SMOOTHNESS AND CLASSICALITY ON EIGENVARIETIES

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Abstract. Let p be a prime number and f an overconvergent p-adic automorphic form on a definite unitary group which is split at p. Assume that f is of "classical weight" and that its Galois representation is crystalline at p, then f is conjectured to be a classical automorphic form. We prove new cases of this conjecture in arbitrary dimensions by making crucial use of the patched eigenvariety constructed in [13].

Contents

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1. Introduction

Let p be a prime number. In this paper we are concerned with classicality of p-adic automorphic forms on some unitary groups, i.e. we are looking for criteria that decide whether a given p-adic automorphic form is classical or not. More precisely we work with p-adic forms of finite slope, that is, in the context of *eigenvarieties*.

Let F^+ be a totally real number field and F be an imaginary quadratic extension of F^+. We fix a unitary group G in n variables over F^+ which splits over F and over all p-adic places of F^+, and which is compact at all infinite places of F^+. Associated to such a group G (and the choice of a tame level, i.e. a compact open subgroup of $G(\mathbb{A}_{F^+}^{p\infty})$) there is a nice Hecke eigenvariety which is an equidimensional rigid analytic space of dimension $n[F^+ : \mathbb{Q}]$, see e.g. [15], [2] or [21]. One may view a *p-adic overconvergent eigenform of finite slope*, or simply overconvergent form, as a point x of such an eigenvariety and one can associate to each overconvergent form a continuous semi-simple representation $\rho_x : \mathrm{Gal}(\overline{F}/F) \to \mathrm{GL}_n(\overline{\mathbb{Q}}_p)$ which is unramified outside a finite set of places of F and which is trianguline in the sense of [19] at all places of F dividing p ([34]).

A natural expectation deduced from the Langlands and Fontaine-Mazur conjectures is that, if ρ_x is de Rham (in the sense of Fontaine) at places of F dividing p, then x is a *classical* automorphic form (see Definition 3.2 and Proposition 3.4 for the precise definition). However, the naive version of this statement fails for two reasons: (1) a classical automorphic form for $G(\mathbb{A}_{F^+})$ can only give Galois representations which have distinct Hodge-Tate weights (in each direction $F \hookrightarrow \overline{\mathbb{Q}}_p$) and (2) the phenomenon of *companion* forms shows that there can exist classical and non-classical forms giving the same Galois representation. However, we can resolve (1) by requiring ρ_x to have distinct Hodge-Tate weights and (2) by requiring x to be of "classical" (or dominant) weight. In fact, since the Hodge-Tate weights of ρ_x are related to the weight of x, requiring the latter automatically implies the former, once ρ_x is assumed to be de Rham. As a conclusion, it seems reasonable to expect that any overconvergent form x of classical weight such that ρ_x is de Rham at places of F dividing p is a classical automorphic form (see Conjecture 3.6 and Remark 3.7).

Such a classicality theorem is due to Kisin ([35]) in the context of Coleman-Mazur's eigencurve, i.e. in the slightly different setting of $\mathrm{GL}_2/\mathbb{Q}$. Note that, at the time of [35], the notion of a trianguline representation was not available, and in fact [35] inspired Colmez to define trianguline representations ([19]).

In the present paper we prove new cases of this classicality conjecture (in the above unitary setting). In particular we are able to deal with cases where the overconvergent form x is *critical*. Throughout, we assume that ρ_x is crystalline at p-adic places. Essentially the same proof should work if ρ_x is only assumed crystabelline, but the crystalline assumption significantly simplifies the notation.

To state our main results, we fix an overconvergent form x of classical weight such that ρ_x is crystalline at all places dividing p. Such an overconvergent form can be described by a pair (ρ_x, δ_x), where ρ_x is as above and $\delta_x = (\delta_{x,v})_{v\in S_p}$ is a locally $\mathbb{Q}_p$-analytic character of the diagonal torus of $G(F^+ \otimes_{\mathbb{Q}} \mathbb{Q}_p) \cong \prod_{v\in S_p} \mathrm{GL}_n(F_v^+)$. Here S_p denotes the set of places of F^+ dividing p. There are nontrivial relations between $\rho_{x,v} := \rho|_{\mathrm{Gal}(\overline{F_v^+}/F_v^+)}$ and $\delta_{x,v}$, in particular the character $\delta_{x,v}$ defines an ordering of the eigenvalues of the crystalline Frobenius on

$D_{\mathrm{cris}}(\rho_{x,v})$. If we assume that these Frobenius eigenvalues are pairwise distinct, then this ordering defines a Frobenius stable flag in $D_{\mathrm{cris}}(\rho_{x,v})$. We can therefore associate to x for each $v \in S_p$ a permutation $w_{x,v}$ that gives the relative position of this flag with respect to the Hodge filtration on $D_{\mathrm{cris}}(\rho_{x,v})$, see §2.3 (where we rather use another equivalent definition of $w_{x,v}$ in terms of triangulations). Following [2, §2.4.3] we say that x is *noncritical* if, for each v, the permutation $w_{x,v}$ is trivial. The invariant $(w_{x,v})_{v\in S_p}$ can thus be seen as "measuring" the criticality of x.

In the statement of our main theorem, we need to assume a certain number of Working Hypotheses (basically the combined hypotheses of all the papers we use). We denote by $\overline{\rho}_x$ the mod p semi-simplification of ρ_x. These Working Hypotheses are:

(i) The field F is unramified over F^+ and G is quasi-split at all finite places of F^+;
(ii) the tame level of x is hyperspecial at all finite places of F^+ inert in F;
(iii) $\overline{\rho}_x(\mathrm{Gal}(\overline{F}/F(\zeta_p))$ is adequate ([48]);
(iv) the eigenvalues of φ on $D_{\mathrm{cris}}(\rho_{x,v})$ are sufficiently generic for any $v \in S_p$ (Definition 2.13).

Our main theorem is:

Theorem 1.1 (Cor. 3.12)**.** *Let $p > 2$ and assume that the group G and the tame level satisfy (i) and (ii). Let x be an overconvergent form of classical weight such that ρ_x is crystalline at p-adic places and satisfies (iii) and (iv). If $w_{x,v}$ is a product of distinct simple reflections for all places v of F^+ dividing p, then x is classical.*

Note that the assumption on the $w_{x,v}$ in Theorem 1.1 is empty when $n = 2$, and already this $n = 2$ case was not previously known (to the knowledge of the authors). The noncritical case of Theorem 1.1, i.e. the special case where all the $w_{x,v}$ are trivial, is already known and due to Chenevier ([16, Prop.4.2]). Thus the main novelty, and difficulty, in Theorem 1.1 is that it deals with possibly critical (though not too critical) points.

In fact we give a more general classicality criterion and prove that it is satisfied under the assumptions of Theorem 1.1. This criterion is formulated in terms of the rigid analytic space of trianguline representations $X_{\mathrm{tri}}^{\square}(\overline{\rho}_{x,v})$ defined in [29] and [13, §2.2]. For every $v \in S_p$ there is a canonical morphism from the eigenvariety to $X_{\mathrm{tri}}^{\square}(\overline{\rho}_{x,v})$.

Theorem 1.2 (Cor. 3.9, Rem. 3.13)**.** *Let $p > 2$ and assume that the group G and the tame level satisfy (i) and (ii). Let x be an overconvergent form of classical weight such that ρ_x is crystalline at p-adic places and satisfies (iii) and (iv). If for any $v \in S_p$ the image x_v of x in $X_{\mathrm{tri}}^{\square}(\overline{\rho}_{x,v})$ is contained in a unique irreducible component of $X_{\mathrm{tri}}^{\square}(\overline{\rho}_{x,v})$, then x is classical.*

According to this theorem we need to understand the local geometry of the space $X_{\mathrm{tri}}^{\square}(\overline{\rho}_{x,v})$ at x_v. It turns out that much of this local geometry is controlled by the Weyl group element $w_{x,v}$ associated to x which only depends on the image x_v of x in $X_{\mathrm{tri}}^{\square}(\overline{\rho}_{x,v})$. For $v \in S_p$ denote by $\lg(w_{x,v})$ the length of the permutation $w_{x,v}$ and by $d_{x,v}$ the rank of the $\mathbb{Z}$-module generated by $w_{x,v}(\alpha) - \alpha$, as α ranges over the roots of $(\mathrm{Res}_{F_v^+/\mathbb{Q}_p}\mathrm{GL}_n) \times_{\mathbb{Q}_p} \overline{\mathbb{Q}}_p \cong \prod_{\tau: F_v^+ \hookrightarrow \overline{\mathbb{Q}}_p} \mathrm{GL}_n$. Then $d_{x,v} \leq \lg(w_{x,v})$, with equality if and only if $w_{x,v}$ is a product of distinct simple reflections (Lemma 2.7).

Theorem 1.3 (Th. 2.15, Cor. 2.16)**.** *Let $v \in S_p$ and let $X \subseteq X_{\mathrm{tri}}^{\square}(\overline{\rho}_{x,v})$ be a union of irreducible components that contain x_v and satisfy the accumulation property of Definition 2.11 at x_v. Then:*

$$\dim T_{X,x_v} \leq \dim X + \lg(w_{x,v}) - d_{x,v} = \dim X_{\mathrm{tri}}^{\square}(\overline{\rho}_{x,v}) + \lg(w_{x,v}) - d_{x,v},$$

where T_{X,x_v} is the tangent space to X at x_v. In particular X is smooth at x_v when $w_{x,v}$ is a product of distinct simple reflections.

The accumulation condition in Theorem 1.3 actually prevents us from directly applying it to $X = X_{\mathrm{tri}}^{\square}(\overline{\rho}_{x,v})$ and thus directly deducing Theorem 1.1 from Theorem 1.2. Hence we have to sharpen Theorem 1.2, see Theorem 3.9.

Assuming the classical modularity lifting conjectures for $\overline{\rho}_x$ (in all weights with trivial inertial type), there is a certain union $\widetilde{X}_{\mathrm{tri}}^{\square}(\overline{\rho}_{x,v})$ of irreducible components of $X_{\mathrm{tri}}^{\square}(\overline{\rho}_{x,v})$ such that $\prod_{v \in S_p} \widetilde{X}_{\mathrm{tri}}^{\square}(\overline{\rho}_{x,v})$ is (essentially) described by the patched eigenvariety $X_p(\overline{\rho}_x)$ defined in [13] (see Remark 5.7). In the last section of the paper (§5), we prove (assuming modularity lifting conjectures) that the inequality in Theorem 1.3 for $X = \widetilde{X}_{\mathrm{tri}}^{\square}(\overline{\rho}_{x,v})$ is an *equality* for all $v \in S_p$:

$$\dim T_{\widetilde{X}_{\mathrm{tri}}^{\square}(\overline{\rho}_{x,v}),x_v} = \dim X_{\mathrm{tri}}^{\square}(\overline{\rho}_{x,v}) + \lg(w_{x,v}) - d_{x,v} \qquad \text{(assuming modularity)}, \tag{1.1}$$

see Corollary 5.17. The precise computation (1.1) of the dimension of the tangent space is intimately related to (and uses in its proof) the existence of many *companion points* on the patched eigenvariety $X_p(\overline{\rho}_x)$. These companion points are provided by the following unconditional theorem, which is of independent interest.

Theorem 1.4 (Th. 5.5)**.** *Let $y = ((\rho_v)_{v \in S_p}, \epsilon)$ be a point on $X_p(\overline{\rho}_x)$. Let T be the diagonal torus in GL_n and let δ be a locally $\mathbb{Q}_p$-analytic character of $T(F^+ \otimes_{\mathbb{Q}} \mathbb{Q}_p)$ such that $\epsilon\delta^{-1}$ is an algebraic character of $T(F^+ \otimes_{\mathbb{Q}} \mathbb{Q}_p)$ and such that ϵ is strongly linked to δ in the sense of* [32, §5.1] *(as modules over the Lie algebra of $T(F^+ \otimes_{\mathbb{Q}} \mathbb{Q}_p)$). Then $((\rho_v)_{v \in S_p}, \delta)$ is also a point on $X_p(\overline{\rho}_x)$.*

We also prove that the equality (1.1) for all $v \in S_p$ (and thus the modularity lifting conjectures) implies that the initial Hecke eigenvariety is itself *singular* at x as soon as the Weyl element $w_{x,v}$ is *not* a product of distinct simple reflections for some $v \in S_p$, see Corollary 5.18.

Let us now outline the strategy of the proofs of Theorems 1.2 and 1.3.

The proof of Theorem 1.3 crucially uses results of Bergdall ([4]) and Liu ([41]), together with a fine analysis of the various conditions on the infinitesimal deformations of $\rho_{x,v}$ carried by vectors in T_{X,x_v}, see §4. Recently, Bergdall proved an analogous bound for the dimension of the tangent space of the initial Hecke eigenvariety at x assuming standard vanishing conjectures on certain Selmer groups ([5]).

The proof of Theorem 1.2 makes use of the patched eigenvariety $X_p(\overline{\rho}_x)$ constructed in [13] by applying Emerton's construction of eigenvarieties [21] to the locally analytic vectors of the patched Banach $G(F^+ \otimes_{\mathbb{Q}} \mathbb{Q}_p)$-representation Π_∞ of [18]. As usual with the patching philosophy, the space $X_p(\overline{\rho}_x)$ can be related to another geometric object which has a much more local flavour, namely the space $X_{\rm tri}^{\square}(\overline{\rho}_{x,p}) := \prod_{v\in S_p} X_{\rm tri}^{\square}(\overline{\rho}_{x,v})$ of trianguline representations. More precisely, by [13, Th.3.21] there is a Zariski closed embedding:

$$X_p(\overline{\rho}_x) \hookrightarrow \mathfrak{X}_{\overline{\rho}_x^p} \times \mathbb{U}^g \times X_{\rm tri}^{\square}(\overline{\rho}_{x,p}), \qquad (1.2)$$

identifying the source with a union of irreducible components of the target. Here $\mathbb{U}^g$ is an open polydisc (related to the patching variables) and $\mathfrak{X}_{\overline{\rho}_x^p}$ is the rigid analytic generic fiber of the framed deformation space of $\overline{\rho}_x$ at all the "bad" places prime to p. Moreover the Hecke eigenvariety containing x can be embedded into the patched eigenvariety $X_p(\overline{\rho}_x)$ (see [13, Th.4.2]). As previously, we denote by x_v the image of x in $X_{\rm tri}^{\square}(\overline{\rho}_{x,v})$ via (1.2).

For $v \in S_p$ let us write $\mathbf{k}_v$ for the set of labelled Hodge-Tate weights of $\rho_{x,v}$, and $R_{\overline{\rho}_{x,v}}^{\square,\mathbf{k}_v-{\rm cr}}$ for the quotient defined in [36] of the framed deformation ring of $\overline{\rho}_{x,v}$ parametrizing crystalline deformations of $\overline{\rho}_{x,v}$ of Hodge-Tate weight $\mathbf{k}_v$, and $\mathfrak{X}_{\overline{\rho}_{x,v}}^{\square,\mathbf{k}_v-{\rm cr}}$ for the rigid space $(\mathrm{Spf}\, R_{\overline{\rho}_{x,v}}^{\square,\mathbf{k}_v-{\rm cr}})^{\rm rig}$. We relate $\mathfrak{X}_{\overline{\rho}_{x,v}}^{\square,\mathbf{k}_v-{\rm cr}}$ to $X_{\rm tri}^{\square}(\overline{\rho}_{x,v})$ by introducing a third rigid analytic space $\widetilde{\mathfrak{X}}_{\overline{\rho}_{x,v}}^{\square,\mathbf{k}_v-{\rm cr}}$ finite over $\mathfrak{X}_{\overline{\rho}_{x,v}}^{\square,\mathbf{k}_v-{\rm cr}}$ parametrizing crystalline deformations ρ_v of $\overline{\rho}_{x,v}$ of Hodge-Tate weights $\mathbf{k}_v$ *together with an ordering* of the Frobenius eigenvalues on $D_{\rm cris}(\rho_v)$, see §2.2 for a precise definition. The space $\widetilde{\mathfrak{X}}_{\overline{\rho}_{x,v}}^{\square,\mathbf{k}_v-{\rm cr}}$ naturally embeds into $X_{\rm tri}^{\square}(\overline{\rho}_{x,v})$ and contains the point x_v (and is smooth at x_v). We prove that there is a unique irreducible component $Z_{\rm tri}(x_v)$ of $X_{\rm tri}^{\square}(\overline{\rho}_{x,v})$ containing the unique irreducible component of $\widetilde{\mathfrak{X}}_{\overline{\rho}_{x,v}}^{\square,\mathbf{k}_v-{\rm cr}}$ passing through x_v (Corollary 2.5). Let $Z_{\rm tri}(x) := \prod_{v\in S_p} Z_{\rm tri}(x_v)$, which is thus an irreducible component of $X_{\rm tri}^{\square}(\overline{\rho}_{x,p})$ containing x. Then Theorem 1.2 easily follows from the following theorem (see (i) of Remark 3.13):

Theorem 1.5 (Th. 3.9)**.** *Assume that $\mathfrak{X}_{\overline{\rho}_x^p} \times \mathbb{U}^g \times Z_{\rm tri}(x) \subseteq X_p(\overline{\rho}_x)$ via (1.2). Then the point x is classical.*

Let us finally sketch the proof of Theorem 1.5 (in fact, for the same reason as above, we have to sharpen Theorem 1.5, see Theorem 3.9). Let R_∞ be the usual patched deformation ring of $\overline{\rho}_x$, there is a canonical morphism of rigid spaces $X_p(\overline{\rho}_x) \longrightarrow \mathfrak{X}_\infty := (\mathrm{Spf}\, R_\infty)^{\rm rig}$. Let $L(\lambda)$ be the finite dimensional algebraic representation of $G(F^+ \otimes_{\mathbb{Q}} \mathbb{Q}_p)$ associated (via the usual shift) to the Hodge-Tate weights $(\mathbf{k}_v)_{v\in S_p}$. Proving classicality of x turns out to be equivalent to proving that the image of x in $\mathfrak{X}_\infty$ is in the support of the R_∞-module $\Pi_\infty(\lambda)'$ which is the continuous dual of:

$$\Pi_\infty(\lambda) := \mathrm{Hom}_{\prod_{v\in S_p} \mathrm{GL}_n(\mathcal{O}_{F_{\tilde v}})}\Big(L(\lambda), \Pi_\infty\Big).$$

By [18, Lem.4.17], the R_∞-module $\Pi_\infty(\lambda)'$ is essentially a Taylor-Wiles-Kisin "usual" patched module for the trivial inertial type and the Hodge-Tate weights $(\mathbf{k}_v)_{v\in S_p}$. Forgetting the factors $\mathfrak{X}_{\overline{\rho}_x^p}$ and $\mathbb{U}^g$ which appear in $\mathfrak{X}_\infty$, its support is a union of irreducible components of the smooth rigid space $\prod_{v\in S_p} \mathfrak{X}_{\overline{\rho}_{x,v}}^{\square,\mathbf{k}_v-{\rm cr}}$. It is thus enough to prove that the unique irreducible component $Z_{\rm cris}(\rho_x)$ of $\prod_{v\in S_p} \mathfrak{X}_{\overline{\rho}_{x,v}}^{\square,\mathbf{k}_v-{\rm cr}}$ passing through $(\rho_{x,v})_{v\in S_p}$ contains a point which is in the support of $\Pi_\infty(\lambda)'$. But it is easy to find a point y in $Z_{\rm tri}(x)$ sufficiently close to x such that $(\rho_{y,v})_{v\in S_p} \in Z_{\rm cris}(\rho_x)$ (in particular $\rho_{y,v}$ is crystalline of the same Hodge-Tate weights as $\rho_{x,v}$) *and* moreover $\rho_{y,v}$ is *generic* in the sense of [13, Def.2.8] for all $v \in S_p$. The assumption in Theorem 1.5 implies $y \in X_p(\overline{\rho}_x)$ and it is now not difficult to prove that such a generic crystalline point of $X_p(\overline{\rho}_x)$ is always classical, i.e. is in the support of $\Pi_\infty(\lambda)'$.

We end this introduction with the main notation of the paper.

If K is a finite extension of $\mathbb{Q}_p$ we denote by $\mathcal{G}_K$ the absolute Galois group $\mathrm{Gal}(\overline{K}/K)$ and by Γ_K the Galois group $\mathrm{Gal}(K(\zeta_{p^n}, n \geq 1)/K)$ where $(\zeta_{p^n})_{n\geq 1}$ is a compatible system of primitive p^n-th roots of 1 in $\overline{K}$. We normalize the reciprocity map $\mathrm{rec}_K : K^\times \to \mathcal{G}_K^{\mathrm{ab}}$ of local class field theory so that the image of a uniformizer of K is a geometric Frobenius element. We denote by ε the p-adic cyclotomic character and recall that its Hodge-Tate weight is 1.

For $a \in L^\times$ (where L is any finite extension of K) we denote by $\mathrm{unr}(a)$ the unramified character of $\mathcal{G}_K$, or equivalently of $\mathcal{G}_K^{\mathrm{ab}}$ or $K^\times$, sending to a (the image by rec_K of) a uniformizer of K. For $z \in L$, we let $|z|_K := p^{-[K:\mathbb{Q}_p]\mathrm{val}(z)}$ where $\mathrm{val}(p) = 1$. We let $K_0 \subseteq K$ be the maximal unramified subfield (we thus have $(|\ |_K)|_{K^\times} = \mathrm{unr}(p^{-[K_0:\mathbb{Q}_p]}) = \mathrm{unr}(q^{-1})$ where q is the cardinality of the residue field of K).

If $X = \mathrm{Sp}\, A$ is an affinoid space, we write $\mathcal{R}_{A,K}$ for the Robba ring associated to K with A-coefficients (see [34, Def.6.2.1] though our notation is slightly different). Given a continuous character $\delta : K^\times \to A^\times$ we write $\mathcal{R}_{A,K}(\delta)$ for the rank one (φ, Γ_K)-module on $\mathrm{Sp}\, A$ defined by δ, see [34, Construction 6.2.4]. If X is a rigid analytic space over L (a finite extension of $\mathbb{Q}_p$) and x is a point on X, we denote by $k(x)$ the residue field of x (a finite extension of L), so that we have $x \in X(k(x))$. If X and Y are two rigid analytic spaces over L, we often write $X \times Y$ instead of $X \times_{\mathrm{Sp}\, L} Y$.

If X is a "geometric object over $\mathbb{Q}_p$" (i.e. a rigid space, a scheme, an algebraic group, etc.), we denote by X_K its base change to K (for instance if X is the algebraic group GL_n we write $\mathrm{GL}_{n,K}$). If H is an abelian p-adic Lie group, we let $\widehat{H}$ be the rigid analytic space over $\mathbb{Q}_p$ which represents the functor mapping an affinoid space $X = \mathrm{Sp}\, A$ to the group $\mathrm{Hom}_{\mathrm{cont}}(H, A^\times)$ of continous group homomorphisms (or equivalently locally $\mathbb{Q}_p$-analytic group homomorphisms) $H \to A^\times$. Finally, if M is an R-module and $I \subseteq R$ an ideal, we denote by $M[I] \subseteq M$ the submodule of elements killed by I, and if S is any finite set, we denote by $|S|$ its cardinality.

2. Crystalline points on the trianguline variety

We give several important definitions and results, including the key local statement bounding the dimension of some tangent spaces on the trianguline variety (Theorem 2.15).

2.1. **Recollections.**

We review some notation and definitions related to the trianguline variety.

We fix two finite extensions K and L of $\mathbb{Q}_p$ such that:

$$|\mathrm{Hom}(K, L)| = [K : \mathbb{Q}_p]$$

and denote by $\mathcal{O}_K$, $\mathcal{O}_L$ their respective rings of integers. We fix a uniformizer $\varpi_K \in \mathcal{O}_K$ and denote by k_L the residue field of $\mathcal{O}_L$. We let $\mathcal{T} := \widehat{K^\times}$ and $\mathcal{W} := \widehat{\mathcal{O}_K^\times}$. The restriction of characters to $\mathcal{O}_K^\times$ induces projections $\mathcal{T} \twoheadrightarrow \mathcal{W}$ and $\mathcal{T}_L \twoheadrightarrow \mathcal{W}_L$. If $\mathbf{k} := (k_\tau)_{\tau: K \hookrightarrow L} \in \mathbb{Z}^{\mathrm{Hom}(K,L)}$, we denote by $z^{\mathbf{k}} \in \mathcal{T}(L)$ the character:

$$z \longmapsto \prod_{\tau \in \mathrm{Hom}(K,L)} \tau(z)^{k_\tau} \tag{2.1}$$

where $z \in K^\times$. For $\mathbf{k} = (k_{\tau,i})_{1\le i\le n, \tau\colon K\hookrightarrow L} \in (\mathbb{Z}^n)^{\mathrm{Hom}(K,L)}$, we denote by $\delta_{\mathbf{k}} \in \mathcal{T}^n(L)$ the character:

$$(z_1,\dots,z_n) \longmapsto \prod_{\substack{1\le i\le n\\ \tau\colon K\hookrightarrow L}} \tau(z_i)^{k_{\tau,i}}$$

where $(z_1,\dots,z_n) \in (K^\times)^n$. We also denote by $\delta_{\mathbf{k}}$ its image in $\mathcal{W}^n(L)$ (i.e. its restriction to $(\mathcal{O}_K^\times)^n$). We say that a point $\delta \in \mathcal{W}_L^n$ is *algebraic* if $\delta = \delta_{\mathbf{k}}$ for some $\mathbf{k} = (k_{\tau,i})_{1\le i\le n,\tau\colon K\hookrightarrow L} \in (\mathbb{Z}^n)^{\mathrm{Hom}(K,L)}$. We say that an algebraic $\delta = \delta_{\mathbf{k}}$ is *dominant* (resp. *strictly dominant*) if moreover $k_{\tau,i} \ge k_{\tau,i+1}$ (resp. $k_{\tau,i} > k_{\tau,i+1}$) for $i \in \{1,\dots,n-1\}$ and $\tau \in \mathrm{Hom}(K,L)$.

We write $\mathcal{T}_{\mathrm{reg}} \subset \mathcal{T}_L$ for the Zariski-open complement of the L-valued points $z^{-\mathbf{k}}, |z|_K z^{\mathbf{k}+\mathbf{1}}$, with $\mathbf{k} = (k_\tau)_{\tau:K\hookrightarrow L} \in \mathbb{Z}_{\ge 0}^{\mathrm{Hom}(K,L)}$. We write $\mathcal{T}_{\mathrm{reg}}^n$ for the Zariski-open subset of characters $(\delta_1,\dots,\delta_n)$ such that $\delta_i/\delta_j \in \mathcal{T}_{\mathrm{reg}}$ for $i \ne j$.

We fix a continuous representation $\overline{r} : \mathcal{G}_K \to \mathrm{GL}_n(k_L)$ and let $R_{\overline{r}}^\square$ be the framed local deformation ring of $\overline{r}$ (a local complete noetherian $\mathcal{O}_L$-algebra of residue field k_L). We write $\mathfrak{X}_{\overline{r}}^\square := (\mathrm{Spf}\, R_{\overline{r}}^\square)^{\mathrm{rig}}$ for the rigid analytic space over L associated to the formal scheme $\mathrm{Spf}\, R_{\overline{r}}^\square$. Recall that a representation r of $\mathcal{G}_K$ on a finite dimensional L-vector space is called *trianguline of parameter* $\delta = (\delta_1,\dots,\delta_n)$ if the (φ,Γ_K)-module $D_{\mathrm{rig}}(r)$ over $\mathcal{R}_{L,K}$ associated to r admits an increasing filtration $\mathrm{Fil}_\bullet$ by sub-(φ,Γ_K)-modules over $\mathcal{R}_{L,K}$ such that the graded piece $\mathrm{Fil}_i/\mathrm{Fil}_{i-1}$ is isomorphic to $\mathcal{R}_{L,K}(\delta_i)$. We let $X_{\mathrm{tri}}^\square(\overline{r})$ be the associated framed trianguline variety, see [13, §2.2] and [29]. Recall that $X_{\mathrm{tri}}^\square(\overline{r})$ is the reduced rigid analytic space over L which is the Zariski closure in $\mathfrak{X}_{\overline{r}}^\square \times \mathcal{T}_L^n$ of:

$$U_{\mathrm{tri}}^\square(\overline{r}) := \{\text{points } (r,\delta) \text{ in } \mathfrak{X}_{\overline{r}}^\square \times \mathcal{T}_{\mathrm{reg}}^n \text{ such that } r \text{ is trianguline of parameter } \delta\} \tag{2.2}$$

(the space $U_{\mathrm{tri}}^\square(\overline{r})$ is denoted $U_{\mathrm{tri}}^\square(\overline{r})^{\mathrm{reg}}$ in [13, §2.2]). The rigid space $X_{\mathrm{tri}}^\square(\overline{r})$ is reduced equidimensional of dimension $n^2 + [K:\mathbb{Q}_p]\frac{n(n+1)}{2}$ and its subset $U_{\mathrm{tri}}^\square(\overline{r}) \subset X_{\mathrm{tri}}^\square(\overline{r})$ turns out to be Zariski-open, see [13, Th.2.6]. Moreover by *loc. cit.* the rigid variety $U_{\mathrm{tri}}^\square(\overline{r})$ is smooth over L and equidimensional, hence there is a bijection between the set of connected components of $U_{\mathrm{tri}}^\square(\overline{r})$ and the set of irreducible components of $X_{\mathrm{tri}}^\square(\overline{r})$.

We denote by $\omega : X_{\mathrm{tri}}^\square(\overline{r}) \to \mathcal{W}_L^n$ the composition $X_{\mathrm{tri}}^\square(\overline{r}) \hookrightarrow \mathfrak{X}_{\overline{r}}^\square \times \mathcal{T}_L^n \twoheadrightarrow \mathcal{T}_L^n \twoheadrightarrow \mathcal{W}_L^n$. If x is a point of $X_{\mathrm{tri}}^\square(\overline{r})$, we write $x = (r,\delta)$ where $r \in \mathfrak{X}_{\overline{r}}^\square$ and $\delta = (\delta_1,\dots,\delta_n) \in \mathcal{T}_L^n$. We say that a point $x = (r,\delta) \in X_{\mathrm{tri}}^\square(\overline{r})$ is *crystalline* if r is a crystalline representation of $\mathcal{G}_K$.

