C_P)) = w(S(P))$ by assumption, from Proposition 2.2.2.6(ii) we obtain

$$fw^{-1}(\theta_G) - \lambda' = \sum_{\alpha \in S(P)} n_\alpha \alpha$$

(for some $n_\alpha \in \mathbb{Q}_{>0}$), which implies $fw^{-1}(\theta_G)|_{Z_{M_P}} = \lambda'|_{Z_{M_P}}$. Since $\lambda|_{Z_{M_P}} = \lambda'|_{Z_{M_P}}$ (see the beginning of the proof of Lemma 2.2.2.4(i)), we deduce that C_P is the isotypic component associated to the character $fw^{-1}(\theta_G)|_{Z_{M_P}}$. $\square$

Note that if C_P is associated to $fw^{-1}(\theta_G)|_{Z_{M_P}}$ (with $w(S(P)) \subseteq S$), we have $W(C_P) = \{w\}$ by Lemma 2.2.3.1.

Example 2.2.3.4. Coming back to Example 2.2.2.9, the isotypic components as in Proposition 2.2.3.3 are the isotypic components $C(\lambda_0)$, $C(\lambda_f)$ when $n = 2$, $P = B$, the isotypic components associated to the six λ_w when $n = 3$, $K = \mathbb{Q}_p$, $P = B$, and the isotypic components associated to λ_0, λ_2 when $n = 3$, $K = \mathbb{Q}_p$, $M_P = \mathrm{GL}_2 \times \mathrm{GL}_1$.

We set for $\alpha = e_j - e_{j+1} \in S(P)$:

$$\lambda_{\alpha,P} \stackrel{\text{def}}{=} \sum_{e_i - e_{j+1} \in R(P)^+} e_i \in X(T). \tag{43}$$

One easily checks that the $\lambda_{\alpha,P}$ for $\alpha \in S(P)$ are fundamental weights for the reductive group M_P and that $\langle \lambda_{\alpha,P}, \beta \rangle \leq 0$ for $\beta \in S \backslash S(P)$. For any $\lambda \in X(T)$, we define $\overline{L}_P(\lambda)$ as in (19) but with $(M_P, M_P \cap B^-)$ instead of (G, B^-). When $S(P) = \emptyset$, we define $\overline{L}_P^{\otimes}$ to be the trivial representation of $T^{\mathrm{Gal}(K/\mathbb{Q}_p)}$ over $\mathbb{F}$ and, when $S(P) \neq \emptyset$, we define similarly to (34) the algebraic representation of $M_P^{\mathrm{Gal}(K/\mathbb{Q}_p)}$ over $\mathbb{F}$:

$$\overline{L}_P^{\otimes} \stackrel{\text{def}}{=} \bigotimes_{\mathrm{Gal}(K/\mathbb{Q}_p)} \Big(\bigotimes_{\alpha \in S(P)} \overline{L}_P(\lambda_{\alpha,P}) \Big). \tag{44}$$

We also define

$$\theta_P \stackrel{\text{def}}{=} \sum_{\alpha \in S(P)} \lambda_{\alpha,P} \in X(T) \quad \text{and} \quad \theta^P \stackrel{\text{def}}{=} \theta_G - \theta_P \in X(T). \tag{45}$$

Since for $\alpha \in S(P)$ we have $\langle \theta^P, \alpha \rangle = \langle \theta_G, \alpha \rangle - \langle \theta_P, \alpha \rangle = 1 - 1 = 0$, we see that θ^P extends to an element of $\mathrm{Hom}_{\mathrm{Gr}}(M_P, \mathbb{G}_{\mathrm{m}})$. Likewise we have for $\alpha \in S(P)$ and $w \in W$ such that $w(S(P)) \subseteq S$:

$$\langle w^{-1}(\theta^{wP}), \alpha \rangle = \langle \theta^{wP}, w(\alpha) \rangle = 0$$

so that $w^{-1}(\theta^{wP})$ also extends to $\mathrm{Hom}_{\mathrm{Gr}}(M_P, \mathbb{G}_{\mathrm{m}})$. Note that, since $\langle \theta_P, \beta \rangle \leq 0$ for $\beta \in S \backslash S(P)$, we get $\langle \theta^P, \beta \rangle = \langle \theta_G, \beta \rangle - \langle \theta_P, \beta \rangle \geq 1$, thus θ^P is a dominant weight.

Example 2.2.3.5. If $G = \mathrm{GL}_6$ and $M_P = \mathrm{GL}_2 \times \mathrm{GL}_3 \times \mathrm{GL}_1$, one gets

$$\begin{array}{rcl}\theta_P : \mathrm{diag}(x_1, \ldots, x_6) & \longmapsto & (x_1)(x_3^2 x_4) \\ \theta^P : \mathrm{diag}(x_1, \ldots, x_6) & \longmapsto & (x_1x_2)^4(x_3x_4x_5).\end{array}$$

Lemma 2.2.3.6. *If $w \in W(P)$, we have $w(\theta^P) = \theta^P$.*

Proof. The character θ^P extends to M_P and factors through M_P/M_P^{der}. But conjugation by $W(P)$ is trivial on M_P/M_P^{der}. □

Lemma 2.2.3.7. *Let $\lambda \in X(T)$ be a dominant weight and denote by $\overline{L}(\lambda)_\mu \subseteq \overline{L}(\lambda)$ for $\mu \in X(T)$ the isotypic component of $\overline{L}(\lambda)|_T$ associated to μ (i.e. the weight space of $\overline{L}(\lambda)$ for μ, see* [Jan03, §I.2.11]*). Then*

$$\bigoplus_{\mu \in \lambda - \sum_{\alpha \in S(P)} \mathbb{Z}_{\geq 0}\alpha} \overline{L}(\lambda)_\mu \quad \subseteq \quad \overline{L}(\lambda)$$

is an M_P-subrepresentation of $\overline{L}(\lambda)|_{M_P}$ which is isomorphic to $\overline{L}_P(\lambda)$.

Proof. Since $\oplus_{\mu \in \lambda - \sum_{\alpha \in S(P)} \mathbb{Z}_{\geq 0}\alpha} \overline{L}(\lambda)_\mu$ is the isotypic component of $\overline{L}(\lambda)|_{Z_{M_P}}$ associated to $\lambda|_{Z_{M_P}}$ (as $\lambda|_{Z_{M_P}} \cong \mu|_{Z_{M_P}} \iff \lambda - \mu \in \sum_{\alpha \in S(P)} \mathbb{Z}\alpha$), it is endowed with an action of M_P by the same proof as for Lemma 2.2.1.2. By [Jan03, II.2.2(1)], [Jan03, I.6.11(2)] and the transitivity of induction ([Jan03, I.3.5(2)]), we have an injection of algebraic representations of M_P over $\mathbb{F}$:

$$H^0(N_P, \overline{L}(\lambda)) \hookrightarrow \overline{L}_P(\lambda) \tag{46}$$

(recall N_P is the unipotent radical of P) and by [Jan03, II.2.11(1)] we have an isomorphism of algebraic representations of M_P over $\mathbb{F}$:

$$\bigoplus_{\mu \in \lambda - \sum_{\alpha \in S(P)} \mathbb{Z}_{\geq 0}\alpha} \overline{L}(\lambda)_\mu \quad \xrightarrow{\sim} \quad H^0(N_P, \overline{L}(\lambda)).$$

It is therefore enough to prove that (46) is an isomorphism, or equivalently that

$$\dim_{\mathbb{F}} \left(\bigoplus_{\mu \in \lambda - \sum_{\alpha \in S(P)} \mathbb{Z}_{\geq 0}\alpha} \overline{L}(\lambda)_\mu \right) = \dim_{\mathbb{F}} \overline{L}_P(\lambda).$$

Let $L(\lambda) \stackrel{\mathrm{def}}{=} \left(\mathrm{ind}_{B^-}^G \lambda\right)_{/\mathbb{Z}} \otimes_{\mathbb{Z}} E$, $L_P(\lambda) \stackrel{\mathrm{def}}{=} \left(\mathrm{ind}_{M_P \cap B^-}^{M_P} \lambda\right)_{/\mathbb{Z}} \otimes_{\mathbb{Z}} E$ and $L(\lambda)_\mu \subseteq L(\lambda)$ the weight space associated to μ, we have $\dim_{\mathbb{F}} \overline{L}(\lambda)_\mu = \dim_E L(\lambda)_\mu$, and thus

$$\dim_{\mathbb{F}} \left(\bigoplus_{\mu \in \lambda - \sum_{\alpha \in S(P)} \mathbb{Z}_{\geq 0}\alpha} \overline{L}(\lambda)_\mu \right) = \dim_E \left(\bigoplus_{\mu \in \lambda - \sum_{\alpha \in S(P)} \mathbb{Z}_{\geq 0}\alpha} L(\lambda)_\mu \right).$$

Likewise, we have $\dim_{\mathbb{F}} \overline{L}_P(\lambda) = \dim_E L_P(\lambda)$. It is therefore enough to have

$$\dim_E \left(\bigoplus_{\mu \in \lambda - \sum_{\alpha \in S(P)} \mathbb{Z}_{\geq 0}\alpha} L(\lambda)_\mu \right) = \dim_E L_P(\lambda).$$

But now, we are over a field of characteristic 0, where it is well known that $L(\lambda)$ and $L_P(\lambda)$ as defined above are *simple modules with highest weight* λ. Then the result follows from [Jan03, Prop.II.2.11]. $\square$

The following lemma is a special case of Lemma 2.2.3.7.

Lemma 2.2.3.8. *Let $\lambda \in X(T)$ be a dominant weight such that $\langle \lambda, \alpha \rangle = 0$ for all $\alpha \in S(P)$ (equivalently λ extends to an element in $\mathrm{Hom}_{\mathrm{Gr}}(M_P, \mathbb{G}_{\mathrm{m}})$). Then any $\mu \in X(T)$ distinct from λ with $\overline{L}(\lambda)_\mu \neq 0$ is such that $\lambda - \mu$ contains at least one root of $S \backslash S(P)$ in its support.*

Proof. Since $\lambda \in \mathrm{Hom}_{\mathrm{Gr}}(M_P, \mathbb{G}_{\mathrm{m}})$, we have $\overline{L}_P(\lambda) \cong \lambda$ by (19) applied with M_P instead of G. By Lemma 2.2.3.7, we deduce $\bigoplus_{\mu \in \lambda - \sum_{\alpha \in S(P)} \mathbb{Z}_{\geq 0}\alpha} \overline{L}(\lambda)_\mu \cong \lambda$ inside $\overline{L}(\lambda)$. This clearly implies the lemma. $\square$

If R is any algebraic representation of M_P or of $M_P^{\mathrm{Gal}(K/\mathbb{Q}_p)}$ and $w \in W$ such that $w(S(P)) \subseteq S$, we define an algebraic representation of $M_{{}^wP} = wM_Pw^{-1}$ or of $M_{{}^wP}^{\mathrm{Gal}(K/\mathbb{Q}_p)} = wM_P^{\mathrm{Gal}(K/\mathbb{Q}_p)}w^{-1}$ (w acting diagonally via $W \hookrightarrow W^{\mathrm{Gal}(K/\mathbb{Q}_p)}$) by ($g \in M_{{}^wP}$ or $M_{{}^wP}^{\mathrm{Gal}(K/\mathbb{Q}_p)}$):

$$w(R)(g) \overset{\mathrm{def}}{=} R(w^{-1}gw). \tag{47}$$

If $\alpha \in S(P)$, one then easily checks that $w(\lambda_{\alpha,P}) = \lambda_{w(\alpha),{}^wP}$ and $w(\overline{L}_P(\lambda_{\alpha,P})) = \overline{L}_{{}^wP}(\lambda_{w(\alpha),{}^wP})$, from which one gets

$$w(\overline{L}_P^{\otimes}) = \overline{L}_{{}^wP}^{\otimes}. \tag{48}$$

Theorem 2.2.3.9. *Let C_P be an isotypic component of $\overline{L}^{\otimes}|_{Z_{M_P}}$, associated to $\lambda|_{Z_{M_P}}$ for $\lambda \in X(T)$. For any $w \in W(C_P)$, there is an isomorphism of algebraic representations of $M_P^{\mathrm{Gal}(K/\mathbb{Q}_p)}$ over $\mathbb{F}$:*

$$C_P \cong w^{-1}\Big(C_{P(C_P),{}^wP}\Big) \otimes \Big(\underbrace{w^{-1}(\theta^{P(C_P)}) \otimes \cdots \otimes w^{-1}(\theta^{P(C_P)})}_{\mathrm{Gal}(K/\mathbb{Q}_p)}\Big), \tag{49}$$

where $C_{P(C_P),{}^wP}$ is the isotypic component of $\overline{L}_{P(C_P)}^{\otimes}|_{Z_{M_{{}^wP}}}$ associated to $(w(\lambda) - f\theta^{P(C_P)})|_{Z_{M_{{}^wP}}}$ (thus an $M_{{}^wP}^{\mathrm{Gal}(K/\mathbb{Q}_p)}$-representation, recall ${}^wP \subseteq P(C_P)$) and $w^{-1}(C_{P(C_P),{}^wP})$ is defined in (47).

Proof. Step 1: Assuming the result holds if $w = \mathrm{Id}$, we prove it for any w. For $\mu \in X(T)$ we have $\mu|_{Z_{M_P}} = \lambda|_{Z_{M_P}}$ if and only if $w(\mu)|_{Z_{M_{wP}}} = w(\lambda)|_{Z_{M_{wP}}}$, therefore the image $w(C_P)$ of C_P for the diagonal action of $w \in W$ on $\overline{L}^{\otimes}$ is the isotypic component of $\overline{L}^{\otimes}|_{Z_{M_{wP}}}$ associated to $w(\lambda)|_{Z_{M_{wP}}}$. Note that, as an algebraic $M_{wP}^{\mathrm{Gal}(K/\mathbb{Q}_p)}$-subrepresentation of $\overline{L}^{\otimes}|_{M_{wP}^{\mathrm{Gal}(K/\mathbb{Q}_p)}}$, $w(C_P)$ is indeed isomorphic to $g \mapsto C_P(w^{-1}gw)$ if $g \in M_{wP}^{\mathrm{Gal}(K/\mathbb{Q}_p)}$, so the notation is consistent with (47). By Remark 2.2.2.3(ii) we have $w(\lambda') = (w(\lambda))'$ in $(X(T) \otimes_{\mathbb{Z}} \mathbb{Q})^{W({}^wP)}$. Recall that $w(\lambda')$, and hence $(w(\lambda))'$, are dominant since $w \in W(C_P)$ (see Proposition 2.2.2.6(i)). Therefore $\mathrm{Id} \in W(w(C_P))$ and by the case $w = \mathrm{Id}$ for the parabolic subgroup ${}^wP$ and the isotypic component $w(C_P)$, we have $w(C_P) \cong C_{P(w(C_P)),{}^wP} \otimes \left(\theta^{P(w(C_P))} \otimes \cdots \otimes \theta^{P(w(C_P))}\right)$. Moreover $S(P(w(C_P)))$, which is the support of $f\theta_G - (w(\lambda))'$ by Proposition 2.2.2.6(ii) (applied to $w = \mathrm{Id}$), is the same as $S(P_{C_P})$, which is the support of $f\theta_G - w(\lambda')$ by *loc.cit.* (applied to w), i.e. we have $P(w(C_P)) = P(C_P)$. We thus deduce $w(C_P) \cong C_{P(C_P),{}^wP} \otimes \left(\theta^{P(C_P)} \otimes \cdots \otimes \theta^{P(C_P)}\right)$ which gives (49) by applying w^{-1}.

Step 2: From now on we assume $w = \mathrm{Id}$ (so in particular $P \subseteq P(C_P)$). Writing

$$\overline{L}^{\otimes} = \Big(\bigotimes_{\mathrm{Gal}(K/\mathbb{Q}_p)} \Big(\bigotimes_{\alpha \in S(P(C_P))} \overline{L}(\lambda_\alpha)\Big)\Big) \bigotimes \Big(\bigotimes_{\mathrm{Gal}(K/\mathbb{Q}_p)} \Big(\bigotimes_{\alpha \in S \backslash S(P(C_P))} \overline{L}(\lambda_\alpha)\Big)\Big),$$

we prove that any $(\mu_1, \mu_2) \in X(T) \times X(T)$ such that

(i) μ_1 occurs in $\left(\bigotimes_{\mathrm{Gal}(K/\mathbb{Q}_p)} \left(\bigotimes_{\alpha \in S(P(C_P))} \overline{L}(\lambda_\alpha)\right)\right)|_T$ (for the diagonal action of T);

(ii) μ_2 occurs in $\left(\bigotimes_{\mathrm{Gal}(K/\mathbb{Q}_p)} \left(\bigotimes_{\alpha \in S \backslash S(P(C_P))} \overline{L}(\lambda_\alpha)\right)\right)|_T$ (idem);

(iii) $\mu_1|_{Z_{M_P}} + \mu_2|_{Z_{M_P}} = \lambda|_{Z_{M_P}}$

must be such that $\mu_2 = f \sum_{\alpha \in S \backslash S(P(C_P))} \lambda_\alpha$ (note that $\mu_2 \leq f \sum_{\alpha \in S \backslash S(P(C_P))} \lambda_\alpha$ and $\mu_1 \leq f \sum_{\alpha \in S(P(C_P))} \lambda_\alpha$). Let λ', μ_1', μ_2' as in (37) for $P(C_P)$ and the respective characters λ, μ_1, μ_2, we have $\lambda' = \mu_1' + \mu_2'$ from (iii) and Remark 2.2.2.3(i), and thus

$$f\theta_G - \lambda' = f\Big(\sum_{\alpha \in S(P(C_P))} \lambda_\alpha\Big) - \mu_1' + f\Big(\sum_{\alpha \in S \backslash S(P(C_P))} \lambda_\alpha\Big) - \mu_2'. \tag{50}$$

Assume μ_2 is not $f \sum_{\alpha \in S \backslash S(P(C_P))} \lambda_\alpha$. Then writing $\mu_2 = \sum_{j,\alpha} \mu_{2,j,\alpha}$ where $(j, \alpha) \in \mathrm{Gal}(K/\mathbb{Q}_p) \times S \backslash S(P(C_P))$ and $\mu_{2,j,\alpha}$ occurs in $L(\lambda_\alpha)$ and applying Lemma 2.2.3.8 with $P = P(C_P)$, $\lambda = \lambda_\alpha$ and $\mu = \mu_{2,j,\alpha}$ for $\alpha \in S \backslash S(P(C_P))$ (the assumptions in Lemma 2.2.3.8 are satisfied since the λ_α, $\alpha \in S$ are fundamental weights), we get that $f \sum_{\alpha \in S \backslash S(P(C_P))} \lambda_\alpha - \mu_2$ has at least one root of $S \backslash S(P(C_P))$ in its support. Averaging over $w \in W(P(C_P))$ as in (37) and using $w(\lambda_\alpha) = \lambda_\alpha$ for $w \in W(P(C_P))$ and $\alpha \in S \backslash S(P(C_P))$ (same proof as for Lemma 2.2.3.6), we get applying Lemma 2.2.2.2

to $P = P(C_P)$ that $f\sum_{\alpha\in S\backslash S(P(C_P))}\lambda_\alpha - \mu_2'$ has still at least one root of $S\backslash S(P(C_P))$ in its support (and that $\mu_2' \leq f\sum_{\alpha\in S\backslash S(P(C_P))}\lambda_\alpha$). Since $\mu_1' \leq f\sum_{\alpha\in S(P(C_P))}\lambda_\alpha$ by the proof of Step 3 below, this root doesn't vanish in the sum (50). But by Proposition 2.2.2.6(ii), $S(P(C_P))$ *is* the support of (50), which is a contradiction. Therefore, we must have $\mu_2 = f\sum_{\alpha\in S\backslash S(P(C_P))}\lambda_\alpha$ and thus from (iii) that

$$C_P \cong C'_{P(C_P),P} \otimes \bigotimes_{\mathrm{Gal}(K/\mathbb{Q}_p)}\Big(\sum_{\alpha\in S\backslash S(P(C_P))}\lambda_\alpha\Big), \tag{51}$$

where $C'_{P(C_P),P}$ is the isotypic component of $\Big(\bigotimes_{\mathrm{Gal}(K/\mathbb{Q}_p)}\Big(\bigotimes_{\alpha\in S(P(C_P))}\overline{L}(\lambda_\alpha)\Big)\Big)|_{Z_{M_P}}$ associated to $\Big(\lambda - f\sum_{\alpha\in S\backslash S(P(C_P))}\lambda_\alpha\Big)|_{Z_{M_P}}$ $(= (\lambda-\mu_2)|_{Z_{M_P}} = \mu_1|_{Z_{M_P}})$.

Step 3: We prove that

$$f\Big(\sum_{\alpha\in S(P(C_P))}\lambda_\alpha\Big) - \mu_1 \in \sum_{\alpha\in S(P(C_P))}\mathbb{Z}_{\geq 0}\alpha$$

(i.e. no root of $S\backslash S(P(C_P))$ is in the support). Since λ_α is dominant, we have $\lambda_\alpha \geq \lambda_\alpha'$, where λ_α' is defined as in (37) for $P = P(C_P)$ and the character λ_α. This implies (with obvious notation)

$$f\Big(\sum_{\alpha\in S(P(C_P))}\lambda_\alpha\Big) - \mu_1' \geq f\Big(\sum_{\alpha\in S(P(C_P))}\lambda_\alpha'\Big) - \mu_1' = \Big(f\Big(\sum_{\alpha\in S(P(C_P))}\lambda_\alpha\Big) - \mu_1\Big)' \geq 0, \tag{52}$$

where the last inequality follows from Lemma 2.2.2.2 (applied with $P = P(C_P)$). If $f\Big(\sum_{\alpha\in S(P(C_P))}\lambda_\alpha\Big)-\mu_1$ has roots of $S\backslash S(P(C_P))$ in its support, then by Lemma 2.2.2.2 again so is the case of $\Big(f\Big(\sum_{\alpha\in S(P(C_P))}\lambda_\alpha\Big) - \mu_1\Big)'$, and thus of $f\Big(\sum_{\alpha\in S(P(C_P))}\lambda_\alpha\Big) - \mu_1'$ by (52). As in Step 2, this is again a contradiction by (50) and the definition of $P(C_P)$.

Step 4: We prove the statement for $w = \mathrm{Id}$. By Lemma 2.2.3.7 applied with $P = P(C_P)$ and the various $\overline{L}(\lambda_\alpha)$ for $\alpha \in S(P(C_P))$, we deduce from Step 3 that μ_1 is a weight of $\bigotimes_{\mathrm{Gal}(K/\mathbb{Q}_p)}\Big(\bigotimes_{\alpha\in S(P(C_P))}\overline{L}_{P(C_P)}(\lambda_\alpha)\Big)$ inside $\bigotimes_{\mathrm{Gal}(K/\mathbb{Q}_p)}\Big(\bigotimes_{\alpha\in S(P(C_P))}\overline{L}(\lambda_\alpha)\Big)$ (see just after (43)). Let $\alpha \in S(P(C_P))$, for each $\beta \in S(P(C_P))$ we have $\langle\lambda_\alpha,\beta\rangle = \langle\lambda_{\alpha,P(C_P)},\beta\rangle$ (a straightforward check from (43)), thus $\lambda_\alpha - \lambda_{\alpha,P(C_P)}$ extends to $\mathrm{Hom}_{\mathrm{Gr}}(M_{P(C_P)},\mathbb{G}_{\mathrm{m}})$ which implies $\overline{L}_{P(C_P)}(\lambda_\alpha) \cong \overline{L}_{P(C_P)}(\lambda_{\alpha,P(C_P)}) \otimes (\lambda_\alpha - \lambda_{\alpha,P(C_P)})$. Thus $\mu_1 - f\sum_{\alpha\in S(P(C_P))}(\lambda_\alpha - \lambda_{\alpha,P(C_P)})$ is a weight of

$$\bigotimes_{\mathrm{Gal}(K/\mathbb{Q}_p)}\Big(\bigotimes_{\alpha\in S(P(C_P))}\overline{L}_{P(C_P)}(\lambda_{\alpha,P(C_P)})\Big) = \overline{L}^{\otimes}_{P(C_P)},$$

or in other terms:

$$C'_{P(C_P),P} \cong C_{P(C_P),P} \otimes \bigotimes_{\mathrm{Gal}(K/\mathbb{Q}_p)}\Big(\sum_{\alpha\in S(P(C_P))}(\lambda_\alpha - \lambda_{\alpha,P(C_P)})\Big),$$

where $C_{P(C_P),P}$ is the isotypic component of $\overline{L}^{\otimes}_{P(C_P)}|_{Z_{M_P}}$ associated to $\Big(\lambda - f\sum_{\alpha\in S\backslash S(P(C_P))}\lambda_\alpha - f\sum_{\alpha\in S(P(C_P))}(\lambda_\alpha - \lambda_{\alpha,P(C_P)})\Big)|_{Z_{M_P}}$. But by (45):

$$\sum_{\alpha\in S\backslash S(P(C_P))}\lambda_\alpha \;+\; \sum_{\alpha\in S(P(C_P))}(\lambda_\alpha - \lambda_{\alpha,P(C_P)}) = \;\theta_G - \sum_{\alpha\in S(P(C_P))}\lambda_{\alpha,P(C_P)} \;=\; \theta^{P(C_P)},$$

so together with (51) we are done. □

Remark 2.2.3.10. The character $w^{-1}(\theta^{P(C_P)})$ of M_P doesn't depend on $w \in W(C_P)$, as follows from Lemma 2.2.2.10 and Lemma 2.2.3.6 (the latter applied with P there being $P(C_P)$). In particular, by (49) we see that the representation $w^{-1}\Big(C_{P(C_P),{}^wP}\Big)$ of $M_P^{\mathrm{Gal}(K/\mathbb{Q}_p)}$ is also independent of $w \in W(C_P)$.

When C_P is as in Proposition 2.2.3.3, its underlying $M_P^{\mathrm{Gal}(K/\mathbb{Q}_p)}$-representation looks like $\overline{L}^{\otimes}$ but for the reductive group M_P instead of G.

Corollary 2.2.3.11. *Let C_P be an isotypic component of $\overline{L}^{\otimes}|_{Z_{M_P}}$ such that $P(C_P) = {}^wP$ for some (unique) $w \in W$ such that $w(S(P)) \subseteq S$. Then there is an isomorphism*

$$C_P \cong \overline{L}^{\otimes}_P \otimes \Big(\underbrace{w^{-1}(\theta^{{}^wP}) \otimes \cdots \otimes w^{-1}(\theta^{{}^wP})}_{\mathrm{Gal}(K/\mathbb{Q}_p)}\Big)$$

of algebraic representations of $M_P^{\mathrm{Gal}(K/\mathbb{Q}_p)}$ over $\mathbb{F}$.

Proof. If $P(C_P) = {}^wP$, then $\overline{L}^{\otimes}_{P(C_P)}|_{Z_{M_{{}^wP}}} = \overline{L}^{\otimes}_{{}^wP}|_{Z_{M_{{}^wP}}}$ has only one isotypic component, corresponding to $f\theta_{{}^wP}|_{Z_{M_{{}^wP}}}$. So the corollary follows from Theorem 2.2.3.9 together with (48). Note that, by Proposition 2.2.3.3, C_P corresponds to $\lambda = fw^{-1}(\theta_G)$, which is consistent with Theorem 2.2.3.9 since

$$\begin{aligned}(w(\lambda) - f\theta^{P(C_P)})|_{Z_{M_{{}^wP}}} = \Big(w(fw^{-1}(\theta_G)) - f\theta^{{}^wP}\Big)|_{Z_{M_{{}^wP}}} &= f(\theta_G - \theta^{{}^wP})|_{Z_{M_{{}^wP}}}\\ &= f\theta_{{}^wP}|_{Z_{M_{{}^wP}}}.\end{aligned}$$

□

Remark 2.2.3.12. In this remark, we use that we are working with $G = \mathrm{GL}_n$. We write $M_{P(C_P)} = \mathrm{diag}(M_1, \dots, M_d)$ for some $d > 0$ with $M_i \cong \mathrm{GL}_{n_i}$, and correspondingly $T = \mathrm{diag}(T_1, \dots, T_d)$, where T_i is the diagonal torus of GL_{n_i}, so that we have $X(T) = \oplus_{i=1}^d X(T_i)$ and $S(P(C_P)) = \amalg_{i=1}^d S(M_i)$, where $X(T_i) \overset{\mathrm{def}}{=} \mathrm{Hom}_{\mathrm{Gr}}(T_i, \mathbb{G}_{\mathrm{m}})$ and $S(M_i) \overset{\mathrm{def}}{=} S(P(C_P)) \cap X(T_i)$ is the set of simple roots of M_i (for the Borel subgroup of upper-triangular matrices). Note that $S(M_i) = \emptyset$ if $M_i \cong \mathrm{GL}_1$. For $i \in \{1, \dots, d\}$ such that $n_i > 1$, one easily checks that $\lambda_{\alpha,P(C_P)} \in X(T_i) \subseteq X(T)$ if $\alpha \in S(M_i)$ and that the $\lambda_{\alpha,P(C_P)} \in X(T_i)$ for $\alpha \in S(M_i)$ are fundamental weights for the reductive group M_i. For $i \in \{1,\dots,d\}$ and $\lambda_i \in X(T_i)$, we define $\overline{L}_{M_i}(\lambda_i)$ as in (19) but for the reductive group M_i instead of G. When $n_i = 1$, we define $\overline{L}_i^{\otimes}$ to be the trivial representation of $M_i^{\mathrm{Gal}(K/\mathbb{Q}_p)} \cong \mathbb{G}_{\mathrm{m}}^{\mathrm{Gal}(K/\mathbb{Q}_p)}$, and when $n_i > 1$, we define as in (34) the algebraic representation of $M_i^{\mathrm{Gal}(K/\mathbb{Q}_p)}$ over $\mathbb{F}$ (seeing $\lambda_{\alpha,P(C_P)}$ in $X(T_i)$):

$$\overline{L}_i^{\otimes} \stackrel{\text{def}}{=} \bigotimes_{\mathrm{Gal}(K/\mathbb{Q}_p)} \Big(\bigotimes_{\alpha \in S(M_i)} \overline{L}_{M_i}(\lambda_{\alpha,P(C_P)}) \Big). \tag{53}$$

We then clearly have $\overline{L}_{P(C_P)}^{\otimes} \cong \bigotimes_{i=1}^{d} \overline{L}_i^{\otimes}$. Likewise, we have $\theta^{P(C_P)} = \otimes_{i=1}^{d} (\theta^{P(C_P)})_i$, where $(\theta^{P(C_P)})_i \in X(T_i)$ extends to $\mathrm{Hom}_{\mathrm{Gr}}(M_i, \mathbb{G}_{\mathrm{m}})$ and where we denote by μ_i the image in $X(T_i)$ of a character $\mu \in X(T)$.

For any $w \in W(C_P)$, we define $({}^{w}P)_i$ as the standard parabolic subgroup of M_i which is the image of ${}^{w}P$ under

$$ {}^{w}P \hookrightarrow P(C_P) \twoheadrightarrow M_{P(C_P)} \twoheadrightarrow M_i $$

(in particular its Levi $M_{({}^{w}P)_i}$ is the image of $M_{{}^{w}P}$ under $M_{{}^{w}P} \hookrightarrow M_{P(C_P)} \twoheadrightarrow M_i$). Applying w to (49), it is not difficult to deduce from Theorem 2.2.3.9 an isomorphism of algebraic representations of $M_{{}^{w}P}^{\mathrm{Gal}(K/\mathbb{Q}_p)} \cong \prod_{i=1}^{d} M_{({}^{w}P)_i}^{\mathrm{Gal}(K/\mathbb{Q}_p)}$ over $\mathbb{F}$:

$$ w(C_P) \cong \bigotimes_{i=1}^{d} \Big(C_{w,i} \otimes \Big(\underbrace{(\theta^{P(C_P)})_i \otimes \cdots \otimes (\theta^{P(C_P)})_i}_{\mathrm{Gal}(K/\mathbb{Q}_p)} \Big) \Big), \tag{54}$$

where $C_{w,i}$ is the isotypic component of $\overline{L}_i^{\otimes}|_{Z_{M_{({}^{w}P)_i}}}$ associated to $(w(\lambda) - f\theta^{P(C_P)})_i|_{Z_{M_{({}^{w}P)_i}}}$ (thus an $M_{({}^{w}P)_i}^{\mathrm{Gal}(K/\mathbb{Q}_p)}$-representation, note that $C_{w,i}$ is trivial if $n_i = 1$). If w' is another element in $W(C_P)$, writing $w' = w_{P(C_P)} w$ with $w_{P(C_P)} \in W(P(C_P))$ (Lemma 2.2.2.10), we have $M_{w'P} = w_{P(C_P)} M_{{}^{w}P} w_{P(C_P)}^{-1}$, and thus $w'(C_p) \cong w_{P(C_P)}(w(C_P))$ and $C_{P(C_P),{}^{w'}P} \cong w_{P(C_P)}\big(C_{P(C_P),{}^{w}P}\big)$ (as the twist by $\theta^{P(C_P)} \otimes \cdots \otimes \theta^{P(C_P)}$ doesn't involve the choice of w). Since $w_{P(C_P)} M_i w_{P(C_P)}^{-1} = M_i$ for all i, we get $M_{({}^{w'}P)_i} = w_{P(C_P)} M_{({}^{w}P)_i} w_{P(C_P)}^{-1}$ (inside M_i) and deduce for $i \in \{1,\dots,d\}$ an isomorphism of algebraic representations of $M_{({}^{w'}P)_i}^{\mathrm{Gal}(K/\mathbb{Q}_p)}$ over $\mathbb{F}$ (with notation similar to (47)):

$$ C_{w',i} \cong w_{P(C_P)}(C_{w,i}). \tag{55}$$

We will avoid applying w^{-1} to $C_{w,i}$ since $w^{-1} M_{P(C_P)} w$ is not in general the Levi subgroup of a standard parabolic subgroup of G (see Remark 2.2.3.2(ii)), although it indeed contains M_P.

2.2.4 From one isotypic component to another

We let P be a standard parabolic subgroup of G. We show that, if C_P is an isotypic component of $\overline{L}^{\otimes}|_{Z_{M_P}}$, then one can associate to C_P in a natural way another isotypic component $w \cdot C_P$ of $\overline{L}^{\otimes}|_{Z_{M_P}}$ for any $w \in W$ such that $w\Big(S(P(C_P))\Big) \subseteq S$ (see Proposition 2.2.4.2). Note that, on the contrary to $w(C_P)$, $w \cdot C_P$ is an isotypic component of $\overline{L}^{\otimes}|_{Z_{M_P}}$ for the *same* standard parabolic subgroup P as C_P.

Lemma 2.2.4.1. *Let $\mu \in X(T)$ be a dominant weight. Then μ occurs in $\overline{L}^{\otimes}|_T$ (for the diagonal embedding of T analogous to* (36)*) if and only if $\mu \leq f\theta_G$ in $X(T)$.*

Proof. Since this statement only concerns weights, we can work in characteristic 0, i.e. with $L^{\otimes} \stackrel{\text{def}}{=} \bigotimes_{\mathrm{Gal}(K/\mathbb{Q}_p)} \Big(\bigotimes_{\alpha \in S} L(\lambda_\alpha)\Big)$, where $L(\lambda_\alpha) \stackrel{\text{def}}{=} \Big(\mathrm{ind}_{B^-}^G \lambda_\alpha\Big)_{/\mathbb{Z}} \otimes_{\mathbb{Z}} E$ (see (19)). Arguing as in the proof of [BH15, Lemma 2.2.3], it is equivalent to prove that μ is a weight of the algebraic representation $L(f\theta_G)$ of G. The result then follows from the inequalities $w(\mu) \leq \mu \leq f\theta_G$ for all $w \in W$ (the left ones hold since μ is dominant and the right ones since $f\theta_G$ is the highest weight) combined with [Hum78, Prop.21.3]. □

Proposition 2.2.4.2. *Let $\lambda_P \in X(Z_{M_P})$ be a character of Z_{M_P} which occurs in $\overline{L}^{\otimes}|_{Z_{M_P}}$ (for the diagonal embedding, as usual) with associated isotypic component C_P of $\overline{L}^{\otimes}|_{Z_{M_P}}$, and let $w \in W$ such that $w\Big(S(P(C_P))\Big) \subseteq S$.*

(i) *For $w_{C_P} \in W(C_P)$ the character of Z_{M_P}:*

$$\lambda_P - \Big(f w_{C_P}^{-1}(\theta_G) + f(ww_{C_P})^{-1}(\theta_G)\Big)|_{Z_{M_P}} \tag{56}$$

doesn't depend on w_{C_P}.

(ii) *The character* (56) *corresponds to an isotypic component $w \cdot C_P$ of $\overline{L}^{\otimes}|_{Z_{M_P}}$, i.e. occurs in $\overline{L}^{\otimes}|_{Z_{M_P}}$.*

(iii) *We have $P(w \cdot C_P) = {}^wP(C_P)$.*

Proof. (i) For any $\alpha \in S(P(C_P))$ we have (since $w(\alpha)$ is still in S)

$$\langle w^{-1}(\theta_G) - \theta_G, \alpha\rangle = \langle \theta_G, w(\alpha)\rangle - \langle \theta_G, \alpha\rangle = 1 - 1 = 0 \tag{57}$$

which implies $s_\alpha(w^{-1}(\theta_G) - \theta_G) = w^{-1}(\theta_G) - \theta_G$, and thus for all $w' \in W(P(C_P))$:

$$w'(w^{-1}(\theta_G) - \theta_G) = w^{-1}(\theta_G) - \theta_G. \tag{58}$$

Let $w'_{C_P} \in W(C_P)$, by Lemma 2.2.2.10 we have $w'_{C_P} w_{C_P}^{-1} \in W(P(C_P))$ and thus by (58):

$$(w'_{C_P} w_{C_P}^{-1})(w^{-1}(\theta_G) - \theta_G) = w^{-1}(\theta_G) - \theta_G.$$

Applying ${w'_{C_P}}^{-1}$ we get in particular

$$\Big(w_{C_P}^{-1}(w^{-1}(\theta_G) - \theta_G)\Big)|_{Z_{M_P}} = \Big({w'_{C_P}}^{-1}(w^{-1}(\theta_G) - \theta_G)\Big)|_{Z_{M_P}}$$

from which (i) follows.
(ii) Let $\lambda \in X(T)$ such that $\lambda|_{Z_{M_P}} = \lambda_P$. Applying ww_{C_P} to (56), it is sufficient to prove that $f\theta_G - w\big(f\theta_G - w_{C_P}(\lambda)\big)$ occurs in $\overline{L}^{\otimes}|_T$ (since $\overline{L}^{\otimes}|_T$ is acted on by the diagonal action of $W \hookrightarrow W^{\mathrm{Gal}(K/\mathbb{Q}_p)}$). Recall from Lemma 2.2.2.4(ii) (and the definition of $P(C_P)$) that

$$f\theta_G - w_{C_P}(\lambda) \quad \in \sum_{\alpha \in S(P(C_P))} \mathbb{Z}_{\geq 0}\alpha. \tag{59}$$

For $\beta = w(\alpha) \in w(S(P(C_P)))$ and *any* $w' \in W$, we have

$$\begin{aligned} \langle f\theta_G - w(f\theta_G - w'(\lambda)), \beta\rangle &= \langle ww'(\lambda), \beta\rangle + f\langle \theta_G - w(\theta_G), \beta\rangle \\ &= \langle ww'(\lambda), \beta\rangle + f\langle w^{-1}(\theta_G) - \theta_G, \alpha\rangle \\ &= \langle ww'(\lambda), \beta\rangle, \end{aligned} \tag{60}$$

where the last equality follows from (57). This can be rewritten as

$$\begin{aligned} s_\beta\Big(f\theta_G - w(f\theta_G - w'(\lambda))\Big) &= f\theta_G - w(f\theta_G - w'(\lambda)) - \langle ww'(\lambda), \beta\rangle\beta \\ &= f\theta_G - w(f\theta_G - s_\alpha w'(\lambda)). \end{aligned} \tag{61}$$

Iterating (61), we see that for any $w_{P(C_P)} \in W(P(C_P))$, we have for $w' \in W$ that

$$ww_{P(C_P)}w^{-1}\Big(f\theta_G - w(f\theta_G - w'(\lambda))\Big) = f\theta_G - w(f\theta_G - w_{P(C_P)}w'(\lambda)). \tag{62}$$

Choose $w_{P(C_P)} \in W(P(C_P))$ such that $w_{P(C_P)}(w_{C_P}(\lambda))$ is dominant for the root subsystem generated by $S(P(C_P))$, equivalently

$$\langle ww_{P(C_P)}w_{C_P}(\lambda), \beta\rangle \geq 0 \quad \forall\ \beta \in w(S(P(C_P))). \tag{63}$$

As λ occurs in $\overline{L}^{\otimes}|_T$, we get that $w_{P(C_P)}(w_{C_P}(\lambda)) \in w_{C_P}(\lambda) + \sum_{\alpha \in S(P(C_P))} \mathbb{Z}\alpha$ occurs in $\overline{L}^{\otimes}|_T$ ($\overline{L}^{\otimes}$ is stable under W), and thus $w_{P(C_P)}(w_{C_P}(\lambda)) \leq f\theta_G$. Since on the other hand by (59):

$$f\theta_G - w_{P(C_P)}(w_{C_P}(\lambda)) = (f\theta_G - w_{C_P}(\lambda)) \quad + \sum_{\alpha \in S(P(C_P))} \mathbb{Z}\alpha \quad \in \sum_{\alpha \in S(P(C_P))} \mathbb{Z}\alpha,$$

we see that we must have

$$f\theta_G - w_{P(C_P)}w_{C_P}(\lambda) \quad \in \sum_{\alpha \in S(P(C_P))} \mathbb{Z}_{\geq 0}\alpha. \tag{64}$$

Since $w(S(P(C_P))) \subseteq S$, we deduce $\langle w(f\theta_G - w_{P(C_P)}w_{C_P}(\lambda)), \beta\rangle \leq 0$ for $\beta \in S\backslash w(S(P(C_P)))$. In particular we have for such β:

$$\begin{aligned} \langle f\theta_G - w(f\theta_G - w_{P(C_P)}w_{C_P}(\lambda)), \beta\rangle &= f - \langle w(f\theta_G - w_{P(C_P)}w_{C_P}(\lambda)), \beta\rangle \\ &\geq f. \end{aligned} \tag{65}$$

Combining (60) for $w' = w_{P(C_P)}w_{C_P}$ with (63) and (65), we obtain that $f\theta_G - w(f\theta_G - w_{P(C_P)}w_{C_P}(\lambda))$ is a dominant weight. Applying w to (64), we also get since $w(S(P(C_P))) \subseteq S$:

$$f\theta_G - w(f\theta_G - w_{P(C_P)}w_{C_P}(\lambda)) \leq f\theta_G.$$

Lemma 2.2.4.1 then implies that $f\theta_G - w(f\theta_G - w_{P(C_P)}w_{C_P}(\lambda))$ occurs in $\overline{L}^{\otimes}|_T$. By (62) applied with $w' = w_{C_P}$, we finally deduce that $f\theta_G - w(f\theta_G - w_{C_P}(\lambda))$ also occurs in $\overline{L}^{\otimes}|_T$.
(iii) By definition $S(P(w \cdot C_P)) \subseteq S$ is the union of $w'(S(P))$ and of the support of

$$f\theta_G - w'\Big(\lambda - fw_{C_P}^{-1}(\theta_G) + f(ww_{C_P})^{-1}(\theta_G)\Big) \tag{66}$$

for any $w' \in W$ such that $w'(S(P)) \subseteq S$ and $w'\Big(\lambda - fw_{C_P}^{-1}(\theta_G) + f(ww_{C_P})^{-1}(\theta_G)\Big)$ is the restriction to $Z_{M_{w'P}}$ of a dominant weight of $X(T)\otimes_{\mathbb{Z}}\mathbb{Q}$. Consider the case $w' \overset{\text{def}}{=} ww_{C_P}$, since $w_{C_P}(S(P)) \subseteq S(P(C_P))$ and $w(S(P(C_P))) \subseteq S$, we get $w'(S(P)) \subseteq S$. Let us check that

$$w'\Big(\lambda - fw_{C_P}^{-1}(\theta_G) + f(ww_{C_P})^{-1}(\theta_G)\Big) = ww_{C_P}(\lambda) - fw(\theta_G) + f\theta_G$$

is the restriction to $Z_{M_{w'P}}$ of a dominant weight of $X(T) \otimes_{\mathbb{Z}} \mathbb{Q}$. Let λ' as in (37), since $\lambda|_{Z_{M_P}} = \lambda'|_{Z_{M_P}}$, we have $w'(\lambda)|_{Z_{M_{w'P}}} = w'(\lambda')|_{Z_{M_{w'P}}}$ and it is enough to prove that $ww_{C_P}(\lambda') - fw(\theta_G) + f\theta_G$ is dominant. As in (60) we have if $\alpha \in w(S(P(C_P)))$:

$$\begin{aligned}\langle ww_{C_P}(\lambda') - fw(\theta_G) + f\theta_G, \alpha\rangle &= \langle ww_{C_P}(\lambda'), \alpha\rangle + f\langle \theta_G - w(\theta_G), \alpha\rangle \\ &= \langle w_{C_P}(\lambda'), w^{-1}(\alpha)\rangle \geq 0\end{aligned}$$

since $w_{C_P}(\lambda')$ is dominant in $X(T) \otimes_{\mathbb{Z}} \mathbb{Q}$ by Proposition 2.2.2.6(i), and as in (65) we have if $\alpha \in S\backslash w(S(P(C_P)))$:

$$\langle ww_{C_P}(\lambda') - fw(\theta_G) + f\theta_G, \alpha\rangle = f - \langle w(f\theta_G - w_{C_P}(\lambda')), \alpha\rangle \geq f$$

since $w\Big(f\theta_G - w_{C_P}(\lambda')\Big) \in \sum_{\beta \in S(P(C_P))} \mathbb{Q}_{\geq 0} w(\beta)$ from Proposition 2.2.2.6(ii). Now all that remains is to compute (66) for $w' = ww_{C_P}$, which gives $w(f\theta_G - w_{C_P}(\lambda))$, the support of which is $w(\text{support}(f\theta_G - w_{C_P}(\lambda)))$. Therefore we obtain

$$S(P(w \cdot C_P)) = w\Big(w_{C_P}(S(P)) \cup \text{support}\Big(f\theta_G - w_{C_P}(\lambda)\Big)\Big) = w\Big(S(P(C_P))\Big)$$

which finishes the proof. □

Remark 2.2.4.3. If C_P is one of the isotypic components of Proposition 2.2.3.3, say associated to $fw_{C_P}^{-1}(\theta_G)|_{Z_{M_P}}$ for some $w_{C_P} \in W$ such that $w_{C_P}(S(P)) \subseteq S$, and if $w \in W$ is such that $w(S(P(C_P))) \subseteq S$, i.e. $ww_{C_P}(S(P)) \subseteq S$, we see from (56) that $w \cdot C_P$ is the isotypic component associated to $f(ww_{C_P})^{-1}(\theta_G)|_{Z_{M_P}}$.

Example 2.2.4.4. Let us consider Example 2.2.2.9(ii) (Example 2.2.2.9(i) only provides components C_P which are either as in Remark 2.2.4.3 or such that $P(C_P) = G$). If $P = B$ and C_P is associated to $\lambda_{\mathrm{Id}} = \theta_G$, then $w \cdot C_P$ for $w \in \mathcal{S}_3$ gives the isotypic component associated to λ_w (and there is no $w \cdot C_P \neq C_P$ if C_P corresponds to det since $P(C_P)$ is the whole G). If $M_P = \mathrm{GL}_2 \times \mathrm{GL}_1$, consider C_P associated to λ_0 and $w \in \mathcal{S}_3$ the unique permutation $e_1 \mapsto e_2$, $e_2 \mapsto e_3$, $e_3 \mapsto e_1$ (so that $w(S(P(C_P))) = w(e_1 - e_2) \subseteq S$). Then $w \cdot C_P$ is the isotypic component associated to λ_2 (here again, there is no $w \cdot C_P \neq C_P$ for C_P corresponding to λ_1).

2.3 Good conjugates of $\overline{\rho}$

Following and generalizing the mod p variant of [BH15, §3.2], we define and study *good conjugates* of a continuous $\overline{\rho} : \mathrm{Gal}(\overline{\mathbb{Q}}_p/K) \to G(\mathbb{F})$ under a mild assumption on $\overline{\rho}$ (see Definition 2.3.2.3) and still assuming K unramified. Though some of the results might hold for more general split reductive groups, we use here in the proofs that we work with GL_n.

2.3.1 Some preliminaries

We start with a few group-theoretic preliminaries.

We fix a standard parabolic subgroup P of G. Recall that a subset $C \subseteq R^+$ is closed if $\alpha \in C$, $\beta \in C$ with $\alpha + \beta \in R^+$ implies $\alpha + \beta \in C$. For instance $R(P)^+ \subseteq R^+$ is closed.

Definition 2.3.1.1. A subset $X \subseteq R^+$ is a *closed subset relative to* P if it satisfies the following three conditions:

(i) it contains $R(P)^+$;

(ii) $X \backslash R(P)^+$ is a closed subset of R^+;

(iii) for any $w \in W(P)$, $w(X \backslash R(P)^+) = X \backslash R(P)^+$.

Note that a closed subset relative to B is the same thing as a closed subset and that R^+ is the only closed subset relative to G.

Lemma 2.3.1.2. *Let $X \subseteq R^+$ be a closed subset relative to P. Then X is a closed subset of R^+.*

Proof. Since we already know that both $R(P)^+$ and $X \backslash R(P)^+$ are closed, it remains to show that if $\alpha \in R(P)^+$ and $\beta \in X \backslash R(P)^+$ are such that $\alpha + \beta \in R^+$, then

$\alpha + \beta \in X$. We work with GL_n, and it is then easy to check that $\alpha + \beta = s_\alpha(\beta)$. Since $s_\alpha \in W(P)$, we have $\alpha + \beta \in X \backslash R(P)^+ \subseteq X$ by Definition 2.3.1.1(iii). $\square$

Remark 2.3.1.3. Note that Lemma 2.3.1.2 doesn't hold for an arbitrary split connected reductive algebraic group (for instance it doesn't work for GSp_4). An alternative definition would be to consider closed subsets Y of $R^+ \backslash R(P)^+$ such that $Y \cup R(P)$ is also closed.

If $X \subseteq R^+$ is any closed subset, we let $N_X \subseteq N$ be the Zariski closed algebraic subgroup generated by the root subgroups N_α for $\alpha \in X$ (see [Jan03, §II.1.7]). Thanks to Lemma 2.3.1.2, we can thus consider N_X for any $X \subseteq R^+$ closed relative to P.

Lemma 2.3.1.4.

(i) *Let X be a closed subset of R^+ relative to P. Then $M_P N_X$ is a Zariski closed algebraic subgroup of P containing M_P.*

(ii) *Let $\widetilde{P} \subseteq P$ be a Zariski closed algebraic subgroup containing M_P. Then there exists a unique closed subset X relative to P such that $\widetilde{P} = M_P N_X$.*

Proof. (i) Since $M_P N_X = M_P N_{X \backslash R(P)^+}$, it is enough to prove that M_P normalizes $N_{X \backslash R(P)^+}$. Let $\alpha \in R(P)^+$, $\beta \in X \backslash R(P)^+$ and let $n_\alpha \in N_\alpha$, $n_\beta \in N_\beta$. Then

$$n_\alpha n_\beta n_\alpha^{-1} = \Big(\prod_{i,j>0} n_{i\alpha + j\beta} \Big) n_\beta, \tag{67}$$

where the product is over all integers $i, j > 0$ such that $i\alpha + j\beta \in R^+$ (see [Jan03, §II.1.2]). Since $X \subseteq R^+$ is closed, all these $i\alpha + j\beta$ are in X, and since $\beta \notin R(P)^+$, they are all in $X \backslash R(P)^+$. Therefore $n_\alpha n_\beta n_\alpha^{-1} \in N_{X \backslash R(P)^+}$. Let $w \in W(P)$, $\beta \in X \backslash R(P)^+$ and $n_\beta \in N_\beta$. Then $w(\beta) \in X \backslash R(P)^+$ implies $w n_\beta w^{-1} \in N_{X \backslash R(P)^+}$. The Bruhat decomposition for the reductive group M_P then shows that M_P normalizes $N_{X \backslash R(P)^+}$.
(ii) Let $\widetilde{P} \subseteq P$ be a closed algebraic subgroup containing M_P. Then $\widetilde{P} = M_P(\widetilde{P} \cap B) = M_P(\widetilde{P} \cap N)$ (since $T \subseteq M_P \subseteq \widetilde{P}$). By [BH15, Lemma 3.4.1] applied to $\widetilde{P} \cap B \subseteq B$, we deduce $\widetilde{P} \cap N = N_X$ for a (unique) closed subset $X \subseteq R^+$. Since $M_P \cap N \subseteq \widetilde{P} \cap N$, the set X contains $R(P)^+$. Since $\widetilde{P} \cap N_P = N_{X \backslash R(P)^+}$, the set $X \backslash R(P)^+$ is closed, and moreover $\widetilde{P} = M_P N_{X \backslash R(P)^+}$. Since M_P normalizes N_P and $\widetilde{P}$, it normalizes $\widetilde{P} \cap N_P = N_{X \backslash R(P)^+}$, from which Definition 2.3.1.1(iii) easily follows. $\square$

Remark 2.3.1.5. (i) The sets $R(P)^+$ and R^+ are closed with respect to P (they correspond respectively to $\widetilde{P} = M_P$ and $\widetilde{P} = P$ in Lemma 2.3.1.4). In particular, if X is closed with respect to P, from $w(R^+ \backslash R(P)^+) = R^+ \backslash R(P)^+$ and $w(X \backslash R(P)^+) = X \backslash R(P)^+$, we also get $w(R^+ \backslash X) = R^+ \backslash X$ for all $w \in W(P)$.
(ii) If $X \subseteq R^+$ is a closed subset relative to P, it follows from the proof of Lemma 2.3.1.4(i) that M_P normalizes $N_{X \backslash R(P)^+}$.

Lemma 2.3.1.6. *Let $X \subseteq R^+$ be a closed subset relative to P. Then there are roots $\alpha_1, \dots, \alpha_m \in R^+ \backslash X$ such that we have a partition*

$$R^+ = X \amalg \{w(\alpha_1) : w \in W(P)\} \amalg \cdots \amalg \{w(\alpha_m) : w \in W(P)\}$$

and such that, for all i, α_i is not in the smallest closed subset relative to P containing X and the α_j for $1 \leq j \leq i-1$.

Proof. Since $w(R^+ \backslash X) = R^+ \backslash X$ for all $w \in W(P)$ (Remark 2.3.1.5(i)), we have a partition $R^+ = X \amalg \{w(\alpha_1) : w \in W(P)\} \amalg \cdots \amalg \{w(\alpha_m) : w \in W(P)\}$ for some $\alpha_1, \dots, \alpha_m \in R^+ \backslash X$. Denote by $h(\cdot)$ the height of a positive root (see e.g. [BH15, Rem.2.5.3]). Replacing each α_i by a suitable $w(\alpha_i)$ for $w \in W(P)$, we can assume $h(\alpha_i)$ maximal among the $h(w(\alpha_i))$, $w \in W(P)$. Permuting the α_i if necessary, we can assume $h(\alpha_1) \geq h(\alpha_2) \geq \cdots \geq h(\alpha_m)$. It is enough to prove that each set $X \amalg \{w(\alpha_1) : w \in W(P)\} \amalg \cdots \amalg \{w(\alpha_i) : w \in W(P)\}$ for $1 \leq i \leq m$ is closed relative to P, or equivalently that $X_i \overset{\text{def}}{=} (X \backslash R(P)^+) \amalg \{w(\alpha_1) : w \in W(P)\} \amalg \cdots \amalg \{w(\alpha_i) : w \in W(P)\}$ satisfies conditions (ii) and (iii) in Definition 2.3.1.1 for $1 \leq i \leq m$. Since (iii) is clear, let us prove (ii), i.e. that each of the X_i is closed in R^+.
This is obvious if $i = m$ since $R^+ \backslash R(P)^+$ is closed, so we can assume $i < m$. If X_i is not closed for some $i < m$, then its complement in R^+ contains an element x which is the sum of at least two roots of X_i, at least one being in $\{w'(\alpha_j) : w' \in W(P), 1 \leq j \leq i\}$ (since $R^+ \backslash R(P)^+$ is closed). Such an element x is in $R(P)^+ \amalg \{w(\alpha_j) : w \in W(P), i+1 \leq j \leq m\}$ and, since $w'(X_i) = X_i$ for $w' \in W(P)$, it also satisfies $w'(x) \in R^+$ for any $w' \in W(P)$. In particular x can't be in $R(P)^+$, and is thus of the form $x = w(\alpha_k)$ for some $k \in \{i+1, \dots, m\}$ and some $w \in W(P)$. Thus $w(\alpha_k)$ is the sum of at least two roots of X_i, one at least being in $\{w'(\alpha_j) : w' \in W(P), 1 \leq j \leq i\}$. Applying a convenient $w' \in W(P)$ and using again $w'(X_i) = X_i$, we can modify w if necessary and assume that α_j for some $j \in \{1, \dots, i\}$ appears in the sum of $w(\alpha_k)$. This implies in particular $h(w(\alpha_k)) > h(\alpha_j)$ for some $j \leq i$ (see the argument in the proof of [BH15, Lemma 3.2.1]), which is impossible since by assumption $h(w(\alpha_k)) \leq h(\alpha_k) \leq h(\alpha_j)$. Hence X_i is closed for all i. □

Lemma 2.3.1.7. *Let $X \subseteq R^+$ be a closed subset relative to P, $\widetilde{P} \overset{\text{def}}{=} M_P N_X$ and let $w \in W$ such that $w(S(P)) \subseteq S$. Then the following assertions are equivalent:*

(i) *$w\widetilde{P}w^{-1}$ is contained in ${}^wP$;*

(ii) *$w(X \backslash R(P)^+) \subseteq R^+$.*

Proof. We have

$$w\widetilde{P}w^{-1} = (wM_Pw^{-1})(wN_{X \backslash R(P)^+}w^{-1}) = (wM_Pw^{-1})N_{w(X \backslash R(P)^+)}.$$

As ${}^wP = (wM_Pw^{-1})N$, we deduce $w\widetilde{P}w^{-1} \subseteq {}^wP$ if and only if $w(X \backslash R(P)^+) \subseteq R^+$. □

2.3.2 Good conjugates of a generic $\overline{\rho}$

We define good conjugates of a $\mathrm{Gal}(\overline{\mathbb{Q}}_p/K)$-representation $\overline{\rho}$ under a mild genericity assumption and show how two good conjugates are related (Theorem 2.3.2.5). The intuitive idea is that conjugating a good conjugate of $\overline{\rho}$ can only increase the image in $G(\mathbb{F})$.

We fix a continuous homomorphism

$$\overline{\rho} : \mathrm{Gal}(\overline{\mathbb{Q}}_p/K) \longrightarrow P_{\overline{\rho}}(\mathbb{F}) \subseteq G(\mathbb{F}), \tag{68}$$

where $P_{\overline{\rho}} \subseteq G$ is a standard parabolic subgroup. We consider

$$\overline{\rho}^{P_{\overline{\rho}}-\mathrm{ss}} : \mathrm{Gal}(\overline{\mathbb{Q}}_p/K) \xrightarrow{\overline{\rho}} P_{\overline{\rho}}(\mathbb{F}) \twoheadrightarrow M_{P_{\overline{\rho}}}(\mathbb{F}),$$

and assume that the image of $\overline{\rho}^{P_{\overline{\rho}}-\mathrm{ss}}$ is *not* contained in the $\mathbb{F}$-points of a proper (not necessarily standard) parabolic subgroup of $M_{P_{\overline{\rho}}}$. This implies in particular that $P_{\overline{\rho}}$ is uniquely determined by the homomorphism $\overline{\rho}$. Finally we let $\overline{\rho}^{\mathrm{ss}}$ be the homomorphism $\mathrm{Gal}(\overline{\mathbb{Q}}_p/K) \to G(\mathbb{F})$ obtained by composing $\overline{\rho}^{P_{\overline{\rho}}-\mathrm{ss}}$ with the inclusion $M_{P_{\overline{\rho}}}(\mathbb{F}) \subseteq G(\mathbb{F})$ (so $\overline{\rho}^{\mathrm{ss}}$ is the usual semisimplification of $\overline{\rho}$). We let $X_{\overline{\rho}}$ be the smallest closed subset of R^+ relative to $P_{\overline{\rho}}$ such that $\widetilde{P}_{\overline{\rho}}(\mathbb{F}) \stackrel{\mathrm{def}}{=} M_{P_{\overline{\rho}}}(\mathbb{F})N_{X_{\overline{\rho}}}(\mathbb{F})$ contains all the $\overline{\rho}(g)$, $g \in \mathrm{Gal}(\overline{\mathbb{Q}}_p/K)$. By Lemma 2.3.1.4, $\widetilde{P}_{\overline{\rho}}$ is the smallest closed algebraic subgroup of $P_{\overline{\rho}}$ containing $M_{P_{\overline{\rho}}}$ such that $\overline{\rho}$ takes values in $\widetilde{P}_{\overline{\rho}}(\mathbb{F})$, i.e. $\overline{\rho} : \mathrm{Gal}(\overline{\mathbb{Q}}_p/K) \to \widetilde{P}_{\overline{\rho}}(\mathbb{F}) \hookrightarrow P_{\overline{\rho}}(\mathbb{F}) \hookrightarrow G(\mathbb{F})$. Note that $X_{\overline{\rho}^{\mathrm{ss}}} = R(P)^+$ and $\widetilde{P}_{\overline{\rho}^{P_{\overline{\rho}}-\mathrm{ss}}} = M_{P_{\overline{\rho}}}$.

Lemma 2.3.2.1. *Assume that the irreducible constituents of $\overline{\rho}^{\mathrm{ss}}$ of dimension* 1 (*i.e. the characters of* $\mathrm{Gal}(\overline{\mathbb{Q}}_p/K)$ *occurring in* $\overline{\rho}^{\mathrm{ss}}$) *are all distinct. Let $\alpha \in R^+\backslash X_{\overline{\rho}}$ and $n_\alpha \in N_\alpha(\mathbb{F})\backslash\{1\}$. Then $X_{n_\alpha\overline{\rho}n_\alpha^{-1}}$ is the smallest closed subset relative to $P_{\overline{\rho}}$ containing $X_{\overline{\rho}}$ and α.*

Proof. The proof of this lemma is quite technical, but is no more than simple computations in GL_n. We denote by $X_{\overline{\rho},\alpha} \subseteq R^+$ the smallest closed subset relative to $P_{\overline{\rho}}$ containing $X_{\overline{\rho}}$ and α and by $\widetilde{X_{\overline{\rho}}} \subseteq X_{\overline{\rho}}$ the subset of roots which are *not* the sum of at least two roots of $X_{\overline{\rho},\alpha}$. For $g \in \mathrm{Gal}(\overline{\mathbb{Q}}_p/K)$ we can write

$$\overline{\rho}(g) = \overline{\rho}^{P_{\overline{\rho}}-\mathrm{ss}}(g) \prod_{\beta \in X_{\overline{\rho}}\backslash R(P_{\overline{\rho}})^+} n_\beta(g), \tag{69}$$

where $\overline{\rho}^{P_{\overline{\rho}}-\mathrm{ss}}(g) \in M_{P_{\overline{\rho}}}(\mathbb{F})$ and $n_\beta(g) \in N_\beta(\mathbb{F})$. Using (67), we see that

$$n_\alpha\Big(\prod_{\beta \in X_{\overline{\rho}}\backslash R(P_{\overline{\rho}})^+} n_\beta(g)\Big)n_\alpha^{-1} \in \prod_{\gamma} N_\gamma(\mathbb{F}), \tag{70}$$

where γ runs among the roots in R^+ of the form $\mathbb{Z}_{\geq 0}\alpha + \mathbb{Z}_{>0}\beta_1 + \cdots + \mathbb{Z}_{>0}\beta_s$ for $s \geq 1$ and $\beta_i \in X_{\overline{\rho}}\backslash R(P_{\overline{\rho}})^+$. This clearly implies $X_{n_\alpha\overline{\rho}n_\alpha^{-1}} \subseteq X_{\overline{\rho},\alpha}$. To prove the reverse inclusion, it is enough to prove $\widetilde{X_{\overline{\rho}}} \subseteq X_{n_\alpha \overline{\rho} n_\alpha^{-1}}$ and $w(\alpha) \in X_{n_\alpha \overline{\rho} n_\alpha^{-1}}$ for some $w \in W(P_{\overline{\rho}})$ (as then $\alpha \in X_{n_\alpha \overline{\rho} n_\alpha^{-1}}$ by Remark 2.3.1.5(i)).

An easy explicit matrix computation in GL_n (that we leave to the reader) gives that $n_\alpha \overline{\rho}^{P_{\overline{\rho}}-\mathrm{ss}}(g) n_\alpha^{-1}$ is of the form in $\mathrm{GL}_n(\mathbb{F})$:

$$n_\alpha \overline{\rho}^{P_{\overline{\rho}}-\mathrm{ss}}(g) n_\alpha^{-1} \in \overline{\rho}^{P_{\overline{\rho}}-\mathrm{ss}}(g) \prod_{\beta \in \{w(\alpha) : w \in W(P_{\overline{\rho}})\}} m_\beta(g) \tag{71}$$

with $m_\beta(g) \in N_\beta(\mathbb{F})$ (note that, as $w \in W(P_{\overline{\rho}})$, $w(\alpha)$ is of the form $\alpha + n_1\alpha_1 + \cdots + n_t\alpha_t$ for some $t \geq 0$, $\alpha_i \in S(P_{\overline{\rho}})$, $n_i \in \mathbb{Z}$). It then follows from (70) and (71) that, for $\beta \in \widetilde{X_{\overline{\rho}}} \backslash (\widetilde{X_{\overline{\rho}}} \cap R(P_{\overline{\rho}})^+)$, the entry $n_\beta(g)$ in (69) is not affected by the conjugation by n_α. In particular, we have $\widetilde{X_{\overline{\rho}}} \subseteq X_{n_\alpha \overline{\rho} n_\alpha^{-1}}$.

We now prove that $w(\alpha) \in X_{n_\alpha \overline{\rho} n_\alpha^{-1}}$ for some $w \in W(P_{\overline{\rho}})$. We first claim that none of the roots γ in (70) are in $\{w(\alpha) : w \in W(P_{\overline{\rho}})\}$. Indeed, assume $w(\alpha) = m\alpha + m_1\beta_1 + \cdots + m_s\beta_s$ for some $s \geq 0$, $m \geq 0$, $\beta_i \in X_{\overline{\rho}} \backslash R(P_{\overline{\rho}})^+$, $m_i > 0$. If $m = 0$, then we get $w(\alpha) = m_1\beta_1 + \cdots + m_s\beta_s \in X_{\overline{\rho}} \backslash R(P_{\overline{\rho}})^+$ since $X_{\overline{\rho}} \backslash R(P_{\overline{\rho}})^+$ is closed in R^+, which implies $\alpha \in X_{\overline{\rho}} \backslash R(P_{\overline{\rho}})^+$ by Definition 2.3.1.1(iii), a contradiction. If $m > 0$, then we get $(m-1)\alpha + m_1\beta_1 + \cdots + m_s\beta_s = n_1\alpha_1 + \cdots + n_t\alpha_t$ (writing $w(\alpha)$ as in the above form), which implies in particular all $\beta_i \in R(P_{\overline{\rho}})^+$, a contradiction. We deduce from this that for all $g \in \mathrm{Gal}(\overline{\mathbb{Q}}_p/K)$:

$$n_\alpha \overline{\rho}(g) n_\alpha^{-1} \in n_\alpha \overline{\rho}^{P_{\overline{\rho}}-\mathrm{ss}}(g) n_\alpha^{-1} \prod_\gamma N_\gamma(\mathbb{F})$$

with γ in $R^+ \backslash \Big(R(P_{\overline{\rho}})^+ \amalg \{w(\alpha) : w \in W(P_{\overline{\rho}})\}\Big)$.

We can see $\overline{\rho}^{P_{\overline{\rho}}-\mathrm{ss}}(g)$ as a block matrix $\mathrm{diag}(\overline{\rho}_1(g), \ldots, \overline{\rho}_d(g))$, where $\overline{\rho}_i : \mathrm{Gal}(\overline{\mathbb{Q}}_p/K) \to \mathrm{GL}_{n_i}(\mathbb{F})$ is irreducible. Assume that $\{w(\alpha) : w \in W(P_{\overline{\rho}})\} \supsetneq \{\alpha\}$. Then using that, for fixed i, the $\overline{\rho}_i(g)$ for $g \in \mathrm{Gal}(\overline{\mathbb{Q}}_p/K)$ do not take all values in the $\mathbb{F}$-points of a strict (not necessarily standard) parabolic subgroup of GL_{n_i}, one can check that at least one $m_\beta(g)$ in (71) is nontrivial for some $g \in \mathrm{Gal}(\overline{\mathbb{Q}}_p/K)$. If $\{w(\alpha) : w \in W(P_{\overline{\rho}})\} = \{\alpha\}$, then there are integers $1 \leq i < j \leq d$ such that $n_i = n_j = 1$ and the non-diagonal entry in $m_\alpha(g)$ is $(\overline{\rho}_i(g) - \overline{\rho}_j(g))x_\alpha$, where $x_\alpha \in \mathbb{F}^\times$ is the non-diagonal entry in n_α. By assumption, there is at least one $g \in \mathrm{Gal}(\overline{\mathbb{Q}}_p/K)$ such that $\overline{\rho}_i(g) \neq \overline{\rho}_j(g)$, which implies $m_\alpha(g) \neq 1$ for that g.

Hence we finally deduce that

$$n_\alpha \overline{\rho}(g) n_\alpha^{-1} \in \overline{\rho}^{P_{\overline{\rho}}-\mathrm{ss}}(g) \Bigg(\prod_{\beta \in \{w(\alpha) : w \in W(P_{\overline{\rho}})\}} m_\beta(g) \Bigg) \prod_\gamma N_\gamma(\mathbb{F})$$

with γ in $R^+ \backslash \Big(R(P_{\overline{\rho}})^+ \amalg \{w(\alpha) : w \in W(P_{\overline{\rho}})\}\Big)$ and at least one $m_\beta(g)$ being nontrivial for some $g \in \mathrm{Gal}(\overline{\mathbb{Q}}_p/K)$ and some $\beta \in \{w(\alpha) : w \in W(P_{\overline{\rho}})\}$. This implies that this β is in $X_{n_\alpha \overline{\rho} n_\alpha^{-1}}$ and finishes the proof. □

Proposition 2.3.2.2. *Let $\overline{\rho} : \mathrm{Gal}(\overline{\mathbb{Q}}_p/K) \to P_{\overline{\rho}}(\mathbb{F})$ and $X_{\overline{\rho}}$ as below* (68)*, and assume that the irreducible constituents of $\overline{\rho}^{\mathrm{ss}}$ of dimension* 1 *are all distinct. Then there is $h_0 \in P_{\overline{\rho}}(\mathbb{F})$* (*non unique in general*) *such that $X_{h_0 \overline{\rho} h_0^{-1}} \subseteq X_{h \overline{\rho} h^{-1}}$ for all $h \in P_{\overline{\rho}}(\mathbb{F})$.*

Proof. The proof is modelled on that of [BH15, Prop.3.2.3]. Since $M_{P_{\overline{\rho}}}$ normalizes $N_{X_{\overline{\rho}}\backslash R(P_{\overline{\rho}})^+}$ (Remark 2.3.1.5(ii)), it is enough to prove the same statement with $h_0, h \in N_{P_{\overline{\rho}}}(\mathbb{F})$. Using that $\overline{\rho}^{P_{\overline{\rho}}-\mathrm{ss}}(g)^{-1}h\overline{\rho}^{P_{\overline{\rho}}-\mathrm{ss}}(g) \in N_{X_{\overline{\rho}}\backslash R(P_{\overline{\rho}})^+}(\mathbb{F})$ for $h \in N_{X_{\overline{\rho}}\backslash R(P_{\overline{\rho}})^+}(\mathbb{F}) \subseteq N_{P_{\overline{\rho}}}(\mathbb{F})$ by Remark 2.3.1.5(ii) again, and that $N_{X_{\overline{\rho}}\backslash R(P_{\overline{\rho}})^+}(\mathbb{F})$ is a group, we deduce $X_{h\overline{\rho}h^{-1}} \subseteq X_{\overline{\rho}}$ for all $h \in N_{X_{\overline{\rho}}\backslash R(P_{\overline{\rho}})^+}(\mathbb{F})$. Replacing $\overline{\rho}$ by a suitable conjugate $h_0\overline{\rho}h_0^{-1}$ with $h_0 \in N_{X_{\overline{\rho}}\backslash R(P_{\overline{\rho}})^+}(\mathbb{F})$, we can assume $X_{h\overline{\rho}h^{-1}} = X_{\overline{\rho}}$ for all $h \in N_{X_{\overline{\rho}}\backslash R(P_{\overline{\rho}})^+}(\mathbb{F})$. It is enough to prove $X_{\overline{\rho}} \subseteq X_{h\overline{\rho}h^{-1}}$ for all $h \in N_{P_{\overline{\rho}}}(\mathbb{F})$. Choosing roots $\alpha_1, \dots, \alpha_m \in R^+\backslash X_{\overline{\rho}}$ as in Lemma 2.3.1.6 (for $P = P_{\overline{\rho}}$ and $X = X_{\overline{\rho}}$), we can write any $h \in N_{P_{\overline{\rho}}}(\mathbb{F})$ as $h = h_m h_{m-1} \cdots h_1 h_{\overline{\rho}}$, where $h_i \in \prod_{\beta\in\{w(\alpha_i):w\in W(P_{\overline{\rho}})\}} N_\beta(\mathbb{F})$ and $h_{\overline{\rho}} \in N_{X_{\overline{\rho}}\backslash R(P_{\overline{\rho}})^+}(\mathbb{F})$. We have $X_{h_{\overline{\rho}}\overline{\rho}h_{\overline{\rho}}^{-1}} = X_{\overline{\rho}}$ and a straightforward induction applying successively Lemma 2.3.2.1 to $X_{h_{\overline{\rho}}\overline{\rho}h_{\overline{\rho}}^{-1}}$ and $\alpha = \alpha_1$, $X_{h_1h_{\overline{\rho}}\overline{\rho}(h_1h_{\overline{\rho}})^{-1}}$ and $\alpha = \alpha_2$, etc. (which we can do thanks to Lemma 2.3.1.6) gives that $X_{h\overline{\rho}h^{-1}}$ is the smallest closed subset of R^+ relative to $P_{\overline{\rho}}$ containing $X_{\overline{\rho}}$ and the α_i, $i = 1, \dots, m$. In particular $X_{\overline{\rho}} \subseteq X_{h\overline{\rho}h^{-1}}$ for all $h \in N_{P_{\overline{\rho}}}(\mathbb{F})$. $\square$

Definition 2.3.2.3. Let $\overline{\rho} : \mathrm{Gal}(\overline{\mathbb{Q}}_p/K) \longrightarrow G(\mathbb{F})$ be a continuous homomorphism such that the irreducible constituents of $\overline{\rho}^{\mathrm{ss}}$ of dimension 1 are all distinct. A *good conjugate* of $\overline{\rho}$ is a conjugate $\overline{\rho}'$ of $\overline{\rho}$ in $G(\mathbb{F})$ which satisfies the two conditions:

(i) it is of the form $\overline{\rho}' : \mathrm{Gal}(\overline{\mathbb{Q}}_p/K) \to P_{\overline{\rho}'}(\mathbb{F}) \subseteq G(\mathbb{F})$ for a standard parabolic subgroup $P_{\overline{\rho}'}$ of G such that the image of $\overline{\rho}'^{P_{\overline{\rho}'}-\mathrm{ss}} : \mathrm{Gal}(\overline{\mathbb{Q}}_p/K) \xrightarrow{\overline{\rho}'} P_{\overline{\rho}'}(\mathbb{F}) \twoheadrightarrow M_{P_{\overline{\rho}'}}(\mathbb{F})$ is not contained in the $\mathbb{F}$-points of a proper parabolic subgroup of $M_{P_{\overline{\rho}'}}$;

(ii) $X_{\overline{\rho}'} \subseteq X_{h\overline{\rho}'h^{-1}}$ for all $h \in P_{\overline{\rho}'}(\mathbb{F})$.

From Proposition 2.3.2.2, we easily deduce that good conjugates always exist. If $\overline{\rho}$ is irreducible, then any conjugate of $\overline{\rho}$ in $G(\mathbb{F})$ is a good conjugate.

For $\overline{\rho} : \mathrm{Gal}(\overline{\mathbb{Q}}_p/K) \longrightarrow \widetilde{P}_{\overline{\rho}}(\mathbb{F}) \subseteq P_{\overline{\rho}}(\mathbb{F})$ as in (68), set

$$\begin{aligned} W_{\overline{\rho}} &\stackrel{\mathrm{def}}{=} \{w \in W : w(S(P_{\overline{\rho}})) \subseteq S \text{ and } w(X_{\overline{\rho}}\backslash R(P_{\overline{\rho}})^+) \subseteq R^+\} \\ &= \{w \in W : w(S(P_{\overline{\rho}})) \subseteq S \text{ and } w\widetilde{P}_{\overline{\rho}}w^{-1} \subseteq {}^wP_{\overline{\rho}}\}, \end{aligned} \tag{72}$$

where the second equality follows from Lemma 2.3.1.7. Using the definition of $X_{\overline{\rho}}$ we see that, for any $w \in W_{\overline{\rho}}$, we have $X_{w\overline{\rho}w^{-1}} = w(X_{\overline{\rho}})$, where

$$w\overline{\rho}w^{-1} : \mathrm{Gal}(\overline{\mathbb{Q}}_p/K) \longrightarrow w\widetilde{P}_{\overline{\rho}}(\mathbb{F})w^{-1} = \widetilde{P}_{w\overline{\rho}w^{-1}}(\mathbb{F}) \subseteq ({}^wP_{\overline{\rho}})(\mathbb{F}).$$

(and note that the set $X_{w\overline{\rho}w^{-1}}$ is relative to ${}^wP_{\overline{\rho}}$, while the set $X_{\overline{\rho}}$ is relative to $P_{\overline{\rho}}$).

Lemma 2.3.2.4. *Let $\overline{\rho} : \mathrm{Gal}(\overline{\mathbb{Q}}_p/K) \to G(\mathbb{F})$ as in Definition 2.3.2.3 and $\overline{\rho}' : \mathrm{Gal}(\overline{\mathbb{Q}}_p/K) \to \widetilde{P}_{\overline{\rho}'}(\mathbb{F}) \subseteq P_{\overline{\rho}'}(\mathbb{F})$ a good conjugate of $\overline{\rho}$ (where $\widetilde{P}_{\overline{\rho}'} \stackrel{\mathrm{def}}{=} M_{P_{\overline{\rho}'}}N_{X_{\overline{\rho}'}} = M_{P_{\overline{\rho}'}}N_{X_{\overline{\rho}'}\backslash R(P_{\overline{\rho}'})^+}$). Then any $h\overline{\rho}'h^{-1}$ for $h \in \widetilde{P}_{\overline{\rho}'}(\mathbb{F})$ and any $w\overline{\rho}'w^{-1}$ for $w \in W_{\overline{\rho}'}$ is a good conjugate of $\overline{\rho}$. Moreover we have $X_{h\overline{\rho}'h^{-1}} = X_{\overline{\rho}'}$ and $X_{w\overline{\rho}'w^{-1}} = w(X_{\overline{\rho}'})$.*

Proof. Again, the proof is formally the same as that of [BH15, Lemma 3.2.5]. The statement is obvious for $h \in \widetilde{P}_{\overline{\rho}'}(\mathbb{F})$ (as $hN_{X\backslash R(P)^+}h^{-1} = N_{X\backslash R(P)^+}$ for any X closed subset relative P and any $h \in N_{X\backslash R(P)^+}$) and the very last equality follows from the discussion just above. Following the argument in the proof of Proposition 2.3.2.2, it is enough to check

$$X_{h(w\overline{\rho}'w^{-1})h^{-1}} = X_{w\overline{\rho}'w^{-1}}$$

for all $h \in N_{X_{w\overline{\rho}'w^{-1}}\backslash R(P_{w\overline{\rho}'w^{-1}})^+}(\mathbb{F}) = N_{w(X_{\overline{\rho}'}\backslash R(P_{\overline{\rho}'})^+)}(\mathbb{F})$. We have

$$h(w\overline{\rho}'w^{-1})h^{-1} = w(w^{-1}hw)\overline{\rho}'(w^{-1}h^{-1}w)w^{-1}.$$

Since $w^{-1}hw \in N_{X_{\overline{\rho}'}\backslash R(P_{\overline{\rho}'})^+}(\mathbb{F})$, we have $X_{(w^{-1}hw)\overline{\rho}'(w^{-1}h^{-1}w)} \subseteq X_{\overline{\rho}'}$ and since $\overline{\rho}'$ is a good conjugate, we have $X_{\overline{\rho}'} \subseteq X_{(w^{-1}hw)\overline{\rho}'(w^{-1}h^{-1}w)}$, hence $X_{\overline{\rho}'} = X_{(w^{-1}hw)\overline{\rho}'(w^{-1}h^{-1}w)}$. Applying the discussion just before this lemma to $(w^{-1}hw)\overline{\rho}'(w^{-1}h^{-1}w)$ and then to $\overline{\rho}'$, we thus get $X_{h(w\overline{\rho}'w^{-1})h^{-1}} = w(X_{(w^{-1}hw)\overline{\rho}'(w^{-1}h^{-1}w)}) = w(X_{\overline{\rho}'}) = X_{w\overline{\rho}'w^{-1}}$. $\square$

We now state and prove the main result of this section (see [BH15, Prop.3.2.6]).

Theorem 2.3.2.5. *Let $\overline{\rho} : \mathrm{Gal}(\overline{\mathbb{Q}}_p/K) \to G(\mathbb{F})$ be a continuous homomorphism such that the irreducible constituents of $\overline{\rho}^{\mathrm{ss}}$ of dimension 1 are all distinct. Let $\overline{\rho}'$ and $\overline{\rho}''$ be two good conjugates of $\overline{\rho}$. Then there exist $h \in \widetilde{P}_{\overline{\rho}'}(\mathbb{F})$ and $w \in W_{\overline{\rho}'}$ such that $\overline{\rho}'' = w(h\overline{\rho}'h^{-1})w^{-1}$. In particular we have $X_{\overline{\rho}''} = w(X_{\overline{\rho}'})$.*

Proof. By assumption there is $x \in G(\mathbb{F})$ such that $\overline{\rho}''(g) = x\overline{\rho}'(g)x^{-1}$ for all $g \in \mathrm{Gal}(\overline{\mathbb{Q}}_p/K)$. We can write $x = h''wh'$ with $h' \in P_{\overline{\rho}'}(\mathbb{F})$, $h'' \in P_{\overline{\rho}''}(\mathbb{F})$ and $w \in W$ such that $w(R(P_{\overline{\rho}'})^+) \subseteq R^+$.