Lemma 2.1. *Let $x = (r,\delta) \in X_{\mathrm{tri}}^\square(\overline{r})$ be a crystalline point. Then for $i \in \{1,\dots,n\}$ there exist $\mathbf{k}_i = (k_{\tau,i})_{\tau\colon K\hookrightarrow L} \in \mathbb{Z}^{\mathrm{Hom}(K,L)}$ and $\varphi_i \in k(x)^\times$ such that:*

$$\delta_i = z^{\mathbf{k}_i}\mathrm{unr}(\varphi_i).$$

Moreover the $(k_{\tau,i})_{i,\tau}$ are the labelled Hodge-Tate weights of r and the φ_i are the eigenvalues of the geometric Frobenius on the (unramified) Weil-Deligne representation $\mathrm{WD}(r)$ *associated to r (cf.* [25]*).*

Proof. The fact that the $(k_{\tau,i})_{i,\tau}$ are the Hodge-Tate weights of r follows for instance from [13, Prop.2.9]. By [34, Th.6.3.13] there exists for each i a continuous character $\delta_i' : K^\times \to k(x)^\times$ such that r is trianguline of parameter $\delta' := (\delta_1',\dots,\delta_n')$ and such that δ_i/δ_i' is an algebraic character of $K^\times$ (i.e. of the form $z^{\mathbf{k}}$ for some $\mathbf{k} \in \mathbb{Z}^{\mathrm{Hom}(K,L)}$). It thus suffices to prove that each δ_i' is of the form $z^{\mathbf{k}_i'}\mathrm{unr}(\varphi_i)$ for some $\mathbf{k}_i' \in \mathbb{Z}^{\mathrm{Hom}(K,L)}$ where the $\varphi_i \in k(x)^\times$ are the eigenvalues of the geometric Frobenius on $\mathrm{WD}(r)$, or equivalently (using the definition of

$\mathrm{WD}(r)$) are the eigenvalues of the linearized Frobenius $\varphi^{[K_0:\mathbb{Q}_p]}$ on the $K_0 \otimes_{\mathbb{Q}_p} k(x)$-module $D_{\mathrm{cris}}(r) := (B_{\mathrm{cris}} \otimes_{\mathbb{Q}_p} r)^{\mathcal{G}_K}$. By [6, Th.3.6] there is an isomorphism (recall t is "Fontaine's $2i\pi$"):

$$D_{\mathrm{cris}}(r) \cong D_{\mathrm{rig}}(r)[\tfrac{1}{t}]^{\Gamma_K}, \tag{2.3}$$

and a triangulation $\mathrm{Fil}_\bullet$ of $D_{\mathrm{rig}}(r)$ with graded pieces giving the parameter δ' induces a complete φ-stable filtration $\mathcal{F}_\bullet$ on $D_{\mathrm{cris}}(r)$ such that $\mathcal{F}_i/\mathcal{F}_{i-1}$ is the filtered φ-module associated to $\mathcal{R}_{L,K}(\delta'_i) = \mathrm{Fil}_i/\mathrm{Fil}_{i-1}$ by the same recipee as (2.3). It follows from this and from [34, Example 6.2.6(3)] that δ'_i is of the form $z^{\mathbf{k}'_i}\mathrm{unr}(a)$ where $a \in k(x)^\times$ is the unique element such that $\varphi^{[K_0:\mathbb{Q}_p]}$ acts on the underlying φ-module of $\mathcal{F}_i/\mathcal{F}_{i-1}$ by multiplication by $1 \otimes a \in K_0 \otimes_{\mathbb{Q}_p} k(x)$. This finishes the proof. $\square$

Note that Lemma 2.1 implies that if $x = (r, \delta) \in X_{\mathrm{tri}}^\square(\overline{r})$ is a crystalline point, then $\omega(x)$ is algebraic ($= \delta_{\mathbf{k}}$ for $\mathbf{k} := (k_{\tau,i})_{1\leq i\leq n, \tau: K\hookrightarrow L}$ where the $k_{\tau,i}$ are as in Lemma 2.1). We say that a point $x = (r, \delta) \in X_{\mathrm{tri}}^\square(\overline{r})$ such that $\omega(x)$ is algebraic is *dominant* (resp. *strictly dominant*) if $\omega(x)$ is dominant (resp. strictly dominant).

2.2. **A variant of the crystalline deformation space.** We define a certain irreducible component $Z_{\mathrm{tri},U}(x)$ of a sufficiently small open neighbouhood $U \subseteq X_{\mathrm{tri}}^\square(\overline{r})$ containing a crystalline strictly dominant point x (Corollary 2.5).

We fix $\mathbf{k} = (k_{\tau,i})_{1\leq i\leq n, \tau: K\hookrightarrow L} \in (\mathbb{Z}^n)^{\mathrm{Hom}(K,L)}$ such that $k_{\tau,i} > k_{\tau,i+1}$ for all i, τ and write $R_{\overline{r}}^{\square,\mathbf{k}-\mathrm{cr}}$ for the crystalline deformation ring of $\overline{r}$ with Hodge-Tate weights $\mathbf{k}$, i.e. the reduced and $\mathbb{Z}_p$-flat quotient of $R_{\overline{r}}^\square$ such that, for any finite extension L' of L, a morphism x : $\mathrm{Spec}\, L' \to \mathrm{Spec}\, R_{\overline{r}}^\square$ factors through $\mathrm{Spec}\, R_{\overline{r}}^{\square,\mathbf{k}-\mathrm{cr}}$ if and only if the representation $\mathcal{G}_K \to \mathrm{GL}_n(L')$ defined by x is crystalline with labelled Hodge-Tate weights $(k_{\tau,i})_{1\leq i\leq n, \tau: K\hookrightarrow L}$. That this ring exists is the main result of [36]. We write $\mathfrak{X}_{\overline{r}}^{\square,\mathbf{k}-\mathrm{cr}}$ for the rigid analytic space associated to $\mathrm{Spf}\, R_{\overline{r}}^{\square,\mathbf{k}-\mathrm{cr}}$. By [36], it is smooth over L.

Let $\tilde{r} : \mathcal{G}_K \to \mathrm{GL}_n(R_{\overline{r}}^{\square,\mathbf{k}-\mathrm{cr}})$ be the corresponding universal deformation. By [36, Th.2.5.5] or [7, Cor.6.3.3] there is a coherent $K_0 \otimes_{\mathbb{Q}_p} \mathcal{O}_{\mathfrak{X}_{\overline{r}}^{\square,\mathbf{k}-\mathrm{cr}}}$-module $\mathcal{D}$ that is locally on $\mathfrak{X}_{\overline{r}}^{\square,\mathbf{k}-\mathrm{cr}}$ free over $K_0 \otimes_{\mathbb{Q}_p} \mathcal{O}_{\mathfrak{X}_{\overline{r}}^{\square,\mathbf{k}-\mathrm{cr}}}$ together with a $\varphi \otimes \mathrm{id}$-linear automorphism Φ_{cris} such that:

$$(\mathcal{D}, \Phi_{\mathrm{cris}}) \otimes_{\mathcal{O}_{\mathfrak{X}_{\overline{r}}^{\square,\mathbf{k}-\mathrm{cr}}}} k(x) \cong D_{\mathrm{cris}}\Big(\tilde{r} \otimes_{R_{\overline{r}}^{\square,\mathbf{k}-\mathrm{cr}}} k(x)\Big)$$

for all $x \in \mathfrak{X}_{\overline{r}}^{\square,\mathbf{k}-\mathrm{cr}}$. Fixing an embedding $\tau_0 : K_0 \hookrightarrow L$ we can define the associated family of Weil-Deligne representations:

$$(\mathrm{WD}(\tilde{r}), \Phi) := \Big(\mathcal{D} \otimes_{K_0 \otimes_{\mathbb{Q}_p} \mathcal{O}_{\mathfrak{X}_{\overline{r}}^{\square,\mathbf{k}-\mathrm{cr}}}, \tau_0 \otimes \mathrm{id}} \mathcal{O}_{\mathfrak{X}_{\overline{r}}^{\square,\mathbf{k}-\mathrm{cr}}}, \Phi_{\mathrm{cris}}^{[K_0:\mathbb{Q}_p]} \otimes \mathrm{id}\Big)$$

on $\mathfrak{X}_{\overline{r}}^{\square,\mathbf{k}-\mathrm{cr}}$ whose isomorphism class does not depend on the choice of the embedding τ_0.

Let $T^{\mathrm{rig}} \cong (\mathbb{G}_{\mathrm{m}}^{\mathrm{rig}})^n$ be the rigid analytic space over $\mathbb{Q}_p$ associated to the diagonal torus $T \subset \mathrm{GL}_n$ and let $\mathcal{S}_n$ be the Weyl group of (GL_n, T) acting on T, and thus on T^{rig}, in the usual way. Recall that the map:

$$\mathrm{diag}(\varphi_1, \varphi_2, \ldots, \varphi_n) \mapsto \text{coefficients of } (X - \varphi_1)(X - \varphi_2) \ldots (X - \varphi_n)$$

induces an isomorphism of schemes over $\mathbb{Q}_p$:

$$T/\mathcal{S}_n \xrightarrow{\sim} \mathbb{G}_{\mathrm{a}}^{n-1} \times_{\mathrm{Spec}\,\mathbb{Q}_p} \mathbb{G}_{\mathrm{m}}$$

and also of the associated rigid analytic spaces. We deduce that the coefficients of the characteristic polynomial of the Frobenius Φ on $\mathrm{WD}(\tilde{r})$ determine a morphism of rigid analytic spaces over L:

$$\mathfrak{X}_{\overline{r}}^{\square,\mathbf{k}-\mathrm{cr}} \longrightarrow T_L^{\mathrm{rig}}/\mathcal{S}_n.$$

Let us define:

$$\widetilde{\mathfrak{X}}_{\overline{r}}^{\square,\mathbf{k}-\mathrm{cr}} := \mathfrak{X}_{\overline{r}}^{\square,\mathbf{k}-\mathrm{cr}} \times_{T_L^{\mathrm{rig}}/\mathcal{S}_n} T_L^{\mathrm{rig}}.$$

Concretely $\widetilde{\mathfrak{X}}_{\overline{r}}^{\square,\mathbf{k}-\mathrm{cr}}$ parametrizes crystalline framed $\mathcal{G}_K$-deformations r of $\overline{r}$ of labelled Hodge-Tate weights $\mathbf{k}$ together with an ordering $(\varphi_1, \ldots, \varphi_n)$ of the eigenvalues of the geometric Frobenius on $\mathrm{WD}(r)$.

Lemma 2.2. *The rigid analytic space* $\widetilde{\mathfrak{X}}_{\overline{r}}^{\square,\mathbf{k}-\mathrm{cr}}$ *is reduced.*

Proof. It is sufficient to prove this result locally. Let $\mathrm{Sp}\,C$ be an admissible irreducible affinoid open subspace of $\mathfrak{X}_{\overline{r}}^{\square,\mathbf{k}-\mathrm{cr}}$ whose image in $T_L^{\mathrm{rig}}/\mathcal{S}_n$ is contained in an admissible affinoid open irreducible subspace $\mathrm{Sp}\,A$ of $T_L^{\mathrm{rig}}/\mathcal{S}_n$. As both $\mathfrak{X}_{\overline{r}}^{\square,\mathbf{k}-\mathrm{cr}}$ and $T_L^{\mathrm{rig}}/\mathcal{S}_n$ are smooth over L we can find an admissible open affinoid covering of $\mathfrak{X}_{\overline{r}}^{\square,\mathbf{k}-\mathrm{cr}}$ by such $\mathrm{Sp}\,C$. The map $T_L^{\mathrm{rig}} \to T_L^{\mathrm{rig}}/\mathcal{S}_n$ is finite flat being the rigidification of a map of affine schemes $T_L \to T_L/\mathcal{S}_n$ which is finite flat. Consequently the inverse image of $\mathrm{Sp}\,A$ in T_L^{rig} is an admissible affinoid open subspace $\mathrm{Sp}\,B$ with B an affinoid algebra which is finite flat over A. As B is a finite A-algebra, we have an isomorphism $C\widehat{\otimes}_A B \simeq C \otimes_A B$. It follows, by definition of the fiber product of rigid analytic spaces, that the rigid analytic spaces of the form $\mathrm{Sp}(C \otimes_A B)$ form an admissible open covering of $\widetilde{\mathfrak{X}}_{\overline{r}}^{\square,\mathbf{k}-\mathrm{cr}} = \mathfrak{X}_{\overline{r}}^{\square,\mathbf{k}-\mathrm{cr}} \times_{T_L^{\mathrm{rig}}/\mathcal{S}_n} T_L^{\mathrm{rig}}$. It is sufficient to prove that rings $C \otimes_A B$ as above are reduced. From Lemma 2.3 below it is sufficient to prove that $C \otimes_A B$ is a finite flat generically étale C-algebra. As B is finite flat over A, the C-algebra $C \otimes_A B$ is clearly finite flat. It is sufficient to prove that it is a generically étale C-algebra. As B is generically étale over A, it is sufficient to prove that the map $\mathrm{Spec}\,C \to \mathrm{Spec}\,A$ is dominant. It is thus sufficient to prove that the map of rigid analytic spaces $\mathfrak{X}_{\overline{r}}^{\square,\mathbf{k}-\mathrm{cr}} \to T_L^{\mathrm{rig}}/\mathcal{S}_n$ is open. This follows from the fact that it has, locally on $\mathfrak{X}_{\overline{r}}^{\square,\mathbf{k}-\mathrm{cr}}$, a factorization:

$$\mathfrak{X}_{\overline{r}}^{\square,\mathbf{k}-\mathrm{cr}} \longrightarrow (\mathrm{Res}_{K_0/\mathbb{Q}_p}\mathrm{GL}_{n,K_0} \times_{\mathrm{Sp}\,\mathbb{Q}_p} \mathrm{Flag})^{\mathrm{rig}} \times_{\mathrm{Sp}\,\mathbb{Q}_p} \mathrm{Sp}\,L \longrightarrow T_L^{\mathrm{rig}}/\mathcal{S}_n$$

where the first map is the smooth map in the proof of Lemma 2.4 below, and the second is the projection on $(\mathrm{Res}_{K_0/\mathbb{Q}_p}\mathrm{GL}_{n,K_0})_L^{\mathrm{rig}}$ followed by the base change to L of the rigidification of the morphism $\mathrm{Res}_{K_0/\mathbb{Q}_p}\mathrm{GL}_{n,K_0} \to T/\mathcal{S}_n$ defined in [28, (9.1)]. The first map being smooth is flat and thus open by [9, Cor.9.4.2], and the last two are easily seen to be open. □

The following (well-known) lemma was used in the proof of Lemma 2.2.

Lemma 2.3. *Let A a commutative noetherian domain and B a finite flat A-algebra. Then the ring B has no embedded component, i.e. all its associated ideals are minimal prime ideals. Moreover if B is generically étale over A, i.e.* $\mathrm{Frac}(A) \otimes_A B$ *is a finite étale* $\mathrm{Frac}(A)$*-algebra, then the ring B is reduced.*

Proof. As B is flat over A, the map $\operatorname{Spec} B \to \operatorname{Spec} A$ has an open image, and A being a domain it contains the unique generic point of $\operatorname{Spec} A$, which implies that the natural map $A \to B$ is injective. Moreover B being finite over A, the image of $\operatorname{Spec} B \to \operatorname{Spec} A$ is closed, hence it is $\operatorname{Spec} A$ since $\operatorname{Spec} A$ is connected. In particular B is a faithfully flat A-algebra. As B is a flat A-module, it follows from [10, §IV.2.6 Lem.1] applied with $E = A$ and $F = B$ that $\mathfrak{p} \in \operatorname{Ass}(B)$ implies $\mathfrak{p} \cap A = 0$ (A is a domain, so $\operatorname{Ass}(A) = \{0\}$). It then follows from [10, §V.2.1 Cor.1] that if $\mathfrak{p} \in \operatorname{Ass}(B)$, then $\mathfrak{p}$ is a minimal prime of B. Indeed, A being noetherian and B a finite A-module, B is an integral extension of A. We can apply *loc. cit.* to the inclusion $\mathfrak{q} \subseteq \mathfrak{p}$ where $\mathfrak{q}$ is a minimal prime ideal of B (both ideals $\mathfrak{q}$ and $\mathfrak{p}$ being above the prime ideal (0) of A since $\mathfrak{p} \cap A = \mathfrak{q} \cap A = 0$).

Let $\mathfrak{d}_{B/A}$ be the discriminant of B/A (its existence comes from the fact that B is a finite faithfully flat A-algebra, hence a finite projective A-module). As the extension is generically étale, we can find $f \in \mathfrak{d}_{B/A}$ such that B_f is étale over A_f. As A_f is a domain, B_f is then reduced. Thus the nilradical $\mathfrak{n}$ of B_f is killed by some power of f. Replacing f by this power, we can assume that the vanishing ideal of $\mathfrak{n}$ contains f. Assume that $\mathfrak{n}$ is nonzero and let $\mathfrak{p}$ be a prime ideal of B minimal among prime ideals containing $\operatorname{Ann}_B(\mathfrak{n})$. It follows from [10, §IV.1.3 Cor.1] that $\mathfrak{p}$ is an associated prime of the B-module $\mathfrak{n}$ and consequently of B. But we have $f \in \mathfrak{p}$ which contradicts the fact that $\mathfrak{p} \cap A = 0$. □

We now embed this "refined" crystalline deformation space $\widetilde{\mathfrak{X}}_{\overline{r}}^{\square,\mathbf{k}-\mathrm{cr}}$ into the space $X_{\mathrm{tri}}^{\square}(\overline{r})$ as follows. We define a morphism of rigid spaces over L:

$$\begin{array}{rcl} \mathfrak{X}_{\overline{r}}^{\square,\mathbf{k}-\mathrm{cr}} \times_{\operatorname{Sp} L} T_L^{\mathrm{rig}} & \longrightarrow & \mathfrak{X}_{\overline{r}}^{\square} \times_{\operatorname{Sp} L} \mathcal{T}_L^n \\ (r, \varphi_1, \dots, \varphi_n) & \longmapsto & (r, z^{\mathbf{k}_1}\mathrm{unr}(\varphi_1), \dots, z^{\mathbf{k}_n}\mathrm{unr}(\varphi_n)). \end{array} \tag{2.4}$$

This morphism is a closed embedding of reduced rigid spaces as both maps $r \mapsto r$ and $(\varphi_1, \dots, \varphi_n) \mapsto (z^{\mathbf{k}_1}\mathrm{unr}(\varphi_1), \dots, z^{\mathbf{k}_n}\mathrm{unr}(\varphi_n))$ respectively define closed embeddings $\mathfrak{X}_{\overline{r}}^{\square,\mathbf{k}-\mathrm{cr}} \hookrightarrow \mathfrak{X}_{\overline{r}}^{\square}$ and $T_L^{\mathrm{rig}} \hookrightarrow \mathcal{T}_L^n$. We claim that the restriction of the morphism (2.4) to:

$$\widetilde{\mathfrak{X}}_{\overline{r}}^{\square,\mathbf{k}-\mathrm{cr}} \hookrightarrow \mathfrak{X}_{\overline{r}}^{\square,\mathbf{k}-\mathrm{cr}} \times_{\operatorname{Sp} L} T_L^{\mathrm{rig}} \tag{2.5}$$

factors through $X_{\mathrm{tri}}^{\square}(\overline{r}) \subset \mathfrak{X}_{\overline{r}}^{\square} \times_{\operatorname{Sp} L} \mathcal{T}_L^n$. As the source of this restriction is reduced by Lemma 2.2, it is enough to check it on a Zariski-dense set of points of $\widetilde{\mathfrak{X}}_{\overline{r}}^{\square,\mathbf{k}-\mathrm{cr}}$.

Let r be an n-dimensional crystalline representation of $\mathcal{G}_K$ over a finite extension L' of L of Hodge-Tate weights $\mathbf{k}$ and let $\varphi_1, \dots, \varphi_n$ be an ordering of the eigenvalues of a geometric Frobenius on $\mathrm{WD}(r)$, equivalently of the eigenvalues of $\varphi^{[K_0:\mathbb{Q}_p]}$ on $D_{\mathrm{cris}}(r)$ (that are assumed to be in $L'^{\times}$). Assuming moreover that the φ_i are pairwise distinct, this datum gives rise to a unique complete φ-stable flag of free $K_0 \otimes_{\mathbb{Q}_p} L'$-modules:

$$0 = \mathcal{F}_0 \subset \mathcal{F}_1 \subset \cdots \subset \mathcal{F}_n = D_{\mathrm{cris}}(r)$$

on $D_{\mathrm{cris}}(r)$ such that $\varphi^{[K_0:\mathbb{Q}_p]}$ acts on $\mathcal{F}_i/\mathcal{F}_{i-1}$ by multiplication by φ_i (this is a *refinement* in the sense of [2, Def.2.4.1]). By the same argument as in the proof of Lemma 2.1 using Berger's dictionary between crystalline (φ, Γ_K)-modules and filtered φ-modules (see e.g. (2.3)), the filtration $\mathcal{F}_\bullet$ induces a triangulation $\mathrm{Fil}_\bullet$ on $D_{\mathrm{rig}}(r)$. If we assume that $\mathcal{F}_\bullet$ is *noncritical* in the sense of [2, Def.2.4.5], i.e. the filtration $\mathcal{F}_\bullet$ is in general position with respect to the Hodge filtration $\mathrm{Fil}^\bullet D_{\mathrm{dR}}(r)$ on $D_{\mathrm{dR}}(r)$, that is, for all embeddings $\tau : K \hookrightarrow L$ and all

$i = 1, \dots, n-1$ we have:

$$(2.6)\quad \Big(\mathcal{F}_i \otimes_{K_0\otimes_{\mathbb{Q}_p}L',\tau\otimes\mathrm{id}} L'\Big) \oplus \Big(\mathrm{Fil}^{-k_{\tau,i+1}} D_{\mathrm{dR}}(r) \otimes_{K\otimes_{\mathbb{Q}_p}L',\tau\otimes\mathrm{id}} L'\Big) = D_{\mathrm{cris}}(r) \otimes_{K_0\otimes_{\mathbb{Q}_p}L,\tau\otimes\mathrm{id}} L' = D_{\mathrm{dR}}(r) \otimes_{K\otimes_{\mathbb{Q}_p}L',\tau\otimes\mathrm{id}} L',$$

then $\mathcal{F}_i/\mathcal{F}_{i-1}$ is a filtered φ-module of Hodge-Tate weights $\mathbf{k}_i$, or equivalently $\mathrm{Fil}_i/\mathrm{Fil}_{i-1} \cong \mathcal{R}_{L',K}(\delta_i)$ with $\delta_i = z^{\mathbf{k}_i}\mathrm{unr}(\varphi_i)$.

Lemma 2.4. *There are smooth (over L) Zariski-open and Zariski-dense subsets in $\widetilde{\mathfrak{X}}_{\overline{r}}^{\square,\mathbf{k}-\mathrm{cr}}$:*

$$\widetilde{V}_{\overline{r}}^{\square,\mathbf{k}-\mathrm{cr}} \subset \widetilde{U}_{\overline{r}}^{\square,\mathbf{k}-\mathrm{cr}} \subset \widetilde{\mathfrak{X}}_{\overline{r}}^{\square,\mathbf{k}-\mathrm{cr}}$$

such that:

(i) *a point $(r, \varphi_1, \dots, \varphi_n) \in \widetilde{\mathfrak{X}}_{\overline{r}}^{\square,\mathbf{k}-\mathrm{cr}}$ lies in $\widetilde{U}_{\overline{r}}^{\square,\mathbf{k}-\mathrm{cr}}$ if and only if the φ_i are pairwise distinct;*

(ii) *a point $(r, \varphi_1, \dots, \varphi_n) \in \widetilde{U}_{\overline{r}}^{\square,\mathbf{k}-\mathrm{cr}}$ lies in $\widetilde{V}_{\overline{r}}^{\square,\mathbf{k}-\mathrm{cr}}$ if and only if it satisfies assumption* (2.6) *above and $z^{\mathbf{k}_i-\mathbf{k}_j}\mathrm{unr}(\varphi_i\varphi_j^{-1}) \in \mathcal{T}_{\mathrm{reg}}$ for $i \neq j$.*

Moreover the image of $\widetilde{V}_{\overline{r}}^{\square,\mathbf{k}-\mathrm{cr}}$ via (2.4) composed with (2.5) lies in $U_{\mathrm{tri}}^{\square}(\overline{r})$.

Proof. The idea of the proof is the same as that of [17, Lem.4.4]. It is enough to show that all the statements are true locally on $\widetilde{\mathfrak{X}}_{\overline{r}}^{\square,\mathbf{k}-\mathrm{cr}}$. Let us (locally) fix a basis of the coherent locally free $K_0 \otimes_{\mathbb{Q}_p} \mathcal{O}_{\mathfrak{X}_{\overline{r}}^{\square,\mathbf{k}-\mathrm{cr}}}$-module $\mathcal{D}$ on $\mathfrak{X}_{\overline{r}}^{\square,\mathbf{k}-\mathrm{cr}}$. By the choice of such a basis, the matrix of the crystalline Frobenius Φ_{cris} and the Hodge filtration define (locally) a morphism:

$$\mathfrak{X}_{\overline{r}}^{\square,\mathbf{k}-\mathrm{cr}} \longrightarrow (\mathrm{Res}_{K_0/\mathbb{Q}_p}\mathrm{GL}_{n,K_0} \times_{\mathrm{Sp}\,\mathbb{Q}_p} \mathrm{Flag})^{\mathrm{rig}} \times_{\mathrm{Sp}\,\mathbb{Q}_p} \mathrm{Sp}\,L$$

where $\mathrm{Flag} := (\mathrm{Res}_{K/\mathbb{Q}_p}\mathrm{GL}_{n,K})/(\mathrm{Res}_{K/\mathbb{Q}_p}B)$ (compare [28, §8]). By [28, Prop.8.12] and the discussion preceding it, it follows that this morphism is smooth, hence so is the morphism:

$$(2.7)\qquad \widetilde{\mathfrak{X}}_{\overline{r}}^{\square,\mathbf{k}-\mathrm{cr}} \longrightarrow \Big((\mathrm{Res}_{K_0/\mathbb{Q}_p}\mathrm{GL}_{n,K_0})_L^{\mathrm{rig}} \times_{T_L^{\mathrm{rig}}/\mathcal{S}_n} T_L^{\mathrm{rig}}\Big) \times_{\mathrm{Sp}\,L} \mathrm{Flag}_L^{\mathrm{rig}}$$

where $\mathrm{Res}_{K_0/\mathbb{Q}_p}\mathrm{GL}_{n,K_0} \to T/\mathcal{S}_n$ is the morphism defined in [28, (9.1)]. On the other hand, using that the morphism $T \to T/\mathcal{S}_n$ is obviously smooth in the neighbourhood of a point $(\varphi_1, \dots, \varphi_n) \in T$ where the φ_i are pairwise distinct, we see that the conditions of (i), resp. (ii), in the statement cut out smooth (over L) Zariski-open and Zariski-dense subspaces of:

$$(2.8)\qquad \Big((\mathrm{Res}_{K_0/\mathbb{Q}_p}\mathrm{GL}_{n,K_0})_L^{\mathrm{rig}} \times_{T_L^{\mathrm{rig}}/\mathcal{S}_n} T_L^{\mathrm{rig}}\Big) \times_{\mathrm{Sp}\,L} \mathrm{Flag}_L^{\mathrm{rig}}.$$

Their inverse images in $\widetilde{\mathfrak{X}}_{\overline{r}}^{\square,\mathbf{k}-\mathrm{cr}}$ via (2.7) are thus smooth over L and Zariski-open in $\widetilde{\mathfrak{X}}_{\overline{r}}^{\square,\mathbf{k}-\mathrm{cr}}$. Let us prove that these inverse images are also Zariski-dense in $\widetilde{\mathfrak{X}}_{\overline{r}}^{\square,\mathbf{k}-\mathrm{cr}}$. It is enough to prove that they intersect nontrivially every irreducible component of $\widetilde{\mathfrak{X}}_{\overline{r}}^{\square,\mathbf{k}-\mathrm{cr}}$. Let $\mathrm{Sp}\,A$ be any affinoid open subset of $\widetilde{\mathfrak{X}}_{\overline{r}}^{\square,\mathbf{k}-\mathrm{cr}}$, it follows from [9, Cor.9.4.2] that the image of $\mathrm{Sp}\,A$ by the smooth, hence flat, morphism (2.7) is admissible open in (2.8). In particular its intersection with one of the above Zariski-open and Zariski-dense subspaces of (2.8) can't be empty, which proves the statement. The final claim of the lemma follows from (ii), the discussion preceding Lemma 2.4 and the definition (2.2) of $U_{\mathrm{tri}}^{\square}(\overline{r})$. □

Note that $\widetilde{\mathfrak{X}}_{\overline{r}}^{\square,\mathbf{k}-\mathrm{cr}}$ is equidimensional of the same dimension as $\mathfrak{X}_{\overline{r}}^{\square,\mathbf{k}-\mathrm{cr}}$. Indeed, by Lemma 2.4 it is enough to prove the same statement for $\widetilde{U}_{\overline{r}}^{\square,\mathbf{k}-\mathrm{cr}}$. But this is clear since the map $\widetilde{U}_{\overline{r}}^{\square,\mathbf{k}-\mathrm{cr}} \to \mathfrak{X}_{\overline{r}}^{\square,\mathbf{k}-\mathrm{cr}}$ is smooth of relative dimension 0, hence étale, and since $\mathfrak{X}_{\overline{r}}^{\square,\mathbf{k}-\mathrm{cr}}$ is equidimensional ([36]). Lemma 2.4 also implies that (2.4) induces (as claimed above) a morphism:

$$\iota_{\mathbf{k}} : \widetilde{\mathfrak{X}}_{\overline{r}}^{\square,\mathbf{k}-\mathrm{cr}} \longrightarrow X_{\mathrm{tri}}^{\square}(\overline{r}) \tag{2.9}$$

which is obviously a closed immersion as (2.4) is.