Step 1: We prove that $w(S(P_{\overline{\rho}'})) = S(P_{\overline{\rho}''})$. We have $wh'\overline{\rho}'(g)h'^{-1}w^{-1} \in P_{\overline{\rho}''}(\mathbb{F})$ for all $g \in \mathrm{Gal}(\overline{\mathbb{Q}}_p/K)$, which implies $h'\overline{\rho}'(g)h'^{-1} \in (w^{-1}P_{\overline{\rho}''}w \cap P_{\overline{\rho}'})(\mathbb{F}) \subseteq P_{\overline{\rho}'}(\mathbb{F})$ for all $g \in \mathrm{Gal}(\overline{\mathbb{Q}}_p/K)$. In particular, using for instance [DM91, Prop.2.1(iii)], the image of $h'\overline{\rho}'h'^{-1}$ in $M_{P_{\overline{\rho}'}}(\mathbb{F})$ is contained in the $\mathbb{F}$-points of the parabolic subgroup $w^{-1}P_{\overline{\rho}''}w \cap M_{P_{\overline{\rho}'}}$ of $M_{P_{\overline{\rho}'}}$. But since $(h'\overline{\rho}'h'^{-1})^{P_{\overline{\rho}'}-\mathrm{ss}}$ is conjugate to $\overline{\rho}'^{P_{\overline{\rho}'}-\mathrm{ss}}$ (recall $h' \in P_{\overline{\rho}'}(\mathbb{F})$), the image of $h'\overline{\rho}'h'^{-1}$ in $M_{P_{\overline{\rho}'}}(\mathbb{F})$ is not contained in the $\mathbb{F}$-points of a proper parabolic subgroup of $M_{P_{\overline{\rho}'}}$. Thus we must have $w^{-1}P_{\overline{\rho}''}w \cap M_{P_{\overline{\rho}'}} = M_{P_{\overline{\rho}'}}$ which implies $M_{P_{\overline{\rho}'}} \subseteq w^{-1}M_{P_{\overline{\rho}''}}w$. The same argument starting with $w^{-1}h''^{-1}\overline{\rho}''(g)h''w \in P_{\overline{\rho}'}(\mathbb{F})$ yields $M_{P_{\overline{\rho}''}} \subseteq wM_{P_{\overline{\rho}'}}w^{-1}$, i.e. we have $M_{P_{\overline{\rho}'}} = w^{-1}M_{P_{\overline{\rho}''}}w$. Since by assumption $w(R(P_{\overline{\rho}'})^+) \subseteq R^+$, this forces $w(S(P_{\overline{\rho}'})) = S(P_{\overline{\rho}''})$ (and thus $w(R(P_{\overline{\rho}'})^+) = R(P_{\overline{\rho}''})^+$).

Step 2: We choose roots $\alpha'_1, \ldots, \alpha'_{m'} \in R^+\backslash X_{\overline{\rho}'}$ as in Lemma 2.3.1.6 (for $P = P_{\overline{\rho}'}$ and $X = X_{\overline{\rho}'}$) and we write $h' = h'_{m'}h'_{m'-1}\cdots h'_1h'_{\overline{\rho}}$, where $h'_i \in \prod_{\beta\in\{w'(\alpha'_i):w'\in W(P_{\overline{\rho}'})\}}N_\beta(\mathbb{F})$ and $h'_{\overline{\rho}'} \in \widetilde{P}_{\overline{\rho}'}(\mathbb{F})$. By Lemma 2.3.2.4, we can replace $\overline{\rho}'$ by $h'_{\overline{\rho}'}\overline{\rho}'h'^{-1}_{\overline{\rho}'}$ and thus assume $h'_{\overline{\rho}'} = 1$. By Lemma 2.3.2.1 and an induction as in the proof of Proposition 2.3.2.2, $X_{h'\overline{\rho}'h'^{-1}}$ is the smallest closed subset relative to $P_{\overline{\rho}'}$ containing $X_{\overline{\rho}'}$ and those α'_i such that $h'_i \neq 1$. Since $w(h'\overline{\rho}'h'^{-1})w^{-1}$ takes values in $P_{\overline{\rho}''}(\mathbb{F})$ and $w(R(P_{\overline{\rho}'})) = R(P_{\overline{\rho}''})$ (by Step 1), we must also have $w(X_{h'\overline{\rho}'h'^{-1}}\backslash R(P_{\overline{\rho}'})^+) \subseteq R^+\backslash R(P_{\overline{\rho}''})^+$. This implies $ww'(\alpha'_i) \in R^+$ if $w' \in W(P_{\overline{\rho}'})$ and $h'_i \neq 1$, and $w(X_{\overline{\rho}'}\backslash R(P_{\overline{\rho}'})^+) \subseteq R^+$. In particular $w \in W_{\overline{\rho}'}$ together with Step 1.

Step 3: We prove that $X_{\overline{\rho}''} = w(X_{\overline{\rho}'})$. Setting

$$h_i \stackrel{\text{def}}{=} wh'_iw^{-1} \quad \in \prod_{\beta\in\{ww'(\alpha'_i):w'\in W(P_{\overline{\rho}'})\}} N_\beta(\mathbb{F}) \quad \subseteq \quad P_{\overline{\rho}''}(\mathbb{F})$$

(we proved $ww'(\alpha'_i) \in R^+$ in Step 2), we have

$$\overline{\rho}'' = h''(h_{m'}\cdots h_1)(w\overline{\rho}'w^{-1})(h_1^{-1}\cdots h_{m'}^{-1})h''^{-1}, \tag{73}$$

where $h''h_{m'}\cdots h_1 \in P_{\overline{\rho}''}(\mathbb{F})$ and where $\overline{\rho}''$ and $w\overline{\rho}'w^{-1}$ are good conjugates of $\overline{\rho}$ (the latter by Lemma 2.3.2.4). Applying Definition 2.3.2.3 to both $\overline{\rho}''$ and $w\overline{\rho}'w^{-1}$, we get $X_{\overline{\rho}''} = X_{w\overline{\rho}'w^{-1}} = w(X_{\overline{\rho}'})$ (and thus $w^{-1}\widetilde{P}_{\overline{\rho}''}w = \widetilde{P}_{\overline{\rho}'}$).

Step 4 : We complete the proof. We choose again roots $\alpha''_1,\dots,\alpha''_{m''} \in R^+\backslash X_{w\overline{\rho}'w^{-1}}$ as in Lemma 2.3.1.6 for $P = P_{w\overline{\rho}'w^{-1}} = P_{\overline{\rho}''}$ (this latter equality from Remark 2.2.1.4) and $X = X_{w\overline{\rho}'w^{-1}} = X_{\overline{\rho}''}$ and we write

$$h''(h_{m'}\cdots h_1) = h''_{m''}h''_{m''-1}\cdots h''_1h''_{X_{\overline{\rho}''}},$$

where $h''_i \in \prod_{\beta\in\{w''(\alpha''_i):w''\in W(P_{\overline{\rho}''})\}} N_\beta(\mathbb{F})$ and $h''_{X_{\overline{\rho}''}} \in \widetilde{P}_{w\overline{\rho}'w^{-1}}(\mathbb{F}) = \widetilde{P}_{\overline{\rho}''}(\mathbb{F})$. From (73) and Lemma 2.3.2.1, we see that we must have $h''_i = 1$ for all $i \in \{1,\dots,m''\}$ otherwise $X_{\overline{\rho}''}$ would be strictly bigger that $X_{w\overline{\rho}'w^{-1}}$. Thus we deduce

$$\overline{\rho}'' = h''_{X_{\overline{\rho}''}}w\overline{\rho}'w^{-1}h''^{-1}_{X_{\overline{\rho}''}} = w(w^{-1}h''_{X_{\overline{\rho}''}}w)\overline{\rho}'(w^{-1}h''^{-1}_{X_{\overline{\rho}''}}w)w^{-1}.$$

Setting $h \stackrel{\text{def}}{=} w^{-1}h''_{X_{\overline{\rho}''}}w \in w^{-1}\widetilde{P}_{\overline{\rho}''}(\mathbb{F})w = \widetilde{P}_{\overline{\rho}'}(\mathbb{F})$, this finishes the proof. □

2.4 The definition of compatibility

Given a sufficiently generic n-dimensional representation of $\mathrm{Gal}(\overline{\mathbb{Q}}_p/K)$ over $\mathbb{F}$ (where $K = \mathbb{Q}_{p^f}$ is still unramified) and a good conjugate $\overline{\rho}$ of this representation as in Definition 2.3.2.3, we define what it means for a smooth representation of $G(K)$ over $\mathbb{F}$ to be *compatible with* $\widetilde{P}_{\overline{\rho}}$ (Definition 2.4.1.5, see the beginning of §2.3.2 for $\widetilde{P}_{\overline{\rho}}$) and to be *compatible with* $\overline{\rho}$ (Definition 2.4.2.7).

2.4.1 Compatibility with $\widetilde{P}$

We first define what it means for a smooth representation of $G(K)$ over $\mathbb{F}$ to be compatible with a Zariski closed subgroup $\widetilde{P}$ of a standard parabolic subgroup P as in Definition 2.2.1.3. We keep the notation of §§2.2, 2.3.

We fix a Zariski closed algebraic subgroup $\widetilde{P}$ of a standard parabolic subgroup P of G as in Definition 2.2.1.3 (by Remark 2.2.1.4, P is in fact determined by $\widetilde{P}$). We let X be the unique closed subset of R^+ relative to P such that $\widetilde{P} = M_P N_X$ (Lemma 2.3.1.4) and define

$$W_{\widetilde{P}} \stackrel{\text{def}}{=} \{w \in W : w(S(P)) \subseteq S,\ w(X \backslash R(P)^+) \subseteq R^+\}.$$

Note that $W_{\widetilde{P}}$ is analogous to $W_{\widetilde{\rho}}$ in (72) with $\widetilde{P}_{\widetilde{\rho}}$ replaced by $\widetilde{P}$.

Let Q be a parabolic subgroup containing ${}^{w_{\widetilde{P}}}P$ for some $w_{\widetilde{P}} \in W_{\widetilde{P}}$, w_Q an element of W such that $w_Q(S(Q)) \subseteq S$ and Q' a parabolic subgroup containing ${}^{w_Q}Q$ (note that both Q and Q' are standard). So we have inclusions of standard parabolic subgroups ${}^{w_Q w_{\widetilde{P}}}P \subseteq {}^{w_Q}Q \subseteq Q'$ and likewise for the Levi subgroups

$$M_{{}^{w_Q w_{\widetilde{P}}}P} = w_Q w_{\widetilde{P}} M_P (w_Q w_{\widetilde{P}})^{-1} \subseteq M_{{}^{w_Q}Q} = w_Q M_Q w_Q^{-1} \subseteq M_{Q'}.$$

Using that we work with GL_n, we write

$$M_{Q'} = \mathrm{diag}(M_1, \dots, M_d)$$

with $M_i \cong \mathrm{GL}_{n_i}$ and we define the standard parabolic subgroup $({}^{w_Q}Q)_i$ of M_i as

$$({}^{w_Q}Q)_i \stackrel{\text{def}}{=} \mathrm{Im}\Big({}^{w_Q}Q \hookrightarrow Q' \twoheadrightarrow M_{Q'} \twoheadrightarrow M_i\Big).$$

We define a standard parabolic subgroup $({}^{w_Q w_{\widetilde{P}}}P)_Q$ of $M_{{}^{w_Q}Q}$, resp. a standard parabolic subgroup $({}^{w_Q w_{\widetilde{P}}}P)_{Q,i}$ of $M_{({}^{w_Q}Q)_i}$, as the image of ${}^{w_Q w_{\widetilde{P}}}P$ via ${}^{w_Q w_{\widetilde{P}}}P \subseteq {}^{w_Q}Q \twoheadrightarrow M_{{}^{w_Q}Q}$, resp. via ${}^{w_Q w_{\widetilde{P}}}P \subseteq {}^{w_Q}Q \twoheadrightarrow M_{{}^{w_Q}Q} \twoheadrightarrow M_{({}^{w_Q}Q)_i}$. Equivalently,

$$\begin{aligned} ({}^{w_Q w_{\widetilde{P}}}P)_Q &= w_Q({}^{w_{\widetilde{P}}}P \cap M_Q)w_Q^{-1} \subseteq w_Q M_Q w_Q^{-1} = M_{{}^{w_Q}Q} \\ ({}^{w_Q w_{\widetilde{P}}}P)_{Q,i} &= \mathrm{Im}\Big(w_Q({}^{w_{\widetilde{P}}}P \cap M_Q)w_Q^{-1} \subseteq M_{{}^{w_Q}Q} \twoheadrightarrow M_{({}^{w_Q}Q)_i}\Big). \end{aligned}$$

Note that

$$M_{({}^{w_Q w_{\widetilde{P}}}P)_Q} = w_Q M_{({}^{w_{\widetilde{P}}}P) \cap M_Q} w_Q^{-1} = w_Q w_{\widetilde{P}} M_P (w_Q w_{\widetilde{P}})^{-1}.$$

We finally define a Zariski closed algebraic subgroup $({}^{w_Q w_{\widetilde{P}}}\widetilde{P})_Q$ of $({}^{w_Q w_{\widetilde{P}}}P)_Q$ containing $M_{({}^{w_Q w_{\widetilde{P}}}P)_Q}$, resp. a Zariski closed algebraic subgroup $({}^{w_Q w_{\widetilde{P}}}\widetilde{P})_{Q,i}$ of $({}^{w_Q w_{\widetilde{P}}}P)_{Q,i}$ containing $M_{({}^{w_Q w_{\widetilde{P}}}P)_{Q,i}}$, as

$$\begin{aligned} ({}^{w_Q w_{\widetilde{P}}}\widetilde{P})_Q &\stackrel{\text{def}}{=} w_Q\Big((w_{\widetilde{P}}\widetilde{P}w_{\widetilde{P}}^{-1}) \cap M_Q\Big)w_Q^{-1} \subseteq w_Q({}^{w_{\widetilde{P}}}P \cap M_Q)w_Q^{-1} = ({}^{w_Q w_{\widetilde{P}}}P)_Q \\ ({}^{w_Q w_{\widetilde{P}}}\widetilde{P})_{Q,i} &\stackrel{\text{def}}{=} \mathrm{Im}\Big(w_Q\Big((w_{\widetilde{P}}\widetilde{P}w_{\widetilde{P}}^{-1}) \cap M_Q\Big)w_Q^{-1} \subseteq M_{{}^{w_Q}Q} \twoheadrightarrow M_{({}^{w_Q}Q)_i}\Big). \end{aligned}$$

We also define the continuous group homomorphism

$$\omega^{-1} \circ \theta^{Q'} \ : \ Q'^{-}(K) \longrightarrow M_{Q'}(K) \xrightarrow{\theta^{Q'}} K^\times \xrightarrow{\omega^{-1}} \mathbb{F}_p^\times \hookrightarrow \mathbb{F}^\times,$$

where $\theta^{Q'}$ is defined in (45) (applied with $P = Q'$).

We need a quite formal and easy lemma.

Lemma 2.4.1.1. *Let Π be a smooth representation of a p-adic analytic group over $\mathbb{F}$ which has finite length and distinct absolutely irreducible constituents. Let H be a split connected reductive algebraic group over $\mathbb{Z}$, $P_H \subseteq H$ a parabolic subgroup with Levi M_{P_H}, $\widetilde{P}_H \subseteq P_H$ a Zariski closed algebraic subgroup containing M_{P_H} and R a (finite-dimensional) algebraic representation of $P_H^{\mathrm{Gal}(K/\mathbb{Q}_p)}$ over $\mathbb{F}$. Assume that there exist*

(a) a filtration on R by good subrepresentations for the $P_H^{\mathrm{Gal}(K/\mathbb{Q}_p)}$-action (see Definition 2.2.1.3) such that the graded pieces exhaust the isotypic components of $R|_{Z_{M_{P_H}}}$;

(b) a bijection Φ of partially ordered finite sets between the set of subrepresentations of Π and the set of good subrepresentations of $R|_{\widetilde{P}_H^{\mathrm{Gal}(K/\mathbb{Q}_p)}}$ (both being ordered by inclusion).

Then the following hold:

(i) *The bijection Φ uniquely extends to bijections between subquotients of Π and good subquotients of $R|_{\widetilde{P}_H^{\mathrm{Gal}(K/\mathbb{Q}_p)}}$, and between irreducible constituents of Π and isotypic components of $R|_{Z_{M_{P_H}}}$.*

(ii) *If Π' is a subquotient of Π, then Φ induces a bijection of partially ordered finite sets between the set of subrepresentations of Π' and the set of good subrepresentations of $\Phi(\Pi')|_{\widetilde{P}_H^{\mathrm{Gal}(K/\mathbb{Q}_p)}}$.*

Proof. Formal and left to the reader. □

Remark 2.4.1.2. (i) Let Π and Φ as in Lemma 2.4.1.1, Π' a subquotient of Π and $\Pi'' \subseteq \Pi'$ a subrepresentation. Then the bijection Φ also induces a short exact sequence $0 \to \Phi(\Pi'') \to \Phi(\Pi') \to \Phi(\Pi'/\Pi'') \to 0$ of algebraic representation of $\widetilde{P}_H^{\mathrm{Gal}(K/\mathbb{Q}_p)}$ over $\mathbb{F}$.
(ii) By Lemma 2.2.1.5 applied with P there being the parabolic ${}^{w_{\widetilde{P}}}P$ above, we see that Lemma 2.4.1.1 can be applied with $H = G$, $P_H = {}^{w_{\widetilde{P}}}P$, $\widetilde{P}_H = w_{\widetilde{P}}\widetilde{P}w_{\widetilde{P}}^{-1}$ and $R = \overline{L}^{\otimes}$. Using moreover Lemma 2.2.1.6, one easily sees that Lemma 2.4.1.1 can also be applied with $H = M_Q$, $P_H = {}^{w_{\widetilde{P}}}P \cap M_Q$, $\widetilde{P}_H = (w_{\widetilde{P}}\widetilde{P}w_{\widetilde{P}}^{-1}) \cap M_Q$ and R any isotypic component C_Q of $\overline{L}^{\otimes}|_{Z_{M_Q}}$ (recall from (the proof of) Lemma 2.2.1.5 applied with P there being Q that the action of $Q^{\mathrm{Gal}(K/\mathbb{Q}_p)}$ on the subquotient C_Q of $\overline{L}^{\otimes}|_{Q^{\mathrm{Gal}(K/\mathbb{Q}_p)}}$ factors through $Q^{\mathrm{Gal}(K/\mathbb{Q}_p)} \twoheadrightarrow M_Q^{\mathrm{Gal}(K/\mathbb{Q}_p)}$).
(iii) Let Q as above, C_Q an isotypic component of $\overline{L}^{\otimes}|_{Z_{M_Q}}$, $Q' \stackrel{\mathrm{def}}{=} P(C_Q)$ (see §2.2.2) and $w_Q \in W(C_Q)$ (see (39) and note that ${}^{w_Q}Q \subseteq Q'$ by (40)). Lemma 2.4.1.1 can

also be applied with $H = M_{({}^{w_Q}Q)_i}$, $P_H = ({}^{w_Q w_{\widetilde{P}}}P)_{Q,i}$, $\widetilde{P}_H = ({}^{w_Q w_{\widetilde{P}}}\widetilde{P})_{Q,i}$ and $R = C_{w_Q,i}$, where $C_{w_Q,i}$ is the algebraic representation of $M_{({}^{w_Q}Q)_i}^{\mathrm{Gal}(K/\mathbb{Q}_p)}$ defined in Remark 2.2.3.12 with P there being Q (it is an isotypic component of $\overline{L}_i^{\otimes}|_{Z_{M_{({}^{w_Q}Q)_i}}}$). To prove that assumption (a) of Lemma 2.4.1.1 is satisfied in that case, note that $C_{w_Q,i}$ is a good subquotient of $\overline{L}_i^{\otimes}|_{({}^{w_Q}Q)_i^{\mathrm{Gal}(K/\mathbb{Q}_p)}}$, and thus *a fortiori* a good subquotient of $\overline{L}_i^{\otimes}|_{({}^{w_Q w_{\widetilde{P}}}P)_{Q',i}^{\mathrm{Gal}(K/\mathbb{Q}_p)}}$ (Lemma 2.2.1.6), where $({}^{w_Q w_{\widetilde{P}}}P)_{Q',i} \subseteq ({}^{w_Q}Q)_i \subseteq M_i$ is the standard parabolic subgroup of M_i with the same Levi as $({}^{w_Q w_{\widetilde{P}}}P)_{Q,i}$. We have

$$({}^{w_Q w_{\widetilde{P}}}\widetilde{P})_{Q,i} \subseteq ({}^{w_Q w_{\widetilde{P}}}P)_{Q,i} \subseteq ({}^{w_Q w_{\widetilde{P}}}P)_{Q',i} \subseteq M_i$$

and $({}^{w_Q w_{\widetilde{P}}}\widetilde{P})_{Q,i}$ is a closed algebraic subgroup of $({}^{w_Q w_{\widetilde{P}}}P)_{Q',i}$ containing $M_{({}^{w_Q w_{\widetilde{P}}}P)_{Q',i}} = M_{({}^{w_Q w_{\widetilde{P}}}P)_{Q,i}}$. One then applies Lemma 2.2.1.5 with $\overline{L}_i^{\otimes}$ and with

$$({}^{w_Q w_{\widetilde{P}}}\widetilde{P})_{Q,i} \subseteq ({}^{w_Q w_{\widetilde{P}}}P)_{Q',i} \subseteq M_i$$

instead of $\widetilde{P} \subseteq P \subseteq G$, which implies that there is a filtration on $C_{w_Q,i}|_{({}^{w_Q w_{\widetilde{P}}}\widetilde{P})_{Q,i}^{\mathrm{Gal}(K/\mathbb{Q}_p)}}$ (or on $C_{w_Q,i}|_{({}^{w_Q w_{\widetilde{P}}}P)_{Q',i}^{\mathrm{Gal}(K/\mathbb{Q}_p)}}$, and thus on $C_{w_Q,i}|_{({}^{w_Q w_{\widetilde{P}}}P)_{Q,i}^{\mathrm{Gal}(K/\mathbb{Q}_p)}}$) by good subrepresentations such that the graded pieces exhaust the isotypic components of $C_{w_Q,i}|_{Z_{M_{({}^{w_Q w_{\widetilde{P}}}P)_{Q,i}}}} = C_{w_Q,i}|_{Z_{M_{({}^{w_Q w_{\widetilde{P}}}P)_{Q',i}}}}$.

Lemma 2.4.1.3. *Let $\widetilde{P} \subseteq P$, $w_{\widetilde{P}} \in W_{\widetilde{P}}$ and Q containing ${}^{w_{\widetilde{P}}}P$ as above. Let C_Q be an isotypic component of $\overline{L}^{\otimes}|_{Z_{M_Q}}$ and $Q' \stackrel{\mathrm{def}}{=} P(C_Q)$.*

(i) *For any $w_Q \in W(C_Q)$, there is a canonical bijection of partially ordered finite sets between the set of good subrepresentations of*

$$C_Q|_{(w_{\widetilde{P}}\widetilde{P}w_{\widetilde{P}}^{-1})^{\mathrm{Gal}(K/\mathbb{Q}_p)}} = C_Q|_{((w_{\widetilde{P}}\widetilde{P}w_{\widetilde{P}}^{-1})\cap M_Q)^{\mathrm{Gal}(K/\mathbb{Q}_p)}}$$

(*where the equality follows from Remark 2.4.1.2(ii)*) *and the set of good subrepresentations of $w_Q(C_Q)|_{({}^{w_Q w_{\widetilde{P}}}\widetilde{P})_Q^{\mathrm{Gal}(K/\mathbb{Q}_p)}}$.*

(ii) *For any $w_Q, w'_Q \in W(C_Q)$ and $i \in \{1, \ldots, d\}$, there is a canonical bijection of partially ordered finite sets between the set of good subrepresentations of $C_{w_Q,i}|_{({}^{w_Q w_{\widetilde{P}}}\widetilde{P})_{Q,i}^{\mathrm{Gal}(K/\mathbb{Q}_p)}}$ and the set of good subrepresentations of $C_{w'_Q,i}|_{({}^{w'_Q w_{\widetilde{P}}}\widetilde{P})_{Q,i}^{\mathrm{Gal}(K/\mathbb{Q}_p)}}$.*

Proof. (i) follows from the definition of $w_Q(C_Q)$ in (47) and the fact that $({}^{w_Q w_{\widetilde{P}}}\widetilde{P})_Q = w_Q\big((w_{\widetilde{P}}\widetilde{P}w_{\widetilde{P}}^{-1}) \cap M_Q\big)w_Q^{-1}$.

(ii) We have $w'_Q = w_{Q'}w_Q$ with $w_{Q'} \in W(P(C_Q)) = W(Q')$ by Lemma 2.2.2.10 (applied with P there being Q). In particular $w_{Q'}(w_Q(S(Q))) \subseteq S$ which implies $({}^{w'_Q w_{\widetilde{P}}}\widetilde{P})_{Q,i} = w_{Q'}({}^{w_Q w_{\widetilde{P}}}\widetilde{P})_{Q,i}w_{Q'}^{-1}$ inside $M_{({}^{w'_Q}Q)_i} = w_{Q'}M_{({}^{w_Q}Q)_i}w_{Q'}^{-1}$ (viewing $w_{Q'}$ as an element in $W(M_i)$ by abuse of notation). By (55) (applied with P there being Q) we have $C_{w'_Q,i} = w_{Q'}(C_{w_Q,i})$, where the conjugation by $w_{Q'}^{-1}$ intertwines the actions of $({}^{w'_Q w_{\widetilde{P}}}\widetilde{P})_{Q,i}$ and of $({}^{w_Q w_{\widetilde{P}}}\widetilde{P})_{Q,i}$. The result follows. □

Remark 2.4.1.4. The bijections in Lemma 2.4.1.3 all extend to bijections between good subquotients or isotypic components on both sides, as for Lemma 2.4.1.1.

Let Π, H, P_H, $\widetilde{P}_H$, R and Φ be as in Lemma 2.4.1.1. For any $w_H \in W_H$ (the Weyl group of H) such that $w_H\widetilde{P}_Hw_H^{-1}$ is contained in a standard parabolic subgroup of H, we can define another bijection $w_H(\Phi)$ between the set of subquotients of Π and the set of good subquotients of $R|_{(w_H\widetilde{P}_Hw_H^{-1})^{\mathrm{Gal}(K/\mathbb{Q}_p)}}$ as follows: $w_H(\Phi)(\Pi')$ is the algebraic representation $w_H\Big(\Phi(\Pi')\Big)$ of $(w_H\widetilde{P}_Hw_H^{-1})^{\mathrm{Gal}(K/\mathbb{Q}_p)}$, where $w_H\Big(\Phi(\Pi')\Big)(g) \stackrel{\text{def}}{=} \Phi(\Pi')(w_H^{-1}gw_H)$ if $g \in (w_H\widetilde{P}_Hw_H^{-1})^{\mathrm{Gal}(K/\mathbb{Q}_p)}$, see (47).

Here is now the first crucial definition.

Definition 2.4.1.5. An admissible smooth representation Π of $G(K)$ over $\mathbb{F}$ which has finite length and distinct absolutely irreducible constituents is *compatible with* $\widetilde{P}$ if there exists a bijection Φ of partially ordered finite sets between the set of subrepresentations of Π and the set of good subrepresentations of $\overline{L}^{\otimes}|_{\widetilde{P}^{\mathrm{Gal}(K/\mathbb{Q}_p)}}$ (both being ordered by inclusion) which satisfies the following conditions (once extended to all subquotients as in Lemma 2.4.1.1):

(i) (**form of subquotients**) for any $w_{\widetilde{P}} \in W_{\widetilde{P}}$, any parabolic subgroup Q containing ${}^{w_{\widetilde{P}}}P$ and any isotypic component C_Q of $\overline{L}^{\otimes}|_{Z_{M_Q}}$, writing $M_{P(C_Q)} = M_1 \times \cdots \times M_d$ with $M_i \cong \mathrm{GL}_{n_i}$ we have
$$w_{\widetilde{P}}(\Phi)^{-1}(C_Q) \cong \mathrm{Ind}_{P(C_Q)^-(K)}^{G(K)} \Big(\pi(C_Q) \otimes (\omega^{-1} \circ \theta^{P(C_Q)})\Big), \tag{74}$$
where $P(C_Q)$ is defined in §2.2.2, $\theta^{P(C_Q)}$ is defined in (45) and where $\pi(C_Q)$ is a $M_{P(C_Q)}$-representation of the form $\pi(C_Q) \cong \pi_1(C_Q) \otimes \cdots \otimes \pi_d(C_Q)$ for some (finite length) admissible smooth representations $\pi_i(C_Q)$ of $M_i(K)$ over $\mathbb{F}$;

(ii) (**compatibility between subquotients**) for any $w_{\widetilde{P}} \in W_{\widetilde{P}}$, any parabolic subgroup Q containing ${}^{w_{\widetilde{P}}}P$, any isotypic component C_Q of $\overline{L}^{\otimes}|_{Z_{M_Q}}$ and any $w \in W$ such that $w\Big(S(P(C_Q))\Big) \subseteq S$, let $w(\pi(C_Q))$ be the representation of $M_{{}^wP(C_Q)}(K) = wM_{P(C_Q)}(K)w^{-1}$ defined by
$$w(\pi(C_Q))(g) \stackrel{\text{def}}{=} \pi(C_Q)(w^{-1}gw)$$

for $\pi(C_Q)$ as in (74) and $g \in M_{^wP(C_Q)}(K)$. Then we have

$$\pi\Big(w \cdot C_Q\Big) \cong w\Big(\pi(C_Q)\Big),$$

where $w \cdot C_Q$ is the isotypic component of $\overline{L}^{\otimes}|_{Z_{M_Q}}$ in Proposition 2.2.4.2(ii) (applied with P there being Q) and where $\pi(w \cdot C_Q)$ is as in (74) for the isotypic component $w \cdot C_Q$ instead of C_Q (note that $P(w \cdot C_Q) = {}^wP(C_Q)$ by Proposition 2.2.4.2(iii));

(iii) (**product structure**) for any $w_{\widetilde{P}} \in W_{\widetilde{P}}$, any parabolic subgroup Q containing ${}^{w_{\widetilde{P}}}P$, any isotypic component C_Q of $\overline{L}^{\otimes}|_{Z_{M_Q}}$, and one, or equivalently any by Lemma 2.4.1.3(ii), element $w_Q \in W(C_Q)$, writing $M_{P(C_Q)} = \mathrm{diag}(M_1, \dots, M_d)$ with $M_i \cong \mathrm{GL}_{n_i}$, the restriction of $w_{\widetilde{P}}(\Phi)$ to the set of subquotients of $w_{\widetilde{P}}(\Phi)^{-1}(C_Q)$ comes from d bijections $w_{\widetilde{P}}(\Phi)_{w_Q,i}$ of partially ordered sets between the set of $M_i(K)$-subrepresentations of $\pi_i(C_Q)$ (where $\pi_i(C_Q)$ is as in (i)) and the set of good subrepresentations of $C_{w_Q,i}|_{({}^{w_Q w_{\widetilde{P}}}\widetilde{P})^{\mathrm{Gal}(K/\mathbb{Q}_p)}_{Q,i}}$ (where $C_{w_Q,i}$ is the isotypic component of $\overline{L}_i^{\otimes}|_{Z_{M_{({}^{w_Q}Q)_i}}}$ with its $M^{\mathrm{Gal}(K/\mathbb{Q}_p)}_{({}^{w_Q}Q)_i}$-action in (54) applied with P there being Q) in the following sense: for any subquotient Π' of $\Phi^{-1}(C_Q)$ of the form

$$\Pi' \cong \mathrm{Ind}_{P(C_Q)^-(K)}^{G(K)} \Big((\pi'_1 \otimes \cdots \otimes \pi'_d) \otimes (\omega^{-1} \circ \theta^{P(C_Q)})\Big)$$

with π'_i a subquotient of $\pi_i(C_Q)$, the good subquotient $w_{\widetilde{P}}(\Phi)(\Pi')$ of

$$C_Q|_{(w_{\widetilde{P}}\widetilde{P}w_{\widetilde{P}}^{-1})^{\mathrm{Gal}(K/\mathbb{Q}_p)}} = C_Q|_{((w_{\widetilde{P}}\widetilde{P}w_{\widetilde{P}}^{-1}) \cap M_Q)^{\mathrm{Gal}(K/\mathbb{Q}_p)}}$$

corresponds via Lemma 2.4.1.3(i) and Remark 2.4.1.4 to the following algebraic representation of $({}^{w_Q w_{\widetilde{P}}}\widetilde{P})_Q^{\mathrm{Gal}(K/\mathbb{Q}_p)} = \prod_{i=1}^d ({}^{w_Q w_{\widetilde{P}}}\widetilde{P})_{Q,i}^{\mathrm{Gal}(K/\mathbb{Q}_p)}$:

$$\bigotimes_{i=1}^{d} \Big(w_{\widetilde{P}}(\Phi)_{w_Q,i}(\pi'_i) \otimes \Big(\underbrace{(\theta^{P(C_Q)})_i \otimes \cdots \otimes (\theta^{P(C_Q)})_i}_{\mathrm{Gal}(K/\mathbb{Q}_p)} \Big)\Big);$$

(iv) (**supersingular**) for any isotypic component C_P of $\overline{L}^{\otimes}|_{Z_{M_P}}$, the (absolutely irreducible) $M_{P(C_P)}(K)$-representation $\pi(C_P)$ of (74) is supersingular (cf. [Her11, Def.4.7, Def.9.12, Cor.9.13]).

If (Π, Φ) is as in Definition 2.4.1.5, then we have in particular $\Phi(\Pi) = \overline{L}^{\otimes}$ and $w_{\widetilde{P}}(\Phi)_{w_Q,i}(\pi_i(C_Q)) = C_{w_Q,i}$. If $\widetilde{P} = G$, then Π is compatible with $\widetilde{P}$ if and only if Π is absolutely irreducible supersingular. Also it is clear from Definition 2.4.1.5 that, for a fixed $w_{\widetilde{P}} \in W_{\widetilde{P}}$, Π is compatible with $\widetilde{P}$ if and only if Π is compatible with $w_{\widetilde{P}}\widetilde{P}w_{\widetilde{P}}^{-1}$ (replace Φ by $w_{\widetilde{P}}(\Phi)$).

Remark 2.4.1.6. (i) In Definition 2.4.1.5, we have used Lemma 2.4.1.1 everywhere (see Remark 2.4.1.2(ii)(iii)). In Definition 2.4.1.5(iii), we have used Remark 2.4.1.4. Also, Definition 2.4.1.5 is somewhat redundant since a parabolic subgroup Q can contain ${}^{w_{\widetilde{P}}}P$ for several $w_{\widetilde{P}} \in W_{\widetilde{P}}$, but we found it too tedious to make it "non-redundant".
(ii) The representations $\pi(C_Q)$ and $\pi_i(C_Q)$ in Definition 2.4.1.5(i) are uniquely defined since there are no nontrivial intertwinings between parabolic inductions (by [Eme10a]).
(iii) When $Q = {}^{w_{\widetilde{P}}}P$, $\pi(C_{{}^{w_{\widetilde{P}}}P})$ in (74) is absolutely irreducible, and is thus automatically of the form $\pi(C_{{}^{w_{\widetilde{P}}}P}) \cong \pi_1(C_{{}^{w_{\widetilde{P}}}P}) \otimes \cdots \otimes \pi_d(C_{{}^{w_{\widetilde{P}}}P})$. It is then not difficult to deduce from this, together with Lemma 2.2.2.8 and [Eme10a] (and the properties of Φ), that each $\pi_i(C_Q)$ as in (74) has distinct (absolutely) irreducible constituents and that each irreducible constituent of (74) is of the form $\mathrm{Ind}_{P(C_Q)^-(K)}^{G(K)}\big((\pi'_1 \otimes \cdots \otimes \pi'_d) \otimes (\omega^{-1} \circ \theta^{P(C_Q)})\big)$, where π'_i is an irreducible constituent of $\pi_i(C_Q)$. This also justifies the terminology "comes from d bijections $w_{\widetilde{P}}(\Phi)_{w_Q,i}$" in Definition 2.4.1.5(iii).
(iv) It is in fact possible that Definition 2.4.1.5(i) for parabolic subgroups Q *strictly* containing some ${}^{w_{\widetilde{P}}}P$ and Definition 2.4.1.5(iii) both *automatically follow* from the other conditions in Definition 2.4.1.5. See for instance how the results of [Hau18] are used in Example 2, Example 4, Example 5 and Example 6 of §2.4.3 below to show that several conditions of Definition 2.4.1.5 are automatic in special cases.
(v) In Definition 2.4.1.5(iii), we have to use some element w_Q of $W(C_Q)$ and "pass through $w_Q(C_Q)$" because of Remark 2.2.3.2(ii) (see also the end of Remark 2.2.3.12). Nothing in here and what follows depends on the choice of such a w_Q.
(vi) For a given Π compatible with $\widetilde{P}$, a bijection Φ as in Definition 2.4.1.5 is not unique in general (consider the case $\widetilde{P} = M_P$).
(vii) In Definition 2.4.1.5, it is necessary in general to consider *all* elements $w_{\widetilde{P}} \in W_{\widetilde{P}}$, note just $w_{\widetilde{P}} = 1$, otherwise one misses some condition, see for instance (98) below (note that this is also quite natural in view of Theorem 2.3.2.5).

Example 2.4.1.7. Let us consider the case $n = 3$, $K = \mathbb{Q}_p$ and $\widetilde{P} = P$ with $M_P = \mathrm{GL}_2 \times \mathrm{GL}_1$ in the last part of Example 2.2.2.9(ii) (see also Example 2.2.4.4). We denote by P' the standard parabolic subgroup of Levi $\mathrm{GL}_1 \times \mathrm{GL}_2$. Then Π is compatible with $\widetilde{P}$ if and only Π has 3 irreducible constituents and the following form (a line means a *nonsplit* extension of length 2 as a subquotient and the constituent on the left-hand side is the *socle*):

$$\mathrm{Ind}_{P^-(\mathbb{Q}_p)}^{\mathrm{GL}_3(\mathbb{Q}_p)}\big(\pi \cdot (\omega^{-1} \circ \det) \otimes \chi\big) \text{ —— } \mathrm{SS} \text{ —— } \mathrm{Ind}_{P'^-(\mathbb{Q}_p)}^{\mathrm{GL}_3(\mathbb{Q}_p)}\big(\chi\omega^{-2} \otimes \pi\big)$$

where $\chi : \mathbb{Q}_p^\times \to \mathbb{F}^\times$ is a smooth character, π is a supersingular representation of $\mathrm{GL}_2(\mathbb{Q}_p)$ and SS is a supersingular representation of $\mathrm{GL}_3(\mathbb{Q}_p)$. The case $\widetilde{P} = M_P$ is analogous but with a semisimple Π (instead of nonsplit extensions). See also §2.4.3 below for more examples.

The following proposition shows that a representation Π as in Definition 2.4.1.5 has internal symmetries.

Proposition 2.4.1.8. *Assume Π is compatible with $\widetilde{P}$ and let Φ be a bijection as in Definition 2.4.1.5. Let $w_{\widetilde{P}} \in W_{\widetilde{P}}$, Q a parabolic subgroup containing ${}^{w_{\widetilde{P}}}P$ and C_Q an isotypic component of $\overline{L}^{\otimes}|_{Z_{M_Q}}$ such that $P(C_Q) = {}^{w_Q}Q$ for some (unique) $w_Q \in W$ with $w_Q(S(Q)) \subseteq S$. Then $\pi_i(C_Q)$ is compatible with $({}^{w_Q w_{\widetilde{P}}}\widetilde{P})_{Q,i}$ for $i \in \{1, \dots, d\}$, where $\pi_i(C_Q)$ is as in Definition 2.4.1.5(i).*

Proof. The proof is long but essentially formal. Replacing $\widetilde{P}$ by $w_{\widetilde{P}}\widetilde{P}w_{\widetilde{P}}^{-1}$ and Φ by $w_{\widetilde{P}}(\Phi)$ (see the discussion following Definition 2.4.1.5), we can assume $w_{\widetilde{P}} = \mathrm{Id}$. We write for simplicity w instead of w_Q. Recall from Proposition 2.2.3.3 that C_Q is the isotypic component of $fw^{-1}(\theta_G)|_{Z_{M_Q}}$ in $\overline{L}^{\otimes}|_{Z_{M_Q}}$. More precisely, by (48), Corollary 2.2.3.11 and Remark 2.2.3.12 (especially (54)), we have an isomorphism of algebraic representations of $M_{{}^wQ}^{\mathrm{Gal}(K/\mathbb{Q}_p)} \cong \prod_{i=1}^d M_i^{\mathrm{Gal}(K/\mathbb{Q}_p)} \cong \prod_{i=1}^d \mathrm{GL}_{n_i}^{\mathrm{Gal}(K/\mathbb{Q}_p)}$:

$$w(C_Q) \cong \overline{L}_{{}^wQ}^{\otimes} \otimes \left(\theta^{{}^wQ} \otimes \cdots \otimes \theta^{{}^wQ}\right) \cong \bigotimes_{i=1}^{d} \left(\overline{L}_i^{\otimes} \otimes \left((\theta^{{}^wQ})_i \otimes \cdots \otimes (\theta^{{}^wQ})_i\right)\right). \quad (75)$$

Thus the map $\Phi_{w,i}$ in Definition 2.4.1.5(iii) (recall $w_{\widetilde{P}} = \mathrm{Id}$ and $w = w_Q$) is a bijection of partially ordered sets between the set of $M_i(K)$-subrepresentations of $\pi_i(C_Q)$ and the set of good subrepresentations of $C_{w,i}|_{({}^w\widetilde{P})_{Q,i}^{\mathrm{Gal}(K/\mathbb{Q}_p)}} = \overline{L}_i^{\otimes}|_{({}^w\widetilde{P})_{Q,i}^{\mathrm{Gal}(K/\mathbb{Q}_p)}}$ (recall that $({}^wP)_{Q,i}$ is here a standard parabolic subgroup of M_i and $({}^w\widetilde{P})_{Q,i}$ a Zariski closed subgroup of $({}^wP)_{Q,i}$ containing $M_{({}^wP)_{Q,i}}$). We have to check that $\Phi_{w,i}$ satisfies conditions (i) to (iv) in Definition 2.4.1.5 (with M_i instead of G and $({}^w\widetilde{P})_{Q,i}$ instead of $\widetilde{P}$). We will only check condition (i) below, leaving the others, which are again essentially formal, to the (motivated) reader.

We can assume $i = 1$. Let $P_1 \stackrel{\text{def}}{=} ({}^wP)_{Q,1}$, $\widetilde{P}_1 \stackrel{\text{def}}{=} ({}^w\widetilde{P})_{Q,1}$ (so $M_{P_1} \subseteq \widetilde{P}_1 \subseteq P_1 \subseteq M_1 \subseteq M_{{}^wQ}$) and recall that T_1 is the torus of diagonal matrices in M_1. Let $w_{\widetilde{P}_1} \in W_{\widetilde{P}_1} \subseteq W(M_1)$, Q_1 a parabolic subgroup of M_1 containing ${}^{w_{\widetilde{P}_1}}P_1$ and C_{Q_1} an isotypic component of $\overline{L}_1^{\otimes}|_{Z_{M_{Q_1}}}$, we have to prove that $w_{\widetilde{P}_1}(\Phi_{w,1})^{-1}(C_{Q_1})$ is of the form (74).

Step 1: Let $\widetilde{w}_1 \stackrel{\text{def}}{=} w_{\widetilde{P}_1} \times \mathrm{Id} \times \cdots \times \mathrm{Id} \in W(M_1) \times \cdots \times W(M_d) = W({}^wQ) \subseteq W$ and set $w_{\widetilde{P}} \stackrel{\text{def}}{=} w^{-1}\widetilde{w}_1 w \in W(Q)$. Then $w_{\widetilde{P}} \in W_{\widetilde{P}}$ and Q contains ${}^{w_{\widetilde{P}}}P$. Indeed, since $w_{\widetilde{P}_1} \in W_{\widetilde{P}_1}$ and the simple roots of P_1 are contained in $w(S(Q)) \subseteq S$, we see that $w_{\widetilde{P}} = w^{-1}\widetilde{w}_1 w$ sends the simple (resp. positive) roots of $\widetilde{P} \cap M_Q$ to simple (resp. positive) roots of M_Q and the roots of $\widetilde{P} \cap N_Q$ to positive roots (using that $W(Q)$ normalizes N_Q). Moreover, one easily checks that ${}^{w_{\widetilde{P}_1}}P_1 = ({}^{ww_{\widetilde{P}}}P)_{Q,1} = ({}^w({}^{w_{\widetilde{P}}}P))_{Q,1}$. Replacing P by ${}^{w_{\widetilde{P}}}P$ and Φ by $w_{\widetilde{P}}(\Phi)$, we can thus assume $w_{\widetilde{P}_1} = \mathrm{Id}$.

Step 2: Let $\lambda_1 \in X(T_1)$ be a weight of $\overline{L}_1^{\otimes}|_{T_1}$ such that C_{Q_1} is the isotypic component of $\lambda_1|_{Z_{M_{Q_1}}}$ and recall that $\lambda_1|_{Z_{M_1}} = f\theta_{M_1}|_{Z_{M_1}} = f\theta_{^wQ}|_{Z_{M_1}}$, where θ_{M_i} for $i \in \{1, \dots, d\}$ is defined as in (35) replacing $G = \mathrm{GL}_n$ by $M_i = \mathrm{GL}_{n_i}$. Let $\lambda_{^wQ} \in X(T)$ be the unique character such that $\lambda_{^wQ}|_{T_1} = \lambda_1$ and $\lambda_{^wQ}|_{T_i} = f\theta_{M_i} = f\theta_{^wQ}|_{T_i}$ if $i \in \{2, \dots, d\}$ (here, we use the convention in Remark 2.2.3.12 and recall that θ_{M_i} is trivial if $M_i = \mathrm{GL}_1$). Then $\lambda_{^wQ}$ is a weight of $\bigotimes_{i=1}^d \overline{L}_i^{\otimes}|_{T_i}$. We set

$$\lambda \stackrel{\text{def}}{=} \lambda_{^wQ} + f\theta^{^wQ} \in X(T)$$

which is a weight of $\overline{L}^{\otimes}|_T$ (use (75)). We have

$$\begin{aligned}\lambda|_{Z_{M_1}} = \lambda_1|_{Z_{M_1}} + f\theta^{^wQ}|_{Z_{M_1}} &= f\theta_{M_1}|_{Z_{M_1}} + f\theta^{^wQ}|_{Z_{M_1}} \\ &= f(\theta_{^wQ} + \theta^{^wQ})|_{Z_{M_1}} = f\theta_G|_{Z_{M_1}} \quad (76)\end{aligned}$$

and if $i \geq 2$:

$$\lambda|_{T_i} = f\theta_{M_i} + f\theta^{^wQ}|_{T_i} = f(\theta_{^wQ} + \theta^{^wQ})|_{T_i} = f\theta_G|_{T_i}. \quad (77)$$

In particular $\lambda|_{Z_{M_{^wQ}}} = f\theta_G|_{Z_{M_{^wQ}}}$ and thus

$$w^{-1}(\lambda)|_{Z_{M_Q}} = fw^{-1}(\theta_G)|_{Z_{M_Q}}. \quad (78)$$

Let $Q_{(1)} \subseteq Q$ be the standard parabolic subgroup of G such that ${}^wQ_{(1)} \subseteq {}^wQ$ has Levi $M_{Q_1} \times M_2 \times \cdots \times M_d$. As $P_1 \subseteq Q_1$ by Step 1, we note that ${}^wQ_{(1)}$ contains ${}^wP$ and hence $Q_{(1)}$ contains P, $W({}^wQ_{(1)}) = W(Q_1) \times W(M_2) \times \cdots \times W(M_d)$ and $w(S(Q_{(1)})) = S(Q_1) \amalg S(M_2) \amalg \cdots \amalg S(M_d)$. Let $C_{Q_{(1)}}$ be the isotypic component of $\overline{L}^{\otimes}|_{Z_{M_{Q_{(1)}}}}$ associated to $w^{-1}(\lambda)|_{Z_{M_{Q_{(1)}}}}$. From (78) we get $C_{Q_{(1)}} \subseteq C_Q$ (inside $\overline{L}^{\otimes}|_{Z_{M_{Q_{(1)}}}}$) and from (76), (77) an isomorphism of algebraic representations of $M_{Q_1}^{\mathrm{Gal}(K/\mathbb{Q}_p)} \otimes \prod_{i=2}^d M_i^{\mathrm{Gal}(K/\mathbb{Q}_p)}$:

$$w(C_{Q_{(1)}}) \cong \Big(C_{Q_1} \otimes \big((\theta^{^wQ})_1 \otimes \cdots \otimes (\theta^{^wQ})_1\big)\Big) \otimes \bigotimes_{i=2}^d \Big(\overline{L}_i^{\otimes} \otimes \big((\theta^{^wQ})_i \otimes \cdots \otimes (\theta^{^wQ})_i\big)\Big). \quad (79)$$

Step 3: Define λ', $\lambda'_{^wQ}$ and θ'_G by the formula (37) for $P = {}^wQ_{(1)}$ and the respective characters λ, $\lambda_{^wQ}$ and θ_G. Set $\lambda'_1 \stackrel{\text{def}}{=} \frac{1}{|W(Q_1)|} \sum_{w'_1 \in W(Q_1)} w'_1(\lambda_1) \in (X(T_1) \otimes_{\mathbb{Z}} \mathbb{Q})^{W(Q_1)}$. From (the proof of) Lemma 2.2.3.6, we easily get $\lambda' = \lambda'_{^wQ} + f\theta^{^wQ}$ with $\lambda'_{^wQ}|_{T_1} = \lambda'_1$. Let $w_1 \in W(M_1)$ such that $w_1(S(Q_1)) \subseteq S(M_1)$ and $w_1(\lambda'_1)$ is dominant (w_1 exists by Proposition 2.2.2.6(i)). We prove that $w_1(\lambda') = w_1(\lambda'_{^wQ}) + f\theta^{^wQ}$ is also dominant (we consider here w_1 as an element of $W({}^wQ)$ in the obvious way and use that $W({}^wQ)$ acts trivially on $\theta^{^wQ}$). From (77) we easily get $\lambda'|_{T_i} = f\theta'_G|_{T_i}$ if $i \geq 2$. But θ'_G is dominant since θ_G is (see the proof of Lemma 2.2.2.4(i)), thus $\langle w_1(\lambda'), \alpha\rangle = \langle \lambda', \alpha\rangle = \langle f\theta'_G, \alpha\rangle \geq 0$ if $\alpha \in \{e_j - e_{j+1} : n_1 + 1 \leq j \leq n-1\}$. Since $w_1(\lambda'_{^wQ})|_{T_1} = w_1(\lambda'_1)$ is dominant by assumption and $\langle f\theta^{^wQ}, \alpha\rangle = 0$ if $\alpha \in \{e_j - e_{j+1} : 1 \leq j \leq n_1 - 1\}$

(see after (45)), we are left to check that $\langle w_1(\lambda'), e_{n_1} - e_{n_1+1}\rangle \geq 0$. But an explicit computation gives

$$\begin{aligned}\langle w_1(\lambda'), e_{n_1} - e_{n_1+1}\rangle &= \langle w_1(\lambda'_{wQ}), e_{n_1} - e_{n_1+1}\rangle + \langle f\theta^{wQ}, e_{n_1} - e_{n_1+1}\rangle \\ &= \langle w_1(\lambda'_{wQ}), e_{n_1}\rangle - \langle w_1(\lambda'_{wQ}), e_{n_1+1}\rangle + fn_2 \\ &= \langle w_1(\lambda'_{wQ}), e_{n_1}\rangle - f\frac{n_2-1}{2} + fn_2 \\ &\geq f\frac{n_2+1}{2},\end{aligned}$$

where the last inequality follows $\langle w_1(\lambda'_{wQ}), e_{n_1}\rangle \geq 0$ by Remark 2.2.1.1(ii) applied to $\overline{L}_1^{\otimes}|_{T_1}$ (instead of $\overline{L}^{\otimes}|_T$) together with formula (37).

Step 4: By definition, $S(P(C_{Q_1}))$ is the support of $f\theta_{M_1} - w_1(\lambda'_1)$ (see Proposition 2.2.2.6(ii)). By Remark 2.2.2.3(ii) we have $w^{-1}(\lambda') = (w^{-1}(\lambda))'$ in $(X(T)\otimes_{\mathbb{Z}}\mathbb{Q})^{W(Q_{(1)})}$, where the latter is given by (37) applied to the parabolic $Q_{(1)}$ and the character $w^{-1}(\lambda)$. Since $w_1w(S(Q_{(1)})) \subseteq S$ and $w_1w((w^{-1}(\lambda))') = w_1w(w^{-1}(\lambda')) = w_1(\lambda')$ is dominant (Step 4), $S(P(C_{Q_{(1)}}))$ is by definition the support of

$$\begin{aligned}f\theta_G - w_1(\lambda') = f\theta_G - \Big(w_1(\lambda'_{wQ}) + f\theta^{wQ}\Big) &= f\theta_{wQ} - w_1(\lambda'_{wQ}) \\ &= (f\theta_{M_1} - w_1(\lambda'_1)) + \sum_{i=2}^{d}(f\theta_{M_i} - f\theta'_{M_i}),\end{aligned} \tag{80}$$

where θ'_{M_i} is defined by (37) applied to $P = M_i = G$ and the character θ_{M_i} of T_i. In fact, θ'_{M_i} is the character $\det^{\frac{n_i-1}{2}}$ of T_i, from which we easily see that the support of (80) is exactly $S(P(C_{Q_1})) \amalg S(M_2) \amalg \cdots \amalg S(M_d)$. This implies

$$M_{P(C_{Q_{(1)}})} = \mathrm{diag}(M_{P(C_{Q_1})}, M_2, \ldots, M_d). \tag{81}$$

Step 5: We now finally prove that $\Phi_{w,1}$ satisfies condition (i) in Definition 2.4.1.5. Write $M_{P(C_{Q_1})} = M_{1,1} \times \cdots \times M_{1,d_1}$ (for some $d_1 \geq 1$), by condition (i) in Definition 2.4.1.5 for the map Φ we have using (81):

$$\Phi^{-1}(C_{Q_{(1)}}) \cong \mathrm{Ind}_{P(C_{Q_{(1)}})^-(K)}^{G(K)}\Big(\Big(\pi_1(C_{Q_{(1)}}) \otimes \cdots \otimes \pi_d(C_{Q_{(1)}})\Big) \otimes (\omega^{-1} \circ \theta^{P(C_{Q_{(1)}})})\Big), \tag{82}$$

where $\pi_1(C_{Q_{(1)}}) = \pi_{1,1}(C_{Q_{(1)}}) \otimes \cdots \otimes \pi_{1,d_1}(C_{Q_{(1)}})$ (with obvious notation). Let

$$\pi'_1 \stackrel{\text{def}}{=} \mathrm{Ind}_{P(C_{Q_1})^-(K)}^{M_1(K)}\Big(\pi_1(C_{Q_{(1)}}) \otimes (\omega^{-1} \circ \theta^{P(C_{Q_1})})\Big), \tag{83}$$

it is enough to prove that π'_1 is a subquotient of $\pi_1(C_Q)$ and that

$$\Phi_{w,1}(\pi'_1) = C_{Q_1}|_{\widetilde{P}_1^{\mathrm{Gal}(K/\mathbb{Q}_p)}}\ (= C_{Q_1}|_{(\widetilde{P}_1 \cap M_{Q_1})^{\mathrm{Gal}(K/\mathbb{Q}_p)}}).$$

Note first that

$$\theta^{P(C_{Q_{(1)}})} = \theta^{P(C_Q)} + \theta^{P(C_{Q_1})}, \tag{84}$$

where we view $\theta^{P(C_{Q_1})}$ as a character of T (not just T_1) by sending the coordinates in T_i to 1 for $i \geq 2$ (this is straightforward to check from (45)). From (82), (83) and (84), we get

$$\Phi^{-1}(C_{Q_{(1)}}) \cong \mathrm{Ind}_{P(C_Q)^-(K)}^{G(K)} \Big(\big(\pi_1' \otimes \pi_2(C_{Q_{(1)}}) \otimes \cdots \otimes \pi_d(C_{Q_{(1)}})\big) \otimes (\omega^{-1} \circ \theta^{P(C_Q)}) \Big).$$

Since $C_{Q_{(1)}}$ is a subquotient of C_Q (both being good subquotients of $\overline{L}^{\otimes}|_{\widetilde{P}^{\mathrm{Gal}(K/\mathbb{Q}_p)}}$), $\Phi^{-1}(C_{Q_{(1)}})$ is a subquotient of $\Phi^{-1}(C_Q)$. This implies in particular (using the ordinary functor of [Eme10a] together with Remark 2.4.1.6(iii)) that π_1' (resp. $\pi_i(C_{Q_{(1)}})$ for $i \geq 2$) is a subquotient of $\pi_1(C_Q)$ (resp. of $\pi_i(C_Q)$ for $i \geq 2$). By condition (iii) for Φ (in Definition 2.4.1.5) applied to $\Pi' = \Phi^{-1}(C_{Q_{(1)}})$ (together with $P(C_Q) = {}^wQ$), we also get an isomorphism of algebraic representations of $\prod_{i=1}^d ({}^w\widetilde{P})_{Q,i}^{\mathrm{Gal}(K/\mathbb{Q}_p)}$ over $\mathbb{F}$:

$$w(C_{Q_{(1)}}) = \Big(\Phi_{w,1}(\pi_1') \otimes \big((\theta^{{}^wQ})_1 \otimes \cdots \otimes (\theta^{{}^wQ})_1\big)\Big) \otimes \bigotimes_{i=2}^{d} \Big(\Phi_{w,i}(\pi_i(C_{Q_{(1)}})) \otimes \big((\theta^{{}^wQ})_i \otimes \cdots \otimes (\theta^{{}^wQ})_i\big)\Big), \tag{85}$$

where $\Phi_{w,1}(\pi_1')$ and $\Phi_{w,i}(\pi_i(C_{Q_{(1)}}))$ $(i \geq 2)$ are good subquotients of $\overline{L}_i^{\otimes}|_{({}^w\widetilde{P})_{Q,i}^{\mathrm{Gal}(K/\mathbb{Q}_p)}}$. Since we have good subquotients of $\overline{L}_i^{\otimes}|_{({}^w\widetilde{P})_{Q,i}^{\mathrm{Gal}(K/\mathbb{Q}_p)}}$ in each factor of (79) and (85), (79) and (85) imply $\Phi_{w,1}(\pi_1') = C_{Q_1}|_{\widetilde{P}_1^{\mathrm{Gal}(K/\mathbb{Q}_p)}}$ and $\Phi_{w,i}(\pi_i(C_{Q_{(1)}})) = \overline{L}_i^{\otimes}|_{({}^w\widetilde{P})_{Q,i}^{\mathrm{Gal}(K/\mathbb{Q}_p)}}$ for $i \geq 2$ (recall isotypic components of $\overline{L}_i^{\otimes}|_{M_{({}^wP)_{Q,i}}^{\mathrm{Gal}(K/\mathbb{Q}_p)}}$ tautology occur with multiplicity 1, so there is no multiplicity issue). This finishes the proof of condition (i) in Definition 2.4.1.5 for $\Phi_{w,1}$. □

Remark 2.4.1.9. When $P(C_Q)$ is strictly bigger than ${}^{w_Q}Q$ for one, or equivalently any by Lemma 2.2.3.1, $w_Q \in W(C_Q)$, there is no real analogue of Proposition 2.4.1.8 since $\overline{L}_i^{\otimes}$ has to be replaced by $C_{w_Q,i}$ in (54) which is not $\overline{L}_i^{\otimes}$ in general.

2.4.2 Compatibility with $\overline{\rho}$

We define what it means for a representation of $G(K)$ over $\mathbb{F}$ to be compatible with a good conjugate $\overline{\rho} : \mathrm{Gal}(\overline{\mathbb{Q}}_p/K) \to \widetilde{P}_{\overline{\rho}}(\mathbb{F})$ as in §2.3.2. Essentially, an admissible smooth representation Π is compatible with $\overline{\rho}$ if it is compatible with $\widetilde{P}_{\overline{\rho}}$ in the sense of Definition 2.4.1.5 and if the bijection Φ of *loc.cit.* satisfies some natural compatibilities with the functor V_G in (16) (see Definition 2.4.2.7).

We now fix a continuous homomorphism

$$\overline{\rho} : \mathrm{Gal}(\overline{\mathbb{Q}}_p/K) \longrightarrow G(\mathbb{F})$$

and recall that $\overline{\rho}^{\mathrm{ss}}$ denotes the semisimplification of the associated representation of $\mathrm{Gal}(\overline{\mathbb{Q}}_p/K)$ (see §2.3.2). We assume that $\overline{\rho}$ is *generic* in the following sense:

(a) $\overline{\rho}^{\mathrm{ss}}$ has distinct irreducible constituents;

(b) the ratio of any two irreducible constituents of $\overline{\rho}^{\mathrm{ss}}$ of dimension 1 is not in $\{\omega, \omega^{-1}\}$.

By Proposition 2.3.2.2, conjugating $\overline{\rho}$ by an element of $G(\mathbb{F})$ if necessary, we can assume that $\overline{\rho}$ is a good conjugate in the sense of Definition 2.3.2.3, that is we have

$$\overline{\rho} : \mathrm{Gal}(\overline{\mathbb{Q}}_p/K) \longrightarrow \widetilde{P}_{\overline{\rho}}(\mathbb{F}) \subseteq P_{\overline{\rho}}(\mathbb{F}) \subseteq G(\mathbb{F}),$$

where $P_{\overline{\rho}}$ is a standard parabolic subgroup of G such that $\overline{\rho}^{\mathrm{ss}}$ is given by the composition $\mathrm{Gal}(\overline{\mathbb{Q}}_p/K) \xrightarrow{\overline{\rho}} P_{\overline{\rho}}(\mathbb{F}) \twoheadrightarrow M_{P_{\overline{\rho}}}(\mathbb{F})$ (see (68)), $\widetilde{P}_{\overline{\rho}} \subseteq P_{\overline{\rho}}$ is the smallest closed algebraic subgroup of $P_{\overline{\rho}}$ containing $M_{P_{\overline{\rho}}}$ and the $\overline{\rho}(g)$ for $g \in \mathrm{Gal}(\overline{\mathbb{Q}}_p/K)$ (in its $\mathbb{F}$-points), and where, for any $h \in P_{\overline{\rho}}(\mathbb{F})$, if we define $\widetilde{P}_{h\overline{\rho}h^{-1}} \subseteq P_{\overline{\rho}}$ as for $\overline{\rho}$, then we have $\widetilde{P}_{\overline{\rho}} \subseteq \widetilde{P}_{h\overline{\rho}h^{-1}}$. Good conjugates are not unique, see Theorem 2.3.2.5, but we fix such a good conjugate $\overline{\rho}$ (and the associated pair $(\widetilde{P}_{\overline{\rho}}, P_{\overline{\rho}})$) for the moment.

For any $\widetilde{w} \in W_{\overline{\rho}} = W_{\widetilde{P}_{\overline{\rho}}}$ (see (72)) and any parabolic subgroup Q containing ${}^{\widetilde{w}}P_{\overline{\rho}}$, we define the Q-semisimplification $\overline{\rho}^{Q-\mathrm{ss}}$ of $\overline{\rho}$ as the continuous homomorphism

$$\overline{\rho}^{Q-\mathrm{ss}} : \mathrm{Gal}(\overline{\mathbb{Q}}_p/K) \xrightarrow{\widetilde{w}\overline{\rho}\widetilde{w}^{-1}} {}^{\widetilde{w}}P_{\overline{\rho}}(\mathbb{F}) \hookrightarrow Q(\mathbb{F}) \twoheadrightarrow M_Q(\mathbb{F})$$

(strictly speaking, it also depends on $\widetilde{w}$). More generally, for any $w \in W$ such that $w(S(Q)) \subseteq S$, we define the continuous homomorphisms

$$w(\overline{\rho}^{Q-\mathrm{ss}}) : \mathrm{Gal}(\overline{\mathbb{Q}}_p/K) \xrightarrow{w\overline{\rho}^{Q-\mathrm{ss}}w^{-1}} wM_Q(\mathbb{F})w^{-1} = M_{{}^wQ}(\mathbb{F})$$

and note that $w(\overline{\rho}^{Q-\mathrm{ss}})$ actually takes values in

$$({}^{w\widetilde{w}}\widetilde{P}_{\overline{\rho}})_Q(\mathbb{F}) \subseteq ({}^{w\widetilde{w}}P_{\overline{\rho}})_Q(\mathbb{F}) \subseteq M_{{}^wQ}(\mathbb{F})$$

(recall from the beginning of §2.4.1 that $({}^{w\widetilde{w}}P_{\overline{\rho}})_Q = w({}^{\widetilde{w}}P_{\overline{\rho}} \cap M_Q)w^{-1}$ and $({}^{w\widetilde{w}}\widetilde{P}_{\overline{\rho}})_Q = w\big((\widetilde{w}\widetilde{P}_{\overline{\rho}}\widetilde{w}^{-1}) \cap M_Q\big)w^{-1})$.

Let $\widetilde{w} \in W_{\overline{\rho}}$, Q a parabolic subgroup containing ${}^{\widetilde{w}}P_{\overline{\rho}}$, $w \in W$ such that $w(S(Q)) \subseteq S$ and Q' a parabolic subgroup containing ${}^wQ$. We write $M_{Q'} = \mathrm{diag}(M_1, \dots, M_d)$ with $M_i \cong \mathrm{GL}_{n_i}$ and we set for $i \in \{1, \dots, d\}$:

$$w(\overline{\rho}^{Q-\mathrm{ss}})_i : \mathrm{Gal}(\overline{\mathbb{Q}}_p/K) \xrightarrow{w(\overline{\rho}^{Q-\mathrm{ss}})} M_{{}^wQ}(\mathbb{F}) \hookrightarrow M_{Q'}(\mathbb{F}) \twoheadrightarrow M_i(\mathbb{F}).$$

We also have (recall from §2.4.1 that $({}^wQ)_i$ is a standard parabolic subgroup of M_i):

$$w(\overline{\rho}^{Q-\mathrm{ss}})_i : \mathrm{Gal}(\overline{\mathbb{Q}}_p/K) \xrightarrow{w(\overline{\rho}^{Q-\mathrm{ss}})} M_{{}^wQ}(\mathbb{F}) \twoheadrightarrow M_{({}^wQ)_i}(\mathbb{F}) \hookrightarrow M_i(\mathbb{F}). \tag{86}$$

Composing $w(\overline{\rho}^{Q-\mathrm{ss}})_i$ with $M_i(\mathbb{F}) \twoheadrightarrow (M_i/M_i^{\mathrm{der}})(\mathbb{F}) \cong \mathbb{F}^\times$, we obtain by class field theory for K a continuous group homomorphism

$$\det(w(\overline{\rho}^{Q-\mathrm{ss}})_i) : K^\times \longrightarrow \mathbb{F}^\times. \tag{87}$$

Lemma 2.4.2.1. *Let $\overline{\rho}$, Q as above, C_Q an isotypic component of $\overline{L}^{\otimes}|_{Z_{M_Q}}$ and $Q' \stackrel{\mathrm{def}}{=} P(C_Q)$. Then the characters (87) for $i \in \{1, \dots, d\}$ and $w \in W(C_Q)$ (see (39)) don't depend on the choice of $w \in W(C_Q)$. Moreover, we have $\prod_{i=1}^d \det(w(\overline{\rho}^{Q-\mathrm{ss}})_i) = \det(\overline{\rho})$.*

Proof. This follows from Lemma 2.2.2.10 (applied to $P = Q$) together with the fact that conjugation by $W(P(C_Q))$ (seen in $M_{P(C_Q)}(\mathbb{F})$) is trivial on $M_{P(C_Q)}/M^{\mathrm{der}}_{P(C_Q)}$, and thus on each M_i/M_i^{der}. The last assertion is obvious. □

As previously, $w(\overline{\rho}^{Q-\mathrm{ss}})_i$ in (86) takes values in

$$({}^{w\widetilde{w}}\widetilde{P}_{\overline{\rho}})_{Q,i}(\mathbb{F}) \subseteq ({}^{w\widetilde{w}}P_{\overline{\rho}})_{Q,i}(\mathbb{F}) \subseteq M_{({}^wQ)_i}(\mathbb{F}) \subseteq M_i(\mathbb{F}) \cong \mathrm{GL}_{n_i}(\mathbb{F})$$

(recall from the beginning of §2.4.1 that $({}^{w\widetilde{w}}P_{\overline{\rho}})_{Q,i}$ is a standard parabolic subgroup of $M_{({}^wQ)_i}$ and that $({}^{w\widetilde{w}}\widetilde{P}_{\overline{\rho}})_{Q,i}$ is a Zariski closed algebraic subgroup of $({}^{w\widetilde{w}}P_{\overline{\rho}})_{Q,i}$ containing $M_{({}^{w\widetilde{w}}P_{\overline{\rho}})_{Q,i}}$).

Proposition 2.4.2.2. *Let $\overline{\rho}$, Q as above, $w \in W$ such that $w(S(Q)) \subseteq S$ and $Q' \stackrel{\mathrm{def}}{=} {}^wQ$. Then $w(\overline{\rho}^{Q-\mathrm{ss}})_i : \mathrm{Gal}(\overline{\mathbb{Q}}_p/K) \longrightarrow M_i(\mathbb{F})$ is a good conjugate with values in $({}^{w\widetilde{w}}\widetilde{P}_{\overline{\rho}})_{Q,i}(\mathbb{F})$ for $i \in \{1, \dots, d\}$.*

Proof. Note that $\widetilde{w}\overline{\rho}\widetilde{w}^{-1}$ is a good conjugate (with values in $\widetilde{w}\widetilde{P}_{\overline{\rho}}(\mathbb{F})\widetilde{w}^{-1} \subseteq {}^{\widetilde{w}}P_{\overline{\rho}}(\mathbb{F})$) by Lemma 2.3.2.4. Since $w(\overline{\rho}^{Q-\mathrm{ss}})$ is obtained from $\overline{\rho}^{Q-\mathrm{ss}}$ by permuting the blocs $M_i \cong \mathrm{GL}_{n_i}$ of M_Q, it is equivalent to prove the statement for $w = \mathrm{Id}$. Assume that $\overline{\rho}_i \stackrel{\mathrm{def}}{=} (\overline{\rho}^{Q-\mathrm{ss}})_i : \mathrm{Gal}(\overline{\mathbb{Q}}_p/K) \to M_i(\mathbb{F})$ is *not* a good conjugate. Then it follows from Proposition 2.3.2.2 that there is $h_i \in ({}^{\widetilde{w}}P_{\overline{\rho}})_{Q,i}(\mathbb{F})$ such that $h_i\overline{\rho}_i h_i^{-1}$ is a good conjugate, and thus $X_{h_i\overline{\rho}_i h_i^{-1}} \subsetneq X_{\overline{\rho}_i}$ (with the notation of §2.3.2). Let α_i be a positive root of GL_{n_i} in $X_{\overline{\rho}_i}\backslash X_{h_i\overline{\rho}_i h_i^{-1}}$ and note that, if α_i is a sum of roots in R^+ (viewing α_i in R^+), then all of these roots are positive roots of GL_{n_i}. Set $h_j \stackrel{\mathrm{def}}{=} \mathrm{Id}_{\mathrm{GL}_{n_j}} \in \mathrm{GL}_{n_j}(\mathbb{F})$ if $j \neq i$ and define $h = (h_1, \dots, h_d) \in \mathrm{diag}(M_1, \dots, M_d) = M_Q(\mathbb{F}) \subseteq Q(\mathbb{F})$. If we had $\alpha_i \in X_{h\widetilde{w}\overline{\rho}\widetilde{w}^{-1}h^{-1}}$, then from what we just said necessarily we would have $\alpha_i \in X_{(h\overline{\rho}^{Q-\mathrm{ss}}h^{-1})_i} = X_{h_i\overline{\rho}_i h_i^{-1}}$ which is impossible. Therefore $\alpha_i \notin X_{h\widetilde{w}\overline{\rho}\widetilde{w}^{-1}h^{-1}}$. But since $\alpha_i \in X_{\overline{\rho}_i} \subseteq X_{\widetilde{w}\overline{\rho}\widetilde{w}^{-1}}$ (viewing the positive roots of GL_{n_i} as a subset of R^+) we deduce $X_{h\widetilde{w}\overline{\rho}\widetilde{w}^{-1}h^{-1}} \subsetneq X_{\widetilde{w}\overline{\rho}\widetilde{w}^{-1}}$ which is impossible as $\widetilde{w}\overline{\rho}\widetilde{w}^{-1}$ is a good conjugate. □

For $\sigma \in \mathrm{Gal}(K/\mathbb{Q}_p) = \mathrm{Gal}(\mathbb{Q}_{p^f}/\mathbb{Q}_p)$ consider

$$\overline{\rho}^\sigma : \mathrm{Gal}(\overline{\mathbb{Q}}_p/K) \to \widetilde{P}_{\overline{\rho}}(\mathbb{F}) \subseteq P_{\overline{\rho}}(\mathbb{F}) \subseteq G(\mathbb{F}),$$

where $\overline{\rho}^\sigma(g) \stackrel{\mathrm{def}}{=} \overline{\rho}(\sigma g \sigma^{-1})$. Here $g \in \mathrm{Gal}(\overline{\mathbb{Q}}_p/K)$ and σ is any lift of σ in $\mathrm{Gal}(\overline{\mathbb{Q}}_p/\mathbb{Q}_p)$. Since $\mathrm{Gal}(\overline{\mathbb{Q}}_p/K)$ is normal in $\mathrm{Gal}(\overline{\mathbb{Q}}_p/\mathbb{Q}_p)$, $\overline{\rho}^\sigma(g)$ is well defined up to conjugation (by elements in $\widetilde{P}_{\overline{\rho}}(\mathbb{F})$). If C is a good subquotient of $\overline{L}^{\otimes}|_{\widetilde{P}_{\overline{\rho}}^{\mathrm{Gal}(K/\mathbb{Q}_p)}}$ (Definition 2.2.1.3), we can view in particular C as a continuous homomorphism

$$\underbrace{\widetilde{P}_{\overline{\rho}}(\mathbb{F}) \times \cdots \times \widetilde{P}_{\overline{\rho}}(\mathbb{F})}_{\mathrm{Gal}(K/\mathbb{Q}_p)} \longrightarrow \mathrm{Aut}\Big(C(\mathbb{F})\Big) \tag{88}$$

(denoting by $C(\mathbb{F})$ the underlying $\mathbb{F}$-vector space of the algebraic representation C) and define a $\mathrm{Gal}(\overline{\mathbb{Q}}_p/K)$-representation $C(\overline{\rho})$ as

$$\mathrm{Gal}(\overline{\mathbb{Q}}_p/K) \xrightarrow{\prod \overline{\rho}^\sigma} \widetilde{P}_{\overline{\rho}}(\mathbb{F}) \times \cdots \times \widetilde{P}_{\overline{\rho}}(\mathbb{F}) \xrightarrow{C} \mathrm{Aut}\Big(C(\mathbb{F})\Big),$$

where, in the first arrow, we choose any order on the elements σ of $\mathrm{Gal}(K/\mathbb{Q}_p)$.

Lemma 2.4.2.3. *The $\mathrm{Gal}(\overline{\mathbb{Q}}_p/K)$-representation $C(\overline{\rho})$ is well-defined up to isomorphism and canonically extends to a $\mathrm{Gal}(\overline{\mathbb{Q}}_p/\mathbb{Q}_p)$-representation.*

Proof. The algebraic representation C of $\widetilde{P}_{\overline{\rho}}^{\mathrm{Gal}(K/\mathbb{Q}_p)}$ over $\mathbb{F}$ doesn't depend up to isomorphism on the order of the copies of $\widetilde{P}_{\overline{\rho}}$, i.e. any permutation of the $\widetilde{P}_{\overline{\rho}}$'s yields an algebraic representation which is conjugate by an element of $\mathrm{Aut}(C(\mathbb{F}))$. Indeed, this clearly holds when C is an isotypic component of $\overline{L}^{\otimes}|_{Z_{M_{P_{\overline{\rho}}}}}$ as $Z_{M_{P_{\overline{\rho}}}}$ embeds diagonally into $\widetilde{P}_{\overline{\rho}}^{\mathrm{Gal}(K/\mathbb{Q}_p)}$. Thus, for a general good subquotient C, any permutation of the $\widetilde{P}_{\overline{\rho}}$'s gives a representation C' which contains the same isotypic components of $\overline{L}^{\otimes}|_{Z_{M_{P_{\overline{\rho}}}}}$ as those of C. Assume now that C is a good subrepresentation of $\overline{L}^{\otimes}|_{\widetilde{P}_{\overline{\rho}}^{\mathrm{Gal}(K/\mathbb{Q}_p)}}$. Then C' must be isomorphic to C since isotypic components of $\overline{L}^{\otimes}|_{Z_{M_{P_{\overline{\rho}}}}}$ tautologically occur with multiplicity 1. In general, one writes C as the quotient of two good subrepresentations of $\overline{L}^{\otimes}|_{\widetilde{P}_{\overline{\rho}}^{\mathrm{Gal}(K/\mathbb{Q}_p)}}$. All this implies that $C(\overline{\rho})$ is well-defined.

We now prove that it extends to $\mathrm{Gal}(\overline{\mathbb{Q}}_p/\mathbb{Q}_p)$. First, if $C = \overline{L}^{\otimes}|_{\widetilde{P}_{\overline{\rho}}^{\mathrm{Gal}(K/\mathbb{Q}_p)}}$, then $C(\overline{\rho})$ is the tensor induction (21) and thus canonically extends to $\mathrm{Gal}(\overline{\mathbb{Q}}_p/\mathbb{Q}_p)$. Let us recall explicitly how it extends. Fix $\sigma_1, \dots, \sigma_f$ some representatives in $\mathrm{Gal}(\overline{\mathbb{Q}}_p/\mathbb{Q}_p)$ of the elements of $\mathrm{Gal}(K/\mathbb{Q}_p) = \mathrm{Gal}(\mathbb{Q}_{p^f}/\mathbb{Q}_p)$ and define permutations $w_1, \dots, w_f$ on $\{1, \dots, f\}$ by $\sigma_i \sigma_j^{-1} = \sigma_{w_i(j)}^{-1} h_{i,j}$, where $h_{i,j} \in \mathrm{Gal}(K/\mathbb{Q}_p)$. The underlying $\mathbb{F}$-vector space $\overline{L}^{\otimes}(\mathbb{F})$ of $\overline{L}^{\otimes}$ is

$$\bigotimes_{i=1}^{f} \Big(\Big(\bigotimes_{\alpha \in S} \overline{L}(\lambda_\alpha) \Big)(\mathbb{F}) \Big),$$

where $\left(\bigotimes_{\alpha\in S}\overline{L}(\lambda_\alpha)\right)(\mathbb{F})$ is the underlying vector space of $\bigotimes_{\alpha\in S}\overline{L}(\lambda_\alpha)$, and the action of σ_i then sends $v_1\otimes v_2\otimes\cdots\otimes v_f\in\overline{L}^{\otimes}(\mathbb{F})$ to $u_1\otimes u_2\otimes\cdots\otimes u_f$, where:

$$u_{w_i(j)}\overset{\mathrm{def}}{=}\left(\left(\bigotimes_{\alpha\in S}\overline{L}(\lambda_\alpha)\right)(\overline{\rho}(h_{i,j}))\right)(v_j). \tag{89}$$

This yields an action of $\mathrm{Gal}(\overline{\mathbb{Q}}_p/\mathbb{Q}_p)$ which doesn't depend on any choice (up to isomorphism). It is enough to prove that this action of $\mathrm{Gal}(\overline{\mathbb{Q}}_p/\mathbb{Q}_p)$ preserves the subspaces $C(\mathbb{F})\subseteq\overline{L}^{\otimes}(\mathbb{F})$, where C is any good subrepresentation of $\overline{L}^{\otimes}|_{\widetilde{P}_{\overline{\rho}}^{\mathrm{Gal}(K/\mathbb{Q}_p)}}$. But this is clear from (89) since $C(\mathbb{F})$ is preserved by the action of $\mathrm{Gal}(\overline{\mathbb{Q}}_p/K)$ *and* by any permutation of the v_i (as we have seen at the beginning). □

Remark 2.4.2.4. One could also use L-groups as in §2.1.4 in order to have more intrinsic definitions (see Remark 2.2.1.1(i)). However the above pedestrian approach will be sufficient for our purpose.

The following lemma is in the same spirit as Lemma 2.4.2.1.

Lemma 2.4.2.5. *Let $\overline{\rho}$, Q as above, C_Q an isotypic component of $\overline{L}^{\otimes}|_{Z_{M_Q}}$ and $Q'\overset{\mathrm{def}}{=}P(C_Q)$. For $w\in W(C_Q)$ and $i\in\{1,\dots,d\}$, let*

- *$C_{w,i}$ be the isotypic component of $\overline{L}_i^{\otimes}|_{Z_{M_{({}^wQ)_i}}}$ defined in* (54) (*applied with P there being Q*)*;*
- *$w(\overline{\rho}^{Q-\mathrm{ss}})_i$ the representation of* $\mathrm{Gal}(\overline{\mathbb{Q}}_p/K)$ *with values in $M_{({}^wQ)_i}(\mathbb{F})$ defined in* (86) (*applied to $Q'=P(C_Q)$*)*;*
- *$C_{w,i}\left(w(\overline{\rho}^{Q-\mathrm{ss}})_i\right)$ the representation of* $\mathrm{Gal}(\overline{\mathbb{Q}}_p/\mathbb{Q}_p)$ *defined in Lemma 2.4.2.3* (*applied to $\overline{\rho}=w(\overline{\rho}^{Q-\mathrm{ss}})_i$, $\overline{L}_i^{\otimes}$ and $C=C_{w,i}$*)*.*

Then the $\mathrm{Gal}(\overline{\mathbb{Q}}_p/\mathbb{Q}_p)$*-representation $C_{w,i}\left(w(\overline{\rho}^{Q-\mathrm{ss}})_i\right)$ doesn't depend on $w\in W(C_Q)$.*

Proof. Let w' be another element in $W(C_Q)$. Then $w'=w_{P(C_Q)}w$ with $w_{P(C_Q)}\in W(P(C_Q))$ by Lemma 2.2.2.10 (with P there being Q). Since $w_{P(C_Q)}$ respects M_i, we have

$$w'(\overline{\rho}^{Q-\mathrm{ss}})_i=w_{P(C_Q)}w(\overline{\rho}^{Q-\mathrm{ss}})_i w_{P(C_Q)}^{-1}.$$

The result then follows from (55) (applied with $P=Q$). □

Remark 2.4.2.6. Lemma 2.4.2.5 still holds replacing $C_{w,i}$ by any good subquotient of $C_{w,i}|_{({}^{w\widetilde{w}}\widetilde{P}_{\overline{\rho}})_{Q,i}^{\mathrm{Gal}(K/\mathbb{Q}_p)}}$ and using the proof of Lemma 2.4.1.3(ii) and Remark 2.4.1.4 to compare with the corresponding good subquotient of $C_{w',i}|_{({}^{w'\widetilde{w}}\widetilde{P}_{\overline{\rho}})_{Q,i}^{\mathrm{Gal}(K/\mathbb{Q}_p)}}$. The proof is the same as for Lemma 2.4.2.5 using that $w(\overline{\rho}^{Q-\mathrm{ss}})_i$ takes values in $({}^{w\widetilde{w}}\widetilde{P}_{\overline{\rho}})_{Q,i}(\mathbb{F})$.