Corollary 2.5. *Let $x = (r, \delta) \in X_{\mathrm{tri}}^{\square}(\overline{r})$ be a crystalline strictly dominant point such that $\omega(x) = \delta_{\mathbf{k}}$ and the Frobenius eigenvalues $(\varphi_1, \ldots, \varphi_n)$ (cf. Lemma 2.1) are pairwise distinct and let U be an open subset of $X_{\mathrm{tri}}^{\square}(\overline{r})$ containing x.*
(i) The point x belongs to $\iota_{\mathbf{k}}(\widetilde{U}_{\overline{r}}^{\square,\mathbf{k}-\mathrm{cr}})$ and there is a unique irreducible component $\widetilde{Z}_{\mathrm{cris}}(x)$ of $\widetilde{\mathfrak{X}}_{\overline{r}}^{\square,\mathbf{k}-\mathrm{cr}}$ containing $\iota_{\mathbf{k}}^{-1}(x)$.
(ii) If U is small enough there is a unique irreducible component $Z_{\mathrm{tri},U}(x)$ of U containing $\iota_{\mathbf{k}}(\widetilde{Z}_{\mathrm{cris}}(x)) \cap U$, and it is such that $Z_{\mathrm{tri},U}(x) \cap U' = Z_{\mathrm{tri},U'}(x)$ for any open $U' \subseteq U$ containing x.

Proof. (i) The assumptions and Lemma 2.1 imply that x is in the image of the map $\iota_{\mathbf{k}}$ in (2.9) and the fact that the φ_i are pairwise distinct implies that $x \in \iota_{\mathbf{k}}(\widetilde{U}_{\overline{r}}^{\square,\mathbf{k}-\mathrm{cr}})$. In particular $\widetilde{\mathfrak{X}}_{\overline{r}}^{\square,\mathbf{k}-\mathrm{cr}}$ is smooth at $\iota_{\mathbf{k}}^{-1}(x)$ by Lemma 2.4 and thus $\iota_{\mathbf{k}}^{-1}(x)$ belongs to a unique irreducible component $\widetilde{Z}_{\mathrm{cris}}(x)$ of $\widetilde{\mathfrak{X}}_{\overline{r}}^{\square,\mathbf{k}-\mathrm{cr}}$.

(ii) We have that $\iota_{\mathbf{k}}(\widetilde{Z}_{\mathrm{cris}}(x)) \cap U$ is a Zariski-closed subset of U, and it is easy to see that it is still irreducible if U is small enough since $\iota_{\mathbf{k}}(\widetilde{Z}_{\mathrm{cris}}(x))$ is smooth at x. Hence there exists at least one irreducible component of U containing the irreducible Zariski-closed subset $\iota_{\mathbf{k}}(\widetilde{Z}_{\mathrm{cris}}(x)) \cap U$. If there are two such irreducible components, then in particular *any* point of $\iota_{\mathbf{k}}(\widetilde{Z}_{\mathrm{cris}}(x)) \cap U$ is singular in U, hence in $X_{\mathrm{tri}}^{\square}(\overline{r})$. But Lemma 2.4 implies $\widetilde{Z}_{\mathrm{cris}}(x) \cap \widetilde{V}_{\overline{r}}^{\square,\mathbf{k}-\mathrm{cr}} \neq \emptyset$ is Zariski-open and Zariski-dense in $\widetilde{Z}_{\mathrm{cris}}(x)$, hence:

$$\Big(\widetilde{Z}_{\mathrm{cris}}(x) \cap \widetilde{V}_{\overline{r}}^{\square,\mathbf{k}-\mathrm{cr}}\Big) \cap \Big(\widetilde{Z}_{\mathrm{cris}}(x) \cap \iota_{\mathbf{k}}^{-1}(U)\Big) \neq \emptyset$$

from which we get $\iota_{\mathbf{k}}(\widetilde{Z}_{\mathrm{cris}}(x) \cap \widetilde{V}_{\overline{r}}^{\square,\mathbf{k}-\mathrm{cr}}) \cap U \neq \emptyset$. The last statement of Lemma 2.4 also implies $\iota_{\mathbf{k}}(\widetilde{Z}_{\mathrm{cris}}(x) \cap \widetilde{V}_{\overline{r}}^{\square,\mathbf{k}-\mathrm{cr}}) \cap U \subseteq U_{\mathrm{tri}}^{\square}(\overline{r})$, which is then a contradiction since $U_{\mathrm{tri}}^{\square}(\overline{r})$ is smooth over L.

Finally, shrinking U again if necessary, we can assume that, for any open subset $U' \subseteq U$ containing x, the map $Z \mapsto Z \cap U'$ induces a bijection between the irreducible components of U containing x and the irreducible components of U' containing x. It then follows from the definition of $Z_{\mathrm{tri},U}(x)$ that $Z_{\mathrm{tri},U}(x) \cap U' = Z_{\mathrm{tri},U'}(x)$. □

Remark 2.6. (i) Since the map $\widetilde{\mathfrak{X}}_{\overline{r}}^{\square,\mathbf{k}-\mathrm{cr}} \to \mathfrak{X}_{\overline{r}}^{\square,\mathbf{k}-\mathrm{cr}}$ is finite, hence closed, and since $\widetilde{\mathfrak{X}}_{\overline{r}}^{\square,\mathbf{k}-\mathrm{cr}}$, $\mathfrak{X}_{\overline{r}}^{\square,\mathbf{k}-\mathrm{cr}}$ are both equidimensional (of the same dimension), the image of any irreducible component of $\widetilde{\mathfrak{X}}_{\overline{r}}^{\square,\mathbf{k}-\mathrm{cr}}$ is an irreducible component of $\mathfrak{X}_{\overline{r}}^{\square,\mathbf{k}-\mathrm{cr}}$. In particular the image of $\widetilde{Z}_{\mathrm{cris}}(x)$ in (i) of Corollary 2.5 is the unique irreducible component of $\mathfrak{X}_{\overline{r}}^{\square,\mathbf{k}-\mathrm{cr}}$ containing r.
(ii) Either by the same proof as that for (ii) of Corollary 2.5 or as a consequence of (ii) of Corollary 2.5, we see that there is also a unique irreducible component $Z_{\mathrm{tri}}(x)$ of the whole $X_{\mathrm{tri}}^{\square}(\overline{r})$ which contains the irreducible closed subset $\iota_{\mathbf{k}}(\widetilde{Z}_{\mathrm{cris}}(x))$.

2.3. **The Weyl group element associated to a crystalline point.** We review the definition of the Weyl group element associated to certain crystalline points on $X_{\mathrm{tri}}^{\square}(\overline{r})$ (measuring their "criticality") and state a local conjecture (Conjecture 2.8).

We keep the notation of §2.2. We let $W \cong \prod_{\tau\colon K\hookrightarrow L} \mathcal{S}_n$ be the Weyl group of the algebraic group:

$$(\mathrm{Res}_{K/\mathbb{Q}_p}\mathrm{GL}_{n,K}) \times_{\mathrm{Spec}\,\mathbb{Q}_p} \mathrm{Spec}\,L \cong \prod_{\tau\colon K\hookrightarrow L} \mathrm{GL}_{n,L}$$

and $X^*((\mathrm{Res}_{K/\mathbb{Q}_p}T_K) \times_{\mathrm{Spec}\,\mathbb{Q}_p} \mathrm{Spec}\,L) \cong \prod_{\tau\colon K\hookrightarrow L} X^*(T_L)$ be the $\mathbb{Z}$-module of algebraic characters of $(\mathrm{Res}_{K/\mathbb{Q}_p}T_K) \times_{\mathrm{Spec}\,\mathbb{Q}_p} \mathrm{Spec}\,L$ (recall T is the diagonal torus in GL_n and T_K, T_L its base change to K, L). We write $\lg(w)$ for the length of w in the Coxeter group W (for the set of simple reflections associated to the simple roots of the upper triangular matrices).

Let $x = (r,\delta) = (r,\delta_1,\dots,\delta_n)$ be a crystalline strictly dominant point on $X_{\mathrm{tri}}^{\square}(\overline{r})$. Then by Lemma 2.1 the characters δ_i are of the form $\delta_i = z^{\mathbf{k}_i}\mathrm{unr}(\varphi_i)$ where $\mathbf{k}_i = (k_{\tau,i})_{\tau\colon K\hookrightarrow L}$ and the $\varphi_i \in k(x)^\times$ are the eigenvalues of the geometric Frobenius on $\mathrm{WD}(r)$. Assume that the φ_i are pairwise distinct, then as in §2.2 the ordering $(\varphi_1,\dots,\varphi_n)$ defines a complete φ-stable flag of free $K_0 \otimes_{\mathbb{Q}_p} k(x)$-modules $0 = \mathcal{F}_0 \subset \mathcal{F}_1 \subset \cdots \subset \mathcal{F}_n = D_{\mathrm{cris}}(r)$ on $D_{\mathrm{cris}}(r)$ such that $\varphi^{[K_0:\mathbb{Q}_p]}$ acts on $\mathcal{F}_i/\mathcal{F}_{i-1}$ by multiplication by φ_i. We view $\mathcal{F}_i$ as a filtered φ-module with the induced Hodge filtration. If we write $(k'_{\tau,i})_{\tau:K\hookrightarrow L}$ for the Hodge-Tate weights of $\mathcal{F}_i/\mathcal{F}_{i-1}$, we find that there is a unique $w_x = (w_{x,\tau})_{\tau\colon K\hookrightarrow L} \in W = \prod_{\tau:K\hookrightarrow L} \mathcal{S}_n$ such that:

$$k'_{\tau,i} = k_{\tau,w_{x,\tau}^{-1}(i)} \tag{2.10}$$

for all $i \in \{1,\dots,n\}$ and each $\tau : K \hookrightarrow L$. We call w_x the *Weyl group element associated to* x. Note that $\mathcal{F}_\bullet$ is noncritical (see §2.2) if and only if $w_{x,\tau} = 1$ for all $\tau : K \hookrightarrow L$, in which case we say that the crystalline strictly dominant point $x = (r,\delta)$ is *noncritical*.

For $w \in W$ we denote by $d_w \in \mathbb{Z}_{\geq 0}$ the rank of the $\mathbb{Z}$-submodule of $X^*((\mathrm{Res}_{K/\mathbb{Q}_p}T_K)\times_{\mathrm{Spec}\,\mathbb{Q}_p} \mathrm{Spec}\,L)$ generated by the $w(\alpha)-\alpha$ where α runs among the roots of $(\mathrm{Res}_{K/\mathbb{Q}_p}\mathrm{GL}_{n,K}) \times_{\mathrm{Spec}\,\mathbb{Q}_p} \mathrm{Spec}\,L$.

Lemma 2.7. *With the above notations we have:*

$$d_w \leq \lg(w) \leq [K:\mathbb{Q}_p]\frac{n(n-1)}{2}$$

and $\lg(w) = d_w$ *if and only if* w *is a product of distinct simple reflections.*

Proof. Note first that the right hand side inequality is obvious. Let us write in this proof $X := X^*((\mathrm{Res}_{K/\mathbb{Q}_p}T_K) \times_{\mathrm{Spec}\,\mathbb{Q}_p} \mathrm{Spec}\,L)$, $X_{\mathbb{Q}} := X \otimes_{\mathbb{Z}} \mathbb{Q}$, and let us denote by S the subset of simple reflections in W (thus $\dim_{\mathbb{Q}}(X_{\mathbb{Q}}) = [K:\mathbb{Q}_p]n$ and $|S| = [K:\mathbb{Q}_p](n-1)$). The rank of the subgroup of X generated by the $w(\alpha)-\alpha$ for α as above, or equivalently by the $w(\alpha)-\alpha$ for $\alpha \in X$, is equal to the dimension of the $\mathbb{Q}$-vector space $(w-\mathrm{id})X_{\mathbb{Q}}$ which, by the rank formula, is equal to $\dim_{\mathbb{Q}}(X_{\mathbb{Q}}) - \dim_{\mathbb{Q}}(\ker(w-\mathrm{id}))$. Let I be the set of simple reflections appearing in w, we have $|I| \leq \lg(w)$ and $|I| = \lg(w)$ if and only if w is a product of distinct simple reflections. It is thus enough to prove $\dim_{\mathbb{Q}}(\ker(w-\mathrm{id})) \geq \dim_{\mathbb{Q}}(X_{\mathbb{Q}}) - |I|$ with equality when w is a product of distinct simple reflections. Note that $\ker(w-\mathrm{id})$ obviously contains the $\mathbb{Q}$-subvector space of $X_{\mathbb{Q}}$ of fixed points by the subgroup W_I of W generated by the elements of I, and it follows from [31, Th.1.12(c)] that, when w is a product of distinct simple reflections, then $\ker(w-\mathrm{id})$ is exactly this $\mathbb{Q}$-subvector space. It is thus enough to prove that this $\mathbb{Q}$-subvector space of $X_{\mathbb{Q}}$, which is just the intersection of the hyperplanes $\ker(s-\mathrm{id})$ for $s\in I$, has dimension $\dim_{\mathbb{Q}}(X_{\mathbb{Q}})-|I|$. However we know that for any $\mathbb{Q}$-subvector space $V\subset X_{\mathbb{Q}}$ and any reflection s of $X_{\mathbb{Q}}$, we have $\dim_{\mathbb{Q}}(V\cap\ker(s-\mathrm{id}))\geq \dim_{\mathbb{Q}}(V)-1$ and thus by induction:

$$\dim_{\mathbb{Q}}\Big(V\bigcap\Big(\bigcap_{s\in S}\ker(s-\mathrm{id})\Big)\Big)\geq \dim_{\mathbb{Q}}(V)-|S|$$

with equality if and only if $\dim_{\mathbb{Q}}\Big(V\cap\Big(\bigcap_{s\in J}\ker(s-\mathrm{id})\Big)\Big)=\dim_{\mathbb{Q}}(V)-|J|$ for all $J\subseteq S$. As the $\mathbb{Q}$-subvector space $X_{\mathbb{Q}}^W$ of fixed points by W has dimension $[K:\mathbb{Q}_p]$ (it is generated by the characters $\tau\circ\det$ for $\tau: K\hookrightarrow L$), we have:

$$\dim_{\mathbb{Q}}(X_{\mathbb{Q}}^W)=\bigcap_{s\in S}\ker(s-\mathrm{id})=[K:\mathbb{Q}_p]=\dim_{\mathbb{Q}}(X_{\mathbb{Q}})-|S|.$$

Consequently we deduce (taking $V=X_{\mathbb{Q}}$):

$$\dim_{\mathbb{Q}}\Big(\bigcap_{s\in I}\ker(s-\mathrm{id})\Big)=\dim_{\mathbb{Q}}(X_{\mathbb{Q}})-|I|$$

which is the desired formula. □

Recall that, if X is a rigid analytic variety over L and $x\in X$, the tangent space to X at x is the $k(x)$-vector space:

$$T_{X,x}:=\mathrm{Hom}_{k(x)}\Big(\mathfrak{m}_{X,x}/\mathfrak{m}_{X,x}^2,k(x)\Big)=\mathrm{Hom}_{k(x)-\mathrm{alg}}\Big(\mathcal{O}_{X,x},k(x)[\varepsilon]/(\varepsilon^2)\Big)\tag{2.11}$$

where $\mathfrak{m}_{X,x}$ is the maximal ideal of the local ring $\mathcal{O}_{X,x}$ at x to X. If X is equidimensional, recall also that $\dim_{k(x)}T_{X,x}\geq\dim X$ and that X is smooth at x if and only if $\dim_{k(x)}T_{X,x}=\dim X$.

We let $\widetilde{X}_{\mathrm{tri}}^{\square}(\overline{r})\subseteq X_{\mathrm{tri}}^{\square}(\overline{r})$ be the union of the irreducible components C of $X_{\mathrm{tri}}^{\square}(\overline{r})$ such that $C\cap U_{\mathrm{tri}}^{\square}(\overline{r})$ contains a crystalline point. For instance it follows from Lemma 2.4 that all the closed embeddings (2.9) factor as closed embeddings $\iota_{\mathbf{k}}:\widetilde{\mathfrak{X}}_{\overline{r}}^{\square,\mathbf{k}-\mathrm{cr}}\hookrightarrow\widetilde{X}_{\mathrm{tri}}^{\square}(\overline{r})\subseteq X_{\mathrm{tri}}^{\square}(\overline{r})$. In particular any point $x\in X_{\mathrm{tri}}^{\square}(\overline{r})$ which is crystalline strictly dominant is in $\widetilde{X}_{\mathrm{tri}}^{\square}(\overline{r})$.

The following statement is our local conjecture.

Conjecture 2.8. *Let $x\in X_{\mathrm{tri}}^{\square}(\overline{r})$ be a crystalline strictly dominant point such that the Frobenius eigenvalues $(\varphi_1,\dots,\varphi_n)$ (cf. Lemma 2.1) are pairwise distinct, let w_x be the Weyl group element associated to x (cf. (2.10)) and let $d_x:=d_{w_x}$. Then:*

$$\dim_{k(x)}T_{\widetilde{X}_{\mathrm{tri}}^{\square}(\overline{r}),x}=\lg(w_x)-d_x+\dim X_{\mathrm{tri}}^{\square}(\overline{r})=\lg(w_x)-d_x+n^2+[K:\mathbb{Q}_p]\frac{n(n+1)}{2}.$$

In particular, since $\dim\widetilde{X}_{\mathrm{tri}}^{\square}(\overline{r})=\dim X_{\mathrm{tri}}^{\square}(\overline{r})=n^2+[K:\mathbb{Q}_p]\frac{n(n+1)}{2}$, we see by Lemma 2.7 that $\widetilde{X}_{\mathrm{tri}}^{\square}(\overline{r})$ should be smooth at x if and only if w_x is a product of distinct simple reflections.

Remark 2.9. The reader can wonder why we don't state Conjecture 2.8 with $X_{\mathrm{tri}}^{\square}(\overline{r})$ instead of $\widetilde{X}_{\mathrm{tri}}^{\square}(\overline{r})$. The reason is that Conjecture 2.8 with $\widetilde{X}_{\mathrm{tri}}^{\square}(\overline{r})$ is actually implied by other conjectures, see §5, and we don't know if this is the case with $X_{\mathrm{tri}}^{\square}(\overline{r})$.

2.4. **Accumulation properties.** We define and prove some accumulation properties concerning $X_{\rm tri}^{\square}(\overline{r})$ (Definition 2.11, Proposition 2.12). The definition is needed in order to state our main local results (Theorem 2.15, Corollary 2.16) and the accumulation property we prove is needed in §3.2 below (more precisely in the proof of Corollary 3.12).

We keep the previous notation. We call a point $x = (r, \delta_1, \ldots, \delta_n) \in X_{\rm tri}^{\square}(\overline{r})$ *saturated* if there exists a triangulation of the (φ, Γ_K)-module $D_{\rm rig}(r)$ with parameter $(\delta_1, \ldots, \delta_n)$. Note that, if x is crystalline strictly dominant with pairwise distinct Frobenius eigenvalues, then x is saturated if and only if x is noncritical. Recall from §2.1 that if x is saturated and if $(\delta_1, \ldots, \delta_n) \in \mathcal{T}_{\rm reg}^n$ then $x \in U_{\rm tri}^{\square}(\overline{r})$.

Lemma 2.10. *Let $x = (r, \delta_1, ..., \delta_n) \in X_{\rm tri}^{\square}(\overline{r})$ with $\omega(x) = \delta_{\mathbf{k}}$ for some $\mathbf{k} = (k_{\tau,i})_{1\leq i\leq n, \tau: K\hookrightarrow L} \in (\mathbb{Z}^n)^{\mathrm{Hom}(K,L)}$. Assume that:*

$$k_{\tau,i} - k_{\tau,i+1} > [K : K_0]\mathrm{val}\Big(\delta_1(\varpi_K) \cdots \delta_i(\varpi_K)\Big) \tag{2.12}$$

for $i \in \{1, \ldots, n-1\}$, $\tau \in \mathrm{Hom}(K, L)$. Then x is saturated and r is semi-stable. If moreover $(\delta_1, \ldots, \delta_n) \in \mathcal{T}_{\rm reg}^n$, then r is crystalline strictly dominant noncritical.

Proof. By [34, Th.6.3.13] and [13, Prop.2.9] the representation r is trianguline with parameter $(\delta'_1, \ldots, \delta'_n)$ where $\delta'_i = \delta_i z^{\mathbf{k}_{w^{-1}(i)} - \mathbf{k}_i}$ for some $w = (w_\tau)_{\tau: K\hookrightarrow L} \in W = \prod_{\tau: K\hookrightarrow L} \mathcal{S}_n$. As $D_{\rm rig}(r)$, and hence $\wedge^i_{\mathcal{R}_{k(x),K}} D_{\rm rig}(r)$, are φ-modules over $\mathcal{R}_{k(x),K}$ which are pure of slope zero (being étale (φ, Γ_K)-modules), it follows that for all i:

$$1 \leq \Big|\delta'_1(\varpi_K) \cdots \delta'_i(\varpi_K)\Big|_K.$$

Since $\delta'_1(\varpi_K) \cdots \delta'_i(\varpi_K) = \delta_1(\varpi_K) \cdots \delta_i(\varpi_K) \cdot \prod_{j=1}^i \prod_\tau (\tau(\varpi_K)^{k_{\tau,w_\tau^{-1}(j)} - k_{\tau,j}})$ we obtain:

$$\mathrm{val}\Big(\delta_1(\varpi_K) \cdots \delta_i(\varpi_K)\Big) \geq \tfrac{1}{[K:K_0]} \sum_{j=1}^{i} \sum_\tau (k_{\tau,j} - k_{\tau,w_\tau^{-1}(j)}). \tag{2.13}$$

We now prove by induction on i that $w_\tau^{-1}(i) = i$ for all τ. The inequality (2.13) for $i = 1$ gives $\mathrm{val}(\delta_1(\varpi_K)) \geq \frac{1}{[K:K_0]} \sum_\tau (k_{\tau,1} - k_{\tau,w_\tau^{-1}(1)})$. But assumption (2.12) with $i = 1$ implies $\mathrm{val}(\delta_1(\varpi_K)) < \frac{1}{[K:K_0]} \sum_\tau (k_{\tau,1} - k_{\tau,j})$ for $j \in \{2, \ldots, n\}$ which forces $w_\tau^{-1}(1) = 1$ for all τ. Assume by induction that $w_\tau^{-1}(j) = j$ for all $j \leq i-1$ and all τ. Then (2.13) gives:

$$\mathrm{val}(\delta_1(\varpi_K) \cdots \delta_i(\varpi_K)) \geq \tfrac{1}{[K:K_0]} \sum_\tau (k_{\tau,i} - k_{\tau,w_\tau^{-1}(i)})$$

and again (2.12) implies $\mathrm{val}(\delta_1(\varpi_K) \cdots \delta_i(\varpi_K)) < \frac{1}{[K:K_0]} \sum_\tau (k_{i,\tau} - k_{j,\tau})$ for $j \in \{i, \ldots, n\}$ which forces $w_\tau^{-1}(i) = i$ for all τ. We thus have $(\delta_1, \ldots, \delta_n) = (\delta'_1, \ldots, \delta'_n)$ which implies that the point $x = (r, \delta_1, \ldots, \delta_n)$ is saturated. Since δ is strictly dominant, we obtain that r is semi-stable by the argument in the proof of [17, Th.3.14] (see also the proof of [30, Cor.2.7(i)]). By a slight generalisation of the proof of Lemma 2.1 (that we leave to the reader), we have $\delta_i = z^{\mathbf{k}_i}\mathrm{unr}(\varphi_i)$ where the φ_i are the eigenvalues of the linearized Frobenius $\varphi^{[K_0:\mathbb{Q}_p]}$ on the $K_0 \otimes_{\mathbb{Q}_p} k(x)$-module $D_{\rm st}(r) := (B_{\rm st} \otimes_{\mathbb{Q}_p} r)^{\mathcal{G}_K}$. If in addition $(\delta_1, \ldots, \delta_n) \in \mathcal{T}_{\rm reg}^n$, then it follows from Remark 2.14 that $\varphi_i\varphi_j^{-1} \neq p^{-[K_0:\mathbb{Q}_p]}$ for $1 \leq i \leq j \leq n$ and the argument of [17, Th.3.14], [30, Cor.2.7(i)] then shows that the monodromy operator N on $D_{\rm st}(r)$ must be zero, i.e. that r is crystalline. This finishes the proof. ☐

Definition 2.11. *Let X be a union of irreducible components of an open subset of $X_{\mathrm{tri}}^{\square}(\overline{r})$ (over L) and let $x \in X_{\mathrm{tri}}^{\square}(\overline{r})$ such that $\omega(x)$ is algebraic. Then X satisfies the* accumulation property *at x if $x \in X$ and if, for any positive real number $C > 0$, the set of crystalline strictly dominant points $x' = (r', \delta')$ such that:*

(i) *the eigenvalues of $\varphi^{[K_0:\mathbb{Q}_p]}$ on $D_{\mathrm{cris}}(r')$ are pairwise distinct;*

(ii) *x' is noncritical;*

(iii) *$\omega(x') = \delta'|_{(\mathcal{O}_K^\times)^n} = \delta_{\mathbf{k}'}$ with $k'_{\tau,i} - k'_{\tau,i+1} > C$ for $i \in \{1, \ldots, n-1\}$, $\tau \in \mathrm{Hom}(K, L)$;*

accumulate at x in X in the sense of [2, §3.3.1].

It easily follows from Definition 2.11 that X satisfies the accumulation property at x if and only if each irreducible component of X containing x satisfies the accumulation property at x. In particular, if x belongs to each irreducible component of X, we see that for every $C > 0$ the set of points x' in the statement of Definition 2.11 is also Zariski-dense in X. Since $U_{\mathrm{tri}}^{\square}(\overline{r}) \cap X$ is Zariski-open and Zariski-dense in X, we also see that each irreducible component of X containing x also contains points x' as in Definition 2.11 which are in $U_{\mathrm{tri}}^{\square}(\overline{r})$, hence each irreducible component of X containing x is in $\widetilde{X}_{\mathrm{tri}}^{\square}(\overline{r})$.

Proposition 2.12. *Let $x = (r, \delta) \in X_{\mathrm{tri}}^{\square}(\overline{r})$ be a crystalline strictly dominant point such that the eigenvalues of the geometric Frobenius on* $\mathrm{WD}(r)$ *are pairwise distinct. Then there exists a sufficiently small open neighbourhood U of x in $X_{\mathrm{tri}}^{\square}(\overline{r})$ such that the irreducible component $Z_{\mathrm{tri},U}(x)$ of U in (ii) of Corollary 2.5 is defined and satisfies the accumulation property at x.*

Proof. We have to prove that, for any positive real number C, the set of points $x' = (r', \delta') \in Z_{\mathrm{tri},U}(x)$ such that r' is crystalline with pairwise distinct geometric Frobenius eigenvalues on $\mathrm{WD}(r')$ and x' is noncritical with $\omega(x') = \delta_{\mathbf{k}'}$ strictly dominant satisfying:

$$k'_{\tau,i} - k'_{\tau,i+1} > C \tag{2.14}$$

for all $i = 1, \ldots, n-1$, $\tau : K \hookrightarrow L$ accumulates at x.

Let U be an open subset of x in $X_{\mathrm{tri}}^{\square}(\overline{r})$ as in (iii) of Corollary 2.5, i.e. such that for any open $U' \subseteq U$ containing x we have $Z_{\mathrm{tri},U}(x) \cap U' = Z_{\mathrm{tri},U'}(x)$. Let $\widetilde{Z}_{\mathrm{cris}}(x)$ as in (i) of Corollary 2.5, by Lemma 2.4, the space $V := \iota_{\mathbf{k}}(\widetilde{Z}_{\mathrm{cris}}(x) \cap \widetilde{V}_{\overline{r}}^{\square,\mathbf{k}-\mathrm{cr}})$ is Zariski-open and Zariski-dense in $\iota_{\mathbf{k}}(\widetilde{Z}_{\mathrm{cris}}(x))$, hence accumulates in $\iota_{\mathbf{k}}(\widetilde{Z}_{\mathrm{cris}}(x))$ at any point of $\iota_{\mathbf{k}}(\widetilde{Z}_{\mathrm{cris}}(x))$, in particular at x. We claim that it is enough to prove that the points $x' \in U$ as above accumulate in U at every point of $V \cap U$. Indeed, if $U' \subseteq U$ is an open neighbourhood containing x, then U' also contains a point $v \in V$. By the accumulation statement at $v \in V \cap U$, the Zariski closure in U' of the points x' contains a small neighbourhood around v, hence contains an irreducible component of U' containing v. But since v is a smooth point of U' (over L) as $v \in U_{\mathrm{tri}}^{\square}(\overline{r})$ by the last statement of Lemma 2.4, there is only one such irreducible component, and since $v \in \iota_{\mathbf{k}}(\widetilde{Z}_{\mathrm{cris}}(x)) \cap U' \subseteq Z_{\mathrm{tri},U'}(x)$, we see that this irreducible component must be $Z_{\mathrm{tri},U'}(x)$. Thus the Zariski closure in U' of the points x' always contains $Z_{\mathrm{tri},U'}(x)$. This easily implies the proposition since $Z_{\mathrm{tri},U'}(x) = Z_{\mathrm{tri},U}(x) \cap U'$.