We now state the second crucial definition. We use the functor V_H defined in §2.1.1 in the case $H = \mathrm{GL}_m$, $m \geq 1$ (with the convention of Example 2.1.1.3). If a smooth representation π of $H(K)$ has a central character, we denote it by $Z(\pi)$ (so writing $Z(\pi)$ in the sequel implicitly means that π has a central character). We also define

$$\omega^{-1} \circ \theta_{M_i} : Z_{M_i}(K) = K^\times \xrightarrow{\theta_{M_i}|_{Z_{M_i}}} K^\times \xrightarrow{\omega^{-1}} \mathbb{F}_p^\times \hookrightarrow \mathbb{F}^\times \tag{90}$$

(θ_{M_i} as in (35) replacing G by M_i).

Definition 2.4.2.7. An admissible smooth representation Π of $G(K)$ over $\mathbb{F}$ which has finite length and distinct absolutely irreducible constituents is *compatible with* $\overline{\rho}$ if there exists a bijection Φ as in Definition 2.4.1.5 for $\widetilde{P} = \widetilde{P}_{\overline{\rho}}$ (in particular Π is compatible with $\widetilde{P}_{\overline{\rho}}$) which satisfies the following extra conditions:

(i) for any subquotient Π' of Π, we have an isomorphism of $\mathrm{Gal}(\overline{\mathbb{Q}}_p/\mathbb{Q}_p)$-representations over $\mathbb{F}$:

$$V_G(\Pi') \cong \Phi(\Pi')(\overline{\rho}), \tag{91}$$

where $\Phi(\Pi')(\overline{\rho})$ is the associated representation of $\mathrm{Gal}(\overline{\mathbb{Q}}_p/\mathbb{Q}_p)$ defined in Lemma 2.4.2.3;

(ii) for any $\widetilde{w} \in W_{\overline{\rho}}$, any parabolic subgroup Q containing ${}^{\widetilde{w}}P_{\overline{\rho}}$ and any isotypic component C_Q of $\overline{L}^{\otimes}|_{Z_{M_Q}}$, writing $M_{P(C_Q)} = \mathrm{diag}(M_1, \dots, M_d)$ with $M_i \cong \mathrm{GL}_{n_i}$ we have for one, or equivalently any, element $w \in W(C_Q)$ and for any subquotient π_i' of $\pi_i(C_Q)$:

$$\begin{aligned} Z\Big(\pi_i'\Big) &\cong \det(w(\overline{\rho}^{Q-\mathrm{ss}})_i) \cdot \omega^{-1} \circ \theta_{M_i} \\ V_{M_i}\Big(\pi_i'\Big) &\cong \widetilde{w}(\Phi)_{w,i}(\pi_i')\Big(w(\overline{\rho}^{Q-\mathrm{ss}})_i\Big), \end{aligned} \tag{92}$$

where

- $\pi_i(C_Q)$ is the admissible smooth representation of $M_i(K)$ over $\mathbb{F}$ in Definition 2.4.1.5(i);
- $\det(w(\overline{\rho}^{Q-\mathrm{ss}})_i)$ (resp. $\omega^{-1} \circ \theta_{M_i}$) is the character of $K^\times$ defined in (87) (resp. in (90));
- $\widetilde{w}(\Phi)_{w,i}(\pi_i')$ is the good subquotient of $C_{w,i}|_{({}^{w\widetilde{w}}\widetilde{P}_{\overline{\rho}})_{Q,i}}$ defined in Definition 2.4.1.5(iii);
- $w(\overline{\rho}^{Q-\mathrm{ss}})_i$ is the representation of $\mathrm{Gal}(\overline{\mathbb{Q}}_p/K)$ with values in $({}^{w\widetilde{w}}\widetilde{P}_{\overline{\rho}})_{Q,i}(\mathbb{F}) \subseteq M_{({}^wQ)_i}(\mathbb{F})$ defined in (86) (applied to $Q' = P(C_Q)$);
- $\widetilde{w}(\Phi)_{w,i}(\pi_i')\Big(w(\overline{\rho}^{Q-\mathrm{ss}})_i\Big)$ is the representation of $\mathrm{Gal}(\overline{\mathbb{Q}}_p/\mathbb{Q}_p)$ defined in Lemma 2.4.2.3 (applied to $\overline{\rho} = w(\overline{\rho}^{Q-\mathrm{ss}})_i$, $\overline{L}_i^{\otimes}$ and $C = \widetilde{w}(\Phi)_{w,i}(\pi_i')$).

If Π is compatible with $\overline{\rho}$, then we have in particular $V_G(\Pi) \cong \overline{L}^{\otimes}(\overline{\rho})$ and $V_{M_i}\Big(\pi_i(C_Q)\Big) \cong C_{w,i}\Big(w(\overline{\rho}^{Q-\mathrm{ss}})_i\Big)$ for Q, w, i as in Definition 2.4.2.7(ii) (recall that $V_{M_i}(\pi_i(C_Q))$ is always the trivial representation of $\mathrm{Gal}(\overline{\mathbb{Q}}_p/\mathbb{Q}_p)$ when $n_i = 1$). If $\overline{\rho}$ is (absolutely) irreducible, then $\widetilde{P}_{\overline{\rho}} = P_{\overline{\rho}} = G$, $W_{\overline{\rho}} = \{\mathrm{Id}\}$ and Π is compatible with $\overline{\rho}$ if and only if Π is absolutely irreducible supersingular, $Z(\Pi) \cong \det(\overline{\rho}) \cdot \omega^{-1} \circ (\theta_G|_{Z_G})$ and $V_G(\Pi) \cong \overline{L}^{\otimes}(\overline{\rho})$.

Remark 2.4.2.8. (i) The isomorphisms in (92) are consistent with Lemma 2.4.2.1, Lemma 2.4.2.5 and Remark 2.4.2.6 since their left-hand sides don't depend on $w \in W(C_Q)$.
(ii) Let Π be compatible with $\overline{\rho}$. From (74) applied with $w_{\widetilde{P}} = 1$ and $Q = P$, (92) applied with $\widetilde{w} = 1$ and $Q = P_{\overline{\rho}}$, the last assertion in Lemma 2.4.2.1, and from

$$\theta_G|_{Z_G} = \theta^{P(C_Q)}|_{Z_G}\theta_{P(C_Q)}|_{Z_G} = \theta^{P(C_Q)}|_{Z_G}\Big(\prod_{i=1}^{d}\theta_{M_i}|_{Z_{M_i}}\Big)$$

(which follows from (45)), we deduce that each irreducible constituent Π' of Π is such that $Z(\Pi') = \det(\overline{\rho}) \cdot \omega^{-1} \circ (\theta_G|_{Z_G})$. Since these irreducible constituents are all distinct by assumption, we obtain that Π has a central character $Z(\Pi) = \det(\overline{\rho}) \cdot \omega^{-1} \circ (\theta_G|_{Z_G}) = \det(\overline{\rho}) \cdot \omega^{\frac{-n(n-1)}{2}}$.
(iii) Let Π be compatible with $\overline{\rho}$, Π' a subquotient of Π and $\Pi'' \subseteq \Pi'$ a subrepresentation. Then from Remark 2.4.1.2(i) we have an exact sequence of $\mathrm{Gal}(\overline{\mathbb{Q}}_p/\mathbb{Q}_p)$-representations:

$$0 \longrightarrow \Phi(\Pi'')(\overline{\rho}) \longrightarrow \Phi(\Pi')(\overline{\rho}) \longrightarrow \Phi(\Pi'/\Pi'')(\overline{\rho}) \longrightarrow 0.$$

Thus (91) implies that the sequence $0 \to V_G(\Pi'') \to V_G(\Pi') \to V_G(\Pi'/\Pi'') \to 0$ is exact. In other terms, when applied to Π and its subquotients V_G behaves like an exact functor.
(iv) Let $\chi : K^\times \to \mathbb{F}^\times$ be a smooth character. Then it easily follows from Remark 2.1.1.4(ii) that Π is compatible with $\overline{\rho}$ if and only if $\Pi \otimes (\chi \circ \det)$ is compatible with $\overline{\rho} \otimes \chi$.
(v) For a given Π compatible with $\overline{\rho}$, a bijection Φ as in Definition 2.4.2.7 is still not unique in general. For instance consider the case $n = 4$, $K = \mathbb{Q}_p$, $\widetilde{P}_{\overline{\rho}} = M_{P_{\overline{\rho}}} = \mathrm{diag}(\mathrm{GL}_2, \mathrm{GL}_2)$ and $\overline{\rho} = \overline{\rho}_1 \oplus \overline{\rho}_2$ with $\overline{\rho}_i : \mathrm{Gal}(\overline{\mathbb{Q}}_p/\mathbb{Q}_p) \to \mathrm{GL}_2(\mathbb{F})$ absolutely irreducible distinct for $i = 1, 2$ but such that $\wedge^2_{\mathbb{F}}\overline{\rho}_1 \cong \wedge^2_{\mathbb{F}}\overline{\rho}_2$.

Definition 2.4.2.7 doesn't depend on the choice of a good conjugate.

Proposition 2.4.2.9. *If $\overline{\rho}' : \mathrm{Gal}(\overline{\mathbb{Q}}_p/K) \to \widetilde{P}_{\overline{\rho}'}(\mathbb{F}) \subseteq P_{\overline{\rho}'}(\mathbb{F})$ is another good conjugate of $\overline{\rho}$, then Π is compatible with $\overline{\rho}$ if and only if Π is compatible with $\overline{\rho}'$.*

Proof. From Theorem 2.3.2.5, we have $\overline{\rho}' = wh\overline{\rho}h^{-1}w^{-1}$ for some $h \in \widetilde{P}_{\overline{\rho}}(\mathbb{F})$ and some $w \in W_{\overline{\rho}}$. By symmetry, it is enough to prove that Π compatible with $\overline{\rho}$ implies Π

compatible with $\overline{\rho}'$. We have first that Π is compatible with $h\overline{\rho}h^{-1}$. Indeed, $\widetilde{P}_{h\overline{\rho}h^{-1}} = \widetilde{P}_{\overline{\rho}}$ and the conditions in Definition 2.4.2.7 for $h\overline{\rho}h^{-1}$ follow from the conditions for $\overline{\rho}$ since $w(\overline{\rho}^{Q-\mathrm{ss}})_i$ and $w((h\overline{\rho}h^{-1})^{Q-\mathrm{ss}})_i$ are conjugate in $({}^{w\widetilde{w}}\widetilde{P}_{\overline{\rho}})_{Q,i}(\mathbb{F})$ (with $\widetilde{w}, w$ here as in Definition 2.4.2.7). Thus we can assume $h = \mathrm{Id}$. But then, it is clear from Definition 2.4.2.7 that Π is compatible with $\overline{\rho}' = w\overline{\rho}w^{-1}$. $\square$

Just as some statements in Definition 2.4.1.5 should follow from others (see Remark 2.4.1.6(iv)), we expect the isomorphisms (91) to follow in many cases from the isomorphisms (92):

Proposition 2.4.2.10. *Assume Π is compatible with $\overline{\rho}$ and let Φ be a bijection as in Definition 2.4.2.7. Let $\widetilde{w} \in W_{\overline{\rho}}$, Q a parabolic subgroup containing ${}^{\widetilde{w}}P_{\overline{\rho}}$, C_Q an isotypic component of $\overline{L}^{\otimes}|_{Z_{M_Q}}$ and Π' a subquotient of $\widetilde{w}(\Phi)^{-1}(C_Q)$ of the form*

$$\Pi' \cong \mathrm{Ind}_{P(C_Q)^-(K)}^{G(K)} \Big((\pi'_1 \otimes \cdots \otimes \pi'_d) \otimes (\omega^{-1} \circ \theta^{P(C_Q)})\Big),$$

where π'_i is a subquotient of the representation $\pi_i(C_Q)$ of $M_i(K)$ over $\mathbb{F}$ defined in Definition 2.4.1.5(i) (so that $\widetilde{w}(\Phi)(\Pi')$ is a good subquotient of $C_Q|_{{}^{\widetilde{w}}\widetilde{P}_{\overline{\rho}}^{\mathrm{Gal}(K/\mathbb{Q}_p)}} = C_Q|_{((\widetilde{w}\widetilde{P}_{\overline{\rho}}\widetilde{w}^{-1})\cap M_Q)^{\mathrm{Gal}(K/\mathbb{Q}_p)}}$). Assume that $V_{M_{P(C_Q)}}(\pi'_1 \otimes \cdots \otimes \pi'_d) \cong \bigotimes_{i=1}^d V_{M_{P(C_Q)},i}(\pi'_i)$ (with the notation used in Lemma 2.1.1.5). Then the isomorphism (91) *for Π' follows from the isomorphisms* (92).

Proof. For $i \in \{1, \ldots, d\}$, we have (easy computation):

$$(\theta^{P(C_Q)})_i = \det{}^{n-\sum_{j=1}^i n_j}. \tag{93}$$

Let $\pi''_i \stackrel{\text{def}}{=} \pi'_i \otimes (\omega^{-1} \circ \det)^{n-\sum_{j=1}^i n_j}$, we have by Lemma 2.1.1.5, (93) and Remark 2.1.1.4(ii):

$$\begin{aligned} V_G(\Pi') &= V_G\Big(\mathrm{Ind}_{P(C_Q)^-(K)}^{G(K)}(\otimes_{i=1}^d \pi'_i) \otimes (\omega^{-1} \circ \theta^{P(C_Q)})\Big) \\ &\cong \left(\bigotimes_{i=1}^d \Big(V_{M_i}(\pi''_i) \otimes \Big(Z(\pi''_i)^{n-\sum_{j=1}^i n_j}\Big)|_{\mathbb{Q}_p^\times}\delta_{M_i}^{-1}\Big)\right) \otimes \delta_G \\ &\cong \left(\bigotimes_{i=1}^d \Big(V_{M_i}(\pi'_i) \otimes \Big(\Big(Z(\pi'_i)\cdot \omega \circ \theta_{M_i}\Big)^{n-\sum_{j=1}^i n_j}\Big)|_{\mathbb{Q}_p^\times}\Big)\right) \otimes \delta, \end{aligned}$$

where $\delta \stackrel{\text{def}}{=} \left(\delta_G \prod_{i=1}^d \delta_{M_i}^{-1}\right) \mathrm{ind}_K^{\otimes \mathbb{Q}_p}(\omega^{-\sum_{i=1}^d c_i})$ with (by an explicit computation):

$$\begin{aligned} c_i &= n_i(n_i-1)\Big(n - \sum_{j=1}^i n_j\Big) + n_i\Big(n - \sum_{j=1}^i n_j\Big)^2 \\ &= n_i\Big(n - \sum_{j=1}^i n_j\Big)\Big(n_i - 1 + n - \sum_{j=1}^i n_j\Big) \\ &= n_i\Big(n - \sum_{j=1}^i n_j\Big)\Big(n - 1 - \sum_{j=1}^{i-1} n_j\Big). \end{aligned} \tag{94}$$

Now, assuming (92) we have for one, or equivalently any, w of $W(C_Q)$:

$$\begin{aligned} \Phi(\Pi')(\overline{\rho}) &\cong \widetilde{w}(\Phi)(\Pi')(\overline{\rho}) \\ &= \widetilde{w}(\Phi)(\Pi')(\overline{\rho}^{Q-\mathrm{ss}}) \\ &\cong \bigotimes_{i=1}^d \bigg(\widetilde{w}(\Phi)_{w,i}(\pi_i')\Big(w(\overline{\rho}^{Q-\mathrm{ss}})_i\Big) \otimes \\ &\qquad\qquad \bigg(\Big((\theta^{P(C_Q)})_i \otimes \cdots \otimes (\theta^{P(C_Q)})_i\Big) \circ \Big(\oplus_\sigma \, (w(\overline{\rho}^{Q-\mathrm{ss}})_i)^\sigma\Big)\bigg)\bigg) \\ &\cong \bigotimes_{i=1}^d \bigg(V_{M_i}(\pi_i') \otimes \bigg(\Big(\det(w(\overline{\rho}^{Q-\mathrm{ss}})_i)\Big)^{n-\sum_{j=1}^i n_j}\bigg)|_{\mathbb{Q}_p^\times}\bigg) \\ &\cong \bigotimes_{i=1}^d \bigg(V_{M_i}(\pi_i') \otimes \bigg(\Big(Z(\pi_i')\cdot \omega \circ \theta_{M_i}\Big)^{n-\sum_{j=1}^i n_j}\bigg)|_{\mathbb{Q}_p^\times}\bigg), \end{aligned}$$

where the first isomorphism follows from $\overline{\rho} \cong \widetilde{w}\overline{\rho}\widetilde{w}^{-1}$, the second equality is obvious ($\widetilde{w}(\Phi)(\Pi')$ being a representation of $M_Q^{\mathrm{Gal}(K/\mathbb{Q}_p)}$ as it is a subquotient of C_Q), the second isomorphism follows from Definition 2.4.1.5(iii), and the last two isomorphisms from (92), (93) and local class field theory for $\mathbb{Q}_p$. So we have to prove $(\delta_G \prod_{i=1}^d \delta_{M_i}^{-1})\,\mathrm{ind}_K^{\otimes \mathbb{Q}_p}(\omega^{-\sum_{i=1}^d c_i}) = 1$, which amounts to checking the following explicit identity (using (94) and Example 2.1.1.3):

$$\sum_{j=1}^{n-1} j^2 = \sum_{i=1}^d \sum_{j=1}^{n_i-1} j^2 + \sum_{i=1}^d \bigg(n_i\Big(n - \sum_{j=1}^i n_j\Big)\Big(n - 1 - \sum_{j=1}^{i-1} n_j\Big)\bigg).$$

This follows easily by induction on d using the case $d = 2$ and the identity

$$(n-m)^2 + (n-m+1)^2 + \cdots + (n-1)^2 = 1 + 2^2 + \cdots + (m-1)^2 + m(n-m)(n-1)$$

for any integers $n \geq m \geq 1$. □

The following proposition is analogous to Proposition 2.4.1.8.

Proposition 2.4.2.11. *Assume* Π *is compatible with* $\overline{\rho}$ *and let* Φ *be a bijection as in Definition 2.4.2.7. Let* $\widetilde{w} \in W_{\overline{\rho}}$, Q *a parabolic subgroup containing* $^{\widetilde{w}}P_{\overline{\rho}}$ *and* C_Q *an isotypic component of* $\overline{L}^{\otimes}|_{Z_{M_Q}}$ *such that* $P(C_Q) = {}^{w}Q$ *for some (unique)* $w \in W$ *with* $w(S(Q)) \subseteq S$. *Then* $\pi_i(C_Q)$ *is compatible with* $w(\overline{\rho}^{Q-\mathrm{ss}})_i$ *for* $i \in \{1, \dots, d\}$, *where* $\pi_i(C_Q)$ *is as in Definition 2.4.1.5(i) and* $w(\overline{\rho}^{Q-\mathrm{ss}})_i$ *as in* (86).

Proof. We use the notation in the proof of Proposition 2.4.1.8. Replacing $\overline{\rho}$ by $\widetilde{w}\overline{\rho}\widetilde{w}^{-1}$ and Φ by $\widetilde{w}(\Phi)$, we can assume $\widetilde{w} = \mathrm{Id}$. We have to prove that the map $\Phi_{w,i}$ satisfies conditions (i) and (ii) of Definition 2.4.2.7 with M_i instead of G and $w(\overline{\rho}^{Q-\mathrm{ss}})_i$ instead of $\overline{\rho}$. Note that this makes sense thanks to Proposition 2.4.2.2. We can assume $i = 1$. Condition (i) clearly follows from the second equality in (92) applied to $\pi'_1 = \pi_1(C_Q)$. Arguing as in Step 1 of Lemma 2.4.1.8, we need only consider a standard parabolic subgroup Q_1 of M_1 containing $({}^{w}P_{\overline{\rho}})_{Q,1}$ and C_{Q_1} an isotypic component of $\overline{L}_1^{\otimes}|_{Z_{M_{Q_1}}}$. Let $C_{Q_{(1)}}$ be the isotypic component of $\overline{L}^{\otimes}|_{Z_{M_{Q_{(1)}}}}$ defined in Step 2 of the proof of Proposition 2.4.1.8. Then it is easy to check that condition (ii) for M_1, $w(\overline{\rho}^{Q-\mathrm{ss}})_1$, C_{Q_1} and an element $w_1 \in W(C_{Q_1})$ follows from condition (ii) with G, $\overline{\rho}$, $C_{Q_{(1)}}$ and $w_1 w \in W(C_{Q_{(1)}})$ (see Step 3, Step 4 and Step 5 of the proof of Proposition 2.4.1.8). □

2.4.3 Explicit examples

We explicitly give the form of a representation Π compatible with $\overline{\rho}$ for various $\overline{\rho}$.

In the examples below, as in Example 2.4.1.7, a line means a nonsplit extension between two irreducible constituents, the constituent on the left being the subobject of the corresponding (length 2) subquotient.

Example 1
We start with $\mathrm{GL}_2(\mathbb{Q}_{p^f})$ and $\widetilde{P}_{\overline{\rho}} = P_{\overline{\rho}} = B$ as in Example 2.2.2.9(i), i.e. we have

$$\overline{\rho} \cong \begin{pmatrix} \chi_1 & * \\ 0 & \chi_2 \end{pmatrix},$$

where χ_i are two smooth characters $\mathbb{Q}_{p^f}^\times \to \mathbb{F}^\times$ (via class field theory) with ratio $\neq 1, \omega^{\pm 1}$ (and where $*$ is nonsplit). Let Π be compatible with $\overline{\rho}$. Then Π has $f+1$ irreducible constituents and the following form:

$$\mathrm{Ind}_{B^-(\mathbb{Q}_{p^f})}^{\mathrm{GL}_2(\mathbb{Q}_{p^f})}(\chi_1\omega^{-1} \otimes \chi_2) \,\text{---}\, \mathrm{SS}_1 \,\text{---}\, \mathrm{SS}_2 \,\text{---} \cdots \text{---}\, \mathrm{SS}_{f-1} \,\text{---}\, \mathrm{Ind}_{B^-(\mathbb{Q}_{p^f})}^{\mathrm{GL}_2(\mathbb{Q}_{p^f})}(\chi_2\omega^{-1} \otimes \chi_1)$$

where the SS_i for $i \in \{1, \dots, f-1\}$ are distinct supersingular representations of $\mathrm{GL}_2(\mathbb{Q}_{p^f})$ over $\mathbb{F}$ such that $Z(\mathrm{SS}_i) = \det(\overline{\rho})\omega^{-1}$ and

$$V_G(\mathrm{SS}_i) \cong \bigoplus_{\substack{I \subseteq \mathrm{Gal}(K/\mathbb{Q}_p) \\ |I| = f-i}} \Big(\Big(\bigotimes_{\sigma \in I} \chi_1^\sigma\Big) \otimes \Big(\bigotimes_{\sigma \notin I} \chi_2^\sigma\Big)\Big)$$

(here $\chi_i^\sigma \overset{\text{def}}{=} \chi_i(\sigma \cdot \sigma^{-1})$ and $V_G(\mathrm{SS}_i)$ is immediately checked to be a representation of $\mathrm{Gal}(\overline{\mathbb{Q}}_p/\mathbb{Q}_p)$). Moreover it follows from Example 2.1.1.6 that

$$V_G\Big(\operatorname{Ind}_{B^-(\mathbb{Q}_p)}^{\mathrm{GL}_2(\mathbb{Q}_{p^f})}(\chi_1\omega^{-1} \otimes \chi_2)\Big) \cong \otimes_{\sigma\in\mathrm{Gal}(K/\mathbb{Q}_p)}\chi_1^\sigma$$

and likewise with $\operatorname{Ind}_{B^-(\mathbb{Q}_p)}^{\mathrm{GL}_2(\mathbb{Q}_{p^f})}(\chi_2\omega^{-1} \otimes \chi_1)$. Finally the conditions in (91) imply that V_G behaves as an exact functor on the (not necessarily irreducible) subquotients of Π (see Remark 2.4.2.8(iii)).

Still with $\mathrm{GL}_2(\mathbb{Q}_{p^f})$ but when $\widetilde{P}_{\overline{\rho}} = T$, i.e. $\overline{\rho} = \chi_1 \oplus \chi_2$, then Π (compatible with $\overline{\rho}$) is semisimple, i.e. has the same form as above but with split extensions everywhere. This is consistent with the discussion at the end of [BP12, §19]. Note that, if we only require Π to be compatible with $\widetilde{P}_{\overline{\rho}}$ (Definition 2.4.1.5), then Π has the same form as above, but with arbitrary distinct supersingular representations of $\mathrm{GL}_2(\mathbb{Q}_{p^f})$ and arbitrary distinct irreducible principal series $\operatorname{Ind}_{B^-(\mathbb{Q}_{p^f})}^{\mathrm{GL}_2(\mathbb{Q}_{p^f})}(\eta_1\omega^{-1} \otimes \eta_2)$ and $\operatorname{Ind}_{B^-(\mathbb{Q}_{p^f})}^{\mathrm{GL}_2(\mathbb{Q}_{p^f})}(\eta_2\omega^{-1} \otimes \eta_1)$. See [HW22, §10.6] and §3.4.4 for instances of representations Π (coming from mod p cohomology) satisfying (special cases of) the above properties.

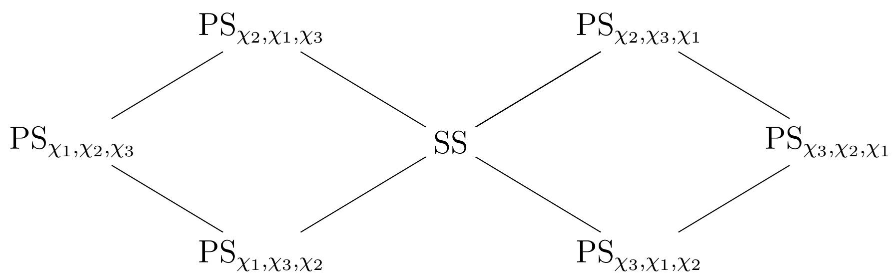
Example 2
We go on with $\mathrm{GL}_3(\mathbb{Q}_p)$ as in Example 2.2.2.9(ii) and $\widetilde{P}_{\overline{\rho}} = P_{\overline{\rho}} = B$, i.e. we have

$$\overline{\rho} \cong \begin{pmatrix} \chi_1 & * & * \\ 0 & \chi_2 & * \\ 0 & 0 & \chi_3 \end{pmatrix},$$

where χ_i are three smooth characters $\mathbb{Q}_p^\times \to \mathbb{F}^\times$ (via class field theory) of ratio $\neq 1, \omega^{\pm 1}$. For $\tau \in W \cong \mathcal{S}_3$, we define

$$\mathrm{PS}_{\chi_{\tau(1)},\chi_{\tau(2)},\chi_{\tau(3)}} \overset{\text{def}}{=} \operatorname{Ind}_{B^-(\mathbb{Q}_p)}^{\mathrm{GL}_3(\mathbb{Q}_p)}(\chi_{\tau(1)}\omega^{-2} \otimes \chi_{\tau(2)}\omega^{-1} \otimes \chi_{\tau(3)}).$$

Let Π be compatible with $\overline{\rho}$. Then Π has 7 irreducible constituents and the following form:



where SS is a supersingular representation of $\mathrm{GL}_3(\mathbb{Q}_p)$ over $\mathbb{F}$ such that $Z(\mathrm{SS}) = \det(\overline{\rho}) \cdot \omega^{-3}$ and $V_G(\mathrm{SS}) \cong (\chi_1\chi_2\chi_3)^{\oplus 3} = \det(\overline{\rho})^{\oplus 3}$. It follows from the proof of

[Hau16, Thm.5.2.1], or from [Hau18, Thm.1.4(i)], combined with [Eme10a, Cor.4.3.5], that the nonsplit extensions between two principal series in subquotient are *automatically* parabolic inductions as required in condition (i) of Definition 2.4.1.5 (looking at isotypic components of $\overline{L}^{\otimes}|_{Z_{M_Q}}$ with $M_Q \in \{\mathrm{GL}_2 \times \mathrm{GL}_1, \mathrm{GL}_1 \times \mathrm{GL}_2\}$, see Example 2.2.2.9(ii)). Conditions (ii) to (iv) in Definition 2.4.1.5 are then easily checked. Concerning Definition 2.4.2.7, the subquotients involving only principal series do satisfy (91) and (92) by [Bre15, Rem.9.9]. The reader can then easily work out the remaining conditions in (91) which all involve the supersingular representation SS, and also work out the shape of a Π which is compatible with $\widetilde{P}_{\overline{\rho}} = B$ only (but not necessarily with $\overline{\rho}$).

Example 3
We stay with $\mathrm{GL}_3(\mathbb{Q}_p)$ but where $\widetilde{P}_{\overline{\rho}} = P_{\overline{\rho}} = P$ with $M_P = \mathrm{diag}(\mathrm{GL}_2, \mathrm{GL}_1)$, i.e. we have

$$\overline{\rho} \cong \begin{pmatrix} \overline{\rho}_1 & * \\ 0 & \chi_2 \end{pmatrix},$$

where $\overline{\rho}_1 : \mathrm{Gal}(\overline{\mathbb{Q}}_p/\mathbb{Q}_p) \to \mathrm{GL}_2(\mathbb{F})$ is any absolutely irreducible representation and χ_2 is any smooth character $\mathbb{Q}_p^\times \to \mathbb{F}^\times$ (via class field theory). Note that such a $\overline{\rho}$ is always generic (see the beginning of §2.4.2). Then Π is compatible with $\overline{\rho}$ if and only Π has the same form as in Example 2.4.1.7:

$$\mathrm{Ind}_{P^-(\mathbb{Q}_p)}^{\mathrm{GL}_3(\mathbb{Q}_p)}\Big(\pi_1 \cdot (\omega^{-1} \circ \det) \otimes \chi_2\Big) \text{——} \mathrm{SS} \text{——} \mathrm{Ind}_{P'^-(\mathbb{Q}_p)}^{\mathrm{GL}_3(\mathbb{Q}_p)}\Big(\chi_2\omega^{-2} \otimes \pi_1\Big)$$

and where moreover

- π_1 is the supersingular representation of $\mathrm{GL}_2(\mathbb{Q}_p)$ over $\mathbb{F}$ corresponding to $\overline{\rho}_1$ by the mod p local Langlands correspondence for $\mathrm{GL}_2(\mathbb{Q}_p)$, i.e. we have $Z(\pi_1) = \det(\overline{\rho}_1)\omega^{-1}$ (via class field theory) and $V_{\mathrm{GL}_2}(\pi_1) \cong \overline{\rho}_1$;

- $Z(\mathrm{SS}) = \det(\overline{\rho})\omega^{-3}$;

- $V_G(\Pi) \cong \overline{\rho} \otimes_{\mathbb{F}} \wedge_{\mathbb{F}}^2 \overline{\rho}$;

- $V_G\Big(\mathrm{Ind}_{P^-(\mathbb{Q}_p)}^{\mathrm{GL}_3(\mathbb{Q}_p)}\Big(\pi_1 \cdot (\omega^{-1} \circ \det) \otimes \chi_2\Big) \text{—} \mathrm{SS}\Big) \cong \mathrm{Ker}(\overline{\rho} \otimes_{\mathbb{F}} \wedge_{\mathbb{F}}^2 \overline{\rho} \twoheadrightarrow \chi_2^2 \otimes \overline{\rho}_1)$.

The properties of V_G in §2.1.1 (in particular Lemma 2.1.1.5 which can be applied here thanks to Remark 2.1.1.7) then automatically give the remaining conditions in (91):

$$\begin{aligned}
V_G\Big(\operatorname{Ind}_{P^-(\mathbb{Q}_p)}^{\mathrm{GL}_3(\mathbb{Q}_p)}\big(\pi_1\cdot(\omega^{-1}\circ\det)\otimes\chi_2\big)\Big) &\cong \overline{\rho}_1\otimes_{\mathbb{F}}\wedge_{\mathbb{F}}^2\overline{\rho}_1\cong\overline{\rho}_1\otimes\det(\overline{\rho}_1)\\
V_G\Big(\mathrm{SS}\longrightarrow\operatorname{Ind}_{P'^-(\mathbb{Q}_p)}^{\mathrm{GL}_3(\mathbb{Q}_p)}\big(\chi_2\omega^{-2}\otimes\pi_1\big)\Big) &\cong (\overline{\rho}\otimes_{\mathbb{F}}\wedge_{\mathbb{F}}^2\overline{\rho})/(\overline{\rho}_1\otimes_{\mathbb{F}}\wedge_{\mathbb{F}}^2\overline{\rho}_1)\\
V_G\Big(\operatorname{Ind}_{P'^-(\mathbb{Q}_p)}^{\mathrm{GL}_3(\mathbb{Q}_p)}\big(\chi_2\omega^{-2}\otimes\pi_1\big)\Big) &\cong \overline{\rho}_1\otimes\chi_2^2\\
V_G(\mathrm{SS}) &\cong (\overline{\rho}_1^{\otimes 2}\otimes\chi_2)\oplus\det(\overline{\rho}_1)\chi_2.
\end{aligned}$$

The case $\widetilde{P}_{\overline{\rho}}=M_P$, i.e. $\overline{\rho}\cong\begin{pmatrix}\overline{\rho}_1 & 0\\ 0 & \chi_2\end{pmatrix}$, is analogous and easier since Π is then semisimple.

Example 4
We consider $\mathrm{GL}_4(\mathbb{Q}_p)$ and $\widetilde{P}_{\overline{\rho}}=P_{\overline{\rho}}=P$, where $M_P=\mathrm{diag}(\mathrm{GL}_2,\mathrm{GL}_1,\mathrm{GL}_1)$, that is we have a good conjugate

$$\overline{\rho}\cong\begin{pmatrix}\overline{\rho}_1 & * & *\\ 0 & \chi_2 & *\\ 0 & 0 & \chi_3\end{pmatrix},$$

where $\overline{\rho}_1:\mathrm{Gal}(\overline{\mathbb{Q}}_p/\mathbb{Q}_p)\to\mathrm{GL}_2(\mathbb{F})$ is any absolutely irreducible representation and χ_i two smooth characters $\mathbb{Q}_p^\times\to\mathbb{F}^\times$ (via class field theory) of ratio $\neq 1,\omega^{\pm1}$. If $1\le i\le 4$ and $\sum_{j=1}^i n_j=4$ with $1\le n_j\le 4$, we write $P_{n_1,\dots,n_i}$ for the standard parabolic subgroup of GL_4 of Levi $\mathrm{diag}(\mathrm{GL}_{n_1},\dots,\mathrm{GL}_{n_i})$ (so $P_{2,1,1}=P$, $P_{1,1,1,1}=B$, etc.). As in Example 3 above, we let π_1 be the supersingular representation of $\mathrm{GL}_2(\mathbb{Q}_p)$ over $\mathbb{F}$ corresponding to $\overline{\rho}_1$ by the mod p local Langlands correspondence for $\mathrm{GL}_2(\mathbb{Q}_p)$ (so $Z(\pi_1)=\det(\overline{\rho}_1)\cdot\omega^{-1}$ and $V_{\mathrm{GL}_2}(\pi_1)\cong\overline{\rho}_1$). We define the following parabolic inductions:

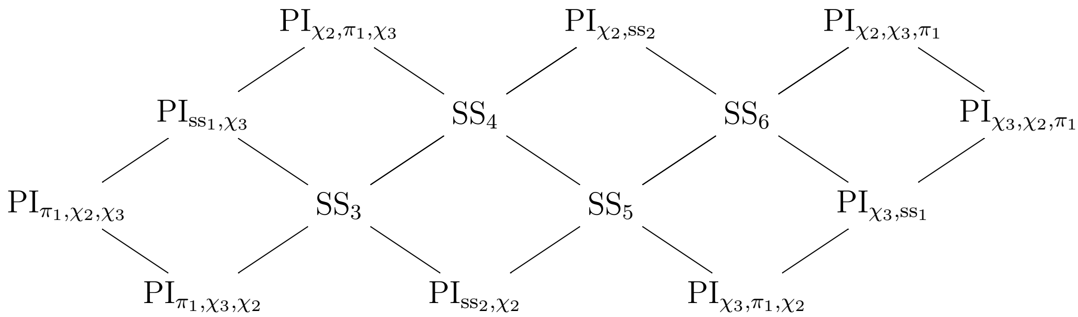
$$\begin{aligned}
\mathrm{PI}_{\pi_1,\chi_2,\chi_3} &\stackrel{\mathrm{def}}{=} \operatorname{Ind}_{P_{2,1,1}^-(\mathbb{Q}_p)}^{\mathrm{GL}_4(\mathbb{Q}_p)}\big(\pi_1\cdot(\omega^{-2}\circ\det)\otimes\chi_2\omega^{-1}\otimes\chi_3\big)\\
\mathrm{PI}_{\pi_1,\chi_3,\chi_2} &\stackrel{\mathrm{def}}{=} \operatorname{Ind}_{P_{2,1,1}^-(\mathbb{Q}_p)}^{\mathrm{GL}_4(\mathbb{Q}_p)}\big(\pi_1\cdot(\omega^{-2}\circ\det)\otimes\chi_3\omega^{-1}\otimes\chi_2\big)\\
\mathrm{PI}_{\chi_2,\pi_1,\chi_3} &\stackrel{\mathrm{def}}{=} \operatorname{Ind}_{P_{1,2,1}^-(\mathbb{Q}_p)}^{\mathrm{GL}_4(\mathbb{Q}_p)}\big(\chi_2\omega^{-3}\otimes\pi_1\cdot(\omega^{-1}\circ\det)\otimes\chi_3\big)\\
\mathrm{PI}_{\chi_3,\pi_1,\chi_2} &\stackrel{\mathrm{def}}{=} \operatorname{Ind}_{P_{1,2,1}^-(\mathbb{Q}_p)}^{\mathrm{GL}_4(\mathbb{Q}_p)}\big(\chi_3\omega^{-3}\otimes\pi_1\cdot(\omega^{-1}\circ\det)\otimes\chi_2\big)\\
\mathrm{PI}_{\chi_2,\chi_3,\pi_1} &\stackrel{\mathrm{def}}{=} \operatorname{Ind}_{P_{1,1,2}^-(\mathbb{Q}_p)}^{\mathrm{GL}_4(\mathbb{Q}_p)}\big(\chi_2\omega^{-3}\otimes\chi_3\omega^{-2}\otimes\pi_1\big)\\
\mathrm{PI}_{\chi_3,\chi_2,\pi_1} &\stackrel{\mathrm{def}}{=} \operatorname{Ind}_{P_{1,1,2}^-(\mathbb{Q}_p)}^{\mathrm{GL}_4(\mathbb{Q}_p)}\big(\chi_3\omega^{-3}\otimes\chi_2\omega^{-2}\otimes\pi_1\big)
\end{aligned}$$

and also, for $\mathrm{ss}_1,\mathrm{ss}_2$ two (not necessarily distinct) supersingular representations of

$\mathrm{GL}_3(\mathbb{Q}_p)$ over $\mathbb{F}$:

$$\begin{aligned}
\mathrm{PI}_{\mathrm{ss}_1,\chi_3} &\stackrel{\mathrm{def}}{=} \mathrm{Ind}_{P^-_{3,1}(\mathbb{Q}_p)}^{\mathrm{GL}_4(\mathbb{Q}_p)}\Big(\mathrm{ss}_1\cdot(\omega^{-1}\circ\det)\otimes\chi_3\Big)\\
\mathrm{PI}_{\mathrm{ss}_2,\chi_2} &\stackrel{\mathrm{def}}{=} \mathrm{Ind}_{P^-_{3,1}(\mathbb{Q}_p)}^{\mathrm{GL}_4(\mathbb{Q}_p)}\Big(\mathrm{ss}_2\cdot(\omega^{-1}\circ\det)\otimes\chi_2\Big)\\
\mathrm{PI}_{\chi_2,\mathrm{ss}_2} &\stackrel{\mathrm{def}}{=} \mathrm{Ind}_{P^-_{1,3}(\mathbb{Q}_p)}^{\mathrm{GL}_4(\mathbb{Q}_p)}\Big(\chi_2\omega^{-3}\otimes\mathrm{ss}_2\Big)\\
\mathrm{PI}_{\chi_3,\mathrm{ss}_1} &\stackrel{\mathrm{def}}{=} \mathrm{Ind}_{P^-_{1,3}(\mathbb{Q}_p)}^{\mathrm{GL}_4(\mathbb{Q}_p)}\Big(\chi_3\omega^{-3}\otimes\mathrm{ss}_1\Big).
\end{aligned}$$

We then let $\mathrm{SS}_3,\mathrm{SS}_4,\mathrm{SS}_5,\mathrm{SS}_6$ be 4 distinct supersingular representations of $\mathrm{GL}_4(\mathbb{Q}_p)$ over $\mathbb{F}$. If Π is compatible with $\overline{\rho}$, then it has the following form:



where we have

$$\begin{aligned}
\mathrm{PI}_{\pi_1,\chi_2,\chi_3} - \mathrm{PI}_{\mathrm{ss}_1,\chi_3} - \mathrm{PI}_{\chi_2,\pi_1,\chi_3} &\cong \mathrm{Ind}_{P^-_{3,1}(\mathbb{Q}_p)}^{\mathrm{GL}_4(\mathbb{Q}_p)}(\Pi_1\cdot(\omega^{-1}\circ\det)\otimes\chi_3)\\
\mathrm{PI}_{\chi_3,\pi_1,\chi_2} - \mathrm{PI}_{\chi_3,\mathrm{ss}_1} - \mathrm{PI}_{\chi_3,\chi_2,\pi_1} &\cong \mathrm{Ind}_{P^-_{1,3}(\mathbb{Q}_p)}^{\mathrm{GL}_4(\mathbb{Q}_p)}(\chi_3\omega^{-3}\otimes\Pi_1)
\end{aligned} \tag{95}$$

for $\Pi_1 \cong \mathrm{Ind}_{P^-_{2,1}(\mathbb{Q}_p)}^{\mathrm{GL}_3(\mathbb{Q}_p)}\Big(\pi_1\cdot(\omega^{-1}\circ\det)\otimes\chi_2\Big) - \mathrm{ss}_1 - \mathrm{Ind}_{P^-_{1,2}(\mathbb{Q}_p)}^{\mathrm{GL}_3(\mathbb{Q}_p)}\Big(\chi_2\omega^{-2}\otimes\pi_1\Big)$, and also

$$\begin{aligned}
&\mathrm{PI}_{\pi_1,\chi_2,\chi_3} - \mathrm{PI}_{\pi_1,\chi_3,\chi_2} \cong\\
&\mathrm{Ind}_{P^-_{2,2}(\mathbb{Q}_p)}^{\mathrm{GL}_4(\mathbb{Q}_p)}\Big(\pi_1\cdot(\omega^{-2}\circ\det)\otimes\Big(\mathrm{Ind}_{B^-(\mathbb{Q}_p)}^{\mathrm{GL}_2(\mathbb{Q}_p)}(\chi_2\omega^{-1}\otimes\chi_3) - \mathrm{Ind}_{B^-(\mathbb{Q}_p)}^{\mathrm{GL}_2(\mathbb{Q}_p)}(\chi_3\omega^{-1}\otimes\chi_2)\Big)\Big)
\end{aligned} \tag{96}$$

and an analogous isomorphism for $\mathrm{PI}_{\chi_2,\chi_3,\pi_1} - \mathrm{PI}_{\chi_3,\chi_2,\pi_1}$. It actually easily follows from [Hau18, Thm.1.4(i)] (see also [Hey, Thm.B(b)(ii)]) together with [Eme10a, Cor.4.3.5] that the isomorphism in (96) and the analogous isomorphism with $\mathrm{PI}_{\chi_2,\chi_3,\pi_1} - \mathrm{PI}_{\chi_3,\chi_2,\pi_1}$ hold (i.e. are not conjectural). It also follows from [Hau18, Thm.1.2(ii)] and [Hau18, Thm.1.2(iii)] that we automatically have isomorphisms

$$\begin{aligned}
\mathrm{PI}_{\pi_1,\chi_2,\chi_3} - \mathrm{PI}_{\mathrm{ss}_1,\chi_3} &\cong \mathrm{Ind}_{P^-_{3,1}(\mathbb{Q}_p)}^{\mathrm{GL}_4(\mathbb{Q}_p)}\Big(\mathrm{Ind}_{P^-_{2,1}(\mathbb{Q}_p)}^{\mathrm{GL}_3(\mathbb{Q}_p)}\Big(\pi_1\cdot(\omega^{-1}\circ\det)\otimes\chi_2\Big) - \mathrm{ss}_1\Big)\\
\mathrm{PI}_{\mathrm{ss}_1,\chi_3} - \mathrm{PI}_{\chi_2,\pi_1,\chi_3} &\cong \mathrm{Ind}_{P^-_{3,1}(\mathbb{Q}_p)}^{\mathrm{GL}_4(\mathbb{Q}_p)}\Big(\mathrm{ss}_1 - \mathrm{Ind}_{P^-_{1,2}(\mathbb{Q}_p)}^{\mathrm{GL}_3(\mathbb{Q}_p)}\Big(\chi_2\omega^{-2}\otimes\pi_1\Big)\Big)
\end{aligned}$$

and likewise with the two "halves" of $\mathrm{PI}_{\chi_3,\pi_1,\chi_2} - \mathrm{PI}_{\chi_3,\mathrm{ss}_1} - \mathrm{PI}_{\chi_3,\chi_2,\pi_1}$. It is likely that the full isomorphisms (95) are in fact also automatic.

We must have moreover $Z(\mathrm{ss}_1) = \det(\overline{\rho}_1)\chi_2\omega^{-3}$, $Z(\mathrm{ss}_2) = \det(\overline{\rho}_1)\chi_3\omega^{-3}$, $Z(\mathrm{SS}_i) = \det(\overline{\rho})\omega^{-6}$ for $i \in \{3, 4, 5, 6\}$ and

$$\begin{array}{rcl} V_{\mathrm{GL}_3}(\mathrm{ss}_1) & \cong & (\overline{\rho}_1^{\otimes 2} \otimes \chi_2) \oplus \det(\overline{\rho}_1)\chi_2 \\ V_{\mathrm{GL}_3}(\mathrm{ss}_2) & \cong & (\overline{\rho}_1^{\otimes 2} \otimes \chi_3) \oplus \det(\overline{\rho}_1)\chi_3 \\ V_{\mathrm{GL}_4}(\mathrm{SS}_3) & \cong & \Big(\overline{\rho}_1^{\otimes 2} \otimes \det(\overline{\rho}_1)\chi_2\chi_3\Big)^{\oplus 3} \oplus \Big(\det(\overline{\rho}_1)^2\chi_2\chi_3\Big)^{\oplus 2} \\ V_{\mathrm{GL}_4}(\mathrm{SS}_4) & \cong & \Big(\overline{\rho}_1 \otimes \det(\overline{\rho}_1)\chi_2^2\chi_3\Big)^{\oplus 5} \oplus \Big(\overline{\rho}_1^{\otimes 3} \otimes \chi_2^2\chi_3\Big) \\ V_{\mathrm{GL}_4}(\mathrm{SS}_5) & \cong & \Big(\overline{\rho}_1 \otimes \det(\overline{\rho}_1)\chi_2\chi_3^2\Big)^{\oplus 5} \oplus \Big(\overline{\rho}_1^{\otimes 3} \otimes \chi_2\chi_3^2\Big) \\ V_{\mathrm{GL}_4}(\mathrm{SS}_6) & \cong & \Big(\overline{\rho}_1^{\otimes 2} \otimes \chi_2^2\chi_3^2\Big)^{\oplus 3} \oplus \Big(\det(\overline{\rho}_1)\chi_2^2\chi_3^2\Big)^{\oplus 2}. \end{array}$$

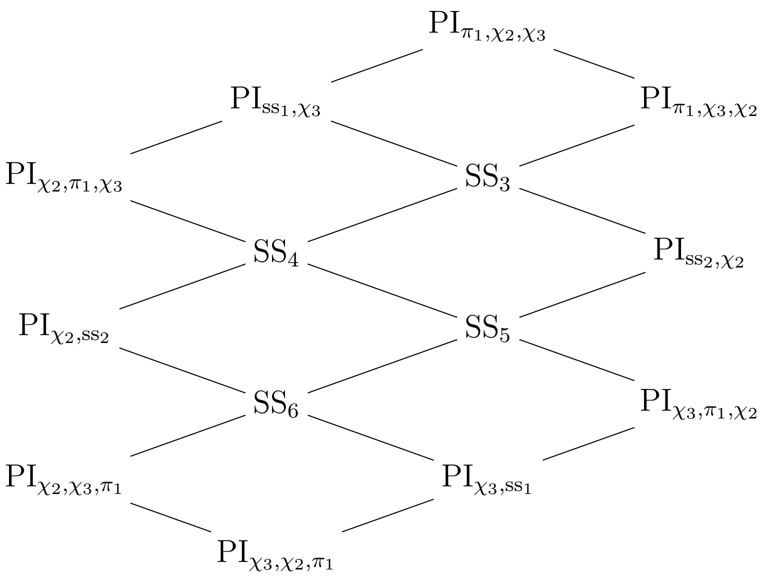
The reader can work out all the other conditions of Definition 2.4.2.7 (applying V_G to subquotients of Π). Note that by Proposition 2.4.2.11 the $\mathrm{GL}_3(\mathbb{Q}_p)$-representation Π_1 is compatible with the subrepresentation $\left(\begin{smallmatrix} \overline{\rho}_1 & * \\ 0 & \chi_2 \end{smallmatrix}\right)$ of $\overline{\rho}$ (see the last part in Example 2).

Example 5

We stay with $\mathrm{GL}_4(\mathbb{Q}_p)$ but where $P_{\overline{\rho}} = P$ with $M_P = \mathrm{diag}(\mathrm{GL}_1, \mathrm{GL}_2, \mathrm{GL}_1)$ and a good conjugate of the form

$$\overline{\rho} \cong \begin{pmatrix} \chi_2 & * & * \\ 0 & \overline{\rho}_1 & 0 \\ 0 & 0 & \chi_3 \end{pmatrix},$$

where the $*$ are nonzero, $\overline{\rho}_1 : \mathrm{Gal}(\overline{\mathbb{Q}}_p/\mathbb{Q}_p) \to \mathrm{GL}_2(\mathbb{F})$ is any absolutely irreducible representation and χ_i are two smooth characters $\mathbb{Q}_p^\times \to \mathbb{F}^\times$ (via class field theory) of ratio $\neq 1, \omega^{\pm 1}$. One has (see (72)) $W_{\overline{\rho}} = \{\mathrm{Id}, s_{e_2-e_3}s_{e_3-e_4}\}$ = the set of permutations of the last two blocks GL_2 and GL_1. Using the notation and conventions of the previous case, we can check that any Π compatible with $\overline{\rho}$ has the following form:



(recall the socle is the first layer on the left), where condition (i) in Definition 2.4.1.5 yields, when applied to a suitable C_Q with $M_Q = \mathrm{diag}(\mathrm{GL}_3, \mathrm{GL}_1)$:

$$\begin{array}{rcl} \mathrm{PI}_{\chi_2,\pi_1,\chi_3} - \mathrm{PI}_{\mathrm{ss}_1,\chi_3} - \mathrm{PI}_{\pi_1,\chi_2,\chi_3} & \cong & \mathrm{Ind}_{P^-_{3,1}(\mathbb{Q}_p)}^{\mathrm{GL}_4(\mathbb{Q}_p)}(\Pi_1 \cdot (\omega^{-1} \circ \det) \otimes \chi_3) \\ \mathrm{PI}_{\chi_3,\chi_2,\pi_1} - \mathrm{PI}_{\chi_3,\mathrm{ss}_1} - \mathrm{PI}_{\chi_3,\pi_1,\chi_2} & \cong & \mathrm{Ind}_{P^-_{1,3}(\mathbb{Q}_p)}^{\mathrm{GL}_4(\mathbb{Q}_p)}(\chi_3\omega^{-3} \otimes \Pi_1) \end{array} \tag{97}$$

for $\Pi_1 \cong \mathrm{Ind}_{P^-_{1,2}(\mathbb{Q}_p)}^{\mathrm{GL}_3(\mathbb{Q}_p)}\big(\chi_2\omega^{-2} \otimes \pi_1\big)$ —— ss_1 —— $\mathrm{Ind}_{P^-_{2,1}(\mathbb{Q}_p)}^{\mathrm{GL}_3(\mathbb{Q}_p)}\big(\pi_1 \cdot (\omega^{-1} \circ \det) \otimes \chi_2\big)$, and yields, when applied to a suitable C_Q with $M_Q = \mathrm{diag}(\mathrm{GL}_2, \mathrm{GL}_2)$ (that is, ${}^{s_{e_2-e_3}s_{e_3-e_4}}P_{\overline{\rho}} \subseteq Q$, note that here $P_{\overline{\rho}} \not\subseteq Q$, see Remark 2.4.1.6(vii)):

$$\mathrm{PI}_{\chi_2,\chi_3,\pi_1} \text{ —— } \mathrm{PI}_{\chi_3,\chi_2,\pi_1} \cong$$
$$\mathrm{Ind}_{P^-_{2,2}(\mathbb{Q}_p)}^{\mathrm{GL}_4(\mathbb{Q}_p)}\Big(\Big(\mathrm{Ind}_{B^-(\mathbb{Q}_p)}^{\mathrm{GL}_2(\mathbb{Q}_p)}(\chi_2\omega^{-3} \otimes \chi_3\omega^{-2}) \text{ —— } \mathrm{Ind}_{B^-(\mathbb{Q}_p)}^{\mathrm{GL}_2(\mathbb{Q}_p)}(\chi_3\omega^{-3} \otimes \chi_2\omega^{-2})\Big) \otimes \pi_1\Big) \tag{98}$$

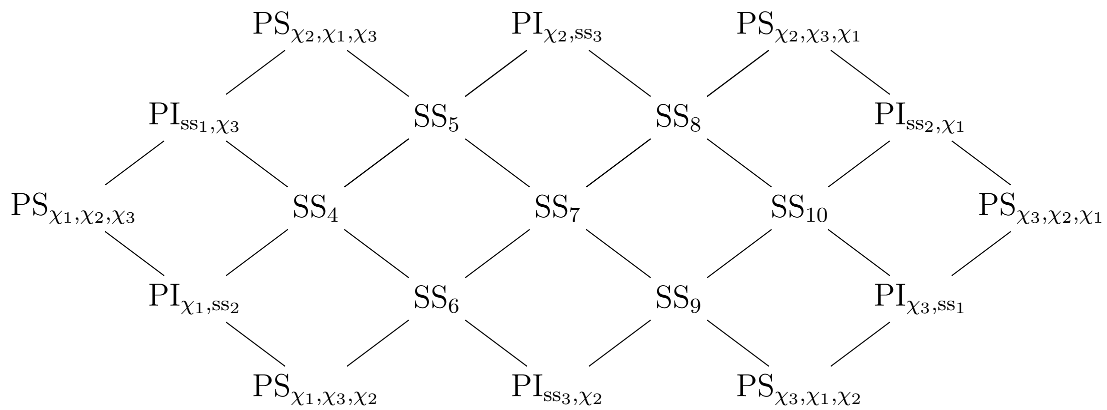
and an analogous isomorphism for $\mathrm{PI}_{\pi_1,\chi_2,\chi_3}$ —— $\mathrm{PI}_{\pi_1,\chi_3,\chi_2}$. As in Example 4, it follows from [Hau18, Thm.1.4(i)] that (98) and the analogous isomorphism are automatic, and from [Hau18, Thm.1.2(ii)], [Hau18, Thm.1.2(iii)] that isomorphisms as in (97) but for every "half" only of the extensions on the left are also automatic.

One can again work out all the conditions of Definition 2.4.2.7 (conditions on $Z(\mathrm{ss}_i)$, $Z(\mathrm{SS}_i)$ and on $V_{\mathrm{GL}_3}(\mathrm{ss}_i)$, $V_{\mathrm{GL}_4}(\mathrm{SS}_i)$ are the same as in Example 4).

Example 6
We consider $\mathrm{GL}_3(\mathbb{Q}_{p^2})$ and $\widetilde{P}_{\overline{\rho}} = P_{\overline{\rho}} = B$, i.e.

$$\overline{\rho} \cong \begin{pmatrix} \chi_1 & * & * \\ 0 & \chi_2 & * \\ 0 & 0 & \chi_3 \end{pmatrix},$$

where χ_i are three smooth characters $\mathbb{Q}_{p^2}^\times \to \mathbb{F}^\times$ (via class field theory) of ratio $\neq 1, \omega^{\pm 1}$. We let $\mathrm{ss}_1, \mathrm{ss}_2, \mathrm{ss}_3$ be 3 (not necessarily distinct) supersingular representations of $\mathrm{GL}_2(\mathbb{Q}_{p^2})$ over $\mathbb{F}$ and SS_i, $i \in \{4, \ldots, 10\}$ be 7 distinct supersingular representations of $\mathrm{GL}_3(\mathbb{Q}_{p^2})$ over $\mathbb{F}$. We use without comment notation for $\mathrm{GL}_3(\mathbb{Q}_{p^2})$ analogous to the ones in Example 2, Example 4 and Example 5 to denote principal series and parabolic inductions. If Π is compatible with $\overline{\rho}$, then it has the following form:



where we have

$$\begin{aligned}
\mathrm{PS}_{\chi_1,\chi_2,\chi_3} - \mathrm{PI}_{\mathrm{ss}_1,\chi_3} - \mathrm{PS}_{\chi_2,\chi_1,\chi_3} &\cong \mathrm{Ind}_{P_{2,1}^-(\mathbb{Q}_{p^2})}^{\mathrm{GL}_3(\mathbb{Q}_{p^2})} \Big(\Pi_1 \cdot (\omega^{-1} \circ \det) \otimes \chi_3\Big) \\
\mathrm{PS}_{\chi_3,\chi_1,\chi_2} - \mathrm{PI}_{\chi_3,\mathrm{ss}_1} - \mathrm{PS}_{\chi_3,\chi_2,\chi_1} &\cong \mathrm{Ind}_{P_{1,2}^-(\mathbb{Q}_{p^2})}^{\mathrm{GL}_3(\mathbb{Q}_{p^2})} \Big(\chi_3\omega^{-2} \otimes \Pi_1\Big) \\
\mathrm{PS}_{\chi_2,\chi_3,\chi_1} - \mathrm{PI}_{\mathrm{ss}_2,\chi_1} - \mathrm{PS}_{\chi_3,\chi_2,\chi_1} &\cong \mathrm{Ind}_{P_{2,1}^-(\mathbb{Q}_{p^2})}^{\mathrm{GL}_3(\mathbb{Q}_{p^2})} \Big(\Pi_2 \cdot (\omega^{-1} \circ \det) \otimes \chi_1\Big) \\
\mathrm{PS}_{\chi_1,\chi_2,\chi_3} - \mathrm{PI}_{\chi_1,\mathrm{ss}_2} - \mathrm{PS}_{\chi_1,\chi_3,\chi_2} &\cong \mathrm{Ind}_{P_{1,2}^-(\mathbb{Q}_{p^2})}^{\mathrm{GL}_3(\mathbb{Q}_{p^2})} \Big(\chi_1\omega^{-2} \otimes \Pi_2\Big)
\end{aligned} \tag{99}$$

for

$$\begin{aligned}
\Pi_1 &\cong \mathrm{Ind}_{B^-(\mathbb{Q}_{p^2})}^{\mathrm{GL}_2(\mathbb{Q}_{p^2})} \Big(\chi_1\omega^{-1} \otimes \chi_2\Big) \text{——} \mathrm{ss}_1 \text{——} \mathrm{Ind}_{B^-(\mathbb{Q}_{p^2})}^{\mathrm{GL}_2(\mathbb{Q}_{p^2})} \Big(\chi_2\omega^{-1} \otimes \chi_1\Big) \\
\Pi_2 &\cong \mathrm{Ind}_{B^-(\mathbb{Q}_{p^2})}^{\mathrm{GL}_2(\mathbb{Q}_{p^2})} \Big(\chi_2\omega^{-1} \otimes \chi_3\Big) \text{——} \mathrm{ss}_2 \text{——} \mathrm{Ind}_{B^-(\mathbb{Q}_{p^2})}^{\mathrm{GL}_2(\mathbb{Q}_{p^2})} \Big(\chi_3\omega^{-1} \otimes \chi_2\Big).
\end{aligned}$$

By a straightforward induction, it follows from [Hau18, Thm.1.3] combined with [Eme10a, Cor.4.3.5] that all isomorphisms (99) are actually true!

We must have moreover $Z(\mathrm{ss}_1) = \chi_1\chi_2\omega^{-1}$, $Z(\mathrm{ss}_2) = \chi_2\chi_3\omega^{-1}$, $Z(\mathrm{ss}_3) = \chi_1\chi_3\omega^{-1}$, $Z(\mathrm{SS}_i) = \det(\overline{\rho})\omega^{-3}$ for $i \in \{4, \dots, 10\}$ and, denoting by σ the only nontrivial element of $\mathrm{Gal}(\mathbb{Q}_{p^2}/\mathbb{Q}_p)$:

$$\begin{aligned}
V_{\mathrm{GL}_2}(\mathrm{ss}_1) &\cong \chi_1\chi_2^\sigma \oplus \chi_1^\sigma\chi_2\\
V_{\mathrm{GL}_2}(\mathrm{ss}_2) &\cong \chi_2\chi_3^\sigma \oplus \chi_3^\sigma\chi_2\\
V_{\mathrm{GL}_2}(\mathrm{ss}_3) &\cong \chi_1\chi_3^\sigma \oplus \chi_3^\sigma\chi_1\\
V_{\mathrm{GL}_3}(\mathrm{SS}_4) &\cong \Big(\chi_1^2\chi_2\det(\overline{\rho})^\sigma \oplus (\chi_1^2\chi_2)^\sigma\det(\overline{\rho})\Big)^{\oplus 3} \oplus \Big(\chi_1^2\chi_3(\chi_2^2\chi_1)^\sigma \oplus (\chi_1^2\chi_3)^\sigma\chi_2^2\chi_1\Big)\\
V_{\mathrm{GL}_3}(\mathrm{SS}_i) &\cong \text{analogous for } i \in \{5,6,8,9,10\} \ \ \text{(left to reader)}\\
V_{\mathrm{GL}_3}(\mathrm{SS}_7) &\cong \Big(\det(\overline{\rho})\det(\overline{\rho})^\sigma\Big)^{\oplus 9} \oplus \Big(\chi_1^2\chi_2(\chi_3^2\chi_2)^\sigma \oplus (\chi_1^2\chi_2)^\sigma\chi_3^2\chi_2\Big) \oplus\\
&\qquad \Big(\chi_1^2\chi_3(\chi_2^2\chi_3)^\sigma \oplus (\chi_1^2\chi_3)^\sigma\chi_2^2\chi_3\Big) \oplus \Big(\chi_2^2\chi_1(\chi_3^2\chi_1)^\sigma \oplus (\chi_2^2\chi_1)^\sigma\chi_3^2\chi_1\Big)
\end{aligned}$$

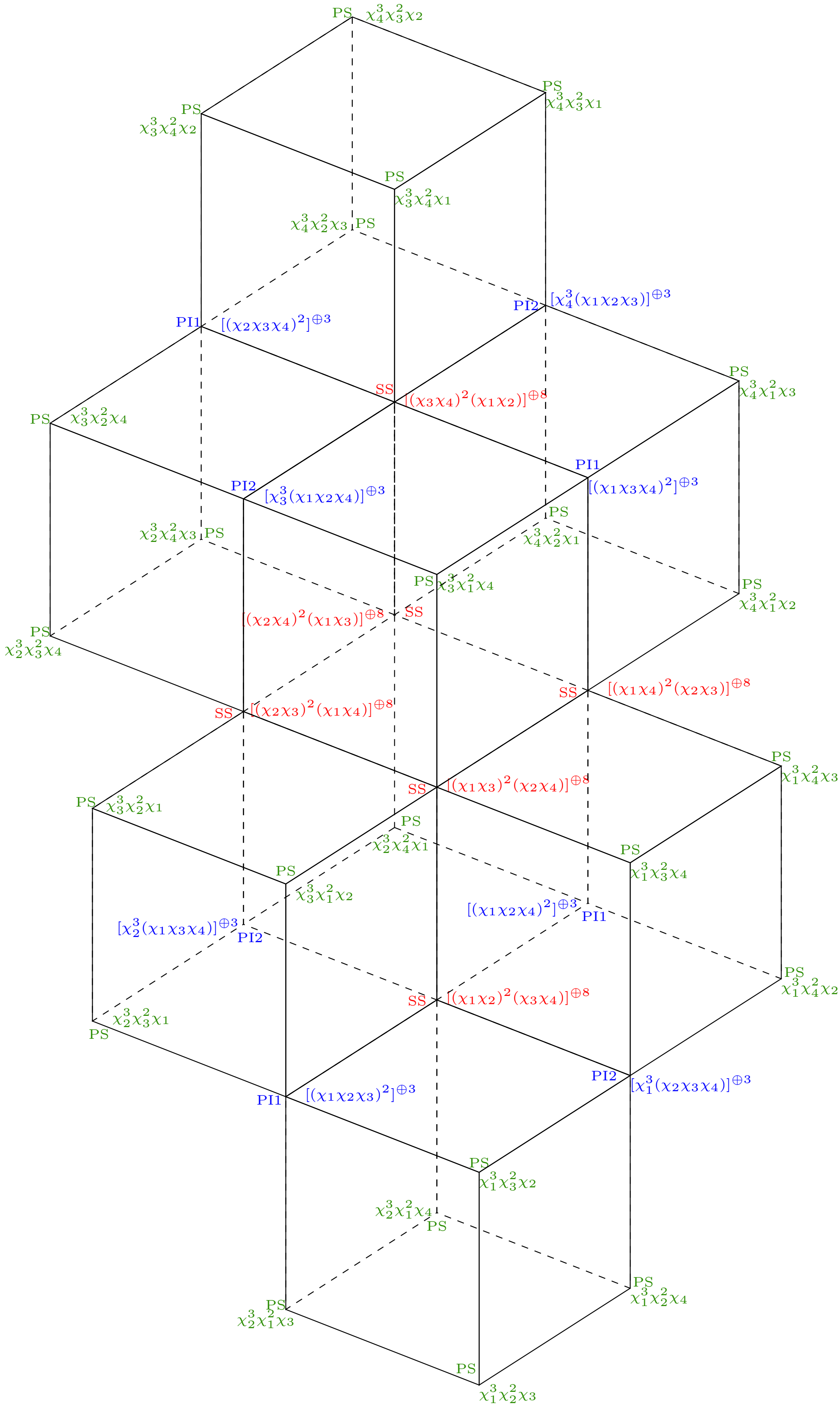
(all obviously representations of $\mathrm{Gal}(\overline{\mathbb{Q}}_p/\mathbb{Q}_p)$ over $\mathbb{F}$). The reader can then work out the conditions in (91) involving the various subquotients of Π. Finally, by Proposition 2.4.2.11 the $\mathrm{GL}_2(\mathbb{Q}_{p^2})$-representation Π_1 (resp. Π_2) is compatible with the subrepresentation $\begin{pmatrix}\chi_1 & * \\ 0 & \chi_2\end{pmatrix}$ (resp. with the quotient $\begin{pmatrix}\chi_2 & * \\ 0 & \chi_3\end{pmatrix}$) of $\overline{\rho}$ (see Example 1).

Example 7
We end up with $\mathrm{GL}_4(\mathbb{Q}_p)$ and $\widetilde{P}_{\overline{\rho}} = P_{\overline{\rho}} = B$, i.e.

$$\overline{\rho} \cong \begin{pmatrix}\chi_1 & * & * & * \\ 0 & \chi_2 & * & * \\ 0 & 0 & \chi_3 & * \\ 0 & 0 & 0 & \chi_4\end{pmatrix},$$

where χ_i are four smooth characters $\mathbb{Q}_p^\times \to \mathbb{F}^\times$ of ratio $\neq 1, \omega^{\pm 1}$. The structure of a Π compatible with $\overline{\rho}$ is given in the next 3D diagram. Just like the previous 2D diagrams look like stacked squares, this 3D diagram looks like stacked cubes: there are 8 cubes, one being entirely "behind". As before, each vertex is an irreducible constituent with PS (in green) meaning principal series, SS (in red) meaning supersingular and PI_1 (resp. PI_2) (in blue) meaning parabolic induction from the standard parabolic subgroup of Levi $\mathrm{GL}_3 \times \mathrm{GL}_1$ (resp. of Levi $\mathrm{GL}_1 \times \mathrm{GL}_3$). The socle is the principal series at the very bottom and the cosocle is the principal series at the very top. Like previously, each edge is a nonsplit extension between two irreducible constituents, the dashed edges being those which are "behind" in the 3D picture. Near each vertex we write the value of V_{GL_4} applied to the corresponding irreducible constituent.

The interested reader can then check all the other conditions and compatibilities in Definition 2.4.1.5 and Definition 2.4.2.7, for instance the two left faces on the bottom correspond to the parabolic induction PI_1 of Example 2 tensored by the character χ_4.


PS $\chi_4^3\chi_3^2\chi_2$
PS $\chi_3^3\chi_4^2\chi_2$
PS $\chi_4^3\chi_3^2\chi_1$
PS $\chi_3^3\chi_4^2\chi_1$
$\chi_4^3\chi_2^2\chi_3$ PS
PI2 $[\chi_4^3(\chi_1\chi_2\chi_3)]^{\oplus 3}$
PI1 $[(\chi_2\chi_3\chi_4)^2]^{\oplus 3}$
PS $\chi_4^3\chi_1^2\chi_3$
SS $[(\chi_3\chi_4)^2(\chi_1\chi_2)]^{\oplus 8}$
PS $\chi_3^3\chi_2^2\chi_4$
PI1 $[(\chi_1\chi_3\chi_4)^2]^{\oplus 3}$
PI2 $[\chi_3^3(\chi_1\chi_2\chi_4)]^{\oplus 3}$
PS $\chi_4^3\chi_2^2\chi_1$
$\chi_2^3\chi_4^2\chi_3$ PS
PS $\chi_3^3\chi_1^2\chi_4$
PS $\chi_4^3\chi_1^2\chi_2$
$(\chi_2\chi_4)^2(\chi_1\chi_3)]^{\oplus 8}$ SS
PS $\chi_2^3\chi_3^2\chi_4$
SS $[(\chi_1\chi_4)^2(\chi_2\chi_3)]^{\oplus 8}$
SS $[(\chi_2\chi_3)^2(\chi_1\chi_4)]^{\oplus 8}$
PS $\chi_1^3\chi_4^2\chi_3$
SS $[(\chi_1\chi_3)^2(\chi_2\chi_4)]^{\oplus 8}$
PS $\chi_3^3\chi_2^2\chi_1$
PS $\chi_2^3\chi_4^2\chi_1$
PS $\chi_1^3\chi_3^2\chi_4$
PS $\chi_3^3\chi_1^2\chi_2$
$[(\chi_1\chi_2\chi_4)^2]^{\oplus 3}$ PI1
$[\chi_2^3(\chi_1\chi_3\chi_4)]^{\oplus 3}$ PI2
PS $\chi_1^3\chi_4^2\chi_2$
SS $[(\chi_1\chi_2)^2(\chi_3\chi_4)]^{\oplus 8}$
$\chi_2^3\chi_3^2\chi_1$ PS
PI2 $[\chi_1^3(\chi_2\chi_3\chi_4)]^{\oplus 3}$
PI1 $[(\chi_1\chi_2\chi_3)^2]^{\oplus 3}$
PS $\chi_1^3\chi_3^2\chi_2$
$\chi_2^3\chi_1^2\chi_4$ PS
PS $\chi_1^3\chi_2^2\chi_4$
PS $\chi_2^3\chi_1^2\chi_3$
PS $\chi_1^3\chi_2^2\chi_3$

2.5 Strong local-global compatibility conjecture

Back to the setting of §2.1 but assuming that F_v^+ is unramified and that $\overline{r}_{\tilde{v}}$ (for $\tilde{v}|v$) is generic as at the beginning of §2.4.2, we conjecture that the $G(F_{\tilde{v}})$-representation $\mathrm{Hom}_{U^v}(\sigma^v, S(V^v, \mathbb{F})[\mathfrak{m}^\Sigma])$ is a direct sum of copies of a $G(F_{\tilde{v}})$-representation which is (up to twist) compatible with any good conjugate of $\overline{r}_{\tilde{v}}$ (Definition 2.4.2.7).

We consider exactly the same global setting as in §2.1.2. We fix $v|p$ in F^+ such that F_v^+ is an unramified extension of $\mathbb{Q}_p$ and consider a continuous representation $\overline{r} : \mathrm{Gal}(\overline{F}/F) \to \mathrm{GL}_n(\mathbb{F})$ such that

(i) $\overline{r}^c \cong \overline{r}^\vee \otimes \omega^{1-n}$ (recall $\overline{r}^c(g) = \overline{r}(cgc)$ for $g \in \mathrm{Gal}(\overline{F}/F)$);

(ii) $\overline{r}$ is an absolutely irreducible representation of $\mathrm{Gal}(\overline{F}/F)$;

(iii) $\overline{r}_{\tilde{v}}$ for $\tilde{v}|v$ has distinct irreducible constituents and the ratio of any two irreducible constituents of dimension 1 is not in $\{\omega, \omega^{-1}\}$

(note that condition (iii) doesn't depend on the place $\tilde{v}$ of F dividing v since $\overline{r}_{\tilde{v}^c} \cong \overline{r}_{\tilde{v}}^\vee \otimes \omega^{1-n}$).

The following is the main conjecture of this paper.

Conjecture 2.5.1. *Let $\overline{r} : \mathrm{Gal}(\overline{F}/F) \to \mathrm{GL}_n(\mathbb{F})$ be a continuous homomorphism that satisfies conditions (i) to (iii) above and fix a place v of F^+ which divides p such that F_v^+ is unramified. Assume that there exist compact open subgroups $V^v \subseteq U^v \subseteq H(\mathbb{A}_{F^+}^{\infty,v})$ with V^v normal in U^v, a finite-dimensional representation σ^v of U^v/V^v over $\mathbb{F}$ and a finite set Σ of finite places of F^+ as in §2.1.3 such that $\mathrm{Hom}_{U^v}(\sigma^v, S(V^v, \mathbb{F})[\mathfrak{m}^\Sigma]) \neq 0$, where $\mathfrak{m}^\Sigma$ is the maximal ideal of $\mathcal{T}^\Sigma$ associated to $\overline{r}$. Let $\tilde{v}|v$ in F and see $\mathrm{Hom}_{U^v}(\sigma^v, S(V^v, \mathbb{F})[\mathfrak{m}^\Sigma])$ as a representation of $H(F_v^+) \cong \mathrm{GL}_n(F_{\tilde{v}}) = G(F_{\tilde{v}})$ via $\iota_{\tilde{v}}$ (cf. §2.1.2). Then there is an integer $d \in \mathbb{Z}_{>0}$ depending only on v, U^v, V^v, σ^v and $\overline{r}$ and an admissible smooth representation $\Pi_{\tilde{v}}$ of $G(F_{\tilde{v}})$ over $\mathbb{F}$ (depending a priori on $\tilde{v}$, U^v, V^v, σ^v and $\overline{r}$) such that*

$$\mathrm{Hom}_{U^v}(\sigma^v, S(V^v, \mathbb{F})[\mathfrak{m}^\Sigma]) \cong \left(\Pi_{\tilde{v}} \otimes (\omega^{n-1} \circ \det)\right)^{\oplus d},$$

where $\Pi_{\tilde{v}}$ is compatible with one (equivalently any by Proposition 2.4.2.9) good conjugate of $\overline{r}_{\tilde{v}}$ in the sense of Definition 2.3.2.3.

Remark 2.5.2. (i) Conjecture 2.5.1 implies in particular that the $G(F_{\tilde{v}})$-representation $\mathrm{Hom}_{U^v}(\sigma^v, S(V^v, \mathbb{F})[\mathfrak{m}^\Sigma])$ is of finite length with all constituents of multiplicity d (under assumptions (i) to (iii) on $\overline{r}$), which is already far from being known in general. See however §3.4 below for nontrivial evidence in the case of GL_2. It also implies that $\mathrm{Hom}_{U^v}(\sigma^v, S(V^v, \mathbb{F})[\mathfrak{m}^\Sigma])$ has a central character, but this is known (at

least under some extra assumptions), see Lemma 2.1.3.3.
(ii) When F_v^+ is unramified and $\overline{r}_{\tilde{v}}$ is as in (iii) above, Conjecture 2.5.1 of course implies (and is in fact much stronger than) Conjecture 2.1.3.1.
(iii) Assuming that p is unramified in F^+ and that $\overline{r}_{\tilde{w}}$ is generic as in (iii) above for all $w|p$, an even stronger conjecture would be as follows.

Conjecture 2.5.3. *For $U^p \subseteq H(\mathbb{A}_{F^+}^{\infty,p})$ such that $S(U^p, \mathbb{F})[\mathfrak{m}^\Sigma] \neq 0$ (where Σ contains the set of places of F^+ that split in F and divide pN, or at which U^p is not unramified, or at which $\overline{r}$ ramifies, and where $S(U^p, \mathbb{F})[\mathfrak{m}^\Sigma]$ is defined as in §2.1.2 replacing U^v by U^p) and for any $\tilde{w}|w$ in F with $w|p$, there is an integer $d \in \mathbb{Z}_{>0}$ depending only on p, U^p and $\overline{r}$ and admissible smooth representations $\Pi_{\tilde{w}}$ of $G(F_{\tilde{w}})$ over $\mathbb{F}$, where $\Pi_{\tilde{w}}$ is compatible with one (equivalently any) good conjugate of $\overline{r}_{\tilde{w}}$ such that*

$$S(U^p, \mathbb{F})[\mathfrak{m}^\Sigma] \cong \Big(\bigotimes_{w|p} \Big(\Pi_{\tilde{w}} \otimes (\omega^{n-1} \circ \det)\Big)\Big)^{\oplus d}.$$

As in §2.1.3, we prove that Conjecture 2.5.1 holds for $\tilde{v}$ if and only if it holds for $\tilde{v}^c$ (we do not need here extra assumptions). We start with two formal lemmas. We use the previous notation and denote by $w_0 \in W$ the unique element with maximal length.

Lemma 2.5.4. *Let $\overline{\rho} : \mathrm{Gal}(\overline{\mathbb{Q}}_p/K) \to \widetilde{P}_{\overline{\rho}}(\mathbb{F}) \subseteq P_{\overline{\rho}}(\mathbb{F}) \subseteq G(\mathbb{F})$ be a good conjugate as in §2.3.2. Then the continuous homomorphism $\mathrm{Gal}(\overline{\mathbb{Q}}_p/K) \to G(\mathbb{F}) = \mathrm{GL}_n(\mathbb{F})$ defined by*

$$g \longmapsto w_0 \tau\Big(\overline{\rho}(g)\Big)^{-1} w_0 \tag{100}$$

is a good conjugate of the dual of the representation associated to $\overline{\rho}$.

Proof. Denote by ${}^{w_0}P_{\overline{\rho}}$ the standard parabolic subgroup of G with set of simple roots $-w_0(S(P_{\overline{\rho}})) \subseteq S$. Using that $W({}^{w_0}P_{\overline{\rho}}) = w_0 W(P_{\overline{\rho}}) w_0$, one checks that $-w_0(X_{\overline{\rho}}) \subseteq R^+$ is a closed subset relative to ${}^{w_0}P_{\overline{\rho}}$ (Definition 2.3.1.1) and thus corresponds to a Zariski-closed algebraic subgroup ${}^{w_0}\widetilde{P}_{\overline{\rho}} \overset{\mathrm{def}}{=} w_0 M_{P_{\overline{\rho}}} w_0 N_{-w_0(X_{\overline{\rho}})}$ of ${}^{w_0}P_{\overline{\rho}}$ (Lemma 2.3.1.4). Denote by $w_0\tau(\overline{\rho})^{-1}w_0$ the homomorphism (100), its associated representation is the dual of the representation associated to $\overline{\rho}$. Moreover one has $\widetilde{P}_{w_0\tau(\overline{\rho})^{-1}w_0} = {}^{w_0}\widetilde{P}_{\overline{\rho}}$ and $X_{hw_0\tau(\overline{\rho})^{-1}w_0h^{-1}} = -w_0(X_{w_0\tau(h)^{-1}w_0\overline{\rho}w_0\tau(h)w_0})$ for any $h \in {}^{w_0}P_{\overline{\rho}}(\mathbb{F})$ (note that $w_0\tau(h)^{-1}w_0 \in P_{\overline{\rho}}(\mathbb{F})$). The result follows from Definition 2.3.2.3. □

As in §2.1.3, if π is a smooth representation of $G(K)$ over $\mathbb{F}$ we denote by $\pi^\star$ the smooth representation of $G(K)$ with the same underlying vector space as π but where $g \in G(K) = \mathrm{GL}_n(K)$ acts by $\tau(g)^{-1}$.

Lemma 2.5.5. *Let $\overline{\rho} : \mathrm{Gal}(\overline{\mathbb{Q}}_p/K) \to G(\mathbb{F})$ be a continuous homomorphism such that $\overline{\rho}^{\mathrm{ss}}$ has distinct irreducible constituents and the ratio of any two irreducible constituents of dimension 1 is not in $\{\omega, \omega^{-1}\}$. Let Π be a smooth representation of $G(K)$ over $\mathbb{F}$. Then Π is compatible with one (equivalently any by Proposition 2.4.2.9) good conjugate of $\overline{\rho}$ if and only if $\Pi^\star$ is compatible with one (ibid.) good conjugate of $\overline{\rho}^\vee \otimes \omega^{n-1}$ (denoting by $\overline{\rho}^\vee$ the dual of the representation associated to $\overline{\rho}$).*

Proof. We use the notation in the proof of Lemma 2.5.4. Assuming $\overline{\rho}$ is a good conjugate, it is enough to show that if Π is compatible with $\overline{\rho}$, then $\Pi^\star$ is compatible with $w_0\tau(\overline{\rho})^{-1}w_0\otimes\omega^{n-1}$. If R is a (finite-dimensional) algebraic representation of $G^{\mathrm{Gal}(K/\mathbb{Q}_p)}$ over $\mathbb{F}$, let $R^\star$ be the algebraic representation where $g \in G^{\mathrm{Gal}(K/\mathbb{Q}_p)}$ acts by $\tau(g)^{-1}$ (inverse transpose on each factor). Then one checks that $\overline{L}^{\otimes\star} \cong \overline{L}^{\otimes} \otimes (\det^{-(n-1)})^{\otimes[K:\mathbb{Q}_p]}$. Let Φ be a bijection as in Definition 2.4.2.7 and define $\Phi^\star$ from the set of subquotients $\Pi'^\star$ of $\Pi^\star$ (where Π' is a subquotient of Π) to the set of good subquotients of $\overline{L}^{\otimes}|_{({}^{w_0}\widetilde{P}_{\overline{\rho}})^{\mathrm{Gal}(K/\mathbb{Q}_p)}}$ as follows: $\Phi^\star(\Pi'^\star)$ is the algebraic representation of $({}^{w_0}\widetilde{P}_{\overline{\rho}})^{\mathrm{Gal}(K/\mathbb{Q}_p)}$ given by $\Phi^\star(\Pi'^\star)(g) \stackrel{\mathrm{def}}{=} \Phi(\Pi')(w_0\tau(g)^{-1}w_0)\det(g)^{n-1}$ for $g \in ({}^{w_0}\widetilde{P}_{\overline{\rho}})^{\mathrm{Gal}(K/\mathbb{Q}_p)}$ (with obvious notation). We leave to the reader the tedious but formal task to check that $\Phi^\star$ satisfies all conditions of Definitions 2.4.1.5 and 2.4.2.7 with ${}^{w_0}\widetilde{P}_{\overline{\rho}}$ and $w_0\tau(\overline{\rho})^{-1}w_0 \otimes \omega^{n-1}$ instead of $\widetilde{P}_{\overline{\rho}}$ and $\overline{\rho}$ using (for Q any standard parabolic subgroup of G):

$$\Big(\mathrm{Ind}_{Q^-(K)}^{G(K)}(\pi_1 \otimes \cdots \otimes \pi_d)\Big)^\star \cong \mathrm{Ind}_{({}^{w_0}Q)^-(K)}^{G(K)}({\pi_d}^\star \otimes \cdots \otimes {\pi_1}^\star)$$

and Lemma 2.1.3.4. □

Proposition 2.5.6. *Conjecture 2.5.1 holds for $\tilde{v}$ if and only if it holds for $\tilde{v}^c$.*

Proof. This follows from Lemma 2.5.5 together with $\overline{r}_{\tilde{v}^c} \cong \overline{r}_{\tilde{v}}^\vee \otimes \omega^{1-n}$, Remark 2.4.2.8(iv) and an easy computation. □

There is an obvious analogous statement with Conjecture 2.5.3 instead of Conjecture 2.5.1.

Remark 2.5.7. Let π be an admissible smooth representation of $G(K)$ over $\mathbb{F}$ with a central character. In [Koh17, Cor.3.15], Kohlhaase associates higher smooth duals $S^i(\pi)$, $i \geq 0$ to π which are also admissible (smooth) representations of $G(K)$ over $\mathbb{F}$ with a central character. In view of the results when $n = 2$ (see condition (iii) in §3.3.5 below and [HW22, Thm.8.2]), it is natural to expect that, when $K = F_{\tilde{v}}$ and $\Pi_{\tilde{v}}$ is as in Conjecture 2.5.1, we have $S^i(\Pi_{\tilde{v}}) \neq 0$ if and only if $i = i_0 \stackrel{\mathrm{def}}{=} [K : \mathbb{Q}_p]\frac{n(n-1)}{2}$ and that $S^{i_0}(\Pi_{\tilde{v}})$ is compatible with (a good conjugate of) $\overline{r}_{\tilde{v}}^\vee \otimes \omega^{n-1}$ (when $n = 2$, this is indeed consistent with *loc.cit.* since $\overline{r}_{\tilde{v}}^\vee \cong \overline{r}_{\tilde{v}} \otimes \det(\overline{r}_{\tilde{v}})^{-1}$). It is also natural to ask if we have $S^{i_0}(\Pi_{\tilde{v}}) \cong \Pi_{\tilde{v}}^\star$ (see Lemma 2.5.5).

From the results of [BH15, §4.4] and [Enn], we can at least give some very weak evidence for Conjecture 2.5.1, more precisely for the stronger Conjecture 2.5.3 in Remark 2.5.2(iii), when p is totally split in F^+ and $\overline{r}_{\tilde{w}}$ is upper-triangular sufficiently generic for all $w|p$ in F^+.

If Π is an admissible smooth representation of $G(K)$ over $\mathbb{F}$, we denote by $\Pi^{\mathrm{ord}} \subseteq \Pi$ the maximal $G(K)$-subrepresentation such that all its irreducible constituents are isomorphic to irreducible subquotients of principal series of $G(K)$ over $\mathbb{F}$. The following lemma is not difficult using Proposition 2.2.3.3, [BH15, Thm.2.2.4] and the results of [BH15, §3.3], [BH15, §3.4] (the proof is left to the reader).

Lemma 2.5.8. *Assume $K = \mathbb{Q}_p$ and let $\overline{\rho} : \mathrm{Gal}(\overline{\mathbb{Q}}_p/\mathbb{Q}_p) \to B(\mathbb{F}) \subseteq G(\mathbb{F})$ be generic (as at the beginning of §2.4.2) and a good conjugate (as in Definition 2.3.2.3). Let Π be compatible with $\overline{\rho}$ (as in Definition 2.4.2.7). Then $\Pi^{\mathrm{ord}} \cong \Pi(\overline{\rho})^{\mathrm{ord}}$, where $\Pi(\overline{\rho})^{\mathrm{ord}}$ is the representation of $G(\mathbb{Q}_p)$ over $\mathbb{F}$ defined in* [BH15, §3.4].

Note that one can explicitly determine $V_G(\Pi(\overline{\rho})^{\mathrm{ord}})$ inside $\overline{L}^{\otimes}(\overline{\rho})$, see [Bre15, §9].

We let S_p be the set of places of F^+ dividing p. Recall that an injection between two representations of a group is called *essential* if it induces an isomorphism on the respective socles.

Theorem 2.5.9 ([Enn])**.** *Assume that F/F^+ is unramified at finite places, that H is defined over $\mathcal{O}_{F^+}$ with $H \times_{\mathcal{O}_{F^+}} F^+$ quasi-split at finite places of F^+, and that p is totally split in F. Assume that $\overline{r} : \mathrm{Gal}(\overline{F}/F) \to \mathrm{GL}_n(\mathbb{F})$ satisfies assumptions A1 to A6 of* [Enn, §3.1]*, let v_1 be a finite place of F^+ as in* [Enn, Lemma 3.1.2] *and $\Sigma \stackrel{\mathrm{def}}{=} S_p \cup \{v_1\}$. Choose $\widetilde{v_1}|v_1$ in F and let $U^p = \prod_{w\nmid p} U_w \subseteq H(\mathbb{A}_{F^+}^{\infty,p})$ such that $U_w = H(\mathcal{O}_{F_w^+})$ if w splits in F, U_w is maximal hyperspecial in $H(F_w^+)$ if w is inert in F and $\iota_{\widetilde{v_1}}(U_{v_1})$ is the Iwahori subgroup of $\mathrm{GL}_n(F_{\widetilde{v_1}})$. Then for any $\tilde{w}|w$ in F and any good conjugates $\overline{r}_{\tilde{w}}$ (where $w \in S_p$), we have an* essential *injection of admissible smooth representations of $\prod_{w|p} H(F_w^+)$ over $\mathbb{F}$:*

$$\Big(\bigotimes_{w|p}\Big(\Pi(\overline{r}_{\tilde{w}})^{\mathrm{ord}} \otimes \omega^{n-1} \circ \det\Big)\Big)^{\oplus n!} \hookrightarrow S(U^p, \mathbb{F})[\mathfrak{m}^{\Sigma}]^{\mathrm{ord}},$$

where $S(U^p, \mathbb{F})[\mathfrak{m}^{\Sigma}]^{\mathrm{ord}} \subseteq S(U^p, \mathbb{F})[\mathfrak{m}^{\Sigma}]$ is defined as $\Pi^{\mathrm{ord}} \subseteq \Pi$ above replacing $G(K)$ by $\prod_{w|p} H(F_w^+)$.

Proof. This follows from [Enn, Thm.3.3.3] (which itself improves [BH15, Thm.4.4.7]) and its proof (see just before [Enn, Lemma 3.2.1] for the $n!$). □

Remark 2.5.10. The cokernel of the injection in Theorem 2.5.9 is an admissible smooth representation of $\prod_{w|p} H(F_w^+)$ over $\mathbb{F}$, and its $\prod_{w|p} H(F_w^+)$-socle is by construction a direct sum of finitely many irreducible subquotients of principal series. If we could prove that all these irreducible subquotients are irreducible principal series which do not appear in the $\prod_{w|p} H(F_w^+)$-socle of $\bigotimes_{w|p}(\Pi(\overline{r}_{\tilde{w}})^{\mathrm{ord}} \otimes \omega^{n-1} \circ \det)$, then it would follow from the mod p version of [Hau19, Cor.1.4] that the essential injection in Theorem 2.5.9 is an isomorphism.

3 The case of $\mathrm{GL}_2(\mathbb{Q}_{p^f})$

We give evidence for Conjecture 2.1.3.1 and Conjecture 2.5.1 when F_v^+ is unramified and $G = \mathrm{GL}_2$. We now assume $K = \mathbb{Q}_{p^f}$ and $n = 2$ till the end. We fix an embedding $\sigma_0 : \mathbb{F}_{p^f} = \mathbb{F}_q \hookrightarrow \mathbb{F}$ and let $\sigma_i \overset{\text{def}}{=} \sigma_0 \circ \varphi^i$ for φ the arithmetic Frobenius and $i \geq 0$.

3.1 $(\varphi, \mathcal{O}_K^\times)$-modules and (φ, Γ)-modules

We associate étale $(\varphi, \mathcal{O}_K^\times)$-modules to certain admissible smooth representations of $\mathrm{GL}_2(K)$ over $\mathbb{F}$ and relate them to the étale (φ, Γ)-modules of §2.1.1.

We assume $p > 2$. We let $I \overset{\text{def}}{=} \left(\begin{smallmatrix} \mathcal{O}_K^\times & \mathcal{O}_K \\ p\mathcal{O}_K & \mathcal{O}_K^\times \end{smallmatrix}\right)$ be the Iwahori subgroup of $\mathrm{GL}_2(\mathcal{O}_K)$, $K_1 \overset{\text{def}}{=} \left(\begin{smallmatrix} 1+p\mathcal{O}_K & p\mathcal{O}_K \\ p\mathcal{O}_K & 1+p\mathcal{O}_K \end{smallmatrix}\right)$ the pro-p radical of $\mathrm{GL}_2(\mathcal{O}_K)$, $I_1 \overset{\text{def}}{=} \left(\begin{smallmatrix} 1+p\mathcal{O}_K & \mathcal{O}_K \\ p\mathcal{O}_K & 1+p\mathcal{O}_K \end{smallmatrix}\right)$ the pro-p radical of I, $N_0 \overset{\text{def}}{=} \left(\begin{smallmatrix} 1 & \mathcal{O}_K \\ 0 & 1 \end{smallmatrix}\right) \subseteq I_1$, $N_0^- \overset{\text{def}}{=} \left(\begin{smallmatrix} 1 & 0 \\ p\mathcal{O}_K & 1 \end{smallmatrix}\right) \subseteq I_1$ and $T_0 \overset{\text{def}}{=} \left(\begin{smallmatrix} 1+p\mathcal{O}_K & 0 \\ 0 & 1+p\mathcal{O}_K \end{smallmatrix}\right) \subseteq I_1$. We denote by Z_1 the center of I_1. If C is a pro-p group then $\mathbb{F}[\![C]\!]$ denotes its Iwasawa algebra over $\mathbb{F}$, which is a local ring, and $\mathfrak{m}_C$ the maximal ideal of $\mathbb{F}[\![C]\!]$. If R (resp. M) is a filtered ring (resp. filtered module) in the sense of [LvO96, §I.2], we denote by F_nR (resp. F_nM) for $n \in \mathbb{Z}$ its ascending filtration and $\mathrm{gr}(R) \overset{\text{def}}{=} \oplus_{n\in\mathbb{Z}} F_nR/F_{n-1}R$ (resp. with M) the associated graded ring (resp. module). When $R = \mathbb{F}[\![C]\!]$, we set $F_nR \overset{\text{def}}{=} \mathfrak{m}_R^{-n}$ if $n \leq 0$ and $F_nR \overset{\text{def}}{=} R$ if $n \geq 0$. If M is an R-module, the filtration defined by $F_nM = \mathfrak{m}_R^{-n}M$ if $n \leq 0$ and $F_nM = M$ if $n \geq 0$ is called *the* $\mathfrak{m}_R$*-adic filtration on* M.

3.1.1 The ring A

We describe some properties of a complete noetherian ring A which will be a coefficient ring for some multivariable $(\psi, \mathcal{O}_K^\times)$-modules and $(\varphi, \mathcal{O}_K^\times)$-modules.

Let v_{N_0} be the $\mathfrak{m}_{N_0}$-adic valuation on the ring $\mathbb{F}[\![N_0]\!]$ defined by the $\mathfrak{m}_{N_0}$-adic filtration (i.e. $F_n\mathbb{F}[\![N_0]\!] = \{x \in \mathbb{F}[\![N_0]\!] : v_{N_0}(x) \geq -n\}$ for $n \in \mathbb{Z}$). We use the same notation to denote the unique extension of v_{N_0} to a valuation of the fraction field of $\mathbb{F}[\![N_0]\!]$. For $i \in \{0, \ldots, f-1\}$ let

$$Y_i \overset{\text{def}}{=} \sum_{a\in\mathbb{F}_q^\times} \sigma_0(a)^{-p^i} \begin{pmatrix} 1 & \tilde{a} \\ 0 & 1 \end{pmatrix} \in \mathfrak{m}_{N_0} \backslash \mathfrak{m}_{N_0}^2 \tag{101}$$

(where $\tilde{a} \in \mathcal{O}_K^\times$ denotes the Teichmüller lift of a) and write $y_i \overset{\text{def}}{=} \mathrm{gr}(Y_i)$ for the image of Y_i in $\mathfrak{m}_{N_0}/\mathfrak{m}_{N_0}^2 \subseteq \mathrm{gr}(\mathbb{F}[\![N_0]\!])$. Then $\mathbb{F}[\![N_0]\!]$ is isomorphic to the power series ring $\mathbb{F}[\![Y_0, \ldots, Y_{f-1}]\!]$ and $\mathrm{gr}(\mathbb{F}[\![N_0]\!])$ to the polynomial algebra $\mathbb{F}[y_0, \ldots, y_{f-1}]$. Let S be the multiplicative subset of $\mathbb{F}[\![N_0]\!]$ whose elements are the $(Y_0\cdots Y_{f-1})^n$ for $n\geq 0$, $\mathbb{F}[\![N_0]\!]_S$ the corresponding localization and $F_n\mathbb{F}[\![N_0]\!]_S \overset{\text{def}}{=} \{x\in \mathbb{F}[\![N_0]\!]_S : v_{N_0}(x)\geq -n\}$. We define the ring A as the completion of the filtered ring $\mathbb{F}[\![N_0]\!]_S$ ([LvO96, §I.3.4]). Note that v_{N_0} extends to A, which is thus a complete filtered ring. As A is complete, an element $x\in A$ is invertible in A if and only if $\mathrm{gr}(x)$ is invertible in $\mathrm{gr}(A)$ (as is easily checked, here $\mathrm{gr}(x)$ is the "principal part" of x as in [LvO96, §I.4.2]).

Let M be a filtered $\mathbb{F}[\![N_0]\!]$-module. The tensor product $A\otimes_{\mathbb{F}[\![N_0]\!]} M$ is then a filtered A-module for the tensor product filtration as defined in [LvO96, p.57]. We let $A\widehat{\otimes}_{\mathbb{F}[\![N_0]\!]}M$ be its completion. This filtered A-module can also be described as the completion of the localization M_S endowed with the tensor product filtration associated to the isomorphism $M_S\cong \mathbb{F}[\![N_0]\!]_S\otimes_{\mathbb{F}[\![N_0]\!]} M$.

Lemma 3.1.1.1. *We have an isomorphism*

$$\mathrm{gr}(A\widehat{\otimes}_{\mathbb{F}[\![N_0]\!]}M)\cong \mathrm{gr}(M_S)\cong \mathrm{gr}(M)[(y_0\cdots y_{f-1})^{-1}]. \tag{102}$$

Proof. As $A\widehat{\otimes}_{\mathbb{F}[\![N_0]\!]}M$ is the completion of M_S, it is sufficient to prove that $\mathrm{gr}(M_S)\cong \mathrm{gr}(M)[(y_0\cdots y_{f-1})^{-1}]$. Note that we have an isomorphism of $\mathbb{F}[\![N_0]\!]$-algebras$\mathbb{F}[\![N_0]\!]_S\cong \mathbb{F}[\![N_0]\!][T]/((Y_0\cdots Y_{f-1})T-1)$. Moreover if we endow the ring $\mathbb{F}[\![N_0]\!][T]$ with the filtration

$$F_n(\mathbb{F}[\![N_0]\!][T])=\sum_{k\geq 0}\mathfrak{m}_{N_0}^{kf-n}T^k$$

(with the convention $\mathfrak{m}_{N_0}^i=\mathbb{F}[\![N_0]\!]$ for $i\leq 0$), the filtration on $\mathbb{F}[\![N_0]\!]_S$ is the quotient filtration via $\mathbb{F}[\![N_0]\!][T]\twoheadrightarrow \mathbb{F}[\![N_0]\!]_S$. Therefore the filtration on M_S is the quotient filtration of the tensor product filtration on $M[T]\overset{\text{def}}{=}\mathbb{F}[\![N_0]\!][T]\otimes_{\mathbb{F}[\![N_0]\!]} M$.

As the filtered $\mathbb{F}[\![N_0]\!]$-module $\mathbb{F}[\![N_0]\!][T]$ is filtered-free by definition (see [LvO96, Def.I.6.1]), it follows from [LvO96, Lemma I.6.14] that $\mathrm{gr}(M[T])\cong \mathrm{gr}(M)[T]$ with $\deg(T)=f$. We claim that the following sequence of filtered modules is strict exact:

$$M[T]\xrightarrow{(Y_0\cdots Y_{f-1})T-1} M[T]\longrightarrow M_S\longrightarrow 0.$$

Namely the exactness of the second arrow follows from the definition of the quotient filtration. As $(Y_0\cdots Y_{f-1})T$ and 1 have degree 0 in $\mathbb{F}[\![N_0]\!][T]$, the multiplication by $(Y_0\cdots Y_{f-1})T-1$ induces the multiplication by $(y_0\cdots y_{f-1})T-1$ on $\mathrm{gr}(M[T])\cong \mathrm{gr}(M)[T]$ which is injective. It follows from [LvO96, Thm.I.4.2.4(2)] (applied with $L=0$, $M=N=M[T]$, $f=0$ and g being the multiplication by $(Y_0\cdots Y_{f-1})T-1$) that the multiplication by $(Y_0\cdots Y_{f-1})T-1$ is a strict map.

It then follows from [LvO96, Thm.I.4.2.4(1)] that the following sequence is exact:

$$\mathrm{gr}(M[T])\xrightarrow{(y_0\cdots y_{f-1})T-1}\mathrm{gr}(M[T])\longrightarrow \mathrm{gr}(M_S)\longrightarrow 0. \tag{103}$$

Finally, since $\mathrm{gr}(M[T])\cong \mathrm{gr}(M)[T]$, we have $\mathrm{gr}(M_S)\cong \mathrm{gr}(M)[(y_0\cdots y_{f-1})^{-1}]$. □

Corollary 3.1.1.2. *We have an isomorphism* $\mathrm{gr}(A) \cong \mathbb{F}[y_0, \dots, y_{f-1}, (y_0 \cdots y_{f-1})^{-1}]$. *As a consequence the ring* A *is a regular domain, i.e. a noetherian domain which has a finite global dimension* ([Ser00, §IV.D]).

Proof. The first sentence is a direct consequence of Lemma 3.1.1.1 applied with $M = \mathbb{F}[\![N_0]\!]$. This implies that the ring $\mathrm{gr}(A)$ is a noetherian domain. Then the noetherianity of A follows from [LvO96, Thm.I.5.7] applied to the ideals of A, and the fact that A is a domain follows easily from $\mathrm{gr}(x)\,\mathrm{gr}(y) = \mathrm{gr}(xy)$ if $x, y \in A\backslash\{0\}$ (using $\mathrm{gr}(x)\,\mathrm{gr}(y) \neq 0$). As $\mathrm{gr}(A)$ is a regular commutative ring, it follows from [LvO96, Thm.III.2.2.5] that A is an Auslander regular ring (note that A is Zariskian by [LvO96, Prop.II.2.2.1]) and a fortiori has finite global dimension ([LvO96, Def.III.2.1.7]). □

Remark 3.1.1.3. (i) The ring A can also be defined as the microlocalization of $\mathbb{F}[\![N_0]\!]$ along the set $\{(y_0 \cdots y_{f-1})^n, n \geq 0\} \subseteq \mathrm{gr}(\mathbb{F}[\![N_0]\!])$ (see [LvO96, Cor.IV.1.20]). This shows that the ring A does not depend on our choice of elements Y_i but rather on the elements y_i.
(ii) If M is a filtered $\mathbb{F}[\![N_0]\!]$-module, the filtration on M_S is given explicitly by the following formula:

$$F_n(M_S) = \sum_{k\geq 0} (Y_0 \cdots Y_{f-1})^{-k} F_{n-kf}(M), \quad n \in \mathbb{Z}.$$

As $(Y_0 \cdots Y_{f-1})^m F_n(M) \subseteq F_{n-mf}(M)$ for all $n \in \mathbb{Z}$ and $m \in \mathbb{N}$, we have

$$(Y_0 \cdots Y_{f-1})^{-k} F_{n-kf}(M) \subseteq (Y_0 \cdots Y_{f-1})^{-k-1} F_{n-(k+1)f}(M)$$

so that $F_n(M_S)$ can also be described as the increasing union

$$F_n(M_S) = \bigcup_{k\geq 0} (Y_0 \cdots Y_{f-1})^{-k} F_{n-kf}(M).$$

Note that the filtration on M_S is not necessarily separated even if the filtration on M is separated.
(iii) The ring A can also be defined as the set of series

$$A = \left\{ \sum_{d \gg -\infty} \frac{P_d}{(Y_0 \cdots Y_{f-1})^{n_d}}, P_d \in (Y_0, \dots, Y_{f-1})^{d+fn_d},\ n_d \geq 0,\ d + fn_d \geq 0 \right\},$$

equivalently, A is the set of infinite sums of monomials in the Y_i with $\mathbb{F}$-coefficients such that the total degree of the monomials tends to $+\infty$.

Let $n \geq 0$ be an integer and let $N_0^{p^n} \subseteq N_0$ be the subgroup of p^n-th powers (which is $p^n\mathcal{O}_K$ under the identification $N_0 \cong \mathcal{O}_K$). Let S^{p^n} be the set of p^n-th powers of S and let A^{p^n} be the completion of $\mathbb{F}[\![N_0^{p^n}]\!]_{S^{p^n}}$ for the filtration coming from the valuation $v_{N_0}|_{\mathbb{F}[\![N_0^{p^n}]\!]} = p^n v_{N_0^{p^n}}$. As the saturation of S^{p^n} (see [LvO96, §IV.1]) contains S, we have by [LvO96, Cor.IV.1.20]

$$\mathbb{F}[\![N_0]\!]_S = \mathbb{F}[\![N_0]\!]_{S^{p^n}} \cong \mathbb{F}[\![N_0^{p^n}]\!]_{S^{p^n}} \otimes_{\mathbb{F}[\![N_0^{p^n}]\!]} \mathbb{F}[\![N_0]\!]. \tag{104}$$

It is easy to check that $\mathbb{F}[\![N_0]\!]$ is a filtered free $\mathbb{F}[\![N_0^{p^n}]\!]$-module with respect to the basis $(Y_0^{i_0}\cdots Y_{f-1}^{i_{f-1}})_{\substack{0\leq i_j\leq p^n-1\\ 0\leq j\leq f-1}}$. Hence, by [LvO96, Lemma I.6.15] and (104), we conclude that $\mathbb{F}[\![N_0]\!]_S$ is a filtered free $\mathbb{F}[\![N_0^{p^n}]\!]_{S^{p^n}}$-module with respect to the same basis, and thus by [LvO96, Lemma I.6.13(3)] that A is a filtered free A^{p^n}-module with respect to the same basis again. Moreover, by [LvO96, Lemma I.6.14], we have an isomorphism of graded modules

$$\operatorname{gr}(A) \cong \operatorname{gr}(A^{p^n}) \otimes_{\operatorname{gr}(\mathbb{F}[\![N_0^{p^n}]\!])} \operatorname{gr}(\mathbb{F}[\![N_0]\!]). \tag{105}$$

Note that the p^n-power Frobenius map $x\mapsto x^{p^n}$ induces an isomorphism of filtered rings $(\mathbb{F}[\![N_0]\!]_S, v_{N_0}) \xrightarrow{\sim} (\mathbb{F}[\![N_0^{p^n}]\!]_{S^{p^n}}, v_{N_0^{p^n}})$ and thus, as $v_{N_0}|_{\mathbb{F}[\![N_0^{p^n}]\!]} = p^n v_{N_0^{p^n}}$, an isomorphism of topological rings $(\mathbb{F}[\![N_0]\!]_S, v_{N_0}) \xrightarrow{\sim} (\mathbb{F}[\![N_0^{p^n}]\!]_{S^{p^n}}, v_{N_0}|_{\mathbb{F}[\![N_0^{p^n}]\!]})$. It induces an isomorphism of complete topological rings $A \xrightarrow{\sim} A^{p^n}$ such that the composite map $A \xrightarrow{\sim} A^{p^n} \hookrightarrow A$ is the p^n-power Frobenius. This implies that the image of A^{p^n} in A is the subring of p^n-th powers of A.

The group $\mathcal{O}_K^\times$ acts on the group N_0 via $a\cdot \left(\begin{smallmatrix}1 & b\\ 0 & 1\end{smallmatrix}\right) = \left(\begin{smallmatrix}1 & ab\\ 0 & 1\end{smallmatrix}\right)$ and thus on $\mathbb{F}[\![N_0]\!]$, preserving the valuation v_{N_0}, and hence the filtration. This induces an action of $\mathcal{O}_K^\times$ on the graded ring $\operatorname{gr}(\mathbb{F}[\![N_0]\!])$, where it is immediately checked that $1+p\mathcal{O}_K$ acts trivially. Moreover if $a\in\mathbb{F}_q^\times$ and $0\leq i\leq f-1$, we have $\tilde{a}\cdot y_i = \sigma_i(a)y_i$.

Lemma 3.1.1.4. *There is a unique continuous action of $\mathcal{O}_K^\times$ on the ring A extending the action of $\mathcal{O}_K^\times$ on $\mathbb{F}[\![N_0]\!]$.*

Proof. As $\mathcal{O}_K^\times$ acts by ring endomorphisms on $\mathbb{F}[\![N_0]\!]$ and as $\mathbb{F}[\![N_0]\!]_S$ is dense in A, the uniqueness is clear.

For the existence, let $a\in\mathcal{O}_K^\times$ and consider the composition $\mathbb{F}[\![N_0]\!] \xrightarrow{a} \mathbb{F}[\![N_0]\!] \subseteq A$ which extends to a ring homomorphism $\mathbb{F}[\![N_0]\!]_S \to A$ since the elements of $a(S)$ are invertible in A (because they are invertible in $\operatorname{gr}(A)$ as $\operatorname{gr}(a(S)) = \operatorname{gr}(S)$). The precomposition of the valuation v_{N_0} on A with this map is a valuation on $\mathbb{F}[\![N_0]\!]_S$ which coincides with v_{N_0} on $\mathbb{F}[\![N_0]\!]$ since the multiplication by a preserves the valuation on $\mathbb{F}[\![N_0]\!]$. Therefore the map $\mathbb{F}[\![N_0]\!]_S \to A$ is isometric and extends to a filtered ring homomorphism $A\to A$ ([LvO96, Thm.I.3.4.5]). This defines an action of $\mathcal{O}_K^\times$ on A. □

We recall that ξ is the cocharacter $x\mapsto \left(\begin{smallmatrix}x & 0\\ 0 & 1\end{smallmatrix}\right)$ of GL_2. The conjugation by the matrix $\xi(p)$ in $\mathrm{GL}_2(K)$ induces a group endomorphism of N_0 and a continuous endomorphism ϕ of $\mathbb{F}[\![N_0]\!]$. We have $\phi(Y_i) = Y_{i-1}^p$ for $1\leq i\leq f-1$ and $\phi(Y_0) = Y_{f-1}^p$.

This implies that ϕ is the composite of the (relative) Frobenius endomorphism with a permutation of the variables Y_i. It follows that ϕ extends to a continuous injective endomorphism of the ring A with image A^p. More generally, for $n \geq 0$, the subring A^{p^n} is the image of ϕ^n.

Proposition 3.1.1.5. *Let $H \subseteq \mathcal{O}_K^\times$ be an open subgroup and let $\mathfrak{a} \subseteq A$ be an ideal of A which is H-stable. Then $\mathfrak{a}$ is controlled by A^p, which means*

$$\mathfrak{a} = A(\mathfrak{a} \cap A^p).$$

Proof. As H is open in $\mathcal{O}_K^\times$ it contains a subgroup of the form $1 + p^m\mathcal{O}_K$ for $m \geq 1$ so that we can assume that $H = 1 + p^m\mathcal{O}_K$.

The proof follows closely the strategy of [AW09].

We note that the pair (A, A^p) is a *Frobenius pair* in the sense of [AW09, Def.2.1] (to see this use [AWZ08, Prop.6.6] applied to $G = N_0$ together with [AW09, Lemma 2.2.(a)] and Remark 3.1.1.3(i)). We endow A^p with the filtration $F_nA^p \stackrel{\text{def}}{=} A^p \cap F_nA$ induced by the filtration of A.

Let $F \stackrel{\text{def}}{=} \mathfrak{a}/A(\mathfrak{a} \cap A^p)$. Endow $A(\mathfrak{a} \cap A^p)$ and $\mathfrak{a}$ with the filtration induced by A, and F with the quotient filtration. Then by [LvO96, Rk.I.5.2(2)] and [LvO96, Cor.I.5.5(1)] all these filtrations are good in the sense of [LvO96, Def.I.5.1]. Moreover $\mathfrak{a}$ and $A(\mathfrak{a} \cap A^p)$ are complete filtered A-modules by [LvO96, Cor.I.6.3(2))] and thus so is F by [LvO96, Prop.I.3.15].

We want to prove that $F = 0$. Assume for a contradiction that $F \neq 0$, or equivalently $\mathrm{gr}(F) \neq 0$ by [LvO96, Prop.I.4.2(1)].

Let $\Gamma \stackrel{\text{def}}{=} H = 1 + p^m\mathcal{O}_K$ (this not the Γ of the (φ, Γ)-modules!). This is a uniform pro-p-group. Note that the action of Γ on N_0 is *uniform* in the sense of [AW09, §4.1]. In the notation of [AW09, §4.2], we have $L_{N_0} = \mathcal{O}_K$, $\mathfrak{g} = \mathbb{F}_q$ and the action of $\mathbb{F}_q$ on L_{N_0}/pL_{N_0} is given by the multiplication in $\mathbb{F}_q$.

Let P be a (homogeneous) prime ideal in the support of the $\mathrm{gr}(A)$-module $\mathrm{gr}(F)$ (which exists since $\mathrm{gr}(F) \neq 0$).

Let $x \in \mathbb{F}_q^\times$ and $\gamma_x \stackrel{\text{def}}{=} \exp(p^m[x]) \in \mathrm{Aut}(N_0) \hookrightarrow \mathrm{End}(A)$. It follows from [AW09, Prop.4.4] and [AW09, Prop.3.2(a)] that the family

$$\mathrm{a}(x) \stackrel{\text{def}}{=} (\gamma_x, \gamma_x^p, \gamma_x^{p^2}, \dots)$$

is a source of derivations of (A, A^p) in the sense of [AW09, Def.3.2]. Let $T_P \subseteq \mathrm{gr}(A)$ be the set of homogeneous elements of $\mathrm{gr}(A)$ which are not in P and let $T_P^{(p)} \stackrel{\text{def}}{=} T_P \cap \mathrm{gr}(A^p)$. It follows again from [AW09, Prop.3.2(a)] that $\mathrm{a}(x)$ induces on $(Q_{T_P}(A), Q_{T_P^{(p)}}(A^p))$ a source of derivations $\mathrm{a}_{T_P}(x)$, where $Q_{T_P}(A)$ (resp. $Q_{T_P^{(p)}}(A^p)$)

is the microlocalization of A (resp. A^p) with respect to T_P (resp. $T_P^{(p)}$). Let $\mathcal{S} \stackrel{\text{def}}{=} \{\mathrm{a}(x), x \in \mathbb{F}_q^\times\}$ and $\mathcal{S}_P \stackrel{\text{def}}{=} \{\mathrm{a}_{T_P}(x), x \in \mathbb{F}_q^\times\}$.

As $\mathfrak{a}$ is Γ-invariant, $\mathfrak{a}$ is also $\mathcal{S}$-invariant, i.e. for all $x \in \mathbb{F}_q^\times$ and $r \geq 0$, we have $\gamma_x^{p^r}\mathfrak{a} \subseteq \mathfrak{a}$. Then $\mathfrak{a}_P \stackrel{\text{def}}{=} Q_{T_P}(\mathfrak{a}) \cong Q_{T_P}(A) \otimes_A \mathfrak{a}$ ([LvO96, Cor.IV.1.18(2)], though here everything is simpler as all rings are commutative) is an ideal of $Q_{T_P}(A)$ which is $\mathcal{S}_P$-invariant.

Let $P_0 \stackrel{\text{def}}{=} P \cap \mathrm{gr}(\mathbb{F}[\![N_0]\!])$ (inside $\mathrm{gr}(A)$). We prove that P_0 contains L_{N_0}/pL_{N_0}, where the latter is seen in $\mathrm{gr}_{-1}(\mathbb{F}[\![N_0]\!])$ (recall $L_{N_0} \cong N_0$). Assume this is not true. Let $J \stackrel{\text{def}}{=} \mathrm{gr}(\mathfrak{a}_P) \cong \mathrm{gr}(\mathfrak{a})_P$ ([AWZ08, Lemma 4.4]), which is a graded ideal of the localization $\mathrm{gr}(A)_P$ of $\mathrm{gr}(A)$ with respect to the set of homogeneous elements which are not in P, and let $Y \in \mathrm{gr}(A)_P$ such that $Y \in J^{\mathcal{S}_P}$ (see [AW09, Def.3.4] for the definition of $J^{\mathcal{S}_P}$). Noticing that $\mathrm{gr}(A)_P = \mathrm{gr}(\mathbb{F}[\![N_0]\!])_{P_0}$ and that L_{N_0}/pL_{N_0} is a 1-dimensional $\mathbb{F}_q$-vector space, we can apply [AW09, Prop.4.3] (together with [AW09, Prop.4.4(c)]) to the graded prime ideal P_0 of $B = \mathrm{gr}(\mathbb{F}[\![N_0]\!])$ and the graded ideal J of $\mathrm{gr}(\mathbb{F}[\![N_0]\!])_{P_0}$. We deduce $\mathcal{D}_P(Y) \subseteq J$ (see [AW09, §4.3] for the definition of $\mathcal{D}_P$). It follows from [AW09, Thm.3.5] applied to the Frobenius pair $(Q_{T_P}(A), Q_{T_P^{(p)}}(A^p))$ and the ideal $\mathfrak{a}_P$ that $\mathfrak{a}_P$ is controlled by $Q_{T_P^{(p)}}(A^p)$. Then [AW09, Lemma 2.3] shows that $\mathrm{gr}(F)_P = 0$. This is a contradiction.

As L_{N_0}/pL_{N_0} generates the $\mathbb{F}$-vector space $\mathrm{gr}_{-1}(\mathbb{F}[\![N_0]\!]) = \oplus_{i=0}^{f-1}\mathbb{F}y_i$, it follows that $y_i \in P$ for all $0 \leq i \leq f-1$ and then that $\mathrm{gr}(A) = P$. This is a contradiction so that $F = 0$ i.e. $\mathfrak{a} = A(\mathfrak{a} \cap A^p)$. □

Lemma 3.1.1.6. *Let $\mathfrak{a} \subsetneq A$ be a proper ideal of A. Then $\cap_{n\geq 0}(A(\mathfrak{a} \cap A^{p^n})) = 0$. In particular, if $\phi(\mathfrak{a}) \subseteq \mathfrak{a}$ we have $\cap_{n\geq 0}A\phi^n(\mathfrak{a}) = 0$.*

Proof. Let $\mathfrak{a}_n \stackrel{\text{def}}{=} A(\mathfrak{a} \cap A^{p^n})$. We endow $\mathfrak{a} \cap A^{p^n}$ with the induced filtration of A^{p^n} (or equivalently A). As A is a finite free A^{p^n}-module, we have $\mathfrak{a}_n \cong A \otimes_{A^{p^n}} (\mathfrak{a} \cap A^{p^n})$. We endow this A-module with the tensor product filtration. Since A is a filtered free A^{p^n}-module, it follows from [LvO96, Lemma I.6.14] that $\mathrm{gr}(\mathfrak{a}_n) \cong \mathrm{gr}(A) \otimes_{\mathrm{gr}(A^{p^n})} \mathrm{gr}(\mathfrak{a} \cap A^{p^n})$. Since $\mathrm{gr}(A)$ is a finite free $\mathrm{gr}(A^{p^n})$-module, the natural map $\mathrm{gr}(\mathfrak{a}_n) \to \mathrm{gr}(A)$ is injective (and the filtration on $\mathfrak{a}_n$ is in fact the one induced from A). Moreover from (105) we deduce

$$\mathrm{gr}(\mathfrak{a}_n) \cong \mathrm{gr}(\mathbb{F}[\![N_0]\!]) \otimes_{\mathrm{gr}(\mathbb{F}[\![N_0^{p^n}]\!])} \mathrm{gr}(\mathfrak{a} \cap A^{p^n}). \tag{106}$$

Assume that $\mathfrak{a} \neq A$. Then as both $\mathfrak{a}$ and A are complete and the injection $\mathfrak{a} \hookrightarrow A$ is strict, it follows as for the A-module F in the proof of Proposition 3.1.1.5 that $\mathrm{gr}(A/\mathfrak{a}) \neq 0$ (with the quotient filtration on $A/\mathfrak{a}$), hence by [LvO96, Thm.I.4.4(1)] that $\mathrm{gr}(\mathfrak{a}) \neq \mathrm{gr}(A)$, and *a fortiori* $\mathrm{gr}(\mathfrak{a}_n) \neq \mathrm{gr}(A)$.

Using (106) and the fact $\mathrm{gr}(\mathbb{F}[\![N_0]\!]) \cong \mathbb{F}[y_0, \ldots, y_{f-1}]$ is free of finite rank over

$\mathrm{gr}(\mathbb{F}[\![N_0^{p^n}]\!]) \cong \mathbb{F}[y_0^{p^n}, \ldots, y_{f-1}^{p^n}]$, we have inside $\mathrm{gr}(A)$ that

$$\mathrm{gr}(\mathfrak{a}_n) \cap \mathrm{gr}(\mathbb{F}[\![N_0]\!]) \cong \mathrm{gr}(\mathbb{F}[\![N_0]\!]) \otimes_{\mathrm{gr}(\mathbb{F}[\![N_0^{p^n}]\!])} (\mathrm{gr}(\mathfrak{a} \cap A^{p^n}) \cap \mathrm{gr}(\mathbb{F}[\![N_0^{p^n}]\!])). \tag{107}$$

The ideal $\mathrm{gr}(\mathfrak{a}_n) \cap \mathrm{gr}(\mathbb{F}[\![N_0]\!])$ is therefore generated by homogeneous elements of $\mathrm{gr}(\mathbb{F}[\![N_0]\!])$ which are of degree $\leq -p^n$ since homogeneous elements of $\mathbb{F}[y_0^{p^n}, \ldots, y_{f-1}^{p^n}]$ of degree zero are invertible and $\mathrm{gr}(\mathfrak{a}_n)$ does not contain invertible elements (as $\mathrm{gr}(\mathfrak{a}_n) \neq \mathrm{gr}(A)$). We conclude that

$$\mathrm{gr}(\mathfrak{a}_n) \cap \mathrm{gr}(\mathbb{F}[\![N_0]\!]) \subseteq F_{-p^n}(\mathrm{gr}(\mathbb{F}[\![N_0]\!])).$$

Consequently (recall $\bigcap_{n\geq 0} \mathfrak{a}_n$ has the induced filtration from A)

$$\mathrm{gr}\Big(\bigcap_{n\geq 0} \mathfrak{a}_n\Big) \cap \mathrm{gr}(\mathbb{F}[\![N_0]\!]) \subseteq \bigcap_{n\geq 0} (\mathrm{gr}(\mathfrak{a}_n) \cap \mathrm{gr}(\mathbb{F}[\![N_0]\!])) = 0. \tag{108}$$

As $\mathrm{gr}(\bigcap_{n\geq 0} \mathfrak{a}_n)$ is an ideal in $\mathrm{gr}(A) \cong \mathbb{F}[y_0, \ldots, y_{f-1}, (y_0 \cdots y_{f-1})^{-1}]$, it follows from (108) that we must have $\mathrm{gr}(\bigcap_{n\geq 0} \mathfrak{a}_n) = 0$, and hence that $\bigcap_{n\geq 0} \mathfrak{a}_n = 0$ by [LvO96, Prop.I.4.2(1)]. □

Corollary 3.1.1.7. *Let $H \subseteq \mathcal{O}_K^\times$ be an open subgroup. The only ideals of A which are H-stable are* 0 *and A.*

Proof. Let $\mathfrak{a}$ be such an ideal and assume that $\mathfrak{a} \neq A$. It follows from Proposition 3.1.1.5 applied recursively with A, A^p, etc. that $\mathfrak{a} = A(\mathfrak{a} \cap A^{p^n})$ for all $n \geq 0$. Then Lemma 3.1.1.6 implies $\mathfrak{a} = 0$. □

If H is an open subgroup of $\mathcal{O}_K^\times$, an *H-module over A* is a finitely generated A-module with a semilinear action of H.

Proposition 3.1.1.8. *Let H be an open subgroup of $\mathcal{O}_K^\times$ and let M be an H-module over A. Then M is a finite projective A-module.*

Proof. (We thank Gabriel Dospinescu for suggesting the following proof which is shorter than our original one.) Let M be an H-module. For $k \geq -1$ let $\mathrm{Fit}_k(M)$ be the k-th Fitting ideal (see for example [Sta19, Def.07Z9]). As M is a finitely generated A-module, it follows from [Sta19, Lemma 07ZA] that there exists some $r \geq 0$ such that $\mathrm{Fit}_r(M) \neq 0$. Let $r \geq 0$ be the smallest integer such that $\mathrm{Fit}_r(M) \neq 0$. Let $\gamma \in H$. It follows easily from the definition of $\mathrm{Fit}_k(M)$ that $\mathrm{Fit}_k(M \otimes_{A,\gamma} A) = \gamma(\mathrm{Fit}_k(M))$ as ideals of A. The action of γ on M induces an A-linear isomorphism $M \otimes_{A,\gamma} A \xrightarrow{\sim} M$, showing that $\gamma(\mathrm{Fit}_k(M)) = \mathrm{Fit}_k(M)$. It follows then from Corollary 3.1.1.7 that all the ideals $\mathrm{Fit}_k(M)$ are zero or A. Therefore we have $\mathrm{Fit}_{r-1}(M) = 0$ and $\mathrm{Fit}_r(M) = A$ and we deduce from [Sta19, Lemma 07ZD] that M is projective of rank r. □

We record one more useful consequence of Corollary 3.1.1.7.

Corollary 3.1.1.9. *Let H be an open subgroup of $\mathcal{O}_K^\times$. We have $A^H = \mathbb{F}$, i.e. the H-invariants in A are given by $\mathbb{F}$.*

Proof. If $x \in A^H$, then xA is an H-stable ideal of A. It follows that $x = 0$ or $x \in A^\times$ by Corollary 3.1.1.7, i.e. A^H is a field. Therefore, the composition $A^H \hookrightarrow A \overset{\mathrm{tr}}{\twoheadrightarrow} \mathbb{F}(\!(T)\!)$ is injective. But tr is also $\mathbb{Z}_p^\times$-equivariant, so A^H injects into $\mathbb{F}(\!(T)\!)^{H\cap\mathbb{Z}_p^\times}$ and it suffices to show that $\mathbb{F}(\!(T)\!)^M = \mathbb{F}$ for any open subgroup $M \subseteq \mathbb{Z}_p^\times$. As the $\mathbb{Z}_p^\times$-action is $\mathbb{F}$-linear, there is no loss in assuming that $\mathbb{F} = \mathbb{F}_p$. To see that $\mathbb{F}_p(\!(T)\!)^M = \mathbb{F}_p$, recall that the $\mathbb{Z}_p^\times$-action on $\mathbb{F}_p(\!(T)\!)$ is given by interpreting $\mathbb{F}_p(\!(T)\!)$ as the field of norms of $\mathbb{Q}_p(\mu_{p^\infty})/\mathbb{Q}_p$ (with Galois group $\mathbb{Z}_p^\times$). Let $L_0 \overset{\mathrm{def}}{=} \mathbb{Q}_p(\mu_{p^\infty})^M$, which is a finite totally ramified extension of $\mathbb{Q}_p$. Thus every $x \in \mathbb{F}_p(\!(X)\!)^M$ is given by a norm-compatible system of elements $x_L \in L_0$, L running through finite subextensions of $\mathbb{Q}_p(\mu_{p^\infty})/L_0$. In particular, if x is nonzero, then x_L is p-divisible in $L_0^\times$, so $x_L \in [\mathbb{F}_p^\times]$. As x is then determined by $x_{L_0(\mu_p)}$, we deduce the claim. □

3.1.2 Multivariable $(\psi, \mathcal{O}_K^\times)$-modules

We define a functor from a certain abelian category of admissible smooth representations of $\mathrm{GL}_2(K)$ over $\mathbb{F}$ to a category of multivariable $(\psi, \mathcal{O}_K^\times)$-modules.

Let R be a noetherian commutative ring of characteristic p endowed with an injective ring endomorphism F_R such that R is a finite free $F_R(R)$-module. If M is an R-module, we define $F_R^*(M) \overset{\mathrm{def}}{=} R \otimes_{F_R,R} M$. Examples of such pairs (R, F_R) are given by $(\mathbb{F}[\![N_0]\!], \phi)$ and (A, ϕ) in §3.1.1.

A *ψ-module* over R is a pair (M, β), where M is an R-module and β is an R-linear homomorphism $M \to F_R^*(M)$. When R is a regular ring, F_R is the Frobenius endomorphism of R and β is an isomorphism, we recover the notion of F_R-module of [Lyu97, Def.1.1]. We say that a ψ-module (M, β) is *étale* if β is injective.

If (M, β) is a ψ-module, the exact functor F_R^* gives us, for each $n \geq 0$, an R-linear map $(F_R^*)^n(\beta) : (F_R^*)^n(M) \to (F_R^*)^{n+1}(M)$ and we can define

$$\beta_n \overset{\mathrm{def}}{=} (F_R^*)^{n-1}(\beta) \circ \cdots \circ (F_R^*)(\beta) \circ \beta : M \longrightarrow (F_R^*)^n(M).$$

The inductive limit of the system $((F_R^*)^n(M), (F_R^*)^n(\beta))_n$ gives rise to a ψ-module $(\mathcal{M}, \underline{\beta})$ with $\underline{\beta}$ an isomorphism. Then (M, β) generates $(\mathcal{M}, \underline{\beta})$ in the sense of [Lyu97, Def.1.9]. Let $M^{\text{ét}}$ be the image of M in $\mathcal{M}$ and M^0 the kernel of $M \to M^{\text{ét}}$. The map β induces a structure of ψ-module on M^0 and $M^{\text{ét}}$ and $M^{\text{ét}}$ is an étale ψ-module. The ψ-module $M^{\text{ét}}$ is called the *étale part* of M and M^0 the *nilpotent part* of M. We note that (M, β) and $(M^{\text{ét}}, \beta^{\text{ét}})$ generate the same F_R-module and (M^0, β^0) generates the trivial F_R-module whose underlying module is zero. Note that the constructions $(M,\beta) \mapsto (M^{\text{ét}}, \beta^{\text{ét}})$ and $(M,\beta) \mapsto (M^0, \beta^0)$ are functorial in (M,β) and that, if β is injective, we have $M^0 = 0$. This implies that if $f : (M,\beta) \to (M', \beta')$ is a morphism of ψ-modules with (M', β') étale, then f factors through $M^{\text{ét}}$.

We are mainly interested in ψ-modules with extra structures, which we call $(\psi, \mathcal{O}_K^\times)$-modules over A. If M is a finitely generated A-module, we always endow it with the topology defined by any good filtration (note that good filtrations generate the same topologies, cf. [LvO96, Lemma I.5.3]). It is also the quotient topology given by any surjection $A^{\oplus d} \twoheadrightarrow M$ (as follows from [LvO96, Rk.I.5.2(2)]), and we call it the canonical topology on M. The group $\mathcal{O}_K^\times$ acts continuously on A and this action commutes with the endomorphism ϕ. If M is an A-module which is endowed with an action of $\mathcal{O}_K^\times$, we consider the diagonal action on $\phi^*(M)$, which is well defined since ϕ commutes with $\mathcal{O}_K^\times$.

Definition 3.1.2.1. A $(\psi, \mathcal{O}_K^\times)$*-module over* A is a ψ-module (M,β) over A such that M is a finitely generated A-module with a continuous semilinear action of $\mathcal{O}_K^\times$ such that β is $\mathcal{O}_K^\times$-equivariant (here, continuity means that the map $\mathcal{O}_K^\times \times M \to M$ is continuous). We say that a $(\psi, \mathcal{O}_K^\times)$-module over A is *étale* if the underlying ψ-module over A is.

We remark that if (M,β) is a $(\psi, \mathcal{O}_K^\times)$-module, then M is an $\mathcal{O}_K^\times$-module and is therefore finite projective as an A-module by Proposition 3.1.1.8.

Proposition 3.1.2.2. *Let (M,β) be an étale $(\psi, \mathcal{O}_K^\times)$-module over A. Then β is an isomorphism.*

Proof. We note that the two A-modules M and $\phi^*(M) = A \otimes_{\phi,A} M$ have the same generic rank. As β is an injective A-linear map between two finitely generated modules of the same generic rank over a noetherian domain, its cokernel is torsion. This cokernel is then an $\mathcal{O}_K^\times$-module which is moreover torsion as an A-module, it follows from Proposition 3.1.1.8 that it is zero and β is an isomorphism. □

We now define a functor from certain representations of $\mathrm{GL}_2(K)$ over $\mathbb{F}$ to $(\psi, \mathcal{O}_K^\times)$-modules over A.

Let π be an admissible smooth representation of $\mathrm{GL}_2(K)$ over $\mathbb{F}$. Its ($\mathbb{F}$-linear) dual $\pi^\vee$ is then a finitely generated $\mathbb{F}[\![I_1]\!]$-module. We fix a good filtration on $\pi^\vee$. As above, we endow $A \otimes_{\mathbb{F}[\![N_0]\!]} \pi^\vee$ with the tensor product filtration and define the filtered A-module

$$D_A(\pi) \overset{\text{def}}{=} A \widehat{\otimes}_{\mathbb{F}[\![N_0]\!]} \pi^\vee \cong \widehat{(\pi^\vee)_S}. \tag{109}$$

Note that the action of $\mathbb{F}[\![N_0]\!]$ on $\pi^\vee$ is given by $\delta_a(f) \overset{\text{def}}{=} f \circ a^{-1}$ for $f \in \pi^\vee$, $a \in N_0$. As all the good filtrations on $\pi^\vee$ are equivalent ([LvO96, Lemma I.5.3]), the underlying topological A-module does not depend on the choice of the good filtration on $\pi^\vee$. An example of a good filtration on $\pi^\vee$ is given by the $\mathfrak{m}_{I_1}$-adic filtration, as follows directly from the definition. It is very important to note that the topology used on $\pi^\vee$ is *not* the $\mathfrak{m}_{N_0}$-adic topology but the $\mathfrak{m}_{I_1}$-adic topology, which is actually coarser.

Proposition 3.1.2.3. *The functor $\pi \longmapsto D_A(\pi)$ is exact.*

Proof. Let $0 \to \pi' \to \pi \to \pi'' \to 0$ be an exact sequence of admissible smooth representations of $\mathrm{GL}_2(K)$ over $\mathbb{F}$. The sequence $0 \to (\pi'')^\vee \to \pi^\vee \to (\pi')^\vee \to 0$ is still exact. We endowed $\pi^\vee$ with a good filtration, $(\pi')^\vee$ with the quotient filtration and $(\pi'')^\vee$ with the induced filtration (which are again good by e.g. [LvO96, Prop.II.1.2.3]). With these choices, the exact sequence remains exact after applying the functor gr (see for example [LvO96, Thm.I.4.2.4(1)]). It follows from Lemma 3.1.1.1, from the exactness of localization and from [LvO96, Thm.I.4.2.4(2)]) that the sequence $0 \to (\pi'')^\vee_S \to (\pi^\vee)_S \to (\pi')^\vee_S \to 0$ is exact and *strict*. The exactness of $0 \to D_A(\pi'') \to D_A(\pi) \to D_A(\pi') \to 0$ then follows from [LvO96, Thm.I.3.4.13]. □

We define a continuous action of $\mathcal{O}_K^\times$ on $\pi^\vee$ as follows, for $f \in \pi^\vee$, $\gamma \in \mathcal{O}_K^\times$ we have

$$(\gamma \cdot f)(x) \stackrel{\text{def}}{=} f\left(\begin{pmatrix}\gamma^{-1} & 0\\ 0 & 1\end{pmatrix} x\right) \quad \forall\, x \in \pi.$$

As $\mathcal{O}_K^\times$ normalizes I_1, the action of $\mathcal{O}_K^\times$ on $\pi^\vee$ is continuous for the $\mathfrak{m}_{I_1}$-adic topology. We use the continuous action of $\mathcal{O}_K^\times$ on A to extend this action diagonally to $A \otimes_{\mathbb{F}[\![N_0]\!]} \pi^\vee$ and, by continuity, to $D_A(\pi)$. The action of $\mathcal{O}_K^\times$ is continuous and A-semilinear in the sense that

$$\gamma \cdot (af) = (\gamma \cdot a)(\gamma \cdot f) \quad \forall\, (\gamma, a, f) \in \mathcal{O}_K^\times \times A \times D_A(\pi).$$

We define an $\mathbb{F}$-linear endomorphism ψ of $\pi^\vee$ by the formula

$$\psi(f)(x) = f(\xi(p)x) \quad \forall (f, x) \in \pi^\vee \times \pi. \tag{110}$$

This endomorphism is continuous, clearly commutes with the action of $\mathcal{O}_K^\times$ and satisfies the relation

$$\psi(\phi(a)f) = a(\psi(f))$$

for all $a \in \mathbb{F}[\![N_0]\!]$, $f \in \pi^\vee$.

Lemma 3.1.2.4. *Let M be some $\mathbb{F}[\![N_0]\!]$-module and let ψ be an $\mathbb{F}$-linear endomorphism of M satisfying the relation*

$$\psi(\phi(a)m) = a\psi(m) \quad \forall\, (a, m) \in \mathbb{F}[\![N_0]\!] \times M.$$

Then for all integers $n \geq 0$, we have

$$\psi(\mathfrak{m}_{N_0}^{pf-(f-1)+pn} M) \subseteq \mathfrak{m}_{N_0}^{n+1} M.$$

As a consequence, for $n \geq pf - (f-1)$, we have

$$\psi(\mathfrak{m}_{N_0}^n M) \subseteq \mathfrak{m}_{N_0}^{\lceil \frac{n}{p} \rceil - f} M.$$

Proof. For $n = 0$, the result follows from the fact that, if $Y_0^{i_0} \cdots Y_{f-1}^{i_{f-1}} \in \mathfrak{m}_{N_0}^{pf-(f-1)}$, there exists some $0 \leq j \leq f-1$ such that $i_j \geq p$. Then, for all $m \in M$, we have

$$\psi(Y_0^{i_0} \cdots Y_{f-1}^{i_{f-1}} m) = Y_{j+1} \psi(Y_0^{i_0} \cdots Y_j^{i_j - p} \cdots Y_{f-1}^{i_{f-1}} m) \in \mathfrak{m}_{N_0} M.$$

The general statement follows from a simple induction on n.

For the last statement, we choose m such that

$$pm + pf - (f-1) \leq n < p(m+1) + pf - (f-1)$$

and we use the first statement to deduce that

$$\psi(\mathfrak{m}_{N_0}^n M) \subseteq \psi(\mathfrak{m}_{N_0}^{pm+pf-(f-1)} M) \subseteq \mathfrak{m}_{N_0}^{m+1} M \subseteq \mathfrak{m}_{N_0}^{\lceil \frac{n}{p} \rceil - f} M. \qquad \square$$

We extend ψ to an $\mathbb{F}$-linear map $(\pi^\vee)_S \to (\pi^\vee)_S$ (recall $(\pi^\vee)_S = \mathbb{F}[\![N_0]\!]_S \otimes_{\mathbb{F}[\![N_0]\!]} \pi^\vee$) by the formula

$$\psi\left(\frac{m}{(Y_0 \cdots Y_{f-1})^{pn}}\right) = \frac{\psi(m)}{(Y_0 \cdots Y_{f-1})^n} \tag{111}$$

for all $m \in \pi^\vee$ and $n \geq 0$. Each element of $(\pi^\vee)_S$ can be written as $(Y_0 \cdots Y_{f-1})^{-pn} m$ for some $m \in \pi^\vee$ and $n \geq 0$, and it follows from the properties of ψ on $\pi^\vee$ that the right-hand side of (111) does not depend on this choice. For any element g in I_1, we denote by δ_g the corresponding element $[g]$ in $\mathbb{F}[\![I_1]\!]$.

Lemma 3.1.2.5. *The map $\psi : (\pi^\vee)_S \to (\pi^\vee)_S$ is continuous.*

Proof. As all the good filtrations on $\pi^\vee$ are equivalent, we choose the $\mathfrak{m}_{I_1}$-adic filtration on $\pi^\vee$ for this proof, i.e. $F_n \pi^\vee = \mathfrak{m}_{I_1}^{-n} \pi^\vee$ for $n \leq 0$ and $F_n \pi^\vee = \pi^\vee$ for $n > 0$. From the proof of [BHH+23, Prop.5.3.3] we have an equality for $n \geq 0$:

$$\mathfrak{m}_{I_1}^n = \sum_{\substack{r,s,t \geq 0 \\ r+2s+t=n}} \mathfrak{m}_{N_0}^r \mathfrak{m}_{T_0}^s \mathfrak{m}_{N_0^-}^t. \tag{112}$$

As $\xi(p)$ commutes with each element in T_0, and $\xi(p)^{-1} \left(\begin{smallmatrix} 1 & 0 \\ z & 1 \end{smallmatrix}\right) \xi(p) = \left(\begin{smallmatrix} 1 & 0 \\ z & 1 \end{smallmatrix}\right)^p$ for any $\left(\begin{smallmatrix} 1 & 0 \\ z & 1 \end{smallmatrix}\right) \in N_0^-$, it is easily checked from the definition of ψ and the $\mathbb{F}[\![I_1]\!]$-action on $\pi^\vee$ that

$$\psi(\delta_h \delta_z \cdot f) = \delta_h \delta_{z^p} \psi(f) \tag{113}$$

for all $h \in T_0$, $z \in N_0^-$. In particular,

$$\psi(\mathfrak{m}_{T_0}^s \mathfrak{m}_{N_0^-}^t \pi^\vee) \subseteq \mathfrak{m}_{T_0}^s \mathfrak{m}_{N_0^-}^{pt} \pi^\vee,$$

and it follows from Lemma 3.1.2.4 that if $r \geq pf - (f-1)$ we have

$$\psi(\mathfrak{m}_{N_0}^r \mathfrak{m}_{T_0}^s \mathfrak{m}_{N_0^-}^t \pi^\vee) \subseteq \mathfrak{m}_{N_0}^{\lceil \frac{r}{p} \rceil - f} \mathfrak{m}_{T_0}^s \mathfrak{m}_{N_0^-}^{pt} \pi^\vee \subseteq \mathfrak{m}_{I_1}^{\lceil \frac{r}{p} \rceil + 2s + pt - f} \pi^\vee \subseteq \mathfrak{m}_{I_1}^{\lceil \frac{r+2s+t}{p} \rceil - f} \pi^\vee. \quad (114)$$

If $r < pf - (f-1)$, we need the following lemma.

Lemma 3.1.2.6. *Let $M \subseteq \pi^\vee$ be a closed $\mathbb{F}[\![N_0^-]\!]$-submodule. Then*

$$\psi(\mathbb{F}[\![N_0]\!] \mathfrak{m}_{N_0^-} M) \subseteq \mathfrak{m}_{I_1} \psi(\mathbb{F}[\![N_0]\!] M).$$

As a consequence, for all $t \geq 0$, $\psi(\mathbb{F}[\![N_0]\!] \mathfrak{m}_{N_0^-}^t \pi^\vee) \subseteq \mathfrak{m}_{I_1}^t \pi^\vee$.

Proof. Note that $\mathfrak{m}_{I_1} \times \mathbb{F}[\![N_0]\!] \times M$ is compact, as M is closed, hence so is the image $\mathfrak{m}_{I_1} \psi(\mathbb{F}[\![N_0]\!] M)$ of the continuous map $\mathfrak{m}_{I_1} \times \mathbb{F}[\![N_0]\!] \times M \to \pi^\vee$, $(a, b, m) \mapsto a\psi(bm)$. As $\mathfrak{m}_{N_0^-}$ is generated as a right $\mathbb{F}[\![N_0^-]\!]$-module by the $\delta_y - 1$ for $y \in N_0^-$ and as ψ is continuous on $\pi^\vee$, it is thus sufficient to prove that, for $y \in N_0^-$, $x \in N_0$ and $m \in M$, we have $\psi(\delta_x(\delta_y - 1)m) \in \mathfrak{m}_{I_1} \psi(\mathbb{F}[\![N_0]\!] M)$. As $N_0^- \subseteq K_1$, K_1 is normalized by N_0 and $K_1 = N_0^p T_0 N_0^-$, we can write $xy = x_1^p t_1 y_1 x$ with $(x_1, t_1, y_1) \in N_0 \times T_0 \times N_0^-$. Therefore

$$\begin{aligned}
\psi(\delta_x(\delta_y - 1)m) &= \psi(\delta_{x_1^p} \delta_{t_1} \delta_{y_1} \delta_x m) - \psi(\delta_x m) \\
&= \delta_{x_1 t_1 y_1^p} \psi(\delta_x m) - \psi(\delta_x m) = (\delta_{x_1 t_1 y_1^p} - 1)\psi(\delta_x m) \\
&\subseteq \mathfrak{m}_{I_1} \psi(\mathbb{F}[\![N_0]\!] M).
\end{aligned}$$

For the second statement, inductively apply the first to $M = \mathfrak{m}_{N_0^-}^{t-1} \pi^\vee$, $M = \mathfrak{m}_{N_0^-}^{t-2} \pi^\vee$, etc. □

When $r < pf - (f-1) = (p-1)f + 1$, we have $2s + t \geq r + 2s + t - (p-1)f$ so that, using Lemma 3.1.2.6 and the fact that T_0 normalizes N_0, we obtain

$$\psi(\mathfrak{m}_{N_0}^r \mathfrak{m}_{T_0}^s \mathfrak{m}_{N_0^-}^t \pi^\vee) \subseteq \mathfrak{m}_{T_0}^s \psi(\mathbb{F}[\![N_0]\!] \mathfrak{m}_{N_0^-}^t \pi^\vee) \subseteq \mathfrak{m}_{I_1}^{2s+t} \pi^\vee \subseteq \mathfrak{m}_{I_1}^{r+2s+t-(p-1)f} \pi^\vee. \quad (115)$$

We deduce from (111), (114) and (115) that, for all $n \in \mathbb{Z}$, $r \geq 0$, $s \geq 0$, $t \geq 0$ and $k \geq 0$ such that $r + 2s + t \geq pf$, we have

$$\psi\left(\frac{1}{(Y_0 \cdots Y_{f-1})^{pk}} \mathfrak{m}_{N_0}^r \mathfrak{m}_{T_0}^s \mathfrak{m}_{N_0^-}^t \pi^\vee\right) \subseteq \frac{1}{(Y_0 \cdots Y_{f-1})^k} \mathfrak{m}_{I_1}^{\lceil \frac{r+2s+t}{p} \rceil - f} \pi^\vee$$

so that, for $n \geq pf$ by (112) we have

$$\psi\left(\frac{1}{(Y_0 \cdots Y_{f-1})^{pk}} \mathfrak{m}_{I_1}^n \pi^\vee\right) \subseteq \frac{1}{(Y_0 \cdots Y_{f-1})^k} \mathfrak{m}_{I_1}^{\lceil \frac{n}{p} \rceil - f} \pi^\vee \subseteq F_{kf+f-\lceil \frac{n}{p} \rceil}((\pi^\vee)_S).$$

From Remark 3.1.1.3(ii), we know that, for $n \in \mathbb{Z}$, $F_n((\pi^\vee)_S)$ is the increasing union over $k \geq \max\{0, \frac{n}{pf}\}$ of the subspaces

$$\frac{1}{(Y_0 \cdots Y_{f-1})^{pk}} \mathfrak{m}_{I_1}^{-n+pkf} \pi^\vee,$$

hence we deduce for all $n \in \mathbb{Z}$ that

$$\psi(F_n((\pi^\vee)_S)) \subseteq \bigcup_{k \geq \max\{0, \frac{n}{pf}\}} F_{kf+f-\lceil \frac{-n+pkf}{p} \rceil}((\pi^\vee)_S) \subseteq F_{f+\lfloor \frac{n}{p} \rfloor}((\pi^\vee)_S).$$

This proves the continuity of ψ. □

We can therefore extend ψ to a continuous $\mathbb{F}$-linear map $\psi : D_A(\pi) \to D_A(\pi)$ such that

$$\psi(\phi(a)m) = a\psi(m) \quad \forall\ (a, m) \in A \times \pi^\vee.$$

We fix $\{a_0, \ldots, a_{q-1}\}$ a system of representatives of the cosets of $N_0^p \cong p\mathcal{O}_K$ in $N_0 \cong \mathcal{O}_K$, so that $\mathbb{F}[\![N_0]\!] = \bigoplus_{i=0}^{q-1} \delta_{a_i} \mathbb{F}[\![N_0^p]\!]$. As $\phi(\mathbb{F}[\![N_0]\!]) = \mathbb{F}[\![N_0^p]\!]$ and $A = \bigoplus_{i=0}^{q-1} \delta_{a_i} \phi(A)$, we have a canonical isomorphism for any A-module M:

$$\phi^*(M) \cong \bigoplus_{i=0}^{q-1} (\mathbb{F}\delta_{a_i} \otimes_{\mathbb{F}} M).$$

We define an $\mathbb{F}$-linear map $\beta : D_A(\pi) \to \phi^*(D_A(\pi)) = A \otimes_{\phi, A} D_A(\pi)$ by

$$\begin{array}{ccc} D_A(\pi) & \longrightarrow & \bigoplus_{i=0}^{q-1} (\mathbb{F}\delta_{a_i} \otimes_{\mathbb{F}} D_A(\pi)) \\ m & \longmapsto & \sum_{i=0}^{q-1} \delta_{a_i} \otimes_\phi \psi(\delta_{a_i}^{-1} m) \end{array} \tag{116}$$

(we write $x \otimes_\phi y$ instead of just $x \otimes y$ in order not to forget the map ϕ in the tensor product).

Remark 3.1.2.7. The definition of the map β does not depend on the choice of the system $\{a_i\}$, namely, replacing a_i with $a_i b^p$ for some $b \in N_0$, we have

$$\begin{aligned} \delta_{a_i b^p} \otimes_\phi \psi(\delta_{a_i b^p}^{-1} m) = \delta_{a_i b^p} \otimes_\phi \psi(\phi(\delta_b)^{-1} \delta_{a_i}^{-1} m) = \delta_{a_i b^p} \otimes_\phi \delta_b^{-1} \psi(\delta_{a_i}^{-1} m) \\ = \delta_{a_i b^p} \delta_{b^p}^{-1} \otimes_\phi \psi(\delta_{a_i}^{-1} m) = \delta_{a_i} \otimes_\phi \psi(\delta_{a_i}^{-1} m). \end{aligned}$$

Using Remark 3.1.2.7, we easily check that β is actually an A-linear map (note that it is enough to check it for an element in $\delta_{a_i}\phi(A)$ using $A = \bigoplus_{i=0}^{q-1} \delta_{a_i}\phi(A)$, and thus for δ_{a_i} and for an element in $\phi(A)$), hence $\beta : D_A(\pi) \to \phi^*(D_A(\pi))$ can be seen as a "linearization" of $\psi : D_A(\pi) \to D_A(\pi)$. Moreover, letting $\mathcal{O}_K^\times$ act diagonally on

$A\otimes_{\phi,A}D_A(\pi)$, the map β is then $\mathcal{O}_K^\times$-equivariant. Indeed, for $a\in\mathcal{O}_K^\times$ and $m\in D_A(\pi)$, we have

$$\begin{aligned} a\cdot\beta(m) &= a\cdot\left(\sum_{i=0}^{q-1}\delta_{a_i}\otimes_\phi\psi(\delta_{a_i}^{-1}m)\right)=\sum_{i=0}^{q-1}\delta_{a\cdot a_i}\otimes_\phi a\cdot\psi(\delta_{a_i}^{-1}m)\\ &=\sum_{i=0}^{q-1}\delta_{a\cdot a_i}\otimes_\phi\psi(a\cdot\delta_{a_i}^{-1}m)=\sum_{i=0}^{q-1}\delta_{a\cdot a_i}\otimes_\phi\psi(\delta_{a\cdot a_i}^{-1}(a\cdot m))\\ &=\beta(a\cdot m),\end{aligned}$$

the last equality coming from Remark 3.1.2.7 and the fact that $\{a\cdot a_0,\dots,a\cdot a_{q-1}\}$ is another system of representatives of N_0^p in N_0.

It is convenient to assume that the admissible smooth representation π has a central character, in which case Z_1 acts trivially on π and $\pi^\vee$ is a finitely generated $\mathbb{F}[\![I_1/Z_1]\!]$-module. We recall from [BHH+23, §5.3] that the graded ring $\operatorname{gr}(\mathbb{F}[\![I_1/Z_1]\!])$ of $\mathbb{F}[\![I_1/Z_1]\!]$ is isomorphic to a tensor product of (noncommutative) graded rings

$$\bigotimes_{i=0}^{f-1}\mathbb{F}[y_i,z_i,h_i], \tag{117}$$

where variables with different indices commute, where $[y_i,z_i]=h_i$, $[h_i,y_i]=[h_i,z_i]=0$, where y_i,z_i are homogeneous of degree -1, and h_i is homogeneous of degree -2. Note that the $\mathfrak{m}_{I_1/Z_1}$-adic topology on $\mathbb{F}[\![I_1/Z_1]\!]$ induces the $\mathfrak{m}_{N_0}$-adic topology on $\mathbb{F}[\![N_0]\!]$ via the inclusion $\mathbb{F}[\![N_0]\!]\subseteq\mathbb{F}[\![I_1/Z_1]\!]$. Therefore the map $\operatorname{gr}(\mathbb{F}[\![N_0]\!])\to\operatorname{gr}(\mathbb{F}[\![I_1/Z_1]\!])$ is injective and its image is $\mathbb{F}[y_0,\dots,y_{f-1}]$ in $\operatorname{gr}(\mathbb{F}[\![I_1/Z_1]\!])$.

Remark 3.1.2.8. The A-module $D_A(\pi)$ can also be defined as the microlocalization of $\pi^\vee$ with respect to the multiplicative subset $T\overset{\text{def}}{=}\{(y_0\cdots y_{f-1})^k,k\in\mathbb{N}\}\subseteq\operatorname{gr}(\mathbb{F}[\![I_1/Z_1]\!])$. This shows that $D_A(\pi)$ can be promoted to a module over the noncommutative ring which is the microlocalization of $\mathbb{F}[\![I_1/Z_1]\!]$ with respect to T.

We now let $\mathcal{C}$ be the category of admissible smooth representations π of $\mathrm{GL}_2(K)$ over $\mathbb{F}$ with a central character *and* such that there exists a good filtration on the $\mathbb{F}[\![I_1/Z_1]\!]$-module $\pi^\vee$ such that $\operatorname{gr}(D_A(\pi))$ is a finitely generated $\operatorname{gr}(A)$-module, or equivalently by Lemma 3.1.1.1 and Corollary 3.1.1.2 $\operatorname{gr}(\pi^\vee)[(y_0\cdots y_{f-1})^{-1}]$ is finitely generated over $\operatorname{gr}(\mathbb{F}[\![N_0]\!])[(y_0\cdots y_{f-1})^{-1}]$. By [LvO96, Thm.I.5.7] this is also equivalent to require that $D_A(\pi)$ is finitely generated over A and that its natural filtration in (109) is good (equivalently gives the canonical topology). In particular, if this holds for one good filtration on $\pi^\vee$, then this holds for all good filtrations. It easily follows from the proof of Proposition 3.1.2.3 and the noetherianity of $\operatorname{gr}(A)$ (Corollary 3.1.1.2) that $\mathcal{C}$ is an abelian subcategory stable under subquotients and extensions in the category of smooth representations of $\mathrm{GL}_2(K)$ over $\mathbb{F}$ with a central character.

For π in $\mathcal{C}$, the pair $(D_A(\pi), \beta)$ is an example of $(\psi, \mathcal{O}_K^\times)$-module over A as in Definition 3.1.2.1. We can in particular consider its étale part $D_A(\pi)^{\text{ét}}$. The action of $\mathcal{O}_K^\times$ on $D_A(\pi)$ preserves its nilpotent part $D_A(\pi)^0$ and thus induces a continuous action of $\mathcal{O}_K^\times$ on $D_A(\pi)^{\text{ét}}$. In particular, $D_A(\pi)^{\text{ét}}$ is an étale $(\psi, \mathcal{O}_K^\times)$-module over A. Note that the canonical topology on the finitely generated A-module $D_A(\pi)^{\text{ét}}$ is also the quotient topology of $D_A(\pi) \twoheadrightarrow D_A(\pi)^{\text{ét}}$.

Corollary 3.1.2.9. *Let π in $\mathcal{C}$. Then the A-modules $D_A(\pi)$ and $D_A(\pi)^{\text{ét}}$ are finite projective over A. Moreover the map $\beta^{\text{ét}} : D_A(\pi)^{\text{ét}} \to \phi^* D_A(\pi)^{\text{ét}}$ is an isomorphism.*

Proof. This is a special case of Propositions 3.1.1.8 and 3.1.2.2. □

Remark 3.1.2.10. If π is 1-dimensional (a character of $\mathrm{GL}_2(K)$), then $D_A(\pi) = D_A(\pi)^{\text{ét}} = 0$.

We give an important condition on an admissible smooth representation π (with a central character) which ensures that π is in $\mathcal{C}$. Let J be the following graded ideal of $\mathrm{gr}(\mathbb{F}[\![I_1/Z_1]\!])$:

$$J \stackrel{\text{def}}{=} (y_i z_i, h_i, 0 \le i \le f-1). \tag{118}$$

From the definition of equivalent filtrations (see [LvO96, §I.3.2]), one easily sees (using [LvO96, Lemma I.5.3]) that if $\mathrm{gr}(\pi^\vee)$ is annihilated by some power of J for one good filtration on π, then it is so for all good filtrations (but note that the power of J which annihilates $\mathrm{gr}(\pi^\vee)$ may depend on the fixed good filtration).

Proposition 3.1.2.11. *Assume that $\mathrm{gr}(\pi^\vee)$ is annihilated by some power of J. Then the A-module $D_A(\pi)$ is finite projective and the $\mathrm{gr}(A)$-module $\mathrm{gr}(D_A(\pi))$ is finitely generated.*

Proof. As the hypothesis does not depend on the choice of the good filtration on $\pi^\vee$, we are free to work with the $\mathfrak{m}_{I_1/Z_1}$-adic topology on $\pi^\vee$. Let us first prove that $\mathrm{gr}(D_A(\pi))$ is a finitely generated $\mathrm{gr}(A)$-module. It follows from the admissibility of π and from the hypothesis that $\mathrm{gr}(\pi^\vee)$ is a finitely generated $\mathrm{gr}(\mathbb{F}[\![I_1/Z_1]\!])/J^N$-module for some $N \ge 1$. Lemma 3.1.1.1 then implies that $\mathrm{gr}(D_A(\pi))$ is a finitely generated $(\mathrm{gr}(\mathbb{F}[\![I_1/Z_1]\!])/J^N)[(y_0 \cdots y_{f-1})^{-1}]$-module. It is therefore sufficient to prove that $(\mathrm{gr}(\mathbb{F}[\![I_1/Z_1]\!])/J^N)[(y_0 \cdots y_{f-1})^{-1}]$ is a finitely generated $\mathrm{gr}(A)$-module. Since $\mathrm{gr}(\mathbb{F}[\![I_1/Z_1]\!])$ is noetherian, we are reduced by dévissage to the case $N = 1$, where we have

$$\begin{aligned}\Big(\mathrm{gr}(\mathbb{F}[\![I_1/Z_1]\!])/J\Big)[(y_0 \cdots y_{f-1})^{-1}] &\cong (\mathbb{F}[y_i, z_i, h_i]/(y_i z_i, h_i))[(y_0 \cdots y_{f-1})^{-1}] \\ &= \mathbb{F}[y_i^{\pm 1}] \cong \mathrm{gr}(A).\end{aligned}$$

Finally, as $D_A(\pi)$ is a complete filtered A-module, it then follows from [LvO96, Thm.I.5.7] that $D_A(\pi)$ is finitely generated over A and from Proposition 3.1.1.8 that it is projective. □

It follows from Proposition 3.1.2.11 that the admissible smooth representations π (with a central character) such that $\mathrm{gr}(\pi^\vee)$ is annihilated by some power of J for at least one good filtration is a full subcategory of the category $\mathcal{C}$. Moreover this full subcategory is abelian and stable under subquotients and extensions in $\mathcal{C}$. Namely, for a short exact sequence $0 \to \pi' \to \pi \to \pi'' \to 0$ in $\mathcal{C}$, the filtrations induced on $(\pi'')^\vee$ and $(\pi')^\vee$ by a good filtration of $\pi^\vee$ are good. For these filtrations we have a short exact sequence $0 \to \mathrm{gr}((\pi'')^\vee) \to \mathrm{gr}(\pi^\vee) \to \mathrm{gr}((\pi')^\vee) \to 0$ which shows that $\mathrm{gr}(\pi^\vee)$ is annihilated by a power of J if and only if $\mathrm{gr}((\pi')^\vee)$ and $\mathrm{gr}((\pi'')^\vee)$ are.

Remark 3.1.2.12. It is natural to consider the image $D_A^\natural(\pi)$ of $\pi^\vee$ in $D_A(\pi) = A\widehat{\otimes}_{\mathbb{F}[\![N_0]\!]}\pi^\vee$. Indeed, as the map $\pi^\vee \to D_A(\pi)$ is continuous and $\pi^\vee$ is compact, it follows that $D_A^\natural(\pi)$ is a compact $\mathbb{F}[\![N_0]\!]$-submodule of $D_A(\pi)$. However, the $\mathbb{F}[\![N_0]\!]$-module $D_A^\natural(\pi)$ is *not* finitely generated when π is an irreducible admissible supersingular representation and $[K : \mathbb{Q}_p] > 1$ (even if $D_A(\pi)$ is finitely generated over A). Namely, if this were the case, this would give us the existence of a nontrivial finitely generated $\mathbb{F}[\![N_0]\!][(\begin{smallmatrix} p & 0 \\ 0 & 1 \end{smallmatrix})]$-submodule of π that is admissible as an $\mathbb{F}[\![N_0]\!]$-module and this would contradict the results of [Sch15] and [Wu21]. Likewise, the image of $\pi^\vee$ in the quotient $D_A(\pi)^{\text{ét}}$ of $D_A(\pi)$ won't be finitely generated over $\mathbb{F}[\![N_0]\!]$ in general (see Remark 3.3.5.4(ii)). Finally, we conjecture in [BHH+22, Conj.1.4] that for those π coming from cohomology we always have $D_A(\pi) \cong D_A(\pi)^{\text{ét}}$.

Remark 3.1.2.13. Recall that the action of $\mathbb{F}[\![N_0]\!]$ on $\pi^\vee$ is defined by $\delta_a(f) = f \circ a^{-1}$ for $f \in \pi^\vee$ and $a \in N_0$. We could have defined it by the formula $\delta_a(f) = f \circ a$ for $f \in \pi^\vee$ and $a \in N_0$ and would have obtained isomorphic $(\psi, \mathcal{O}_K^\times)$-modules $D_A(\pi)$ and $D_A(\pi)^{\text{ét}}$ (for instance, this is the convention used in [Bre15, Lemme 2.6]). Namely the map $f \mapsto \gamma_{-1} \cdot f$, with $\gamma = \begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix}$ induces an intertwining, commuting with ψ and $\mathcal{O}_K^\times$, between the two $\mathbb{F}[\![N_0]\!]$-structures.

3.1.3 Multivariable $(\varphi, \mathcal{O}_K^\times)$-modules

Using the results of §3.1.2, we promote the functor $\pi \mapsto D_A(\pi)^{\text{ét}}$ to an exact functor from $\mathcal{C}$ to a category of étale multivariable $(\varphi, \mathcal{O}_K^\times)$-modules (Theorem 3.1.3.3) and we compare $D_A(\pi)^{\text{ét}}$ with the functor $D_\xi^\vee(\pi)$ of §2.1.1 (Theorem 3.1.3.7).

Let R be a noetherian commutative ring of characteristic p endowed with an injective ring endomorphism F_R such that R is a finite free $F_R(R)$-module (as at the beginning of §3.1.2). A *φ-module* (D, φ) over R is an R-module D with an F_R-semilinear map $\varphi : D \to D$. We say that a φ-module (D, φ) is *étale* if the R-linear map $F_R^*(D) \to D$ defined by $a \otimes d \mapsto a\varphi(d)$ is an isomorphism.

Definition 3.1.3.1. A *$(\varphi, \mathcal{O}_K^\times)$-module over A* is a φ-module (D, φ) over A such that D is a finitely generated A-module, the endomorphism φ is continuous (for the canonical topology on D as at the beginning of §3.1.2) and D is endowed with a continuous A-semilinear action of $\mathcal{O}_K^\times$ commuting with φ. We say that a $(\varphi,\mathcal{O}_K^\times)$-module over A is *étale* if the underlying φ-module over A is.

We note that, by Proposition 3.1.1.8, if (D,φ) is a $(\varphi,\mathcal{O}_K^\times)$-module over A, then D is a finite projective A-module.

If (D,β) is an étale $(\psi,\mathcal{O}_K^\times)$-module over A as in Definition 3.1.2.1, by Proposition 3.1.2.2 we can define a ϕ-semilinear endomorphism φ of D such that $\mathrm{Id}\otimes\varphi=\beta^{-1}$, so that (D,φ) is an étale $(\varphi,\mathcal{O}_K^\times)$-module over A. (Note that φ is continuous, as the topology of D is defined by any good filtration and $\phi : A\to A$ is continuous.)