Since $U_{\mathrm{tri}}^{\square}(\overline{r})$ is open in $X_{\mathrm{tri}}^{\square}(\overline{r})$, it is enough to prove that the crystalline points $x' = (r', \delta')$ in $U \cap U_{\mathrm{tri}}^{\square}(\overline{r})$ satisfying the conditions in the first paragraph of this proof accumulate at any crystalline strictly dominant point x of $U \cap U_{\mathrm{tri}}^{\square}(\overline{r})$. The condition on their Frobenius eigenvalues is automatic using $\delta' \in \mathcal{T}_{\mathrm{reg}}^n$. Shrinking U further if necessary, we can take U to be contained in some quasi-compact open neighbourhood of x in $X_{\mathrm{tri}}^{\square}(\overline{r})$, and thus we may assume that for $i \in \{1, \ldots, n\}$ the functions $y = (r_y, (\delta_{y,1}, \ldots, \delta_{y,n})) \mapsto \delta_{y,i}(\varpi_K)$ are uniformly bounded on U. Hence by Lemma 2.10 we may assume that C is sufficiently large so that the points $x' \in U \cap U_{\mathrm{tri}}^{\square}(\overline{r})$ with $\omega(x) = \delta_{\mathbf{k}'}$ algebraic satisfying (2.14) are in fact also automatically crystalline noncritical. Changing notation, we see that it is finally enough to prove that the points $x' \in U_{\mathrm{tri}}^{\square}(\overline{r})$ satisfying (2.14) for C big enough accumulate at any crystalline strictly dominant point x of $U_{\mathrm{tri}}^{\square}(\overline{r})$.

We now consider the rigid analytic spaces $\mathcal{S}_n$, $\mathcal{S}^{\square}(\overline{r})$ appearing in the proof of [13, Th.2.6] (to which we refer the reader for more details; do not confuse here $\mathcal{S}_n$ with the permutation group!). In *loc. cit.* there is a diagram of rigid spaces over $\mathcal{T}_L^n$:

$$\begin{array}{ccc} & \mathcal{S}^{\square}(\overline{r}) & \\ {\scriptstyle \pi_{\overline{r}}} \swarrow & & \searrow {\scriptstyle g} \\ U_{\mathrm{tri}}^{\square}(\overline{r}) & & \mathcal{S}_n \end{array}$$

where $\pi_{\overline{r}}$ is a $\mathbb{G}_m^n$-torsor and g is a composition $\mathcal{S}^{\square}(\overline{r}) \hookrightarrow \mathcal{S}_n^{\square,\mathrm{adm}} \to \mathcal{S}_n^{\mathrm{adm}} \hookrightarrow \mathcal{S}_n$ where the first and last maps are open embeddings and the middle one is a GL_n-torsor.

Let us choose a point $\tilde{x} \in \pi_{\overline{r}}^{-1}(x)$. As $\pi_{\overline{r}}$ is a $\mathbb{G}_m^n$-torsor, it is enough to prove that the points in $\mathcal{S}^{\square}(\overline{r})$ satisfying (2.14) accumulate at $\tilde{x}$. The same argument shows that it is enough to prove that the points of $\mathcal{S}_n$ satisfying (2.14) accumulate at $g(\tilde{x})$. But the morphism $\mathcal{S}_n \to \mathcal{T}_L^n$ is a composition of open embeddings and structure morphisms of geometric vector bundles (compare the proof of [30, Th.2.4]). It follows that $g(\tilde{x})$ has a basis of neighbourhoods $(U_i)_{i \in I}$ in $\mathcal{S}_n$ such that $V_i := \omega(U_i)$ is a basis of neighbourhoods of $\omega(x)$ in $\mathcal{W}_L^n$ and such that the rigid space U_i is isomorphic to a product $V_i \times \mathbb{B}$ of rigid spaces over L where $\mathbb{B}$ is some closed polydisc (compare [17, Cor.3.5] and [30, Lem.2.18]). Write $\omega(x) = \delta_{\mathbf{k}}$, it is thus enough to prove that the algebraic weights $\delta_{\mathbf{k}'} \in \mathcal{W}_L^n$ satisfying (2.14) accumulate at $\delta_{\mathbf{k}}$ in $\mathcal{W}_L^n$, which is obvious. $\square$

We now state our main local results.

Definition 2.13. *A crystalline strictly dominant point* $x = (r, \delta) \in X_{\mathrm{tri}}^{\square}(\overline{r})$ *is very regular if it satisfies the following conditions (where the* $(\varphi_i)_{1 \leq i \leq n}$ *are the geometric Frobenius eigenvalues on* $\mathrm{WD}(r)$*):*

(i) $\varphi_i \varphi_j^{-1} \notin \{1, q\}$ *for* $1 \leq i \neq j \leq n$*;*

(ii) $\varphi_1 \varphi_2 \ldots \varphi_i$ *is a simple eigenvalue of the geometric Frobenius acting on* $\bigwedge_{k(x)}^i \mathrm{WD}(r)$ *for* $1 \leq i \leq n$*.*

Remark 2.14. If $x = (r, \delta)$ is crystalline strictly dominant, it easily follows from the dominance property that (i) of Definition 2.13 is equivalent to $\delta_i \delta_j^{-1} \notin \{z^{-\mathbf{h}}, |z|_K z^{\mathbf{h}},\ |z|_K^{-1} z^{\mathbf{h}},\ \mathbf{h} \in \mathbb{Z}_{\geq 0}^{\mathrm{Hom}(K,L)}\}$ for $1 \leq i \neq j \leq n$. In particular it implies $\delta \in \mathcal{T}_{\mathrm{reg}}^n$, whence the terminology (compare also [4, §6.1]).

In §4 below we will prove the following theorem in the direction of Conjecture 2.8.

Theorem 2.15. *Let $x \in X_{\rm tri}^{\square}(\overline{r})$ be a crystalline strictly dominant very regular point and let $X \subseteq X_{\rm tri}^{\square}(\overline{r})$ be a union of irreducible components of an open subset of $X_{\rm tri}^{\square}(\overline{r})$ such that X satisfies the accumulation property at x. Then we have:*

$$\dim_{k(x)} T_{X,x} \leq \lg(w_x) - d_x + \dim X_{\rm tri}^{\square}(\overline{r}) = \lg(w_x) - d_x + n^2 + [K:\mathbb{Q}_p]\frac{n(n+1)}{2}.$$

By Lemma 2.7 we thus deduce the following important corollary.

Corollary 2.16. *Let $x \in X_{\rm tri}^{\square}(\overline{r})$ be a crystalline strictly dominant very regular point and let $X \subseteq X_{\rm tri}^{\square}(\overline{r})$ be a union of irreducible components of an open subset of $X_{\rm tri}^{\square}(\overline{r})$ such that X satisfies the accumulation property at x. Assume that w_x is a product of distinct simple reflections. Then X is smooth at x.*

Remark 2.17. Note that for X, x as above we only have $\dim_{k(x)} T_{X,x} \leq \dim_{k(x)} T_{\widetilde{X}_{\rm tri}^{\square}(\overline{r}),x}$, thus Theorem 2.15 doesn't give an upper bound on $\dim_{k(x)} T_{\widetilde{X}_{\rm tri}^{\square}(\overline{r}),x}$ (but Conjecture 2.8 implies Theorem 2.15). However Theorem 2.15 and Corollary 2.16 will be enough for our purpose.

3. Crystalline points on the patched eigenvariety

We state the classicality conjecture (Conjecture 3.6) and prove new cases of it (Corollary 3.12).

3.1. **The classicality conjecture.** We review the definition of classicality (Definition 3.2, Proposition 3.4) and state the classicality conjecture (Conjecture 3.6).

We first recall the global setting, basically the same as [13, §2.4]. We fix a totally real field F^+, we write q_v for the cardinality of the residue field of F^+ at a finite place v and we denote by S_p the set of places of F^+ dividing p . We fix a totally imaginary quadratic extension F of F^+ that splits at all places of S_p and let $\mathcal{G}_F := \mathrm{Gal}(\overline{F}/F)$. We fix a unitary group G in n variables over F^+ (with $n \geq 2$) such that $G \times_{F^+} F \cong \mathrm{GL}_{n,F}$ and $G(F^+ \otimes_{\mathbb{Q}} \mathbb{R})$ is compact. We fix an isomorphism $i : G \times_{F^+} F \xrightarrow{\sim} \mathrm{GL}_{n,F}$ and, for each $v \in S_p$, a place $\tilde{v}$ of F dividing v. The isomorphisms $F_v^+ \xrightarrow{\sim} F_{\tilde{v}}$ and i induce an isomorphism $i_{\tilde{v}} : G(F_v^+) \xrightarrow{\sim} \mathrm{GL}_n(F_{\tilde{v}})$ for $v \in S_p$. We let $G_v := G(F_v^+) \cong \mathrm{GL}_n(F_{\tilde{v}})$ and $G_p := \prod_{v\in S_p} G(F_v^+) \cong \prod_{v \in S_p} \mathrm{GL}_n(F_{\tilde{v}})$. We denote by K_v (resp. B_v, resp. $\overline{B}_v$, resp. T_v) the inverse image of $\mathrm{GL}_n(\mathcal{O}_{F_{\tilde{v}}})$ (resp. of the subgroup of upper triangular matrices of $\mathrm{GL}_n(F_{\tilde{v}})$, resp. of the subgroup of lower triangular matrices of $\mathrm{GL}_n(F_{\tilde{v}})$, resp. of the subgroup of diagonal matrices of $\mathrm{GL}_n(F_{\tilde{v}})$) in G_v under $i_{\tilde{v}}$ and we let $K_p := \prod_{v\in S_p} K_v$ (resp. $B_p := \prod_{v\in S_p} B_v$, resp. $\overline{B}_p := \prod_{v\in S_p} \overline{B}_v$, resp. $T_p := \prod_{v\in S_p} T_v$). We let $T_p^0 := T_p \cap K_p = \prod_{v\in S_p}(T_v \cap K_v)$.

We fix a finite extension L of $\mathbb{Q}_p$ that is assumed to be large enough so that $|\mathrm{Hom}(F_v^+, L)| = [F_v^+ : \mathbb{Q}_p]$ for $v \in S_p$. We let $\widehat{T}_{p,\rm reg} \subset \widehat{T}_{p,L}$ the open subspace of characters $\delta = (\delta_v)_{v\in S_p} = (\delta_{v,1}, \ldots, \delta_{v,n})_{v\in S_p}$ such that $\delta_{v,i}/\delta_{v,j} \in \mathcal{T}_{v,\rm reg}$ for all $v \in S_p$ and all $i \neq j$, where $\mathcal{T}_{v,\rm reg}$ is defined as $\mathcal{T}_{\rm reg}$ of §2.1 but with $F_v^+ = F_{\tilde{v}}$ instead of K.

We fix a tame level $U^p = \prod_v U_v \subset G(\mathbb{A}_{F^+}^{p\infty})$ where U_v is a compact open subgroup of $G(F_v^+)$ and we denote by $\widehat{S}(U^p, L)$ the associated space of p-adic automorphic forms on $G(\mathbb{A}_{F^+})$ of tame level U^p with coefficients in L, that is, the L-vector space of continuous functions $f : G(F^+)\backslash G(\mathbb{A}_{F^+}^\infty)/U^p \longrightarrow L$. Since $G(F^+)\backslash G(\mathbb{A}_{F^+}^\infty)/U^p$ is compact, it is a p-adic Banach space (for the sup norm) endowed with the linear continuous unitary action of G_p by right translation on functions. In particular a unit ball is given by the $\mathcal{O}_L$-submodule $\widehat{S}(U^p,\mathcal{O}_L)$ of continuous functions $f : G(F^+)\backslash G(\mathbb{A}_{F^+}^\infty)/U^p \longrightarrow \mathcal{O}_L$ and the corresponding residual representation is the k_L-vector space $S(U^p,k_L)$ of locally constant functions $f : G(F^+)\backslash G(\mathbb{A}_{F^+}^\infty)/U^p \longrightarrow k_L$ (a smooth admissible representation of G_p). Note that $S(U^p,k_L) = \varinjlim_{U_p} S(U^pU_p,k_L)$ where the inductive limit is taken over compact open subgroups U_p of G_p and where $S(U^pU_p,k_L)$ is the finite dimensional k_L-vector space of functions $f : G(F^+)\backslash G(\mathbb{A}_{F^+}^\infty)/U^pU_p \longrightarrow k_L$. We also denote by $\widehat{S}(U^p,L)^{\mathrm{an}} \subset \widehat{S}(U^p,L)$ the L-subvector space of locally $\mathbb{Q}_p$-analytic vectors for the action of G_p ([46, §7]). This is a very strongly admissible locally $\mathbb{Q}_p$-analytic representation of G_p in the sense of [23, Def.0.12]. It immediately follows from *loc. cit.* that its closed invariant subspaces are also very strongly admissible.

We fix S a finite set of finite places of F^+ that split in F containing S_p and the set of finite places $v \nmid p$ (that split in F) such that U_v is not maximal. We consider the commutative spherical Hecke algebra:

$$\mathbb{T}^S := \varinjlim_{I} \Big(\bigotimes_{v\in I} \mathcal{O}_L[U_v\backslash G(F_v^+)/U_v]\Big),$$

the inductive limit being taken over finite sets I of finite places of F^+ that split in F such that $I\cap S=\emptyset$. This Hecke algebra naturally acts on the spaces $\widehat{S}(U^p,L)$, $\widehat{S}(U^p,L)^{\mathrm{an}}$, $\widehat{S}(U^p,\mathcal{O}_L)$, $S(U^p,k_L)$ and $S(U^pU_p,k_L)$ (for any compact open subgroup U_p). If C is a field, $\theta : \mathbb{T}^S \to C$ a ring homomorphism and $\rho : \mathcal{G}_F \to \mathrm{GL}_n(C)$ a group homomorphism which is unramified at each finite place of F above a place of F^+ which splits in F and is not in S, we refer to [13, §2.4] for what it means for ρ to be *associated to* θ.

Though we could state a more general classicality conjecture, it is convenient for us to assume right now the following two extra hypothesis: $p > 2$ and G quasi-split at each finite place of F^+ (these assumptions will be needed anyway for our partial results, note however that they imply that 4 divides $n[F^+ : \mathbb{Q}]$ which rules out the case $n=2$, $F^+=\mathbb{Q}$). We fix $\mathfrak{m}^S$ a maximal ideal of $\mathbb{T}^S$ of residue field k_L (increasing L if necessary) such that $\widehat{S}(U^p,L)_{\mathfrak{m}^S} \neq 0$, or equivalently $\widehat{S}(U^p,\mathcal{O}_L)_{\mathfrak{m}^S} \neq 0$, or $S(U^p,k_L)_{\mathfrak{m}^S} = \varinjlim_{U_p} S(U^pU_p,k_L)_{\mathfrak{m}^S} \neq 0$, or $S(U^pU_p,k_L)_{\mathfrak{m}^S} \neq 0$ for some U_p (note that $\widehat{S}(U^p,L)_{\mathfrak{m}^S}$ is then a closed subspace of $\widehat{S}(U^p,L)$ preserved by G_p). We denote by $\overline{\rho} = \overline{\rho}_{\mathfrak{m}^S} : \mathcal{G}_F \to \mathrm{GL}_n(k_L)$ the unique absolutely semi-simple Galois representation associated to $\mathfrak{m}^S$ (see [48, Prop.6.6] and note that the running assumption F/F^+ unramified in *loc. cit.* is useless at this point). We assume $\mathfrak{m}^S$ *non-Eisenstein*, that is, $\overline{\rho}$ absolutely irreducible. Then it follows from [48, Prop.6.7] (with the same remark as above) that the spaces $\widehat{S}(U^p,L)_{\mathfrak{m}^S}$, $\widehat{S}(U^p,\mathcal{O}_L)_{\mathfrak{m}^S}$ and $S(U^p,k_L)_{\mathfrak{m}^S}$ become modules over $R_{\overline{\rho},S}$, the complete local noetherian $\mathcal{O}_L$-algebra of residue field k_L pro-representing the functor of deformations ρ of $\overline{\rho}$ that are unramified outside S and such that $\rho'\circ c \cong \rho\otimes\varepsilon^{n-1}$ (where ρ' is the dual of ρ and $c\in\mathrm{Gal}(F/F^+)$ is the complex conjugation).

The continuous dual $(\widehat{S}(U^p,L)^{\mathrm{an}}_{\mathfrak{m}^S})'$ of $\widehat{S}(U^p,L)^{\mathrm{an}}_{\mathfrak{m}^S} := (\widehat{S}(U^p,L)_{\mathfrak{m}^S})^{\mathrm{an}} = (\widehat{S}(U^p,L)^{\mathrm{an}})_{\mathfrak{m}^S}$ becomes a module over the global sections $\Gamma(\mathfrak{X}_{\overline{\rho},S},\mathcal{O}_{\mathfrak{X}_{\overline{\rho},S}})$ where $\mathfrak{X}_{\overline{\rho},S} := (\mathrm{Spf}\, R_{\overline{\rho},S})^{\mathrm{rig}}$ (see for instance [13, §3.1]). We denote by $Y(U^p,\overline{\rho})$ the *eigenvariety of tame level* U^p (over L) defined in [21] (see also [13, §4.1]) associated to $\widehat{S}(U^p,L)^{\mathrm{an}}_{\mathfrak{m}^S}$, that is, the support of the coherent $\mathcal{O}_{\mathfrak{X}_{\overline{\rho},S}\times\widehat{T}_{p,L}}$-module $(J_{B_p}(\widehat{S}(U^p,L)^{\mathrm{an}}_{\mathfrak{m}^S}))'$ on $\mathfrak{X}_{\overline{\rho},S}\times\widehat{T}_{p,L}$ where J_{B_p} is Emerton's locally $\mathbb{Q}_p$-analytic Jacquet functor with respect to the Borel B_p and $(\cdot)'$ means the continuous dual. This is a reduced closed analytic subset of $\mathfrak{X}_{\overline{\rho},S}\times\widehat{T}_{p,L}$ of dimension $n[F^+:\mathbb{Q}]$ whose points are:

$$\Big\{x=(\rho,\delta)\in\mathfrak{X}_{\overline{\rho},S}\times\widehat{T}_{p,L}\text{ such that }\mathrm{Hom}_{T_p}\Big(\delta,J_{B_p}\big(\widehat{S}(U^p,L)^{\mathrm{an}}_{\mathfrak{m}^S}[\mathfrak{p}_\rho]\otimes_{k(\mathfrak{p}_\rho)}k(x)\big)\Big)\neq0\Big\}\tag{3.1}$$

where $\mathfrak{p}_\rho\subset R_{\overline{\rho},S}$ denotes the prime ideal corresponding to the point $\rho\in\mathfrak{X}_{\overline{\rho},S}$ under the identification of the sets underlying $\mathfrak{X}_{\overline{\rho},S}=(\mathrm{Spf}\,R_{\overline{\rho},S})^{\mathrm{rig}}$ and $\mathrm{Spm}\,R_{\overline{\rho},S}[1/p]$ ([33, Lem.7.1.9]) and where $k(\mathfrak{p}_\rho)$ is its residue field. We denote by $\omega:Y(U^p,\overline{\rho})\to\widehat{T}^0_{p,L}$ the composition $Y(U^p,\overline{\rho})\hookrightarrow\mathfrak{X}_{\overline{\rho},S}\times\widehat{T}_{p,L}\twoheadrightarrow\widehat{T}_{p,L}\twoheadrightarrow\widehat{T}^0_{p,L}$.

Remark 3.1. If $U'^p\subseteq U^p$ (and S contains S_p and the set of finite places $v\nmid p$ that split in F such that U'_v is not maximal), then a point $x=(\rho,\delta)$ of $Y(U^p,\overline{\rho})$ is also in $Y(U'^p,\overline{\rho})$ since $\widehat{S}(U^p,L)^{\mathrm{an}}_{\mathfrak{m}^S}[\mathfrak{p}_\rho]\subseteq\widehat{S}(U'^p,L)^{\mathrm{an}}_{\mathfrak{m}^S}[\mathfrak{p}_\rho]$ and J_{B_p} is left exact.

We let $X^\square_{\mathrm{tri}}(\overline{\rho}_p)$ be the product rigid analytic variety $\prod_{v\in S_p}X^\square_{\mathrm{tri}}(\overline{\rho}_{\tilde{v}})$ (over L) where $\overline{\rho}_{\tilde{v}}$ is the restriction of ρ to the decomposition subgroup of $\mathcal{G}_F$ at $\tilde{v}$ (that we identify with $\mathcal{G}_{F_{\tilde{v}}}=\mathrm{Gal}(\overline{F}_{\tilde{v}}/F_{\tilde{v}})$) and where $X^\square_{\mathrm{tri}}(\overline{\rho}_{\tilde{v}})$ is as in §2.1. This is a reduced closed analytic subvariety of $(\mathrm{Spf}\,R^\square_{\overline{\rho}_p})^{\mathrm{rig}}\times\widehat{T}_{p,L}$ where $R^\square_{\overline{\rho}_p}:=\widehat{\bigotimes}_{v\in S_p}R^\square_{\overline{\rho}_{\tilde{v}}}$. Identifying B_v (resp. T_v) with the upper triangular (resp. diagonal) matrices of $\mathrm{GL}_n(F_{\tilde{v}})$ via $i_{\tilde{v}}$, we let $\delta_{B_v}:=|\cdot|^{n-1}_{F_{\tilde{v}}}\otimes|\cdot|^{n-3}_{F_{\tilde{v}}}\otimes\cdots\otimes|\cdot|^{1-n}_{F_{\tilde{v}}}$ be the modulus character of B_v and define as in [13, §2.3] an automorphism $\imath_v:\widehat{T}_v\xrightarrow{\sim}\widehat{T}_v$ by:

$$\imath_v(\delta_1,\dots,\delta_n):=\delta_{B_v}\cdot(\delta_1,\dots,\delta_i\cdot(\varepsilon\circ\mathrm{rec}_{F_{\tilde{v}}})^{i-1},\dots,\delta_n\cdot(\varepsilon\circ\mathrm{rec}_{F_{\tilde{v}}})^{n-1})$$

(the twist by δ_{B_v} here ultimately comes from the same twist appearing in the definition of J_{B_v}). It then follows from [13, Th.4.2] that the morphism of rigid spaces:

$$\begin{array}{rcl}(\mathrm{Spf}\,R_{\overline{\rho},S})^{\mathrm{rig}}\times\widehat{T}_{p,L}&\longrightarrow&(\mathrm{Spf}\,R^\square_{\overline{\rho}_p})^{\mathrm{rig}}\times\widehat{T}_{p,L}\\ \big(\rho,(\delta_v)_{v\in S_p}\big)=\big(\rho,(\delta_{v,1},\dots,\delta_{v,n})_{v\in S_p}\big)&\longmapsto&\big((\rho|_{\mathcal{G}_{F_{\tilde{v}}}})_{v\in S_p},(\imath_v^{-1}(\delta_{v,1},\dots,\delta_{v,n}))_{v\in S_p}\big)\end{array}\tag{3.2}$$

induces a morphism of reduced rigid spaces over L:

$$Y(U^p,\overline{\rho})\longrightarrow X^\square_{\mathrm{tri}}(\overline{\rho}_p)=\prod_{v\in S_p}X^\square_{\mathrm{tri}}(\overline{\rho}_{\tilde{v}})\tag{3.3}$$

(note that (3.3) is thus *not* compatible with the weight maps ω on both sides). We say that a point $x=(\rho,\delta)=(\rho,(\delta_v)_{v\in S_p})\in Y(U^p,\overline{\rho})$ is crystalline (resp. dominant, resp. strictly dominant, resp. crystalline strictly dominant very regular etc.) if for each $v\in S_p$ its image in $X^\square_{\mathrm{tri}}(\overline{\rho}_{\tilde{v}})$ via (3.3) is (see §2.1 and Definition 2.13). Due to the twist $\imath_v$ beware that $x=(\rho,\delta)\in Y(U^p,\overline{\rho})$ is strictly dominant if and only if $\delta|_{T_v\cap K_v}$ is (algebraic) dominant for each $v\in S_p$.

Let $\delta\in\widehat{T}_{p,L}$ be any locally algebraic character. Then we can write $\delta=\delta_\lambda\delta_{\mathrm{sm}}$ in $\widehat{T}_{p,L}$ where $\lambda=(\lambda_v)_{v\in S_p}\in\prod_{v\in S_p}(\mathbb{Z}^n)^{\mathrm{Hom}(F_{\tilde{v}},L)}$, $\delta_\lambda:=\prod_{v\in S_p}\delta_{\lambda_v}$ (see §2.1 for $\delta_{\lambda_v}\in\widehat{T}_{v,L}$) and δ_{sm} is a smooth character of T_p with values in $k(\delta)$ (the residue field of the point $\delta\in\widehat{T}_{p,L}$). Using the theory of Orlik and Strauch ([44]), we define as in [13, (3.7)] the following strongly admissible locally $\mathbb{Q}_p$-analytic representation of G_p over $k(\delta)$:

$$\mathcal{F}^{G_p}_{\overline{B}_p}(\delta):=\mathcal{F}^{G_p}_{\overline{B}_p}\Big(\big(U(\mathfrak{g}_L)\otimes_{U(\overline{\mathfrak{b}}_L)}(-\lambda)\big)^\vee,\delta_{\mathrm{sm}}\delta_{B_p}^{-1}\Big)$$

where $\delta_{B_p} := \prod_{v\in S_p}\delta_{B_v}$ and where we refer to [13, §3.5] for the details and notation. Recall that $\mathcal{F}_{\overline{B}_p}^{G_p}(\delta)$ has the same constituents as the locally $\mathbb{Q}_p$-analytic principal series $(\mathrm{Ind}_{\overline{B}_p}^{G_p}\delta_\lambda\delta_{\mathrm{sm}}\delta_{B_p}^{-1})^{\mathrm{an}} = (\mathrm{Ind}_{\overline{B}_p}^{G_p}\delta\delta_{B_p}^{-1})^{\mathrm{an}}$ but in the "reverse order" (at least generically). If λ is dominant (that is λ_v is dominant for each v in the sense of §2.1), we denote by $\mathrm{LA}(\delta)$ the locally algebraic representation:

$$\text{(3.4)}\quad \mathrm{LA}(\delta) := \mathcal{F}_{\overline{B}_p}^{G_p}(L(\lambda)',\delta_{\mathrm{sm}}\delta_{B_p}^{-1}) = \mathcal{F}_{G_p}^{G_p}\Big(L(\lambda)',(\mathrm{Ind}_{\overline{B}_p}^{G_p}\delta_{\mathrm{sm}}\delta_{B_p}^{-1})^{\mathrm{sm}}\Big) = L(\lambda)\otimes_L\Big(\mathrm{Ind}_{\overline{B}_p}^{G_p}\delta_{\mathrm{sm}}\delta_{B_p}^{-1}\Big)^{\mathrm{sm}}$$

where $L(\lambda)$ is the simple $U(\mathfrak{g}_L)$-module over L of highest weight λ relative to the Lie algebra of B_p (which is finite dimensional over L since λ is dominant) that we see as an irreducible algebraic representation of G_p over L, where $L(\lambda)'$ is its dual, and where $(-)^{\mathrm{sm}}$ denotes the smooth Borel induction over $k(\delta)$ (the second equality in (3.4) following from [44, Prop.4.9(b)]). Arguing as in [44, §6] (note that $L(\lambda)'$ is the unique irreducible subobject of $(U(\mathfrak{g}_L)\otimes_{U(\overline{\mathfrak{b}}_L)}(-\lambda))^\vee$), it easily follows from [44, Th.5.8] (see also [11, Th.2.3(iii)]) and [32, §5.1] that $\mathrm{LA}(\delta)$ is identified with the maximal locally $\mathbb{Q}_p$-algebraic quotient of $\mathcal{F}_{\overline{B}_p}^{G_p}(\delta)$ (or the maximal locally algebraic subobject of $(\mathrm{Ind}_{\overline{B}_p}^{G_p}\delta\delta_{B_p}^{-1})^{\mathrm{an}}$).

It follows from (3.1) together with [12, Th.4.3] that a point $x=(\rho,\delta)\in\mathfrak{X}_{\overline{\rho},S}\times\widehat{T}_{p,L}$ lies in $Y(U^p,\overline{\rho})$ if and only if:

$$\text{(3.5)}\quad \mathrm{Hom}_{T_p}\Big(\delta, J_{B_p}(\widehat{S}(U^p,L)^{\mathrm{an}}_{\mathfrak{m}^S}[\mathfrak{p}_\rho]\otimes_{k(\mathfrak{p}_\rho)}k(x))\Big)\cong\mathrm{Hom}_{G_p}\Big(\mathcal{F}_{\overline{B}_p}^{G_p}(\delta),\widehat{S}(U^p,L)^{\mathrm{an}}_{\mathfrak{m}^S}[\mathfrak{p}_\rho]\otimes_{k(\mathfrak{p}_\rho)}k(x)\Big)\neq 0.$$

Definition 3.2. *A point $x=(\rho,\delta)\in Y(U^p,\overline{\rho})$ is called* classical *if there exists a nonzero continuous G_p-equivariant morphism:*

$$\mathcal{F}_{\overline{B}_p}^{G_p}(\delta)\longrightarrow\widehat{S}(U^p,L)^{\mathrm{an}}_{\mathfrak{m}^S}[\mathfrak{p}_\rho]\otimes_{k(\mathfrak{p}_\rho)}k(x)$$

that factors through the locally $\mathbb{Q}_p$-algebraic quotient $\mathrm{LA}(\delta)$ *of $\mathcal{F}_{\overline{B}_p}^{G_p}(\delta)$ (equivalently (ρ,δ) is classical if $\mathrm{Hom}_{G_p}(\mathrm{LA}(\delta),\widehat{S}(U^p,L)_{\mathfrak{m}^S}[\mathfrak{p}_\rho]\otimes_{k(\mathfrak{p}_\rho)}k(x))\neq 0$).*

Remark 3.3. (i) This definition is [13, Def.3.15] when δ_{sm} is unramified.
(ii) It seems reasonable to expect that if $x=(\rho,\delta)\in Y(U^p,\overline{\rho})$ is classical, then in fact *any* continuous G_p-equivariant morphism $\mathcal{F}_{\overline{B}_p}^{G_p}(\delta)\longrightarrow\widehat{S}(U^p,L)^{\mathrm{an}}_{\mathfrak{m}^S}[\mathfrak{p}_\rho]\otimes_{k(\mathfrak{p}_\rho)}k(x)$ factors through $\mathrm{LA}(\delta)$. See the last statement of Corollary 3.12 below for a partial result in that direction. Note however that such a statement can't be expected for more general algebraic groups, as already follows from the main result of [42] in the case of SL_2.