We now go back to representations π of $\mathrm{GL}_2(K)$, but we first need some more notation. The trace map $\mathrm{tr} : N_0\cong\mathcal{O}_K\to\mathbb{Z}_p$ induces a ring homomorphism $\mathrm{tr} : \mathbb{F}[\![N_0]\!]\to\mathbb{F}[\![\mathbb{Z}_p]\!]\cong\mathbb{F}[\![X]\!]$, where we recall that $X=\left(\begin{smallmatrix}1&1\\0&1\end{smallmatrix}\right)-1$. Moreover, for Y_i as in (101), we have $\mathrm{tr}(Y_i)\equiv -X \mod X^2$ (see Lemma 3.2.2.2 and the last statement in Lemma 3.2.2.4 below) and the universal property of the ring A shows that this map extends to a continuous ring homomorphism $\mathrm{tr} : A\to\mathbb{F}(\!(X)\!)$. We let

$$\mathfrak{p}\stackrel{\mathrm{def}}{=}\mathrm{Ker}(\mathrm{tr} : A\to\mathbb{F}(\!(X)\!)).$$

Then $\mathfrak{p}$ is a closed maximal ideal of A. Note that

$$\mathfrak{p}\cap\mathbb{F}[\![N_0]\!]=\mathrm{Ker}(\mathrm{tr} : \mathbb{F}[\![N_0]\!]\to\mathbb{F}[\![X]\!])=\mathfrak{m}_{N_1}\mathbb{F}[\![N_0]\!]=(Y_0-Y_1,\dots,Y_0-Y_{f-1}),$$

where $N_1\subseteq N_0$ is as in (11) (for the second isomorphism write $N_0\cong N_1\oplus\mathbb{Z}_pe$, where $\mathrm{tr}(e)=1$, noting that $\mathrm{tr} : \mathcal{O}_K\to\mathbb{Z}_p$ is surjective, as K is unramified, and for the third use the first statement of Lemma 3.2.2.4 below).

Remark 3.1.3.2. Let B be the completion of $\mathbb{F}[\![N_0]\!]_S$ along the prime ideal generated by $(Y_0-Y_1,\dots,Y_0-Y_{f-1})$ (see the beginning of §3.1.1 for S). Expanding $Y_i^n=(Y_0-(Y_0-Y_i))^n$ if $n\geq 0$, and writing $Y_i^n=(\sum_{m=0}^{+\infty}\frac{(Y_0-Y_i)^m}{Y_0^{m+1}})^{-n}$ and expanding everything if $n<0$, one can see using Remark 3.1.1.3(iii) that the ring A embeds into B. The endomorphism ϕ on A extends to B but only the action of $\mathbb{Z}_p^\times\subseteq\mathcal{O}_K^\times$ extends to B, as $(Y_0-Y_1,\dots,Y_0-Y_{f-1})$ is not preserved by all of $\mathcal{O}_K^\times$. Then from Corollary 3.1.2.9 and as B is a local ring, we see that $D_A(\pi)^{\text{ét}}\otimes_A B$ is a *finite free étale* $(\varphi,\mathbb{Z}_p^\times)$*-module over* B, which is similar to the generalized (φ,Γ)-modules defined in [SV11] (though *loc.cit.* only considers split algebraic groups over $\mathbb{Q}_p$).

Let π be in the category $\mathcal{C}$. Using Corollary 3.1.2.9, we can define a ϕ-semilinear endomorphism φ of $D_A(\pi)^{\text{ét}}$ such that $\mathrm{Id}\otimes\varphi=(\beta^{\text{ét}})^{-1}$, so that $D_A(\pi)^{\text{ét}}$ is an étale $(\varphi,\mathcal{O}_K^\times)$-module over A. As $\mathfrak{p}$ is a ϕ-stable ideal of A, we deduce that $D_A(\pi)^{\text{ét}}/\mathfrak{p}\cong D_A(\pi)^{\text{ét}}\otimes_A\mathbb{F}(\!(X)\!)$ is an étale $(\varphi,\mathbb{Z}_p^\times)$-module over $\mathbb{F}(\!(X)\!)$.

Theorem 3.1.3.3.

(i) *The functor* $\pi \longmapsto D_A(\pi)^{\text{ét}}$ *is exact from the category* $\mathcal{C}$ *to the category of étale* $(\varphi, \mathcal{O}_K^\times)$*-modules over* A.

(ii) *The functor* $\pi \longmapsto D_A(\pi)^{\text{ét}} \otimes_A \mathbb{F}(\!(X)\!)$ *is exact from the category* $\mathcal{C}$ *to the category of étale* $(\varphi, \mathbb{Z}_p^\times)$*-modules over* $\mathbb{F}(\!(X)\!)$.

Proof. (i) is a consequence of Proposition 3.1.2.3, of the exactness of ϕ^* and of the exactness of direct limits, together with the description (see the beginning of §3.1.2)

$$D_A(\pi)^{\text{ét}} \cong \varinjlim_{(\phi^*)^n(\beta^{\text{ét}})} (\phi^*)^n(D_A(\pi)^{\text{ét}}) \cong \varinjlim_{(\phi^*)^n(\beta)} (\phi^*)^n(D_A(\pi)).$$

(ii) is a consequence of (i), of Corollary 3.1.2.9 and of the exactness of $(-) \otimes_A \mathbb{F}(\!(X)\!)$ on short exact sequences of finite projective A-modules. $\square$

Remark 3.1.3.4. One can prove that if $\pi \in \mathcal{C}$ then the endomorphism $\psi : D_A(\pi) \to D_A(\pi)$ (defined right after Lemma 3.1.2.6) is always surjective. (This follows ultimately from the fact that the image of the natural map $A \otimes_{\mathbb{F}[\![N_0]\!]} \pi^\vee \to D_A(\pi)$ is surjective since A is complete and Noetherian, and $A \otimes_{\mathbb{F}[\![N_0]\!]} \pi^\vee$ is endowed with a surjective endomorphism that is compatible with ψ on $D_A(\pi)$.) In particular, this implies that $D_A(\pi)^{\text{ét}} \neq 0$ as soon as $D_A(\pi) \neq 0$, since ψ cannot be nilpotent if it is surjective on $D_A(\pi)$ and the latter is nonzero. Note that for the representations π of particular interest for us here, we will actually have $D_A(\pi) = D_A(\pi)^{\text{ét}}$; see Remark 3.3.5.4(ii).

We now compare the étale $(\varphi, \mathbb{Z}_p^\times)$-module $D_A(\pi)^{\text{ét}}/\mathfrak{p}$ with $D_\xi^\vee(\pi)$ (15).

Let $\overline{\psi}$ be the $\mathbb{F}$-linear endomorphism of $\pi^\vee/\mathfrak{m}_{N_1} \cong (\pi^{N_1})^\vee$ defined by

$$\overline{\psi}(x) \stackrel{\text{def}}{=} \sum_{b \in N_1/N_1^p} \psi(\delta_{\tilde{b}}\tilde{x}) \mod \mathfrak{m}_{N_1}, \tag{119}$$

where $\tilde{b} \in N_1$ is a lift of b, $\tilde{x} \in \pi^\vee$ is a lift of x and ψ is as in (110) (it is easy to check that the definition of $\overline{\psi}$ does not depend on the choice of these lifts). We have $\overline{\psi}(S(X^p)m) = S(X)\overline{\psi}(m)$ for all $S(X) \in \mathbb{F}[\![X]\!]$ and $m \in \pi^\vee/\mathfrak{m}_{N_1}$, and $\overline{\psi}$ is the dual of the endomorphism F of π^{N_1} in §2.1.1. We define an endomorphism $\overline{\psi}$ of $D_A(\pi)/\mathfrak{p}$ (resp. $D_A(\pi)^{\text{ét}}/\mathfrak{p}$) by the same formula replacing $\pi^\vee$ by $D_A(\pi)$ (resp. $D_A(\pi)^{\text{ét}}$) and $\mathfrak{m}_{N_1}$ by $\mathfrak{p}$, it is then clear that the following diagram commutes:

$$\begin{array}{ccc} \pi^\vee/\mathfrak{m}_{N_1} & \xrightarrow{\overline{\psi}} & \pi^\vee/\mathfrak{m}_{N_1} \\ \downarrow & & \downarrow \\ D_A(\pi)/\mathfrak{p} & \xrightarrow{\overline{\psi}} & D_A(\pi)/\mathfrak{p}, \end{array} \tag{120}$$

together with an analogous diagram with $D_A(\pi)/\mathfrak{p} \twoheadrightarrow D_A(\pi)^{\text{ét}}/\mathfrak{p}$ that we leave to the reader.

Let $\overline{\beta} : D_A(\pi)/\mathfrak{p} \to \phi^*(D_A(\pi)/\mathfrak{p}) \cong \mathbb{F}[\![X]\!] \otimes_{\varphi,\mathbb{F}[\![X]\!]} (D_A(\pi)/\mathfrak{p}) \cong \phi^*(D_A(\pi))/\mathfrak{p}$ be the $\mathbb{F}(\!(X)\!)$-linear map defined by

$$\overline{\beta}(m) \stackrel{\text{def}}{=} \sum_{i=0}^{p-1} (1+X)^{-i} \otimes_\phi \overline{\psi}((1+X)^i m).$$

Lemma 3.1.3.5. *The following diagram is commutative* (*where the horizontal maps are the canonical surjections*)*:*

$$\begin{array}{ccc} D_A(\pi) & \twoheadrightarrow & D_A(\pi)/\mathfrak{p} \\ \downarrow \beta & & \downarrow \overline{\beta} \\ \phi^*(D_A(\pi)) & \twoheadrightarrow & \phi^*(D_A(\pi)/\mathfrak{p}). \end{array}$$

Proof. We choose a system of representatives $(g^{-i}b_j)_{\substack{0\le i\le p-1 \\ 1\le j\le p^{f-1}}}$ of N_0/N_0^p such that $g \stackrel{\text{def}}{=} \left(\begin{smallmatrix} 1 & 1 \\ 0 & 1 \end{smallmatrix}\right) \in N_0$ and $b_1, \dots, b_{p^{f-1}}$ are in N_1. We then have for $m \in D_A(\pi)$ that

$$\begin{aligned} \beta(m) &= \sum_{i=0}^{p-1} \sum_{j=1}^{p^{f-1}} \left(\delta_{g^i}^{-1}\delta_{b_j} \otimes_\phi \psi(\delta_{b_j}^{-1}\delta_{g^i}m)\right) \\ &\equiv \sum_{i=0}^{p-1} \left(\delta_{g^i}^{-1} \otimes_\phi \sum_{j=1}^{p^{f-1}} \psi(\delta_{b_j}^{-1}(\delta_{g^i}m))\right) \mod \mathfrak{p}\phi^*(D_A(\pi)) \\ &\equiv \sum_{i=0}^{p-1} \delta_{g^i}^{-1} \otimes_\phi \overline{\psi}(\delta_{g^i}m) \mod \mathfrak{p}\phi^*(D_A(\pi)), \end{aligned}$$

where the first equality follows from (116), the second from $\delta_{b_j} - 1 \in \mathfrak{p} \subseteq A$ (and the commutativity of N_0), and the third from the analog of (119) for $D_A(\pi)/\mathfrak{p}$. Noting that the image of δ_{g^i} in $\mathbb{F}[\![X]\!]$ is $(1+X)^i$, we obtain the desired compatibility. □

Lemma 3.1.3.6. *Let $M \subseteq \pi^{N_1}$ be an $\mathbb{F}[\![X]\!]$-submodule that is admissible as an $\mathbb{F}[\![X]\!]$-module. Then the surjective map $\pi^\vee \twoheadrightarrow M^\vee$ is continuous for the $\mathfrak{m}_{I_1}$-adic topology on $\pi^\vee$ and the X-adic topology on $M^\vee$.*

Proof. The map $\pi^\vee \twoheadrightarrow M^\vee$ is continuous with respect to the natural profinite topologies arising from Pontryagin duality. As M is admissible as an $\mathbb{F}[\![X]\!]$-module, the natural topology on $M^\vee$ is the X-adic topology. It thus suffices to show that the $\mathfrak{m}_{I_1}$-adic topology is at least as fine as the natural topology on $\pi^\vee$. Dually this means that any finite-dimensional subspace of π is contained in $\pi[\mathfrak{m}_{I_1}^N]$ for some sufficiently large integer N, which is true by smoothness. □

Recall that we defined in (15) a projective limit $D_\xi^\vee(\pi)$ of étale $(\varphi, \mathbb{Z}_p^\times)$-modules over $\mathbb{F}(\!(X)\!)$ associated to π.

Theorem 3.1.3.7. *We have an isomorphism of étale $(\varphi, \mathbb{Z}_p^\times)$-modules over $\mathbb{F}(\!(X)\!)$:*

$$D_A(\pi)^{\text{ét}}/\mathfrak{p} \xrightarrow{\sim} D_\xi^\vee(\pi).$$

In particular, $D_\xi^\vee(\pi)$ is finite-dimensional over $\mathbb{F}(\!(X)\!)$ and the functor $\pi \longmapsto D_\xi^\vee(\pi)$ is exact on $\mathcal{C}$.

Proof. For the purpose of this proof it is convenient to use the action of $\mathbb{F}[\![N_0]\!]$ on $\pi^\vee$ given by $\delta_a(f) = f \circ a$ for $f \in \pi^\vee$ and $a \in N_0$. This does not change $D_A(\pi)^{\text{ét}}$ up to isomorphism by Remark 3.1.2.13.

As a first step we construct the map. Let $M \subseteq \pi^{N_1}$ be a finitely generated $\mathbb{F}[\![X]\!][F]$-submodule that is admissible as an $\mathbb{F}[\![X]\!]$-module and $\mathbb{Z}_p^\times$-stable. By Lemma 3.1.3.6, the map $\pi^\vee \twoheadrightarrow M^\vee$ is continuous. It extends to a surjection of $\mathbb{F}[\![N_0]\!]_S$-modules $(\pi^\vee)_S \twoheadrightarrow M^\vee[X^{-1}]$. By definition of the tensor product filtration on $(\pi^\vee)_S$, this surjection is continuous if $M^\vee[X^{-1}]$ is endowed with its natural topology of finite-dimensional $\mathbb{F}(\!(X)\!)$-vector space. As $M^\vee[X^{-1}]$ is complete for this topology, by completion we obtain a continuous surjection of topological A-modules $\zeta_M : D_A(\pi) \twoheadrightarrow M^\vee[X^{-1}]$. Since N_1 acts trivially on M, ζ_M factors through a surjection of $\mathbb{F}(\!(X)\!)$-vector spaces $\overline{\zeta_M} : D_A(\pi)/\mathfrak{p} \twoheadrightarrow M^\vee[X^{-1}]$. By definition of $\overline{\psi}$, we obtain a commutative diagram (where $F^\vee$ is the $\mathbb{F}$-linear dual of $F : M \to M$ that we extend to $M^\vee[X^{-1}]$ using $F^\vee(X^{-i}f) = X^{-i}F(X^{i(p-1)}f)$)

$$\begin{array}{ccc} D_A(\pi)/\mathfrak{p} & \xrightarrow{\overline{\zeta_M}} & M^\vee[X^{-1}] \\ \downarrow{\scriptstyle\overline{\psi}} & & \downarrow{\scriptstyle F^\vee} \\ D_A(\pi)/\mathfrak{p} & \xrightarrow{\overline{\zeta_M}} & M^\vee[X^{-1}]. \end{array}$$

It then follows from Lemma 3.1.3.5 that, identifying $\phi^*(M^\vee) \cong \mathbb{F}[\![X]\!] \otimes_{\varphi, \mathbb{F}[\![X]\!]} M^\vee$ with $(\mathbb{F}[\![X]\!] \otimes_{\varphi, \mathbb{F}[\![X]\!]} M)^\vee$ via (14), the following diagram is commutative:

$$\begin{array}{ccccc} D_A(\pi) & \twoheadrightarrow & D_A(\pi)/\mathfrak{p} & \xrightarrow{\overline{\zeta_M}} & M^\vee[X^{-1}] \\ \downarrow{\scriptstyle\beta} & & \downarrow{\scriptstyle\overline{\beta}} & & \downarrow{\scriptstyle(\mathrm{Id}\otimes F)^\vee} \\ \phi^*(D_A(\pi)) & \twoheadrightarrow & \phi^*(D_A(\pi)/\mathfrak{p}) & \xrightarrow{\mathrm{Id}\otimes\overline{\zeta_M}} & \phi^*(M^\vee[X^{-1}]), \end{array} \tag{121}$$

where $(\mathrm{Id} \otimes F)^\vee$ comes from $\mathbb{F}$-linear dual of $\mathrm{Id} \otimes F : \mathbb{F}[\![X]\!] \otimes_{\varphi, \mathbb{F}[\![X]\!]} M \to M$. As $(\mathrm{Id} \otimes F)^\vee$ is an isomorphism (see just after (14)), the map $\zeta_M : D_A(\pi) \twoheadrightarrow M^\vee[X^{-1}]$ factors through $D_A(\pi)^{\text{ét}}$ and the map $\overline{\zeta_M} : D_A(\pi)/\mathfrak{p} \twoheadrightarrow M^\vee[X^{-1}]$ factors through $D_A(\pi)^{\text{ét}}/\mathfrak{p}$. The map $\overline{\zeta_M} : D_A(\pi)^{\text{ét}}/\mathfrak{p} \twoheadrightarrow M^\vee[X^{-1}]$ clearly commutes with the action of $\mathbb{Z}_p^\times$ and the commutative diagram (121) shows that it is a morphism φ-modules. These maps are obviously compatible when M is varying among the finitely generated $\mathbb{F}[\![X]\!][F]$-submodules of π^{N_1} that are admissible as $\mathbb{F}[\![X]\!]$-modules and $\mathbb{Z}_p^\times$-stable so that we obtain a map

$$\zeta : D_A(\pi)^{\text{ét}}/\mathfrak{p} \longrightarrow \varprojlim_M M^\vee[X^{-1}] = D_\xi^\vee(\pi).$$

We prove that the map ζ is surjective. Since $D_A(\pi)^{\text{ét}}/\mathfrak{p}$ is a finite-dimensional $\mathbb{F}(\!(X)\!)$-vector space, the dimension of the vector spaces $M^\vee[X^{-1}]$ when M is varying is bounded. This implies that there exists some M such that $D_\xi^\vee(\pi) = M^\vee[X^{-1}]$ and that the map $\zeta : D_A(\pi)^{\text{ét}}/\mathfrak{p} \to D_\xi^\vee(\pi)$ is surjective. In particular, $\dim_{\mathbb{F}(\!(X)\!)} D_\xi^\vee(\pi) < +\infty$.

We prove that the map ζ is an isomorphism. Let $D^\natural(\pi)^{\text{ét}}$ be the image of $\pi^\vee$ in $D_A(\pi)^{\text{ét}}/\mathfrak{p}$. This is a compact $\mathbb{F}[\![X]\!]$-module in the finite-dimensional $\mathbb{F}(\!(X)\!)$-vector space $D_A(\pi)^{\text{ét}}/\mathfrak{p}$, hence a finite free $\mathbb{F}[\![X]\!]$-module. Since the maps $\pi^\vee \to D_A(\pi)/\mathfrak{p} \twoheadrightarrow D_A(\pi)^{\text{ét}}/\mathfrak{p}$ commute with the action of $\mathbb{Z}_p^\times$, $D^\natural(\pi)^{\text{ét}}$ is preserved by $\mathbb{Z}_p^\times$. The image of $(\pi^\vee)_S$ in $D_A(\pi)^{\text{ét}}/\mathfrak{p}$ coincides with $D^\natural(\pi)^{\text{ét}}[X^{-1}]$. As $(\pi^\vee)_S$ has a dense image in $D_A(\pi)$ by definition, $D^\natural(\pi)^{\text{ét}}[X^{-1}]$ is a dense $\mathbb{F}(\!(X)\!)$-vector subspace of $D_A(\pi)^{\text{ét}}/\mathfrak{p}$ and thus equal to $D_A(\pi)^{\text{ét}}/\mathfrak{p}$ by finiteness of the dimension. The surjective map $\pi^\vee \twoheadrightarrow D^\natural(\pi)^{\text{ét}}$ factors through $\pi^\vee/\mathfrak{m}_{N_1} \cong (\pi^{N_1})^\vee$ so that the topological $\mathbb{F}$-linear dual $(D^\natural(\pi)^{\text{ét}})^\vee$ of $D^\natural(\pi)^{\text{ét}}$ is identified with an $\mathbb{F}[\![X]\!]$-submodule of π^{N_1} (endowed with the discrete topology) preserved by $\mathbb{Z}_p^\times$. As $D^\natural(\pi)^{\text{ét}}$ is stable by $\overline{\psi}$ by (120), $(D^\natural(\pi)^{\text{ét}})^\vee$ is actually an $\mathbb{F}[\![X]\!][F]$-submodule of π^{N_1}. Since $\beta^{\text{ét}} : D_A(\pi)^{\text{ét}} \xrightarrow{\sim} \phi^*(D_A(\pi)^{\text{ét}})$ is an isomorphism, it easily follows from Lemma 3.1.3.5 that the map $\overline{\beta}$ induces a surjective map of finite-dimensional $\mathbb{F}(\!(X)\!)$-vector spaces $\overline{\beta}^{\text{ét}} : D_A(\pi)^{\text{ét}}/\mathfrak{p} \twoheadrightarrow \phi^*(D_A(\pi)^{\text{ét}}/\mathfrak{p})$. As these spaces have the same dimension, $\overline{\beta}^{\text{ét}}$ is actually an isomorphism, and in particular $\overline{\beta}^{\text{ét}}|_{D^\natural(\pi)^{\text{ét}}} : D^\natural(\pi)^{\text{ét}} \to \mathbb{F}[\![X]\!] \otimes_{\varphi,\mathbb{F}[\![X]\!]} D^\natural(\pi)^{\text{ét}}$ is an injection and becomes an isomorphism after inverting X.

We claim that $(D^\natural(\pi)^{\text{ét}})^\vee$ is finitely generated as an $\mathbb{F}[\![X]\!][F]$-module. Note that $(D^\natural(\pi)^{\text{ét}})^\vee$ is admissible as an $\mathbb{F}[\![X]\!]$-module since $D^\natural(\pi)^{\text{ét}}$ is a finitely generated $\mathbb{F}[\![X]\!]$-module. Hence, the claim follows from [Bre15, Lemma 5.2] using the last statement of the previous paragraph.

We now give another proof of the claim using results of [Lyu97, §4]. In fact, we even prove that $(D^\natural(\pi)^{\text{ét}})^\vee$ is of finite length as an $\mathbb{F}[\![X]\!][F]$-module. As $\mathbb{F}$ is a finite extension of $\mathbb{F}_p$, the $\mathbb{F}_p[\![X]\!]$-module $(D^\natural(\pi)^{\text{ét}})^\vee$ is artinian so that the $\mathbb{F}_p[\![X]\!][F]$-module $(D^\natural(\pi)^{\text{ét}})^\vee$ is a cofinite $\mathbb{F}_p[\![X]\!][F]$-module in the sense of [Lyu97, §4] (the ring $\mathbb{F}_p[\![X]\!][F]$ is isomorphic to the ring $A\{f\}$ of *loc. cit.* where $A = \mathbb{F}_p[\![X]\!]$). It follows from Theorem 4.7 in *loc. cit.* that $(D^\natural(\pi)^{\text{ét}})^\vee$ has a filtration

$$0 = M_0 \subseteq M_1 \subseteq \cdots \subseteq M_r = (D^\natural(\pi)^{\text{ét}})^\vee$$

by $\mathbb{F}_p[\![X]\!][F]$-submodules such that M_{i+1}/M_i is a simple $\mathbb{F}_p[\![X]\!][F]$-module or a nilpotent $\mathbb{F}_p[\![X]\!][F]$-module, i.e. such that some power of F is zero on M_{i+1}/M_i. Let $M_i^\perp$ be the kernel of $D^\natural(\pi)^{\text{ét}} \to M_i^\vee$ for all i. As $\overline{\beta}^{\text{ét}}|_{D^\natural(\pi)^{\text{ét}}}$ coincides with $(\text{Id} \otimes F)^\vee$ (this is analogous to (121) using (14) with $M = (D^\natural(\pi)^{\text{ét}})^\vee$), the map $\overline{\beta}^{\text{ét}}$ induces an isomorphism of $\mathbb{F}_p(\!(X)\!) \otimes_{\mathbb{F}_p[\![X]\!]} M_i^\perp$ onto $\mathbb{F}_p(\!(X)\!) \otimes_{\varphi,\mathbb{F}_p[\![X]\!]} M_i^\perp$. In particular, if M_{i+1}/M_i is nilpotent then $F^\vee$ induces a nilpotent endomorphism of $M_i^\perp/M_{i+1}^\perp$ so that $\mathbb{F}_p(\!(X)\!) \otimes_{\mathbb{F}_p[\![X]\!]} M_i^\perp = \mathbb{F}_p(\!(X)\!) \otimes_{\mathbb{F}_p[\![X]\!]} M_{i+1}^\perp$ (as $\mathbb{F}_p(\!(X)\!) \otimes_{\varphi,\mathbb{F}_p[\![X]\!]} (M_i^\perp/M_{i+1}^\perp) = 0$ in this case) and hence $M_i^\perp/M_{i+1}^\perp$ is a torsion $\mathbb{F}_p[\![X]\!]$-module. As $D^\natural(\pi)^{\text{ét}}$ is a finitely generated $\mathbb{F}_p[\![X]\!]$-module, we conclude that when M_{i+1}/M_i is nilpotent the $\mathbb{F}_p[\![X]\!]$-module $M_i^\perp/M_{i+1}^\perp$ is finite-dimensional over $\mathbb{F}_p$, in particular it is an $\mathbb{F}_p[\![X]\!][F]$-module of finite length. Since M_{i+1}/M_i is obviously of finite length when M_{i+1}/M_i is irreducible, the claim follows.

The claim implies that $(D^\natural(\pi)^{\text{ét}})^\vee$ is one of the modules $M \subseteq \pi^{N_1}$ in §2.1.1, in particular

$$\dim_{\mathbb{F}(\!(X)\!)} D_\xi^\vee(\pi) \geq \dim_{\mathbb{F}(\!(X)\!)}(D^\natural(\pi)^{\text{ét}}[X^{-1}]) = \dim_{\mathbb{F}(\!(X)\!)}(D_A(\pi)^{\text{ét}}/\mathfrak{p}).$$

This implies that the map ζ is an isomorphism (and that $D_A(\pi)^{\text{ét}}/\mathfrak{p} = D^\natural(\pi)^{\text{ét}}[X^{-1}] \cong D_\xi^\vee(\pi)$). The very last statement follows from Theorem 3.1.3.3(ii). □

3.1.4 An upper bound for the ranks of $D_A(\pi)^{\text{ét}}$ and $D_\xi^\vee(\pi)$

For π in $\mathcal{C}$ we bound the dimension of $D_\xi^\vee(\pi)$ in terms of $\text{gr}(\pi^\vee)$. When $\text{gr}(\pi^\vee)$ is killed by some J^n, we give an interpretation of this bound as a certain multiplicity.

We keep all previous notation. We start with the following lemma.

Lemma 3.1.4.1. *Let M be a finitely generated A-module endowed with a good filtration. Then the generic rank of the A-module M and the generic rank of the $\text{gr}(A)$-module $\text{gr}(M)$ coincide.*

Proof. We first note that if N is an A-module of generic rank 0, then $N \otimes_A \text{Frac}(A) = 0$ and N is a torsion module. This implies that $\text{gr}(N)$ is a torsion module and that its generic rank is 0.

Let d be the generic rank of M and $f : A^{\oplus d} \to M \otimes_A \text{Frac}(A)$ be a morphism of A-modules sending an A-basis of the left-hand side to a $\text{Frac}(A)$-basis of the right-hand side. The kernel of f is then a torsion A-submodule of $A^{\oplus d}$ and is zero since A is a domain. Moreover there exists $a \in A\backslash\{0\}$ such that the image of af is contained in M. As $\text{Frac}(A)$ is a flat A-module, the generic rank is an additive map on the abelian category of finitely generated A-modules. As af is injective and $A^{\oplus d}$ and M have identical generic ranks, this implies that the cokernel Q of af has generic rank

0. We fix a good filtration on M: it induces good filtrations on $af(A^{\oplus d})$ and on Q. For these filtrations we have a short exact sequence

$$0 \longrightarrow \mathrm{gr}(af(A^{\oplus d})) \longrightarrow \mathrm{gr}(M) \longrightarrow \mathrm{gr}(Q) \longrightarrow 0.$$

As Q has generic rank 0, so does $\mathrm{gr}(Q)$ so that it suffices to prove that $\mathrm{gr}(af(A^{\oplus d}))$ has generic rank d. It follows from the second paragraph after [Bjö89, Def.4.2] that, for a finitely generated A-module N, the generic rank of $\mathrm{gr}(N)$ does not depend on the choice of good filtration. We can thus choose a good filtration $af(A^{\oplus d}) \cong A^{\oplus d}$ which is filtered free with respect to the canonical basis of $A^{\oplus d}$, for which the result is obvious. □

Let π be in the category $\mathcal{C}$ and choose a good filtration on the $\mathbb{F}[\![I_1/Z_1]\!]$-module $\pi^\vee$. Since the finitely generated A-module $D_A(\pi)$ doesn't depend up to isomorphism on the choice of this good filtration (see §3.1.2), it follows from Lemma 3.1.4.1 (applied to $M = D_A(\pi)$) and Lemma 3.1.1.1 (applied to $M = \pi^\vee$) that the generic rank of $\mathrm{gr}(A) \otimes_{\mathrm{gr}(\mathbb{F}[\![N_0]\!])} \mathrm{gr}(\pi^\vee)$ also doesn't depend on this choice.

Proposition 3.1.4.2. *Let* $\pi \in \mathcal{C}$. *Then* $\mathrm{rk}_A(D_A(\pi)^{\text{ét}}) = \dim_{\mathbb{F}(\!(X)\!)} D_\xi^\vee(\pi)$ *is bounded by the generic rank of the* $\mathrm{gr}(A)$*-module* $\mathrm{gr}(A) \otimes_{\mathrm{gr}(\mathbb{F}[\![N_0]\!])} \mathrm{gr}(\pi^\vee)$.

Proof. As $D_A(\pi)^{\text{ét}}$ is a quotient of $D_A(\pi)$, the result follows from Lemma 3.1.4.1, Lemma 3.1.1.1 and Theorem 3.1.3.7. □

When $\mathrm{gr}(\pi^\vee)$ is moreover killed by the ideal J^n for some $n \geq 1$ (here J is as in (118) and recall this doesn't depend on the good filtration), the generic rank of $\mathrm{gr}(A) \otimes_{\mathrm{gr}(\mathbb{F}[\![N_0]\!])} \mathrm{gr}(\pi^\vee)$ has a nice and useful interpretation that we give now.

We define $\overline{R} \stackrel{\text{def}}{=} \mathrm{gr}(\mathbb{F}[\![I_1/Z_1]\!])/J$. Recall using (117) that we have

$$\overline{R} \cong \mathbb{F}[y_i, z_i, 0 \leq i \leq f-1]/(y_i z_i, 0 \leq i \leq f-1). \tag{122}$$

Therefore $\overline{R}$ has 2^f minimal prime ideals which are the ideals $(y_i, z_j, i \in \mathcal{J}, j \notin \mathcal{J})$ with $\mathcal{J}$ a subset of $\{0, \dots, f-1\}$. Let

$$\mathfrak{p}_0 \stackrel{\text{def}}{=} (z_j, 0 \leq j \leq f-1)$$

be the minimal prime ideal corresponding to the choice of $\mathcal{J} = \emptyset$.

If N is a finitely generated module over $\overline{R}$ and $\mathfrak{q}$ is a minimal prime ideal of $\overline{R}$, we denote by $m_{\mathfrak{q}}(N)$ the length of $N_{\mathfrak{q}}$ over $\overline{R}_{\mathfrak{q}}$. More generally, if N is a finitely generated $\mathrm{gr}(\mathbb{F}[\![I_1/Z_1]\!])$-module annihilated by J^n for some $n \geq 1$, we define the multiplicity of N at $\mathfrak{q}$ to be

$$m_{\mathfrak{q}}(N) = \sum_{i=0}^{n-1} m_{\mathfrak{q}}(J^i N / J^{i+1} N). \tag{123}$$

Lemma 3.1.4.3. *If* $0 \to N_1 \to N \to N_2 \to 0$ *is a short exact sequence of finitely generated* $\mathrm{gr}(\mathbb{F}[\![I_1/Z_1]\!])/J^n$*-modules, then* $m_{\mathfrak{q}}(N) = m_{\mathfrak{q}}(N_1) + m_{\mathfrak{q}}(N_2)$.

Proof. This is checked by a standard dévissage. If $n = 1$, the statement is obvious since $\mathrm{gr}(\mathbb{F}[\![I_1/Z_1]\!])/J = \overline{R}$ is commutative (and noetherian). Assume $n \geq 2$ and by induction we assume that the result holds if N is annihilated by J^{n-1}.

Assume first that N_1 and N_2 are both annihilated by J^{n-1} (but not necessarily N). Then N_2 is a quotient of $N/J^{n-1}N$. Let $\mathrm{Ker} \stackrel{\text{def}}{=} \mathrm{Ker}(N/J^{n-1}N \twoheadrightarrow N_2)$ be the corresponding kernel. Then we have two short exact sequences

$$0 \to \mathrm{Ker} \to N/J^{n-1}N \to N_2 \to 0$$

$$0 \to J^{n-1}N \to N_1 \to \mathrm{Ker} \to 0. \tag{124}$$

By definition of $m_{\mathfrak{q}}(N)$ and the inductive hypothesis, we then obtain

$$m_{\mathfrak{q}}(N) = m_{\mathfrak{q}}(J^{n-1}N) + m_{\mathfrak{q}}(N/J^{n-1}N) = m_{\mathfrak{q}}(N_1) + m_{\mathfrak{q}}(N_2).$$

Assume now that N_2 is annihilated by J^{n-1} (but not necessarily for N_1). Then the surjection $N \twoheadrightarrow N_2$ factors through the quotient $N/J^{n-1}N$ of N. Again let $\mathrm{Ker} \stackrel{\text{def}}{=} \mathrm{Ker}(N/J^{n-1}N \twoheadrightarrow N_2)$. Then $m_{\mathfrak{q}}(N/J^{n-1}N) = m_{\mathfrak{q}}(\mathrm{Ker}) + m_{\mathfrak{q}}(N_2)$ by the inductive hypothesis. On the other hand, both $J^{n-1}N$ and Ker are annihilated by J^{n-1}, thus $m_{\mathfrak{q}}(\cdot)$ is additive for the short exact sequence (124) by the discussion in last paragraph. The result also holds in this case.

To finish the proof it suffices to decompose further N as $0 \to \mathrm{Ker}' \to N \to N_2/J^{n-1}N_2 \to 0$, with Ker' sitting in the exact sequence $0 \to N_1 \to \mathrm{Ker}' \to J^{n-1}N_2 \to 0$, and apply the above discussion. □

If N is a finitely generated module over $\mathrm{gr}(\mathbb{F}[\![I_1/Z_1]\!])/J^n$ for some $n \geq 1$ recall that the $\mathrm{gr}(A)$-module $\mathrm{gr}(A) \otimes_{\mathrm{gr}(\mathbb{F}[\![N_0]\!])} N$ is finitely generated by Proposition 3.1.2.11.

Lemma 3.1.4.4. *Let* N *be a finitely generated module over* $\mathrm{gr}(\mathbb{F}[\![I_1/Z_1]\!])/J^n$ *for some* $n \geq 1$. *Then the generic rank of the* $\mathrm{gr}(A)$*-module* $\mathrm{gr}(A) \otimes_{\mathrm{gr}(\mathbb{F}[\![N_0]\!])} N$ *is equal to* $m_{\mathfrak{p}_0}(N)$.

Proof. By Corollary 3.1.1.2, $\mathrm{gr}(A)$ is flat over $\mathrm{gr}(\mathbb{F}[\![N_0]\!])$, so $\mathrm{gr}(A) \otimes_{\mathrm{gr}(\mathbb{F}[\![N_0]\!])} N$ has a finite filtration with graded pieces given by $\mathrm{gr}(A) \otimes_{\mathrm{gr}(\mathbb{F}[\![N_0]\!])} (J^iN/J^{i+1}N)$ for $0 \leq i \leq n-1$. Since taking generic rank and taking $m_{\mathfrak{p}_0}(\cdot)$ are both additive in short exact sequences (by Lemma 3.1.4.3 for the latter), we are reduced to the case where N is killed by J.

In that case we have

$$\mathrm{gr}(A) \otimes_{\mathrm{gr}(\mathbb{F}[\![N_0]\!])} N \cong (\mathrm{gr}(A) \otimes_{\mathrm{gr}(\mathbb{F}[\![N_0]\!])} \overline{R}) \otimes_{\overline{R}} N.$$

Since the image of $\mathrm{gr}(\mathbb{F}[\![N_0]\!])$ in $\overline{R}$ is $\mathbb{F}[y_0,\dots,y_{f-1}]$, we have

$$\mathrm{gr}(A)\otimes_{\mathrm{gr}(\mathbb{F}[\![N_0]\!])}\overline{R}\cong\overline{R}[(y_0\cdots y_{f-1})^{-1}]\cong\mathrm{gr}(A).$$

Since the fraction field of $\overline{R}[(y_0\cdots y_{f-1})^{-1}]$ is just $\overline{R}_{\mathfrak{p}_0}$, we see that the generic rank of the $\overline{R}[(y_0\cdots y_{f-1})^{-1}]$-module $\mathrm{gr}(A)\otimes_{\mathrm{gr}(\mathbb{F}[\![N_0]\!])}N$ is equal to $m_{\mathfrak{p}_0}(N)$. □

We finally deduce from Proposition 3.1.4.2 and Lemma 3.1.4.4:

Corollary 3.1.4.5. *Let π be an admissible smooth representation of $\mathrm{GL}_2(K)$ over $\mathbb{F}$ with a central character having at least one good filtration such that the $\mathrm{gr}(\mathbb{F}[\![I_1/Z_1]\!])$-module $\mathrm{gr}(\pi^\vee)$ is killed by some power of J. Then we have*

$$\mathrm{rk}_A(D_A(\pi)^{\text{ét}})=\dim_{\mathbb{F}(\!(X)\!)}D_\xi^\vee(\pi)\le m_{\mathfrak{p}_0}(\mathrm{gr}(\pi^\vee)).$$

3.2 Tensor induction for $\mathrm{GL}_2(\mathbb{Q}_{p^f})$

We prove that $V_{\mathrm{GL}_2}(\pi)$ (as defined in (16)) contains some copies of a tensor induction as in Example 2.1.2.1 for certain admissible smooth representations π of $\mathrm{GL}_2(K)$ over $\mathbb{F}$ (Theorem 3.2.1.1).

We recall that the definition of the functor V_{GL_2} depends on the choice of a cocharacter ξ_{GL_2}, which we have fixed to be $\xi_{\mathrm{GL}_2}(x)=\mathrm{diag}(x,1)$, and depends on a normalizing character $\delta_{\mathrm{GL}_2}=\mathrm{ind}_K^{\otimes\mathbb{Q}_p}(\omega)$ (cf. Example 2.1.1.3).

3.2.1 Lower bound for $V_{\mathrm{GL}_2}(\pi)$: statement

We state the main theorem of this section on $V_{\mathrm{GL}_2}(\pi)$ for certain admissible smooth representations π of $\mathrm{GL}_2(K)$ over $\mathbb{F}$ (Theorem 3.2.1.1). After some simple reductions, this theorem will be proved in §§3.2.2 to 3.2.4.

We keep all the previous notation and denote by I_K the inertia subgroup of $\mathrm{Gal}(\overline{\mathbb{Q}}_p/K)$. We fix an embedding $\sigma'_0:\mathbb{F}_{p^{2f}}\hookrightarrow\mathbb{F}$ such that $\sigma'_0|_{\mathbb{F}_{p^f}}=\sigma_0$ (see the very beginning of §3), and denote by $\omega_f,\omega_{2f}:I_K\to\mathbb{F}^\times$ Serre's corresponding fundamental characters of level f and $2f$.

We consider $\overline{\rho}:\mathrm{Gal}(\overline{\mathbb{Q}}_p/K)\to\mathrm{GL}_2(\mathbb{F})$ of the following form *up to twist*:

$$\overline{\rho}|_{I_K}\cong\begin{cases}\omega_f^{\sum_{j=0}^{f-1}(r_j+1)p^j}\oplus 1 & \text{if } \overline{\rho} \text{ is reducible,}\\ \omega_{2f}^{\sum_{j=0}^{f-1}(r_j+1)p^j}\oplus\omega_{2f}^{\sum_{j=0}^{f-1}(r_j+1)p^{j+f}} & \text{if } \overline{\rho} \text{ is irreducible,}\end{cases}\tag{125}$$

where the integers r_i satisfy the following (strong) genericity condition:

$$\begin{aligned} 2f-1 \leq r_j \leq p-2-2f \qquad & \text{if } j>0 \text{ or } \overline{\rho} \text{ is reducible,} \\ 2f \leq r_0 \leq p-1-2f \qquad & \text{if } \overline{\rho} \text{ is irreducible} \end{aligned} \tag{126}$$

(note that this implies in particular $p \geq 4f+1$). Let $\chi : \mathrm{Gal}(\overline{\mathbb{Q}}_p/K) \to \mathbb{F}^\times$ such that $(\overline{\rho} \otimes \chi)|_{I_K}$ is as in (125) and moreover $\det(\overline{\rho} \otimes \chi) = \omega_f^{\sum_j (r_j+1)p^j}$.

We refer to [Paš04] and [BP12, §§9,13] (and the references therein) for the background and definitions about *diagrams*.

We choose *one* diagram $D(\overline{\rho} \otimes \chi) = (D_1 \hookrightarrow D_0)$ associated to $\overline{\rho} \otimes \chi$ in [Bre11, §5], and we set

$$D(\overline{\rho}) = (D_1(\overline{\rho}) \hookrightarrow D_0(\overline{\rho})) \stackrel{\text{def}}{=} \Big(D_1 \otimes (\chi^{-1} \circ \det) \hookrightarrow D_0 \otimes (\chi^{-1} \circ \det)\Big), \tag{127}$$

where the actions of $\mathrm{GL}_2(\mathcal{O}_K)$ and the center $K^\times$ on $D_0(\overline{\rho})$ (resp. of I, $\left(\begin{smallmatrix} 0 & 1 \\ p & 0 \end{smallmatrix}\right)$ and $K^\times$ on $D_1(\overline{\rho})$) are multiplied by $\chi^{-1} \circ \det$ via local class field theory for K (note that χ is trivial on K_1 and I_1 and recall that $\left(\begin{smallmatrix} 0 & 1 \\ p & 0 \end{smallmatrix}\right)$ normalizes I and I_1). Recall that the action of $\mathrm{GL}_2(\mathcal{O}_K)$ on $D_0(\overline{\rho})$ factors through $\mathrm{GL}_2(\mathcal{O}_K) \twoheadrightarrow \mathrm{GL}_2(\mathbb{F}_q)$. More precisely, denoting by $W(\overline{\rho})$ the set of Serre weights of $\overline{\rho}$ defined in [BDJ10, §3], $D_0(\overline{\rho})$ is the (unique) maximal finite-dimensional representation of $\mathrm{GL}_2(\mathbb{F}_q)$ over $\mathbb{F}$ with socle isomorphic to $\oplus_{\sigma \in W(\overline{\rho})} \sigma$ such that each $\sigma \in W(\overline{\rho})$ occurs with multiplicity 1 in $D_0(\overline{\rho})$. (For instance, recall that the Serre weight $(r_0, r_1, \dots, r_{f-1})$ with the notation of (129) below is in $W(\overline{\rho})$ if $\overline{\rho}$ is an extension of 1 by $\omega_f^{\sum_{j=0}^{f-1}(r_j+1)p^j}$.) Finally $K^\times$ acts on $D_0(\overline{\rho})$ by the character $\det(\overline{\rho})\omega^{-1}$.

If π is an admissible smooth representation of $\mathrm{GL}_2(K)$ over $\mathbb{F}$, recall that $(\pi^{I_1} \hookrightarrow \pi^{K_1})$ is naturally a diagram. We aim to prove the following theorem.

Theorem 3.2.1.1. *Let π be an admissible smooth representation of $\mathrm{GL}_2(K)$ over $\mathbb{F}$. Assume that there exists an integer $r \geq 1$ such that one has an isomorphism of diagrams*

$$D(\overline{\rho})^{\oplus r} \xrightarrow{\sim} (\pi^{I_1} \hookrightarrow \pi^{K_1}).$$

Then one has an $I_{\mathbb{Q}_p}$-equivariant injection $\Big(\mathrm{ind}_K^{\otimes \mathbb{Q}_p}(\overline{\rho})\Big)|_{I_{\mathbb{Q}_p}}^{\oplus r} \hookrightarrow V_{\mathrm{GL}_2}(\pi)|_{I_{\mathbb{Q}_p}}$. If we assume moreover that the constants ν_i associated to $D(\overline{\rho} \otimes \chi)$ at the beginning of [Bre11, §6] *are as in* [Bre11, Thm.6.4]*, then one has a* $\mathrm{Gal}(\overline{\mathbb{Q}}_p/\mathbb{Q}_p)$*-equivariant injection $\Big(\mathrm{ind}_K^{\otimes \mathbb{Q}_p}(\overline{\rho})\Big)^{\oplus r} \hookrightarrow V_{\mathrm{GL}_2}(\pi)$.*

Let us first make some straightforward reductions. In order not to repeat arguments, we assume from now on that the constants ν_i associated to $D(\overline{\rho} \otimes \chi)$ in [Bre11, §6] are as in [Bre11, Thm.6.4] and we will prove the last statement of Theorem 3.2.1.1

(the proof for the first one being the same up to some trivial modifications). It is enough to prove Theorem 3.2.1.1 for the $\mathrm{GL}_2(K)$-subrepresentation of π generated by $D_0(\overline{\rho})^{\oplus r}$. Hence we can assume that π has a central character which is $\chi_\pi \overset{\text{def}}{=} \det(\overline{\rho})\omega^{-1}$. Using Remark 2.1.1.4(ii) (for $n=2$), it is also enough to prove Theorem 3.2.1.1 for $\overline{\rho}\otimes\chi$ as above and replacing π by $\pi\otimes\chi\circ\det$, i.e. we can assume $\overline{\rho}|_{I_K}$ is as in (125) and $\det(\overline{\rho}) = \omega_f^{\sum_j (r_j+1)p^j}$.

In the sequel, for any $\mathbb{F}[\![X]\!][F]$-submodule M of π^{N_1} which is stable under $\mathbb{Z}_p^\times$, denote by $M\otimes\chi_\pi^{-1}$ the same $\mathbb{F}[\![X]\!]$-module but where the action of F is multiplied by $\chi_\pi(p)^{-1}$ and the action of $x\in\mathbb{Z}_p^\times$ is multiplied by $\chi_\pi(x)^{-1}$.

Lemma 3.2.1.2. *With the notation in §2.1.1, in order to prove Theorem 3.2.1.1 it is enough to prove that $(\pi\otimes\chi_\pi^{-1})^{N_1}$ contains a finite type $\mathbb{F}[\![X]\!][F]$-submodule M which is admissible as an $\mathbb{F}[\![X]\!]$-module and stable under $\mathbb{Z}_p^\times$ such that $\mathbf{V}(M^\vee[1/X]) \cong \Big(\mathrm{ind}_K^{\otimes\mathbb{Q}_p}(\overline{\rho})\Big)^{\oplus r}$.*

Proof. As $(\pi\otimes\chi_\pi^{-1})^{N_1} = \pi^{N_1}$ as $\mathbb{F}$-vector subspaces of π, it is equivalent to assume that π^{N_1} contains a finite type $\mathbb{F}[\![X]\!][F]$-submodule M which is admissible as an $\mathbb{F}[\![X]\!]$-module and stable under $\mathbb{Z}_p^\times$ such that $\mathbf{V}((M\otimes\chi_\pi^{-1})^\vee[1/X]) \cong \Big(\mathrm{ind}_K^{\otimes\mathbb{Q}_p}(\overline{\rho})\Big)^{\oplus r}$. From the definition of V_{GL_2} in (16), it is enough to prove $\mathbf{V}^\vee(M^\vee[1/X])\otimes\delta_{\mathrm{GL}_2} \cong \Big(\mathrm{ind}_K^{\otimes\mathbb{Q}_p}(\overline{\rho})\Big)^{\oplus r}$. From Example 2.1.1.3 and as in Remark 2.1.1.4(ii) (both for $n=2$), we have

$$\begin{aligned}
\mathbf{V}^\vee(M^\vee[1/X])\otimes\delta_{\mathrm{GL}_2} &= \mathbf{V}\Big((M\otimes\chi_\pi^{-1})^\vee[1/X]\Big)^\vee \otimes (\chi_\pi\omega)|_{\mathbb{Q}_p^\times}\\
&= \Big(\Big(\mathrm{ind}_K^{\otimes\mathbb{Q}_p}(\overline{\rho})\Big)^{\oplus r}\Big)^\vee \otimes \mathrm{ind}_K^{\otimes\mathbb{Q}_p}\Big(\det(\overline{\rho})\Big)\\
&\overset{(22)}{=} \Big(\mathrm{ind}_K^{\otimes\mathbb{Q}_p}(\overline{\rho})\Big)^{\oplus r}
\end{aligned}$$

which finishes the proof. $\square$

The sections that follow will be devoted to the proof that there exists a certain finite type $\mathbb{F}[\![X]\!][F]$-submodule M_π of π^{N_1} which is admissible as an $\mathbb{F}[\![X]\!]$-module and stable under $\mathbb{Z}_p^\times$ such that $\mathbf{V}((M_\pi\otimes\chi_\pi^{-1})^\vee[1/X]) \cong \Big(\mathrm{ind}_K^{\otimes\mathbb{Q}_p}(\overline{\rho})\Big)^{\oplus r}$ (see Proposition 3.2.4.6). Note that the assumption $\det(\overline{\rho}) = \omega_f^{\sum_j (r_j+1)p^j}$ implies $\chi_\pi(p)=1$, so that the operator F on $M_\pi\otimes\chi_\pi^{-1}$ is the same as on M_π, but the action of $\gamma\in\mathbb{Z}_p^\times$ now comes from the action of $\begin{pmatrix}1&0\\0&\gamma^{-1}\end{pmatrix}$ on π^{N_1}.

3.2.2 Preliminaries

We give some technical results on $\mathbb{F}[\![N_0]\!]$, $\mathbb{F}[\![N_0/N_1]\!]$ and on certain modules over these rings coming from Serre weights.

We let $H \overset{\text{def}}{=} \begin{pmatrix} \mathbb{F}_q^\times & 0 \\ 0 & \mathbb{F}_q^\times \end{pmatrix} \cong I/I_1 \subseteq \mathrm{GL}_2(\mathbb{F}_q)$ (this finite group H shouldn't be confused with the algebraic group H in §2.1.1 or in §2.1). Note that the trace $\mathrm{Tr}_{K/\mathbb{Q}_p} : \mathcal{O}_K \to \mathbb{Z}_p$ is surjective (using that K is unramified) hence directly induces an isomorphism $N_0/N_1 \xrightarrow{\sim} \mathbb{Z}_p$. Recall we defined the elements Y_i for $i \in \{0, \dots, f-1\}$ in (101). We define analogously

$$Y \overset{\text{def}}{=} \sum_{a \in \mathbb{F}_p^\times} a^{-1} \begin{pmatrix} 1 & \tilde{a} \\ 0 & 1 \end{pmatrix} \in \mathbb{F}[\![\mathbb{Z}_p]\!] = \mathbb{F}[\![N_0/N_1]\!].$$

We write $\underline{i}$ for an element $(i_0, \dots, i_{f-1})$ in $\mathbb{Z}^f$, $\underline{Y}^{\underline{i}}$ for $Y_0^{i_0} \cdots Y_{f-1}^{i_{f-1}}$ and set $\|\underline{i}\| \overset{\text{def}}{=} \sum_{j=0}^{f-1} i_j$. We also write $\underline{i} \le \underline{i}'$ to mean $i_j \le i'_j$ for all $0 \le j \le f-1$.

Lemma 3.2.2.1. *We have the following isomorphisms and equalities:*

(i) $\mathbb{F}[\![N_0]\!] \cong \mathbb{F}[\![Y_0, \dots, Y_{f-1}]\!]$ *and* $\mathbb{F}[N_0/N_0^p] \cong \mathbb{F}[\![Y_0, \dots, Y_{f-1}]\!]/(Y_0^p, \dots, Y_{f-1}^p)$;

(ii) $Y_i^p \begin{pmatrix} p & 0 \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} p & 0 \\ 0 & 1 \end{pmatrix} Y_{i+1}$ *and* $\begin{pmatrix} \tilde{\lambda} & 0 \\ 0 & \tilde{\mu} \end{pmatrix} Y_i = (\lambda\mu^{-1})^{p^i} Y_i \begin{pmatrix} \tilde{\lambda} & 0 \\ 0 & \tilde{\mu} \end{pmatrix}$ *for* $\lambda, \mu \in \mathbb{F}_q^\times$;

(iii) $\mathbb{F}[\![N_0/N_1]\!] \cong \mathbb{F}[\![Y]\!]$ *and* $\begin{pmatrix} \tilde{\lambda} & 0 \\ 0 & \tilde{\mu} \end{pmatrix} Y = (\lambda\mu^{-1}) Y \begin{pmatrix} \tilde{\lambda} & 0 \\ 0 & \tilde{\mu} \end{pmatrix}$ *for* $\lambda, \mu \in \mathbb{F}_p^\times$.

Proof. Note that $\mathbb{F}[N_0/N_0^p] \cong \mathbb{F}\left[\begin{pmatrix} 1 & \mathbb{F}_q \\ 0 & 1 \end{pmatrix}\right]$. The first equality in (i) and the explicit action of $\begin{pmatrix} \tilde{\lambda} & 0 \\ 0 & \tilde{\mu} \end{pmatrix}$ on $\underline{Y}^{\underline{i}}$ in (ii) are immediately obtained from [Mor17, Lemma 3.2] (after conjugating by the element $\begin{pmatrix} 0 & 1 \\ p & 0 \end{pmatrix}$). The second equality in (i) follows from the first by dimension reasons, as $Y_i^p = 0$ in $\mathbb{F}[N_0/N_0^p]$. The action of $\begin{pmatrix} p & 0 \\ 0 & 1 \end{pmatrix}$ on Y_{i+1} in (ii) is a direct computation (see also [Mor17, Lemma 5.1]). Finally, (iii) is a special case of (i) and (ii). □

Note that $\mathbb{F}[\![N_0/N_1]\!] \cong \mathbb{F}[\![X]\!] \cong \mathbb{F}[\![Y]\!]$ with $X = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix} - 1$ as in §2.1.1, but it is more convenient in the computations to use the "H-eigenvariable" Y rather than the variable X. To compare them the following lemma will be useful.

Lemma 3.2.2.2. *We have* $X \in -Y(1 + Y\mathbb{F}[\![Y]\!])$ *and* $Y \in -X(1 + X\mathbb{F}[\![X]\!])$ *in* $\mathbb{F}[\![N_0/N_1]\!]$.

Proof. Equivalently, we have to prove $Y = -X$ in $\mathfrak{m}/\mathfrak{m}^2$, where $\mathfrak{m}$ is the maximal ideal of $\mathbb{F}[\![N_0/N_1]\!]$. We can work modulo $\mathfrak{m}^p$, i.e. in $\mathbb{F}[N_0/N_1N_0^p] \cong \mathbb{F}\left[\begin{pmatrix} 1 & \mathbb{F}_p \\ 0 & 1 \end{pmatrix}\right]$. In that group ring we have

$$Y = \sum_{a \in \mathbb{F}_p^\times} a^{-1} \begin{pmatrix} 1 & a \\ 0 & 1 \end{pmatrix} = \sum_{a=1}^{p-1} a^{-1}(1+X)^a \equiv -X$$

(where the last congruence is taken modulo the image of $\mathfrak{m}^2$ in that group ring). □

For $\lambda, \mu \in \mathbb{F}_q^\times$ we set

$$\alpha\Big(\big(\begin{smallmatrix} \tilde{\lambda} & 0 \\ 0 & \tilde{\mu} \end{smallmatrix}\big)\Big) \stackrel{\text{def}}{=} \lambda\mu^{-1} \in \mathbb{F}^\times.$$

Remark 3.2.2.3. By Lemma 3.2.2.1(ii), if V is a representation of $\mathrm{GL}_2(\mathbb{F}_q)$ and $v \in V^{H=\chi}$, then $\underline{Y}^{\underline{i}}v \in V^{H=\chi\alpha^{\underline{i}}}$, where $\alpha^{\underline{i}} \stackrel{\text{def}}{=} \alpha^{\sum_{j=0}^{f-1} i_j p^j}$.

Lemma 3.2.2.4. *Assume $p > 2$. The kernel of the map $h : \mathbb{F}[\![N_0]\!] \twoheadrightarrow \mathbb{F}[\![N_0/N_1]\!]$ is generated by the elements $Y_i - Y_j$ ($i \neq j$). Moreover, there exists $f(Y) \in \mathbb{F}[\![N_0/N_1]\!] \cong \mathbb{F}[\![Y]\!]$ such that $h(Y_i) = Y + Y^p f(Y)$.*

Proof. Note that $\mathrm{Tr}_{K/\mathbb{Q}_p}(\tilde{\lambda}^{p^i}) = \mathrm{Tr}_{K/\mathbb{Q}_p}(\tilde{\lambda})$ for all $\lambda \in \mathbb{F}_q^\times$ and $i \in \mathbb{Z}$, hence $Y_i - Y_j \in \mathrm{Ker}(h)$. As $\mathbb{F}[\![N_0]\!]/(Y_i - Y_j, i \neq j)$ and $\mathbb{F}[\![N_0/N_1]\!]$ are both isomorphic to power series rings in one variable, the quotient map $\mathbb{F}[\![N_0]\!]/(Y_i - Y_j, i \neq j) \twoheadrightarrow \mathbb{F}[\![N_0/N_1]\!]$ has to be an isomorphism. To establish the final claim it suffices to prove that the image of Y_0 in $\mathbb{F}[\![Y]\!]/(Y^p) \cong \mathbb{F}[N_0/N_1N_0^p] \cong \mathbb{F}\Big[\big(\begin{smallmatrix} 1 & \mathbb{F}_p \\ 0 & 1 \end{smallmatrix}\big)\Big]$ is Y. We compute

$$\sum_{\lambda \in \mathbb{F}_q^\times} \lambda^{-1} \begin{pmatrix} 1 & \mathrm{Tr}_{\mathbb{F}_q/\mathbb{F}_p}(\lambda) \\ 0 & 1 \end{pmatrix} = \sum_{a \in \mathbb{F}_p} \Bigg(\sum_{\substack{\lambda \in \mathbb{F}_q^\times \\ \mathrm{Tr}_{\mathbb{F}_q/\mathbb{F}_p}(\lambda) = a}} \lambda^{-1} \Bigg) \begin{pmatrix} 1 & a \\ 0 & 1 \end{pmatrix}. \tag{128}$$

If $a \neq 0$, the index λ runs over the distinct roots of $Y^{p^{f-1}} + Y^{p^{f-2}} + \cdots + Y - a = 0$, so the inside sum on the right hand side of (128) equals $1/a$ (from the last two coefficients). If $a = 0$ the index λ runs over the distinct roots of $Y^{p^{f-1}-1} + \cdots + Y^{p-1} + 1 = 0$, so the inside sum in (128) equals 0 as $p > 2$. Hence the right-hand side of (128) is just Y. □

By Lemma 3.2.2.4, if V is a representation of $\mathrm{GL}_2(\mathbb{F}_q)$, then $Y_i = Y$ on V^{N_1}.

For $0 \leq i \leq q-1$, we set

$$\theta_i \stackrel{\text{def}}{=} \sum_{\lambda \in \mathbb{F}_q} \lambda^i \begin{pmatrix} 1 & \lambda \\ 0 & 1 \end{pmatrix} \in \mathbb{F}[N_0/N_0^p] \cong \mathbb{F}\Big[\big(\begin{smallmatrix} 1 & \mathbb{F}_q \\ 0 & 1 \end{smallmatrix}\big)\Big].$$

So $Y_i = \theta_{q-1-p^i}$ in $\mathbb{F}[N_0/N_0^p]$. In what follows we write $\underline{p-1}$ for the constant f-tuple $(p-1, p-1, \ldots, p-1) \in \mathbb{Z}^f$.

Lemma 3.2.2.5. *Suppose $\underline{i} \in \{0, \ldots, p-1\}^f$ and let $i \stackrel{\text{def}}{=} \sum_{j=0}^{f-1} i_j p^j$.*

(i) *We have*

$$\theta_i = (-1)^{f-1} \Bigg(\prod_{j=0}^{f-1} i_j! \Bigg) \underline{Y}^{\underline{p-1}-\underline{i}}$$

in $\mathbb{F}[N_0/N_0^p]$ for $0 \leq i < q-1$.

(ii) *For $f_0, \dots, f_{q-1}$ and ϕ as defined in* [BP12, §2] *we have*

$$f_i = (-1)^{f-1}\Big(\prod_{j=0}^{f-1} i_j!\Big)\underline{Y}^{\underline{p-1}-\underline{i}}\left(\begin{smallmatrix}0&1\\1&0\end{smallmatrix}\right)\phi$$

for $0 \le i < q-1$.

Proof. Part (i) follows from [Mor, Lemma 0.2] after conjugation by $\left(\begin{smallmatrix}0&1\\p&0\end{smallmatrix}\right)$. Indeed, in the notation of *loc.cit.* we take $m = n = 1$ (so that $A_{1,1}$ is the group algebra of $\left(\begin{smallmatrix}1&0\\p\mathcal{O}_K/p^2\mathcal{O}_K&1\end{smallmatrix}\right)$): we see that θ_i corresponds (under conjugation) to $F_{\underline{i}}$ if $0 \le i \le q-1$, and the constant $\kappa_{\underline{p-1}-\underline{i}}$ equals $(-1)^{f-1}\Big(\prod_{j=0}^{f-1} i_j!\Big)^{-1}$. Part (ii) follows immediately from (i) and the definition of θ_i. □

As in [BP12] we write $(s_0, s_1, \dots, s_{f-1}) \otimes \eta$ for the Serre weight

$$\mathrm{Sym}^{s_0}\mathbb{F}^2 \otimes_{\mathbb{F}} (\mathrm{Sym}^{s_1}\mathbb{F}^2)^{\mathrm{Fr}} \otimes \cdots \otimes_{\mathbb{F}} (\mathrm{Sym}^{s_{f-1}}\mathbb{F}^2)^{\mathrm{Fr}^{f-1}} \otimes_{\mathbb{F}} \eta \circ \det, \tag{129}$$

where the s_i are integers between 0 and $p-1$, η is a character $\mathbb{F}_q^\times \to \mathbb{F}^\times$ and $\mathrm{GL}_2(\mathbb{F}_q)$ acts on $(\mathrm{Sym}^{s_i}\mathbb{F}^2)^{\mathrm{Fr}^i}$ via $\sigma_i : \mathbb{F}_q \hookrightarrow \mathbb{F}$. If $\chi = \chi_1 \otimes \chi_2$ is a character of $H = \left(\begin{smallmatrix}\mathbb{F}_q^\times&0\\0&\mathbb{F}_q^\times\end{smallmatrix}\right)$, we let $\chi^s \overset{\text{def}}{=} \chi_2 \otimes \chi_1$.

Lemma 3.2.2.6. *Let $\sigma \overset{\text{def}}{=} (s_0, \dots, s_{f-1}) \otimes \eta$, $\underline{s} \overset{\text{def}}{=} (s_0, s_1, \dots, s_{f-1}) \in \{0, \dots, p-1\}^f$, and fix $v \in \sigma^{N_0}$, $v \ne 0$. Let χ_σ denote the H-eigencharacter on σ^{N_0}.*

(i) *The $\mathbb{F}[\![N_0/N_1]\!] = \mathbb{F}[\![Y]\!]$-module σ^{N_1} is cyclic of dimension* $\min\{s_0, \dots, s_{f-1}\}+1$.

(ii) *If $\underline{0} \le \underline{i} \le \underline{s}$ and $\underline{i} < \underline{p-1}$ then σ contains a unique H-eigenvector, which we call $\underline{Y}^{-\underline{i}}v$, that is sent by $\underline{Y}^{\underline{i}}$ to v. The corresponding H-eigencharacter is $\chi_\sigma\alpha^{-\underline{i}}$. Also, $Y_j\underline{Y}^{-\underline{i}}v = 0$ if $i_j = 0$.*

(iii) *If $0 \le i \le \min\{s_0, \dots, s_{f-1}\}$ and $i < p-1$, then σ^{N_1} contains a unique $\left(\begin{smallmatrix}\mathbb{F}_p^\times&0\\0&\mathbb{F}_p^\times\end{smallmatrix}\right)$-eigenvector $Y^{-i}v$ that is sent by Y^i to v. The corresponding eigencharacter is $\chi_\sigma\alpha^{-i}$. We have $Y^{-i}v = \sum_{\underline{i}, \|\underline{i}\|=i}\underline{Y}^{-\underline{i}}v$.*

Proof. (i) Note that σ^{N_1} is a torsion module over $\mathbb{F}[\![N_0/N_1]\!] = \mathbb{F}[\![Y]\!]$ as σ^{N_1} is finite-dimensional. To show cyclicity it suffices to note that $\sigma^{N_0} = \sigma^{N_1}[X]$ is 1-dimensional. Then from [Mor17, Prop.3.3] applied with $n = 1$ we have an isomorphism

$$\begin{aligned}\mathbb{F}[\![Y_0, \dots, Y_{f-1}]\!]/(Y_j^{s_j+1}, 0 \le j \le f-1) &\xrightarrow{\sim} \sigma\\ g(\underline{Y}) &\longmapsto g(\underline{Y})\left(\begin{smallmatrix}0&1\\1&0\end{smallmatrix}\right)v.\end{aligned} \tag{130}$$

(Restrict equation (9) in [Mor17] to $\left(\begin{smallmatrix} 1 & 0 \\ p\mathcal{O}_K & 1 \end{smallmatrix}\right)$ and conjugate by $\left(\begin{smallmatrix} 0 & 1 \\ p & 0 \end{smallmatrix}\right)$. Note that σ is self-dual up to twist.) In particular, $\{\underline{Y}^{\underline{k}}\left(\begin{smallmatrix} 0 & 1 \\ 1 & 0 \end{smallmatrix}\right)v : \underline{0} \le \underline{k} \le \underline{s}\}$ is a basis of σ consisting of H-eigenvectors.

Let $m \overset{\text{def}}{=} \min\{s_0, \dots, s_{f-1}\}$. We claim that the vectors

$$v_i \overset{\text{def}}{=} \sum_{\substack{\underline{0} \le \underline{k} \le \underline{s} \\ \|\underline{k}\| = \|\underline{s}\| - i}} \underline{Y}^{\underline{k}}\left(\begin{smallmatrix} 0 & 1 \\ 1 & 0 \end{smallmatrix}\right)v, \qquad 0 \le i \le m \tag{131}$$

form a basis of σ^{N_1}. If $i < m$ and $\|\underline{k}\| = \|\underline{s}\| - i$, then $k_j > 0$ for all j. By using also (130) we see that $v_i = Y_j v_{i+1}$ for all j. Also, $Y_j v_0 = 0$ for all j. In particular, $Y_j - Y_{j'}$ annihilates v_i for all i, so $v_i \in \sigma^{N_1}$ by Lemma 3.2.2.4. Moreover, $Xv_{i+1} = v_i$ $(0 \le i < m)$ and $Xv_0 = 0$. It remains to show that $v_m \notin X\sigma^{N_1}$. Choose j_0 such that $s_{j_0} = m$. Then $\prod_{j \ne j_0} Y_j^{s_j}\left(\begin{smallmatrix} 0 & 1 \\ 1 & 0 \end{smallmatrix}\right)v$ is the only term appearing in the sum (131) for $i = m$ that is not divisible by Y_{j_0}. Hence $v_m \notin Y_{j_0}\sigma$, and thus $v_m \notin X\sigma^{N_1}$.

(ii) Let $v' \overset{\text{def}}{=} \underline{Y}^{\underline{s}}\left(\begin{smallmatrix} 0 & 1 \\ 1 & 0 \end{smallmatrix}\right)v$, which is a scalar multiple of v. By (130), $\left(\underline{Y}^{\underline{k}}\left(\begin{smallmatrix} 0 & 1 \\ 1 & 0 \end{smallmatrix}\right)v\right)_{\underline{0} \le \underline{k} \le \underline{s}}$ forms a basis of σ consisting of H-eigenvectors with eigencharacters $\chi_\sigma^s \alpha^{\underline{k}} = \chi_\sigma \alpha^{\underline{k} - \underline{r}}$. The eigencharacters are pairwise distinct, except if $\underline{s} = \underline{p-1}$ where $\underline{Y}^{\underline{p-1}}\left(\begin{smallmatrix} 0 & 1 \\ 1 & 0 \end{smallmatrix}\right)v$ and $\left(\begin{smallmatrix} 0 & 1 \\ 1 & 0 \end{smallmatrix}\right)v$ have the same eigencharacter. Hence, as $\underline{i} < \underline{p-1}$, the unique H-eigenvector in the preimage $(\underline{Y}^{\underline{i}})^{-1}(v')$ is $\underline{Y}^{\underline{s}-\underline{i}}\left(\begin{smallmatrix} 0 & 1 \\ 1 & 0 \end{smallmatrix}\right)v$. Note also that $Y_j \underline{Y}^{\underline{s}-\underline{i}}\left(\begin{smallmatrix} 0 & 1 \\ 1 & 0 \end{smallmatrix}\right)v = 0$ if $i_j = 0$ by (130).

(iii) Using the notation in (ii), we have $v_i = \sum_{\|\underline{i}\| = i} \underline{Y}^{-\underline{i}} v'$ for $0 \le i \le m$ and it is a $\left(\begin{smallmatrix} \mathbb{F}_p^\times & 0 \\ 0 & \mathbb{F}_p^\times \end{smallmatrix}\right)$-eigenvector with eigencharacter $\chi_\sigma \alpha^{-i}$. These characters for $0 \le i \le m$ are pairwise distinct, except if $\underline{s} = \underline{p-1}$, in which case v_0 and v_{p-1} have the same eigencharacter. As we assume $i < p-1$ the claim follows. ☐

Lemma 3.2.2.7. *Suppose V is a representation of $\mathrm{GL}_2(\mathbb{F}_q)$ generated by some vector $v \in V^{N_0}$ that is an eigenvector for the action of H. If $\dim_{\mathbb{F}} V \le q$, then the map*

$$\begin{aligned} \mathbb{F}[\![Y_0, \dots, Y_{f-1}]\!] &\longrightarrow V \\ f(\underline{Y}) &\mapsto f(\underline{Y})\left(\begin{smallmatrix} 0 & 1 \\ 1 & 0 \end{smallmatrix}\right)v \end{aligned}$$

is surjective and its kernel is generated by monomials. In particular, if $\underline{Y}^{\underline{i}}\left(\begin{smallmatrix} 0 & 1 \\ 1 & 0 \end{smallmatrix}\right)v = \underline{Y}^{\underline{j}}\left(\begin{smallmatrix} 0 & 1 \\ 1 & 0 \end{smallmatrix}\right)v \ne 0$, then $\underline{i} = \underline{j}$.

Proof. Let χ denote the eigencharacter of H on v. Then we have a $\mathrm{GL}_2(\mathbb{F}_q)$-equivariant surjection $S : \mathrm{Ind}_I^{\mathrm{GL}_2(\mathcal{O}_K)}(\chi) \twoheadrightarrow V$ sending ϕ to v, where ϕ is the unique function supported on I which sends 1 to 1. Consider $i : \mathbb{F}[Y_0, \dots, Y_{f-1}]/(Y_0^p, \dots, Y_{f-1}^p) \to \mathrm{Ind}_I^{\mathrm{GL}_2(\mathcal{O}_K)}(\chi)$ sending $f(\underline{Y})$ to $f(\underline{Y})\left(\begin{smallmatrix} 0 & 1 \\ 1 & 0 \end{smallmatrix}\right)\phi$. By Lemma 3.2.2.5, $f_j \in \mathrm{Im}(i)$ for all j

(even if $j = q-1$), so by [BP12, Lemma 2.5], $\mathrm{Ind}_I^{\mathrm{GL}_2(\mathcal{O}_K)}(\chi) = \mathrm{Im}(i) \oplus \mathbb{F}\phi$ (as $\mathbb{F}$-vector spaces) and i is injective.

Suppose first $\chi \not\cong \chi^s$. By [BP12, Lemma 2.7(i)] and as $\dim V \leq q$ we have $f_r \pm \phi \in \mathrm{Ker}(S)$ for some $r = \sum_{j=0}^{f-1} p^j s_j \in \{0, \dots, q-2\}$ and some sign $\pm$ (both depending on χ), so $S \circ i$ is surjective. If $\mathrm{Ker}(S)$ is irreducible (as a $\mathrm{GL}_2(\mathbb{F}_q)$-representation), then by [BP12, Lemma 2.7], $\mathrm{Ker}(S) = \langle f_{\sum p^j d_j}, 0 \leq d_j \leq s_j \text{ (not all equal)}, \ f_r \pm \phi \rangle_{\mathbb{F}}$. Intersecting with $\mathrm{Im}(i) = \langle f_{\sum p^j d_j}, 0 \leq d_j \leq p-1 \rangle_{\mathbb{F}}$ we get

$$\mathrm{Ker}(S) \cap \mathrm{Im}(i) = \Big\langle f_{\sum p^j d_j}, 0 \leq d_j \leq s_j \text{ (not all equal)} \Big\rangle_{\mathbb{F}}.$$

By Lemma 3.2.2.5(ii), it follows in particular that $\mathrm{Ker}(S \circ i)$ is generated by monomials. If $\mathrm{Ker}(S)$ is reducible, the argument is analogous using [BP12, Lemma 2.7(ii)]. If $\chi = \chi^s$, it is again almost identical, using [BP12, Lemma 2.6] instead. ☐

Lemma 3.2.2.8. *Suppose $f > 1$. In $\mathbb{F}[N_0/N_0^p]$ we have*

$$\sum_{\lambda \in \mathbb{F}_q, \mathrm{Tr}_{\mathbb{F}_q/\mathbb{F}_p}(\lambda) = 0} \left(\begin{smallmatrix} 1 & \lambda \\ 0 & 1 \end{smallmatrix}\right) = (-1)^{f-1} \Bigg(\underline{Y}^{\underline{p-1}} + \sum_{\substack{\|\underline{i}\| = (p-1)(f-1) \\ 0 \leq i_j \leq p-1}} \underline{Y}^{\underline{i}} \Bigg).$$

Proof. First we have (using $x^{p-1} = 1$ if $x \in \mathbb{F}_p^\times$):

$$\begin{aligned}
\sum_{\lambda \in \mathbb{F}_q, \mathrm{Tr}_{\mathbb{F}_q/\mathbb{F}_p}(\lambda) \neq 0} \left(\begin{smallmatrix} 1 & \lambda \\ 0 & 1 \end{smallmatrix}\right) &= \sum_{\lambda \in \mathbb{F}_q} (\mathrm{Tr}_{\mathbb{F}_q/\mathbb{F}_p}(\lambda))^{p-1} \left(\begin{smallmatrix} 1 & \lambda \\ 0 & 1 \end{smallmatrix}\right) \\
&= \sum_{\lambda \in \mathbb{F}_q} (\lambda + \lambda^p + \dots + \lambda^{p^{f-1}})^{p-1} \left(\begin{smallmatrix} 1 & \lambda \\ 0 & 1 \end{smallmatrix}\right) \\
&= \sum_{\lambda \in \mathbb{F}_q} \sum_{\substack{\underline{i} \in \mathbb{Z}_{\geq 0}^f \\ \|\underline{i}\| = p-1}} \frac{(p-1)!}{\prod_j i_j!} \lambda^{i_0 + i_1 p + \dots + i_{f-1} p^{f-1}} \left(\begin{smallmatrix} 1 & \lambda \\ 0 & 1 \end{smallmatrix}\right) \\
&= \sum_{\substack{\underline{i} \in \mathbb{Z}_{\geq 0}^f \\ \|\underline{i}\| = p-1}} \frac{(p-1)!}{\prod_j i_j!} (-1)^{f-1} \Bigg(\prod_j i_j! \Bigg) \underline{Y}^{\underline{p-1} - \underline{i}},
\end{aligned}$$

where the last equality follows from Lemma 3.2.2.5(i), noting that $\sum_{j=0}^{f-1} i_j p^j < q-1$ since $f > 1$. Letting $\underline{i}' \stackrel{\text{def}}{=} \underline{p-1} - \underline{i}$ we get (as $(p-1)! = -1$ in $\mathbb{F}_p$):

$$\sum_{\lambda \in \mathbb{F}_q, \mathrm{Tr}_{\mathbb{F}_q/\mathbb{F}_p}(\lambda) \neq 0} \left(\begin{smallmatrix} 1 & \lambda \\ 0 & 1 \end{smallmatrix}\right) = (-1)^f \sum_{\substack{\underline{i}' \in \mathbb{Z}_{\geq 0}^f \\ \|\underline{i}'\| = (p-1)(f-1)}} \underline{Y}^{\underline{i}'}.$$

On the other hand, Lemma 3.2.2.5(i) gives

$$\sum_{\lambda \in \mathbb{F}_q} \left(\begin{smallmatrix} 1 & \lambda \\ 0 & 1 \end{smallmatrix}\right) = (-1)^{f-1} \underline{Y}^{\underline{p-1}}.$$

The result follows. ☐

Proposition 3.2.2.9. *Fix $j_0 \in \{0, \dots, f-1\}$. In*

$$\mathbb{F}[\![N_0/N_1^p]\!] \cong \mathbb{F}[\![Y_0, \dots, Y_{f-1}]\!]/\big((Y_i - Y_j)^p, i \neq j\big)$$

we have

$$\sum_{n \in N_1/N_1^p} n = (-1)^{f-1} \prod_{j \neq j_0} (Y_j - Y_{j_0})^{p-1}$$

modulo terms of degree $\geq f(p-1)$.

Proof. The statement being trivial if $f = 1$, we can assume $f > 1$. We prove the first isomorphism. As $Y_i - Y_j \in \operatorname{Ker}\big(\mathbb{F}[\![N_0]\!] \to \mathbb{F}[\![N_0/N_1]\!]\big)$ by Lemma 3.2.2.4, we deduce that $(Y_i - Y_j)^p \in \operatorname{Ker}\big(\mathbb{F}[\![N_0]\!] \to \mathbb{F}[\![N_0/N_1^p]\!]\big)$, and we thus have a surjection $\mathbb{F}[\![Y_0, \dots, Y_{f-1}]\!]/\big((Y_i - Y_j)^p, i \neq j\big) \twoheadrightarrow \mathbb{F}[\![N_0/N_1^p]\!]$. Since both terms are free modules of rank $p(f-1)$ over a power series ring in one variable over $\mathbb{F}$, the surjection has to be an isomorphism.

Let $A \overset{\text{def}}{=} \mathbb{F}[\![N_1/N_1^p]\!]$, $B \overset{\text{def}}{=} \mathbb{F}[\![N_0/N_1^p]\!]$ and $\overline{B} \overset{\text{def}}{=} \mathbb{F}[\![N_0/N_0^p]\!]$, they are complete local commutative rings of respective maximal ideals denoted by $\mathfrak{m}_A$, $\mathfrak{m}_B$, $\mathfrak{m}_{\overline{B}}$. Let $Z \overset{\text{def}}{=} \sum_{n \in N_1/N_1^p} n \in A$. Note that $\mathfrak{m}_A$ is the augmentation ideal of A, hence the $\mathfrak{m}_A$-torsion $A[\mathfrak{m}_A]$ in A equals $\mathbb{F}Z$. As $N_1/N_1^p \cong (\mathbb{Z}/p\mathbb{Z})^{f-1}$, we have an isomorphism $A \cong \mathbb{F}[Z_1, \dots, Z_{f-1}]/(Z_1^p, \dots, Z_{f-1}^p)$, so $\mathfrak{m}_A^{(p-1)(f-1)+1} = 0$ and hence $Z \in \mathfrak{m}_A^{(p-1)(f-1)}$.

Let $\imath : A \hookrightarrow B$ denote the inclusion and denote by $\operatorname{gr}^m(\imath)$ the induced map $\mathfrak{m}_A^m/\mathfrak{m}_A^{m+1} \to \mathfrak{m}_B^m/\mathfrak{m}_B^{m+1}$ for $m \geq 0$. We claim that $\operatorname{gr}^1(\imath)$ is injective with image generated by all $Y_j - Y_{j_0}$ ($j \neq j_0$) in $\mathfrak{m}_B/\mathfrak{m}_B^2$. If so, then $\operatorname{gr}^{(p-1)(f-1)}(\imath)$ has to send the 1-dimensional $\mathbb{F}$-vector space $\mathfrak{m}_A^{(p-1)(f-1)}$ to a multiple of $\prod_{j \neq j_0}(Y_j - Y_{j_0})^{p-1}$ modulo $\mathfrak{m}_B^{(p-1)(f-1)+1}$. But $\left(\begin{smallmatrix} \tilde{\lambda} & 0 \\ 0 & \tilde{\mu} \end{smallmatrix}\right) Z = Z \left(\begin{smallmatrix} \tilde{\lambda} & 0 \\ 0 & \tilde{\mu} \end{smallmatrix}\right)$ for $\lambda, \mu \in \mathbb{F}_p^\times$, and considering the action of H, it follows from the sentence following Lemma 3.2.2.1 that we must have

$$\imath(Z) = c \prod_{j \neq j_0} (Y_j - Y_{j_0})^{p-1} + (\text{element of } \mathfrak{m}_B^{f(p-1)})$$

for some $c \in \mathbb{F}$ (note that every element of B can be written uniquely as $\sum_{\underline{i}} c_{\underline{i}} \underline{Y}^{\underline{i}}$ with $i_j < p$ for all $j \neq j_0$ and that $\mathfrak{m}_B$ is generated by the $\underline{Y}^{\underline{i}}$, $\underline{i} \neq \underline{0}$). By passing to $\overline{B}$ and using Lemma 3.2.2.8, we deduce that we must have $c = (-1)^{f-1}$.

It remains to prove the claim. As $\overline{B} \cong B/(Y_0^p, \dots, Y_{f-1}^p)$, we have $\mathfrak{m}_B/\mathfrak{m}_B^2 \xrightarrow{\sim} \mathfrak{m}_{\overline{B}}/\mathfrak{m}_{\overline{B}}^2$ and it is equivalent to prove the claim with $\bar{\imath} : A \to \overline{B}$. We first note that $\operatorname{gr}^1(\bar{\imath})$ is injective with 1-dimensional cokernel, because for any finite abelian p-group U the cotangent space of $\operatorname{Spec} \mathbb{F}[U]$ at its closed point is identified with $\mathbb{F} \otimes_{\mathbb{Z}} U$. Consider the natural map $s : \overline{B} \twoheadrightarrow C \overset{\text{def}}{=} \mathbb{F}[N_0/N_1 N_0^p] \cong \mathbb{F}[Y]/(Y^p)$. As $\operatorname{gr}^1(s \circ \bar{\imath}) = 0$ and $s(Y_i) = Y$ by Lemma 3.2.2.4, we deduce from *loc.cit.* that the image of $\operatorname{gr}^1(\bar{\imath})$ is indeed spanned by all $Y_j - Y_{j_0}$ ($j \neq j_0$). $\square$

3.2.3 A computation for the operator F

We give a crucial computation for the operator F on π^{N_1} for π as at the end of §3.2.1. The main result of this section is Proposition 3.2.3.1(ii).

We keep the notation of §3.2.2. For $\sigma = (t_0, \dots, t_{f-1}) \otimes \eta \in W(\overline{\rho})$, recall we have $t_j \in \{r_j, r_j + 1, p-2-r_j, p-3-r_j\}$ if $j > 0$ or $\overline{\rho}$ is reducible and $t_0 \in \{r_0 - 1, r_0, p-1-r_0, p-2-r_0\}$ if $\overline{\rho}$ is irreducible (see e.g. [Bre11, §2]). We deduce from (126) that

$$t_j \in \{2f-1, \dots, p-1-2f\} \quad \text{for all } j. \tag{132}$$

We identify $W(\overline{\rho})$ with the subsets of $\{0, 1, \dots, f-1\}$ as in [Bre11, §2] and let $J_\sigma \subseteq \{0, \dots, f-1\}$ be the subset associated to σ. We have $t_j \in \{p-2-r_j, p-3-r_j\}$ for $j \in J_\sigma$ if $j > 0$ or $\overline{\rho}$ is reducible, $t_0 \in \{p-2-r_0, p-1-r_0\}$ if $0 \in J_\sigma$ and $\overline{\rho}$ is irreducible.

Let $\sigma = (t_0, \dots, t_{f-1}) \otimes \eta \in W(\overline{\rho})$. Denote $\delta(\sigma) \overset{\text{def}}{=} \delta_{\text{red}}(\sigma)$ if $\overline{\rho}$ is reducible and $\delta(\sigma) \overset{\text{def}}{=} \delta_{\text{irr}}(\sigma)$ if $\overline{\rho}$ is irreducible the Serre weights $\delta_{\text{red}}(\sigma)$, $\delta_{\text{irr}}(\sigma)$ defined in [Bre11, §5]. We write $\delta(\sigma) = (s_0, \dots, s_{f-1}) \otimes \eta'$. Let $x_\sigma \in \sigma^{N_0} \setminus \{0\}$ and let $\chi_\sigma : H \to \mathbb{F}^\times$ denote the H-eigencharacter of x_σ. We also identify the irreducible constituents of $\operatorname{Ind}_I^{\mathrm{GL}_2(\mathcal{O}_K)}(\chi_\sigma^s)$ with the subsets of $\{0, \dots, f-1\}$ as in [BP12, §2] (for instance $\emptyset$ corresponds to the socle σ of $\operatorname{Ind}_I^{\mathrm{GL}_2(\mathcal{O}_K)}(\chi_\sigma^s)$). For any $J \subseteq \{0, \dots, f-1\}$ let $Q(\chi_\sigma^s, J)$ denote the unique quotient of $\operatorname{Ind}_I^{\mathrm{GL}_2(\mathcal{O}_K)}(\chi_\sigma^s)$ with irreducible $\mathrm{GL}_2(\mathcal{O}_K)$-socle parametrized by J (see [BP12, Thm.2.4(iv)]). We know that the Serre weight $\delta(\sigma)$ occurs in $\operatorname{Ind}_I^{\mathrm{GL}_2(\mathcal{O}_K)}(\chi_\sigma^s)$ (see the proof of [Bre11, Prop.5.1]) and we denote by $J^{\max}(\sigma) \subseteq \{0, \dots, f-1\}$ the associated subset. We thus have

$$\operatorname{soc}_{\mathrm{GL}_2(\mathcal{O}_K)} Q(\chi_\sigma^s, J^{\max}(\sigma)) \cong \delta(\sigma)$$

(by definition of $\delta(\sigma)$, it is the only constituent of $Q(\chi_\sigma^s, J^{\max}(\sigma))$ that is in $W(\overline{\rho})$). We also have from [BP12, §2] (with $-1 = f-1$):

$$\begin{aligned} s_j &= p-2-t_j + \mathbf{1}_{J^{\max}(\sigma)}(j-1) && \text{if } j \in J^{\max}(\sigma), \\ s_j &= t_j - \mathbf{1}_{J^{\max}(\sigma)}(j-1) && \text{if } j \notin J^{\max}(\sigma). \end{aligned} \tag{133}$$

(Above, we write $\mathbf{1}_{J^{\max}(\sigma)}$ for the indicator function of $J^{\max}(\sigma)$.) Moreover, using [BP12, Lemma 2.7] it is a combinatorial exercise (left to the reader) to prove

$$J^{\max}(\sigma) = (J_\sigma \cup J_{\delta(\sigma)}) \setminus (J_\sigma \cap J_{\delta(\sigma)}). \tag{134}$$

We define

$$m \overset{\text{def}}{=} |J^{\max}(\sigma)| \in \{0, \dots, f\}.$$

We have $m = 0$ if and only if $\delta(\sigma) \cong \sigma$, and this occurs precisely if $\overline{\rho}$ is reducible and σ is an "ordinary" Serre weight of $\overline{\rho}$, i.e. such that $J_\sigma = \emptyset$ or $J_\sigma = \{0, \dots, f-1\}$ (this follows, for example, from the proof of Lemma 3.2.3.2 below).

We consider a $\mathrm{GL}_2(K)$-representation π as at the end of §3.2.1, and fix an embedding $\sigma \hookrightarrow \mathrm{soc}_{\mathrm{GL}_2(\mathcal{O}_K)}(\pi)$ (recall there are r copies of σ inside $\mathrm{soc}_{\mathrm{GL}_2(\mathcal{O}_K)}(\pi)$). From the assumption on π, we know that $\left(\begin{smallmatrix} 0 & 1 \\ p & 0 \end{smallmatrix}\right) x_\sigma$ generates $Q(\chi_\sigma^s, J^{\max}(\sigma))$ as a $\mathrm{GL}_2(\mathcal{O}_K)$-subrepresentation of $\pi|_{\mathrm{GL}_2(\mathcal{O}_K)}$, in particular $\delta(\sigma)$ can also be seen in $\mathrm{soc}_{\mathrm{GL}_2(\mathcal{O}_K)}(\pi)$ (its embedding being determined by that of σ up to a scalar).

Proposition 3.2.3.1.

(i) *The vector*

$$x_{\delta(\sigma)} \overset{\mathrm{def}}{=} \prod_{j \in J^{\max}(\sigma)} Y_j^{s_j} \prod_{j \notin J^{\max}(\sigma)} Y_j^{p-1} \left(\begin{smallmatrix} p & 0 \\ 0 & 1 \end{smallmatrix}\right) x_\sigma \tag{135}$$

spans $\delta(\sigma)^{N_0}$ *as an* $\mathbb{F}$*-vector space.*

(ii) *We have in* π^{N_1} *that*

$$\begin{aligned} Y^{\sum_{j \in J^{\max}(\sigma)} s_j} F(Y^{1-m} x_\sigma) &= (-1)^{f-1} Y^{1-m} x_{\delta(\sigma)} \quad \textit{if } m > 0, \\ Y^{p-1} F(x_\sigma) &= (-1)^{f-1} x_{\delta(\sigma)} \qquad\qquad \textit{if } m = 0. \end{aligned}$$

Proof of Proposition 3.2.3.1(i). Suppose first $m > 0$. From [BP12, Lemma 2.7(ii)] and Lemma 3.2.2.5(ii) we see that $\delta(\sigma)$ has basis $\underline{Y}^{\underline{i}} \left(\begin{smallmatrix} p & 0 \\ 0 & 1 \end{smallmatrix}\right) x_\sigma$, where $0 \le i_j \le s_j$ if $j \in J^{\max}(\sigma)$ and $p-1-s_j \le i_j \le p-1$ if $j \notin J^{\max}(\sigma)$. Hence the only vectors in $\delta(\sigma)$ that are killed by all Y_j are the multiples of $x_{\delta(\sigma)}$. The statement follows by an inspection of the H-action on this basis (which is formed by H-eigenvectors), see Remark 3.2.2.3.

If $m = 0$, then $\delta(\sigma)$ is the socle of $\mathrm{Ind}_I^{\mathrm{GL}_2(\mathcal{O}_K)}(\chi_\sigma^s)$. By [BP12, Lemma 2.7(i)], f_0 is the unique I-invariant element of $\delta(\sigma) \subseteq \mathrm{Ind}_I^{\mathrm{GL}_2(\mathcal{O}_K)}(\chi_\sigma^s)$. The statement follows from Lemma 3.2.2.5(ii). $\square$

In order to prove Proposition 3.2.3.1(ii), we first need several lemmas.

Lemma 3.2.3.2. *We have* $|J^{\max}(\sigma)| = |J^{\max}(\delta(\sigma))|$.

Proof. If $\overline{\rho}$ is reducible, identifying $\{0, \dots, f-1\}$ with $\mathbb{Z}/f$ we have $J_{\delta(\sigma)} = J_\sigma - 1$ as subsets of $\mathbb{Z}/f$ by [Bre11, §5], and the statement follows in that case by (134). If $\overline{\rho}$ is irreducible, let $J'_\sigma \overset{\mathrm{def}}{=} J_\sigma \coprod (\overline{J_\sigma} + f) \subseteq \{0, \dots, 2f-1\}$ as in [Bre11, §5], where $\overline{J_\sigma}$ is the complement of J_σ in $\{0, \dots, f-1\}$. It follows from (134) that $|J^{\max}(\sigma)| = \frac{1}{2}|(J'_\sigma \cup J'_{\delta(\sigma)}) \setminus (J'_\sigma \cap J'_{\delta(\sigma)})|$. Identifying $\{0, \dots, 2f-1\}$ with $\mathbb{Z}/2f$, we again have $J'_{\delta(\sigma)} = J'_\sigma - 1$ as subsets of $\mathbb{Z}/(2f)$ by [Bre11, §5], and the statement follows. $\square$

The three lemmas that follow only apply to $m > 0$ and require the strong genericity assumption. In these three lemmas, we identify without comment $\{0, \dots, f-1\}$ with $\mathbb{Z}/f\mathbb{Z}$ (so $-1 = f-1$, $f = 0$, etc.).

Lemma 3.2.3.3. *Assume $m > 0$ and let $\underline{i} \in \mathbb{Z}_{\geq 0}^f$ with $\|\underline{i}\| \leq m-1$. Then we have*

$$\Big\langle \mathrm{GL}_2(\mathcal{O}_K)\Big(\begin{smallmatrix} p & 0 \\ 0 & 1 \end{smallmatrix}\Big)\underline{Y}^{-\underline{i}}x_\sigma \Big\rangle \Big/ \sum_{\underline{0}\leq \underline{j} < \underline{i}} \Big\langle \mathrm{GL}_2(\mathcal{O}_K)\Big(\begin{smallmatrix} p & 0 \\ 0 & 1 \end{smallmatrix}\Big)\underline{Y}^{-\underline{j}}x_\sigma \Big\rangle \\ \cong \quad Q\Big(\chi_\sigma^s \alpha^{\underline{i}}, \{j \in J^{\max}(\sigma) : i_{j+1} = 0\}\Big). \quad (136)$$

Proof. Note first that $t_j \in \{2i_j+1, \dots, p-2\}$ for all j by (132) and the assumption on $\underline{i}$, so that the vectors $\underline{Y}^{-\underline{i}}x_\sigma$ and $\underline{Y}^{-\underline{j}}x_\sigma$ are well-defined elements of σ by Lemma 3.2.2.6(ii). We rewrite $\langle \mathrm{GL}_2(\mathcal{O}_K)\big(\begin{smallmatrix} p & 0 \\ 0 & 1 \end{smallmatrix}\big)\underline{Y}^{-\underline{j}}x_\sigma\rangle = \langle \mathrm{GL}_2(\mathcal{O}_K)\big(\begin{smallmatrix} 0 & 1 \\ p & 0 \end{smallmatrix}\big)\underline{Y}^{-\underline{j}}x_\sigma\rangle$ and, using notation from [BHH+23, §§2.1,2.2], $\sigma \cong F(\lambda)$ where $\lambda = (\lambda_0, \dots, \lambda_{f-1})$ with $\lambda_j = (\lambda_{j,1}, \lambda_{j,2}) \in \{0, \dots, p-1\}^2$. We have $\lambda_{j,1} - \lambda_{j,2} = t_j$ for all j.

Let W' (resp. W) be the I-subrepresentation of π generated by $\underline{Y}^{-\underline{i}}x_\sigma$ (resp. $\big(\begin{smallmatrix} 0 & 1 \\ p & 0 \end{smallmatrix}\big)\underline{Y}^{-\underline{i}}x_\sigma$). We deduce from Lemma 3.2.2.6(ii) that $W' = \langle N_0 \underline{Y}^{-\underline{i}}x_\sigma\rangle$ has $\mathbb{F}$-basis $\underline{Y}^{-\underline{j}}x_\sigma$ for all $\underline{0} \leq \underline{j} \leq \underline{i}$, and $\mathrm{soc}_I(W') = \mathbb{F}x_\sigma$. We moreover have $W = \big(\begin{smallmatrix} 0 & 1 \\ p & 0 \end{smallmatrix}\big)W'$ since I is normalized by $\big(\begin{smallmatrix} 0 & 1 \\ p & 0 \end{smallmatrix}\big)$. In particular we see that W injects into the I-representation $\mathcal{J}_{\chi_\sigma}$ of [BHH+23, Cor.6.1.4] and that W has Jordan–Hölder factors $\chi_\sigma^s\alpha^{\underline{j}}$ for $\underline{0} \leq \underline{j} \leq \underline{i}$, each occurring with multiplicity 1. Let $V \overset{\mathrm{def}}{=} \mathrm{Ind}_I^{\mathrm{GL}_2(\mathcal{O}_K)}(W)$. Then V is the representation appearing in the first paragraph of the proof of [BHH+23, Prop.6.2.2], with B_j taken to be $2i_j+1$ for all j (and note the bounds on $\lambda_{j,1} - \lambda_{j,2}$ which let us invoke *loc.cit.*). Hence, by [BHH+23, Prop.6.2.2] and its proof in the case $\varepsilon_j = -1$ and $B_j = 2i_j+1$ for all j, we get that V is multiplicity-free, has Jordan–Hölder factors $\sigma_{\underline{a}} \overset{\mathrm{def}}{=} F(\mathfrak{t}_\lambda(-\sum a_j\overline{\eta}_j))$ for $\underline{0} \leq \underline{a} \leq 2\underline{i}+1$ with the notation of [BHH+23, §2.4], and $\mathrm{GL}_2(\mathcal{O}_K)$-socle σ. Moreover, the unique subrepresentation of V with cosocle $\sigma_{\underline{a}}$ has constituents $\sigma_{\underline{b}}$ for $\underline{0} \leq \underline{b} \leq \underline{a}$. On the other hand, $\mathrm{Ind}_I^{\mathrm{GL}_2(\mathcal{O}_K)}(W)$ has a filtration with subquotients $\mathrm{Ind}_I^{\mathrm{GL}_2(\mathcal{O}_K)}(\chi_\sigma^s\alpha^{\underline{j}})$ for $\underline{0} \leq \underline{j} \leq \underline{i}$, and by [BHH+23, Lemma 6.2.1(i)] the constituents of $\mathrm{Ind}_I^{\mathrm{GL}_2(\mathcal{O}_K)}(\chi_\sigma^s\alpha^{\underline{j}})$ are the Serre weights $\sigma_{\underline{a}}$ with $2\underline{j} \leq \underline{a} \leq 2\underline{j}+1$. By the proof of [BHH+23, Lemma 6.2.1(i)], one easily checks that the constituent $\sigma_{\underline{a}}$ of $\mathrm{Ind}_I^{\mathrm{GL}_2(\mathcal{O}_K)}(\chi_\sigma^s\alpha^{\underline{j}})$ corresponds to the subset $\{\ell : a_{\ell+1} \text{ is odd}\} \subseteq \{0, \dots, f-1\}$ in the parametrization of [BP12, §2] (note that twisting χ_σ^s by $\alpha^{\underline{j}}$ corresponds to shifting by $-2\sum j_\ell\overline{\eta}_\ell$ in the extension graph).

By Frobenius reciprocity $\overline{V} \overset{\mathrm{def}}{=} \langle \mathrm{GL}_2(\mathcal{O}_K)\big(\begin{smallmatrix} 0 & 1 \\ p & 0 \end{smallmatrix}\big)\underline{Y}^{-\underline{i}}x_\sigma\rangle$ is the image of a nonzero map $\mathrm{Ind}_I^{\mathrm{GL}_2(\mathcal{O}_K)}(W) \to \pi$ and any Serre weight in its $\mathrm{GL}_2(\mathcal{O}_K)$-socle has to be in $W(\overline{\rho})$. By [BHH+23, Prop.2.4.2] if $\sigma_{\underline{a}} \in W(\overline{\rho})$, then $\underline{0} \leq \underline{a} \leq \underline{1}$, so $\sigma_{\underline{a}}$ is a constituent of $\mathrm{Ind}_I^{\mathrm{GL}_2(\mathcal{O}_K)}(\chi_\sigma^s) \subseteq V$. Thus by the definition of $\delta(\sigma)$ and as $\pi^{K_1}/\mathrm{soc}_{\mathrm{GL}_2(\mathcal{O}_K)}\pi$ does not contain any Serre weight of $W(\overline{\rho})$ it follows that $\overline{V}$ is the unique quotient of V with $\mathrm{GL}_2(\mathcal{O}_K)$-socle $\delta(\sigma)$. By the previous paragraph and the definition of $J^{\max}(\sigma)$, we have $\delta(\sigma) \cong \sigma_{\underline{b}}$, where $b_j = \mathbf{1}_{J^{\max}(\sigma)+1}(j)$ for all j, and $\overline{V}$ has constituents $\sigma_{\underline{a}}$ with $\mathbf{1}_{J^{\max}(\sigma)+1}(j) \leq a_j \leq 2i_j+1$ for all j. By construction, the left-hand side of (136) is a quotient of $\mathrm{Ind}_I^{\mathrm{GL}_2(\mathcal{O}_K)}(\chi_\sigma^s\alpha^{\underline{i}})$. Moreover, by what is before, it must have constituents $\sigma_{\underline{a}}$ with $\max(\mathbf{1}_{J^{\max}(\sigma)+1}(j), 2i_j) \le a_j \le 2i_j+1$ for all j. It follows that its $\mathrm{GL}_2(\mathcal{O}_K)$-socle is irreducible and isomorphic to $\sigma_{\underline{c}}$, where $c_j \overset{\mathrm{def}}{=} \max(\mathbf{1}_{J^{\max}(\sigma)+1}(j), 2i_j)$ for all j. Since $2i_{j+1}$ is even and >1 as soon as $i_{j+1}\neq 0$, we see that c_{j+1} is odd if and only if $i_{j+1}=0$ and $j\in J^{\max}(\sigma)$. Hence the $\mathrm{GL}_2(\mathcal{O}_K)$-socle of this quotient of $\mathrm{Ind}_I^{\mathrm{GL}_2(\mathcal{O}_K)}(\chi_\sigma^s\alpha^{\underline{i}})$ corresponds to the subset $\{j\in J^{\max}(\sigma): i_{j+1}=0\}$, as required. □

Lemma 3.2.3.4. *Assume $m>0$ and let $\underline{i}\in\mathbb{Z}_{\ge 0}^f$, $\ell\in J^{\max}(\sigma)$ such that $\|\underline{i}\|\le m-1$ and $i_{\ell+1}=0$. Then*

$$Y_\ell^{p-t_\ell+2i_\ell}\begin{pmatrix} p & 0\\ 0 & 1\end{pmatrix}\underline{Y}^{-\underline{i}}x_\sigma = 0.$$

Proof. Recall $p-t_\ell+2i_\ell\ge 0$ by (132), so that $Y_\ell^{p-t_\ell+2i_\ell}\left(\begin{smallmatrix} p & 0\\ 0 & 1\end{smallmatrix}\right)\underline{Y}^{-\underline{i}}x_\sigma$ is well-defined. Suppose on the contrary that $Y_\ell^{p-t_\ell+2i_\ell}\left(\begin{smallmatrix} p & 0\\ 0 & 1\end{smallmatrix}\right)\underline{Y}^{-\underline{i}}x_\sigma\neq 0$ for some $\ell\in J^{\max}(\sigma)$ such that $i_{\ell+1}=0$ and $\|\underline{i}\|\le m-1$. By Lemma 3.2.2.1(ii) and Lemma 3.2.2.6(ii) this is an eigenvector for $\{\left(\begin{smallmatrix} \tilde\lambda & 0\\ 0 & \tilde\mu\end{smallmatrix}\right): \lambda,\mu\in\mathbb{F}_q^\times\}$ with eigencharacter $\chi_\sigma\alpha^{-\underline{i}}\alpha^{(p-t_\ell+2i_\ell)p^\ell}$. By Lemma 3.2.3.3 it suffices to show that the H-eigencharacter $\chi_\sigma\alpha^{-\underline{i}}\alpha^{(p-t_\ell+2i_\ell)p^\ell}$ does not occur in

$$V_{\underline{i}'} \overset{\mathrm{def}}{=} Q(\chi_\sigma^s\alpha^{\underline{i}'}, J_{\underline{i}'})$$

for any $\underline{i}'$ such that $\underline{0}\le\underline{i}'\le\underline{i}$, where $J_{\underline{i}'}\overset{\mathrm{def}}{=}\{j\in J^{\max}(\sigma): i'_{j+1}=0\}$.

Using the notation $\lambda=(\lambda_0(x_0),\dots,\lambda_{f-1}(x_{f-1}))$ and $\mathcal{P}(x_0,\dots,x_{f-1})$ of [BP12, Thm.2.4], the irreducible constituents of $V_{\underline{i}'}$ are given by the Serre weights $(\lambda_0(t_0-2i'_0),\dots,\lambda_{f-1}(t_{f-1}-2i'_{f-1}))$ (up to twist) for those $\lambda\in\mathcal{P}(x_0,\dots,x_{f-1})$ such that $J(\lambda)\supseteq J_{\underline{i}'}$. Recall that $\lambda_j(x)=p-2-x+\mathbf{1}_{J(\lambda)}(j-1)$ if $j\in J(\lambda)$ and $\lambda_j(x)=x-\mathbf{1}_{J(\lambda)}(j-1)$ if $j\notin J(\lambda)$. By [BP12, Lemma 2.5(i)] and [BP12, Lemma 2.7], the H-eigencharacters that occur in $V_{\underline{i}'}$ are $\chi_\sigma\alpha^{-\underline{i}'}\alpha^{\underline{k}}$, where $\underline{k}$ is such that there exists $\lambda\in\mathcal{P}(x_0,\dots,x_{f-1})$ with $J(\lambda)\supseteq J_{\underline{i}'}$ and

$$\begin{aligned} 0\le k_j&\le p-2-(t_j-2i'_j)+\mathbf{1}_{J(\lambda)}(j-1) &&\text{if } j\in J(\lambda),\\ p-1-(t_j-2i'_j-\mathbf{1}_{J(\lambda)}(j-1))\le k_j&\le p-1 &&\text{if } j\notin J(\lambda).\end{aligned} \tag{137}$$

(Note that $J_{\underline{i}'}\neq\emptyset$ as $\ell\in J_{\underline{i}'}$, noting that $\ell\in J^{\max}(\sigma)$ and $0\le i'_{\ell+1}\le i_{\ell+1}=0$.)

Assume $\chi_\sigma\alpha^{-\underline{i}}\alpha^{(p-t_\ell+2i_\ell)p^\ell}=\chi_\sigma\alpha^{-\underline{i}'}\alpha^{\underline{k}}$ for some λ and $\underline{k}$ as above. Then

$$-\sum_{j=0}^{f-1} i_jp^j+(p-t_\ell+2i_\ell)p^\ell\equiv-\sum_{j=0}^{f-1}i'_jp^j+\sum_{j=0}^{f-1}k_jp^j \pmod{q-1}$$

or equivalently

$$(p-t_\ell+2i_\ell)p^\ell-\sum_{j=0}^{f-1}(i_j-i'_j)p^j\equiv\sum_{j=0}^{f-1}k_jp^j \pmod{q-1}. \tag{138}$$

Note that, since $\ell \in J_{\underline{i}'}$, we have in particular $\ell \in J(\lambda)$.

If $i'_j = i_j$ for all $j \neq \ell$ (for example if $\underline{i}' = \underline{i}$ or if $f = 1$), then (138) gives $(p - t_\ell + i_\ell + i'_\ell)p^\ell \equiv \sum_j k_j p^j$, so $k_\ell = p - t_\ell + i_\ell + i'_\ell$ as (using (132) for t_ℓ and $0 \leq i'_\ell \leq i_\ell \leq m-1 \leq f-1$):

$$p - t_\ell + i_\ell + i'_\ell \in \{2f+1, \ldots, p-1\}. \tag{139}$$

This contradicts (137) as $\ell \in J(\lambda)$ and $i'_\ell \leq i_\ell$. Therefore $f > 1$ and $i'_j < i_j$ for some $j \neq \ell$. For $m \in \mathbb{Z}_{\geq 0}$, let $[m]$ the unique element of $\{0, \ldots, f-1\}$ which is congruent to m modulo f. In particular $p^m \equiv p^{[m]} \pmod{q-1}$. Let $h \in \{\ell+1, \ldots, \ell+f-1\}$ be minimal such that $i'_{[h]} < i_{[h]}$. Then modulo $q-1$:

$$\sum_{j=0}^{f-1}(i_j - i'_j)p^j \equiv \sum_{j=\ell+1}^{\ell+f}(i_{[j]} - i'_{[j]})p^{[j]} = \sum_{j=h}^{\ell+f}(i_{[j]} - i'_{[j]})p^{[j]}$$

and we deduce the following congruences modulo $q-1$:

$$\begin{aligned}
(p-t_\ell + 2i_\ell)p^\ell &- \sum_{j=0}^{f-1}(i_j - i'_j)p^j \\
&\equiv (p-1-t_\ell+2i_\ell)p^\ell + p^\ell - \sum_{j=h}^{\ell+f}(i_{[j]} - i'_{[j]})p^{[j]} \\
&\equiv (p-1-t_\ell+2i_\ell)p^\ell + \sum_{j=h+1}^{\ell+f-1}(p-1)p^j + p^{h+1} - \sum_{j=h}^{\ell+f}(i_{[j]} - i'_{[j]})p^{[j]} \\
&\equiv (p-1-t_\ell+2i_\ell)p^\ell + \sum_{j=h+1}^{\ell+f-1}(p-1)p^{[j]} + p^{[h]+1} - \sum_{j=h}^{\ell+f}(i_{[j]} - i'_{[j]})p^{[j]} \\
&\equiv (p-1-t_\ell+i_\ell+i'_\ell)p^\ell + \sum_{j=h+1}^{\ell+f-1}(p-1-(i_{[j]}-i'_{[j]}))p^{[j]} + (p-(i_{[h]}-i'_{[h]}))p^{[h]}.
\end{aligned} \tag{140}$$

Note that all powers of p in (140) are distinct in $\{0, \ldots, f-1\}$ and all coefficients are in $\{0, \ldots, p-1\}$. Moreover these coefficients cannot all equal 0 as $p - (i_{[h]} - i'_{[h]}) \neq 0$, nor $p-1$ by (139). Hence by (138) we get $k_\ell = p - 1 - t_\ell + i_\ell + i'_\ell$. As $\ell \in J(\lambda)$ and $i'_\ell \leq i_\ell$, we get from (137) that $i_\ell = i'_\ell$ and $\ell - 1 \in J(\lambda)$. By (137) for $j = \ell - 1$ and by (140), (138) we get

$$p - 1 - (i_{\ell-1} - i'_{\ell-1}) \leq k_{\ell-1} \leq p - 1 - t_{\ell-1} + 2i'_{\ell-1}$$

(note that by (140) the left-hand side is an equality as soon as $\ell - 1 \not\equiv h \bmod f$ which can only occur if $f > 2$). This implies $t_{\ell-1} \leq i_{\ell-1} + i'_{\ell-1} \leq 2(m-1) \leq 2f-2$, which contradicts genericity (132). This finishes the proof. □

Lemma 3.2.3.5. *Assume $m > 0$ and let $\underline{k} \in \mathbb{Z}^f_{\geq 0}$.*

(i) *If* $\underline{Y}^{\underline{k}}\left(\begin{smallmatrix} p & 0 \\ 0 & 1 \end{smallmatrix}\right)Y^{1-m}x_\sigma \neq 0$*, then*

$$\|\underline{k}\| \leq (f-1)(p-1) + (m-1) + \sum_{j \in J^{\max}(\sigma)} s_j.$$

If moreover equality holds, then $\underline{Y}^{\underline{k}}\left(\begin{smallmatrix} p & 0 \\ 0 & 1 \end{smallmatrix}\right)Y^{1-m}x_\sigma = x_{\delta(\sigma)}$ *(see* (135)*) and*

$$\begin{aligned} k_j &\equiv s_j \pmod p \quad \text{if } j \in J^{\max}(\sigma), \\ k_j &\equiv -1 \pmod p \quad \text{if } j \notin J^{\max}(\sigma). \end{aligned}$$

(ii) *If* $\|\underline{k}\| = (f-1)(p-1) + \sum_{J^{\max}(\sigma)} s_j$ *then* $\underline{Y}^{\underline{k}}\left(\begin{smallmatrix} p & 0 \\ 0 & 1 \end{smallmatrix}\right)Y^{1-m}x_\sigma \in \delta(\sigma)$*, more precisely:*

$$\underline{Y}^{\underline{k}}\left(\begin{smallmatrix} p & 0 \\ 0 & 1 \end{smallmatrix}\right)Y^{1-m}x_\sigma \in \left\langle \underline{Y}^{-\underline{\ell}}x_{\delta(\sigma)}, \|\underline{\ell}\| = m-1 \right\rangle_{\mathbb{F}}.$$

Proof. We prove the following statements inductively on $\|\underline{i}\| \leq m-1$ for $\underline{i} \in \mathbb{Z}_{\geq 0}^f$:

(a) If $\underline{Y}^{\underline{k}}\left(\begin{smallmatrix} p & 0 \\ 0 & 1 \end{smallmatrix}\right)\underline{Y}^{-\underline{i}}x_\sigma \neq 0$ then

$$\|\underline{k}\| \leq (f-1)(p-1) + (m-1) + \sum_{j \in J^{\max}(\sigma)} s_j - (m-1-\|\underline{i}\|)p.$$

If moreover equality holds, then $\underline{Y}^{\underline{k}}\left(\begin{smallmatrix} p & 0 \\ 0 & 1 \end{smallmatrix}\right)\underline{Y}^{-\underline{i}}x_\sigma = x_{\delta(\sigma)}$ and

$$\begin{aligned} k_j &= i_{j+1}p + s_j \quad && \text{if } j \in J^{\max}(\sigma), \\ k_j &= i_{j+1}p + (p-1) \quad && \text{if } j \notin J^{\max}(\sigma). \end{aligned}$$

(b) If $\|\underline{k}\| = (f-1)(p-1) + \sum_{j \in J^{\max}(\sigma)} s_j - (m-1-\|\underline{i}\|)p$ then

$$\underline{Y}^{\underline{k}}\left(\begin{smallmatrix} p & 0 \\ 0 & 1 \end{smallmatrix}\right)\underline{Y}^{-\underline{i}}x_\sigma = \underline{Y}^{-\underline{\ell}}x_{\delta(\sigma)}$$

for some $\|\underline{\ell}\| = m-1$, or it is zero.

By Lemma 3.2.2.6(iii) we have

$$Y^{1-m}x_\sigma = \sum_{\substack{\underline{i} \in \mathbb{Z}_{\geq 0}^f \\ \|\underline{i}\| = m-1}} \underline{Y}^{-\underline{i}}x_\sigma$$

and we see that (a) and (b) for $\|\underline{i}\| = m-1$ imply (i) and (ii) (note that in (a) if $\underline{Y}^{\underline{k}}\left(\begin{smallmatrix} p & 0 \\ 0 & 1 \end{smallmatrix}\right)\underline{Y}^{-\underline{i}}x_\sigma \neq 0$ and equality holds, then $\underline{i}$ is uniquely determined by $\underline{k}$ and $J^{\max}(\sigma)$).

We first prove by induction on $\|\underline{i}\| \leq m-1$ for $\underline{i} \in \mathbb{Z}_{\geq 0}^f$ that if $\|\underline{k}\| \geq (f-1)(p-1)+\sum_{J^{\max}(\sigma)} s_j-(m-1-\|\underline{i}\|)p$ and $\underline{Y}^{\underline{k}}\left(\begin{smallmatrix} p & 0 \\ 0 & 1 \end{smallmatrix}\right)\underline{Y}^{-\underline{i}}x_\sigma \neq 0$, then $\underline{Y}^{\underline{k}}\left(\begin{smallmatrix} p & 0 \\ 0 & 1 \end{smallmatrix}\right)\underline{Y}^{-\underline{i}}x_\sigma = \underline{Y}^{\underline{k}'}\left(\begin{smallmatrix} p & 0 \\ 0 & 1 \end{smallmatrix}\right)x_\sigma$ for $\underline{k}' \in \mathbb{Z}_{\geq 0}^f$ such that $k'_j = k_j - i_{j+1}p$ for all j. A examination of (a) and (b) shows it will then be enough to prove them for $\underline{i} = \underline{0}$ (replacing $\underline{k}$ by $\underline{k}'$).

There is nothing to prove for $\underline{i} = \underline{0}$, so we can assume $\underline{i} \neq \underline{0}$. If $k_{j_0} \geq p$ for some j_0, then using Lemma 3.2.2.1(ii):

$$\underline{Y}^{\underline{k}}\left(\begin{smallmatrix} p & 0 \\ 0 & 1 \end{smallmatrix}\right)\underline{Y}^{-\underline{i}}x_\sigma = \underline{Y}^{\underline{k}-p\underline{\varepsilon}_{j_0}}Y_{j_0}^p\left(\begin{smallmatrix} p & 0 \\ 0 & 1 \end{smallmatrix}\right)\underline{Y}^{-\underline{i}}x_\sigma = \underline{Y}^{\underline{k}-p\underline{\varepsilon}_{j_0}}\left(\begin{smallmatrix} p & 0 \\ 0 & 1 \end{smallmatrix}\right)\underline{Y}^{-(\underline{i}-\underline{\varepsilon}_{j_0+1})}x_\sigma,$$

where $\underline{\varepsilon}_j \stackrel{\text{def}}{=} (0,\dots,0,1,0,\dots,0)$ with 1 in position j and 0 elsewhere (note that $Y_{j_0+1}\underline{Y}^{-\underline{i}}x_\sigma = \underline{Y}^{-(\underline{i}-\underline{\varepsilon}_{j_0+1})}x_\sigma$ is nonzero by assumption, and hence $\underline{i}-\underline{\varepsilon}_{j_0+1} \in \mathbb{Z}_{\geq 0}^f$ by the last statement in Lemma 3.2.2.6(ii)). As $\|\underline{i}-\underline{\varepsilon}_{j_0+1}\| = \|\underline{i}\|-1$ and $\|\underline{k}-p\underline{\varepsilon}_{j_0}\| = \|\underline{k}\|-p \geq (f-1)(p-1)+\sum_{J^{\max}(\sigma)} s_j - (m-1-\|\underline{i}-\underline{\varepsilon}_{j_0+1}\|)p$, we can apply the induction hypothesis and a small computation shows that $\underline{k}'$ is the right one, so we are done in that case.

We assume $k_j < p$ for all j and derive below a contradiction (so this case can't happen). Define

$$J \stackrel{\text{def}}{=} \{j \in J^{\max}(\sigma) : i_{j+1} = 0\},$$

then by Lemma 3.2.3.4 (applied to $\ell = j$ and using $Y_j^{k_j}\left(\begin{smallmatrix} p & 0 \\ 0 & 1 \end{smallmatrix}\right)\underline{Y}^{-\underline{i}}x_\sigma \neq 0$):

$$\begin{aligned} k_j &\leq p-1-t_j+2i_j && \text{if } j \in J, \\ k_j &\leq p-1 && \text{if } j \notin J, \end{aligned}$$

which implies $\|\underline{k}\| \leq (f-|J|)(p-1)+\sum_{j\in J}(p-1-t_j+2i_j)$. From (133) we deduce

$$\|\underline{k}\| \leq (f-|J|)(p-1)+\sum_{j\in J}(s_j+2i_j)+|J \setminus (J^{\max}(\sigma)+1)|.$$

So to get a contradiction it is enough to show that

$$(f-|J|)(p-1)+\sum_{j\in J}(s_j+2i_j)+|J\setminus(J^{\max}(\sigma)+1)| < (f-1)(p-1) + \sum_{j\in J^{\max}(\sigma)} s_j - (m-1-\|\underline{i}\|)p,$$

or equivalently

$$\begin{aligned} pm+|J\backslash(J^{\max}(\sigma)+1)| &\leq (p-1)|J|+p\sum_{j\notin J} i_j+(p-2)\sum_{j\in J} i_j+\sum_{j\in J^{\max}(\sigma)\backslash J} s_j \\ &= (p-2)\|\underline{i}\|+(p-1)|J|+\Big(2\sum_{j\notin J} i_j+\sum_{j\in J^{\max}(\sigma)\backslash J} s_j\Big) \quad (141) \end{aligned}$$

Case 1: assume $|J^{\max}(\sigma) \setminus J| > 0$.
If $j \in J^{\max}(\sigma) \setminus J$, then $i_{j+1} > 0$, so $|J^{\max}(\sigma) \setminus J| \leq \|\underline{i}\|$. As $|J^{\max}(\sigma) \setminus J| = m - |J|$, this means $m \leq \|\underline{i}\| + |J|$, hence (141) is implied by

$$2m + |J \setminus (J^{\max}(\sigma) + 1)| \leq |J| + \Big(2\sum_{j \notin J} i_j + \sum_{j \in J^{\max}(\sigma)\setminus J} s_j\Big). \tag{142}$$

Using $|J \setminus (J^{\max}(\sigma) + 1)| \leq |J|$, (142) is implied by

$$2m \leq \sum_{j \in J^{\max}(\sigma)\setminus J} s_j. \tag{143}$$

Genericity (132) with (133) give $s_j \geq 2f - 1 \geq 2m - 1$ for $j \in J^{\max}(\sigma)$, hence (143) holds if either $s_j \geq 2m$ for at least one $j \in J^{\max}(\sigma) \setminus J$ or if $|J^{\max}(\sigma) \setminus J| \geq 2$ (using $2m - 2 \geq 0$ for the latter). Therefore, the only way inequality (142) may fail is when $J^{\max}(\sigma) \setminus J = \{j_0\}$ (for some j_0) and moreover $J \setminus (J^{\max}(\sigma) + 1) = J$ and $i_j = 0$ for all $j \notin J$. But then $i_{j_0+1} > 0$ so we have $j_0 + 1 \in J \cap (J^{\max}(\sigma) + 1)$, which contradicts $J \cap (J^{\max}(\sigma) + 1) = \emptyset$. Hence inequality (142) holds.
Case 2: assume $J^{\max}(\sigma) = J$.
Then using

$$|J \setminus (J^{\max}(\sigma)+1)| \leq |\{0, \dots, f-1\} \setminus (J^{\max}(\sigma)+1)| = |\{0, \dots, f-1\} \setminus J^{\max}(\sigma)| = f - m$$

and $|J| = m$, we see that (141) is implied by $(p-1)m + f \leq (p-2)\|\underline{i}\| + (p-1)m$ which is true as $\|\underline{i}\| > 0$ and $f \leq p - 2$ by (126).

To prove (a) and (b), it therefore suffices to consider the case $\underline{i} = \underline{0}$, which we prove now.

Recall $\left\langle \mathrm{GL}_2(\mathcal{O}_K)\left(\begin{smallmatrix} 0 & 1 \\ p & 0 \end{smallmatrix}\right)x_\sigma \right\rangle \cong Q(\chi_\sigma^s, J^{\max}(\sigma))$. By [BP12, Thm.2.4(iv)] the constituents of this $\mathrm{GL}_2(\mathcal{O}_K)$-representation are the Serre weights $(\lambda_0(t_0), \dots, \lambda_{f-1}(t_{f-1}))$ up to twist, where $\lambda \in \mathcal{P}(x_0, \dots, x_{f-1})$, $J(\lambda) \supseteq J^{\max}(\sigma)$ and $\lambda_j(t_j) = p - 2 - t_j + \mathbf{1}_{J(\lambda)}(j-1)$ if $j \in J(\lambda)$ (we use the notation of [BP12, §2] as in the proof of Lemma 3.2.3.4). By [BP12, Lemma 2.7, Lemma 2.6] and Lemma 3.2.2.5(ii), $Q(\chi_\sigma^s, J^{\max}(\sigma))$ has $\mathbb{F}$-basis $\underline{Y}^{\underline{k}}\left(\begin{smallmatrix} p & 0 \\ 0 & 1 \end{smallmatrix}\right)x_\sigma$, where

$$\begin{aligned} 0 \leq k_j \leq \lambda_j(t_j) \qquad & \text{if } j \in J(\lambda), \\ p - 1 - \lambda_j(t_j) \leq k_j \leq p - 1 \qquad & \text{if } j \notin J(\lambda) \end{aligned} \tag{144}$$

for some $\lambda \in \mathcal{P}(x_0, \dots, x_{f-1})$ with $J(\lambda) \supseteq J^{\max}(\sigma)$. We see that (144) implies

$$\|\underline{k}\| \ \leq \ (f - |J(\lambda)|)(p-1) + \sum_{j \in J(\lambda)} (p - 2 - t_j + \mathbf{1}_{J(\lambda)}(j-1)) \tag{145}$$

with equality if and only if $k_j = \lambda_j(t_j)$ if $j \in J(\lambda)$ and $k_j = p - 1$ otherwise. Moreover, $\underline{Y}^{\underline{k}}\left(\begin{smallmatrix} p & 0 \\ 0 & 1 \end{smallmatrix}\right)x_\sigma \in \delta(\sigma) \setminus \{0\}$ if and only if (144) holds with $J(\lambda) = J^{\max}(\sigma)$.

Hence if $\underline{Y}^{\underline{k}}\begin{pmatrix} p & 0 \\ 0 & 1 \end{pmatrix} x_\sigma \notin \delta(\sigma)$ we deduce that (144) holds for some $\lambda \in \mathcal{P}(x_0, \ldots, x_{f-1})$ with $J(\lambda) \supsetneq J^{\max}(\sigma)$.

We claim that the right-hand side of (145) is smaller or equal than $(f-1)(p-1) + m - 1 + \sum_{J^{\max}(\sigma)} s_j - p(m-1)$ if $J(\lambda) = J^{\max}(\sigma)$ and strictly smaller than $(f-1)(p-1) + \sum_{J^{\max}(\sigma)} s_j - p(m-1)$ if $J(\lambda) \supsetneq J^{\max}(\sigma)$. Recalling that $s_j = p - 2 - t_j + \mathbf{1}_{J^{\max}(\sigma)}(j-1)$ for $j \in J^{\max}(\sigma)$, the first case follows from $(f - |J^{\max}(\sigma)|)(p-1) = (f-1)(p-1) + m - 1 - p(m-1)$. For the second case, as $(f-1)(p-1) - p(m-1) = (f - |J^{\max}(\sigma)|)(p-1) - (m-1)$, it is enough to prove

$$\begin{aligned}(f - |J(\lambda)|)(p-1) + \sum_{j \in J(\lambda)} (p - 2 - t_j) + |J(\lambda) \cap (J(\lambda)+1)| \\ < (f - |J^{\max}(\sigma)|)(p-1) + \sum_{j \in J^{\max}(\sigma)} (p - 2 - t_j) + |J^{\max}(\sigma) \cap (J^{\max}(\sigma)+1)| \\ - (m-1),\end{aligned}$$

or equivalently (by an easy calculation):

$$(m-1) + |J(\lambda) \cap (J(\lambda)+1)| - |J^{\max}(\sigma) \cap (J^{\max}(\sigma)+1)| < \sum_{j \in J(\lambda) \setminus J^{\max}(\sigma)} (t_j + 1).$$

This is true, as $m - 1 \leq f - 1$ (so the left-hand side is at most $(f-1) + f$), $J(\lambda) \setminus J^{\max}(\sigma) \neq \emptyset$ and $t_j + 1 \geq 2f$ for any j by genericity (132).

Therefore $\|\underline{k}\| \leq (f-1)(p-1) + (m-1) + \sum_{J^{\max}(\sigma)} s_j - p(m-1)$ if $\underline{Y}^{\underline{k}}\begin{pmatrix} p & 0 \\ 0 & 1 \end{pmatrix} x_\sigma \neq 0$ and $\underline{Y}^{\underline{k}}\begin{pmatrix} p & 0 \\ 0 & 1 \end{pmatrix} x_\sigma \in \delta(\sigma)$ if $\|\underline{k}\| \geq (f-1)(p-1) + \sum\limits_{J^{\max}(\sigma)} s_j - p(m-1)$.

We prove the remaining statements in (a) and (b) (for $\underline{i} = \underline{0}$). If $\|\underline{k}\| \geq (f-1)(p-1) + \sum\limits_{J^{\max}(\sigma)} s_j - p(m-1)$ and $\underline{Y}^{\underline{k}}\begin{pmatrix} p & 0 \\ 0 & 1 \end{pmatrix} x_\sigma \neq 0$, we know by above that $J(\lambda) = J^{\max}(\sigma)$. By (144) we then have $k_j \leq s_j$ if $j \in J^{\max}(\sigma)$ and $k_j \leq p-1$ if $j \notin J^{\max}(\sigma)$. By the definition of $x_{\delta(\sigma)}$ in (135) and by Lemma 3.2.2.6(ii) (and Remark 3.2.2.3) we deduce $\underline{Y}^{\underline{k}}\begin{pmatrix} p & 0 \\ 0 & 1 \end{pmatrix} x_\sigma = \underline{Y}^{-\underline{\ell}} x_{\delta(\sigma)}$, where $\ell_j = s_j - k_j$ if $j \in J^{\max}(\sigma)$ and $\ell_j = p - 1 - k_j$ if $j \notin J^{\max}(\sigma)$. This implies $\|\underline{\ell}\| = (f-m)(p-1) + \sum\limits_{J^{\max}(\sigma)} s_j - \|\underline{k}\|$, and in particular $\|\underline{\ell}\| = 0$ if $\|\underline{k}\| = (f-1)(p-1) + (m-1) + \sum_{J^{\max}(\sigma)} s_j - (m-1)p$ and $\|\underline{\ell}\| = m-1$ if $\|\underline{k}\| = (f-1)(p-1) + \sum\limits_{J^{\max}(\sigma)} s_j - p(m-1)$. This finishes the proof of (a) and (b). □

Now we can finally complete the proof of Proposition 3.2.3.1.

Proof of Proposition 3.2.3.1(ii). Suppose first that $m > 0$ and fix $j_0 \in J^{\max}(\sigma)$. By

Lemma 3.2.2.4, Proposition 3.2.2.9 and the definition of F (see (ii) in §2.1.1), we have

$$\begin{aligned}&Y^{\sum_{j\in J^{\max}(\sigma)} s_j} F(Y^{1-m}x_\sigma)\\ &\qquad = \left[(-1)^{f-1}\prod_{j\in J^{\max}(\sigma)} Y_j^{s_j}\prod_{j\neq j_0}(Y_j-Y_{j_0})^{p-1} + f(\underline{Y})\right]\left(\begin{smallmatrix} p & 0\\ 0 & 1\end{smallmatrix}\right)Y^{1-m}x_\sigma\end{aligned}$$

for some $f(\underline{Y}) \in \mathbb{F}[\![Y_0,\dots,Y_{f-1}]\!]$ of $\mathfrak{m}_{N_0}$-adic valuation (i.e. total degree) $\geq \sum_{J^{\max}(\sigma)} s_j + (p-1)f$. As $p > f \geq m$ we have $(p-1)f > (p-1)(f-1)+m-1$ and by Lemma 3.2.3.5(i) we get $f(\underline{Y})\left(\begin{smallmatrix} p & 0\\ 0 & 1\end{smallmatrix}\right)Y^{1-m}x_\sigma = 0$, hence

$$Y^{\sum_{J^{\max}(\sigma)} s_j} F(Y^{1-m}x_\sigma) = (-1)^{f-1}\prod_{j\in J^{\max}(\sigma)} Y_j^{s_j}\prod_{j\neq j_0}(Y_j-Y_{j_0})^{p-1}\left(\begin{smallmatrix} p & 0\\ 0 & 1\end{smallmatrix}\right)Y^{1-m}x_\sigma.$$

Moreover, the right-hand side is contained in $\langle \underline{Y}^{-\underline{\ell}}x_{\delta(\sigma)}, \|\underline{\ell}\| = m-1\rangle_{\mathbb{F}} \subseteq \delta(\sigma)$ by Lemma 3.2.3.5(ii). As it is also N_1-invariant, it is contained in $\mathbb{F}Y^{1-m}x_{\delta(\sigma)}$ by Lemma 3.2.2.6(iii). It is therefore enough to show that $Y^{m-1+\sum_{J^{\max}(\sigma)} s_j}F(Y^{1-m}x_\sigma) = (-1)^{f-1}x_{\delta(\sigma)}$, or again by Lemma 3.2.2.4, Proposition 3.2.2.9 and Lemma 3.2.3.5(i) that

$$Y_{j_0}^{m-1}\prod_{j\in J^{\max}(\sigma)} Y_j^{s_j}\prod_{j\neq j_0}(Y_j-Y_{j_0})^{p-1}\left(\begin{smallmatrix} p & 0\\ 0 & 1\end{smallmatrix}\right)Y^{1-m}x_\sigma = x_{\delta(\sigma)}.$$

As $\binom{p-1}{i} = (-1)^i$ for $0 \leq i \leq p-1$, the left-hand side equals

$$Y_{j_0}^{m-1}\prod_{J^{\max}(\sigma)} Y_j^{s_j}\sum_{\substack{\|\underline{k}'\|=(p-1)(f-1)\\ k'_j\leq p-1 \text{ if } j\neq j_0}} \underline{Y}^{\underline{k}'}\left(\begin{smallmatrix} p & 0\\ 0 & 1\end{smallmatrix}\right)Y^{1-m}x_\sigma. \tag{146}$$

By Lemma 3.2.3.5(i), as $k'_j + s_j$ can never be congruent to s_j modulo p when $k'_j \in \{1,\dots,p-1\}$, only the terms with $k'_j = 0$ for $j \in J^{\max}(\sigma)\setminus\{j_0\}$ and $k'_j = p-1$ for $j \notin J^{\max}(\sigma)$ survive. As $\|\underline{k}'\| = (p-1)(f-1)$, we must have $k'_{j_0} = (p-1)(m-1)$, and by Lemma 3.2.3.5(i) again it follows that (146) equals $x_{\delta(\sigma)}$, as required.

Finally suppose $m = 0$. As $Y_j^p\left(\begin{smallmatrix} p & 0\\ 0 & 1\end{smallmatrix}\right)x_\sigma = 0$ for all j, we get again by Lemma 3.2.2.4, Proposition 3.2.2.9 and (135):

$$\begin{aligned}Y^{p-1}F(x_\sigma) &= (-1)^{f-1}Y_0^{p-1}\prod_{j\neq 0}(Y_j-Y_0)^{p-1}\left(\begin{smallmatrix} p & 0\\ 0 & 1\end{smallmatrix}\right)x_\sigma\\ &= (-1)^{f-1}\prod_{j=0}^{f-1} Y_j^{p-1}\left(\begin{smallmatrix} p & 0\\ 0 & 1\end{smallmatrix}\right)x_\sigma = (-1)^{f-1}x_{\delta(\sigma)}.\end{aligned}$$

□

3.2.4 Lower bound for $V_{\mathrm{GL}_2}(\pi)$: proof

We prove Theorem 3.2.1.1.

We keep the notation of §§3.2.1, 3.2.2, 3.2.3. Fix $\sigma \in W(\overline{\rho})$ and define $\sigma_i \in W(\overline{\rho})$ inductively by $\sigma_1 \stackrel{\text{def}}{=} \sigma$ and $\sigma_i \stackrel{\text{def}}{=} \delta(\sigma_{i-1})$ for $i > 1$ (σ_i here shouldn't be confused with the embedding $\sigma_i = \sigma_0 \circ \varphi^i$). Let $n \geq 1$ be the smallest integer such that $\sigma_{n+1} \cong \sigma_1$ and write $\sigma_i = (s_0^{(i)}, \dots, s_{f-1}^{(i)}) \otimes \eta_i$. Recall that $n = 1$ if and only if $J^{\max}(\sigma) = \emptyset$ if and only if $\overline{\rho}$ is reducible and σ corresponds to $J_\sigma = \emptyset$ or $J_\sigma = S$ (see the beginning of §3.2.3). We set $m \stackrel{\text{def}}{=} |J^{\max}(\sigma_i)|$ if $n > 1$ (this doesn't depend on $i \in \{1, \dots, n\}$ by Lemma 3.2.3.2) and $m \stackrel{\text{def}}{=} 1$ if $n = 1$, so that $m \in \{1, \dots, f\}$. For $i \in \{1, \dots, n\}$ we let χ_i denote the H-eigencharacter on $\sigma_i^{N_0} = \sigma_i^{I_1}$. We also define for $i \in \{1, \dots, n\}$:

$$\begin{aligned} s_i &\stackrel{\text{def}}{=} \sum_{j \in J^{\max}(\sigma_i)} s_j^{(i+1)} && \text{if } n > 1, \\ s_1 &\stackrel{\text{def}}{=} p - 1 && \text{if } n = 1. \end{aligned}$$

The following lemma will be useful later.

Lemma 3.2.4.1. *We have* $\sum_{i=1}^n s_i \equiv 0 \pmod{p-1}$.

Proof. Let $s(\chi_i) \in \{0, \dots, q-1\}$ such that $\chi_{i+1} = \chi_i \alpha^{-s(\chi_i)}$ and denote by $|s(\chi_i)| \in \{0, \dots, (p-1)f\}$ the sum of the digits of $s(\chi_i)$ in its p-expansion. Then it follows from (153) below that we have

$$\alpha^{\sum_{j \in J^{\max}(\sigma_i)} s_j^{(i+1)} p^j + \sum_{j \notin J^{\max}(\sigma_i)} (p-1) p^j} \chi_i = \chi_{i+1}$$

and so

$$s(\chi_i) = \sum_{j \in J^{\max}(\sigma_i)} (p - 1 - s_j^{(i+1)}) p^j \tag{147}$$

which implies $|s(\chi_i)| = (p-1)m - s_i$. As $\chi_{n+1} = \chi_1 = \chi_1 \alpha^{-\sum_{i=1}^n s(\chi_i)}$, we have $\sum_{i=1}^n s(\chi_i) \equiv 0 \pmod{q-1}$, hence $\sum_{i=1}^n |s(\chi_i)| \equiv 0 \pmod{p-1}$ and the result follows. ☐

Recall π is as at the end of §3.2.1. In [Bre11, §4] there is defined an $\mathbb{F}$-linear isomorphism

$$S : (\operatorname{soc}_{\mathrm{GL}_2(\mathcal{O}_K)} \pi)^{I_1} \xrightarrow{\sim} (\operatorname{soc}_{\mathrm{GL}_2(\mathcal{O}_K)} \pi)^{I_1}. \tag{148}$$

Fixing an embedding $\sigma \hookrightarrow \operatorname{soc}_{\mathrm{GL}_2(\mathcal{O}_K)} \pi$, for $i \in \{2, \dots, n\}$ there are unique embeddings $\sigma_i \hookrightarrow \operatorname{soc}_{\mathrm{GL}_2(\mathcal{O}_K)} \pi$ such that the morphism S cyclically permutes the lines $\sigma_i^{I_1}$. In particular there exists $\nu \in \mathbb{F}^\times$ (which depends on σ but not on the fixed embedding $\sigma \hookrightarrow \operatorname{soc}_{\mathrm{GL}_2(\mathcal{O}_K)} \pi$) such that $S^n|_{\sigma_i^{I_1}}$ is the multiplication by ν for all $i \in \{1, \dots, n\}$. We define $\mu_i \in \mathbb{F}^\times$ for $1 \leq i \leq n$ by $\mu_1 \stackrel{\text{def}}{=} \nu$ if $n = 1$ and if $n > 1$:

$$\mu_i \stackrel{\text{def}}{=} \begin{cases} \left(\prod_{1 \leq i' \leq n} \prod_{j \in J^{\max}(\sigma_{i'})} (p - 1 - s_j^{(i'+1)})! \right)^{-1} \nu & \text{if } i = n, \\ 1 & \text{otherwise.} \end{cases}$$

We let M_σ be the $\mathbb{F}[\![X]\!][F]$-submodule of π^{N_1}, or equivalently the $\mathbb{F}[\![Y]\!][F]$-submodule, generated by $Y^{1-m}\sigma_i^{N_0} = Y^{1-m}\sigma_i^{I_1}$ for $1 \le i \le n$. Recall $\gamma \in \mathbb{Z}_p^\times$ acts on $M_\sigma \otimes \chi_\pi^{-1}$ by the action of $\left(\begin{smallmatrix} 1 & 0 \\ 0 & \gamma^{-1} \end{smallmatrix}\right)$ (see the end of §3.2.1).

Proposition 3.2.4.2. *The module $M_\sigma \otimes \chi_\pi^{-1}$ is admissible as an $\mathbb{F}[\![X]\!]$-module (see §2.1.1), $\mathbb{Z}_p^\times$-stable, and such that $(M_\sigma \otimes \chi_\pi^{-1})^\vee$ is free of rank n as an $\mathbb{F}[\![X]\!]$-module. Moreover the étale (φ, Γ)-module $(M_\sigma \otimes \chi_\pi^{-1})^\vee[1/X]$ admits a basis $(e_1, \dots, e_n)$ over $\mathbb{F}[\![X]\!][1/X]$ such that for $i \in \{1, \dots, n\}$ (with $e_{n+1} \stackrel{\text{def}}{=} e_1$):*

$$\varphi(e_i) = \mu_i^{-1} X^{s_i} e_{i+1}, \tag{149}$$

$$\gamma(e_i) \in \chi_i\Big(\left(\begin{smallmatrix} 1 & 0 \\ 0 & \gamma \end{smallmatrix}\right)\Big)\overline{\gamma}^m (1 + X\mathbb{F}[\![X]\!]) e_i \text{ for all } \gamma \in \mathbb{Z}_p^\times, \tag{150}$$

where $\overline{\gamma}$ is the image of $\gamma \in \mathbb{Z}_p^\times$ in $\mathbb{F}$. Moreover $\gamma(e_i)$ is uniquely determined by (149) *and* (150)*.*

To prepare for the proof, fix $x_1 \in \sigma_1^{N_0} \setminus \{0\}$ and define for $1 \le i \le n-1$:

$$x_{i+1} \stackrel{\text{def}}{=} (-1)^{f-1} \prod_{j \in J^{\max}(\sigma_i)} Y_j^{s_j^{(i+1)}} \prod_{j \notin J^{\max}(\sigma_i)} Y_j^{p-1} \left(\begin{smallmatrix} p & 0 \\ 0 & 1 \end{smallmatrix}\right) x_i \in \sigma_{i+1}^{N_0} \setminus \{0\}$$

and $x_{n+1} \stackrel{\text{def}}{=} x_1$ (note that this formula is (135) multiplied by $(-1)^{f-1}$).

Lemma 3.2.4.3. *For $i \in \{1, \dots, n\}$ we have*

$$S(x_i) = \Big(\prod_{j \in J^{\max}(\sigma_i)} (p - 1 - s_j^{(i+1)})!\Big) \mu_i x_{i+1} \tag{151}$$

and

$$Y^{s_i} F(Y^{1-m} x_i) = \mu_i Y^{1-m} x_{i+1}. \tag{152}$$

Proof. If $i \in \{1, \dots, n\}$ we have

$$\begin{aligned} (-1)^{f-1} & \prod_{j \in J^{\max}(\sigma_i)} Y_j^{s_j^{(i+1)}} \prod_{j \notin J^{\max}(\sigma_i)} Y_j^{p-1} \left(\begin{smallmatrix} p & 0 \\ 0 & 1 \end{smallmatrix}\right) x_i \\ &= \Big(\prod_{j \in J^{\max}(\sigma_i)} (p - 1 - s_j^{(i+1)})!\Big)^{-1} \theta_{\sum_{J^{\max}(\sigma_i)} (p-1-s_j^{(i+1)})p^j} \left(\begin{smallmatrix} p & 0 \\ 0 & 1 \end{smallmatrix}\right) x_i \\ &= \Big(\prod_{j \in J^{\max}(\sigma_i)} (p - 1 - s_j^{(i+1)})!\Big)^{-1} S(x_i), \end{aligned} \tag{153}$$

where the first equality follows from Lemma 3.2.2.5(i) and the second from the definition of the function S in [Bre11, §4]. From the definition of x_{i+1}, we obtain (151) for $i < n$. For $i = n$, using inductively

$$x_{i+1} = \Big(\prod_{j \in J^{\max}(\sigma_i)} (p - 1 - s_j^{(i+1)})!\Big)^{-1} S(x_i)$$

for $i = n-1$, $i = n-2$ till $i = 1$ we obtain (as S is $\mathbb{F}$-linear):

$$\begin{aligned} S(x_n) &= \Big(\prod_{j \in J^{\max}(\sigma_{n-1})} (p-1-s_j^{(n)})!\Big)^{-1} S^2(x_{n-1}) \\ &= \cdots \\ &= \Big(\prod_{1 \le i \le n-1} \prod_{j \in J^{\max}(\sigma_i)} (p-1-s_j^{(i+1)})!\Big)^{-1} S^n(x_1). \end{aligned}$$

Since $S^n(x_1) = \nu x_1$ and from the definition of μ_n, we get (151) for $i = n$. The last part follows from Proposition 3.2.3.1 combined with (153) and (151). $\square$

The following lemma is stated with the variable Y, but remains the same with the variable X.

Lemma 3.2.4.4. *Suppose M is a torsion $\mathbb{F}[\![Y]\!]$-module. Let $\Sigma \subseteq M$ be a subset spanning M as an $\mathbb{F}$-vector space and set $\widetilde{\Sigma} \overset{\text{def}}{=} \bigcup_{v \in \Sigma} \mathbb{F}^\times v$. If*

(i) $Y\Sigma \subseteq \widetilde{\Sigma} \cup \{0\}$;

(ii) $\mathbb{F}Yv_1 = \mathbb{F}Yv_2 \neq 0 \Longrightarrow v_1 = v_2$ *for* $v_1, v_2 \in \Sigma$;

(iii) $\Sigma \cap M[Y]$ *is a finite set of* $\mathbb{F}$*-linearly independent vectors,*

then Σ is an $\mathbb{F}$-basis of M and M is an admissible $\mathbb{F}[\![Y]\!]$-module. If moreover $Y\widetilde{\Sigma} = \widetilde{\Sigma} \cup \{0\}$, then $M^\vee$ is a finite free $\mathbb{F}[\![Y]\!]$-module of rank $\dim_{\mathbb{F}} M[Y]$.

Proof. Write $\Sigma \cap M[Y] = \{v_1, \ldots, v_d\}$ (assuming $\Sigma \cap M[Y] \neq \emptyset$ otherwise $M = 0$ and there is nothing to prove). For $\ell \in \{1, \ldots, d\}$ let $\Sigma_\ell \overset{\text{def}}{=} \{v \in \Sigma : Y^j v \in \mathbb{F}^\times v_\ell \text{ for some } j \geq 0\}$. Then $M_\ell \overset{\text{def}}{=} \oplus_{v \in \Sigma_\ell} \mathbb{F}v$ is an $\mathbb{F}[\![Y]\!]$-module using (i). If $v, v' \in \Sigma_\ell$, then using (ii) there is $j \geq 0$ such that either $\mathbb{F}v = \mathbb{F}Y^j v'$, or $\mathbb{F}v' = \mathbb{F}Y^j v$, from which one easily deduces $M_\ell[Y] = \mathbb{F}v_\ell$, in particular M_ℓ is admissible. Since Σ spans M over $\mathbb{F}$ and $\Sigma = \coprod_{\ell=1}^n \Sigma_\ell$, the natural map $f : \bigoplus_{\ell=1}^d M_\ell \to M$ is surjective, and thus M is also admissible. Since $\bigoplus_\ell M_\ell[Y] = \bigoplus_\ell \mathbb{F}v_\ell \hookrightarrow M[Y]$ (the last injection following from (iii)), we deduce that $\mathrm{Ker}(f)[Y] = 0$, hence $\mathrm{Ker}(f) = 0$ and f is an isomorphism. This proves the first part of the statement. It follows from $Y\widetilde{\Sigma} = \widetilde{\Sigma} \cup \{0\}$ that the multiplication by Y is surjective on each M_ℓ, i.e. we have exact sequences $0 \to \mathbb{F}v_\ell \to M_\ell \xrightarrow{Y} M_\ell \to 0$. Dualizing, this gives $0 \to M_\ell^\vee \xrightarrow{Y} M_\ell^\vee \to (\mathbb{F}v_\ell)^\vee \to 0$, which shows $M_\ell^\vee$ is free of rank 1 over $\mathbb{F}[\![Y]\!]$. The last statement follows. $\square$

Recall that M_σ is the $\mathbb{F}[\![Y]\!][F]$-submodule of π^{N_1} generated by $Y^{1-m}x_i$ for $1 \leq i \leq n$. Let

$$\Sigma \overset{\text{def}}{=} \left\{ \begin{array}{ll} Y^j F^k(Y^{1-m}x_i) : 1 \leq i \leq n,\ k \geq 0, & 0 \leq j < p^{k-1}s_i \quad \text{if } k \geq 1 \\ & 0 \leq j < m \quad\quad\ \ \text{if } k = 0 \end{array} \right\}.$$

We now check that M_σ and Σ satisfy all the assumptions in Lemma 3.2.4.4. Define for $\ell \in \mathbb{Z}_{\geq 1}$:

$$\Sigma_\ell \stackrel{\text{def}}{=} \left\{Y^j F^k(Y^{1-m}x_i) \in \Sigma : k+i \equiv \ell \pmod n\right\}$$

and $M_{\ell,\sigma} \stackrel{\text{def}}{=} \bigoplus_{v\in\Sigma_\ell} \mathbb{F}v$. We have $\Sigma = \coprod_{\ell=1}^n \Sigma_\ell$. Applying F^{k-1} to (152) for $k \geq 1$ we get (recall that $F \circ Y = Y^p \circ F$ on π^{N_1}):

$$Y^{p^{k-1}s_i} F^k(Y^{1-m}x_i) \in \mathbb{F}^\times F^{k-1}(Y^{1-m}x_{i+1}), \tag{154}$$

hence Σ spans M_σ and condition (i) of Lemma 3.2.4.4 holds for Σ. Using (154) we also see that the multiplication by Y induces an injection $\Sigma_\ell \hookrightarrow \widetilde{\Sigma}_\ell \cup \{0\}$ and that $Y\widetilde{\Sigma}_\ell = \widetilde{\Sigma}_\ell \cup \{0\}$, hence $M_{\ell,\sigma}$ is an $F[\![Y]\!]$-submodule of M_σ and condition (ii) of Lemma 3.2.4.4 holds for Σ_ℓ and Σ. Moreover, $Y\widetilde{\Sigma} = \widetilde{\Sigma} \cup \{0\}$. Finally, $\Sigma \cap M_\sigma[Y] = \{x_1, \dots, x_n\}$ (and $\Sigma \cap M_{\ell,\sigma}[Y] = x_\ell$). By Lemma 3.2.4.4 and its proof, we deduce that Σ is an $\mathbb{F}$-basis of M_σ, that $M_\sigma = \bigoplus_{\ell=1}^n M_{\ell,\sigma}$ and that each $M_{\ell,\sigma}^\vee$ is free of rank 1 over $F[\![Y]\!]$. In fact one can visualize the "Y-divisible line" $M_{i+1,\sigma}$ as follows using (152):

$$\mathbb{F}x_{i+1} \stackrel{Y^{m-1}}{\longleftarrow} \mathbb{F}Y^{1-m}x_{i+1} \stackrel{Y^{s_i}}{\longleftarrow} \mathbb{F}F(Y^{1-m}x_i) \stackrel{Y^{ps_{i-1}}}{\longleftarrow} \mathbb{F}F^2(Y^{1-m}x_{i-1}) \stackrel{Y^{p^2 s_{i-2}}}{\longleftarrow} \mathbb{F}F^3(Y^{1-m}x_{i-2}) \longleftarrow \cdots,$$

where $\mathbb{F}x_{i+1} = M_{i+1,\sigma}[Y]$ and the arrows mean "multiplication by the power of Y just above". In particular we see that if $d(v) \stackrel{\text{def}}{=} \min\{j \geq 1 : Y^j v = 0\}$ for $v \in \Sigma$, then $v \in \Sigma_{i+1}$ is contained in $F(\widetilde{\Sigma})$ if and only if $d(v) \equiv s_i + m \pmod p$.

Define a basis $f_1, \dots, f_n$ of the free $F[\![Y]\!]$-module $M_\sigma^\vee$ by

$$f_i(x_i) \stackrel{\text{def}}{=} 1 \text{ and } f_i(\Sigma \setminus \{x_i\}) \stackrel{\text{def}}{=} 0,\ i \in \{1, \dots, n\}.$$

From what is above we then easily deduce the following formula, where $F(f)(v) \stackrel{\text{def}}{=} f(F(v))$ for $f \in M_\sigma^\vee$ and $v \in M_\sigma$ (and using conventions as in §2.1.1):

$$F(Y^{\ell+(s_i+m-1)}f_{i+1}) = \begin{cases} \mu_i Y^{m-1} f_i & \text{if } \ell = 0, \\ 0 & \text{if } 1 \leq \ell \leq p-1. \end{cases} \tag{155}$$

Lemma 3.2.4.5. *The module $M_\sigma \otimes \chi_\pi^{-1}$ is $\mathbb{Z}_p^\times$-stable, hence $\mathbb{Z}_p^\times$ acts on $(M_\sigma \otimes \chi_\pi^{-1})^\vee$. Moreover we have for $\gamma \in \mathbb{Z}_p^\times$ (recall $\gamma(f)(v) = f\left(\left(\begin{smallmatrix} 1 & 0 \\ 0 & \gamma \end{smallmatrix}\right)v\right)$ for $f \in (M_\sigma \otimes \chi_\pi^{-1})^\vee$, $v \in M_\sigma$):*

$$\gamma(f_i) \in \chi_i\left(\left(\begin{smallmatrix} 1 & 0 \\ 0 & \gamma \end{smallmatrix}\right)\right)(1 + Y\mathbb{F}[\![Y]\!])f_i$$

for $1 \leq i \leq n$.

Proof. As $M_\sigma = \bigoplus_{i=1}^n \mathbb{F}[\![Y]\!][F]Y^{1-m}x_i$ and $Y^{1-m}x_i$ is a $\mathbb{Z}_p^\times$-eigenvector by Lemma 3.2.2.6(iii) we deduce that M_σ, and hence $M_\sigma \otimes \chi_\pi^{-1}$, are $\mathbb{Z}_p^\times$-stable.

From $\gamma\circ X = ((1+X)^\gamma - 1)\circ\gamma$ and Lemma 3.2.2.2 it is easy to deduce that $\gamma\circ Y = f_\gamma(Y)\circ\gamma$ for some $f_\gamma(Y)\in\gamma Y + Y^2\mathbb{F}[\![Y]\!]$, hence $\mathbb{Z}_p^\times$ preserves the decomposition of $\mathbb{F}[\![Y]\!]$-modules $M_\sigma\otimes\chi_\pi^{-1} = \bigoplus_{i=1}^n M_{i,\sigma}\otimes\chi_\pi^{-1}$. In particular, $\gamma(f_i)$ annihilates $M_{i',\sigma}\otimes\chi_\pi^{-1}$ for all $i'\neq i$. Let $Y^{-j}x_i$ for $j\geq 0$ denote the unique element of $\widetilde{\Sigma}_i$ such that $Y^j(Y^{-j}x_i) = x_i$ (this is compatible with our previous notation in Lemma 3.2.2.6(iii)). Then

$$\gamma(f_i) = \sum_{j\geq 0}(\gamma(f_i)(Y^{-j}x_i))Y^j f_i \in \chi_i\Big(\left(\begin{smallmatrix}1&0\\0&\gamma\end{smallmatrix}\right)\Big)(1+Y\mathbb{F}[\![Y]\!])f_i. \qquad \square$$

Proof of Proposition 3.2.4.2. We have already seen above that $M_\sigma\otimes\chi_\pi^{-1}$ is admissible, $\mathbb{Z}_p^\times$-stable, and that $(M_\sigma\otimes\chi_\pi^{-1})^\vee$ is free of rank n as an $\mathbb{F}[\![N_0/N_1]\!]$-module. To find the basis $(e_i)_i$, first note from Lemma 3.2.2.2 and (155) that (using $F\circ Y^p = Y\circ F$ on $(M_\sigma\otimes\chi_\pi^{-1})^\vee$):

$$\begin{aligned} F(X^{s_i+m-1}f_{i+1}) &= F\Big(\sum_{j\geq 0} c_j Y^{s_i+m-1+j} f_{i+1}\Big) \\ &= \mu_i \sum_{j\geq 0} c_{jp} Y^{m-1+j} f_i \\ &\in (-1)^{s_i}\mu_i(1+X\mathbb{F}[\![X]\!])X^{m-1}f_i \end{aligned} \tag{156}$$

for some $c_j\in\mathbb{F}$ with $c_0 = (-1)^{s_i+m-1}$. Similarly for $\ell\in\{1,\dots,p-1\}$:

$$F\Big(X^{s_i+m-1+\ell}f_{i+1}\Big) \in \mathbb{F}[\![X]\!]X^m f_i. \tag{157}$$

It easily follows from (14) that

$$\sum_{\ell=0}^{p-1}(1+X)^{-\ell}\varphi\Big(F((1+X)^\ell f)\Big) = f \tag{158}$$

for all $f\in(M_\sigma\otimes\chi_\pi^{-1})^\vee[1/X]$. Let $f \overset{\text{def}}{=} X^{s_i+m-1}f_{i+1}$, by (156) and (157) we have for $\ell\in\{0,\dots,p-1\}$:

$$F((1+X)^\ell f) \in (-1)^{s_i}\mu_i(1+X\mathbb{F}[\![X]\!])X^{m-1}f_i,$$

and so

$$\varphi\Big(F((1+X)^\ell f)\Big) \in (-1)^{s_i}\mu_i(1+X^p\mathbb{F}[\![X]\!])\varphi(X^{m-1}f_i).$$

Using

$$\sum_{\ell=0}^{p-1}(1+X)^{-\ell} = \Big(\frac{X}{1+X}\Big)^{p-1} \equiv X^{p-1} \pmod{X^p},$$

we see that (158) applied to $f = X^{s_i+m-1}f_{i+1}$ becomes

$$(-1)^{s_i}\mu_i X^{p-1}\varphi(X^{m-1}f_i) \in (1+X\mathbb{F}[\![X]\!])X^{s_i+m-1}f_{i+1}$$

or equivalently in $(M_\sigma \otimes \chi_\pi^{-1})^\vee[1/X]$:

$$\varphi(X^m f_i) = (-1)^{s_i}\mu_i^{-1} g_i(X) X^{s_i+m} f_{i+1} \tag{159}$$

for some $g_i(X) \in 1 + X\mathbb{F}[\![X]\!]$.

Let $e_i \overset{\text{def}}{=} (-1)^{\sum_{j=1}^{i-1} s_j} h_i(X) X^m f_i$ for some $h_i(X) \in 1 + X\mathbb{F}[\![X]\!]$ and note that the sign doesn't change if i is replaced by $i+n$ by Lemma 3.2.4.1. Then (149) is equivalent to

$$h_i(X^p)\varphi(X^m f_i) = (-1)^{s_i}\mu_i^{-1} h_{i+1}(X) X^{s_i} X^m f_{i+1},$$

or equivalently $h_i(X^p) g_i(X) = h_{i+1}(X)$ by (159). This system has the unique solution

$$h_i(X) = \prod_{j=1}^{\infty} g_{i-j}(X^{p^{j-1}})$$

in $1 + X\mathbb{F}[\![X]\!]$, where the indices are considered modulo n. Then (150) follows from Lemma 3.2.4.5. The final uniqueness assertion follows from $\gamma \circ \varphi = \varphi \circ \gamma$ and is left as an exercise (similar to [Bre11, Lemma 4.5]). □

Let $\mathcal{O}(\pi)$ (resp. $\mathcal{O}(\overline{\rho})$) be a set of representatives for the orbits of δ on the set of Serre weights in $\mathrm{soc}_{\mathrm{GL}_2(\mathcal{O}_K)}\,\pi$ counted with their multiplicity r (resp. on the set $W(\overline{\rho})$). We define $M_\pi \overset{\text{def}}{=} \bigoplus_{\sigma \in \mathcal{O}(\pi)} M_\sigma$ (with M_σ as above). It follows from the assumptions on π that we have

$$M_\pi \cong \bigoplus_{\sigma \in \mathcal{O}(\overline{\rho})} M_\sigma^{\oplus r}.$$

In particular $(M_\pi \otimes \chi_\pi^{-1})^\vee[1/X]$ is an étale (φ, Γ)-module over $\mathbb{F}(\!(X)\!)$ of rank $r|W(\overline{\rho})| = r2^f$. From the description of $M_\sigma[X]$, we also see that the natural map $M_\pi \to \pi^{N_1}$ of torsion $F[\![X]\!]$-modules is injective as the following composition is injective:

$$M_\pi[X] \cong \oplus \sigma^{I_1} \hookrightarrow \pi^{I_1} \subseteq \pi^{N_1}[X],$$

where the direct sum is over all Serre weights σ in $\mathrm{soc}_{\mathrm{GL}_2(\mathcal{O}_K)}\,\pi$ (counting their multiplicity r).

Proposition 3.2.4.6. *We have an isomorphism of representations of* $\mathrm{Gal}(\overline{\mathbb{Q}}_p/\mathbb{Q}_p)$ *over* $\mathbb{F}$*:*

$$\mathbf{V}((M_\pi \otimes \chi_\pi^{-1})^\vee[1/X]) \cong \left(\mathrm{ind}_K^{\otimes \mathbb{Q}_p}(\overline{\rho})\right)^{\oplus r}.$$

Proof. We are going to use a computation of [Bre11, §4]. Associated to the diagram $D \overset{\text{def}}{=} D(\overline{\rho})^{\oplus r}$ of §3.2.1, there is defined in *loc.cit.* an étale (φ, Γ)-module over $\mathbb{F}(\!(X)\!)$ denoted there $M(D)$ and which is of the form $M(D) = \oplus_{\sigma \in \mathcal{O}(\pi)} M(D)_\sigma$[2], where $M(D)_\sigma$

[2]A more consistent notation with the ones of this article would have been $M(D)^\vee$ and $M(D)_\sigma^\vee \ldots$

is a rank n étale (φ,Γ)-module over $\mathbb{F}(\!(X)\!)$ associated to the orbit of σ, i.e. to the cycle $\sigma=\sigma_1,\dots,\sigma_n$ as above (so in fact one has $M(D)=\oplus_{\sigma\in\mathcal{O}(\overline{\rho})}M(D)_\sigma^{\oplus r}$).

Let $N\stackrel{\text{def}}{=}\mathbb{F}(\!(X)\!)e$ be the rank 1 étale (φ,Γ)-module over $\mathbb{F}(\!(X)\!)$ defined by

$$\begin{aligned}\varphi(e)&=X^{-(p-1)\sum_j(r_j+1)}e,\\ \gamma(e)&=\left(\frac{\overline{\gamma}X}{(1+X)^\gamma-1}\right)^{\sum_j(r_j+1)}e.\end{aligned}$$

We have $\mathbf{V}(N)\cong\omega^{\sum_j(r_j+1)}=\mathrm{ind}_K^{\otimes\mathbb{Q}_p}(\det\overline{\rho})$ (using $\mathrm{ind}_K^{\otimes\mathbb{Q}_p}(\omega_f)\cong\omega$) by [Bre11, Prop.3.5] and

$$\mathbf{V}(M(D))\cong\Big(\mathrm{ind}_K^{\otimes\mathbb{Q}_p}(\overline{\rho}\otimes(\det\overline{\rho})^{-1})\Big)^{\oplus r}\cong\Big(\mathrm{ind}_K^{\otimes\mathbb{Q}_p}(\overline{\rho})\otimes\mathrm{ind}_K^{\otimes\mathbb{Q}_p}(\det\overline{\rho}^{-1})\Big)^{\oplus r}$$

by [Bre11, Thm.6.4]. We therefore deduce

$$\mathbf{V}(M(D)\otimes_{\mathbb{F}(\!(X)\!)}N)\cong\Big(\mathrm{ind}_K^{\otimes\mathbb{Q}_p}(\overline{\rho})\Big)^{\oplus r}.$$

Therefore it suffices to show that $M(D)_\sigma\otimes_{\mathbb{F}(\!(X)\!)}N\cong M_\sigma^\vee[1/X]$ for each $\sigma\in\mathcal{O}(\pi)$, or equivalently each $\sigma\in\mathcal{O}(\overline{\rho})$.