We fix an algebraic closure $\overline{\mathbb{Q}}_p$ of L and embeddings $j_\infty:\overline{\mathbb{Q}}\hookrightarrow\mathbb{C}$, $j_p:\overline{\mathbb{Q}}\hookrightarrow\overline{\mathbb{Q}}_p$. Recall that, if $\pi=\pi_\infty\otimes_{\mathbb{C}}\pi_f$ is an automorphic representation of $G(\mathbb{A}_{F^+})$ over $\mathbb{C}$ where π_∞ (resp. π_f) is a representation of $G(F^+\otimes_{\mathbb{Q}}\mathbb{R})$ (resp. of $G(\mathbb{A}_{F^+}^\infty)$), then due to the compactness of $G(F^+\otimes_{\mathbb{Q}}\mathbb{R})$, we have that π_∞ is a finite dimensional irreducible representation that comes from an algebraic representation of $\mathrm{Res}_{F^+/\mathbb{Q}}G$ over $\mathbb{C}$ (argue as in [2, §§6.2.3,6.7]). Moreover, arguing again as in *loc. cit.*, π_∞ (resp. π_f) has a $\overline{\mathbb{Q}}$-structure given by j_∞ which is stable under the action of $(\mathrm{Res}_{F^+/\mathbb{Q}}G)(\overline{\mathbb{Q}})$ (resp. of $G(\mathbb{A}_{F^+}^\infty)$). Hence the scalar extension of the $\overline{\mathbb{Q}}$-structure of π_∞ to $\overline{\mathbb{Q}}_p$ via j_p is endowed with an action of $(\mathrm{Res}_{F^+/\mathbb{Q}}G)(\overline{\mathbb{Q}}_p)$, thus in particular of $(\mathrm{Res}_{F^+/\mathbb{Q}}G)(\mathbb{Q}_p) = G(F^+ \otimes_{\mathbb{Q}} \mathbb{Q}_p) = G_p$. This latter representation of G_p is easily seen to be defined over L and of the form $L(\lambda)$ for a dominant λ as above. We say that π_∞ *is of weight* λ if the resulting representation of G_p is $L(\lambda)$.

For the sake of completeness, we recall the following proposition showing that Definition 3.2 coincides with the usual classicality definition.

Proposition 3.4. *A strictly dominant point $x = (\rho, \delta) \in Y(U^p, \overline{\rho})$, that is such that $\omega(x) = \delta_\lambda$ for some dominant $\lambda \in \prod_{v\in S_p}(\mathbb{Z}^n)^{\mathrm{Hom}(F_{\tilde{v}},L)}$, is classical if and only if there exists an automorphic representation $\pi = \pi_\infty \otimes_{\mathbb{C}} \pi_f^p \otimes_{\mathbb{C}} \pi_p$ of $G(\mathbb{A}_{F^+})$ over $\mathbb{C}$ such that the following conditions hold:*

(i) *the $G(F^+ \otimes_{\mathbb{Q}} \mathbb{R})$-representation π_∞ is of weight λ in the above sense;*

(ii) *the $\mathcal{G}_F$-representation ρ is the Galois representation associated to π (see proof below);*

(iii) *the invariant subspace $(\pi_f^p)^{U^p}$ is nonzero;*

(iv) *the G_p-representation π_p is a quotient of $\left(\mathrm{Ind}_{\overline{B}_p}^{G_p}\delta\delta_\lambda^{-1}\delta_{B_p}^{-1}\right)^{\mathrm{sm}} \otimes_{k(\delta)} \overline{\mathbb{Q}}_p$.*

If moreover F is unramified over F^+ and U_v is hyperspecial when v is inert in F, then such a π is unique and appears with multiplicity 1 in $L^2(G(F^+)\backslash G(\mathbb{A}_{F^+}), \mathbb{C})$.

Proof. Let W be any linear representation of G_p over an L-vector space and U any compact open subgroup of $G(\mathbb{A}_{F^+}^\infty)$, we define $S(U, W)$ to be the L-vector space of functions $f : G(F^+)\backslash G(\mathbb{A}_{F^+}^\infty) \longrightarrow W$ such that $f(gu) = u_p^{-1}(f(g))$ for $g \in G(\mathbb{A}_{F^+}^\infty)$ and $u \in U$, where u_p is the projection of u in G_p. Fixing U^p as previously, we define $S(U^p, W) := \varinjlim_{U_p} S(U^pU_p, W)$ (inductive limit taken over compact open subgroups U_p of G_p) endowed with the linear left action of G_p given by $(h_p \cdot f)(g) := h_p(f(gh_p))$ ($h_p \in G_p$, $g \in G(\mathbb{A}_{F^+}^\infty)$) where the second h_p is seen in $G(\mathbb{A}_{F^+}^\infty)$ in the obvious way. Note that $\mathbb{T}^S$ also naturally acts on $S(U^p, W)$ (the representation W here playing no role since this action is "outside p"). Then it follows from [24, §7.1.4] that there is an isomorphism of smooth representations of G_p over $\overline{\mathbb{Q}}_p$:

$$S(U^p, L(\lambda)') \otimes_L \overline{\mathbb{Q}}_p \cong \bigoplus_\pi \left[\left((\pi_f^p)^{U^p} \otimes_{\overline{\mathbb{Q}}} \pi_p\right) \otimes_{\overline{\mathbb{Q}},j_p} \overline{\mathbb{Q}}_p\right]^{\oplus m(\pi)} \tag{3.6}$$

where the direct sum is over the automorphic representations $\pi = \pi_\infty \otimes_{\mathbb{C}} \pi_p$ of $G(\mathbb{A}_{F^+})$ such that π_∞ is of weight λ and $(\pi_f^p)^{U^p} \neq 0$ (we take the $\overline{\mathbb{Q}}$-structures) and where $m(\pi)$ is the multiplicity of π in $L^2(G(F^+)\backslash G(\mathbb{A}_{F^+}), \mathbb{C})$. We then say that a point $\rho \in \mathfrak{X}_{\overline{\rho},S}$ is the Galois representation associated to π (with π_∞ of weight λ) if we have:

$$\left[\left((\pi_f^p)^{U^p} \otimes_{\overline{\mathbb{Q}}} \pi_p\right) \otimes_{\overline{\mathbb{Q}},j_p} \overline{\mathbb{Q}}_p\right]^{\oplus m(\pi)} \subseteq S(U^p, L(\lambda)')_{\mathfrak{m}^S}[\mathfrak{p}_\rho] \otimes_{k(\mathfrak{p}_\rho)} \overline{\mathbb{Q}}_p$$

where $\mathfrak{p}_\rho$ is as in (3.1) (and $R_{\overline{\rho},S}$ acts on $S(U^p, L(\lambda)')_{\mathfrak{m}^S}$ again using [48, Prop.6.7]). Note that $S(U^p, L(\lambda)')_{\mathfrak{m}^S}[\mathfrak{p}_\rho] \otimes_{k(\mathfrak{p}_\rho)} \overline{\mathbb{Q}}_p \neq 0$ (equivalently $S(U^p, L(\lambda)')_{\mathfrak{m}^S}[\mathfrak{p}_\rho] \neq 0$) if and only if there exists an automorphic representation π such that π_∞ is of weight λ, $(\pi_f^p)^{U^p} \neq 0$ and ρ is the Galois representation associated to π.

Let $\widehat{S}(U^p,L)^{\lambda-\mathrm{la}}\subset\widehat{S}(U^p,L)^{\mathrm{an}}$ be the closed G_p-subrepresentation of locally $L(\lambda)$-algebraic vectors, that is the L-subvector space of $\widehat{S}(U^p,L)^{\mathrm{an}}$ (or equivalently of $\widehat{S}(U^p,L)$) of vectors v such that there exists a compact open subgroup U_p of G_p such that the U_p-subrepresentation generated by v in $\widehat{S}(U^p,L)|_{U_p}$ is isomorphic to $(L(\lambda)|_{U_p})^{\oplus d}$ for some positive integer d. Note that the subspace $\widehat{S}(U^p,L)^{\lambda-\mathrm{la}}$ is preserved under the action of $\mathbb{T}^S$ (since the latter commutes with G_p). Then it follows from [21, Prop.3.2.4] and its proof that there is an isomorphism of locally $\mathbb{Q}_p$-algebraic representations of G_p over L which is $\mathbb{T}^S$-equivariant (with the action of $\mathbb{T}^S$ on the right hand side given by its action on $S(U^p,L(\lambda)'))$:

$$\widehat{S}(U^p,L)^{\lambda-\mathrm{la}}\cong L(\lambda)\otimes_L S(U^p,L(\lambda)').$$

We then deduce a G_p-equivariant isomorphism of $R_{\overline{\rho},S}$-modules:

$$\widehat{S}(U^p,L)^{\lambda-\mathrm{la}}_{\mathfrak{m}^S}\cong L(\lambda)\otimes_L S(U^p,L(\lambda)')_{\mathfrak{m}^S} \tag{3.7}$$

where $\widehat{S}(U^p,L)^{\lambda-\mathrm{la}}_{\mathfrak{m}^S}:=(\widehat{S}(U^p,L)^{\lambda-\mathrm{la}})_{\mathfrak{m}^S}=(\widehat{S}(U^p,L)_{\mathfrak{m}^S})^{\lambda-\mathrm{la}}$.

Now let $x=(\rho,\delta)\in Y(U^p,\overline{\rho})$ with $\omega(x)=\delta_\lambda$ for λ dominant and define $\mathfrak{p}_\rho$ as in (3.1). From Definition 3.2 and the definition of $\widehat{S}(U^p,L)^{\lambda-\mathrm{la}}$, we get that the point x is classical if and only if there exists a nonzero G_p-equivariant morphism:

$$L(\lambda)\otimes_L\left(\mathrm{Ind}_{B_p}^{G_p}\delta\delta_\lambda^{-1}\delta_{B_p}^{-1}\right)^{\mathrm{sm}}\longrightarrow\widehat{S}(U^p,L)^{\lambda-\mathrm{la}}_{\mathfrak{m}^S}[\mathfrak{p}_\rho]\otimes_{k(\mathfrak{p}_\rho)}k(x)$$

if and only if by (3.7) there exists a nonzero G_p-equivariant morphism:

$$\left(\mathrm{Ind}_{B_p}^{G_p}\delta\delta_\lambda^{-1}\delta_{B_p}^{-1}\right)^{\mathrm{sm}}\longrightarrow S(U^p,L(\lambda)')_{\mathfrak{m}^S}[\mathfrak{p}_\rho]\otimes_{k(\mathfrak{p}_\rho)}k(x)$$

if and only if by (3.6) there exists an automorphic representation $\pi=\pi_\infty\otimes_{\mathbb{C}}\pi_f^p\otimes_{\mathbb{C}}\pi_p$ of $G(\mathbb{A}_{F^+})$ such that π_∞ is of weight λ, $(\pi_f^p)^{U^p}\neq 0$, ρ is the Galois representation associated to π and π_p is a quotient of $(\mathrm{Ind}_{B_p}^{G_p}\delta\delta_\lambda^{-1}\delta_{B_p}^{-1})^{\mathrm{sm}}\otimes_{k(\delta)}\overline{\mathbb{Q}}_p$.

Now let us prove the last assertion. According to [39, Cor.5.3], there exists an isobaric representation $\Pi=\Pi_1\boxplus\cdots\boxplus\Pi_r$ where $n=m_1+\cdots+m_r$ and Π_i nonzero automorphic representations of $\mathrm{GL}_{m_i}(\mathbb{A}_F)$ occuring in the discrete spectrum such that Π is a weak base change of π in the sense of [39, §4.10]. Since $\overline{\rho}$, hence ρ, are absolutely irreducible, we have $r=1$ and $\Pi=\Pi_1$ cuspidal. The equality $m(\pi)=1$ then follows from [39, Th.5.4] (which uses the extra assumption F/F^+ unramified). The uniqueness of π is a consequence of the strong base change theorem [39, Th.5.9] together with the fact that π_v is unramified at finite places v of F^+ which are inert in F (which uses the extra assumption U_v hyperspecial for v inert) and the fact that the L-packets at finite places of F^+ which are split in F are singletons. □

Remark 3.5. With the notation of Proposition 3.4, write $\delta\delta_\lambda^{-1}=(\delta_{\mathrm{sm},v,1},\dots,\delta_{\mathrm{sm},v,n})_{v\in S_p}$, if moreover $\delta_{\mathrm{sm},v,i}/\delta_{\mathrm{sm},v,j}\notin\{1,|\cdot|_{F_{\tilde v}}^{-2}\}$ for $1\leq i\neq j\leq n$ and $v\in S_p$, then we see from (iv) of Proposition 3.4 that $\pi_p\cong(\mathrm{Ind}_{B_p}^{G_p}\delta\delta_\lambda^{-1})^{\mathrm{sm}}\otimes_{k(\delta)}\overline{\mathbb{Q}}_p$.

We then have the following conjecture, which by Proposition 3.4 is essentially a consequence of the Fontaine-Mazur conjecture and the Langlands philosophy, and which is the natural generalization in the context of definite unitary groups of the main result of [35] for $\mathrm{GL}_2/\mathbb{Q}$ (in the crystalline case).

Conjecture 3.6. *Let $x=(\rho,\delta)\in Y(U^p,\overline{\rho})$ be a crystalline strictly dominant point. Then x is classical.*

Remark 3.7. We didn't seek to state the most general classicality conjecture. Obviously, the assumptions that $p>2$ and G is quasi-split at each finite place of F^+ shouldn't be crucial, and one could replace crystalline by de Rham.

3.2. **Proof of the main classicality result.** We prove a criterion for classicality (Theorem 3.9) in terms of the patched eigenvariety of [13], which itself builds on the construction in [18] of a "big" patching module M_∞. We use it to prove our main classicality result (Corollary 3.12).

We keep the notation of §3.1 and make the following extra assumptions (which are required for the construction of M_∞): F is unramified over F^+, U_v is hyperspecial if v is inert in F and $\overline{\rho}(\mathcal{G}_{F(\zeta_p)})$ is adequate in the sense of [48, Def.2.3]. For instance if $p>2n+1$ and $\overline{\rho}|_{\mathcal{G}_{F(\zeta_p)}}$ is (still) absolutely irreducible, then $\overline{\rho}(\mathcal{G}_{F(\zeta_p)})$ is automatically adequate ([27, Th.9]). We first briefly recall some notation, definitions and statements and refer to [13, §3.2] for more details on what follows.

We let $R^{\overline{\square}}_{\overline{\rho}_{\tilde{v}}}$ be the maximal reduced and $\mathbb{Z}_p$-flat quotient of the framed local deformation ring $R^{\square}_{\overline{\rho}_{\tilde{v}}}$ and set:

$$R^{\mathrm{loc}}:=\widehat{\bigotimes}_{v\in S}R^{\overline{\square}}_{\overline{\rho}_{\tilde{v}}},\quad R_{\overline{\rho}^p}:=\widehat{\bigotimes}_{v\in S\backslash S_p}R^{\overline{\square}}_{\overline{\rho}_{\tilde{v}}},\quad R_{\overline{\rho}_p}:=\widehat{\bigotimes}_{v\in S_p}R^{\overline{\square}}_{\overline{\rho}_{\tilde{v}}},\quad R_\infty:=R^{\mathrm{loc}}[\![x_1\dots,x_g]\!]$$

where $g\geq 1$ is some integer which will be fixed below. We let $\mathfrak{X}_{\overline{\rho}^p}:=(\mathrm{Spf}\,R_{\overline{\rho}^p})^{\mathrm{rig}}$, $\mathfrak{X}_{\overline{\rho}_p}:=(\mathrm{Spf}R_{\overline{\rho}_p})^{\mathrm{rig}}$ and $\mathfrak{X}_\infty:=(\mathrm{Spf}\,R_\infty)^{\mathrm{rig}}$ so that:

$$\mathfrak{X}_\infty=\mathfrak{X}_{\overline{\rho}^p}\times\mathfrak{X}_{\overline{\rho}_p}\times\mathbb{U}^g \tag{3.8}$$

where $\mathbb{U}:=(\mathrm{Spf}\,\mathcal{O}_L[\![y]\!])^{\mathrm{rig}}$ is the open unit disc over L. We also define $S_\infty:=\mathcal{O}_L[\![y_1,\dots,y_t]\!]$ where $t:=g+[F^+:\mathbb{Q}]\frac{n(n-1)}{2}+|S|n^2$ and $\mathfrak{a}:=(y_1,\dots,y_t)$ (an ideal of S_∞).

Thanks to Remark 3.1 and Proposition 3.4 we can (and do) assume that the tame level U^p is small enough so that we have:

$$G(F)\cap(hU^pK_ph^{-1})=\{1\}\ \text{ for all } h\in G(\mathbb{A}^\infty_{F^+}) \tag{3.9}$$

(indeed, let $w\nmid p$ be a finite place of F^+ that splits in F such that U_w is maximal, replace U^p by $U'^p:=U'_w\prod_{v\neq w}U_v$ where U'_w is small enough so that U'^p satisfies (3.9), and use Proposition 3.4 and local-global compatibility at w to deduce classicality in level U^p from classicality in level U'^p). Then there is a quotient $R_{\overline{\rho},S}\twoheadrightarrow R_{\overline{\rho},\mathcal{S}}$ such that the action of $R_{\overline{\rho},S}$ on $\widehat{S}(U^p,L)_{\mathfrak{m}^S}$ factors through $R_{\overline{\rho},\mathcal{S}}$, an integer $g\geq 1$ and:

(i) a continuous R_∞-admissible (see [13, Def.3.1]) unitary representation Π_∞ of G_p over L together with a G_p-stable and R_∞-stable unit ball $\Pi^\circ_\infty\subset\Pi_\infty$;

(ii) a morphism of local $\mathcal{O}_L$-algebras $S_\infty\to R_\infty$ such that $M_\infty:=\mathrm{Hom}_{\mathcal{O}_L}(\Pi^\circ_\infty,\mathcal{O}_L)$ is finite projective as an $S_\infty[\![K_p]\!]$-module;

(iii) a surjection $R_\infty/\mathfrak{a}R_\infty\twoheadrightarrow R_{\overline{\rho},\mathcal{S}}$ and a compatible G_p-equivariant isomorphism $\Pi_\infty[\mathfrak{a}]\cong\widehat{S}(U^p,L)_{\mathfrak{m}^S}$.

We then define the *patched eigenvariety* $X_p(\overline{\rho})$ as the support of the coherent $\mathcal{O}_{\mathfrak{X}_\infty\times\widehat{T}_{p,L}}$-module $\mathcal{M}_\infty=(J_{B_p}(\Pi_\infty^{R_\infty-\mathrm{an}}))'$ on $\mathfrak{X}_\infty\times\widehat{T}_{p,L}$ (see [13, Def.3.2] for $\Pi_\infty^{R_\infty-\mathrm{an}}$; strictly speaking $(J_{B_p}(\Pi_\infty^{R_\infty-\mathrm{an}}))'$ is the global sections of the sheaf $\mathcal{M}_\infty$). This is a reduced closed analytic subset of $\mathfrak{X}_\infty\times\widehat{T}_{p,L}$ ([13, Cor.3.20]) whose points are ([13, Prop.3.7]):

$$\Big\{x=(y,\delta)\in\mathfrak{X}_\infty\times\widehat{T}_{p,L}\text{ such that }\mathrm{Hom}_{T_p}\Big(\delta,J_{B_p}(\Pi_\infty^{R_\infty-\mathrm{an}}[\mathfrak{p}_y]\otimes_{k(\mathfrak{p}_y)}k(x))\Big)\neq 0\Big\}\tag{3.10}$$

where $\mathfrak{p}_y\subset R_\infty$ denotes the prime ideal corresponding to the point $y\in\mathfrak{X}_\infty$ (under the identification of the sets underlying $\mathfrak{X}_\infty$ and $\mathrm{Spm}\,R_\infty[1/p]$) and $k(\mathfrak{p}_y)$ is the residue field of $\mathfrak{p}_y$. It follows from the proof of [13, Th.4.2] that we can recover the eigenvariety $Y(U^p,\overline{\rho})$ as the reduced Zariski-closed subspace of $X_p(\overline{\rho})$ underlying the vanishing locus of the ideal $\mathfrak{a}\Gamma(\mathfrak{X}_\infty,\mathcal{O}_{\mathfrak{X}_\infty})$.

Lemma 3.8. *The coherent sheaf $\mathcal{M}_\infty$ is Cohen-Macaulay over $X_p(\overline{\rho})$.*

Proof. From the proof of [13, Prop.3.11] (to which we refer the reader for more details) we deduce that there exists an admissible affinoid covering $(U_i)_i$ of $X_p(\overline{\rho})$ such that $\Gamma(U_i,\mathcal{M}_\infty)$ is a finite projective module over a ring $\mathcal{O}_{\mathcal{W}_\infty}(W_i)$ whose action on $\Gamma(U_i,\mathcal{M}_\infty)$ factors through a ring homomorphism $\mathcal{O}_{\mathcal{W}_\infty}(W_i)\to\mathcal{O}_{X_p(\overline{\rho})}(U_i)$. Consequently we can deduce from [26, Prop.16.5.3] that $\Gamma(U_i,\mathcal{M}_\infty)$ is a Cohen-Macaulay $\mathcal{O}_{X_p(\overline{\rho})}(U_i)$-module. ☐

It follows from [13, Th.3.21] that the isomorphism of rigid spaces:

$$\begin{array}{rcl}\mathfrak{X}_\infty\times\widehat{T}_{p,L} & \xrightarrow{\sim} & \mathfrak{X}_\infty\times\widehat{T}_{p,L}\\ \Big(x,(\delta_v)_{v\in S_p}\Big)=\Big(x,(\delta_{v,1},\dots,\delta_{v,n})_{v\in S_p}\Big) & \longmapsto & \Big(x,(\imath_v^{-1}(\delta_{v,1},\dots,\delta_{v,n}))_{v\in S_p}\Big)\end{array}$$

induces via (3.8) a morphism of reduced rigid spaces over L:

$$X_p(\overline{\rho})\longrightarrow\mathfrak{X}_{\overline{\rho}^p}\times X_{\mathrm{tri}}^{\square}(\overline{\rho}_p)\times\mathbb{U}^g\tag{3.11}$$

which identifies the source with a union of irreducible components of the target. Note that the composition:

$$Y(U^p,\overline{\rho})\hookrightarrow X_p(\overline{\rho})\xrightarrow{(3.11)}\mathfrak{X}_{\overline{\rho}^p}\times X_{\mathrm{tri}}^{\square}(\overline{\rho}_p)\times\mathbb{U}^g\twoheadrightarrow X_{\mathrm{tri}}^{\square}(\overline{\rho}_p)$$

is the map (3.3). An irreducible component of the right hand side of (3.11) is of the form $\mathfrak{X}^p\times Z\times\mathbb{U}^g$ where $\mathfrak{X}^p$ (resp. Z) is an irreducible component of $\mathfrak{X}_{\overline{\rho}^p}$ (resp. $X_{\mathrm{tri}}^{\square}(\overline{\rho}_p)$). Given an irreducible component $\mathfrak{X}^p\subseteq\mathfrak{X}_{\overline{\rho}^p}$, we denote by $X_{\mathrm{tri}}^{\mathfrak{X}^p-\mathrm{aut}}(\overline{\rho}_p)\subseteq X_{\mathrm{tri}}^{\square}(\overline{\rho}_p)$ the union (possibly empty) of those irreducible components $Z\subseteq X_{\mathrm{tri}}^{\square}(\overline{\rho}_p)$ such that $\mathfrak{X}^p\times Z\times\mathbb{U}^g$ is an irreducible component of $X_p(\overline{\rho})$ via (3.11). The morphism (3.11) thus induces an isomorphism:

$$X_p(\overline{\rho})\xrightarrow{\sim}\bigcup_{\mathfrak{X}^p}\Big(\mathfrak{X}^p\times X_{\mathrm{tri}}^{\mathfrak{X}^p-\mathrm{aut}}(\overline{\rho}_p)\times\mathbb{U}^g\Big)\tag{3.12}$$

the union (inside $\mathfrak{X}_{\overline{\rho}^p}\times X_{\mathrm{tri}}^{\square}(\overline{\rho}_p)\times\mathbb{U}^g$) being over the irreducible components $\mathfrak{X}^p$ of $\mathfrak{X}_{\overline{\rho}^p}$.

We now state and prove the main result of this section, which gives a criterion for classicality on $Y(U^p,\overline{\rho})$. Recall that, given a crystalline strictly dominant point $x_v=(r_v,\delta_v)\in X_{\mathrm{tri}}^{\square}(\overline{\rho}_{\tilde{v}})$ such that the geometric Frobenius eigenvalues on $\mathrm{WD}(r_v)$ are pairwise distinct and $V_v\subseteq X_{\mathrm{tri}}^{\square}(\overline{\rho}_{\tilde{v}})$ a sufficiently small open neighbourhood of x_v, we have constructed in Corollary 2.5 an irreducible component $Z_{\mathrm{tri},V_v}(x_v)$ of V_v containing x_v.

Theorem 3.9. *Let $x = (\rho, \delta) \in Y(U^p, \overline{\rho})$ be a crystalline strictly dominant point such that the eigenvalues $\varphi_{\tilde{v},1}, \dots, \varphi_{\tilde{v},n}$ of the geometric Frobenius on the (unramified) Weil-Deligne representation $\mathrm{WD}(\rho|_{\mathcal{G}_{F_{\tilde{v}}}})$ satisfy $\varphi_{\tilde{v},i}\varphi_{\tilde{v},j}^{-1} \notin \{1, q_v\}$ for all $i \neq j$ and all $v \in S_p$. Let $\mathfrak{X}^p \subset \mathfrak{X}_{\overline{\rho}^p}$ be an irreducible component such that $x \in \mathfrak{X}^p \times X_{\mathrm{tri}}^{\mathfrak{X}^p-\mathrm{aut}}(\overline{\rho}_p) \times \mathbb{U}^g \subseteq X_p(\overline{\rho})$ via (3.12), let $x_v \in X_{\mathrm{tri}}^{\square}(\overline{\rho}_{\tilde{v}})$ (for $v \in S_p$) be the image of x via:*

$$\mathfrak{X}^p \times X_{\mathrm{tri}}^{\mathfrak{X}^p-\mathrm{aut}}(\overline{\rho}_p) \times \mathbb{U}^g \twoheadrightarrow X_{\mathrm{tri}}^{\mathfrak{X}^p-\mathrm{aut}}(\overline{\rho}_p) \hookrightarrow X_{\mathrm{tri}}^{\square}(\overline{\rho}_p) \twoheadrightarrow X_{\mathrm{tri}}^{\square}(\overline{\rho}_{\tilde{v}})$$

and let $V_v \subseteq X_{\mathrm{tri}}^{\square}(\overline{\rho}_{\tilde{v}})$ (for $v \in S_p$) be a sufficiently small open neighbourhood of x_v so that $Z_{\mathrm{tri},V_v}(x_v) \subseteq V_v$ is defined. If we have:

$$\prod_{v \in S_p} Z_{\mathrm{tri},V_v}(x_v) \subseteq X_{\mathrm{tri}}^{\mathfrak{X}^p-\mathrm{aut}}(\overline{\rho}_p)$$

then the point x is classical.

Proof. Let us write $\mathfrak{p}_y \subset R_\infty$ for the prime ideal corresponding to the image y of x in $\mathfrak{X}_\infty$ via $Y(U^p, \overline{\rho}) \hookrightarrow X_p(\overline{\rho}) \hookrightarrow \mathfrak{X}_\infty \times \widehat{T}_{p,L} \twoheadrightarrow \mathfrak{X}_\infty$ and $\mathfrak{p}_\rho \subset R_{\overline{\rho},S}$ for the prime ideal corresponding to the global representation ρ. Then it follows from property (iii) above that we have $\mathfrak{a}R_\infty \subseteq \mathfrak{p}_\rho$ and $\widehat{S}(U^p, L)_{\mathfrak{m}^S}[\mathfrak{p}_\rho] = \Pi_\infty[\mathfrak{p}_y]$. From Definition 3.2 we thus need to show that $\mathrm{Hom}_{G_p}(\mathrm{LA}(\delta), \Pi_\infty[\mathfrak{p}_y] \otimes_{k(\mathfrak{p}_y)} k(x)) \neq 0$.

As in §3.1 let us write $\delta = \delta_\lambda \delta_{\mathrm{sm}}$ with $\lambda = (\lambda_v)_{v \in S_p}$ and:

$$\lambda_v := (\lambda_{v,\tau,i})_{1 \le i \le n, \tau \in \mathrm{Hom}(F_{\tilde{v}},L)} \in (\mathbb{Z}^n)^{\mathrm{Hom}(F_{\tilde{v}},L)}$$

(recall that each λ_v is dominant with respect to B_v). Consider the usual induction with compact support $\mathrm{ind}_{K_p}^{G_p}(L(\lambda)|_{K_p})$ (resp. $\mathrm{ind}_{K_v}^{G_v}(L(\lambda_v)|_{K_v})$) where $L(\lambda_p)$ (resp. $L(\lambda_v)$) is the irreducible algebraic representation of G_p (resp. G_v) over L of highest weight λ (resp. λ_v) with respect to B_p (resp. B_v). Let $\mathcal{H}(\lambda) := \mathrm{End}_{G_p}(\mathrm{ind}_{K_p}^{G_p} L(\lambda))$ and $\mathcal{H}(\lambda_v) := \mathrm{End}_{G_v}(\mathrm{ind}_{K_v}^{G_v} L(\lambda_v))$ be the respective convolution algebras (which are commutative L-algebras), we have $\mathcal{H}(\lambda) \cong \prod_{v \in S_p} \mathcal{H}(\lambda_v)$. Moreover by Frobenius reciprocity:

$$\Pi_\infty(\lambda) := \mathrm{Hom}_{K_p}(L(\lambda), \Pi_\infty) \cong \mathrm{Hom}_{G_p}\Big(\mathrm{ind}_{K_p}^{G_p} L(\lambda), \Pi_\infty\Big)$$

carries an action of $\mathcal{H}(\lambda)$. For $v \in S_p$, set $\mathbf{k}_v := (k_{v,\tau,i})_{1 \le i \le n, \tau \in \mathrm{Hom}(F_{\tilde{v}},L)}$ with $k_{v,\tau,i} := \lambda_{v,\tau,i} - (i-1)$ and note that $\omega(x_v) = \delta_{\mathbf{k}_v}$. By a slight extension of [18, Lem.4.16(1)] (see the proof of [13, Prop.3.16]), the action of $R_{\overline{\rho}_{\tilde{v}}}^{\square}$ on $\Pi_\infty(\lambda)$ via $R_{\overline{\rho}_{\tilde{v}}}^{\square} \to R^{\mathrm{loc}} \hookrightarrow R_\infty$ factors through the quotient $R_{\overline{\rho}_{\tilde{v}}}^{\square,\mathbf{k}_v-\mathrm{cr}}$ of $R_{\overline{\rho}_{\tilde{v}}}^{\square}$.