Let $x_1^\vee,\dots,x_n^\vee\in(\bigoplus_{i=1}^n\sigma_i^{I_1})^\vee$ be the dual basis of the $\mathbb{F}$-basis $(x_i)_i$ of $\bigoplus_{i=1}^n\sigma_i^{I_1}$, it follows from its definition in [Bre11, §4] and from (147) that $M(D)_\sigma$ has basis $x_1^\vee,\dots,x_n^\vee$ as $\mathbb{F}(\!(X)\!)$-module with

$$\varphi(x_i^\vee)=X^{s_i+(p-1)(f-m)}\Bigg(\prod_{j\in J^{\max}(\sigma_i)}(p-1-s_j^{(i+1)})!\Bigg)(x_i^\vee\circ S|^{-1}_{\oplus\sigma_i^{I_1}}),$$

where S^{-1} is the inverse of the bijection S of (148) (which preserves $\bigoplus_{i=1}^n\sigma_i^{I_1}$). By (151) we have

$$x_i^\vee\circ S|^{-1}_{\oplus\sigma_i^{I_1}}=\Bigg(\prod_{J^{\max}(\sigma_i)}(p-1-s_j^{(i+1)})!\Bigg)^{-1}\mu_i^{-1}x_{i+1}^\vee,$$

so we obtain

$$\varphi(x_i^\vee)=\mu_i^{-1}X^{s_i+(p-1)(f-m)}x_{i+1}^\vee.$$

Also we have for $\gamma\in\mathbb{Z}_p^\times$ (using the hypothesis on the central character of π):

$$\begin{aligned}x_i^\vee\circ\left(\begin{smallmatrix}\gamma^{-1}&0\\0&1\end{smallmatrix}\right)&=\overline{\gamma}^{-\sum_j r_j}\left(x_i^\vee\circ\left(\begin{smallmatrix}1&0\\0&\gamma\end{smallmatrix}\right)\right)\\&=\overline{\gamma}^{-\sum_j r_j}\chi_i\Big(\left(\begin{smallmatrix}1&0\\0&\gamma\end{smallmatrix}\right)\Big)x_i^\vee,\end{aligned}$$

hence with the definition of $\gamma(x_i^\vee)$ given in [Bre11, Lemma 4.5]:

$$\gamma(x_i^\vee)\in\chi_i\Big(\left(\begin{smallmatrix}1&0\\0&\gamma\end{smallmatrix}\right)\Big)\overline{\gamma}^{-\sum_j r_j}(1+X\mathbb{F}[\![X]\!])x_i^\vee.$$

We deduce that $M(D)_\sigma \otimes_{\mathbb{F}(\!(X)\!)} N \cong \bigoplus_{i=1}^n \mathbb{F}[\![X]\!](x_i^\vee \otimes e)$ with

$$\varphi(x_i^\vee \otimes e) = \mu_i^{-1} X^{s_i-(p-1)(m+\sum_j r_j)}(x_{i+1}^\vee \otimes e),$$
$$\gamma(x_i^\vee \otimes e) \in \chi_i\Big(\big(\begin{smallmatrix}1 & 0\\ 0 & \gamma\end{smallmatrix}\big)\Big)\overline{\gamma}^{-\sum_j r_j}(1+X\mathbb{F}[\![X]\!])(x_i^\vee \otimes e).$$

Now, let $e_i' \stackrel{\text{def}}{=} X^{m+\sum_j r_j}(x_i^\vee \otimes e)$ for all i. Then $e_1', \ldots, e_n'$ is a basis of $M(D)_\sigma \otimes_{\mathbb{F}(\!(X)\!)} N$ and we have for $i \in \{1, \ldots, n\}$ (with $e_{n+1}' \stackrel{\text{def}}{=} e_1'$):

$$\varphi(e_i') = \mu_i^{-1} X^{s_i} e_{i+1}',$$
$$\gamma(e_i') \in \chi_i\Big(\big(\begin{smallmatrix}1 & 0\\ 0 & \gamma\end{smallmatrix}\big)\Big)\overline{\gamma}^{m}(1+X\mathbb{F}[\![X]\!])e_i'.$$

From Proposition 3.2.4.2 we see that $M(D)_\sigma \otimes_{\mathbb{F}(\!(X)\!)} N \cong M_\sigma^\vee[1/X]$. □

By Lemma 3.2.1.2 this completes the proof of Theorem 3.2.1.1 when the constants ν_i are as in [Bre11, Thm.6.4]. When they are arbitrary, the proof of Proposition 3.2.4.6 gives $\mathbf{V}((M_\pi \otimes \chi_\pi^{-1})^\vee[1/X])|_{I_{\mathbb{Q}_p}} \cong \Big(\mathrm{ind}_K^{\otimes \mathbb{Q}_p}(\overline{\rho})\Big)|_{I_{\mathbb{Q}_p}}^{\oplus r}$ using [Bre11, Cor.5.4], which finishes the proof of Theorem 3.2.1.1.

3.3 On the structure of some representations of $\mathrm{GL}_2(K)$

We prove results on the structure of an admissible smooth representation π of $\mathrm{GL}_2(K)$ over $\mathbb{F}$ associated to a semisimple sufficiently generic representation $\overline{\rho}$ of $\mathrm{Gal}(\overline{\mathbb{Q}}_p/K)$ as in [BP12] when π satisfies a further multiplicity one assumption as in [BHH+23] and a self-duality property. In particular we prove that such a π is irreducible if and only if $\overline{\rho}$ is, and is semisimple when $f=2$ (Corollary 3.3.5.8 and Corollary 3.3.5.6).

We keep the notation at the beginning of §§3, 3.1, and set $\Lambda \stackrel{\text{def}}{=} \mathbb{F}[\![I_1/Z_1]\!]$. We recall that the graded ring $\mathrm{gr}(\Lambda)$ is isomorphic to $\otimes_{i=0}^{f-1}\mathbb{F}[y_i, z_i, h_i]$ with h_i lying in the center (see (117)). We set

$$R \stackrel{\text{def}}{=} \mathrm{gr}(\Lambda)/(h_0, \ldots, h_{f-1}),$$

which is commutative and isomorphic to $\mathbb{F}[y_i, z_i, 0 \leq i \leq f-1]$, and recall that $\overline{R} = R/(y_iz_i, 0 \leq i \leq f-1) = \mathrm{gr}(\Lambda)/J$ (see (122)). Moreover the finite torus H naturally acts on Λ by the conjugation on I_1 (via its Teichmüller lift) and we see (using (101)) that the induced action on $\mathrm{gr}(\Lambda)$ is trivial on h_i and is the multiplication by the character α_i (resp. α_i^{-1}) on y_i (resp. z_i), where $\alpha_i\Big(\big(\begin{smallmatrix}\lambda & 0\\ 0 & \mu\end{smallmatrix}\big)\Big) \stackrel{\text{def}}{=} \sigma_i(\lambda\mu^{-1})$ for $\big(\begin{smallmatrix}\lambda & 0\\ 0 & \mu\end{smallmatrix}\big) \in H$.

Notice that $\mathrm{gr}(\Lambda)$ is an Auslander regular ring (see [LvO96, Def.III.2.1.7], [LvO96, Def.III.2.1.3]) by the first statement in [BHH+23, Thm.5.3.4] and so is Λ itself by [LvO96, Thm.III.2.2.5]. This allows us to apply (many) results of [LvO96, §III.2].

For any ring S and any S-module M, we set $\mathrm{E}_S^i(M) \stackrel{\text{def}}{=} \mathrm{Ext}_S^i(M, S)$ for $i \geq 0$.

3.3.1 Combinatorial results

We define some explicit ideals $\mathfrak{a}(\lambda)$ of R and study some of their properties.

We fix a continuous representation $\overline{\rho} : \mathrm{Gal}(\overline{\mathbb{Q}}_p/K) \to \mathrm{GL}_2(\mathbb{F})$ which is generic in the sense of [BP12, §11] and let $D_0(\overline{\rho})$ be the representation of $\mathrm{GL}_2(\mathbb{F}_q)$ over $\mathbb{F}$ defined in [BP12, §13] (see also §3.2.1 when $\overline{\rho}$ is semisimple). Recall from [BP12, Cor.13.6] that $D_0(\overline{\rho})^{I_1}$ is multiplicity-free as a representation of $H \cong I/I_1$. By [Bre14, §4], there is a bijection between the characters of H appearing in $D_0(\overline{\rho})^{I_1}$ and a certain set of f-tuples, denoted by

$$\mathscr{PID}(x_0, \ldots, x_{f-1}), \text{ resp. } \mathscr{PRD}(x_0, \ldots, x_{f-1}), \text{ resp. } \mathscr{PD}(x_0, \ldots, x_{f-1}),$$

if $\overline{\rho}$ is irreducible, resp. reducible split, resp. reducible nonsplit. We refer to [Bre14, §4] for the precise definition of these sets and we simply write $\mathscr{P}$ for the set associated to $\overline{\rho}$. We write χ_λ for the character of H associated to $\lambda \in \mathscr{P}$ (more precisely, in *loc.cit.* one rather associates a Serre weight σ_λ to λ, and χ_λ is the action of $H = I/I_1$ on the 1-dimensional subspace $\sigma_\lambda^{I_1}$, different σ_λ giving different χ_λ).

On the other hand, the set $W(\overline{\rho})$ is in bijection with another set of f-tuples, denoted by (see [BP12, §11])

$$\mathscr{ID}(x_0, \ldots, x_{f-1}), \text{ resp. } \mathscr{RD}(x_0, \ldots, x_{f-1}), \text{ resp. } \mathscr{D}(x_0, \ldots, x_{f-1}),$$

depending on $\overline{\rho}$ as above. We simply write $\mathscr{D}$ for the set associated to $\overline{\rho}$. Since the socle of $D_0(\overline{\rho})$ is $\oplus_{\sigma \in W(\overline{\rho})}\sigma$, we may view $\mathscr{D}$ as a subset of $\mathscr{P}$. For example, if $\overline{\rho}$ is reducible split, then $\mathscr{D}$ is the subset of $\mathscr{P}$ consisting of λ such that

$$\lambda_j(x_j) \in \{x_j, x_j + 1, p - 2 - x_j, p - 3 - x_j\},$$

while if $\overline{\rho}$ is nonsplit, then we require moreover that $\lambda_j(x_j) \in \{x_j + 1, p - 3 - x_j\}$ implies $j \in J_{\overline{\rho}}$, where $J_{\overline{\rho}}$ is a certain subset of $\{0, \ldots, f - 1\}$ uniquely determined by the Fontaine–Laffaille module of $\overline{\rho}$ (cf. [Bre14, (17)]).

Definition 3.3.1.1. We associate to $\lambda \in \mathscr{P}$ an ideal $\mathfrak{a}(\lambda)$ of R as follows.

- If $\overline{\rho}$ is irreducible, then $\mathfrak{a}(\lambda) = (t_0, \ldots, t_{f-1})$, where

$$t_0 \stackrel{\text{def}}{=} \begin{cases} z_0 & \text{if} \quad \lambda_0(x_0) \in \{x_0 - 1, p - 2 - x_0\} \\ y_0 & \text{if} \quad \lambda_0(x_0) \in \{x_0 + 1, p - x_0\} \\ y_0 z_0 & \text{if} \quad \lambda_0(x_0) \in \{x_0, p - 1 - x_0\}, \end{cases}$$

  and if $j \neq 0$

$$t_j \stackrel{\text{def}}{=} \begin{cases} z_j & \text{if} \quad \lambda_j(x_j) \in \{x_j, p - 3 - x_j\} \\ y_j & \text{if} \quad \lambda_j(x_j) \in \{x_j + 2, p - 1 - x_j\} \\ y_j z_j & \text{if} \quad \lambda_j(x_j) \in \{x_j + 1, p - 2 - x_j\}. \end{cases}$$

- If $\overline{\rho}$ is reducible nonsplit, then $\mathfrak{a}(\lambda) = (t_0, \dots, t_{f-1})$, where

$$t_j \overset{\text{def}}{=} \begin{cases} z_j & \text{if } \lambda_j(x_j) \in \{x_j, p-3-x_j\} \text{ and } j \in J_{\overline{\rho}} \\ y_j & \text{if } \lambda_j(x_j) \in \{x_j+2, p-1-x_j\} \text{ and } j \in J_{\overline{\rho}} \\ y_j z_j & \text{if } \lambda_j(x_j) \in \{x_j, p-1-x_j\} \text{ and } j \notin J_{\overline{\rho}} \\ y_j z_j & \text{if } \lambda_j(x_j) \in \{x_j+1, p-2-x_j\}. \end{cases}$$

- If $\overline{\rho}$ is reducible split, then $\mathfrak{a}(\lambda) = (t_0, \dots, t_{f-1})$ is defined as in the nonsplit case by letting $J_{\overline{\rho}} = \{0, \dots, f-1\}$, namely

$$t_j \overset{\text{def}}{=} \begin{cases} z_j & \text{if } \lambda_j(x_j) \in \{x_j, p-3-x_j\} \\ y_j & \text{if } \lambda_j(x_j) \in \{x_j+2, p-1-x_j\} \\ y_j z_j & \text{if } \lambda_j(x_j) \in \{x_j+1, p-2-x_j\}. \end{cases}$$

In particular, if $\overline{\rho}$ is reducible nonsplit and $J_{\overline{\rho}} = \emptyset$, then $\mathfrak{a}(\lambda) = (y_0 z_0, \dots, y_{f-1} z_{f-1})$ for any $\lambda \in \mathscr{P}$. Note that $R/\mathfrak{a}(\lambda)$ is always a quotient of $\overline{R}$.

Remark 3.3.1.2. An equivalent form of Definition 3.3.1.1 is as follows (compare the proof of Theorem 3.3.2.1). Given $\lambda \in \mathscr{P}$, $t_j = y_j$ (resp. $t_j = z_j$) if and only if the character $\chi_\lambda \alpha_j^{-1}$ (resp. $\chi_\lambda \alpha_j$) occurs in $D_0(\overline{\rho})^{I_1}$ (i.e. has the form $\chi_{\lambda'}$ for some $\lambda' \in \mathscr{P}$), and $t_j = y_j z_j$ if and only if neither of $\chi_\lambda \alpha_j^{\pm 1}$ occurs in $D_0(\overline{\rho})^{I_1}$.

Lemma 3.3.1.3. *Let $\lambda \in \mathscr{P}$.*

(i) *Assume $\overline{\rho}$ is semisimple. Then $\lambda \in \mathscr{D}$ if and only if $y_j \notin \mathfrak{a}(\lambda)$ for any $j \in \{0, \dots, f-1\}$.*

(ii) *Assume $\overline{\rho}$ is reducible nonsplit and let $\overline{\rho}^{\text{ss}}$ be the semisimplification of $\overline{\rho}$. Then there is a bijection between $\mathscr{D}(\overline{\rho}^{\text{ss}})$ (defined as the set $\mathscr{D}$ associated to $\overline{\rho}^{\text{ss}}$) and the set of $\lambda \in \mathscr{P}$ such that $y_j \notin \mathfrak{a}(\lambda)$ for any $j \in \{0, \dots, f-1\}$.*

Proof. (i) It is clear by definition of $\mathscr{D}$ and $\mathfrak{a}(\lambda)$.
(ii) Let $\lambda \in \mathscr{P}$ such that $y_j \notin \mathfrak{a}(\lambda)$ for any $j \in \{0, \dots, f-1\}$. By definition, we have (for $\overline{\rho}$ reducible nonsplit)

$$\lambda_j(x_j) \in \{x_j, x_j+1, p-1-x_j, p-2-x_j, p-3-x_j\}$$

and from the definition of $\mathfrak{a}(\lambda)$ if $\lambda_j(x_j) = p-1-x_j$ then $j \notin J_{\overline{\rho}}$ (note that if $\lambda_j(x_j) = p-3-x_j$ then it is automatic that $j \in J_{\overline{\rho}}$). We define an f-tuple μ by

$$\mu_j(x_j) \overset{\text{def}}{=} \begin{cases} p-3-x_j & \text{if } \lambda_j(x_j) = p-1-x_j \\ \lambda_j(x_j) & \text{otherwise.} \end{cases}$$

It is then easy to see that μ is an element of $\mathscr{D}(\overline{\rho}^{\text{ss}})$ and that any element of $\mathscr{D}(\overline{\rho}^{\text{ss}})$ arises (uniquely) in this way. $\square$

Corollary 3.3.1.4. *The set* $\{\lambda \in \mathscr{P} : y_j \notin \mathfrak{a}(\lambda)\ \forall\ j \in \{0, \ldots, f-1\}\}$ *has cardinality* 2^f.

Proof. This is a direct consequence of Lemma 3.3.1.3 and of $|W(\overline{\rho}^{\mathrm{ss}})| = 2^f$. □

Given $\lambda \in \mathscr{P}$, write $\mathfrak{a}(\lambda) = (t_0, \ldots, t_{f-1})$ as in Definition 3.3.1.1 and define

$$\mathcal{A}(\lambda) \stackrel{\mathrm{def}}{=} \{j \in \{0, \ldots, f-1\} : t_j = y_j z_j\} \subseteq \{0, \ldots, f-1\}. \tag{160}$$

The following proposition will only be used in Corollary 3.3.2.5 below.

Proposition 3.3.1.5. *We have* $\sum_{\lambda \in \mathscr{P}} 2^{|\mathcal{A}(\lambda)|} = 4^f$.

Proof. We will only give the proof in the case $\overline{\rho}$ is reducible (split or not), the irreducible case can be treated similarly.

First assume that $\overline{\rho}$ is split. Given $\lambda \in \mathscr{P}$, we define an element $\overline{\lambda} \in \mathscr{D}$ as follows:

$$\overline{\lambda}_j(x_j) \stackrel{\mathrm{def}}{=} \begin{cases} x_j & \text{if} \quad \lambda_j(x_j) \in \{x_j, x_j+2\} \\ p-3-x_j & \text{if} \quad \lambda_j(x_j) \in \{p-1-x_j, p-3-x_j\} \\ \lambda_j(x_j) & \qquad \text{otherwise.} \end{cases}$$

It is easy to see that $\overline{\lambda} \in \mathscr{D}$. By definition of $\mathscr{P}$ (see [Bre14, §4] and recall $\mathscr{P} = \mathscr{P}\mathscr{R}\mathscr{D}(x_0, \ldots, x_{f-1})$), for each $\overline{\lambda} \in \mathscr{D}$, there are exactly $2^{|\{0,\ldots,f-1\} \setminus \mathcal{A}(\overline{\lambda})|}$ elements λ in $\mathscr{P}$ giving rise to $\overline{\lambda}$ under the above rule. Moreover, it is direct from the definitions that $\mathcal{A}(\lambda) = \mathcal{A}(\overline{\lambda})$. Hence

$$\sum_{\lambda \in \mathscr{P}} 2^{|\mathcal{A}(\lambda)|} = \sum_{\overline{\lambda} \in \mathscr{D}} (2^{f-|\mathcal{A}(\overline{\lambda})|} 2^{|\mathcal{A}(\overline{\lambda})|}) = 2^f |\mathscr{D}| = 2^f 2^f = 4^f.$$

Now assume that $\overline{\rho}$ is nonsplit. Let $\overline{\mathscr{P}}$ be the subset of $\mathscr{P}$ considered in the proof of Lemma 3.3.1.3(ii), namely $\lambda \in \overline{\mathscr{P}}$ if and only if

$$\lambda_j(x_j) \in \{x_j, x_j+1, p-1-x_j, p-2-x_j, p-3-x_j\}$$

and $\lambda_j(x_j) = p-1-x_j$ implies $j \notin J_{\overline{\rho}}$. By the proof of *loc.cit.*, we have $|\overline{\mathscr{P}}| = |\mathscr{D}(\overline{\rho}^{\mathrm{ss}})| = 2^f$. Given $\lambda \in \mathscr{P}$, we define an element $\overline{\lambda} \in \overline{\mathscr{P}}$ as follows:

$$\overline{\lambda}_j(x_j) \stackrel{\mathrm{def}}{=} \begin{cases} x_j & \text{if} \quad \lambda_j(x_j) \in \{x_j, x_j+2\} \\ p-3-x_j & \text{if} \quad \lambda_j(x_j) = p-3-x_j \text{ or } (\lambda_j(x_j) = p-1-x_j \text{ and } j \in J_{\overline{\rho}}) \\ \lambda_j(x_j) & \qquad \text{otherwise.} \end{cases}$$

As in the split case it is easy to see that $\mathcal{A}(\lambda) = \mathcal{A}(\overline{\lambda})$ and that given $\overline{\lambda} \in \overline{\mathscr{P}}$, there exist exactly $2^{|\{0,\ldots,f-1\} \setminus \mathcal{A}(\overline{\lambda})|}$ elements λ in $\mathscr{P}$ giving rise to $\overline{\lambda}$. The result follows as in the split case. □

Definition 3.3.1.6. Given $\lambda \in \mathscr{P}$, we define another f-tuple λ^* as follows:

$$\lambda_j^*(x_j) \overset{\text{def}}{=} \begin{cases} p-3-\lambda_j(x_j) & \text{if} \quad t_j = z_j \\ p+1-\lambda_j(x_j) & \text{if} \quad t_j = y_j \\ p-1-\lambda_j(x_j) & \text{if} \quad t_j = y_j z_j. \end{cases}$$

If $\lambda \in \mathscr{D}$, we define its "length" $\ell(\lambda)$ to be (see [BP12, §4]):

$$\ell(\lambda) \overset{\text{def}}{=} |\{j \in \{0, \dots, f-1\} : \lambda_j(x_j) \in \{p-2-x_j \pm 1, x_j \pm 1\}\}|. \tag{161}$$

Lemma 3.3.1.7. *Let $\lambda \in \mathscr{P}$.*

(i) *We have $\lambda^* \in \mathscr{P}$ and $\mathfrak{a}(\lambda) = \mathfrak{a}(\lambda^*)$.*

(ii) *Assume that $\overline{\rho}$ is semisimple. Then $\lambda \in \mathscr{D}$ if and only if $\lambda^* \in \mathscr{D}$, and in this case $\ell(\lambda^*) = f - \ell(\lambda)$.*

Proof. (i) The first statement can be checked directly using the definition of $\mathscr{P}$ and the second one is obvious from the definitions.

(ii) The first statement follows from (i) and Lemma 3.3.1.3(i). By definition of $\mathscr{D}$ (see [BP12, §11]), $\ell(\lambda)$ can be computed as the cardinality of the following set:

$$\Big\{j \in \{0, \dots, f-1\} : \lambda_j(x_j) \in \{p-1-x_j, p-2-x_j, p-3-x_j\}\Big\}.$$

For example, when $\overline{\rho}$ is reducible split, we have (cf. the beginning of [BP12, §11])

$$\lambda_j(x_j) \in \{p-2-x_j, p-3-x_j\} \Longleftrightarrow \lambda_{j+1}(x_{j+1}) \in \{p-3-x_{j+1}, x_{j+1}+1\}.$$

The second statement of (ii) follows from this and Definition 3.3.1.6. □

Lemma 3.3.1.8. *Let $\lambda \in \mathscr{P}$, χ_λ the character of H associated to λ, $(t_0, \dots, t_{f-1})$ the ideal $\mathfrak{a}(\lambda)$ in Definition 3.3.1.1 and η_λ be the character of H acting on $\prod_{j=0}^{f-1} t_j$. Then we have*

$$\chi_\lambda \chi_{\lambda^*} = \eta_\lambda (\eta \circ \det),$$

where λ^* is as in Definition 3.3.1.6 and $\eta(a) \overset{\text{def}}{=} \chi_\lambda\Big(\left(\begin{smallmatrix} a & 0 \\ 0 & a \end{smallmatrix}\right)\Big)$ for $a \in \mathbb{F}_q^\times$ (η does not depend on $\lambda \in \mathscr{P}$).

Proof. This is an easy computation, but we give some details. Note that $\lambda_j(x_j) + \lambda_j^*(x_j) = (p-1) + 2\varepsilon_j$, where ε_j equals 1, 0 or -1 if t_j equals y_j, $y_j z_j$ or z_j respectively. Moreover, in the notation of [Bre14, §4], we have

$$\begin{aligned} e(\lambda) + e(\lambda^*) &= \frac{1}{2}\Big(p^f - 1 + \sum_{j=0}^{f-1} p^j (x_j - \lambda_j(x_j) + x_j - \lambda_j^*(x_j))\Big) \\ &= \sum_{j=0}^{f-1} p^j (x_j - \varepsilon_j). \end{aligned}$$

The conclusion follows now from a simple computation, noting that for $\left(\begin{smallmatrix} a & 0 \\ 0 & b \end{smallmatrix}\right) \in H$

$$\chi_\lambda\Big(\left(\begin{smallmatrix} a & 0 \\ 0 & b \end{smallmatrix}\right)\Big) = \sigma_0(a)^{\left(\sum_{j=0}^{f-1} p^j \lambda_j(r_j)\right) + e(\lambda)(r_0,\dots,r_{f-1})} \sigma_0(b)^{e(\lambda)(r_0,\dots,r_{f-1})}$$

(see [Bre14, §4]) and that H acts on y_i (resp. z_i) via α_i (resp. α_i^{-1}). □

Note that H acts on I_1/Z_1 by conjugation and hence on Λ and $\mathrm{gr}(\Lambda)$, preserving the filtration and the graded pieces on the former and the latter respectively. This induces H-actions also on R, $\overline{R}$, and $R/\mathfrak{a}(\lambda)$ for any $\lambda \in \mathscr{P}$. We say that a $\mathrm{gr}(\Lambda)$-module M has a *compatible H-action* if it has an H-action such that $h(rm) = h(r)h(m)$ for all $h \in H$, $r \in \mathrm{gr}(\Lambda)$, and $m \in M$. In this case $\mathrm{E}^i_{\mathrm{gr}(\Lambda)}(M)$ is again a $\mathrm{gr}(\Lambda)$-module with compatible H-action for any $i \geq 0$.

Lemma 3.3.1.9. *If M is a $\mathrm{gr}(\Lambda)$-module with compatible H-action that is annihilated by $(h_0, \dots, h_{f-1})$, then we have isomorphisms of $\mathrm{gr}(\Lambda)$-modules with compatible H-action for $i \geq 0$:*

$$\mathrm{E}^{i+f}_{\mathrm{gr}(\Lambda)}(M) \cong \mathrm{E}^i_R(M). \tag{162}$$

If moreover M is annihilated by J, then we have isomorphisms of $\mathrm{gr}(\Lambda)$-modules with compatible H-action for $i \geq 0$:

$$\mathrm{E}^{i+2f}_{\mathrm{gr}(\Lambda)}(M) \cong \mathrm{E}^{i+f}_R(M) \cong \mathrm{E}^i_{\overline{R}}(M). \tag{163}$$

Proof. Since $(h_0, \dots, h_{f-1})$ is a regular sequence of central elements in $\mathrm{gr}(\Lambda)$ and $(y_0z_0, \dots, y_{f-1}z_{f-1})$ is a regular sequence in R (which is commutative), the isomorphisms (162) and (163) as $\mathrm{gr}(\Lambda)$-modules are proved as in the proof of [BHH+23, Lemma 5.1.3]. Moreover, H acts trivially on h_j and y_jz_j (for $0 \leq j \leq f-1$), the isomorphisms are also H-equivariant, from which the results follow. □

We don't use the following proposition in the sequel, but it is consistent with Remark 3.3.2.6(i) and the essential self-duality assumption (iii) in §3.3.5 below (see Proposition 3.3.4.6).

Proposition 3.3.1.10. *For $\lambda \in \mathscr{P}$ there is an isomorphism of $\mathrm{gr}(\Lambda)$-modules with compatible H-action:*

$$\mathrm{E}^{2f}_{\mathrm{gr}(\Lambda)}\Big(\chi_\lambda^{-1} \otimes R/\mathfrak{a}(\lambda)\Big) \cong \Big(\chi_{\lambda^*}^{-1} \otimes R/\mathfrak{a}(\lambda)\Big) \otimes \eta \circ \det.$$

Proof. Applying (163) with $i = 0$ and $M = \chi_\lambda^{-1} \otimes R/\mathfrak{a}(\lambda)$, we are left to prove

$$\mathrm{Hom}_{\overline{R}}(\chi_\lambda^{-1} \otimes R/\mathfrak{a}(\lambda), \overline{R}) \cong \Big(\chi_{\lambda^*}^{-1} \otimes R/\mathfrak{a}(\lambda)\Big) \otimes \eta \circ \det.$$

Using Lemma 3.3.1.8, it suffices to construct an isomorphism of $\mathrm{gr}(\Lambda)$-modules with compatible H-action

$$\mathrm{Hom}_{\overline{R}}(R/\mathfrak{a}(\lambda), \overline{R}) \cong \eta_\lambda^{-1} \otimes R/\mathfrak{a}(\lambda), \tag{164}$$

where η_λ is the character of H acting on $\prod_{j=0}^{f-1} t_j$ if we write $\mathfrak{a}(\lambda) = (t_0, \dots, t_{f-1})$ with $t_j \in \{y_j, z_j, y_j z_j\}$. Put $t' \overset{\mathrm{def}}{=} \prod_{j=0}^{f-1}(y_j z_j / t_j)$. One easily checks that $t'\overline{R} = \overline{R}[\mathfrak{a}(\lambda)]$ and there is an isomorphism of $\overline{R}$-modules

$$\theta : \ \eta_\lambda^{-1} \otimes \overline{R}/\mathfrak{a}(\lambda) \xrightarrow{\sim} t'\overline{R},$$

where the first map sends 1 to t'. As H acts on t' via η_λ^{-1}, θ is also H-equivariant. The isomorphism (164) is then obtained by sending $r \in \eta_\lambda^{-1} \otimes R/\mathfrak{a}(\lambda)$ to $\phi \in \mathrm{Hom}_{\overline{R}}(R/\mathfrak{a}(\lambda), \overline{R})$ such that $\phi(1) \overset{\mathrm{def}}{=} \theta(r)$. $\square$

3.3.2 On the structure of $\mathrm{gr}(\pi^\vee)$

We give a partial result on the structure of $\mathrm{gr}(\pi^\vee)$ for certain admissible smooth representations π of $\mathrm{GL}_2(K)$ over $\mathbb{F}$ associated to $\overline{\rho}$ when $\mathrm{gr}(\pi^\vee)$ comes from the $\mathfrak{m}_{I_1/Z_1}$-adic filtration on $\pi^\vee$.

We let $\overline{\rho}$ be as in §3.3.1 (in particular $\overline{\rho}$ is not necessarily semisimple) and keep the notation of *loc.cit.* As in §3.2.1 when $\overline{\rho}$ is semisimple, we consider $D_0(\overline{\rho})$ as a representation of $\mathrm{GL}_2(\mathcal{O}_K)K^\times$, where $\mathrm{GL}_2(\mathcal{O}_K)$ acts via its quotient $\mathrm{GL}_2(\mathbb{F}_q)$ and the center $K^\times$ acts by the character $\det(\overline{\rho})\omega^{-1}$. We now write $\mathfrak{m}$ for $\mathfrak{m}_{I_1/Z_1}$.

We consider an admissible smooth representation π of $\mathrm{GL}_2(K)$ over $\mathbb{F}$ satisfying the following two conditions:

(i) there is $r \geq 1$ such that $\pi^{K_1} \cong D_0(\overline{\rho})^{\oplus r}$ as a representation of $\mathrm{GL}_2(\mathcal{O}_K)K^\times$ (in particular π has a central character);

(ii) for any $\lambda \in \mathscr{P}$, we have an equality of multiplicities

$$[\pi[\mathfrak{m}^3] : \chi_\lambda] = [\pi[\mathfrak{m}] : \chi_\lambda].$$

Note that (ii) implies that the $\mathrm{gr}(\Lambda)$-module $\mathrm{gr}(\pi^\vee)$ (defined with the $\mathfrak{m}$-adic filtration on $\pi^\vee$) is annihilated by the ideal J in (118) by the proof of [BHH+23, Cor.5.3.5], and in particular is an $\overline{R}$-module.

Theorem 3.3.2.1. *For π as above, there is a surjection of $\mathrm{gr}(\Lambda)$-modules with compatible H-action*

$$\Big(\bigoplus_{\lambda \in \mathscr{P}} \chi_\lambda^{-1} \otimes R/\mathfrak{a}(\lambda)\Big)^{\oplus r} \twoheadrightarrow \mathrm{gr}(\pi^\vee), \tag{165}$$

where $\mathfrak{a}(\lambda)$ is as in Definition 3.3.1.1.

Proof. Consider the $\mathrm{gr}(\Lambda)$-module with compatible H-action:

$$M \stackrel{\text{def}}{=} \Big(\bigoplus_{\lambda\in\mathscr{P}} \chi_\lambda^{-1}\otimes R/\mathfrak{a}(\lambda)\Big)^{\oplus r}.$$

Since there is a bijection $\lambda\mapsto\chi_\lambda$ between $\mathscr{P}$ and the characters of H on $D_0(\overline{\rho})^{I_1}$ (see §3.3.1), we can choose a basis of π^{I_1} over $\mathbb{F}$, say $\{v_{\lambda,k}:\lambda\in\mathscr{P},1\le k\le r\}$, such that each $v_{\lambda,k}$ is an eigenvector for I of character χ_λ. We denote by $\{e_{\lambda,k}:\lambda\in\mathscr{P},1\le k\le r\}$ the basis of $\mathrm{gr}^0(\pi^\vee)$ over $\mathbb{F}$ which is the dual basis of $\{v_{\lambda,k}\}$, and note that $\{e_{\lambda,k}\}$ generates the $\mathrm{gr}(\Lambda)$-module $\mathrm{gr}(\pi^\vee)$. To prove that there exists a surjective morphism $M\twoheadrightarrow\mathrm{gr}(\pi^\vee)$ it suffices to prove that, for any $\lambda\in\mathscr{P}$ and any $k\in\{1,\dots,r\}$, the vector $e_{\lambda,k}$ is annihilated by the ideal $\mathfrak{a}(\lambda)$ of $R=\mathrm{gr}(\Lambda)/(h_0,\dots,h_{f-1})$. Writing $\mathfrak{a}(\lambda)=(t_0,\dots,t_{f-1})$ as in Definition 3.3.1.1, we already see that if $t_j=y_jz_j$, then t_j kills all the $e_{\lambda,k}$ since $\mathrm{gr}(\pi^\vee)$ is annihilated by J.

Let $j\in\{0,\dots,f-1\}$ such that $t_j\in\{y_j,z_j\}$ and define $\chi'\stackrel{\text{def}}{=}\chi_\lambda\alpha_j^{-1}$ if $t_j=y_j$, $\chi'\stackrel{\text{def}}{=}\chi_\lambda\alpha_j$ if $t_j=z_j$. By Definition 3.3.1.1 one checks that $\chi'=\chi_{\lambda'}$, where $\lambda'\in\mathscr{P}$ is defined by $\lambda'_i(x_i)\stackrel{\text{def}}{=}\lambda_i(x_i)$ if $i\ne j$, and $\lambda'_j(x_j)\stackrel{\text{def}}{=}\lambda_j(x_j)+\varepsilon_j$, where ε_j equals either -2 or 2 when t_j equals either y_j or z_j respectively. Note that χ'^{-1} is equal to the character of I acting on $t_je_{\lambda,k}\in\mathrm{gr}^1(\pi^\vee)$. Thus, if $t_je_{\lambda,k}\neq0$, then dually the χ'-isotypic subspace of $\pi[\mathfrak{m}^2]/\pi[\mathfrak{m}]$ would be nonzero. But this contradicts condition (ii) above. Hence $e_{\lambda,k}$ is annihilated by the whole ideal $\mathfrak{a}(\lambda)$ and we are done. □

Corollary 3.3.2.2. *Let π' be a subrepresentation of π and $\mathscr{P}'\subseteq\mathscr{P}$ be the subset corresponding to the characters* (*without multiplicities*) *of H appearing in π'^{I_1}. Then* $\mathrm{gr}(\pi'^\vee)$ (*with the $\mathfrak{m}$-adic filtration on $\pi'^\vee$*) *is a quotient of* $\Big(\bigoplus_{\lambda\in\mathscr{P}'}\chi_\lambda^{-1}\otimes R/\mathfrak{a}(\lambda)\Big)^{\oplus r}$.

Proof. We have a natural quotient map $\pi^\vee\twoheadrightarrow\pi'^\vee$ which induces a quotient map $\mathrm{gr}(\pi^\vee)\twoheadrightarrow\mathrm{gr}(\pi'^\vee)$. It is enough to prove that the composition

$$\Big(\bigoplus_{\lambda\in\mathscr{P}'}\chi_\lambda^{-1}\otimes R/\mathfrak{a}(\lambda)\Big)^{\oplus r}\hookrightarrow\Big(\bigoplus_{\lambda\in\mathscr{P}}\chi_\lambda^{-1}\otimes R/\mathfrak{a}(\lambda)\Big)^{\oplus r}\twoheadrightarrow\mathrm{gr}(\pi^\vee)\twoheadrightarrow\mathrm{gr}(\pi'^\vee)$$

is surjective (where the second map is the surjection of Theorem 3.3.2.1). The assumption implies that it is surjective on $\mathrm{gr}^0(-)$, and we conclude using that $\mathrm{gr}(\pi'^\vee)$ is generated by $\mathrm{gr}^0(\pi'^\vee)$ as a $\mathrm{gr}(\Lambda)$-module. □

If N is a finitely generated $\overline{R}$-module and $\mathfrak{q}$ a minimal prime ideal of $\overline{R}$, recall that $m_{\mathfrak{q}}(N)\in\mathbb{Z}_{\ge0}$ denotes the multiplicity of N at $\mathfrak{q}$, see (123).

Theorem 3.3.2.3. *We have* $\dim_{\mathbb{F}}V_{\mathrm{GL}_2}(\pi)=\dim_{\mathbb{F}(\!(X)\!)}D_\xi^\vee(\pi)\le m_{\mathfrak{p}_0}(\mathrm{gr}(\pi^\vee))\le 2^fr$, *where the minimal ideal $\mathfrak{p}_0$ is as in §3.1.4.*

Proof. This is a direct consequence of (16), of Corollary 3.1.4.5, of Theorem 3.3.2.1 and of Corollary 3.3.1.4, noting that, if $y_j \in \mathfrak{a}(\lambda)$ for some $j \in \{0, \dots, f-1\}$, then $m_{\mathfrak{p}_0}(R/\mathfrak{a}(\lambda)) = 0$ (as $y_j \notin \mathfrak{p}_0$), and if $y_j \notin \mathfrak{a}(\lambda)\ \forall\ j \in \{0, \dots, f-1\}$, then $m_{\mathfrak{p}_0}(R/\mathfrak{a}(\lambda)) = 1$ (as $(R/\mathfrak{a}(\lambda))[(y_0 \cdots y_{f-1})^{-1}] \cong \mathbb{F}[y_0, \dots, y_{f-1}][(y_0 \cdots y_{f-1})^{-1}] \cong \mathrm{gr}(A)$). $\square$

Combined with the results of §3.2, we can deduce the following important corollary.

Corollary 3.3.2.4. *Assume moreover that $\overline{\rho}$ is semisimple, satisfies the genericity condition* (126) *and that condition (i) above can be enhanced into an isomorphism of diagrams $(\pi^{I_1} \hookrightarrow \pi^{K_1}) \cong D(\overline{\rho})^{\oplus r}$, where $D(\overline{\rho})$ is as in* (127)*. Then we have an isomorphism of representations of $I_{\mathbb{Q}_p}$:*

$$V_{\mathrm{GL}_2}(\pi)|_{I_{\mathbb{Q}_p}} \cong \Big(\mathrm{ind}_K^{\otimes \mathbb{Q}_p}(\overline{\rho})\Big)|_{I_{\mathbb{Q}_p}}^{\oplus r}.$$

In particular we have $\dim_{\mathbb{F}} V_{\mathrm{GL}_2}(\pi) = \dim_{\mathbb{F}((X))} D_{\xi}^{\vee}(\pi) = m_{\mathfrak{p}_0}(\pi^{\vee}) = 2^f r$*. If moreover the constants ν_i associated to $D(\overline{\rho} \otimes \chi)$ (χ as in §3.2.1) at the beginning of* [Bre11, §6] *are as in* [Bre11, Thm.6.4]*, then we have an isomorphism of representations of* $\mathrm{Gal}(\overline{\mathbb{Q}}_p/\mathbb{Q}_p)$*:*

$$V_{\mathrm{GL}_2}(\pi) \cong \Big(\mathrm{ind}_K^{\otimes \mathbb{Q}_p}(\overline{\rho})\Big)^{\oplus r}.$$

Proof. It follows from Theorem 3.2.1.1 and Theorem 3.3.2.3 as $\dim_{\mathbb{F}}\Big(\mathrm{ind}_K^{\otimes \mathbb{Q}_p}(\overline{\rho})\Big)^{\oplus r} = 2^f r$. $\square$

It is also worth mentioning the following corollary of Theorem 3.3.2.1.

Corollary 3.3.2.5. *We have $\sum_{\mathfrak{q}} m_{\mathfrak{q}}(\mathrm{gr}(\pi^{\vee})) \leq 4^f r$, where the sum is taken over all minimal prime ideals $\mathfrak{q}$ of $\overline{R}$.*

Proof. By an easy computation, we have $\sum_{\mathfrak{q}} m_{\mathfrak{q}}(R/\mathfrak{a}(\lambda)) = 2^{|\mathcal{A}(\lambda)|}$ (see (160) for $\mathcal{A}(\lambda)$). Thus the result follows from Proposition 3.3.1.5 and Theorem 3.3.2.1. $\square$

Remark 3.3.2.6. (i) It seems possible to us that the surjection in Theorem 3.3.2.1 could actually be an isomorphism, at least for π coming from the global theory as in §3.4.1 below. Note that such an isomorphism implies in particular $\mathrm{E}^i_{\mathrm{gr}(\Lambda)}(\mathrm{gr}(\pi^{\vee})) \neq 0$ if and only if $i = 2f$ (i.e. the $\mathrm{gr}(\Lambda)$-module $\mathrm{gr}(\pi^{\vee})$ is Cohen–Macaulay of grade $2f$), which in turns implies $\mathrm{E}^i_{\Lambda}(\pi^{\vee}) \neq 0$ if and only if $i = 2f$ (use [Ven02, Cor.6.3] and the similar result with $\mathrm{gr}(\Lambda)$ instead of Λ, the first statement in [Ven02, Thm.3.21(ii)] and [LvO96, Thm.I.7.2.11(1)]). Note moreover that by [HW22, Prop.A.8] we know that $\pi^{\vee}$ is Cohen–Macaulay for π coming from the global theory in the so-called *minimal case* (see §3.4.4), but we don't know this for $\mathrm{gr}(\pi^{\vee})$ without extra assumptions (e.g. that

the surjection of Theorem 3.3.2.1 is an isomorphism).
(ii) It is worth recalling here the following implications that we have seen. Consider the following conditions on an admissible smooth representation π of $\mathrm{GL}_2(K)$ over $\mathbb{F}$ with a central character:

(a) $[\pi[\mathfrak{m}^3] : \chi] = [\pi[\mathfrak{m}] : \chi]$ for every character $\chi : I \to \mathbb{F}^\times$ appearing in $\pi[\mathfrak{m}]$;

(b) $\mathrm{gr}(\pi^\vee)$ is killed by J, where $\mathrm{gr}(\pi^\vee)$ is computed with the $\mathfrak{m}$-adic filtration on $\pi^\vee$;

(c) $\mathrm{gr}(\pi^\vee)$ is killed by some power of J, where $\mathrm{gr}(\pi^\vee)$ is computed with any good filtration on the Λ-module $\pi^\vee$;

(d) π is in the category $\mathcal{C}$ of §3.1.2.

Then we have (a) $\Rightarrow$ (b) $\Rightarrow$ (c) $\Rightarrow$ (d). We suspect that every implication is strict.

3.3.3 Examples

We completely compute the $\mathrm{gr}(\mathbb{F}[\![I/Z_1]\!])$-module $\mathrm{gr}(V^\vee)$ for certain irreducible admissible smooth representations V of $\mathrm{GL}_2(K)$ over $\mathbb{F}$ (with $V^\vee$ endowed with the $\mathfrak{m}$-adic filtration). We assume $p \geq 5$ in this section.

We keep the previous notation. If V is a smooth representation of I_1/Z_1 over $\mathbb{F}$, we write $\mathrm{gr}(V^\vee)$ for the graded module associated to the $\mathfrak{m}$-adic filtration on $V^\vee$.

Lemma 3.3.3.1. *Let V be a smooth representation of I_1/Z_1 over $\mathbb{F}$ such that $V|_{N_0}$ is admissible as a representation of N_0 and such that the natural map $\mathrm{gr}_{\mathfrak{m}_{N_0}}(V^\vee) \to \mathrm{gr}(V^\vee)$ (induced by the inclusions $\mathfrak{m}_{N_0}^n V^\vee \subseteq \mathfrak{m}^n V^\vee$ for $n \geq 0$) is surjective. Then this map is an isomorphism.*

Proof. Since $V|_{N_0}^\vee$ is a finite type $\mathbb{F}[\![N_0]\!]$-module by assumption, it is a complete filtered $\mathbb{F}[\![N_0]\!]$-module for the $\mathfrak{m}_{N_0}$-adic filtration. As all the maps $\mathfrak{m}_{N_0}^n V^\vee/\mathfrak{m}_{N_0}^{n+1} V^\vee \to \mathfrak{m}^n V^\vee/\mathfrak{m}^{n+1} V^\vee$ are surjective, any element in $v \in \mathfrak{m}^n V^\vee$ can be written $v = v_0 + w$, where $v_0 \in \sum_{m \geq n} \mathfrak{m}_{N_0}^m V^\vee = \mathfrak{m}_{N_0}^n V^\vee$ (as $V|_{N_0}^\vee$ is complete) and $w \in \cap_{m \geq n} \mathfrak{m}^m V^\vee = 0$ (as the $\mathfrak{m}$-adic filtration is separated since V is smooth). Thus the inclusion $\mathfrak{m}_{N_0}^n V^\vee \subseteq \mathfrak{m}^n V^\vee$ is an equality for $n \geq 0$, and we are done. $\square$

The following two lemmas are motivated by [Paš10, Prop.7.1, Prop.7.2]. We consider the finite group H as subgroup of I via the Teichmüller lift.

Lemma 3.3.3.2. *Let V be an admissible smooth representation of I/Z_1 over $\mathbb{F}$. Assume that $V|_{HN_0}$ is isomorphic to an injective envelope of some character χ in the category of smooth representations of HN_0 over $\mathbb{F}$ (so in particular $\dim_{\mathbb{F}} V^{N_0} = 1$). Then $\mathrm{Ext}^1_{I/Z_1}(\chi\alpha_j^{-1}, V) = 0$ for any $0 \leq j \leq f-1$.*

Proof. Consider an extension class in $\mathrm{Ext}^1_{I/Z_1}(\chi\alpha_j^{-1}, V)$ represented by $0 \to V \to V' \to \chi\alpha_j^{-1} \to 0$. By assumption on V, this extension splits when restricted to HN_0, hence we may find $v' \in V'\backslash V$ on which HN_0 acts via $\chi\alpha_j^{-1}$ (in particular $v' \in V'^{N_0}$). Notice that $(g-1)v' \in V$ for any $g \in I_1$. Let $v \in V^{N_0}$ be a nonzero vector so that $V^{N_0} = \mathbb{F}v$ by assumption.

First take $g \in \begin{pmatrix} 1+p\mathcal{O}_K & 0 \\ 0 & 1+p\mathcal{O}_K \end{pmatrix}$. It is easy to see that $(g-1)v'$ is again fixed by N_0 and H acts on it via $\chi\alpha_j^{-1}$. But, by assumption V^{N_0} is 1-dimensional on which H acts via χ, thus we must have $(g-1)v' = 0$. We deduce that v' is fixed by $I_1 \cap B(\mathcal{O}_K)$.

We claim that v' is fixed by $N_1^- \overset{\text{def}}{=} \begin{pmatrix} 1 & 0 \\ p\mathcal{O}_K & 1 \end{pmatrix}$. This will imply that v' is fixed by I_1 by the Iwahori decomposition, and consequently V' splits as an I-representation. Let $k \geq 1$ be the smallest integer such that v' is fixed by $N_k^- \overset{\text{def}}{=} \begin{pmatrix} 1 & 0 \\ p^k\mathcal{O}_K & 1 \end{pmatrix}$; such an integer always exists, as V is a smooth representation of I. Suppose $k \geq 2$ and take $g \in N_{k-1}^-$. Using the matrix identity (see [Paš10, Eq.(14)])

$$\begin{pmatrix} 1 & b \\ 0 & 1 \end{pmatrix}\begin{pmatrix} 1 & 0 \\ c & 1 \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ c(1+bc)^{-1} & 1 \end{pmatrix}\begin{pmatrix} 1+bc & b \\ 0 & (1+bc)^{-1} \end{pmatrix}$$

and the fact that v' is fixed by $\begin{pmatrix} 1+p\mathcal{O}_K & \mathcal{O}_K \\ p^k\mathcal{O}_K & 1+p\mathcal{O}_K \end{pmatrix}$, one checks that $(g-1)v' \in V^{N_0}$. Consequently, $\mathbb{F}v \oplus \mathbb{F}v'$ gives rise to an extension in $\mathrm{Ext}^1_{HN_{k-1}^-}(\chi\alpha_j^{-1}, \chi)$ which is nonsplit by the choice of k. But, as in [Paš10, Lemma 5.6], one shows that $\mathrm{Ext}^1_{HN_{k-1}^-}(\chi', \chi) \neq 0$ if and only if $\chi' = \chi\alpha_i$ for some $0 \leq i \leq f-1$. Indeed, after conjugating by $\begin{pmatrix} p^{k-2} & 0 \\ 0 & 1 \end{pmatrix}$, we are reduced to the case $k = 2$, in which case the result is proved by determining the H-action on $\mathrm{Hom}(N_1^-, \mathbb{F})$ as in [Paš10, Lemma 5.3] (see the proof of [BP12, Prop.5.1] for the computation). This finishes the proof as $\chi\alpha_j^{-1} \neq \chi\alpha_i$ for any $0 \leq i, j \leq f-1$ (as $p \geq 5$). □

Lemma 3.3.3.3. *Let V be an admissible smooth representation of I/Z_1 over $\mathbb{F}$. Assume that $V|_{HN_0}$ is isomorphic to an injective envelope of some character χ in the category of smooth representations of HN_0 over $\mathbb{F}$ (so in particular $\dim_{\mathbb{F}} V^{N_0} = 1$). Then we have an isomorphism of $\mathrm{gr}(\mathbb{F}[\![I/Z_1]\!])$-modules:*

$$\mathrm{gr}(V^\vee) \cong \chi^{-1} \otimes R/(z_0, \dots, z_{f-1}).$$

Proof. By assumption, $V[\mathfrak{m}] = V[\mathfrak{m}_{N_0}]$ is one-dimensional and isomorphic to χ, hence we may view $\mathrm{gr}(V^\vee)$ as a cyclic module over $\mathrm{gr}(\Lambda)$ generated by $e_\chi \in \mathrm{gr}^0(V^\vee) = V[\mathfrak{m}]^\vee$, where H acts on e_χ by χ^{-1}. Let $\mathfrak{a} \subseteq \mathrm{gr}(\Lambda)$ be the annihilator of e_χ.

We first prove that $z_j \in \mathfrak{a}$ for $0 \le j \le f-1$. Since H acts on z_j via α_j^{-1} (see just above §3.3.1), to prove $z_j e_\chi = 0$ in $\mathrm{gr}^1(V^\vee)$ it is equivalent to prove that

$$\mathrm{Hom}_H(\chi\alpha_j, V[\mathfrak{m}^2]/V[\mathfrak{m}]) = 0 \ \ \forall\, j \in \{0, \dots, f-1\}.$$

If not, then V would admit a subrepresentation isomorphic to $E_{\chi,\chi\alpha_j}$ (for some j), where $E_{\chi,\chi\alpha_j}$ denotes the unique I/Z_1-representation which is a nonsplit extension of $\chi\alpha_j$ by χ. But by [BHH+23, Lemma 6.1.1(ii)] (after conjugating by the element $\left(\begin{smallmatrix} 0 & 1 \\ p & 0 \end{smallmatrix}\right)$), N_0 acts trivially on $E_{\chi,\chi\alpha_j}$, which implies $\dim_{\mathbb{F}} V[\mathfrak{m}_{N_0}] \ge 2$, a contradiction to the assumption on V.

Using [BHH+23, Lemma 6.1.1(ii)], we then deduce an embedding

$$V[\mathfrak{m}^2]/V[\mathfrak{m}] \hookrightarrow \oplus_{j=0}^{f-1} \chi\alpha_j^{-1}. \tag{166}$$

On the other hand, since $\mathrm{Hom}_I(\chi\alpha_j^{-1}, V) = 0$, we deduce from Lemma 3.3.3.2 that

$$\mathrm{Hom}_I(\chi\alpha_j^{-1}, V[\mathfrak{m}^2]/V[\mathfrak{m}]) = \mathrm{Hom}_I(\chi\alpha_j^{-1}, V/V[\mathfrak{m}]) \xrightarrow{\sim} \mathrm{Ext}^1_{I/Z_1}(\chi\alpha_j^{-1}, \chi)$$

which have dimension 1 over $\mathbb{F}$ by [BHH+23, Lemma 6.1.1(ii)] again. Combining this with (166), we obtain

$$0 \to \chi \to V[\mathfrak{m}^2] \to \oplus_{j=0}^{f-1} \chi\alpha_j^{-1} \to 0. \tag{167}$$

and that $V[\mathfrak{m}^2] = V[\mathfrak{m}_{N_0}^2]$.

Next, we prove that $\mathrm{Ext}^1_{I/Z_1}(\chi, E_{\chi,\chi\alpha_j^{-1}})$ has dimension 1 over $\mathbb{F}$ for any $0 \le j \le f-1$. A straightforward dévissage using $\mathrm{Ext}^1_{I/Z_1}(\chi,\chi) = 0$ and $\dim_{\mathbb{F}} \mathrm{Ext}^1_{I/Z_1}(\chi, \chi\alpha_j^{-1}) = 1$ (see [BHH+23, Lemma 6.1.1(ii)]) yields $\dim_{\mathbb{F}} \mathrm{Ext}^1_{I/Z_1}(\chi, E_{\chi,\chi\alpha_j^{-1}}) \le 1$. So it suffices to explicitly construct a nonzero element in this space, as follows. Let $\mathcal{E}_j \overset{\mathrm{def}}{=} \mathbb{F}v_0 \oplus \mathbb{F}v_1 \oplus \mathbb{F}v_2$ equipped with the action of I/Z_1 determined by:

- H acts on v_0, v_1, v_2 by χ, $\chi\alpha_j^{-1}$, χ respectively;
- if $g = \left(\begin{smallmatrix} 1+pa & b \\ pc & 1+pd \end{smallmatrix}\right) \in I_1$, then

$$gv_0 = v_0, \quad gv_1 = v_1 + \sigma_j(\overline{b})v_0,$$

$$gv_2 = v_2 + \sigma_j(\overline{c})v_1 + \frac{1}{2}\Big(\sigma_j(\overline{a}) - \sigma_j(\overline{d}) + \sigma_j(\overline{b}\overline{c})\Big)v_0.$$

One easily checks that $\mathcal{E}_j$ is well defined and yields the desired nonsplit extension class in $\mathrm{Ext}^1_{I/Z_1}(\chi, E_{\chi,\chi\alpha_j^{-1}})$. Moreover one also checks that $\mathcal{E}_j^{N_0} = \mathbb{F}v_0 \oplus \mathbb{F}v_2$.

We prove that $h_j \in \mathfrak{a}$ for $0 \leq j \leq f-1$. Since $\mathrm{Ext}^1_{I/Z_1}(\chi, \chi) = 0$, the sequence (167) induces an embedding

$$\mathrm{Ext}^1_{I/Z_1}(\chi, V[\mathfrak{m}^2]) \hookrightarrow \mathrm{Ext}^1_{I/Z_1}(\chi, \oplus_{j=0}^{f-1} \chi\alpha_j^{-1}).$$

Note that the right-hand side has dimension f over $\mathbb{F}$. Since $\mathcal{E}_j/\chi$ is nonzero in $\mathrm{Ext}^1_{I/Z_1}(\chi, \chi\alpha_j^{-1})$ for $0 \leq j \leq f-1$, we easily see that the above embedding is actually an isomorphism and that $\mathrm{Ext}^1_{I/Z_1}(\chi, V[\mathfrak{m}^2])$ is spanned by the $\mathcal{E}_j$'s. By the last statement of the previous paragraph, if an extension $\mathcal{E} \in \mathrm{Ext}^1_{I/Z_1}(\chi, V[\mathfrak{m}^2])$ is nonzero then $\dim_{\mathbb{F}} \mathcal{E}^{N_0} \geq 2$. Since $\dim_{\mathbb{F}} V^{N_0} = 1$ by assumption, we see that there exists no embedding $\mathcal{E} \hookrightarrow V$. From (167) we then easily deduce

$$\mathrm{Hom}_H(\chi, V[\mathfrak{m}^3]/V[\mathfrak{m}^2]) = 0.$$

Since H acts trivially on h_j and $h_j e_\chi \in \mathrm{gr}^2(V^\vee) \cong (V[\mathfrak{m}^3]/V[\mathfrak{m}^2])^\vee$, we thus must have $h_j e_\chi = 0$, i.e. $h_j \in \mathfrak{a}$ for $0 \leq j \leq f-1$. This proves the claim.

We deduce a surjection $\mathrm{gr}(\Lambda)/(z_j, h_j,\ 0 \leq j \leq f-1) \twoheadrightarrow \mathrm{gr}(V^\vee)$. As the left-hand side is $\mathbb{F}[y_0, \ldots, y_{f-1}] \cong \mathrm{gr}(\mathbb{F}[\![N_0]\!])$ and $(V|_{N_0})^\vee \cong \mathbb{F}[\![N_0]\!]$ from the assumption, we obtain a surjection $\mathrm{gr}_{\mathfrak{m}_{N_0}}(V^\vee) \twoheadrightarrow \mathrm{gr}(V^\vee)$. By Lemma 3.3.3.1 this surjection is an isomorphism (and hence $\mathfrak{a} = (z_j, h_j, 0 \leq j \leq f-1)$). This finishes the proof. $\square$

If $\chi = \chi_1 \otimes \chi_2$ is a character of H or of $T(K)$, recall $\chi^s = \chi_2 \otimes \chi_1$.

Proposition 3.3.3.4. *Let V be an irreducible smooth $\mathbb{F}$-representation of $\mathrm{GL}_2(K)$ with a central character.*

(i) *If $V \cong \psi \circ \det$ for some smooth character $\psi : K^\times \to \mathbb{F}^\times$, then $\mathrm{gr}(V^\vee) \cong (\psi \otimes \psi)^{-1} \otimes \mathbb{F}$, where $\psi \otimes \psi$ is viewed as a character of H.*

(ii) *If $V \cong \mathrm{Ind}_{B(K)}^{\mathrm{GL}_2(K)} \chi$ for some smooth character $\chi : T(K) \to \mathbb{F}^\times$, then*

$$\mathrm{gr}(V^\vee) \cong \Big((\chi^s|_H)^{-1} \otimes R/(z_0, \ldots, z_{f-1})\Big) \oplus \Big((\chi|_H)^{-1} \otimes R/(y_0, \ldots, y_{f-1})\Big).$$

(iii) *If $V \cong (\mathrm{Ind}_{B(K)}^{\mathrm{GL}_2(K)} 1)/1$ is the special series, then $\mathrm{gr}(V^\vee) \cong R/(y_i z_j, 0 \leq i, j \leq f-1)$.*

(iv) *Assume $K = \mathbb{Q}_p$. If V is supersingular, i.e. isomorphic to $(\text{c-Ind}_{\mathrm{GL}_2(\mathbb{Z}_p)\mathbb{Q}_p^\times}^{\mathrm{GL}_2(\mathbb{Q}_p)} \sigma)/T$ for some Serre weight σ (recall that c-Ind here means compact induction and that $\mathrm{End}_{\mathrm{GL}_2(\mathbb{Q}_p)}(\text{c-Ind}_{\mathrm{GL}_2(\mathbb{Z}_p)\mathbb{Q}_p^\times}^{\mathrm{GL}_2(\mathbb{Q}_p)} \sigma) \cong \mathbb{F}[T]$), then*

$$\mathrm{gr}(V^\vee) \cong \Big(\chi_\sigma^{-1} \otimes R/(y_0 z_0)\Big) \oplus \Big((\chi_\sigma^s)^{-1} \otimes R/(y_0 z_0)\Big),$$

where χ_σ is the action of H on σ^{I_1}.

Proof. (i) It is trivial.
(ii) The restriction of V to I admits a decomposition

$$V|_I \cong \mathrm{Ind}_{I\cap B(K)}^{I}\chi \oplus \mathrm{Ind}_{I\cap B^-(K)}^{I}\chi^s, \tag{168}$$

(cf. the proof of [Paš10, Prop.11.1]). By *loc.cit.*, when restricted to HN_0, $\mathrm{Ind}_{I\cap B^-(K)}^{I}\chi^s$ is an injective envelope of χ^s in the category of smooth representations of HN_0 over $\mathbb{F}$, hence

$$\mathrm{gr}((\mathrm{Ind}_{I\cap B^-(K)}^{I}\chi^s)^\vee) \cong (\chi^s|_H)^{-1}\otimes R/(z_0,\dots,z_{f-1})$$

by Lemma 3.3.3.3. One handles the other direct summand by taking conjugation by the element $\left(\begin{smallmatrix}0&1\\p&0\end{smallmatrix}\right)$.
(iii) By assumption we have a short exact sequence $0\to 1\to \mathrm{Ind}_{B(K)}^{\mathrm{GL}_2(K)}1\to V\to 0$. Write $W=(\mathrm{Ind}_{B(K)}^{\mathrm{GL}_2(K)}1)|_I$ and decompose $W=W_1\oplus W_2$ as in (168). The image of $1\hookrightarrow W$ is equal to the subspace of constant functions, hence the composition $1\hookrightarrow W\twoheadrightarrow W_i$ is nonzero for $i\in\{1,2\}$. Consequently, the dual morphism $\mathrm{gr}(W_i^\vee)\to \mathrm{gr}(1^\vee)$ is also nonzero, and using (ii) (applied to W) we obtain an exact sequence of $\mathrm{gr}(\mathbb{F}[\![I/Z_1]\!])$-modules

$$0\to R/(y_iz_j, 0\le i,j\le f-1)\to \mathrm{gr}(W_1^\vee)\oplus\mathrm{gr}(W_2^\vee)\to\mathrm{gr}(1^\vee)\to 0. \tag{169}$$

Denote by F the induced filtration on $V^\vee$ from the $\mathfrak{m}$-adic filtration on $W^\vee$. By (169) we have an isomorphism $\mathrm{gr}_F(V^\vee)\cong R/(y_iz_j,0\le i,j\le f-1)$. To finish the proof, it suffices to prove that F coincides with the $\mathfrak{m}$-adic filtration on $V^\vee$, or equivalently the inclusion $\mathfrak{m}^nV^\vee\subseteq \mathfrak{m}^nW^\vee\cap V^\vee$ (for $n\ge 0$) is an equality. As in the proof of Lemma 3.3.3.1 it suffices to prove that the induced graded morphism $\mathrm{gr}_{\mathfrak{m}}(V^\vee)\to\mathrm{gr}_F(V^\vee)$ is surjective. But, $\mathrm{gr}_F(V^\vee)$ is generated by $\mathrm{gr}_F^0(V^\vee)$, so it suffices to show that $\mathrm{gr}^0_{\mathfrak{m}}(V^\vee)\to\mathrm{gr}^0_F(V^\vee)$ is surjective, which follows from (169) and the exact sequence

$$\mathrm{gr}^0_{\mathfrak{m}}(V^\vee)\to\mathrm{gr}^0_{\mathfrak{m}}(W^\vee)\to\mathrm{gr}^0_{\mathfrak{m}}(1^\vee)\to 0$$

induced by $0\to 1^{I_1}\to W^{I_1}\to V^{I_1}$ (this sequence is actually right exact but we don't need this fact).
(iv) The proof is analogous to (iii), using [Paš10, Thm.1.2] together with [Paš10, Prop.4.7]. □

By the classification of irreducible admissible smooth representations of $\mathrm{GL}_2(\mathbb{Q}_p)$ over $\mathbb{F}$, we deduce from Proposition 3.3.3.4 and the results of §3.1.2:

Corollary 3.3.3.5. *Let V be an admissible smooth representation of $\mathrm{GL}_2(\mathbb{Q}_p)$ over $\mathbb{F}$ which has a central character and is of finite length. Then there is an integer $n\ge 0$ such that $\mathrm{gr}(V^\vee)$ is annihilated by J^n. In particular V is in the category $\mathcal{C}$ of §3.1.2.*

Finally, we give the first example of an explicit $D_A(\pi)$ for arbitrary $f\ge 1$.

Proposition 3.3.3.6. *Suppose* $\pi = \mathrm{Ind}_{B(K)}^{\mathrm{GL}_2(K)} \chi$ *is a principal series for some smooth character* $\chi = \chi_1 \otimes \chi_2$*. Then* π *lies in category* $\mathcal{C}$ *and* $D_A(\pi) = D_A(\pi)^{\text{ét}}$ *is étale and free of rank 1. More precisely, let* $\kappa \in \pi^\vee$ *be the element sending* $f \in \mathrm{Ind}_{B(K)}^{\mathrm{GL}_2(K)} \chi$ *to* $f\left(\left(\begin{smallmatrix} & 1 \\ 1 & \end{smallmatrix}\right)\right) \in \mathbb{F}$*. Then the image of* κ *in* $D_A(\pi)$ *is a basis of* $D_A(\pi)$*, and we have*

$$\varphi(\kappa) = \chi_2(p)^{-1}\kappa, \tag{170}$$

$$a(\kappa) = \chi_2(a)^{-1}\kappa \quad \forall\ a \in \mathcal{O}_K^\times. \tag{171}$$

Proof. Note that $\pi \in \mathcal{C}$ by Proposition 3.3.3.4(ii). The torus $T(K)$ (hence also H) acts on κ by the character $(\chi^s)^{-1}$; in particular, we get (171). On graded pieces the map $\pi^\vee \to D_A(\pi)$ becomes the map $\mathrm{gr}(\pi^\vee) \to \mathrm{gr}(\pi^\vee)[(y_0 \cdots y_{f-1})^{-1}]$ (Lemma 3.1.1.1). As κ does not annihilate π^{I_1}, it induces a nonzero element of $\mathrm{gr}^0(\pi^\vee) = (\pi^{I_1})^\vee$, which is in fact a $\mathrm{gr}(A)$-basis of $\mathrm{gr}(\pi^\vee)[(y_0 \cdots y_{f-1})^{-1}]$ by Proposition 3.3.3.4(ii). (If $\chi = \chi^s$ the argument still works because κ annihilates the first direct summand in the Mackey decomposition (168).) By the proof of [LvO96, Thm.I.5.7] it follows that $D_A(\pi) = A\kappa$. As $D_A(\pi)$ is a projective A-module and $\mathrm{gr}(D_A(\pi)) \neq 0$, it follows that $D_A(\pi)$ is free of rank 1 with basis κ.

It remains to show (170). First, from the definitions we see that $\psi(\kappa) = \chi_2(p)\kappa$. This implies that the $(\psi, \mathcal{O}_K^\times)$-module $D_A(\pi)$ is étale and so by the previous sentence it becomes an étale $(\varphi, \mathcal{O}_K^\times)$-module, cf. (116). Say $\varphi(\kappa) = a\kappa$ for some $a \in A^\times$. As the actions of φ and $\mathcal{O}_K^\times$ commute, equation (171) and Corollary 3.1.1.9 imply that $a \in \mathbb{F}^\times$. We deduce (170). □

3.3.4 Characteristic cycles

We define the characteristic cycle of a finitely generated filtered Λ-module M such that $\mathrm{gr}(M)$ is annihilated by a power of J and prove an important property (Theorem 3.3.4.5).

Recall from §3.1.4 that the minimal prime ideals of $\overline{R} = R/(y_j z_j, 0 \leq j \leq f-1)$ are the $(y_i, z_j, i \in \mathcal{J}, j \notin \mathcal{J})$ with $\mathcal{J}$ a subset of $\{0, \ldots, f-1\}$.

Definition 3.3.4.1. Let N be a finitely generated module over $\mathrm{gr}(\Lambda)$ which is annihilated by some power of J. We define the *characteristic cycle* of N, denoted by $\mathcal{Z}(N)$[3] as follows:

$$\mathcal{Z}(N) \stackrel{\text{def}}{=} \sum_{\mathfrak{q}} m_{\mathfrak{q}}(N)\mathfrak{q} \ \in \oplus_{\mathfrak{q}} \mathbb{Z}_{\geq 0}\mathfrak{q},$$

where $\mathfrak{q}$ runs over all minimal prime ideals of $\overline{R}$.

Lemma 3.3.4.2. *Let* $n \geq 0$*. If* $0 \to N_1 \to N \to N_2 \to 0$ *is a short exact sequence of finitely generated* $\mathrm{gr}(\Lambda)/J^n$*-modules, then* $\mathcal{Z}(N) = \mathcal{Z}(N_1) + \mathcal{Z}(N_2)$ *in* $\oplus_{\mathfrak{q}} \mathbb{Z}_{\geq 0}\mathfrak{q}$.

[3] A more standard notation is $\mathcal{Z}_f(N)$, where f indicates the dimension of the cycles.

Proof. It is a direct consequence of Lemma 3.1.4.3. □

Let M be a finitely generated Λ-module which is equipped with a good filtration $F \stackrel{\text{def}}{=} \{F_n M : n \in \mathbb{Z}\}$ (in the sense of [LvO96, §I.5]) such that $\mathrm{gr}_F(M)$ is annihilated by some power of J. Recall that this condition doesn't depend on the choice of the good filtration F (see just before Proposition 3.1.2.11) and that $\mathrm{gr}_F(M)$ is also finitely generated over $\mathrm{gr}(\Lambda)$ ([LvO96, Lemma I.5.4]).

Lemma 3.3.4.3. *If F, F' are two such good filtrations on M, then*

$$\mathcal{Z}(\mathrm{gr}_F(M)) = \mathcal{Z}(\mathrm{gr}_{F'}(M)).$$

Proof. The proof is (almost) the same as in [Bjö89, §4]. We recall it for the convenience of the reader. Since F and F' are equivalent by [LvO96, Lemma I.5.3], we may find $c \in \mathbb{Z}_{\geq 0}$ such that

$$F_{n-c}M \subseteq F'_n M \subseteq F_{n+c}M, \quad \forall\, n \in \mathbb{Z}.$$

For $i \in \{-c, -c+1, \dots, c\}$ define a sequence of filtrations $F^{(i)} = \{F^{(i)}_n M : n \in \mathbb{Z}\}$ on M by

$$F^{(i)}_n M \stackrel{\text{def}}{=} F_{n+i}M \cap F'_n M.$$

It is clear that $F^{(-c)} = F[-c]$ and $F^{(c)} = F'$, where $F[-c]$ denotes the shifted filtration $F[-c]_n \stackrel{\text{def}}{=} F_{n-c}$, $n \in \mathbb{Z}$. Hence it suffices to show that each $F^{(i)}$ is a good filtration on M such that

$$\mathcal{Z}(\mathrm{gr}_{F^{(i)}}(M)) = \mathcal{Z}(\mathrm{gr}_{F^{(i+1)}}(M)). \tag{172}$$

Put for $-c \leq i \leq c$:

$$T_i \stackrel{\text{def}}{=} \bigoplus_{n\in\mathbb{Z}} (F_{n+i}M \cap F'_n M)/(F_{n+i}M \cap F'_{n-1}M),$$

$$S_i \stackrel{\text{def}}{=} \bigoplus_{n\in\mathbb{Z}} (F_{n+i+1}M \cap F'_n M)/(F_{n+i}M \cap F'_n M).$$

Since T_i is a $\mathrm{gr}(\Lambda)$-submodule of $\mathrm{gr}_{F'}(M)$ and S_i is a $\mathrm{gr}(\Lambda)$-submodule of $\mathrm{gr}_F(M)[i+1]$, both T_i and S_i are finitely generated $\mathrm{gr}(\Lambda)$-modules and are annihilated by some power of J. Moreover, one checks that there are short exact sequences of $\mathrm{gr}(\Lambda)$-modules (annihilated by some power of J):

$$0 \to T_i \to \mathrm{gr}_{F^{(i+1)}}(M) \to S_i \to 0,$$

$$0 \to S_i[-1] \to \mathrm{gr}_{F^{(i)}}(M) \to T_i \to 0.$$

Hence, $\mathrm{gr}_{F^{(i)}}(M)$ is also finitely generated over $\mathrm{gr}(\Lambda)$ and annihilated by a power of J. Consequently, $F^{(i)}$ is a good filtration on M by [LvO96, Thm.I.5.7] and (172) follows from Lemma 3.3.4.2. □

Thanks to Lemma 3.3.4.3, we can define $m_{\mathfrak{q}}(M)$ to be $m_{\mathfrak{q}}(\mathrm{gr}_F(M))$ and $\mathcal{Z}(M)$ to be $\mathcal{Z}(\mathrm{gr}_F(M))$ for any minimal prime ideal $\mathfrak{q}$ of $\overline{R}$ and any good filtration F on M.

Lemma 3.3.4.4. *Let M be as above and let $0 \to M_1 \to M \to M_2 \to 0$ be an exact sequence of Λ-modules. Then we have in $\oplus_{\mathfrak{q}}\mathbb{Z}_{\geq 0}\mathfrak{q}$:*

$$\mathcal{Z}(M) = \mathcal{Z}(M_1) + \mathcal{Z}(M_2).$$

Proof. We may equip M_1 (resp. M_2) with the induced filtration (resp. quotient filtration) from the one of M, which are automatically good by [LvO96, Cor.I.5.5(1)] and [LvO96, Rem.I.5.2(2)]. Moreover the sequence $0 \to \mathrm{gr}(M_1) \to \mathrm{gr}(M) \to \mathrm{gr}(M_2) \to 0$ is again exact. In particular, both $\mathrm{gr}(M_1)$ and $\mathrm{gr}(M_2)$ are finitely generated $\mathrm{gr}(\Lambda)$-modules annihilated by some power of J, and the result follows from Lemma 3.3.4.2. $\square$

If M is a finitely generated Λ-module, recall from [LvO96, Def.III.2.1.1] that the grade of M is by definition the smallest integer $j_\Lambda(M) \geq 0$ such that $\mathrm{E}_\Lambda^{j_\Lambda(M)}(M) \neq 0$ (with $j_\Lambda(M) \overset{\mathrm{def}}{=} +\infty$ if $\mathrm{E}_\Lambda^j(M) = 0$ for all $j \geq 0$). For a good filtration F on M, we define similarly the grade $j_{\mathrm{gr}(\Lambda)}(\mathrm{gr}_F(M))$ of the $\mathrm{gr}(\Lambda)$-module $\mathrm{gr}_F(M)$. By [LvO96, Thm.III.2.5.2] we have $j_{\mathrm{gr}(\Lambda)}(\mathrm{gr}_F(M)) = j_\Lambda(M)$ (note that Λ is a left and right Zariski ring by [LvO96, Prop.II.2.2.1]), in particular $j_{\mathrm{gr}(\Lambda)}(\mathrm{gr}_F(M))$ doesn't depend on the good filtration F.

Recall that the Krull dimension $\dim_R(N)$ of a finitely generated module N over R (which is commutative) is the Krull dimension of $R/\mathrm{Ann}_R(N)$. For such a module N, by the argument in the proof of [BHH+23, Lemma 5.1.3] applied to $A = \mathrm{gr}(\Lambda)$, $I = (h_0, \ldots, h_{f-1})$ and with N instead of $\mathrm{gr}_{\mathfrak{m}} M$ there, we have

$$j_{\mathrm{gr}(\Lambda)}(N) = \dim(I_1/Z_1) - \dim_R(N). \tag{173}$$

Now, for M as above, assume that $\mathrm{gr}_F(M)$ is annihilated by a power of J. Then applying (173) to the $\overline{R}$-modules $N = J^i \mathrm{gr}_F(M)/J^{i+1}\mathrm{gr}_F(M)$ for $i \geq 0$ and by an obvious dévissage using [LvO96, Lemma III.2.1.2(1)], we deduce

$$j_\Lambda(M) \geq \dim(I_1/Z_1) - \dim(\overline{R}) = 3f - f = 2f. \tag{174}$$

Moreover, by the same dévissage using [LvO96, Cor.III.2.1.6] (note that all assumptions are satisfied since $\mathrm{gr}(\Lambda)$ is Auslander regular) and (173), we deduce that if $j_\Lambda(M) = j_{\mathrm{gr}(\Lambda)}(\mathrm{gr}_F(M)) > 2f$, then we have $\dim_R(J^i \mathrm{gr}_F(M)/J^{i+1}\mathrm{gr}_F(M)) < f$ for all i, hence $\mathcal{Z}(J^i \mathrm{gr}_F(M)/J^{i+1}\mathrm{gr}_F(M)) = 0$ for all $i \geq 0$ and $\mathcal{Z}(M) = 0$ (see (123)).

Theorem 3.3.4.5. *Let M be a finitely generated Λ-module such that $\mathrm{gr}(M)$ is annihilated by a power of J for one (equivalently every) good filtration on M. Then $\mathcal{Z}(\mathrm{E}_\Lambda^{2f}(M))$ is well-defined and we have*

$$\mathcal{Z}(M) = \mathcal{Z}(\mathrm{E}_\Lambda^{2f}(M)).$$

Proof. If $j_\Lambda(M) > 2f$, then the result is trivial since both terms are 0 by the sentence just before the proposition. So from (174) we may assume $j_\Lambda(M) = 2f$ in the rest of the proof.

Choose a good filtration F of M so that $\mathcal{Z}(M) = \mathcal{Z}(\mathrm{gr}_F(M))$. We first show that the $\mathrm{gr}(\Lambda)$-module $\mathrm{E}^{2f}_{\mathrm{gr}(\Lambda)}(\mathrm{gr}_F(M))$ is also annihilated by some power of J. Indeed, $\mathrm{gr}_F(M)$ has a finite filtration whose graded pieces are annihilated by J, hence by dévissage it suffices to show that $\mathrm{E}^{2f}_{\mathrm{gr}(\Lambda)}(N)$ is annihilated by J if N is a finitely generated $\overline{R}$-module. As in the proof of Proposition 3.3.1.10 it is equivalent to prove the same property for $\mathrm{E}^f_R(N)$, which is obvious as R is commutative.

As a consequence, by the first statement in Proposition 3.3.4.6 below the graded module associated to the filtration on $\mathrm{E}^{2f}_\Lambda(M)$ in *loc.cit.* is again finitely generated over $\mathrm{gr}(\Lambda)$ and annihilated by some power of J. Hence $\mathcal{Z}(\mathrm{E}^{2f}_\Lambda(M))$ can be defined. By Proposition 3.3.4.6 the cokernel of the injection $\mathrm{gr}(\mathrm{E}^{2f}_\Lambda(M)) \hookrightarrow \mathrm{E}^{2f}_{\mathrm{gr}(\Lambda)}(\mathrm{gr}_F(M))$ has grade $> 2f$, hence its associated characteristic cycle is 0, as explained above. From Lemma 3.3.4.2 we deduce an equality of cycles

$$\mathcal{Z}\Big(\mathrm{gr}(\mathrm{E}^{2f}_\Lambda(M))\Big) = \mathcal{Z}\Big(\mathrm{E}^{2f}_{\mathrm{gr}(\Lambda)}(\mathrm{gr}_F(M))\Big).$$

Hence, we are left to show that

$$\mathcal{Z}(\mathrm{gr}_F(M)) = \mathcal{Z}\Big(\mathrm{E}^{2f}_{\mathrm{gr}(\Lambda)}(\mathrm{gr}_F(M))\Big).$$

As $\mathrm{gr}(\Lambda)$ is an Auslander regular ring, any subquotient N of $\mathrm{gr}_F(M)$ has grade $\geq 2f$ (by [LvO96, Prop.III.2.1.6]) and is such that $\mathrm{E}^j_{\mathrm{gr}(\Lambda)}(N)$ has grade $\geq j$ for any $j \geq 0$, so that $\mathrm{E}^j_{\mathrm{gr}(\Lambda)}(N)$ and all its subquotients have zero cycle if $j < 2f$ or if $j > 2f$ (by Lemma 3.3.4.2 and the discussion before the proposition for the latter). Hence, for n large enough so that J^n annihilates $\mathrm{gr}_F(M)$, we deduce using again Lemma 3.3.4.2:

$$\mathcal{Z}\Big(\mathrm{E}^{2f}_{\mathrm{gr}(\Lambda)}(\mathrm{gr}_F(M))\Big) = \sum_{i=0}^{n-1} \mathcal{Z}\Big(\mathrm{E}^{2f}_{\mathrm{gr}(\Lambda)}(J^i\,\mathrm{gr}_F(M)/J^{i+1}\,\mathrm{gr}_F(M))\Big).$$

By the definition of $\mathcal{Z}$ and of $m_{\mathfrak{q}}(N)$, see (123), it thus suffices to show

$$\mathcal{Z}(N) = \mathcal{Z}(\mathrm{E}^{2f}_{\mathrm{gr}(\Lambda)}(N))$$

for any finitely generated $\overline{R}$-module N. Using Lemma 3.3.1.9 it suffices to show

$$\mathcal{Z}(N) = \mathcal{Z}(\mathrm{Hom}_{\overline{R}}(N, \overline{R})),$$

which is equivalent to show that for any minimal prime ideal $\mathfrak{q}$ of $\overline{R}$,

$$\mathrm{lg}_{\overline{R}_{\mathfrak{q}}}(N_{\mathfrak{q}}) = \mathrm{lg}_{\overline{R}_{\mathfrak{q}}}(\mathrm{Hom}_{\overline{R}}(N, \overline{R})_{\mathfrak{q}}).$$

Using the isomorphism $\mathrm{Hom}_{\overline{R}}(N, \overline{R})_{\mathfrak{q}} \cong \mathrm{Hom}_{\overline{R}_{\mathfrak{q}}}(N_{\mathfrak{q}}, \overline{R}_{\mathfrak{q}})$ and noting that $\overline{R}_{\mathfrak{q}}$ is a field (being artinian, and reduced as $\overline{R}$ is), the result is clear. $\square$

The first part of the following general result was used in the proof of Theorem 3.3.4.5. Recall that a finitely generated $\mathrm{gr}(\Lambda)$-module of grade j is Cohen–Macaulay if all its $\mathrm{E}^i_{\mathrm{gr}(\Lambda)}$ are 0 when $i \neq j$.

Proposition 3.3.4.6. *Let M be a finitely generated Λ-module of grade j_0 with a good filtration. Then there exists a good filtration on $\mathrm{E}^{j_0}_\Lambda(M)$ such that $\mathrm{gr}(\mathrm{E}^{j_0}_\Lambda(M))$ is a submodule of $\mathrm{E}^{j_0}_{\mathrm{gr}(\Lambda)}(\mathrm{gr}(M))$ and the corresponding cokernel has grade (over $\mathrm{gr}(\Lambda)$) $\geq j_0+1$. If $\mathrm{gr}(M)$ is moreover Cohen–Macaulay, then*

$$\mathrm{gr}(\mathrm{E}^{j_0}_\Lambda(M)) \xrightarrow{\sim} \mathrm{E}^{j_0}_{\mathrm{gr}(\Lambda)}(\mathrm{gr}(M)).$$

Proof. See [Bjö89, Prop.3.1] and the remark following it. We explain the proof following the presentation of [LvO96, §III.2.2].

As in [LvO96, §III.2.2], we may construct a filtered free resolution of M

$$\cdots \to L_j \to L_{j-1} \to \cdots \to L_0 \to M \to 0$$

and taking $\mathrm{E}^0_\Lambda(-) = \mathrm{Hom}_\Lambda(-,\Lambda)$ obtain a filtered complex of finitely generated Λ-modules

$$0 \to \mathrm{E}^0_\Lambda(L_0) \to \mathrm{E}^0_\Lambda(L_1) \to \cdots , \tag{175}$$

where each $\mathrm{E}^0_\Lambda(L_j)$ is endowed with a good filtration as in *loc.cit.*. Taking the associated graded complex of (175), we obtain a complex of $\mathrm{gr}(\Lambda)$-modules (denoted $G(*)$ in *loc.cit.*):

$$0 \to \mathrm{gr}(\mathrm{E}^0_\Lambda(L_0)) \to \mathrm{gr}(\mathrm{E}^0_\Lambda(L_1)) \to \cdots$$

and by [LvO96, Lemma III.2.2.2(2)] we have isomorphisms $\mathrm{E}^0_{\mathrm{gr}(\Lambda)}(\mathrm{gr}(L_j)) \cong \mathrm{gr}(\mathrm{E}^0_\Lambda(L_j))$ for $j \geq 0$. Next, as in [LvO96, §III.1] we may associate a spectral sequence $\{E^r_j : r \geq 0, j \geq 0\}$ to the filtered complex (175) and define a good filtration on $\mathrm{E}^j_\Lambda(M)$ for $j \geq 0$ with the following properties (for convenience we have shifted the numbering):

(a) $E^0_j = \mathrm{gr}(\mathrm{E}^0_\Lambda(L_j))$ and $E^1_j = \mathrm{E}^j_{\mathrm{gr}(\Lambda)}(\mathrm{gr}(M))$ for any j;

(b) for any fixed $r \geq 1$, there is a complex

$$0 \to E^r_0 \to \cdots \to E^r_j \to E^r_{j+1} \to \cdots$$

whose homology gives E^{r+1}_j;

(c) for r large enough (depending on j), $E^\infty_j = E^r_j \cong \mathrm{gr}(\mathrm{E}^j_\Lambda(M))$.

Since $j_\Lambda(M) = j_0$ by assumption, we also have $j_{\mathrm{gr}(\Lambda)}(\mathrm{gr}(M)) = j_0$ by [LvO96, Thm.III.2.5.2] and so $E^1_j = 0$ for $j < j_0$. By (b), this implies short exact sequences

$$0 \to E^{r+1}_{j_0} \to E^r_{j_0} \to E^r_{j_0+1}, \quad \forall\, r \geq 1.$$

In particular, by taking r large enough, $\mathrm{gr}(\mathrm{E}_\Lambda^{j_0}(M)) = E_{j_0}^\infty$ is a submodule of $E_{j_0}^1$. Moreover, since $E_{j_0+1}^r$ has grade $\geq j_0+1$ for all r and so do its subquotients, the cokernel of $E_{j_0}^\infty \hookrightarrow E_{j_0}^1$ also has grade $\geq j_0+1$.

If moreover $\mathrm{gr}(M)$ is Cohen–Macaulay, then $E_j^1 = 0$ except for $j = j_0$, hence $E_{j_0}^\infty = E_{j_0}^1$ which implies the last claim. □

3.3.5 On the length of π in the semisimple case

For $\overline{\rho}$ as in §3.3.1 assumed moreover semisimple and strongly generic, and π as in §3.3.2 with moreover $r = 1$ and satisfying one more hypothesis, we prove that π is generated over $\mathrm{GL}_2(K)$ by its $\mathrm{GL}_2(\mathcal{O}_K)$-socle, is irreducible if $\overline{\rho}$ is, and is semisimple of length 3 if $\overline{\rho}$ is reducible split and $f = 2$.

We keep the notation in §3.3.2 and we assume moreover that $\overline{\rho}$ is *semisimple* and satisfies the strong genericity condition (126) (we will use the results of §3.2). We fix an admissible smooth representation π of $\mathrm{GL}_2(K)$ over $\mathbb{F}$ satisfying the conditions (i), (ii) in *loc.cit.* with $r = 1$ in (i), i.e. $\pi^{K_1} \cong D_0(\overline{\rho})$. Recall this implies that $\mathrm{gr}(\pi^\vee)$ is annihilated by J, where $\mathrm{gr}(\pi^\vee)$ is computed with the $\mathfrak{m}$-adic filtration. We assume moreover:

(iii) $\pi^\vee$ is *essentially self-dual* of grade $2f$, i.e. there is a $\mathrm{GL}_2(K)$-equivariant isomorphism of Λ-modules
$$\mathrm{E}_\Lambda^{2f}(\pi^\vee) \cong \pi^\vee \otimes (\det(\overline{\rho})\omega^{-1}) \tag{176}$$
(recall $\det(\overline{\rho})\omega^{-1}$ is the central character of π). Here $\mathrm{E}_\Lambda^j(\pi^\vee)$ is endowed with the action of $\mathrm{GL}_2(K)$ (compatible with the Λ-module structure) defined in [Koh17, Prop.3.2].

(Note that, compared with [HW22, Def.A.7], in the definition of essentially self-dual we do not assume that $\pi^\vee$ is Cohen–Macaulay. However, by [LvO96, Prop.III.4.2.8(1)] $\pi^\vee$ is *pure* in the sense of [LvO96, Def.III.4.2.7].)

Remark 3.3.5.1. Conditions (i) to (iii), with $r = 1$ in (i), will be satisfied for π coming from the global theory in the minimal case (see §3.4.4). The reason to impose the extra assumption $r = 1$ in (i) is that although for general r we have an equality of diagrams
$$(\pi^{I_1} \hookrightarrow \pi^{K_1}) = (D_0(\overline{\rho})^{I_1} \hookrightarrow D_0(\overline{\rho}))^{\oplus r}$$
for the representations π coming from cohomology (see Theorem 3.4.1.1 below), we do not know if this implies that π has the form $\pi'^{\oplus r}$ for some representation π' of $\mathrm{GL}_2(K)$.

Given $\sigma \in W(\overline{\rho})$, we define the length of σ as follows: if $\lambda \in \mathscr{D}$ corresponds to σ (see §3.3.1), then $\ell(\sigma) \stackrel{\text{def}}{=} \ell(\lambda)$, see (161). For $0 \le \ell \le f$, let

$$W_\ell(\overline{\rho}) \stackrel{\text{def}}{=} \{\sigma \in W(\overline{\rho}), \ell(\sigma) = \ell\}$$

and define $\tau_\ell(\overline{\rho}) \stackrel{\text{def}}{=} \oplus_{\sigma \in W_\ell(\overline{\rho})}\sigma$. We call $W_\ell(\overline{\rho})$, or by abuse of notation $\tau_\ell(\overline{\rho})$, an *orbit* in $W(\overline{\rho})$. Note that this is different from an orbit of δ in $W(\overline{\rho})$ as defined in §3.2.4 (see §3.2.3 for δ), i.e. in general $\tau_\ell(\overline{\rho})$ contains several orbits of δ.

Lemma 3.3.5.2. *If π' is a nonzero subrepresentation of π, then $\mathrm{soc}_{\mathrm{GL}_2(\mathcal{O}_K)}(\pi')$ is a direct sum of orbits in $W(\overline{\rho})$.*

Proof. It is clear that $(\pi'^{I_1} \hookrightarrow \pi'^{K_1})$ is a subdiagram of $(\pi^{I_1} \hookrightarrow \pi^{K_1})$. The result follows from this using [BP12, Thm.15.4] together with the proof of [BP12, Thm.19.10]. Actually, when $\overline{\rho}$ is irreducible, we even have $\mathrm{soc}_{\mathrm{GL}_2(\mathcal{O}_K)}(\pi') = \mathrm{soc}_{\mathrm{GL}_2(\mathcal{O}_K)}(\pi)$ by (the proof of) [BP12, Thm.19.10]. □

We use without comment the notation and definitions in §3.1.4 and denote by $\mathrm{lg}(\tau)$ the length of a finite-dimensional representation τ of $\mathrm{GL}_2(\mathcal{O}_K)$ over $\mathbb{F}$.

Proposition 3.3.5.3. *Let π' be a subquotient of π.*

(i) *We have $\dim_{\mathbb{F}(\!(X)\!)} D_\xi^\vee(\pi') = m_{\mathfrak{p}_0}(\pi'^\vee)$.*

(ii) *Assume that π' is a subrepresentation of π. Then*

$$\dim_{\mathbb{F}(\!(X)\!)} D_\xi^\vee(\pi') = m_{\mathfrak{p}_0}(\pi'^\vee) = \mathrm{lg}(\mathrm{soc}_{\mathrm{GL}_2(\mathcal{O}_K)}(\pi')).$$

In particular, if $\pi' \neq 0$, then $D_\xi^\vee(\pi') \neq 0$.

(iii) *Assume that π' is a nonzero quotient of π. Then $D_\xi^\vee(\pi') \neq 0$.*

Proof. (i) First, for any subquotient π' of π, we equip the Λ-module $\pi'^\vee$ with a good filtration F by choosing two submodules $\pi_1^\vee \subseteq \pi_2^\vee$ of $\pi^\vee$ (with filtrations induced from the $\mathfrak{m}$-adic one on $\pi^\vee$) such that $\pi'^\vee \cong \pi_2^\vee/\pi_1^\vee$ and taking the induced filtration.[4] Then $\mathrm{gr}_F(\pi'^\vee)$ is again an $\overline{R}$-module, and $\dim_{\mathbb{F}(\!(X)\!)} D_\xi^\vee(\pi') \le m_{\mathfrak{p}_0}(\pi'^\vee)$ by Corollary 3.1.4.5. Since $\dim_{\mathbb{F}(\!(X)\!)} D_\xi^\vee(\pi) = m_{\mathfrak{p}_0}(\pi^\vee)$ by Corollary 3.3.2.4, since $D_\xi^\vee(-)$ is an exact functor by Theorem 3.1.3.3 and since $\mathcal{Z}(-)$, and in particular $m_{\mathfrak{p}_0}(-)$, are additive by Lemma 3.3.4.4, the result follows.

[4]The filtrations on $\pi_2^\vee$ and $\pi_1^\vee$ might not be the $\mathfrak{m}$-adic ones, and the resulting filtration on $\pi'^\vee$ might depend on the choice of $\pi_1^\vee$ and $\pi_2^\vee$.

(ii) By assumption π' is a subrepresentation of π. Using that $\mathrm{soc}_{\mathrm{GL}_2(\mathcal{O}_K)}(\pi')$ is a union of orbits of δ, or equivalently of S as in (148), by Lemma 3.3.5.2, it follows from Proposition 3.2.4.2 that

$$\dim_{\mathbb{F}(\!(X)\!)} D_\xi^\vee(\pi') \geq \mathrm{lg}(\mathrm{soc}_{\mathrm{GL}_2(\mathcal{O}_K)}(\pi')).$$

On the other hand, by Lemma 3.3.1.3(i) and Corollary 3.3.2.2, we have $m_{\mathfrak{p}_0}(\pi'^\vee) \leq \mathrm{lg}(\mathrm{soc}_{\mathrm{GL}_2(\mathcal{O}_K)}(\pi'))$ (see the proof of Theorem 3.3.2.3). Hence all the three quantities are equal by (i).

(iii) Let π'' be the kernel of the quotient map $\pi \twoheadrightarrow \pi'$ so that we have an exact sequence of Λ-modules:

$$0 \to \pi'^\vee \to \pi^\vee \to \pi''^\vee \to 0.$$

Since $\pi^\vee$ is essentially self-dual of grade $2f$ by assumption, $\pi'^\vee$ also has grade $2f$ by [LvO96, Prop.III.4.2.8(1)] and [LvO96, Prop.III.4.2.9]. Taking $\mathrm{E}^i_\Lambda(-)$, we obtain a long exact sequence of Λ-modules

$$0 \to \mathrm{E}^{2f}_\Lambda(\pi''^\vee) \to \mathrm{E}^{2f}_\Lambda(\pi^\vee) \to \mathrm{E}^{2f}_\Lambda(\pi'^\vee) \to \mathrm{E}^{2f+1}_\Lambda(\pi''^\vee) \tag{177}$$

which gives rise by Pontryagin duality to an exact sequence of admissible smooth representations of $\mathrm{GL}_2(K)$ with central character (see [Koh17, Cor.1.8]). Define $\widetilde{\pi}$ to be the admissible smooth representation of $\mathrm{GL}_2(K)$ such that

$$\widetilde{\pi}^\vee \otimes (\det(\overline{\rho})\omega^{-1}) \cong \mathrm{Im}\Big(\mathrm{E}^{2f}_\Lambda(\pi^\vee) \to \mathrm{E}^{2f}_\Lambda(\pi'^\vee)\Big). \tag{178}$$

Since $\pi^\vee$ is essentially self-dual by assumption (see (176)), $\widetilde{\pi}^\vee$ is a quotient of $\pi^\vee$ and dually $\widetilde{\pi}$ is a subrepresentation of π. Since $\mathrm{E}^{2f+1}_\Lambda(\pi''^\vee)$ has grade $\geq 2f+1$ as Λ is Auslander regular, we have by (177) and the discussion before Theorem 3.3.4.5:

$$\mathcal{Z}(\mathrm{E}^{2f}_\Lambda(\pi'^\vee)) = \mathcal{Z}(\widetilde{\pi}^\vee \otimes (\det(\overline{\rho})\omega^{-1})),$$

hence $\mathcal{Z}(\pi'^\vee) = \mathcal{Z}(\widetilde{\pi}^\vee)$ by Theorem 3.3.4.5 which implies in particular by (i):

$$\dim_{\mathbb{F}(\!(X)\!)} D_\xi^\vee(\pi') = \dim_{\mathbb{F}(\!(X)\!)} D_\xi^\vee(\widetilde{\pi}). \tag{179}$$

Since $j_\Lambda(\pi'^\vee) = 2f$, $\mathcal{Z}(\pi'^\vee)$ is nonzero (using e.g. (173)), hence $\widetilde{\pi}$ is nonzero, thus $D_\xi^\vee(\widetilde{\pi}) \neq 0$ by (ii), and finally $D_\xi^\vee(\pi') \neq 0$ by (179). □

Remark 3.3.5.4. (i) The construction of $\widetilde{\pi}$ in the proof of Proposition 3.3.5.3(iii) does not use the assumption that $\overline{\rho}$ is semisimple. Moreover, items (i) and (ii) of Proposition 3.3.5.3 do not require the essential self-duality of $\pi^\vee$ (equation (176) above).
(ii) It follows from Proposition 3.3.5.3(ii), from Corollary 3.1.4.5, from Lemma 3.1.4.4, from Lemma 3.1.4.1 and from (109) that for $\pi' \subseteq \pi$ as in Proposition 3.3.5.3(ii) we have

$$\mathrm{rk}_A(D_A(\pi')^{\text{ét}}) = \dim_{\mathbb{F}(\!(X)\!)} D_\xi^\vee(\pi') = m_{\mathfrak{p}_0}(\mathrm{gr}(\pi'^\vee)) = \mathrm{rk}_A(D_A(\pi')). \tag{180}$$

By Corollary 3.1.2.9, both $D_A(\pi')$ and $D_A(\pi')^{\text{ét}}$ are finite projective A-modules and it follows from (180) that the surjection of A-modules $D_A(\pi') \twoheadrightarrow D_A(\pi')^{\text{ét}}$ is here an isomorphism.

Theorem 3.3.5.5. *As a* $\mathrm{GL}_2(K)$*-representation,* π *is generated by its* $\mathrm{GL}_2(\mathcal{O}_K)$*-socle.*

Proof. Let $\tau \stackrel{\text{def}}{=} \mathrm{soc}_{\mathrm{GL}_2(\mathcal{O}_K)}(\pi)$, let $\pi' \stackrel{\text{def}}{=} \langle \mathrm{GL}_2(K).\tau\rangle$ be the subrepresentation of π generated by τ and let $\pi'' \stackrel{\text{def}}{=} \pi/\pi'$. Since $D_\xi^\vee(-)$ is exact by Theorem 3.1.3.7, we have

$$\dim_{\mathbb{F}(\!(X)\!)} D_\xi^\vee(\pi) = \dim_{\mathbb{F}(\!(X)\!)} D_\xi^\vee(\pi') + \dim_{\mathbb{F}(\!(X)\!)} D_\xi^\vee(\pi'').$$

However, since π and π' have the same $\mathrm{GL}_2(\mathcal{O}_K)$-socle, we have

$$\dim_{\mathbb{F}(\!(X)\!)} D_\xi^\vee(\pi) = \dim_{\mathbb{F}(\!(X)\!)} D_\xi^\vee(\pi')$$

by Proposition 3.3.5.3(ii), thus $D_\xi^\vee(\pi'') = 0$. If π'' is nonzero this contradicts Proposition 3.3.5.3(iii). ☐

Corollary 3.3.5.6. *Assume that* $\overline{\rho}$ *is irreducible. Then* π *is irreducible and is a supersingular representation.*

Proof. This follows from Theorem 3.3.5.5 and [BP12, Thm.19.10(i)]. ☐

Remark 3.3.5.7. (i) A result analogous to Theorem 3.3.5.5 when $\overline{\rho}$ is not semisimple is proved in [HW22, Thm.1.6].
(ii) While we believe that Proposition 3.3.5.3 and Theorem 3.3.5.5 should be true without assuming $r = 1$, we don't know how to prove a generalization of Corollary 3.3.5.6 (i.e. π is semisimple and has length r in general), as mentioned in Remark 3.3.5.1.

Corollary 3.3.5.8. *Assume that* $\overline{\rho}$ *is reducible split. Then* π *has the form*

$$\pi = \pi_0 \oplus \pi_f \oplus \pi', \tag{181}$$

where

- π_0 *and* π_f *are irreducible principal series such that* $\mathrm{E}_\Lambda^{2f}(\pi_i^\vee) \cong \pi_{f-i}^\vee \otimes (\det(\overline{\rho})\omega^{-1})$, $i \in \{0, f\}$*;*
- π' *is generated by its* $\mathrm{GL}_2(\mathcal{O}_K)$*-socle and* $\pi'^\vee$ *is essentially self-dual* (*as in* (176)). *Moreover,* π' *is irreducible and supersingular when* $f = 2$.

Proof. By the definition of $W(\overline{\rho})$ (see [BP12, §11]), there exists a unique Serre weight $\sigma_0 \in W(\overline{\rho})$ such that $\ell(\sigma_0) = 0$. Let χ_{σ_0} be the character of I acting on $\sigma_0^{I_1}$. It is easy to check that

$$\mathrm{JH}\Big(\mathrm{Ind}_I^{\mathrm{GL}_2(\mathcal{O}_K)} \chi_{\sigma_0}\Big) \cap W(\overline{\rho}) = \{\sigma_0\}.$$

Let $\pi_0 \overset{\text{def}}{=} \langle \mathrm{GL}_2(K).\sigma_0 \rangle$, a subrepresentation of π. We claim that π_0 is an irreducible principal series. Indeed, by [HW22, Lemma 5.14] and its proof, the morphism (induced from $\sigma_0 \hookrightarrow \pi$ by Frobenius reciprocity)

$$\text{c-Ind}_{\mathrm{GL}_2(\mathcal{O}_K)K^\times}^{\mathrm{GL}_2(K)} \sigma_0 \to \pi$$

(where c-Ind means compact induction) factors through $\text{c-Ind}_{\mathrm{GL}_2(\mathcal{O}_K)K^\times}^{\mathrm{GL}_2(K)} \sigma_0/(T-\mu_0)$ for some $\mu_0 \in \mathbb{F}^\times$ (as $\mathrm{soc}_{\mathrm{GL}_2(\mathcal{O}_K)}(\pi)$ is multiplicity-free). Note that the genericity of $\overline{\rho}$ implies that $\dim_{\mathbb{F}} \sigma_0 \geq 2$, hence the representation $\text{c-Ind}_{\mathrm{GL}_2(\mathcal{O}_K)K^\times}^{\mathrm{GL}_2(K)} \sigma_0/(T-\mu_0)$ is irreducible and isomorphic to some principal series by [BL94, Thm.30]. This proves the claim. Moreover, the $\mathrm{GL}_2(\mathcal{O}_K)$-socle of π_0 is exactly σ_0, and if $\pi_0 \cong \mathrm{Ind}_{B(K)}^{\mathrm{GL}_2(K)} \chi_0$ for some smooth character $\chi_0 : T(K) \to \mathbb{F}^\times$ then $\chi_0^s|_H = \chi_{\sigma_0}$. Similarly, there exists a unique Serre weight $\sigma_f \in W(\overline{\rho})$ such that $\ell(\sigma_f) = f$. It satisfies again

$$\mathrm{JH}\Big(\mathrm{Ind}_I^{\mathrm{GL}_2(\mathcal{O}_K)} \chi_{\sigma_f}\Big) \cap W(\overline{\rho}) = \{\sigma_f\}$$

and by the same argument as above the subrepresentation $\pi_f \overset{\text{def}}{=} \langle \mathrm{GL}_2(K).\sigma_f \rangle$ of π is an irreducible principal series with $\mathrm{GL}_2(\mathcal{O}_K)$-socle equal to σ_f, and if $\pi_f \cong \mathrm{Ind}_{B(K)}^{\mathrm{GL}_2(K)} \chi_f$ then $\chi_f^s|_H = \chi_{\sigma_f}$. The map $\pi_0 \oplus \pi_f \to \pi$ is injective since it is injective on the $\mathrm{GL}_2(\mathcal{O}_K)$-socles.

Letting $\pi' \overset{\text{def}}{=} \pi/(\pi_0 \oplus \pi_f)$, we have an exact sequence of Λ-modules:

$$0 \to \pi'^\vee \to \pi^\vee \to \pi_0^\vee \oplus \pi_f^\vee \to 0.$$

As Λ is Auslander regular and $\pi^\vee$ is of grade $2f$, it follows from [LvO96, Cor.III.2.1.6] that $\pi'^\vee$ is of grade $\geq 2f$, hence $\mathrm{E}_\Lambda^{2f-1}(\pi'^\vee) = 0$ and there is an exact sequence of (finitely generated) Λ-modules

$$0 \to \mathrm{E}_\Lambda^{2f}(\pi_0^\vee) \oplus \mathrm{E}_\Lambda^{2f}(\pi_f^\vee) \to \mathrm{E}_\Lambda^{2f}(\pi^\vee) \to \mathrm{E}_\Lambda^{2f}(\pi'^\vee).$$

Since $\pi^\vee$ is essentially self-dual by assumption (see (176)) and since $\mathrm{E}_\Lambda^{2f}(\pi_0^\vee)^\vee$ and $\mathrm{E}_\Lambda^{2f}(\pi_f^\vee)^\vee$ are also irreducible principal series by [Koh17, Prop.5.4], we see that π admits a quotient isomorphic to $\pi_0' \oplus \pi_f'$, where π_i' (for $i \in \{0, f\}$) is the (irreducible) principal series such that

$$\pi_i'^\vee \otimes (\det(\overline{\rho})\omega^{-1}) = \mathrm{E}_\Lambda^{2f}(\pi_{f-i}^\vee). \tag{182}$$

Explicitly, if $\pi_i' \cong \mathrm{Ind}_{B(K)}^{\mathrm{GL}_2(K)} \chi_i'$ for some smooth characters $\chi_i' : T(K) \to \mathbb{F}^\times$, and if we let $\alpha_B \overset{\text{def}}{=} \omega \otimes \omega^{-1} : T(K) \to \mathbb{F}^\times$ and $\eta \overset{\text{def}}{=} \det(\overline{\rho})\omega^{-1}$ (for short), then by [HW22, Lemma 10.7] (which is based on [Koh17, Prop.5.4]):

$$\chi_f' = \chi_0^{-1}\alpha_B(\eta \otimes \eta), \quad \chi_0' = \chi_f^{-1}\alpha_B(\eta \otimes \eta). \tag{183}$$

Let us compute the $\mathrm{GL}_2(\mathcal{O}_K)$-socle of π'_f (the case of π'_0 is similar). Since η is equal to the central character of π_0, we have $\chi_0^{-1}(\eta \otimes \eta) = \chi_0^s$, so that (183) becomes $\chi'_f = \chi_0^s \alpha_B$. Since $\chi_0^s|_H = \chi_{\sigma_0}$ as seen in the first paragraph, we deduce

$$(\chi'_f)^s|_H = \chi_{\sigma_0}^s \alpha_B^{-1} = \chi_{\sigma_f}, \tag{184}$$

where the last equality holds by an easy check using the definition of σ_0 and σ_f (see [BP12, §11]). In particular, our genericity assumption on $\overline{\rho}$ implies that $\chi'_f \neq \chi'^s_f$ when restricted to $T(\mathcal{O}_K)$. Using [BL94, Thm.34(2)], this implies that the $\mathrm{GL}_2(\mathcal{O}_K)$-socle of π'_f is irreducible and actually isomorphic to σ_f by (184). Similarly, the $\mathrm{GL}_2(\mathcal{O}_K)$-socle of π'_0 is isomorphic to σ_0.

We claim that the composite morphism

$$\pi_0 \oplus \pi_f \hookrightarrow \pi \twoheadrightarrow \pi'_0 \oplus \pi'_f$$

is an isomorphism. Since π is generated by its $\mathrm{GL}_2(\mathcal{O}_K)$-socle, namely $\bigoplus_{\sigma \in W(\overline{\rho})} \sigma$, the composite morphism

$$\iota_0 : \bigoplus_{\sigma \in W(\overline{\rho})} \sigma \hookrightarrow \pi \twoheadrightarrow \pi'_0$$

is nonzero. Since the image is contained in $\mathrm{soc}_{\mathrm{GL}_2(\mathcal{O}_K)}(\pi'_0)$, which is equal to σ_0 as seen in the last paragraph, ι_0 is nonzero when restricted to σ_0. But, by construction we have $\langle \mathrm{GL}_2(K).\sigma_0 \rangle = \pi_0$ inside π, hence the composite morphism $\pi_0 \hookrightarrow \pi \twoheadrightarrow \pi'_0$ is nonzero, hence an isomorphism as both π_0 and π'_0 are irreducible. In the same way the composite morphism $\pi_f \hookrightarrow \pi \twoheadrightarrow \pi'_f$ is also an isomorphism. This proves the claim, from which the decomposition (181) immediately follows. From (182) we also deduce the isomorphism $\mathrm{E}_\Lambda^{2f}(\pi_i^\vee) \cong \pi_{f-i}^\vee \otimes \eta$ for $i \in \{0, f\}$.

We now finish the proof. First, π' is generated by its $\mathrm{GL}_2(\mathcal{O}_K)$-socle by Theorem 3.3.5.5. Explicitly, we have

$$\mathrm{soc}_{\mathrm{GL}_2(\mathcal{O}_K)}(\pi') = \bigoplus_{\substack{\sigma \in W(\overline{\rho}) \\ 0 < \ell(\sigma) < f}} \sigma.$$

In particular, if $f = 2$, then π' is irreducible and is a supersingular representation by [BP12, Thm.19.10(ii)]. Finally we prove that $\pi'^\vee$ is essentially self-dual (as in (176)). In fact, using (181) and noting that

$$(\mathrm{E}_\Lambda^{2f}(\pi^\vee))^\vee \otimes \eta \cong \pi_0 \oplus \pi_f \oplus (\mathrm{E}_\Lambda^{2f}(\pi'^\vee))^\vee \otimes \eta,$$

it suffices to prove that the composite morphism

$$\pi' \hookrightarrow \pi \xrightarrow{\sim} (\mathrm{E}_\Lambda^{2f}(\pi^\vee))^\vee \otimes \eta \twoheadrightarrow (\mathrm{E}_\Lambda^{2f}(\pi'^\vee))^\vee \otimes \eta$$

is an isomorphism. Since both the source and the target have the same $\mathrm{GL}_2(\mathcal{O}_K)$-socle, the morphism is injective because it is when restricted to the $\mathrm{GL}_2(\mathcal{O}_K)$-socle of π' and is surjective because $(\mathrm{E}_\Lambda^{2f}(\pi'^\vee))^\vee \otimes \eta$ is generated by its $\mathrm{GL}_2(\mathcal{O}_K)$-socle. □

3.4 Local-global compatibility results for $\mathrm{GL}_2(\mathbb{Q}_{p^f})$

We prove special cases of Conjecture 2.1.3.1 and Conjecture 2.5.1 when $F_v^+ = \mathbb{Q}_{p^f}$ and $n = 2$. We assume $E = W(\mathbb{F})[1/p]$ (thus $\mathcal{O}_E = W(\mathbb{F})$ and $\varpi_E = p$).

3.4.1 Global setting and results

We refine the global setting of §§2.1, 2.5 when $n = 2$ in order to apply the results of [BHH+23] and we state the first main global result.

We come back to the setting of §2.1 when $n = 2$ and we assume $p > 7$. We make the following extra assumptions on the field F and the unitary group H:

(i) F/F^+ is unramified at all finite places;

(ii) p is unramified in F^+;

(iii) H is defined over $\mathcal{O}_{F^+}$ and $H \times_{\mathcal{O}_{F^+}} F^+$ is quasi-split at all finite places of F^+.

Condition (i) (together with the fact that any p-adic place of F^+ splits in F) implies $[F^+ : \mathbb{Q}]$ is even (see [GK14, §3.1]). By [GK14, §3.1.1] such groups H always exist. We denote by $R^{\square}_{\overline{r}_{\tilde{w}}}$ the universal framed deformation ring of $\overline{r}_{\tilde{w}}$ over $W(\mathbb{F})$ ($\tilde{w}$ is any finite place of F). We set $K \overset{\text{def}}{=} F_v^+$ and $f \overset{\text{def}}{=} [K : \mathbb{Q}_p]$.

We let $\overline{r} : \mathrm{Gal}(\overline{F}/F) \to \mathrm{GL}_2(\mathbb{F})$ as in §2.1.3 and make the following extra assumptions on $\overline{r}$ (recall that S_p is the set of places of F^+ dividing p):

(iv) $\overline{r}|_{\mathrm{Gal}(\overline{F}/F(\sqrt[p]{1}))}$ is adequate ([Tho17, Def.2.20]);

(v) $\overline{r}_{\tilde{w}}$ is unramified if $\tilde{w}|_{F^+}$ is inert in F;

(vi) $R^{\square}_{\overline{r}_{\tilde{w}}}$ is formally smooth over $W(\mathbb{F})$ if $\overline{r}_{\tilde{w}}$ is ramified and $\tilde{w}|_{F^+} \notin S_p$;

(vii) $\overline{r}_{\tilde{w}}$ is generic in the sense of [BP12, Def.11.7] if $\tilde{w}|_{F^+} \in S_p \backslash \{v\}$;

(viii) $\overline{r}_{\tilde{v}}$ is, up to twist, of one of the following forms for $\tilde{v}|_{F^+} = v$:

- $\overline{r}_{\tilde{v}}|_{I_K} \cong \begin{pmatrix} \omega_f^{(r_0+1)+\cdots+p^{f-1}(r_{f-1}+1)} & 0 \\ 0 & 1 \end{pmatrix}$ $3 \leq r_i \leq p-6$,
- $\overline{r}_{\tilde{v}}|_{I_K} \cong \begin{pmatrix} \omega_{2f}^{(r_0+1)+\cdots+p^{f-1}(r_{f-1}+1)} & 0 \\ 0 & \omega_{2f}^{p^f(\text{same})} \end{pmatrix}$ $4 \leq r_0 \leq p-5$, $3 \leq r_i \leq p-6$ for $i > 0$.

Note that conditions (iv) to (viii) only depend on $\tilde{w}|_{F^+}$ and $\tilde{v}|_{F^+}$ using condition (i) in §2.1.3 (the genericity conditions in (viii) are satisfied in [DL21, §3.3] *and* don't depend on the choices of σ_0, σ_0'). We denote by $S_{\overline{r}}$ the finite set of finite places of F^+ such that $\tilde{w}|_{F^+} \in S_{\overline{r}}$ if and only if $\overline{r}_{\tilde{w}}$ is ramified. Thus $S_p \subseteq S_{\overline{r}}$ and by (ii) any place in $S_{\overline{r}}$ splits in F^+. We fix a finite place v_1 of F^+ which is not in $S_{\overline{r}}$ and satisfies the assumptions in [EGS15, §6.6], and we choose $\widetilde{v_1}|v_1$ in F.

We choose S a finite set of finite places of F^+ that split in F containing $S_{\overline{r}}$ but not v_1, and a compact open subgroup $U = \prod_w U_w \subseteq H(\mathbb{A}_{F^+}^\infty)$ such that

(ix) $U_w \subseteq H(\mathcal{O}_{F_w^+})$ if w splits in F;

(x) U_w is maximal hyperspecial in $H(F_w^+)$ if w is inert in F;

(xi) $U_w = H(\mathcal{O}_{F_w^+})$ if $w \notin S \cup \{v_1\}$ and w splits in F or if $w \in S_p$;

(xii) $\iota_{\widetilde{v_1}}(U_{v_1})$ is contained in the upper-triangular unipotent matrices mod $\widetilde{v_1}$.

We also define $V \stackrel{\text{def}}{=} U^p \prod_{w\in S_p} V_w$, where $U^p \stackrel{\text{def}}{=} \prod_{w\notin S_p} U_w$ and V_w is a pro-p normal subgroup of U_w if $w \in S_p$ (hence V is normal in U). We set $\Sigma \stackrel{\text{def}}{=} S \cup \{v_1\}$ and assume $S(V, \mathbb{F})[\mathfrak{m}^\Sigma] \neq 0$ (see §2.1.2). Note that $S(V, \mathbb{F})[\mathfrak{m}^\Sigma]$ doesn't depend on S as above by the proof of [BDJ10, Lemma 4.6(a)]. For each place $w \in S_p$ we choose a place $\tilde{w}|w$ in F. For $w \in S_p$ recall from §3.2.1 that $W(\overline{r}_{\tilde{w}}(1))$ is the set of Serre weights associated to $\overline{r}_{\tilde{w}}(1) \stackrel{\text{def}}{=} \overline{r}_{\tilde{w}} \otimes \omega$ defined as in [BDJ10, §3]. Then it follows from [GLS14, Thm.A] and [BLGG13, Def.2.9] that we have

$$\mathrm{Hom}_U\left(\otimes_{w\in S_p} \sigma_{\tilde{w}}, S(V, \mathbb{F})[\mathfrak{m}^\Sigma]\right) \neq 0 \iff \sigma_{\tilde{w}} \in W(\overline{r}_{\tilde{w}}(1)) \ \forall\ w \in S_p, \tag{185}$$

where we consider $\otimes_{w\in S_p}\sigma_{\tilde{w}}$ as a representation of U via $U \twoheadrightarrow U/V \xrightarrow{\sim} \prod_{w\in S_p} U_w/V_w$ and the isomorphisms $\iota_{\tilde{w}}$. Note that the left-hand side of (185) is also isomorphic to $\mathrm{Hom}_U(\otimes_{w\in S_p}\sigma_{\tilde{w}}, S(U^p, \mathbb{F})[\mathfrak{m}^\Sigma])$, where $S(U^p, \mathbb{F})[\mathfrak{m}^\Sigma]$ is defined as in §2.1.2, replacing U^v by U^p.

We freely use the previous local notation (I_1 is the pro-p Iwahori subgroup in $\mathrm{GL}_2(\mathcal{O}_K) = \mathrm{GL}_2(\mathcal{O}_{F_{\tilde{v}}})$ etc.) and set $\overline{\rho} \stackrel{\text{def}}{=} \overline{r}_{\tilde{v}}(1)$.

Theorem 3.4.1.1. *Choose Serre weights $\sigma_{\tilde{w}} \in W(\overline{r}_{\tilde{w}}(1))$ for $w \in S_p\backslash\{v\}$ and set*

$$\pi \stackrel{\text{def}}{=} \mathrm{Hom}_{U^v}(\otimes_{w\in S_p\backslash\{v\}}\sigma_{\tilde{w}}, S(V^v, \mathbb{F})[\mathfrak{m}^\Sigma]).$$

Then there exist an integer $r \geq 1$ only depending on v, U^v, V^v, $\otimes_{w\in S_p\backslash\{v\}}\sigma_{\tilde{w}}$ and $\overline{r}$ and a diagram $D(\overline{\rho}) = (D_1(\overline{\rho}) \hookrightarrow D_0(\overline{\rho}))$ as in §3.2.1 only depending on $\overline{\rho} = \overline{r}_{\tilde{v}}(1)$ (and satisfying the assumptions in loc.cit. on the constants ν_i) such that there is an isomorphism of diagrams

$$D(\overline{\rho})^{\oplus r} \cong (\pi^{I_1} \hookrightarrow \pi^{K_1}).$$

The case $r = 1$ of Theorem 3.4.1.1 is known and due to Dotto and Le ([DL21, Thm.1.3]). We generalize below their proof to the case $r \geq 1$ using the results in [BHH$^+$23, §8.2]. Moreover the diagram $D(\overline{\rho})$ in Theorem 3.4.1.1 is in fact the same as the diagram $\mathcal{D}(\pi_{\mathrm{glob}}(\overline{\rho}))$ of [DL21, Thm.1.3].

3.4.2 Review of patching functors

We recall the patching functors of [EGS15, §6.6] and some results of [BHH$^+$23, §8.2].

We keep the notation of §3.4.1. We choose Serre weights $\sigma_{\tilde{w}} \in W(\overline{r}_{\tilde{w}}(1))$ for $w \in S_p\backslash\{v\}$ and set

$$\sigma^v \stackrel{\mathrm{def}}{=} \bigotimes_{w \in S_p\backslash\{v\}} \sigma_{\tilde{w}}.$$

For each $w \in S_p\backslash\{v\}$ we fix a tame inertial type $\tau_{\tilde{w}}$ at the place $\tilde{w}$ such that, denoting by $\sigma(\tau_{\tilde{w}})$ the irreducible smooth representation of $\mathrm{GL}_2(\mathcal{O}_{F_{\tilde{w}}})$ over E associated by Henniart to $\tau_{\tilde{w}}$ in the appendix to [BM02], $\mathrm{JH}(\overline{\sigma(\tau_{\tilde{w}})})$ contains exactly one Serre weight in $W(\overline{r}_{\tilde{w}}(1))$ (where $\overline{(-)}$ means the mod p semisimplification). The existence of such $\tau_{\tilde{w}}$ follows from [EGS15, Prop.3.5.1], and the fact $\sigma(\tau_{\tilde{w}})$ can be realized over $E = W(\mathbb{F})[1/p]$ follows from [EGS15, Lemma 3.1.1]. For each $w \in S_p\backslash\{v\}$ we also fix a $\mathrm{GL}_2(\mathcal{O}_{F_{\tilde{w}}})$-invariant $W(\mathbb{F})$-lattice $\sigma^0(\tau_{\tilde{w}})$ in $\sigma(\tau_{\tilde{w}})$.

We define

$$\sigma^{0,v} \stackrel{\mathrm{def}}{=} \bigotimes_{w \in S_p\backslash\{v\}} \sigma^0(\tau_{\tilde{w}}),$$

and for any continuous representation $\sigma_{\tilde{v}}$ of $\mathrm{GL}_2(\mathcal{O}_{F_{\tilde{v}}})$ on a finite type $W(\mathbb{F})$-module, we consider $\sigma^{0,v} \otimes_{W(\mathbb{F})} \sigma_{\tilde{v}}$ as a representation of U via $U \twoheadrightarrow \prod_{w \in S_p} U_w$ and the isomorphisms $\iota_{\tilde{w}}$. We define $S(U^p, W(\mathbb{F}))_{\mathfrak{m}^\Sigma}$ exactly as in §2.1.2 replacing $\mathbb{F}$ by $W(\mathbb{F})$ and U^v by U^p. Then, as in [EGS15, §§6.2,6.6], by "patching" $\mathrm{Hom}_U(\sigma^{0,v} \otimes_{W(\mathbb{F})} \sigma_{\tilde{v}}, S(U^p, W(\mathbb{F}))_{\mathfrak{m}^\Sigma})^*$ for various U (where $(-)^* \stackrel{\mathrm{def}}{=} \mathrm{Hom}_{W(\mathbb{F})}((-), E/W(\mathbb{F}))$ as in *loc.cit.*), we obtain a patching functor

$$M_\infty : \ \sigma_{\tilde{v}} \longmapsto M_\infty(\sigma^{0,v} \otimes_{W(\mathbb{F})} \sigma_{\tilde{v}})$$

which is an exact functor from the category of continuous representations $\sigma_{\tilde{v}}$ of $\mathrm{GL}_2(\mathcal{O}_{F_{\tilde{v}}})$ on finite type $W(\mathbb{F})$-modules to the category of finite type R_∞-modules (though this patching functor depends on $\sigma^{0,v}$, we just write $M_\infty(\sigma_{\tilde{v}})$ in the sequel). The local ring R_∞ is (see [GK14, §4.3] or [DL21, §6.2]):

$$R_\infty \stackrel{\mathrm{def}}{=} R^{\mathrm{loc}}[\![X_1, \ldots, X_{q-[F^+:\mathbb{Q}]}]\!],$$

where q is an integer $\geq [F^+ : \mathbb{Q}]$ and

$$R^{\mathrm{loc}} \stackrel{\mathrm{def}}{=} \left(\widehat{\otimes}_{w \in S\backslash S_p} R^{\square}_{\overline{r}_{\tilde{w}}(1)}\right) \widehat{\otimes}_{W(\mathbb{F})} \left(\widehat{\otimes}_{w \in S_p\backslash\{v\}} R^{\square,(1,0),\tau_{\tilde{w}}}_{\overline{r}_{\tilde{w}}(1)}\right) \widehat{\otimes}_{W(\mathbb{F})} R^{\square}_{\overline{r}_v(1)}.$$

Recall $R^{\square,(1,0),\tau_{\tilde{w}}}_{\overline{r}_{\tilde{w}}(1)}$ is the reduced p-torsion free quotient of $R^{\square}_{\overline{r}_{\tilde{w}}(1)}$ parametrizing framed potentially Barsotti–Tate deformations with inertial type $\tau_{\tilde{w}}$ (by local-global compatibility and the inertial Langlands correspondence, for $w \in S_p\backslash\{v\}$ the action of $R^{\square}_{\overline{r}_{\tilde{w}}(1)}$ on $M_\infty(\sigma^{0,v} \otimes_{W(\mathbb{F})} \sigma_v)$ factors through this quotient, see [EGS15, §6.6]). As in [BHH+23, §8.1] (see the discussion before [BHH+23, Rem.8.1.3] but note that we do not need to fix the determinant here) we have isomorphisms $R^{\square}_{\overline{r}_{\tilde{w}}(1)} \cong W(\mathbb{F})[\![X_1, X_2, X_3, X_4]\!]$ for $w \in S\backslash S_p$, and, by genericity of $\overline{r}_v$,

$$R^{\square}_{\overline{r}_v(1)} \cong W(\mathbb{F})[\![X_1, \dots, X_{4+4[F_{\tilde{v}}:\mathbb{Q}_p]}]\!].$$

By [EGS15, Thm.7.2.1(2)] (and [GK14, Rk.5.2.2]) we have

$$R^{\square,(1,0),\tau_{\tilde{w}}}_{\overline{r}_{\tilde{w}}(1)} \cong W(\mathbb{F})[\![X_1, \dots, X_{4+[F_{\tilde{w}}:\mathbb{Q}_p]}]\!],$$

so that we finally get

$$R_\infty \cong R^{\square}_{\overline{r}_v(1)}[\![X_1, \dots, X_{4(|S|-1)+q-[F_v^+:\mathbb{Q}_p]}]\!] \cong W(\mathbb{F})[\![X_1, \dots, X_{4|S|+q+3[F_v^+:\mathbb{Q}_p]}]\!]. \quad (186)$$

Moreover, if $\sigma_{\tilde{v}}$ is free of finite type over $W(\mathbb{F})$, then $M_\infty(\sigma_{\tilde{v}})$ is free of finite type over a subring S_∞ of R_∞, where $S_\infty \cong W(\mathbb{F})[\![x_1, \dots, x_{4|S|+q}]\!]$. Finally, denoting by $\mathfrak{m}_\infty$ the maximal ideal of R_∞, we have

$$M_\infty(\sigma_{\tilde{v}})/\mathfrak{m}_\infty \cong \mathrm{Hom}_U\Big((\otimes_{w\in S_p\backslash v}\sigma_{\tilde{w}}) \otimes_{\mathbb{F}} \overline{\sigma_{\tilde{v}}}, S(U^p, \mathbb{F})[\mathfrak{m}^\Sigma]\Big)^\vee \cong \mathrm{Hom}_{U_v}(\overline{\sigma_{\tilde{v}}}, \pi)^\vee, \quad (187)$$

where π is as in Theorem 3.4.1.1.

Since everything is now at the place $\tilde{v}$, we drop the index $\tilde{v}$. If τ is a tame inertial type, we set $R_\infty^{(1,0),\tau} \overset{\mathrm{def}}{=} R_\infty \otimes_{R^{\square}_{\overline{\rho}}} R^{\square,(1,0),\tau}_{\overline{\rho}}$. If $\sigma \in W(\overline{\rho})$, we denote by P_σ the projective $\mathbb{F}[\mathrm{GL}_2(\mathbb{F}_q)]$-envelope of σ and by $\widetilde{P}_\sigma$ the projective $W(\mathbb{F})[\mathrm{GL}_2(\mathbb{F}_q)]$-module lifting P_σ. We recall that the scheme theoretic support of an R_∞-module M is $R_\infty/\mathrm{Ann}_{R_\infty}(M)$. The following theorem then follows by exactly the same proof as for [BHH+23, Prop.8.2.3] and [BHH+23, Prop.8.2.6].

Theorem 3.4.2.1. *There exists an integer $r \geq 1$ such that*

(i) *for any $\sigma \in W(\overline{\rho})$ the module $M_\infty(\sigma)$ is free of rank r over its scheme-theoretic support which is a domain;*

(ii) *for any $\sigma \in W(\overline{\rho})$ the modules $M_\infty(\widetilde{P}_\sigma)$ and $M_\infty(P_\sigma)$ are free of rank r over their respective scheme-theoretic support;*

(iii) *for any tame inertial type τ such that $\mathrm{JH}(\overline{\sigma(\tau)}) \cap W(\overline{\rho}) \neq \emptyset$ and any $\mathrm{GL}_2(\mathcal{O}_K)$-invariant $W(\mathbb{F})$-lattice $\sigma^0(\tau)$ in $\sigma(\tau)$ with irreducible cosocle, the module $M_\infty(\sigma^0(\tau))$ is free of rank r over its scheme-theoretic support, which is the domain $R_\infty^{(1,0),\tau}$.*

Corollary 3.4.2.2. *Let* π *as in Theorem 3.4.1.1 and* r *as in Theorem 3.4.2.1. We have an isomorphism of* $\mathrm{GL}_2(\mathcal{O}_K)K^\times$*-representations* $D_0(\overline{\rho})^{\oplus r} \cong \pi^{K_1}$.

Proof. The action of the center $K^\times$ being by definition the same on both sides, we can focus on the action of $\mathrm{GL}_2(\mathcal{O}_K)$. It follows from Theorem 3.4.2.1(i) and (ii) and from (187) that the surjection $P_\sigma \twoheadrightarrow \sigma$ induces an isomorphism of r-dimensional $\mathbb{F}$-vector spaces $\mathrm{Hom}_{\mathrm{GL}_2(\mathcal{O}_K)}(\sigma, \pi^{K_1}) \xrightarrow{\sim} \mathrm{Hom}_{\mathrm{GL}_2(\mathcal{O}_K)}(P_\sigma, \pi^{K_1})$. In particular the multiplicity of each $\sigma \in W(\overline{\rho})$ in π^{K_1} is r. It follows from $M_\infty(D_{0,\sigma}(\overline{\rho})/\sigma) = 0$ (recall $D_0(\overline{\rho}) = \oplus_{\sigma\in W(\overline{\rho})} D_{0,\sigma}(\overline{\rho})$) and from (187) that the injection $\sigma \hookrightarrow D_{0,\sigma}(\overline{\rho})$ induces an isomorphism $\mathrm{Hom}_{\mathrm{GL}_2(\mathcal{O}_K)}(D_{0,\sigma}(\overline{\rho}), \pi^{K_1}) \xrightarrow{\sim} \mathrm{Hom}_{\mathrm{GL}_2(\mathcal{O}_K)}(\sigma, \pi^{K_1})$. This gives an inclusion $D_0(\overline{\rho})^{\oplus r} \hookrightarrow \pi^{K_1}$. If this inclusion is strict, then by maximality of $D_0(\overline{\rho})^{\oplus r}$ (an obvious generalization of [BP12, Prop.13.1]) this implies there exists $\sigma \in W(\overline{\rho})$ which appears in $\pi^{K_1}/D_0(\overline{\rho})^{\oplus r}$, and hence has multiplicity $> r$ in π^{K_1}, which is a contradiction. $\square$

Remark 3.4.2.3. In the proof of Theorem 3.4.2.1, and hence also in Corollary 3.4.2.2, one only needs the slightly weaker bounds $1 \le r_i \le p-4$ (and $2 \le r_0 \le p-3$ if $\overline{r}_{\tilde{v}}$ is irreducible) in the genericity conditions (viii) on $\overline{r}_{\tilde{v}}$ (or equivalently $\overline{\rho}$) in §3.4.1 (these bounds are used in [LMS22, §4] which is used in the proof of [BHH$^+$23, Prop.8.2.6]).

3.4.3 Direct sums of diagrams

We prove Theorem 3.4.1.1 using the method of [DL21, §4].

We keep the notation in §§3.4.1, 3.4.2. Everything in this section being at the place $\tilde{v}$, we drop it from the notation. Recall we identify the set of embeddings $\mathbb{F}_q \hookrightarrow \mathbb{F}$ with $\{0, \dots, f-1\}$ via $\sigma_0 \circ \varphi^i \mapsto i$. We denote by $\mathcal{P}$ the set of subsets of $\{0, \dots, f-1\}$ and by $J^c \in \mathcal{P}$ the complement of a subset $J \in \mathcal{P}$.

We start by fixing a tame inertial type τ such that $\mathrm{JH}(\overline{\sigma(\tau)}) \cap W(\overline{\rho}) \neq \emptyset$ and a $\mathrm{GL}_2(\mathcal{O}_K)$-invariant $W(\mathbb{F})$-lattice θ_0 in $\sigma(\tau)$ with irreducible cosocle. With the notation of [EGS15, §5.1] there is $I \in \mathcal{P}$ such that this cosocle is $\overline{\sigma}_I(\tau)$ and $\theta_0 = \sigma_I^\circ(\tau)$. As in [EGS15, p.77] we can reindex the irreducible constituents of θ_0/p by elements J' in $\mathcal{P}$ as follows:

$$\sigma_{J'} \stackrel{\text{def}}{=} \overline{\sigma}_{(J'\cup I^c)\setminus(J'\cap I^c)}(\tau),$$

so that (by [EGS15, Thm.5.1.1]) the j-th layer of the cosocle filtration of θ_0/p consists of the $\sigma_{J'}$ for $|J'| = f - j$, $0 \le j \le f$. By the beginning of the proof of [EGS15, Thm.10.1.1] (see *loc.cit.* p.77), there is $J'_{\min} \subseteq J'_{\max}$ in $\mathcal{P}$ such that $\mathrm{JH}(\overline{\theta_0/p}) \cap W(\overline{\rho}) = \{\sigma_{J'} : J'_{\min} \subseteq J' \subseteq J'_{\max}\}$. By [EGS15, Thm.7.2.1] we have

$$R_\infty^{(1,0),\tau} \cong \Big(W(\mathbb{F})[\![(X'_j, Y'_j)_{j\in J'_{\max}\setminus J'_{\min}}]\!]/(X'_jY'_j - p)_{j\in J'_{\max}\setminus J'_{\min}}\Big)[\![U_1, \dots, U_d]\!]$$

for some integer $d \ge 0$. Up to renumbering the variables we can assume that the irreducible component of $R_\infty^{(1,0),\tau}/p$ corresponding to $\sigma_{J'}$, $J'_{\min} \subseteq J' \subseteq J'_{\max}$, in [EGS15, p.77] (which is the support of $M_\infty(\sigma_{J'})$ by Theorem 3.4.2.1(i)) is given by the ideal $((X'_j)_{j\in J'\backslash J'_{\min}}, (Y'_j)_{j\in J'_{\max}\backslash J'})$.

We first fix $J \in \mathcal{P}$ such that $|J| = f-1$, so that $J^c = \{j\}$ for some $j \in \{0, \ldots, f-1\}$. We let θ be the unique (up to homothety) $\mathrm{GL}_2(\mathcal{O}_K)$-invariant $W(\mathbb{F})$-lattice in $\sigma(\tau)$ with irreducible cosocle σ_J ([EGS15, Lemma 4.1.1]). Up to multiplication by an element in $W(\mathbb{F})^\times$, there is a unique $\mathrm{GL}_2(\mathcal{O}_K)$-equivariant saturated inclusion $\iota : \theta \hookrightarrow \theta_0$, i.e. such that the induced morphism $\bar{\iota} : \theta/p \to \theta_0/p$ is nonzero. Recall that by Theorem 3.4.2.1(iii) both $M_\infty(\theta)$ and $M_\infty(\theta_0)$ are free of rank r over $R_\infty^{(1,0),\tau}$.

Lemma 3.4.3.1. *The image of $M_\infty(\iota) : M_\infty(\theta) \hookrightarrow M_\infty(\theta_0)$ is $xM_\infty(\theta_0)$, where $x = p$ if $j \in J'_{\min}$, $x = X'_j$ if $j \in J'_{\max}\backslash J'_{\min}$ and $x = 1$ if $j \notin J'_{\max}$.*

Proof. It follows from [EGS15, Thm.5.2.4(4)] (up to a reindexation as above) that $p(\theta_0/\iota(\theta)) = 0$ and that the irreducible constituents of $\theta_0/\iota(\theta)$ are the $\sigma_{J'}$ for J' containing j. In particular $\theta_0/\iota(\theta)$ is of the form $\overline{\sigma}^{\mathcal{J}}$ for a capped interval $\mathcal{J}$ as in [EGS15, p.81] (namely $\mathcal{J} = \{J' : j \in J'\}$). By the proof of [BHH+23, Prop.8.2.3] the module $M_\infty(\theta_0/\iota(\theta)) = M_\infty(\overline{\sigma}^{\mathcal{J}})$ is free of rank r over its schematic support, which is the unique reduced quotient of $R_\infty^{(1,0),\tau}/p$ with irreducible components corresponding to the $\sigma_{J'}$ such that $j \in J'$ and $J'_{\min} \subseteq J' \subseteq J'_{\max}$. If $j \notin J'_{\max}$, there are no such J', so this quotient is 0 (i.e. $M_\infty(\theta_0/\iota(\theta)) = 0$). If $j \in J'_{\max}\backslash J'_{\min}$, then this quotient is clearly $(R_\infty^{(1,0),\tau}/p)/(X'_j) = R_\infty^{(1,0),\tau}/(X'_j)$. Finally, if $j \in J'_{\min}$, all irreducible components remain, i.e. this quotient is $R_\infty^{(1,0),\tau}/p$. The lemma follows by exactness of M_∞. □

We now consider an arbitrary $J \in \mathcal{P}$ and let θ be the unique invariant $W(\mathbb{F})$-lattice in $\sigma(\tau)$ with irreducible cosocle σ_J. If $J^c \neq \emptyset$ we set $J^c = \{j_1, \ldots, j_h\}$ and $J_i \stackrel{\text{def}}{=} J \amalg \{j_1, \ldots, j_{h-i}\}$ for $i \in \{0, \ldots, h\}$ (so $J_0 = \{0, \ldots, f-1\}$ and $J_h = J$). As above we then denote by θ_i for $i \in \{0, \ldots, h\}$ the unique (up to homothety) invariant $W(\mathbb{F})$-lattice in $\sigma(\tau)$ with irreducible cosocle σ_{J_i} and $\iota_i : \theta_i \hookrightarrow \theta_{i-1}$ the corresponding saturated inclusion for $i \in \{1, \ldots, h\}$ (so θ_0 is the same as before and $\theta_h = \theta$). The composition

$$\iota_1 \circ \cdots \circ \iota_i : \theta_i \stackrel{\iota_i}{\hookrightarrow} \theta_{i-1} \stackrel{\iota_{i-1}}{\hookrightarrow} \cdots \theta_1 \stackrel{\iota_1}{\hookrightarrow} \theta_0$$

is still saturated since one can check using [EGS15, Thm.5.1.1] that the cosocle σ_{J_h} of θ_h/p remains in the image of $\theta_i/p \to \theta_{i-1}/p$ for all $i \in \{h, h-1, \ldots, 1\}$ (indeed, by *loc. cit.* the Serre weights $\sigma_{J_i} - \sigma_{J_{i-1}}$ in θ_0/p form a nonsplit extension as $J_i \subsetneq J_{i-1}$ and $|J_{i-1}\backslash J_i| = 1$). In particular $\iota \stackrel{\text{def}}{=} \iota_1 \circ \cdots \circ \iota_h$ is the unique (up to scalar) saturated inclusion $\theta \hookrightarrow \theta_0$.

Proposition 3.4.3.2. *There is $x \in R_\infty^{(1,0),\tau}$ such that the image of $M_\infty(\iota) : M_\infty(\theta) \hookrightarrow M_\infty(\theta_0)$ is $xM_\infty(\theta_0)$. Moreover the principal ideal $xR_\infty^{(1,0),\tau}$ only depends on* (*the semisimplification of*) *$\theta_0/\iota(\theta)$.*

Proof. The statement being trivial if $J^c = \emptyset$ (equivalently if $\theta = \theta_0$) we can assume $J^c \neq \emptyset$. For $i \in \{1, \dots, h\}$ we can apply Lemma 3.4.3.1 to $\iota_i : \theta_i \hookrightarrow \theta_{i-1}$ instead of $\iota : \theta \hookrightarrow \theta_0$. Hence there is $x_i \in R_\infty^{(1,0),\tau}$ such that the image of $M_\infty(\iota_i)$ is $x_i M_\infty(\theta_{i-1})$. The image of $M_\infty(\iota)$ is thus $(\prod_{i=1}^h x_i) M_\infty(\theta_0)$, i.e. we can take $x = \prod_{i=1}^h x_i$. It follows that $M_\infty(\theta_0/\iota(\theta)) \cong (R_\infty^{(1,0),\tau}/(x))^{\oplus r}$. Hence the irreducible components of $R_\infty^{(1,0),\tau}/(x)$ are the ones corresponding to the $\sigma_{J'}$ such that $J'_{\min} \subseteq J' \subseteq J'_{\max}$ and $\sigma_{J'}$ appears in $\theta_0/\iota(\theta)$, and their multiplicities are the multiplicities of the $\sigma_{J'}$ in $\theta_0/\iota(\theta)$. The second assertion then follows by the same argument as at the end of the proof of [DL21, Prop.4.17] (it also follows from an explicit computation of x via Lemma 3.4.3.1). □

Till the end of this section, we now extensively use notation and results from [DL21, §4] to which we refer the reader for more details.

Recall that $D_0(\overline{\rho}) = \oplus_{\sigma \in W(\overline{\rho})} D_{0,\sigma}(\overline{\rho})$. If $\chi : I \to \mathbb{F}^\times$ is a character appearing in $D_0(\overline{\rho})^{I_1}$ and $\mathbb{F} v_\chi \subseteq D_0(\overline{\rho})$ is the corresponding eigenspace (which is 1-dimensional), we define as in [DL21, Def.4.1] $R\chi$ as the character of I on $(\mathrm{soc}_{\mathrm{GL}_2(\mathcal{O}_K)} \langle \mathbb{F}\,\mathrm{GL}_2(\mathcal{O}_K) v_\chi \rangle)^{I_1}$, which is also 1-dimensional as it is σ^{I_1} for the unique $\sigma \in W(\overline{\rho})$ such that χ appears in $D_{0,\sigma}(\overline{\rho})^{I_1}$. As in [BP12, p.8] we denote by χ^s the character of I on $\left(\begin{smallmatrix} 0 & 1 \\ p & 0 \end{smallmatrix}\right) v_\chi \in D_0(\overline{\rho})^{I_1}$ and by $\sigma(\chi)$ the Serre weight which is the cosocle of $\mathrm{Ind}_I^{\mathrm{GL}_2(\mathcal{O}_K)} \chi$.

We define as in [DL21, Prop.4.14] an isomorphism

$$\overline{h}_\chi : M_\infty(\sigma(R\chi^s))/\mathfrak{m}_\infty \xrightarrow{\sim} M_\infty(\sigma(R\chi))/\mathfrak{m}_\infty$$

(the "one-dimensional by Theorem 4.6" in the proof of *loc.cit.* can just be replaced by "of the same dimension by Theorem 3.4.2.1"; also note that $\overline{h}_\chi$ is an isomorphism, as it is dual to the isomorphism g_χ in *loc.cit.*).

Proposition 3.4.3.3. *Let $k \geq 1$ and $\chi_0, \dots, \chi_{k-1}$ arbitrary characters of I which occur on π^{I_1} (equivalently on $D_0(\overline{\rho})^{I_1}$) such that $R\chi_i^s = R\chi_{i+1}$ for $i \in \{0, \dots, k-2\}$ and $R\chi_{k-1}^s = R\chi_0$. Then the isomorphism*

$$\overline{h}_{\chi_1} \circ \overline{h}_{\chi_2} \circ \cdots \circ \overline{h}_{\chi_{k-1}} \circ \overline{h}_{\chi_0} : M_\infty(\sigma(R\chi_0^s))/\mathfrak{m}_\infty \xrightarrow{\sim} M_\infty(\sigma(R\chi_0^s))/\mathfrak{m}_\infty$$

is the multiplication by a scalar in $\mathbb{F}^\times$ which depends neither on r nor on M_∞. In particular this scalar is the same as in [DL21, (34)].

Proof. We just indicate the steps in the proofs of [DL21, §§4.4, 4.5], where the assumption $r = 1$ is used, and how one can extend the argument there to $r \geq 1$. We use without comment the notation of *loc.cit.*

- The definition of the isomorphism $\widetilde{h}_\chi : M_\infty(\theta^{R\chi^s}) \xrightarrow{\sim} M_\infty(\theta^{R\chi})$ in [DL21, (28)] holds because one only needs to know that $M_\infty(\theta^{R\chi^s})$ and $M_\infty(\theta^{R\chi})$ are free of the same finite rank over $R_\infty(\tau)$.
- By Proposition 3.4.3.2 there exists $\widetilde{U}_p(\chi) \in R_\infty(\tau)$ such that $M_\infty(\iota)(M_\infty(\theta^{R\chi})) =$

$\widetilde{U}_p(\chi)M_\infty(\theta^{R\chi^s})$, where $\iota : \theta^{R\chi} \hookrightarrow \theta^{R\chi^s}$ is as in the unlabelled commutative diagram below [DL21, (27)]. Since $R_\infty(\tau)$ is a domain by [EGS15, Thm.7.2.1(2)] and $M_\infty(\theta^{R\chi})$, $M_\infty(\theta^{R\chi^s})$ are free of rank r over $R_\infty(\tau)$ by Theorem 3.4.2.1(iii), there is a unique $R_\infty(\tau)$-linear isomorphism $\widetilde{\iota}_\chi : M_\infty(\theta^{R\chi}) \xrightarrow{\sim} M_\infty(\theta^{R\chi^s})$ such that $M_\infty(\iota) = \widetilde{\iota}_\chi \circ \widetilde{U}_p(\chi)$, where $\widetilde{U}_p(\chi)$ here means multiplication by $\widetilde{U}_p(\chi)$ on $M_\infty(\theta^{R\chi})$. Then we have a commutative diagram analogous to [DL21, (29)] replacing the multiplication by $\widetilde{U}_p(\chi)$ in the diagonal map by the map $\widetilde{h}_\chi \circ \widetilde{\iota}_\chi \circ \widetilde{U}_p(\chi) = \widetilde{U}_p(\chi)(\widetilde{h}_\chi \circ \widetilde{\iota}_\chi)$.

• By the commutativity of the right-hand side of (the analog of) [DL21, (28)] and by the isomorphism $M_\infty(Q(\chi^s)^{R\chi}) \cong M_\infty(\theta(\chi^s)^{R\chi})/p$, we deduce that the map

$$h_\chi \circ \iota_Q : M_\infty(Q(\chi^s)^{R\chi}) \longrightarrow M_\infty(Q(\chi^s)^{R\chi})$$

is the multiplication by the image of $p^{-e(\chi)}U_p(\chi)$ in $R_\infty(\tau(\chi^s))/p$. As the image of $h_\chi \circ \iota_Q$ is $\widetilde{U}_p(\chi)M_\infty(Q(\chi^s)^{R\chi})$ by the commutativity of the left-hand side of (the analog of) [DL21, (28)] and the definition of $\widetilde{U}_p(\chi)$, we deduce that

$$\widetilde{U}_p(\chi)(R_\infty(\tau(\chi^s))/p) = (p^{-e(\chi)}U_p(\chi))(R_\infty(\tau(\chi^s))/p).$$

In particular, multiplying $\widetilde{U}_p(\chi)$ by a unit in $R_\infty(\tau)$ we can assume that $\widetilde{U}_p(\chi)$ and $p^{-e(\chi)}U_p(\chi)$ have the same image in the quotient $R_\infty(\tau(\chi^s))/p$ of $R_\infty(\tau)$. As a consequence the analogue of [DL21, Prop.4.17] holds.

• Since by definition $p^{-e(\chi)}U_p(\chi) \in R_\infty(\tau(\chi^s)) \backslash pR_\infty(\tau(\chi^s))$, we have

$$\mathrm{Ann}_{R_\infty(\tau(\chi^s))/p}\Big(\overline{p^{-e(\chi)}U_p(\chi)}\Big) \subseteq \mathfrak{m}_\infty(R_\infty(\tau(\chi^s))/p). \tag{188}$$

As $\widetilde{U}_p(\chi) \mapsto \overline{p^{-e(\chi)}U_p(\chi)} \in R_\infty(\tau(\chi^s))/p$ (previous point), we deduce $\widetilde{U}_p(\chi)(\widetilde{h}_\chi \circ \widetilde{\iota}_\chi - \mathrm{Id}) \mapsto 0$ in $\mathrm{End}_{R_\infty(\tau(\chi^s))/p}(M_\infty(Q(\chi^s)^{R\chi}))$ by the analog of [DL21, (28)]. As $M_\infty(Q(\chi^s)^{R\chi}) \cong M_\infty(\theta(\chi^s)^{R\chi})/p$ is free of rank r over $R_\infty(\tau(\chi^s))/p$ (by Theorem 3.4.2.1(iii)), (188) implies the image of $\widetilde{h}_\chi \circ \widetilde{\iota}_\chi - \mathrm{Id}$ in $\mathrm{End}_{R_\infty(\tau(\chi^s))/p}(M_\infty(Q(\chi^s)^{R\chi}))$ lands in $\mathfrak{m}_\infty \mathrm{End}_{R_\infty(\tau(\chi^s))/p}(M_\infty(Q(\chi^s)^{R\chi}))$. Since $\mathrm{Ker}(R_\infty(\tau) \twoheadrightarrow R_\infty(\tau(\chi^s))/p) \subseteq \mathfrak{m}_\infty R_\infty(\tau)$, we also have

$$\widetilde{h}_\chi \circ \widetilde{\iota}_\chi - \mathrm{Id} \in \mathfrak{m}_\infty \mathrm{End}_{R_\infty(\tau)}(M_\infty(\theta^{R\chi})). \tag{189}$$

• The big unlabelled diagram before [DL21, (33)] still holds but the diagonal maps are not simply multiplication by some $\widetilde{U}_p(\chi_i)$. For instance in the case $k = 3$ (the general case being similar) one has to replace the left diagonal maps in *loc.cit.* by successively (from top to bottom) $\widetilde{U}_p(\chi_0)((\widetilde{\iota}_{\chi_2} \circ \widetilde{\iota}_{\chi_1})^{-1} \circ (\widetilde{h}_{\chi_0} \circ \widetilde{\iota}_{\chi_0}) \circ \widetilde{\iota}_{\chi_2} \circ \widetilde{\iota}_{\chi_1})$, $\widetilde{U}_p(\chi_2)(\widetilde{\iota}_{\chi_1}^{-1} \circ (\widetilde{h}_{\chi_2} \circ \widetilde{\iota}_{\chi_2}) \circ \widetilde{\iota}_{\chi_1})$, and $\widetilde{U}_p(\chi_1)(\widetilde{h}_{\chi_1} \circ \widetilde{\iota}_{\chi_1})$. By (189) and the $R_\infty(\tau)$-linearity of the isomorphisms $\widetilde{\iota}_{\chi_i}$, all these diagonal maps are in $\widetilde{U}_p(\chi_i)(\mathrm{Id} + \mathfrak{m}_\infty \mathrm{End}_{R_\infty(\tau)}(M_\infty(\theta^{R\chi_0^s})))$, and their composition is thus in

$$\Big(\prod_{i=0}^{k-1} \widetilde{U}_p(\chi_i)\Big)(\mathrm{Id} + \mathfrak{m}_\infty \mathrm{End}_{R_\infty(\tau)}(M_\infty(\theta^{R\chi_0^s}))). \tag{190}$$

• For $\nu \geq 1$ defined as above [DL21, (33)], we have from the definition of the $\widetilde{\iota}_{\chi_i}$:

$$\Big(\prod_{i=0}^{k-1} \widetilde{U}_p(\chi_i)\Big)(\widetilde{\iota}_{\chi_0} \circ \widetilde{\iota}_{\chi_{k-1}} \circ \cdots \circ \widetilde{\iota}_{\chi_1}) = p^{\nu} \operatorname{Id} \tag{191}$$

which implies $p^{-\nu}(\prod_{i=0}^{k-1} \widetilde{U}_p(\chi_i)) \in R_\infty(\tau)^\times$ since the $\widetilde{\iota}_{\chi_i}$ are isomorphisms. By the commutativity in the (analog of) the big unlabelled diagram before [DL21, (33)] (see the previous point) together with (190) and (191) we finally obtain

$$\widetilde{h}_{\chi_1} \circ \cdots \circ \widetilde{h}_{\chi_{k-1}} \circ \widetilde{h}_{\chi_0} \in \Big(p^{-\nu} \prod_{i=0}^{k-1} \widetilde{U}_p(\chi_i)\Big)(\operatorname{Id} + \mathfrak{m}_\infty \operatorname{End}_{R_\infty(\tau)}(M_\infty(\theta^{R\chi_0^s})))$$

which is our analog of [DL21, (33)]. Then [DL21, (34)] follows by the same argument. The rest of the proof in [DL21, §5] is unchanged. □

We can now prove Theorem 3.4.1.1.

Proof of Theorem 3.4.1.1. We let $D(\overline{\rho}) = (D_1(\overline{\rho}) \hookrightarrow D_0(\overline{\rho}))$ be the diagram denoted by $\mathcal{D}(\pi_{\mathrm{glob}}(\overline{\rho}))$ in [DL21], which only depends on $\overline{\rho}$. Let $D(\pi) = (D_1(\pi) \hookrightarrow D_0(\pi)) \overset{\mathrm{def}}{=} (\pi^{I_1} \hookrightarrow \pi^{K_1})$ be the diagram defined by π. We will show that $D(\overline{\rho})^{\oplus r} \cong D(\pi)$ as diagrams.

Define first $R : \pi^{I_1} \to (\operatorname{soc}_{\mathrm{GL}_2(\mathcal{O}_K)} \pi)^{I_1}$ as in [DL21, Def.4.1], i.e. $Rv = S_{i(\chi)}v$ with $S_{i(\chi)}$ as in [DL21, Rem.4.2] if $v \in \pi^{I_1}$ is an I-eigenvector with eigencharacter χ. Note that the eigencharacter of Rv is $R\chi$.

Starting from $D(\overline{\rho})$ we define a groupoid $\mathcal{G}$ with objects $\mathbf{x}_\xi$, where ξ is any character of I such that $(\operatorname{soc}_{\mathrm{GL}_2(\mathcal{O}_K)} D_0(\overline{\rho}))^{I_1}[\xi] \neq 0$, and morphisms freely generated by $g_\chi : \mathbf{x}_{R\chi} \xrightarrow{\sim} \mathbf{x}_{R\chi^s}$, where χ is any character of I such that $D_1(\overline{\rho})[\chi] \neq 0$, as in [DL21, Def.4.3].

The diagram $D(\pi)$ defines an r-dimensional representation of $\mathcal{G}$, sending $\mathbf{x}_\xi$ to the vector space $(\operatorname{soc}_{\mathrm{GL}_2(\mathcal{O}_K)} D_0(\pi))^{I_1}[\xi]$ and g_χ to the linear map

$$g_\chi^\pi : (\operatorname{soc}_{\mathrm{GL}_2(\mathcal{O}_K)} D_0(\pi))^{I_1}[R\chi] \xrightarrow{\sim} (\operatorname{soc}_{\mathrm{GL}_2(\mathcal{O}_K)} D_0(\pi))^{I_1}[R\chi^s]$$

as in [DL21, §4]. Similarly, we have an r-dimensional representation of $\mathcal{G}$ defined by the diagram $D(\overline{\rho})^{\oplus r}$; we denote the linear maps by $g_\chi^{\overline{\rho}}$.

To check that the two r-dimensional representations of $\mathcal{G}$ are isomorphic it suffices to check that for each object $\mathbf{x}$ the restrictions of the two representations to the automorphism group $\mathcal{G}_{\mathbf{x}}$ are isomorphic (see [DL21, Prop.4.5]), which is the case by Proposition 3.4.3.3, remembering that g_χ^π is the dual of $\overline{h}_\chi$ by (the analog of) [DL21, Prop.4.14].

Therefore there exists an isomorphism

$$\lambda : (\mathrm{soc}_{\mathrm{GL}_2(\mathcal{O}_K)} D_0(\pi))^{I_1} \xrightarrow{\sim} (\mathrm{soc}_{\mathrm{GL}_2(\mathcal{O}_K)} D_0(\overline{\rho})^{\oplus r})^{I_1}$$

of I-representations such that $\lambda \circ g^{\pi}_{\chi} = g^{\overline{\rho}}_{\chi} \circ \lambda$ on $(\mathrm{soc}_{\mathrm{GL}_2(\mathcal{O}_K)} D_0(\pi))^{I_1}[R\chi]$ for all χ. As $\pi^{K_1} \cong D_0(\overline{\rho})^{\oplus r}$ as K-representations we can extend λ uniquely to an isomorphism $\lambda : D_0(\pi) \xrightarrow{\sim} D_0(\overline{\rho})^{\oplus r}$ of K-representations (extending to the $\mathrm{GL}_2(\mathcal{O}_K)$-socle first). As in the proof of [DL21, Prop.4.4] we deduce that λ restricts to an isomorphism $\lambda : D_1(\pi) \xrightarrow{\sim} D_1(\overline{\rho})^{\oplus r}$ commuting with $\left(\begin{smallmatrix} 0 & 1 \\ p & 0 \end{smallmatrix}\right)$ and I, which completes the proof. □

3.4.4 Local-global compatibility results

We collect our previous results to deduce (together with the results of [HW22]) special cases of Conjecture 2.1.3.1 and Conjecture 2.5.1 when $n = 2$ and K is unramified.

We keep all the previous notation. We also keep the assumptions (i) to (xii) of §3.4.1 (in particular $\overline{r}_{\tilde{v}}$ is semisimple), except that we replace the bounds on the r_i in (viii) by the stronger bounds (which are those of [BHH+23, §1]):

$$\begin{aligned} 12 \le r_j \le p - 15 & \quad \text{if } j > 0 \text{ or } \overline{\rho} \text{ is reducible;} \\ 13 \le r_0 \le p - 14 & \quad \text{if } \overline{\rho} \text{ is irreducible.} \end{aligned}$$

Recall that we choose Serre weights $\sigma_{\tilde{w}} \in W(\overline{r}_{\tilde{w}}(1))$ for $w \in S_p \backslash \{v\}$ and consider $\pi = \mathrm{Hom}_{U^v}(\otimes_{w \in S_p \backslash \{v\}} \sigma_{\tilde{w}}, S(V^v, \mathbb{F})[\mathfrak{m}^{\Sigma}])$ (see Theorem 3.4.1.1).

Theorem 3.4.4.1. *We have* $[\pi[\mathfrak{m}^3_{I_1/Z_1}] : \chi] = [\pi[\mathfrak{m}_{I_1/Z_1}] : \chi]$ *for all smooth characters* $\chi : I \to \mathbb{F}^{\times}$ *appearing in* $\pi[\mathfrak{m}_{I_1/Z_1}]$.

Proof. The statement of [BHH+23, Thm.8.3.11] applies *verbatim* with the same proof to π as above using Theorem 3.4.2.1 and (187). Combining this with Corollary 3.4.2.2, we see that π satisfies all the assumptions of [BHH+23, Thm.1.4], whence the result by [BHH+23, Thm.1.5]. □

Remark 3.4.4.2. A similar argument as in (ii) of the proof of [BHH+23, Thm.8.4.1] (which uses [GN22, App.A]) shows that we also have $\dim_{\mathrm{GL}_2(K)}(\pi) = f$, where $\dim_{\mathrm{GL}_2(K)}(\pi)$ is the Gelfand–Kirillov dimension of π as defined in [BHH+23, §5.1].

The following theorem is one of the main results of this paper.

Theorem 3.4.4.3. *Keep all the previous assumptions and assume that the* r_i *in* $\overline{r}_{\tilde{v}}$ *satisfy the following stronger bounds:*

$$\begin{array}{lllll} \max\{12, 2f-1\} & \le & r_j & \le & p - \max\{15, 2f+2\} \quad \textit{if } j > 0 \textit{ or } \overline{\rho} \textit{ is reducible;} \\ \max\{13, 2f\} & \le & r_0 & \le & p - \max\{14, 2f+1\} \quad \textit{if } \overline{\rho} \textit{ is irreducible.} \end{array} \tag{192}$$

Let $\sigma^v \overset{\text{def}}{=} \otimes_{w\in S_p\backslash\{v\}}\sigma_{\tilde{w}}$*, where the* $\sigma_{\tilde{w}}$ *are Serre weights in* $W(\overline{r}_{\tilde{w}}(1))$ *for* $w \in S_p\backslash\{v\}$*. Then Conjecture 2.1.3.1 holds for* $\mathrm{Hom}_{U^v}(\sigma^v, S(V^v,\mathbb{F})[\mathfrak{m}^\Sigma])$*.*

Proof. This follows from Corollary 3.3.2.4 applied to $\pi = \mathrm{Hom}_{U^v}(\sigma^v, S(V^v,\mathbb{F})[\mathfrak{m}^\Sigma])$, which satisfies all the assumptions there by Theorem 3.4.1.1 and Theorem 3.4.4.1, and by Remark 2.1.1.4(ii). □

We now give some evidence for Conjecture 2.5.1, still assuming (192). As we also need $r = 1$, and to make things as simple as possible, we replace assumptions (v) and (vii) in §3.4.1 by

$\overline{r}$ is unramified at all finite places outside S_p

and we then take $S \overset{\text{def}}{=} S_p$ (hence $\Sigma = S_p \cup \{v_1\}$). We also replace assumption (xii) in §3.4.1 by

$\iota_{\widetilde{v_1}}(U_{v_1})$ is equal to the upper-triangular unipotent matrices mod $\widetilde{v_1}$.

We take $V^v = U^p \prod_{w\in S_p\backslash\{v\}} V_w$ with $\iota_{\tilde{w}}(V_w) = 1 + pM_2(\mathcal{O}_{F_{\tilde{w}}}) \subseteq \mathrm{GL}_2(\mathcal{O}_{F_{\tilde{w}}}) = \iota_{\tilde{w}}(U_w)$. We let $T_{\widetilde{v_1}}$ be the Hecke operator acting on $S(V^v,\mathbb{F})$ by the double coset

$$\iota_{\widetilde{v_1}}^{-1}\left[\iota_{\widetilde{v_1}}(U_{v_1})\begin{pmatrix}\varpi_{\widetilde{v_1}} & \\ & 1\end{pmatrix}\iota_{\widetilde{v_1}}(U_{v_1})\right],$$

where $\varpi_{\widetilde{v_1}}$ is a uniformizer in $\mathcal{O}_{F_{\widetilde{v_1}}}$. Increasing $\mathbb{F}$ if necessary, we fix a choice of eigenvalues $\overline{\alpha}_{\widetilde{v_1}} \in \mathbb{F}$ of $\overline{\rho}(\mathrm{Frob}_{\widetilde{v_1}})$ (the image of a geometric Frobenius at $\widetilde{v_1}$) and consider the ideal

$$\mathfrak{m}^S \overset{\text{def}}{=} (\mathfrak{m}^\Sigma, T_{\widetilde{v_1}} - \alpha_{\widetilde{v_1}}) \subseteq \mathcal{T}^\Sigma[T_{\widetilde{v_1}}],$$

where $\alpha_{\widetilde{v_1}}$ is any element in $W(\mathbb{F})$ lifting $\overline{\alpha}_{\widetilde{v_1}}$ (see §2.1.2 for $\mathcal{T}^\Sigma$). Then, replacing $\mathfrak{m}^\Sigma$ by $\mathfrak{m}^S$ everywhere in §§3.4.1, 3.4.2, 3.4.3, by a multiplicity 1 result analogous to the one in [BD14, Prop.3.5.1] (see for instance the argument in the proof of [Enn, Lemma 3.1.4]) all the previous global results hold with r being 1.

Proposition 3.4.4.4. *Choose Serre weights* $\sigma_{\tilde{w}} \in W(\overline{r}_{\tilde{w}}(1))$ *for* $w \in S_p\backslash\{v\}$ *and let*

$$\pi \overset{\text{def}}{=} \mathrm{Hom}_{U^v}(\otimes_{w\in S_p\backslash\{v\}}\sigma_{\tilde{w}}, S(V^v,\mathbb{F})[\mathfrak{m}^S]).$$

The representation π *satisfies all the assumptions of §3.3.5 (with* $\overline{\rho} = \overline{r}_{\tilde{v}}(1)$*).*

Proof. The only missing assumption is the essential self-duality (176). But it holds by the same proof as for the definite case of [HW22, Thm.8.2] using Remark 3.4.4.2. □

From the results of §3.3.5, we thus deduce the following theorems.

Theorem 3.4.4.5. *The* $\mathrm{GL}_2(F_{\tilde{v}})$*-representation* π *is generated by its* $\mathrm{GL}_2(\mathcal{O}_{F_{\tilde{v}}})$*-socle, in particular is of finite type.*

Theorem 3.4.4.6.

(i) *Assume that* $\overline{r}_{\tilde{v}}$ *is irreducible. Then* π *is irreducible and is a supersingular representation.*

(ii) *Assume that* $\overline{r}_{\tilde{v}}$ *is reducible (split) and write* $\overline{\rho} = \overline{r}_{\tilde{v}}(1) = \begin{pmatrix} \chi_1 & 0 \\ 0 & \chi_2 \end{pmatrix}$*. Then one has*

$$\pi = \mathrm{Ind}_{B^-(F_{\tilde{v}})}^{\mathrm{GL}_2(F_{\tilde{v}})}(\chi_1\omega^{-1} \otimes \chi_2) \oplus \pi' \oplus \mathrm{Ind}_{B^-(F_{\tilde{v}})}^{\mathrm{GL}_2(F_{\tilde{v}})}(\chi_2\omega^{-1} \otimes \chi_1),$$

where π' *is generated by its* $\mathrm{GL}_2(\mathcal{O}_{F_{\tilde{v}}})$*-socle and* $\pi'^{\vee}$ *is essentially self-dual, i.e. satisfies* (176)*. Moreover, when* $f = 2$*,* π' *is irreducible and supersingular* (*and hence* π *is semisimple*)*.*

Proof. Everything is in Corollary 3.3.5.6 and Corollary 3.3.5.8, except the precise form of the irreducible principal series π_0, π_f in *loc.cit.*, but this easily follows from (183) and Theorem 3.4.1.1 (which is [DL21, §5] since $r = 1$). □

Combining Theorem 3.4.4.6 with Theorem 3.4.4.3, we obtain:

Corollary 3.4.4.7. *Keep the same assumptions as just before Proposition 3.4.4.4. If* $\overline{r}_{\tilde{v}}$ *is irreducible or if* $f = 2$*, then* π *is compatible with* $\overline{\rho}$ (*Definition 2.4.2.7*)*. In particular in these cases Conjecture 2.5.1 holds for* $\mathrm{Hom}_{U^v}(\sigma^v, S(V^v, \mathbb{F})[\mathfrak{m}^S])$*.*

Remark 3.4.4.8. When $\overline{r}_{\tilde{v}}$ is reducible nonsplit, a similar proof as for [HW22, Thm.1.6] (with the hypothesis of *loc.cit.* on $\overline{r}_{\tilde{v}}$) implies that π is generated over $\mathrm{GL}_2(F_{\tilde{v}})$ by π^{K_1}. When moreover $f = 2$, a similar proof as for [HW22, Thm.1.7] implies that π is at least compatible with $\widetilde{P}_{\overline{\rho}} = P_{\overline{\rho}} = B$ (Definition 2.4.1.5).

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