These two actions of $\mathcal{H}(\lambda_v)$ and $R_{\overline{\rho}_{\tilde{v}}}^{\square,\mathbf{k}_v-\mathrm{cr}}$ on the L-vector space $\Pi_\infty(\lambda)$ are related. By [18, Th.4.1] and a slight extension of [18, Lem.4.16(2)] (see the proof of [13, Prop.3.16]), there is a unique L-algebra homomorphism $\eta_v : \mathcal{H}(\lambda_v) \to R_{\overline{\rho}_{\tilde{v}}}^{\square,\mathbf{k}_v-\mathrm{cr}}[1/p]$ which interpolates the local Langlands correspondence (in a sense given in [18, Th.4.1]) and such that the above action of $\mathcal{H}(\lambda_v)$ on $\Pi_\infty(\lambda)$ agrees with the action induced by that of $R_{\overline{\rho}_{\tilde{v}}}^{\square,\mathbf{k}_v-\mathrm{cr}}[1/p]$ composed with the morphism η_v.

In order to show that $\mathrm{LA}(\delta)$ admits a nonzero G_p-equivariant morphism to $\Pi_\infty[\mathfrak{p}_y] \otimes_{k(\mathfrak{p}_y)} k(x)$, we claim it is enough to show that $\Pi_\infty(\lambda)[\mathfrak{p}_y] \cong \mathrm{Hom}_{G_p}(\mathrm{ind}_{K_p}^{G_p} L(\lambda), \Pi_\infty[\mathfrak{p}_y])$ is nonzero.

Indeed, by what we just saw, any nonzero G_p-equivariant morphism $\mathrm{ind}_{K_p}^{G_p} L(\lambda) \to \Pi_\infty[\mathfrak{p}_y]$ induces a nonzero G_p-equivariant morphism:

$$\mathrm{ind}_{K_p}^{G_p} L(\lambda) \otimes_L k(x) \longrightarrow \Pi_\infty[\mathfrak{p}_y] \otimes_{k(\mathfrak{p}_y)} k(x)$$

which factors through $\mathrm{ind}_{K_p}^{G_p} L(\lambda) \otimes_{\mathcal{H}(\lambda)} \theta_{\mathfrak{p}_y}$ where $\theta_{\mathfrak{p}_y}$ is the character:

$$\theta_{\mathfrak{p}_y} : \mathcal{H}(\lambda) \xrightarrow{\otimes_{v\in S_p}\eta_v} \widehat{\bigotimes}_{v\in S_p} R_{\overline{\rho}_{\tilde{v}}}^{\square,\mathbf{k}_v-\mathrm{cr}}[1/p] \longrightarrow k(\mathfrak{p}_y) \subseteq k(x),$$

the last morphism being the canonical projection to the residue field $k(\mathfrak{p}_y)$ at $\mathfrak{p}_y$ (the map $R_{\overline{\rho}_p} \hookrightarrow R_\infty \twoheadrightarrow R_\infty/\mathfrak{p}_y$ factoring through $\widehat{\bigotimes}_{v\in S_p} R_{\overline{\rho}_{\tilde{v}}}^{\square,\mathbf{k}_v-\mathrm{cr}}$ by the assumption on ρ). But by the compatibility with the local Langlands correspondence in [18, Th.4.1] together with the assumption $\varphi_{\tilde{v},i}/\varphi_{\tilde{v},j} \neq q_v$ for $1 \leq i,j \leq n$ and $v \in S_p$, we have $\mathrm{ind}_{K_p}^{G_p} L(\lambda) \otimes_{\mathcal{H}(\lambda)} \theta_{\mathfrak{p}_y} \cong \mathrm{LA}(\delta) \otimes_{k(\delta)} k(x)$.

By the same proof as that of [18, Lem.4.17(2)], the $R_\infty \otimes_{R_{\overline{\rho}_p}} \widehat{\bigotimes}_{v\in S_p} R_{\overline{\rho}_{\tilde{v}}}^{\square,\mathbf{k}_v-\mathrm{cr}}$-module $\Pi_\infty(\lambda)'$ is supported on a union of irreducible components of:

$$\mathfrak{X}_{\overline{\rho}^p} \times \prod_{v\in S_p} \mathfrak{X}_{\overline{\rho}_{\tilde{v}}}^{\square,\mathbf{k}_v-\mathrm{cr}} \times \mathbb{U}^g$$

and we have to prove that y is a point on one of these irreducible components. Since $y \in \mathfrak{X}^p \times \prod_{v\in S_p} Z_{\mathrm{cris}}(\rho_{\tilde{v}}) \times \mathbb{U}^g$ where $Z_{\mathrm{cris}}(\rho_{\tilde{v}})$ is the unique irreducible component of $\mathfrak{X}_{\overline{\rho}_{\tilde{v}}}^{\square,\mathbf{k}_v-\mathrm{cr}}$ containing $\rho_{\tilde{v}} := \rho|_{\mathcal{G}_{F_{\tilde{v}}}}$ (recall $\mathfrak{X}_{\overline{\rho}_{\tilde{v}}}^{\square,\mathbf{k}_v-\mathrm{cr}}$ is smooth over L by [36]), it is enough to prove that $\mathfrak{X}^p \times \prod_{v\in S_p} Z_{\mathrm{cris}}(\rho_{\tilde{v}}) \times \mathbb{U}^g$ is one of the irreducible components in the support of $\Pi_\infty(\lambda)'$, or equivalently that $\mathfrak{X}^p \times \prod_{v\in S_p} Z_{\mathrm{cris}}(\rho_{\tilde{v}}) \times \mathbb{U}^g$ contains at least one point which is in the support of $\Pi_\infty(\lambda)'$.

For each $v \in S_p$ let $x'_v = (r'_v, \delta'_v)$ be any point in $\iota_{\mathbf{k}_v}(\widetilde{Z}_{\mathrm{cris}}(x_v)) \cap V_v \subseteq V_v \subseteq X_{\mathrm{tri}}^{\square}(\overline{\rho}_{\tilde{v}})$ where $\widetilde{Z}_{\mathrm{cris}}(x_v)$ is as in (i) of Corollary 2.5 (so in particular x'_v is crystalline strictly dominant of Hodge-Tate weights $\mathbf{k}_v$ and r'_v lies on $Z_{\mathrm{cris}}(\rho_{\tilde{v}})$ by (i) of Remark 2.6). Then we have $x'_v \in Z_{\mathrm{tri},V_v}(x_v)$ for $v \in S_p$ by (ii) of Corollary 2.5. From the assumption:

$$\prod_{v\in S_p} Z_{\mathrm{tri},V_v}(x_v) \subset X_{\mathrm{tri}}^{\mathfrak{X}^p-\mathrm{aut}}(\overline{\rho}_p)$$

it then follows that there exists:

$$x' = (y', \epsilon') \in \mathfrak{X}^p \times \prod_{v\in S_p} Z_{\mathrm{tri},V_v}(x_v) \times \mathbb{U}^g \subseteq \mathfrak{X}^p \times X_{\mathrm{tri}}^{\mathfrak{X}^p-\mathrm{aut}}(\overline{\rho}_p) \times \mathbb{U}^g \overset{(3.12)}{\subseteq} X_p(\overline{\rho}) \subset \mathfrak{X}_\infty \times \widehat{T}_{p,L}$$

(with $y' \in \mathfrak{X}_\infty$, $\epsilon' \in \widehat{T}_{p,L}$) mapping to x'_v via $\mathfrak{X}^p \times X_{\mathrm{tri}}^{\mathfrak{X}^p-\mathrm{aut}}(\overline{\rho}_p) \times \mathbb{U}^g \twoheadrightarrow X_{\mathrm{tri}}^{\mathfrak{X}^p-\mathrm{aut}}(\overline{\rho}_p) \hookrightarrow X_{\mathrm{tri}}^{\square}(\overline{\rho}_p) \twoheadrightarrow X_{\mathrm{tri}}^{\square}(\overline{\rho}_{\tilde{v}})$ (so $\epsilon'_v = \imath_v^{-1}(\delta'_v)$) and where y' still belongs to $\mathfrak{X}^p \times \prod_{v\in S_p} Z_{\mathrm{cris}}(\rho_{\tilde{v}}) \times \mathbb{U}^g$. It is thus enough to prove that y' is in the support of $\Pi_\infty(\lambda)'$, i.e. that $\Pi_\infty(\lambda)[\mathfrak{p}_{y'}] \cong \mathrm{Hom}_{K_p}(L(\lambda), \Pi_\infty[\mathfrak{p}_{y'}])$ is nonzero.

We conclude by a similar argument as in the proof of [13, Prop.3.28]. By (the proof of) [17, Lem.4.4] and the same argument as at the end of the proof of Lemma 2.4 (using the smoothness, hence flatness, of $\widetilde{U}_{\overline{\rho}_{\tilde{v}}}^{\square,\mathbf{k}_v-\mathrm{cr}} \to \mathfrak{X}_{\overline{\rho}_{\tilde{v}}}^{\square,\mathbf{k}_v-\mathrm{cr}}$), we may choose $x'_v \in \iota_{\mathbf{k}_v}(\widetilde{Z}_{\mathrm{cris}}(x_v)) \cap V_v$ such that the crystalline Galois representation r'_v is generic in the sense of [13, Def.2.8]. Then we claim that the nonzero G_p-equivariant morphism $\mathcal{F}_{\overline{B}_p}^{G_p}(\epsilon') \to \Pi_\infty^{R_\infty-\mathrm{an}}[\mathfrak{p}_{y'}] \otimes_{k(\mathfrak{p}_{y'})} k(x')$ corresponding by [12, Th.4.3] to the nonzero T_p-equivariant morphism $\epsilon' \to J_{B_p}(\Pi_\infty^{R_\infty-\mathrm{an}}[\mathfrak{p}_{y'}] \otimes_{k(\mathfrak{p}_{y'})} k(x'))$ given by the point x' factors through its locally $\mathbb{Q}_p$-algebraic quotient $\mathrm{LA}(\epsilon')$ (which provides a nonzero K_p-equivariant morphism $L(\lambda) \to \Pi_\infty[\mathfrak{p}_{y'}]$). Indeed, if it doesn't, then the computation of the Jordan-Hölder factors of $\mathcal{F}_{\overline{B}_p}^{G_p}(\epsilon')$ ([12, Cor.4.6]) together with [11, Cor.3.4] show that there exists a point $x'' = (y', \epsilon'') \in X_p(\overline{\rho})$ such that ϵ'' is locally algebraic of *non-dominant* weight. In particular there is some $v \in S_p$ such that the image of x'' in $X_{\mathrm{tri}}^{\square}(\overline{\rho}_{\tilde{v}})$ is of the form $(r'_v, \imath_v^{-1}(\epsilon''_v))$ with $\imath_v^{-1}(\epsilon''_v)$ locally algebraic *not* strictly dominant. This contradicts [13, Cor.2.12]. □

Before stating our main consequence of Theorem 3.9, we need yet another proposition. Similarly to Definition 2.11, we say that a union X of irreducible components of an open subset of $X_{\mathrm{tri}}^{\square}(\overline{\rho}_p) = \prod_{v \in S_p} X_{\mathrm{tri}}^{\square}(\overline{\rho}_{\tilde{v}})$ satisfies the accumulation property at a point $x \in X$ if, for any positive real number $C > 0$, X contains crystalline strictly dominant points $x' = (x'_v)_{v \in S_p}$ with pairwise distinct Frobenius eigenvalues, which are noncritical, such that $\omega(x'_v) = \delta_{\mathbf{k}'_v}$ with $k'_{v,\tau,i} - k'_{v,\tau,i+1} > C$ for $v \in S_p$, $i \in \{1, \dots, n-1\}$, $\tau \in \mathrm{Hom}(F_{\tilde{v}}, L)$ and that accumulate at x in X.

Proposition 3.10. *Let $\mathfrak{X}^p \subset \mathfrak{X}_{\overline{\rho}^p}$ be an irreducible component and $x \in X_{\mathrm{tri}}^{\mathfrak{X}^p-\mathrm{aut}}(\overline{\rho}_p)$ be a crystalline strictly dominant point. Then $X_{\mathrm{tri}}^{\mathfrak{X}^p-\mathrm{aut}}(\overline{\rho}_p)$ satisfies the accumulation property at x.*

Proof. It is enough to show that, for C large enough, the points of $\mathfrak{X}^p \times X_{\mathrm{tri}}^{\mathfrak{X}^p-\mathrm{aut}}(\overline{\rho}_p) \times \mathbb{U}^g$ such that their projection to $X_{\mathrm{tri}}^{\mathfrak{X}^p-\mathrm{aut}}(\overline{\rho}_p)$ is a point $x' = (x'_v)_{v \in S_p}$ of the same form as above accumulate at any point of $\mathfrak{X}^p \times X_{\mathrm{tri}}^{\mathfrak{X}^p-\mathrm{aut}}(\overline{\rho}_p) \times \mathbb{U}^g$ mapping to x in $X_{\mathrm{tri}}^{\mathfrak{X}^p-\mathrm{aut}}(\overline{\rho}_p)$. Using (3.12) this claim is contained in the proof of [13, Th.3.19] (which itself is a consequence of [13, Prop.3.11]). □

Remark 3.11. It is obvious from the definition that if a union X of irreducible components of $X_{\mathrm{tri}}^{\square}(\overline{\rho}_p) = \prod_{v \in S_p} X_{\mathrm{tri}}^{\square}(\overline{\rho}_{\tilde{v}})$ satisfies the accumulation property at some point $x \in X$, then for each $v \in S_p$ the image of X in $X_{\mathrm{tri}}^{\square}(\overline{\rho}_{\tilde{v}})$ (which is a union of irreducible components of $X_{\mathrm{tri}}^{\square}(\overline{\rho}_{\tilde{v}})$) satisfies the accumulation property at the image of x in $X_{\mathrm{tri}}^{\square}(\overline{\rho}_{\tilde{v}})$.

Let $x = (\rho, \delta) \in Y(U^p, \overline{\rho})$ be a crystalline strictly dominant point such that for all $v \in S_p$ the eigenvalues of the geometric Frobenius on $\mathrm{WD}(\rho|_{\mathcal{G}_{F_{\tilde{v}}}})$ are pairwise distinct. Recall that we have associated in §2.3 a Weyl group element w_{x_v} to the image x_v of x in $X_{\mathrm{tri}}^{\square}(\overline{\rho}_{\tilde{v}})$ via (3.3). We write:

$$w_x := (w_{x_v})_{v \in S_p} \in \prod_{v \in S_p} \Big(\prod_{F_{\tilde{v}} \hookrightarrow L} \mathcal{S}_n \Big) \tag{3.13}$$

for the corresponding element of the Weyl group of $(\mathrm{Res}_{F^+/\mathbb{Q}} G)_L \xrightarrow{\sim} \prod_{v \in S_p} (\mathrm{Res}_{F_{\tilde{v}}/\mathbb{Q}_p} \mathrm{GL}_{n,F_{\tilde{v}}})_L$. We then obtain the following corollary, which is our main classicality result.

Corollary 3.12. *Let $x = (\rho, \delta) \in Y(U^p, \overline{\rho})$ be a crystalline strictly dominant very regular point. Assume that the Weyl group element w_x in (3.13) is a product of pairwise distinct simple reflections. Then x is classical. Moreover all eigenvectors associated to x are classical, that is we have (see the proof of Proposition 3.4 for $\widehat{S}(U^p, L)_{\mathfrak{m}^S}^{\lambda-\mathrm{la}}$):*

$$\mathrm{Hom}_{T_p}\Big(\delta, J_{B_p}(\widehat{S}(U^p, L)_{\mathfrak{m}^S}^{\lambda-\mathrm{la}}[\mathfrak{p}_\rho] \otimes_{k(\mathfrak{p}_\rho)} k(x))\Big) \xrightarrow{\sim} \mathrm{Hom}_{T_p}\Big(\delta, J_{B_p}(\widehat{S}(U^p, L)_{\mathfrak{m}^S}^{\mathrm{an}}[\mathfrak{p}_\rho] \otimes_{k(\mathfrak{p}_\rho)} k(x))\Big).$$

Proof. Keep the notation of Theorem 3.9. By Proposition 2.12, for each $v \in S_p$ there is a sufficiently small open neighbourhood V_v of x_v in $X_{\mathrm{tri}}^{\square}(\overline{\rho}_{\tilde{v}})$ such that the irreducible component $Z_{\mathrm{tri},V_v}(x_v)$ of V_v in (ii) of Corollary 2.5 is defined and satisfies the accumulation property at x_v (Definition 2.11).

Seeing x in $X_p(\overline{\rho})$ via the closed embedding $Y(U^p,\overline{\rho}) \hookrightarrow X_p(\overline{\rho})$, by (3.12) there exist irreducible components $\mathfrak{X}^p$ of $\mathfrak{X}_{\overline{\rho}^p}$ and $Z = \prod_{v\in S_p} Z_v$ of $X_{\mathrm{tri}}^{\square}(\overline{\rho}_p) = \prod_{v\in S_p} X_{\mathrm{tri}}^{\square}(\overline{\rho}_{\tilde{v}})$ such that:

$$x \in \mathfrak{X}^p \times Z \times \mathbb{U}^g \subseteq \mathfrak{X}^p \times X_{\mathrm{tri}}^{\mathfrak{X}^p-\mathrm{aut}}(\overline{\rho}_p) \times \mathbb{U}^g \overset{(3.12)}{\subseteq} X_p(\overline{\rho}).$$

Then it follows from Proposition 3.10 and Remark 3.11 that Z_v satisfies the accumulation property at x_v for all $v \in S_p$. Let $Y_v \subseteq Z_v \cap V_v$ be a nonempty union of irreducible components of V_v, then $X_v := Y_v \cup Z_{\mathrm{tri},V_v}(x_v)$ satisfies the accumulation property at x_v since both Y_v and $Z_{\mathrm{tri},V_v}(x_v)$ do. But X_v is smooth at x_v by the assumption on w_{x_v} and Corollary 2.16 applied with $(X,x) = (X_v,x_v)$, hence there can only be one irreducible component of X_v passing through x_v. We deduce in particular $Z_{\mathrm{tri},V_v}(x_v) \subseteq Y_v \subseteq Z_v$, hence $\prod_{v\in S_p} Z_{\mathrm{tri},V_v}(x_v) \subseteq \prod_{v\in S_p} Z_v \subseteq X_{\mathrm{tri}}^{\mathfrak{X}^p-\mathrm{aut}}(\overline{\rho}_p)$ and x is classical by Theorem 3.9. We also deduce that the only possible $Z = \prod_{v\in S_p} Z_v$ passing through $(x_v)_{v\in S_p}$ is smooth at $(x_v)_{v\in S_p}$, hence that $X_{\mathrm{tri}}^{\mathfrak{X}^p-\mathrm{aut}}(\overline{\rho}_p)$ is smooth at $(x_v)_{v\in S_p}$.

Let us now prove the last statement. From the injection:

$$\mathrm{Hom}_{T_p}\Big(\delta, J_{B_p}(S(U^p,L)^{\lambda-\mathrm{la}}_{\mathfrak{m}^S}[\mathfrak{p}_\rho] \otimes_{k(\mathfrak{p}_\rho)} k(x))\Big) \hookrightarrow \mathrm{Hom}_{T_p}\Big(\delta, J_{B_p}(\widehat{S}(U^p,L)^{\mathrm{an}}_{\mathfrak{m}^S}[\mathfrak{p}_\rho] \otimes_{k(\mathfrak{p}_\rho)} k(x))\Big)$$

it is enough to prove that these two $k(x)$-vector spaces have the same (finite) dimension. Recall from [13, §3.2] that for any $x' = (y',\delta') \in X_p(\overline{\rho})$ we have an isomorphism of $k(x')$-vector spaces:

$$\mathrm{Hom}_{T_p}\Big(\delta', J_{B_p}(\Pi_\infty^{R_\infty-\mathrm{an}}[\mathfrak{p}_{y'}] \otimes_{k(\mathfrak{p}_{y'})} k(x'))\Big) \cong \mathcal{M}_\infty \otimes_{\mathcal{O}_{X_p(\overline{\rho})}} k(x'). \tag{3.14}$$

If moreover $x' = (\rho',\delta') \in Y(U^p,\overline{\rho}) \hookrightarrow X_p(\overline{\rho})$ we have $\widehat{S}(U^p,L)_{\mathfrak{m}^S}[\mathfrak{p}_{\rho'}] = \Pi_\infty[\mathfrak{p}_{y'}]$, hence an isomorphism of $k(x')$-vector spaces:

(3.15)

$$\mathrm{Hom}_{T_p}\Big(\delta', J_{B_p}(\widehat{S}(U^p,L)^{\mathrm{an}}_{\mathfrak{m}^S}[\mathfrak{p}_{\rho'}] \otimes_{k(\mathfrak{p}_{\rho'})} k(x'))\Big) \simeq \mathrm{Hom}_{T_p}\Big(\delta', J_{B_p}(\Pi_\infty^{R_\infty-\mathrm{an}}[\mathfrak{p}_{y'}] \otimes_{k(\mathfrak{p}_{y'})} k(x'))\Big).$$

We first claim that x is a smooth point of $X_p(\overline{\rho})$. Indeed, by what we proved above, it is enough to show that its component $y^p = (y_v)_{v\in S\backslash S_p}$ in $\mathfrak{X}_{\overline{\rho}^p}$ is a smooth point. As x is classical, by Proposition 3.4 (in particular the end of the proof) it corresponds to an automorphic representation π of $G(\mathbb{A}_{F^+})$ with cuspidal strong base change Π to $\mathrm{GL}_n(\mathbb{A}_F)$. It then follows from [14, Th.1.2] that Π is tempered, in particular generic, at all finite places of F. Then [8, Lem.1.3.2(1)] implies that y_v for $v \in S\backslash S_p$ is a smooth point of $(\mathrm{Spf}\, R^{\square}_{\overline{\rho}_{\tilde{v}}})^{\mathrm{rig}}$. As $\mathcal{M}_\infty$ is Cohen-Macaulay (Lemma 3.8) and x is smooth on $X_p(\overline{\rho})$, we conclude from [26, Cor.17.3.5(i)] that $\mathcal{M}_\infty$ is actually locally free at x. Consequently there exists an open affinoid neighbourhood of x in $X_p(\overline{\rho})$ on which the dimension of the fibers of $\mathcal{M}_\infty$ is constant. Intersecting this neighbourhood with $Y(U^p,\overline{\rho})$ and using (3.14) and (3.15), we obtain an open affinoid neighbourhood V_x of x in $Y(U^p,\overline{\rho})$ on which $\dim_{k(x')} \mathrm{Hom}_{T_p}(\delta', J_{B_p}(\widehat{S}(U^p,L)^{\mathrm{an}}_{\mathfrak{m}^S}[\mathfrak{p}_{\rho'}] \otimes_{k(\mathfrak{p}_{\rho'})} k(x')))$ is constant for $x' = (\rho',\delta') \in V_x$.

Now let $x' \in V_x$ be a very classical point in the sense of [13, Def.3.17] and write $\omega(x') = \delta_{\lambda'}$ with dominant $\lambda' \in \prod_{v\in S_p}(\mathbb{Z}^n)^{\mathrm{Hom}(F_{\tilde{v}},L)}$. It follows from *loc. cit.* and [12, Th.4.3] that we have:

$$\begin{aligned}\mathrm{Hom}_{T_p}\Big(\delta',\ J_{B_p}(\widehat{S}(U^p,L)^{\mathrm{an}}_{\mathfrak{m}^S}[\mathfrak{p}_{\rho'}]\otimes_{k(\mathfrak{p}_{\rho'})}k(x'))\Big) &\cong \mathrm{Hom}_{G_p}\Big(\mathrm{LA}(\delta'),\ \widehat{S}(U^p,L)^{\mathrm{an}}_{\mathfrak{m}^S}[\mathfrak{p}_{\rho'}]\otimes_{k(\mathfrak{p}_{\rho'})}k(x')\Big)\\ &\cong \mathrm{Hom}_{G_p}\Big(\mathrm{LA}(\delta'),\ \widehat{S}(U^p,L)^{\lambda'-\mathrm{la}}_{\mathfrak{m}^S}[\mathfrak{p}_{\rho'}]\otimes_{k(\mathfrak{p}_{\rho'})}k(x')\Big)\\ &\cong \mathrm{Hom}_{T_p}\Big(\delta',\ J_{B_p}(\widehat{S}(U^p,L)^{\lambda'-\mathrm{la}}_{\mathfrak{m}^S}[\mathfrak{p}_{\rho'}]\otimes_{k(\mathfrak{p}_{\rho'})}k(x'))\Big).\end{aligned}$$

From what is proved above, it is thus enough to find a very classical point x' in V_x such that:

$$\begin{aligned}(3.16)\quad \dim_{k(x')}\mathrm{Hom}_{T_p}\Big(\delta', J_{B_p}(\widehat{S}(U^p,L)^{\lambda'-\mathrm{la}}_{\mathfrak{m}^S}[\mathfrak{p}_{\rho'}]\otimes_{k(\mathfrak{p}_{\rho'})}k(x'))\Big) &=\\ \dim_{k(x)}\mathrm{Hom}_{T_p}\Big(\delta, J_{B_p}(\widehat{S}(U^p,L)^{\lambda-\mathrm{la}}_{\mathfrak{m}^S}[\mathfrak{p}_{\rho}]\otimes_{k(\mathfrak{p}_{\rho})}k(x))\Big).\end{aligned}$$

Let $x''=(\rho'',\delta'')\in Y(U^p,\overline{\rho})$ be any classical crystalline strictly dominant point and let $\omega(x'')=\delta_{\lambda''}$. By Proposition 3.4 it corresponds to a unique automorphic representation π'' which moreover has multiplicity 1, hence we have (with the notation of the proof of Proposition 3.4):

$$J_{B_p}\Big(S(U^p,L)^{\lambda''-\mathrm{la}}_{\mathfrak{m}^S}[\mathfrak{p}_{\rho''}]\otimes_{k(\mathfrak{p}_{\rho''})}\overline{\mathbb{Q}}_p\Big)\simeq J_{B_p}\Big(L(\lambda'')\otimes_L\bigotimes_{v\in S_p}\pi''_v\Big)\otimes_{\overline{\mathbb{Q}}}(\pi''^{p}_f)^{U^p}\otimes_{\overline{\mathbb{Q}},j_p}\overline{\mathbb{Q}}_p.$$

From the definition of S together with [22, Prop.4.3.6] and property (iv) in Proposition 3.4, it then easily follows that:

$$(3.17)\quad \dim_{k(x'')}\mathrm{Hom}_{T_p}\Big(\delta'', J_{B_p}(\widehat{S}(U^p,L)^{\lambda''-\mathrm{la}}_{\mathfrak{m}^S}[\mathfrak{p}_{\rho''}]\otimes_{k(\mathfrak{p}_{\rho''})}k(x''))\Big)=\dim_{\overline{\mathbb{Q}}}\Big(\bigotimes_{v\in S\setminus S_p}\pi''^{U_v}_v\Big).$$

Let Z be the union of x and of the very classical points in V_x, by [13, Thm.3.19] this set Z accumulates at x. By [14, Th.1.2], we can apply [16, Lem.4.5(ii)] to the intersection of Z with one irreducible component of V_x, and obtain that, for $v\nmid p$, the value $\dim_{\overline{\mathbb{Q}}}\pi''^{U_v}_v$ is constant on this intersection. In particular $\dim_{\overline{\mathbb{Q}}}(\otimes_{v\in S\setminus S_p}\pi''^{U_v}_v)$ is also constant on this intersection, which finishes the proof by (3.17) and (3.16). □

Remark 3.13. (i) Keeping the notation of Theorem 3.9, we actually think that there should always be a *unique* irreducible component Z of $X^{\square}_{\mathrm{tri}}(\overline{\rho}_p)$ (smooth or not) passing through the image of x in $X^{\square}_{\mathrm{tri}}(\overline{\rho}_p)$, or equivalently that for each $v\in S_p$ there should be a unique irreducible component of $X^{\square}_{\mathrm{tri}}(\overline{\rho}_{\tilde{v}})$ passing through x_v. If this is so, then x is classical. Indeed, in that case there is an irreducible component $\mathfrak{X}^p$ of $\mathfrak{X}_{\overline{\rho}^p}$ such that $x\in\mathfrak{X}^p\times Z\times\mathbb{U}^g=\mathfrak{X}^p\times X^{\mathfrak{X}^p-\mathrm{aut}}_{\mathrm{tri}}(\overline{\rho}_p)\times\mathbb{U}^g$. In particular, for a sufficiently small open neighbourhood V_v of x_v in $X^{\square}_{\mathrm{tri}}(\overline{\rho}_{\tilde{v}})$, we have $\prod_{v\in S_p}V_v\subseteq Z=X^{\mathfrak{X}^p-\mathrm{aut}}_{\mathrm{tri}}(\overline{\rho}_p)$ and we see that the assumption in Theorem 3.9 is *a fortiori* satisfied.
(ii) Let us recall the various global hypothesis underlying the statements of Theorem 3.9 and Corollary 3.12: $p>2$, G is quasi-split at all finite places of F^+, F/F^+ is unramified, U_v is hyperspecial if v is inert in F and $\overline{\rho}(\mathcal{G}_{F(\zeta_p)})$ is adequate.

4. On the local geometry of the trianguline variety

This section is entirely local and devoted to the proof of Theorem 2.15 above giving an upper bound on some local tangent spaces. We use the notation of §2.

4.1. **Tangent spaces and local triangulations.** We first recall some of the results of [4] and [41]. Then we use them to prove a technical statement on the image of $T_{X,x}$ in $\mathrm{Ext}^1_{\mathcal{G}_K}(r,r)$ (Corollary 4.3).

We fix a point $x = (r,\delta) = (r,\delta_1,\dots,\delta_n) \in X^{\square}_{\mathrm{tri}}(\overline{r})$ which is crystalline strictly dominant very regular and a union X of irreducible components of an open subset of $X^{\square}_{\mathrm{tri}}(\overline{r})$ such that X satisfies the accumulation property at x (Definition 2.11). It obviously doesn't change the tangent space $T_{X,x}$ of X at x if we replace X by the union of its irreducible components that contain x, hence we may (and do) assume that x belongs to each irreducible component of X. Taking a look at [4, §§5.1,6.1], it is easy to see from the properties of $X^{\square}_{\mathrm{tri}}(\overline{r})$ and from Definition 2.11 (together with the discussion that follows) that one can apply all the results of [4, §7] at X and the point x (called the "center" and denoted by x_0 in *loc. cit.*). We let $w_x = (w_{x,\tau})_{\tau\colon K\hookrightarrow L} \in \prod_{\tau:K\hookrightarrow L}\mathcal{S}_n$ be the Weyl group element associated to x (§2.3).

Recall that $D_{\mathrm{rig}}(r)$ is the étale (φ,Γ_K)-module over $\mathcal{R}_{k(x),K} = k(x)\otimes_{\mathbb{Q}_p}\mathcal{R}_K$ associated to r. We denote by $\mathrm{Ext}^1_{\mathcal{G}_K}(r,r)$ the usual $k(x)$-vector space of $\mathcal{G}_K$-extensions $0\to r\to * \to r\to 0$ and note that $\mathrm{Ext}^1_{\mathcal{G}_K}(r,r)\cong \mathrm{Ext}^1_{(\varphi,\Gamma_K)}(D_{\mathrm{rig}}(r),D_{\mathrm{rig}}(r))$ where the right hand side denotes the extension in the category of (φ,Γ_K)-module over $\mathcal{R}_{k(x),K}$ (see [2, Prop.5.2.6] for $K=\mathbb{Q}_p$, the proof for any K is analogous). We write $\omega(x) = \delta_{\mathbf{k}}$ for $\mathbf{k} = (k_{\tau,i})_{1\le i\le n,\tau\colon K\hookrightarrow L} \in (\mathbb{Z}^n)^{\mathrm{Hom}(K,L)}$. Let $\vec{v}\in T_{X,x}$, seeing the image of $\vec{v}$ in $\mathrm{Ext}^1_{\mathcal{G}_K}(r,r)$ as a $k(x)[\varepsilon]/(\varepsilon^2)$-valued representation of $\mathcal{G}_K$, we can write its Sen weights as $(k_{\tau,i}+\varepsilon d_{\tau,i,\vec{v}})_{1\le i\le n,\tau\colon K\hookrightarrow L}$ for some $d_{\tau,i,\vec{v}}\in k(x)$. The tangent space $T_{\mathcal{W}^n_L,\omega(x)}$ to $\mathcal{W}^n_L$ at $\omega(x)$ is isomorphic to $k(x)^{[K:\mathbb{Q}_p]n}$ and the $k(x)$-linear map of tangent spaces $d\omega : T_{X,x}\longrightarrow T_{\mathcal{W}^n_L,\omega(x)}$ induced by the weight map $\omega|_X$ sends $\vec{v}$ to the tuple $(d_{\tau,i,\vec{v}})_{1\le i\le n,\tau\colon K\hookrightarrow L}$. The following theorem is a direct application of [4, Th.7.1].

Theorem 4.1. *For any $\vec{v}\in T_{X,x}$, we have $d_{\tau,i,\vec{v}} = d_{\tau,w_{x,\tau}^{-1}(i),\vec{v}}$ for $1\le i\le n$ and $\tau : K\hookrightarrow L$.*

Let $\vec{v}\in T_{X,x}$, we can see $\vec{v}$ as a $k(x)[\varepsilon]/(\varepsilon^2)$-valued point of X, and the composition:

$$\mathrm{Sp}\, k(x)[\varepsilon]/(\varepsilon^2) \xrightarrow{\vec{v}} X \hookrightarrow X^{\square}_{\mathrm{tri}}(\overline{r}) \longrightarrow \mathcal{T}^n_L$$

gives rise to continuous characters $\delta_{i,\vec{v}} : K^\times \to (k(x)[\varepsilon]/(\varepsilon^2))^\times$ for $1\le i\le n$. The following theorem is an easy consequence of [41, Prop.4.1.13] (see also the proof of [4, Th.7.1]).

Theorem 4.2. *For any $\vec{v}\in T_{X,x}$ and $1\le i\le n$ we have an injection of (φ,Γ_K)-modules over $\mathcal{R}_{k(x)[\varepsilon]/(\varepsilon^2),K} = k(x)[\varepsilon]/(\varepsilon^2)\otimes_{\mathbb{Q}_p}\mathcal{R}_K$:*

$$\mathcal{R}_{k(x)[\varepsilon]/(\varepsilon^2),K}\Big(\delta_{1,\vec{v}}\delta_{2,\vec{v}}\cdots\delta_{i,\vec{v}}\Big)\hookrightarrow D_{\mathrm{rig}}(\wedge^i_{k(x)[\varepsilon]/(\varepsilon^2)} r_{\vec{v}}) \cong \wedge^i_{\mathcal{R}_{k(x)[\varepsilon]/(\varepsilon^2),K}} D_{\mathrm{rig}}(r_{\vec{v}})$$

where the left hand side is the rank one (φ,Γ_K)-module defined by the character $\delta_{1,\vec{v}}\delta_{2,\vec{v}}\cdots\delta_{i,\vec{v}}$ ([34, Cons.6.2.4]) and where $r_{\vec{v}}$ is the $k(x)[\varepsilon]/(\varepsilon^2)$-valued representation of $\mathcal{G}_K$ associated to $\vec{v}$.

From [2, Prop.2.4.1] (which readily extends to $K\neq\mathbb{Q}_p$) or arguing as in §2.2, the (φ,Γ_K)-module $D_{\mathrm{rig}}(r)$ has a triangulation $\mathrm{Fil}_\bullet$ for $\bullet\in\{1,\cdots,n\}$, the graded pieces being:

$$\mathcal{R}_{k(x),K}\Big(z^{\mathbf{k}_{w_x^{-1}(1)}}\mathrm{unr}(\varphi_1)\Big),\dots,\mathcal{R}_{k(x),K}\Big(z^{\mathbf{k}_{w_x^{-1}(n)}}\mathrm{unr}(\varphi_n)\Big) \tag{4.1}$$

where $\mathbf{k}_{w_x^{-1}(i)} := (k_{\tau,w_{x,\tau}^{-1}(i)})_{\tau\colon K\hookrightarrow L}$ (see (2.1) for $z^{\mathbf{k}_j}$). Note that we have:

$$\delta_i(z) = z^{\mathbf{k}_i-\mathbf{k}_{w_x^{-1}(i)}}(z^{\mathbf{k}_{w_x^{-1}(i)}}\mathrm{unr}(\varphi_i)). \tag{4.2}$$

For $1 \leq i \leq n$ we let $D_{\rm rig}(r)^{\leq i} := {\rm Fil}_i \subseteq D_{\rm rig}(r)$, and we set $D_{\rm rig}(r)^{\leq 0} := 0$. We thus have for $1 \leq i \leq n$:

$$\mathrm{gr}_i D_{\rm rig}(r) := D_{\rm rig}(r)^{\leq i}/D_{\rm rig}(r)^{\leq i-1} = \mathcal{R}_{k(x),K}\Big(z^{\mathbf{k}_{w_x^{-1}(i)}}\mathrm{unr}(\varphi_i)\Big).$$

For $\tau : K \hookrightarrow L$ we fix a Lubin-Tate element $t_\tau \in \mathcal{R}_{L,K}$ as in [34, Not.6.2.7] (recall that the ideal $t_\tau\mathcal{R}_{L,K}$ is uniquely determined). If $\mathbf{k} := (k_\tau)_{\tau: K\hookrightarrow L} \in \mathbb{Z}_{\geq 0}^{\mathrm{Hom}(K,L)}$, we let $t^{\mathbf{k}} := \prod_{\tau: K\hookrightarrow L} t_\tau^{k_\tau}$. We set for $1 \leq i \leq n$:

$$\Sigma_i(\mathbf{k}, w_x) := \sum_{j=1}^{i}(\mathbf{k}_j - \mathbf{k}_{w_x^{-1}(j)}) \in \mathbb{Z}_{\geq 0}^{\mathrm{Hom}(K,L)}$$

(where nonnegativity comes from $k_{\tau,i} \geq k_{\tau,i+1}$ for every i, τ) and we can thus define $t^{\Sigma_i(\mathbf{k},w_x)} \in \mathcal{R}_{L,K}$. In particular we deduce from (4.2) (and the properties of the t_τ):

$$\mathcal{R}_{k(x),K}(\delta_1\cdots\delta_i) \cong t^{\Sigma_i(\mathbf{k},w_x)} \wedge^i_{\mathcal{R}_{k(x),K}} D_{\rm rig}(r)^{\leq i} \hookrightarrow \wedge^i_{\mathcal{R}_{k(x),K}} D_{\rm rig}(r). \tag{4.3}$$

We consider for $1 \leq i \leq n$ the cartesian square (which defines V_i):

$$\begin{array}{ccc}
\mathrm{Ext}^1_{(\varphi,\Gamma_K)}\Big(D_{\rm rig}(r), D_{\rm rig}(r)\Big) & \longrightarrow & \mathrm{Ext}^1_{(\varphi,\Gamma_K)}\Big(t^{\Sigma_i(\mathbf{k},w_x)}D_{\rm rig}(r)^{\leq i}, D_{\rm rig}(r)\Big) \\
\uparrow & & \uparrow \\
V_i & \longrightarrow & \mathrm{Ext}^1_{(\varphi,\Gamma_K)}\Big(t^{\Sigma_i(\mathbf{k},w_x)}D_{\rm rig}(r)^{\leq i}, D_{\rm rig}(r)^{\leq i}\Big)
\end{array}$$

where the first horizontal map is the restriction map and where the injection on the right follows from the very regularity assumption. Indeed, its kernel is $\mathrm{Hom}_{(\varphi,\Gamma_K)}(t^{\Sigma_i(\mathbf{k},w_x)}D_{\rm rig}(r)^{\leq i}, D_{\rm rig}(r)/D_{\rm rig}(r)^{\leq i})$ which is 0 by dévissage since $\mathrm{Hom}_{(\varphi,\Gamma_K)}(z^{\mathbf{k}}\mathrm{unr}(\varphi_j), z^{\mathbf{k}'}\mathrm{unr}(\varphi_{j'})) = 0$ for $\mathbf{k}, \mathbf{k}' \in \mathbb{Z}_{\geq 0}^{\mathrm{Hom}(K,L)}$ and $j \neq j'$ using Definition 2.13 (in a similar way Definition 2.13 will imply many Hom-spaces to vanish in what follows). Equivalently we have:

(4.4)
$$V_i \cong \ker\bigg(\mathrm{Ext}^1_{(\varphi,\Gamma_K)}\Big(D_{\rm rig}(r), D_{\rm rig}(r)\Big) \longrightarrow \mathrm{Ext}^1_{(\varphi,\Gamma_K)}\Big(t^{\Sigma_i(\mathbf{k},w_x)}D_{\rm rig}(r)^{\leq i}, D_{\rm rig}(r)/D_{\rm rig}(r)^{\leq i}\Big)\bigg)$$

where the map is defined by pushforward along $D_{\rm rig}(r) \twoheadrightarrow D_{\rm rig}(r)/D_{\rm rig}(r)^{\leq i}$ and pullback along $t^{\Sigma_i(\mathbf{k},w_x)}D_{\rm rig}(r)^{\leq i} \hookrightarrow D_{\rm rig}(r)$.

Corollary 4.3. *The image of any $\vec{v} \in T_{X,x}$ in $\mathrm{Ext}^1_{\mathcal{G}_K}(r,r) \cong \mathrm{Ext}^1_{(\varphi,\Gamma_K)}(D_{\rm rig}(r), D_{\rm rig}(r))$ is in $V_1 \cap \cdots \cap V_{n-1}$ (where the intersection is within $\mathrm{Ext}^1_{(\varphi,\Gamma_K)}(D_{\rm rig}(r), D_{\rm rig}(r))$).*

Proof. Note that $V_1 \cap V_2 \cap \cdots \cap V_n = V_1 \cap V_2 \cap \cdots \cap V_{n-1}$. Let $\vec{v} \in T_{X,x}$, $r_{\vec{v}}$ the associated $k(x)[\varepsilon]/(\varepsilon^2)$-deformation and see $D_{\rm rig}(r_{\vec{v}})$ as an element of $\mathrm{Ext}^1_{(\varphi,\Gamma_K)}(D_{\rm rig}(r), D_{\rm rig}(r))$. We have to prove that the image of $D_{\rm rig}(r_{\vec{v}})$ in $\mathrm{Ext}^1_{(\varphi,\Gamma_K)}(t^{\Sigma_i(\mathbf{k},w_x)}D_{\rm rig}(r)^{\leq i}, D_{\rm rig}(r)/D_{\rm rig}(r)^{\leq i})$ is zero for any $1 \leq i \leq n$ (see (4.4)). The proof is by induction on $i \geq 1$.

For the case $i = 1$, we deduce from Theorem 4.2 (applied with $i = 1$) a commutative diagram:

$$\begin{array}{ccccccccc} 0 & \longrightarrow & D_{\mathrm{rig}}(r) & \longrightarrow & D_{\mathrm{rig}}(r_{\vec{v}}) & \longrightarrow & D_{\mathrm{rig}}(r) & \longrightarrow & 0 \\ & & \uparrow & & \uparrow & & \uparrow & & \\ 0 & \longrightarrow & \mathcal{R}_{k(x),K}(\delta_1) & \longrightarrow & \mathcal{R}_{k(x)[\varepsilon]/(\varepsilon^2),K}(\delta_{1,\vec{v}}) & \longrightarrow & \mathcal{R}_{k(x),K}(\delta_1) & \longrightarrow & 0 \end{array}$$

where the left and right vertical injections are unique by Definition 2.13. Let $D_{\mathrm{rig}}(r_{\vec{v}})_1$ be the inverse image of (the right hand side) $\mathcal{R}_{k(x),K}(\delta_1)$ in $D_{\mathrm{rig}}(r_{\vec{v}})$, then an obvious diagram chase shows that there is a canonical isomorphism:

$$D_{\mathrm{rig}}(r) \oplus_{\mathcal{R}_{k(x),K}(\delta_1)} \mathcal{R}_{k(x)[\varepsilon]/(\varepsilon^2),K}(\delta_{1,\vec{v}}) \xrightarrow{\sim} D_{\mathrm{rig}}(r_{\vec{v}})_1$$

which exactly means that the image of $D_{\mathrm{rig}}(r_{\vec{v}})$ in $\mathrm{Ext}^1_{(\varphi,\Gamma_K)}(t^{\Sigma_1(\mathbf{k},w_x)}D_{\mathrm{rig}}(r)^{\leq 1}, D_{\mathrm{rig}}(r)) = \mathrm{Ext}^1_{(\varphi,\Gamma_K)}(\mathcal{R}_{k(x),K}(\delta_1), D_{\mathrm{rig}}(r))$ comes from $\mathrm{Ext}^1_{(\varphi,\Gamma_K)}(\mathcal{R}_{k(x),K}(\delta_1), \mathcal{R}_{k(x),K}(\delta_1))$, and is thus 0 in $\mathrm{Ext}^1_{(\varphi,\Gamma_K)}(\mathcal{R}_{k(x),K}(\delta_1), D_{\mathrm{rig}}(r)/\mathcal{R}_{k(x),K}(\delta_1))$ hence in $\mathrm{Ext}^1_{(\varphi,\Gamma_K)}(\mathcal{R}_{k(x),K}(\delta_1), D_{\mathrm{rig}}(r)/D_{\mathrm{rig}}(r)^{\leq 1})$.

Assume $i \geq 2$ and that the image of $D_{\mathrm{rig}}(r_{\vec{v}})$ in:

$$\mathrm{Ext}^1_{(\varphi,\Gamma_K)}\Big(t^{\Sigma_{i-1}(\mathbf{k},w_x)}D_{\mathrm{rig}}(r)^{\leq i-1}, D_{\mathrm{rig}}(r)/D_{\mathrm{rig}}(r)^{\leq i-1}\Big)$$

is zero. Then by Corollary 4.10 below the image of $D_{\mathrm{rig}}(r_{\vec{v}})$ in:

$$\mathrm{Ext}^1_{(\varphi,\Gamma_K)}\Big(t^{\Sigma_i(\mathbf{k},w_x)}D_{\mathrm{rig}}(r)^{\leq i-1}, D_{\mathrm{rig}}(r)/D_{\mathrm{rig}}(r)^{\leq i}\Big)$$

is also zero. From the exact sequence:

$$\begin{aligned} 0 \to \mathrm{Ext}^1_{(\varphi,\Gamma_K)}\Big(t^{\Sigma_i(\mathbf{k},w_x)}\mathrm{gr}_i D_{\mathrm{rig}}(r), D_{\mathrm{rig}}(r)/D_{\mathrm{rig}}(r)^{\leq i}\Big) \to \\ \mathrm{Ext}^1_{(\varphi,\Gamma_K)}\Big(t^{\Sigma_i(\mathbf{k},w_x)}D_{\mathrm{rig}}(r)^{\leq i}, D_{\mathrm{rig}}(r)/D_{\mathrm{rig}}(r)^{\leq i}\Big) \to \\ \mathrm{Ext}^1_{(\varphi,\Gamma_K)}\Big(t^{\Sigma_i(\mathbf{k},w_x)}D_{\mathrm{rig}}(r)^{\leq i-1}, D_{\mathrm{rig}}(r)/D_{\mathrm{rig}}(r)^{\leq i}\Big) \end{aligned}$$

(where the injectivity on the left follows from Definition 2.13), we see that the image of $D_{\mathrm{rig}}(r_{\vec{v}})$ in $\mathrm{Ext}^1_{(\varphi,\Gamma_K)}\Big(t^{\Sigma_i(\mathbf{k},w_x)}D_{\mathrm{rig}}(r)^{\leq i}, D_{\mathrm{rig}}(r)/D_{\mathrm{rig}}(r)^{\leq i}\Big)$ comes from a unique extension:

$$D_{\mathrm{rig}}(r_{\vec{v}})^{(i)} \in \mathrm{Ext}^1_{(\varphi,\Gamma_K)}\Big(t^{\Sigma_i(\mathbf{k},w_x)}\mathrm{gr}_i D_{\mathrm{rig}}(r), D_{\mathrm{rig}}(r)/D_{\mathrm{rig}}(r)^{\leq i}\Big).$$

We thus have to prove that $D_{\mathrm{rig}}(r_{\vec{v}})^{(i)} = 0$.

The twist by the rank one (φ, Γ_K)-module $\wedge^{i-1}_{\mathcal{R}_{k(x),K}} D_{\mathrm{rig}}(r)^{\leq i-1}$ is easily seen (by elementary linear algebra) to induce an isomorphism:

$$\begin{aligned} (4.5) \quad \mathrm{Ext}^1_{(\varphi,\Gamma_K)}\Big(t^{\Sigma_i(\mathbf{k},w_x)}\mathrm{gr}_i D_{\mathrm{rig}}(r), D_{\mathrm{rig}}(r)/D_{\mathrm{rig}}(r)^{\leq i}\Big) \xrightarrow{\sim} \\ \mathrm{Ext}^1_{(\varphi,\Gamma_K)}\Big(t^{\Sigma_i(\mathbf{k},w_x)} \wedge^i D_{\mathrm{rig}}(r)^{\leq i}, (D_{\mathrm{rig}}(r)/D_{\mathrm{rig}}(r)^{\leq i}) \wedge (\wedge^{i-1}D_{\mathrm{rig}}(r)^{\leq i-1})\Big) \end{aligned}$$

where we write $\wedge^{\cdot} D_{\mathrm{rig}}(r)$ for $\wedge^{\cdot}_{\mathcal{R}_{k(x),K}} D_{\mathrm{rig}}(r)$ and where $(D_{\mathrm{rig}}(r)/D_{\mathrm{rig}}(r)^{\leq i}) \wedge (\wedge^{i-1}D_{\mathrm{rig}}(r)^{\leq i-1})$ stands for the quotient:

$$\begin{aligned} \Big(D_{\mathrm{rig}}(r) \wedge (\wedge^{i-1}D_{\mathrm{rig}}(r)^{\leq i-1})\Big)/\Big(D_{\mathrm{rig}}(r)^{\leq i} \wedge (\wedge^{i-1}D_{\mathrm{rig}}(r)^{\leq i-1})\Big) \cong \\ \Big(D_{\mathrm{rig}}(r) \wedge (\wedge^{i-1}D_{\mathrm{rig}}(r)^{\leq i-1})\Big)/ \wedge^i D_{\mathrm{rig}}(r)^{\leq i}. \end{aligned}$$

(here, $D_{\mathrm{rig}}(r)\wedge(\wedge^{i-1}D_{\mathrm{rig}}(r)^{\le i-1})$ and $D_{\mathrm{rig}}(r)^{\le i}\wedge(\wedge^{i-1}D_{\mathrm{rig}}(r)^{\le i-1})$ are seen inside $\wedge^i D_{\mathrm{rig}}(r)$). Moreover the injective map $\wedge^{i-1}D_{\mathrm{rig}}(r)^{\le i-1}\hookrightarrow \wedge^{i-1}D_{\mathrm{rig}}(r)^{\le i}$ still induces an injection (using Definition 2.13):

$$\mathrm{Ext}^1_{(\varphi,\Gamma_K)}\Big(t^{\Sigma_i(\mathbf{k},w_x)}\wedge^i D_{\mathrm{rig}}(r)^{\le i},(D_{\mathrm{rig}}(r)/D_{\mathrm{rig}}(r)^{\le i})\wedge(\wedge^{i-1}D_{\mathrm{rig}}(r)^{\le i-1})\Big)\hookrightarrow \mathrm{Ext}^1_{(\varphi,\Gamma_K)}\Big(t^{\Sigma_i(\mathbf{k},w_x)}\wedge^i D_{\mathrm{rig}}(r)^{\le i},(D_{\mathrm{rig}}(r)/D_{\mathrm{rig}}(r)^{\le i})\wedge(\wedge^{i-1}D_{\mathrm{rig}}(r)^{\le i})\Big).$$

Denote by:

$$\widetilde{D}_{\mathrm{rig}}(r_{\vec{v}})^{(i)} \in \mathrm{Ext}^1_{(\varphi,\Gamma_K)}\Big(t^{\Sigma_i(\mathbf{k},w_x)}\wedge^i D_{\mathrm{rig}}(r)^{\le i},(D_{\mathrm{rig}}(r)/D_{\mathrm{rig}}(r)^{\le i})\wedge(\wedge^{i-1}D_{\mathrm{rig}}(r)^{\le i})\Big) \tag{4.6}$$

the image of $D_{\mathrm{rig}}(r_{\vec{v}})^{(i)}$ (using the isomorphism (4.5)). It is thus equivalent to prove that $\widetilde{D}_{\mathrm{rig}}(r_{\vec{v}})^{(i)}=0$. Note that:

$$(D_{\mathrm{rig}}(r)/D_{\mathrm{rig}}(r)^{\le i})\wedge(\wedge^{i-1}D_{\mathrm{rig}}(r)^{\le i}) \cong \Big(D_{\mathrm{rig}}(r)\wedge(\wedge^{i-1}D_{\mathrm{rig}}(r)^{\le i})\Big)/\wedge^i D_{\mathrm{rig}}(r)^{\le i}. \tag{4.7}$$

For $1\le i\le n$, we have a $k(x)$-linear map $\mathrm{Ext}^1_{\mathcal{G}_K}(r,r)\to \mathrm{Ext}^1_{\mathcal{G}_K}(\wedge^i_{k(x)}r,\wedge^i_{k(x)}r)$ defined by mapping a $k(x)[\varepsilon]/(\varepsilon^2)$-valued representation of $\mathcal{G}_K$ to its i-th exterior power over $k(x)[\varepsilon]/(\varepsilon^2)$. This induces a $k(x)$-linear map:

$$\mathrm{Ext}^1_{(\varphi,\Gamma_K)}(D_{\mathrm{rig}}(r),D_{\mathrm{rig}}(r))\longrightarrow \mathrm{Ext}^1_{(\varphi,\Gamma_K)}\Big(\wedge^i D_{\mathrm{rig}}(r),\wedge^i D_{\mathrm{rig}}(r)\Big).$$

Let $D_{\mathrm{rig}}(\wedge^i r_{\vec{v}})\in \mathrm{Ext}^1_{(\varphi,\Gamma_K)}(\wedge^i D_{\mathrm{rig}}(r),\wedge^i D_{\mathrm{rig}}(r))$ be the image of $D_{\mathrm{rig}}(r_{\vec{v}})$. The pull-back along $\wedge^i D_{\mathrm{rig}}(r)^{\le i}\hookrightarrow \wedge^i D_{\mathrm{rig}}(r)$ sends $D_{\mathrm{rig}}(\wedge^i r_{\vec{v}})$ to an element in $\mathrm{Ext}^1_{(\varphi,\Gamma_K)}(\wedge^i D_{\mathrm{rig}}(r)^{\le i},\wedge^i D_{\mathrm{rig}}(r))$. Elementary linear algebra (recall $\varepsilon^2=0$!) shows this element in fact belongs to:

$$\mathrm{Ext}^1_{(\varphi,\Gamma_K)}\Big(\wedge^i D_{\mathrm{rig}}(r)^{\le i},D_{\mathrm{rig}}(r)\wedge(\wedge^{i-1}D_{\mathrm{rig}}(r)^{\le i})\Big)$$

(which embeds into $\mathrm{Ext}^1_{(\varphi,\Gamma_K)}(\wedge^i D_{\mathrm{rig}}(r)^{\le i},\wedge^i D_{\mathrm{rig}}(r))$ again by Definition 2.13). The pushforward along:

$$D_{\mathrm{rig}}(r)\wedge(\wedge^{i-1}D_{\mathrm{rig}}(r)^{\le i})\twoheadrightarrow (D_{\mathrm{rig}}(r)\wedge(\wedge^{i-1}D_{\mathrm{rig}}(r)^{\le i}))/\wedge^i D_{\mathrm{rig}}(r)^{\le i}$$

now gives by (4.7) an element in:

$$\mathrm{Ext}^1_{(\varphi,\Gamma_K)}\Big(\wedge^i D_{\mathrm{rig}}(r)^{\le i},(D_{\mathrm{rig}}(r)/D_{\mathrm{rig}}(r)^{\le i})\wedge(\wedge^{i-1}D_{\mathrm{rig}}(r)^{\le i})\Big)$$

and further pull-back along $t^{\Sigma_i(\mathbf{k},w_x)}\wedge^i D_{\mathrm{rig}}(r)^{\le i}\hookrightarrow \wedge^i D_{\mathrm{rig}}(r)^{\le i}$ finally gives an element:

$$\widetilde{D}_{\mathrm{rig}}(\wedge^i r_{\vec{v}}) \in \mathrm{Ext}^1_{(\varphi,\Gamma_K)}\Big(t^{\Sigma_i(\mathbf{k},w_x)}\wedge^i D_{\mathrm{rig}}(r)^{\le i},(D_{\mathrm{rig}}(r)/D_{\mathrm{rig}}(r)^{\le i})\wedge(\wedge^{i-1}D_{\mathrm{rig}}(r)^{\le i})\Big). \tag{4.8}$$

Now, again manipulations of elementary linear algebra show we recover the element $\widetilde{D}_{\mathrm{rig}}(r_{\vec{v}})^{(i)}$ of (4.6), that is, we have $\widetilde{D}_{\mathrm{rig}}(\wedge^i r_{\vec{v}})=\widetilde{D}_{\mathrm{rig}}(r_{\vec{v}})^{(i)}$.

But we know from Theorem 4.2 (using (4.3) and Definition 2.13) that the image of $D_{\mathrm{rig}}(\wedge^i r_{\vec{v}})$ (by pullback) in $\mathrm{Ext}^1_{(\varphi,\Gamma_K)}\Big(t^{\Sigma_i(\mathbf{k},w_x)}\wedge^i D_{\mathrm{rig}}(r)^{\le i},\wedge^i D_{\mathrm{rig}}(r)\Big)$ actually sits in:

$$\mathrm{Ext}^1_{(\varphi,\Gamma_K)}\Big(t^{\Sigma_i(\mathbf{k},w_x)}\wedge^i D_{\mathrm{rig}}(r)^{\le i},\wedge^i D_{\mathrm{rig}}(r)^{\le i}\Big)$$

(in fact even in the image of $\mathrm{Ext}^1_{(\varphi,\Gamma_K)}(t^{\Sigma_i(\mathbf{k},w_x)}\wedge^i D_{\mathrm{rig}}(r)^{\le i},t^{\Sigma_i(\mathbf{k},w_x)}\wedge^i D_{\mathrm{rig}}(r)^{\le i})$). In particular its image $\widetilde{D}_{\mathrm{rig}}(\wedge^i r_{\vec{v}})$ in:

$$\mathrm{Ext}^1_{(\varphi,\Gamma_K)}\Big(t^{\Sigma_i(\mathbf{k},w_x)}\wedge^i D_{\mathrm{rig}}(r)^{\le i},(D_{\mathrm{rig}}(r)\wedge(\wedge^{i-1}D_{\mathrm{rig}}(r)^{\le i}))/\wedge^i D_{\mathrm{rig}}(r)^{\le i}\Big)$$

must be zero. Since $\widetilde{D}_{\mathrm{rig}}(\wedge^i r_{\vec{v}}) = \widetilde{D}_{\mathrm{rig}}(r_{\vec{v}})^{(i)}$, we obtain $\widetilde{D}_{\mathrm{rig}}(r_{\vec{v}})^{(i)} = 0$. □

4.2. **Calculation of some dimensions.** We compute various dimensions that will be used in the proof of Theorem 2.15 in §4.4. These computations themselves use some technical results of Galois cohomology which are proven in §4.3 below.

We keep the notation of §4.1. Recall that a (φ, Γ_K)-module D over $\mathcal{R}_{k(x),K}$ is called *crystalline* if $D[1/\prod_{\tau\colon K\hookrightarrow L} t_\tau]^{\Gamma_K}$ is free over $K_0 \otimes_{\mathbb{Q}_p} k(x)$ of the same rank as D. If D, D' are two crystalline (φ, Γ_K)-module over $\mathcal{R}_{k(x),K}$, one can define the $k(x)$-subvector space of *crystalline extensions* $\mathrm{Ext}^1_{\mathrm{cris}}(D, D') \subseteq \mathrm{Ext}^1_{(\varphi,\Gamma_K)}(D, D')$. Note that $\mathrm{Ext}^1_{\mathrm{cris}}(\cdot,\cdot)$ respects surjectivities on the right entry (resp. sends injectivities to surjectivities on the left entry) as there is no $\mathrm{Ext}^2_{\mathrm{cris}}$, see [3, Cor.1.4.6].

Lemma 4.4. *For $1 \le i \le \ell \le n$ we have:*

$$\dim_{k(x)} \mathrm{Ext}^1_{(\varphi,\Gamma_K)}\Big(\mathrm{gr}_i D_{\mathrm{rig}}(r)/(t^{\Sigma_\ell(\mathbf{k},w_x)}), D_{\mathrm{rig}}(r)/D_{\mathrm{rig}}(r)^{\le \ell}\Big) = \sum_{\tau\colon K\hookrightarrow L} \Big|\{i_1 \in \{\ell+1,\dots,n\}, w_{x,\tau}^{-1}(i_1) < w_{x,\tau}^{-1}(i)\}\Big|.$$

Proof. It follows from Proposition 4.9 below (applied with $(i,\ell) = (i,i)$ and $(i,\ell) = (i-1,i)$) together with the two exact sequences:

$$0 \to \mathrm{Ext}^1_{(\varphi,\Gamma_K)}\Big(\mathrm{gr}_i D_{\mathrm{rig}}(r)/(t^{\Sigma_i(\mathbf{k},w_x)}), D_{\mathrm{rig}}(r)/D_{\mathrm{rig}}(r)^{\le i}\Big) \to \mathrm{Ext}^1_{(\varphi,\Gamma_K)}\Big(D_{\mathrm{rig}}^{\le i}(r)/(t^{\Sigma_i(\mathbf{k},w_x)}), D_{\mathrm{rig}}(r)/D_{\mathrm{rig}}(r)^{\le i}\Big) \to \mathrm{Ext}^1_{(\varphi,\Gamma_K)}\Big(D_{\mathrm{rig}}(r)^{\le i-1}/(t^{\Sigma_i(\mathbf{k},w_x)}), D_{\mathrm{rig}}(r)/D_{\mathrm{rig}}(r)^{\le i}\Big),$$

$$0 \to \mathrm{Ext}^1_{\mathrm{cris}}\Big(\mathrm{gr}_i D_{\mathrm{rig}}(r), D_{\mathrm{rig}}(r)/D_{\mathrm{rig}}(r)^{\le i}\Big) \to \mathrm{Ext}^1_{\mathrm{cris}}\Big(D_{\mathrm{rig}}^{\le i}, D_{\mathrm{rig}}(r)/D_{\mathrm{rig}}(r)^{\le i}\Big) \to \mathrm{Ext}^1_{\mathrm{cris}}\Big(D_{\mathrm{rig}}(r)^{\le i-1}, D_{\mathrm{rig}}(r)/D_{\mathrm{rig}}(r)^{\le i}\Big)$$

(injectivity on the left following again from Definition 2.13), that we have:

$$\mathrm{Ext}^1_{(\varphi,\Gamma_K)}\Big(\mathrm{gr}_i D_{\mathrm{rig}}(r)/(t^{\Sigma_\ell(\mathbf{k},w_x)}), D_{\mathrm{rig}}(r)/D_{\mathrm{rig}}(r)^{\le \ell}\Big) \cong \mathrm{Ext}^1_{\mathrm{cris}}\Big(\mathrm{gr}_i D_{\mathrm{rig}}(r), D_{\mathrm{rig}}(r)/D_{\mathrm{rig}}(r)^{\le \ell}\Big). \tag{4.9}$$

By dévissage on $D_{\mathrm{rig}}(r)/D_{\mathrm{rig}}(r)^{\le \ell}$ using that $\mathrm{Ext}^1_{\mathrm{cris}}$ respects here short exact sequences (by Definition 2.13 and the discussion above), we have:

$$\dim_{k(x)} \mathrm{Ext}^1_{\mathrm{cris}}\Big(\mathrm{gr}_i D_{\mathrm{rig}}(r), D_{\mathrm{rig}}(r)/D_{\mathrm{rig}}(r)^{\le \ell}\Big) = \sum_{i_1=\ell+1}^{n} \dim_{k(x)} \mathrm{Ext}^1_{\mathrm{cris}}\Big(\mathrm{gr}_i D_{\mathrm{rig}}(r), \mathrm{gr}_{i_1} D_{\mathrm{rig}}(r)\Big).$$

The result follows from (4.19) below. □

Proposition 4.5. *We have:*

$$\dim_{k(x)}\Big(V_1 \cap V_2 \cap \cdots \cap V_{n-1}\Big) = \dim_{k(x)} \mathrm{Ext}^1_{\mathcal{G}_K}(r,r) - \Big([K:\mathbb{Q}_p]\frac{n(n-1)}{2} - \lg(w_x)\Big).$$

Proof. To lighten notation in this proof, we write $D_{\rm rig}$ instead of $D_{\rm rig}(r)$ and drop the subscript (φ,Γ_K). We first prove that, for $1\le i\le n$, we have an isomorphism of $k(x)$-vector spaces:

$$(4.10)\quad V_1\cap\cdots\cap V_{i-1}/V_1\cap\cdots\cap V_i \xrightarrow{\sim} \mathrm{Ext}^1\Big(\mathrm{gr}_iD_{\rm rig},D_{\rm rig}/D_{\rm rig}^{\le i}\Big)\Big/\mathrm{Ext}^1\Big(\mathrm{gr}_iD_{\rm rig}/(t^{\Sigma_i(\mathbf{k},w_x)}),D_{\rm rig}/D_{\rm rig}^{\le i}\Big)$$

where $V_1\cap\cdots\cap V_{i-1}:=\mathrm{Ext}^1(D_{\rm rig},D_{\rm rig})$ if $i=1$. We first define the map. We have the following commutative diagram:

$$\begin{array}{ccccccccc}
 & & 0 & & 0 & & 0 & & \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
0 & \to & \mathrm{Ext}^1(\mathrm{gr}_iD_{\rm rig}/(t^{\Sigma_i(\mathbf{k},w_x)}),D_{\rm rig}/D_{\rm rig}^{\le i}) & \to & \mathrm{Ext}^1(D_{\rm rig}^{\le i}/(t^{\Sigma_i(\mathbf{k},w_x)}),D_{\rm rig}/D_{\rm rig}^{\le i}) & \to & \mathrm{Ext}^1(D_{\rm rig}^{\le i-1}/(t^{\Sigma_i(\mathbf{k},w_x)}),D_{\rm rig}/D_{\rm rig}^{\le i}) & \to & 0\\
 & & \downarrow & & \downarrow & & \downarrow & & \\
0 & \to & \mathrm{Ext}^1(\mathrm{gr}_iD_{\rm rig},D_{\rm rig}/D_{\rm rig}^{\le i}) & \to & \mathrm{Ext}^1(D_{\rm rig}^{\le i},D_{\rm rig}/D_{\rm rig}^{\le i}) & \to & \mathrm{Ext}^1(D_{\rm rig}^{\le i-1},D_{\rm rig}/D_{\rm rig}^{\le i}) & \to & 0\\
 & & \downarrow & & \downarrow & & \downarrow & & \\
0 & \to & \mathrm{Ext}^1(t^{\Sigma_i(\mathbf{k},w_x)}\mathrm{gr}_iD_{\rm rig},D_{\rm rig}/D_{\rm rig}^{\le i}) & \to & \mathrm{Ext}^1(t^{\Sigma_i(\mathbf{k},w_x)}D_{\rm rig}^{\le i},D_{\rm rig}/D_{\rm rig}^{\le i}) & \to & \mathrm{Ext}^1(t^{\Sigma_i(\mathbf{k},w_x)}D_{\rm rig}^{\le i-1},D_{\rm rig}/D_{\rm rig}^{\le i}) & \to & 0
\end{array}$$

where the injections on top and left and the surjections on the two bottom lines all follow from Definition 2.13, and where the surjection on the top right corner follows from Corollary 4.11 below. Denote by E_i the inverse image of $\mathrm{Ext}^1(D_{\rm rig}^{\le i-1}/(t^{\Sigma_i(\mathbf{k},w_x)}),D_{\rm rig}/D_{\rm rig}^{\le i})$ in $\mathrm{Ext}^1(D_{\rm rig}^{\le i},D_{\rm rig}/D_{\rm rig}^{\le i})$, then we have an isomorphism:

$$(4.11)\quad \mathrm{Ext}^1\Big(\mathrm{gr}_iD_{\rm rig},D_{\rm rig}/D_{\rm rig}^{\le i}\Big)\Big/\mathrm{Ext}^1\Big(\mathrm{gr}_iD_{\rm rig}/(t^{\Sigma_i(\mathbf{k},w_x)}),D_{\rm rig}/D_{\rm rig}^{\le i}\Big)\xrightarrow{\sim} E_i\Big/\mathrm{Ext}^1\Big(D_{\rm rig}^{\le i}/(t^{\Sigma_i(\mathbf{k},w_x)}),D_{\rm rig}/D_{\rm rig}^{\le i}\Big).$$

We consider the composition:

$$(4.12)\quad V_1\cap\cdots\cap V_{i-1}\hookrightarrow \mathrm{Ext}^1\Big(D_{\rm rig},D_{\rm rig}\Big)\twoheadrightarrow \mathrm{Ext}^1\Big(D_{\rm rig}^{\le i},D_{\rm rig}/D_{\rm rig}^{\le i}\Big)\twoheadrightarrow \mathrm{Ext}^1\Big(D_{\rm rig}^{\le i},D_{\rm rig}/D_{\rm rig}^{\le i}\Big)\Big/\mathrm{Ext}^1\Big(D_{\rm rig}^{\le i}/(t^{\Sigma_i(\mathbf{k},w_x)}),D_{\rm rig}/D_{\rm rig}^{\le i}\Big)$$

and note that the image of $V_1\cap\cdots\cap V_{i-1}$ falls in $E_i\big/\mathrm{Ext}^1(D_{\rm rig}^{\le i}/(t^{\Sigma_i(\mathbf{k},w_x)}),D_{\rm rig}/D_{\rm rig}^{\le i})$ by Corollary 4.10 below. If $v\in V_1\cap\cdots\cap V_{i-1}$ is also in V_i, then its image in $\mathrm{Ext}^1(D_{\rm rig}^{\le i},D_{\rm rig}/D_{\rm rig}^{\le i})$ maps to 0 in $\mathrm{Ext}^1(t^{\Sigma_i(\mathbf{k},w_x)}D_{\rm rig}^{\le i},D_{\rm rig}/D_{\rm rig}^{\le i})$, hence belongs to $\mathrm{Ext}^1(D_{\rm rig}^{\le i}/(t^{\Sigma_i(\mathbf{k},w_x)}),D_{\rm rig}/D_{\rm rig}^{\le i})$. By (4.11), we thus have a canonical induced map:

$$(4.13)\quad V_1\cap\cdots\cap V_{i-1}/V_1\cap\cdots\cap V_i\to \mathrm{Ext}^1\Big(\mathrm{gr}_iD_{\rm rig},D_{\rm rig}/D_{\rm rig}^{\le i}\Big)\Big/\mathrm{Ext}^1\Big(\mathrm{gr}_iD_{\rm rig}/(t^{\Sigma_i(\mathbf{k},w_x)}),D_{\rm rig}/D_{\rm rig}^{\le i}\Big).$$

Let us prove that (4.13) is surjective. One easily checks that $\mathrm{Ext}^1(D_{\rm rig}/D_{\rm rig}^{\le i-1},D_{\rm rig})\subseteq V_1\cap\cdots\cap V_{i-1}$ and that the natural map $\mathrm{Ext}^1(D_{\rm rig}/D_{\rm rig}^{\le i-1},D_{\rm rig})\to\mathrm{Ext}^1\Big(\mathrm{gr}_iD_{\rm rig},D_{\rm rig}/D_{\rm rig}^{\le i}\Big)$ is surjective (again by Definition 2.13). This implies that *a fortiori* (4.13) must also be surjective. Let us prove that (4.13) is injective. If $v\in V_1\cap\cdots\cap V_{i-1}$ maps to zero, then the image of v in $\mathrm{Ext}^1(D_{\rm rig}^{\le i},D_{\rm rig}/D_{\rm rig}^{\le i})$ belongs to $\mathrm{Ext}^1(D_{\rm rig}^{\le i}/(t^{\Sigma_i(\mathbf{k},w_x)}),D_{\rm rig}/D_{\rm rig}^{\le i})$ by (4.11), i.e. maps to zero in $\mathrm{Ext}^1(t^{\Sigma_i(\mathbf{k},w_x)}D_{\rm rig}^{\le i},D_{\rm rig}/D_{\rm rig}^{\le i})$, i.e. $v\in V_i$ by (4.4), hence $v\in V_1\cap\cdots\cap V_i$.

We now prove the statement of the proposition. From (4.10) and Lemma 4.4, we obtain for $1 \leq i \leq n$:

$$\begin{aligned}
&\dim_{k(x)} \Big(V_1 \cap \cdots \cap V_{i-1}/V_1 \cap \cdots \cap V_i\Big) = \\
&\qquad [K:\mathbb{Q}_p](n-i) - \sum_{\tau\colon K \hookrightarrow L} \Big|\{j \in \{i+1,\ldots,n\}, w_{x,\tau}^{-1}(j) < w_{x,\tau}^{-1}(i)\}\Big| = \\
&\qquad\qquad \sum_{\tau\colon K \hookrightarrow L} \Big|\{j \in \{i+1,\ldots,n\}, w_{x,\tau}^{-1}(i) < w_{x,\tau}^{-1}(j)\}\Big|.
\end{aligned}$$

Summing up $\dim_{k(x)}(V_1 \cap \cdots \cap V_{i-1}/V_1 \cap \cdots \cap V_i)$ for $i = 1$ to $n-1$ thus yields:

$$\begin{aligned}
&\dim_{k(x)} \mathrm{Ext}^1\Big(D_{\mathrm{rig}}, D_{\mathrm{rig}}\Big) - \dim_{k(x)}(V_1 \cap \cdots \cap V_{n-1}) = \\
&\qquad\qquad \sum_{\tau\colon K \hookrightarrow L} \Big|\{1 \leq i_1 < i_2 \leq n,\ w_{x,\tau}^{-1}(i_1) < w_{x,\tau}^{-1}(i_2)\}\Big|.
\end{aligned}$$

But $|\{1 \leq i_1 < i_2 \leq n,\ w_{x,\tau}^{-1}(i_1) < w_{x,\tau}^{-1}(i_2)\}| = \frac{n(n-1)}{2} - \lg(w_{x,\tau})$ (see e.g. [32, §0.3]), and thus we get:

$$\begin{aligned}
\dim_{k(x)}(V_1 \cap \cdots \cap V_{n-1}) &= \dim_{k(x)} \mathrm{Ext}^1\Big(D_{\mathrm{rig}}, D_{\mathrm{rig}}\Big) - \sum_{\tau\colon K \hookrightarrow L} \Big(\frac{n(n-1)}{2} - \lg(w_{x,\tau})\Big) \\
&= \dim_{k(x)} \mathrm{Ext}^1\Big(D_{\mathrm{rig}}, D_{\mathrm{rig}}\Big) - \Big([K:\mathbb{Q}_p]\frac{n(n-1)}{2} - \lg(w_x)\Big)
\end{aligned}$$

which finishes the proof. □

Seeing an element of $\mathrm{Ext}^1_{\mathcal{G}_K}(r,r)$ as a $k(x)[\varepsilon]/(\varepsilon^2)$-valued representation of $\mathcal{G}_K$, we can write its Sen weights as $(k_{\tau,i} + \varepsilon d_{\tau,i})_{1 \leq i \leq n, \tau\colon K \hookrightarrow L}$ for some $d_{\tau,i} \in k(x)$. We let V be the $k(x)$-subvector space of $\mathrm{Ext}^1_{\mathcal{G}_K}(r,r)$ (or of $\mathrm{Ext}^1_{(\varphi,\Gamma_K)}(D_{\mathrm{rig}}(r), D_{\mathrm{rig}}(r))$) of extensions such that $d_{\tau,i} = d_{\tau, w_{x,\tau}^{-1}(i)}$ for $1 \leq i \leq n$ and $\tau : K \hookrightarrow L$.

Proposition 4.6. *We have* $\dim_{k(x)} V = \dim_{k(x)} \mathrm{Ext}^1_{\mathcal{G}_K}(r,r) - d_x$.

Proof. The Sen map $\mathrm{Ext}^1_{\mathcal{G}_K}(r,r) \cong \mathrm{Ext}^1_{(\varphi,\Gamma_K)}(D_{\mathrm{rig}}(r), D_{\mathrm{rig}}(r)) \longrightarrow k(x)^{[K:\mathbb{Q}_p]n}$ sending an extension to $(d_{\tau,i})_{1 \leq i \leq n, \tau\colon K \hookrightarrow L}$ is easily checked to be surjective (by a dévissage argument using Definition 2.13, we are reduced to the rank one case where it is obvious). The $k(x)$-subvector space of $k(x)^{[K:\mathbb{Q}_p]n}$ of tuples $(d_{\tau,i})_{1 \leq i \leq n, \tau\colon K \hookrightarrow L}$ such that $d_{\tau,i} = d_{\tau, w_{x,\tau}^{-1}(i)}$ for $1 \leq i \leq n$ and $\tau : K \hookrightarrow L$ has dimension $[K:\mathbb{Q}_p]n - d_x$ (argue as in the beginning of the proof of Lemma 2.7). The result follows. □

Proposition 4.7. *We have:*

$$\dim_{k(x)}\Big(V \cap (V_1 \cap \cdots \cap V_{n-1})\Big) = \dim_{k(x)} \mathrm{Ext}^1_{\mathcal{G}_K}(r,r) - d_x - \Big([K:\mathbb{Q}_p]\frac{n(n-1)}{2} - \lg(w_x)\Big).$$

Proof. Consider the following cartesian diagram which defines W_i:

$$\begin{array}{ccc}
\mathrm{Ext}^1_{(\varphi,\Gamma_K)}\Big(D_{\mathrm{rig}}(r), D_{\mathrm{rig}}(r)\Big) & \longrightarrow & \mathrm{Ext}^1_{(\varphi,\Gamma_K)}\Big(D_{\mathrm{rig}}(r)^{\leq i}, D_{\mathrm{rig}}(r)\Big) \\
\uparrow & & \uparrow \\
W_i & \longrightarrow & \mathrm{Ext}^1_{(\varphi,\Gamma_K)}\Big(D_{\mathrm{rig}}(r)^{\leq i}, D_{\mathrm{rig}}(r)^{\leq i}\Big)
\end{array},$$

then $W_i \subseteq V_i$, hence $W_1 \cap \cdots \cap W_{n-1} \subseteq V_1 \cap \cdots \cap V_{n-1}$. In fact, $W_1 \cap \cdots \cap W_{n-1}$ is the $k(x)$-subvector space of $\mathrm{Ext}^1_{(\varphi,\Gamma_K)}(D_{\mathrm{rig}}(r), D_{\mathrm{rig}}(r))$ of extensions which respect the triangulation $(D_{\mathrm{rig}}(r)^{\leq i})_{1\leq i\leq n}$ on $D_{\mathrm{rig}}(r)$. A dévissage argument (using Definition 2.13) that we leave to the reader then shows that the composition:

$$W_1 \cap \cdots \cap W_{n-1} \hookrightarrow \mathrm{Ext}^1_{(\varphi,\Gamma_K)}\Big(D_{\mathrm{rig}}(r), D_{\mathrm{rig}}(r)\Big) \longrightarrow k(x)^{[K:\mathbb{Q}_p]n}$$

(where the second map is the Sen map in the proof of Proposition 4.6) remains surjective. *A fortiori*, $V_1 \cap \cdots \cap V_{n-1} \hookrightarrow \mathrm{Ext}^1_{(\varphi,\Gamma_K)}(D_{\mathrm{rig}}(r), D_{\mathrm{rig}}(r)) \to k(x)^{[K:\mathbb{Q}_p]n}$ is also surjective. By the same proof as that of Proposition 4.6 we get:

$$\dim_{k(x)}\Big(V \cap (V_1 \cap \cdots \cap V_{n-1})\Big) = \dim_{k(x)}(V_1 \cap \cdots \cap V_{n-1}) - d_x$$

and the result follows from Proposition 4.5. □

To sum up the result of Proposition 4.7, we see that $[K:\mathbb{Q}_p]\frac{n(n-1)}{2} - \lg(w_x)$ corresponds to the "splitting conditions" and d_x corresponds to the "weight conditions" .

4.3. **Calculation of some Ext groups.** We prove several technical but crucial results of Galois cohomology that were used in §4.2.

For a continuous character $\delta : K^\times \to L^\times$ and $\tau : K \hookrightarrow L$, we define its (Sen) weight $\mathrm{wt}_\tau(\delta) \in L$ in the direction τ by taking the *opposite* of the weight defined in [4, §2.3]. For instance $\mathrm{wt}_\tau(\tau(z)^{k_\tau}) = k_\tau$ ($k_\tau \in \mathbb{Z}$).

Lemma 4.8. *Let $\tau : K \hookrightarrow L$ and $k_\tau \in \mathbb{Z}_{>0}$.*
(i) For $j \in \{0,1\}$ we have $\mathrm{Ext}^j_{(\varphi,\Gamma_K)}\Big(\mathcal{R}_{L,K}, \mathcal{R}_{L,K}(\delta)/(t_\tau^{k_\tau})\Big) \neq 0$ if and only if $\mathrm{wt}_\tau(\delta) \in \{-(k_\tau-1),\dots,0\}$ and we have $\mathrm{Ext}^2_{(\varphi,\Gamma_K)}\Big(\mathcal{R}_{L,K}, \mathcal{R}_{L,K}(\delta)/(t_\tau^{k_\tau})\Big) = 0$ for all δ.
(ii) For $j \in \{1,2\}$ we have $\mathrm{Ext}^j_{(\varphi,\Gamma_K)}\Big(\mathcal{R}_{L,K}(\delta)/(t_\tau^{k_\tau}), \mathcal{R}_{L,K}\Big) \neq 0$ if and only if $\mathrm{wt}_\tau(\delta) \in \{-k_\tau,\dots,-1\}$ and we have $\mathrm{Ext}^0_{(\varphi,\Gamma_K)}\Big(\mathcal{R}_{L,K}(\delta)/(t_\tau^{k_\tau}), \mathcal{R}_{L,K}\Big) = 0$ for all δ.
(iii) When one of these spaces is nonzero, it has dimension 1 over L.

Proof. The first part of (i) is in [4, Prop.2.14] (and initially in [19, Prop.2.18] for $K = \mathbb{Q}_p$, see also [43, Lem.2.16]) and the second part in [40, Th.3.7(2)]. The second part of (ii) is obvious, let us prove the first. We have an exact sequence:

$$0 \longrightarrow \mathcal{R}_{L,K}(\delta^{-1}) \longrightarrow \mathcal{R}_{L,K}(\tau(z)^{-k_\tau}\delta^{-1}) \longrightarrow \mathcal{R}_{L,K}(\tau(z)^{-k_\tau}\delta^{-1})/(t_\tau^{k_\tau}) \longrightarrow 0. \tag{4.14}$$

The cup product with (4.14) yields canonical morphisms of L-vector spaces:

$$\begin{aligned}\mathrm{Ext}^0_{(\varphi,\Gamma_K)}\Big(\mathcal{R}_{L,K}/(t_\tau^{k_\tau}), \mathcal{R}_{L,K}(\tau(z)^{-k_\tau}\delta^{-1})/(t_\tau^{k_\tau})\Big) &\to \mathrm{Ext}^1_{(\varphi,\Gamma_K)}\Big(\mathcal{R}_{L,K}/(t_\tau^{k_\tau}), \mathcal{R}_{L,K}(\delta^{-1})\Big)\\ \mathrm{Ext}^1_{(\varphi,\Gamma_K)}\Big(\mathcal{R}_{L,K}/(t_\tau^{k_\tau}), \mathcal{R}_{L,K}(\tau(z)^{-k_\tau}\delta^{-1})/(t_\tau^{k_\tau})\Big) &\to \mathrm{Ext}^2_{(\varphi,\Gamma_K)}\Big(\mathcal{R}_{L,K}/(t_\tau^{k_\tau}), \mathcal{R}_{L,K}(\delta^{-1})\Big).\end{aligned}$$

There is an obvious isomorphism of L-vector spaces:

$$\mathrm{Ext}^0_{(\varphi,\Gamma_K)}\Big(\mathcal{R}_{L,K}, \mathcal{R}_{L,K}(\tau(z)^{-k_\tau}\delta^{-1})/(t_\tau^{k_\tau})\Big) \cong \mathrm{Ext}^0_{(\varphi,\Gamma_K)}\Big(\mathcal{R}_{L,K}/(t_\tau^{k_\tau}), \mathcal{R}_{L,K}(\tau(z)^{-k_\tau}\delta^{-1})/(t_\tau^{k_\tau})\Big)$$

and an analysis of the cokernel of the multiplication by $t_\tau^{k_\tau}$ map on a short exact sequence $0 \to \mathcal{R}_{L,K}(\tau(z)^{-k_\tau}\delta^{-1})/(t_\tau^{k_\tau}) \to \mathcal{E} \to \mathcal{R}_{L,K} \to 0$ of (φ,Γ_K)-module over $\mathcal{R}_{L,K}$ yields a canonical morphism of L-vector spaces:

$$\mathrm{Ext}^1_{(\varphi,\Gamma_K)}\Big(\mathcal{R}_{L,K},\mathcal{R}_{L,K}(\tau(z)^{-k_\tau}\delta^{-1})/(t_\tau^{k_\tau})\Big)\to \mathrm{Ext}^1_{(\varphi,\Gamma_K)}\Big(\mathcal{R}_{L,K}/(t_\tau^{k_\tau}),\mathcal{R}_{L,K}(\tau(z)^{-k_\tau}\delta^{-1})/(t_\tau^{k_\tau})\Big).$$

Thus we have canonical morphisms of L-vector spaces:

$$\begin{aligned}(4.15)\quad &\mathrm{Ext}^0_{(\varphi,\Gamma_K)}\Big(\mathcal{R}_{L,K},\mathcal{R}_{L,K}(\tau(z)^{-k_\tau}\delta^{-1})/(t_\tau^{k_\tau})\Big)\to \mathrm{Ext}^1_{(\varphi,\Gamma_K)}\Big(\mathcal{R}_{L,K}/(t_\tau^{k_\tau}),\mathcal{R}_{L,K}(\delta^{-1})\Big)\\ &\mathrm{Ext}^1_{(\varphi,\Gamma_K)}\Big(\mathcal{R}_{L,K},\mathcal{R}_{L,K}(\tau(z)^{-k_\tau}\delta^{-1})/(t_\tau^{k_\tau})\Big)\to \mathrm{Ext}^2_{(\varphi,\Gamma_K)}\Big(\mathcal{R}_{L,K}/(t_\tau^{k_\tau}),\mathcal{R}_{L,K}(\delta^{-1})\Big).\end{aligned}$$

It is then a simple exercise of linear algebra to check that the morphisms in (4.15) fit into a natural morphism of complexes of L-vector spaces from the long exact sequence of $\mathrm{Ext}^j_{(\varphi,\Gamma_K)}(\mathcal{R}_{L,K},\cdot)$ applied to the short exact sequence (4.14) to the long exact sequence of $\mathrm{Ext}^j_{(\varphi,\Gamma_K)}(\cdot,\mathcal{R}_{L,K}(\delta^{-1}))$ applied to the short exact sequence $0\to t_\tau^{k_\tau}\mathcal{R}_{L,K}\to\mathcal{R}_{L,K}\to\mathcal{R}_{L,K}/(t_\tau^{k_\tau})\to 0$ (note that there is a shift in this map of complexes). Since all the morphisms are obviously isomorphisms except possibly the morphisms (4.15), we deduce that the latter are also isomorphisms. Twisting by $\mathcal{R}_{L,K}(\delta)$ on the right hand side of (4.15) and using (i) applied to the left hand side, the first part of (ii) easily follows. Finally (iii) follows from [4, Prop.2.14] and from the previous isomorphisms (4.15). ☐

Recall that for $i,\ell\in\{1,\dots,n\}$ we have an exact sequence:

$$\begin{aligned}(4.16)\quad 0\to \mathrm{Ext}^1_{(\varphi,\Gamma_K)}\Big(D_{\mathrm{rig}}(r)^{\le i}/t^{\Sigma_\ell(\mathbf{k},w_x)}D_{\mathrm{rig}}(r)^{\le i},D_{\mathrm{rig}}(r)/D_{\mathrm{rig}}(r)^{\le \ell}\Big)\to\\ \mathrm{Ext}^1_{(\varphi,\Gamma_K)}\Big(D_{\mathrm{rig}}(r)^{\le i},D_{\mathrm{rig}}(r)/D_{\mathrm{rig}}(r)^{\le \ell}\Big)\to\\ \mathrm{Ext}^1_{(\varphi,\Gamma_K)}\Big(t^{\Sigma_\ell(\mathbf{k},w_x)}D_{\mathrm{rig}}(r)^{\le i},D_{\mathrm{rig}}(r)/D_{\mathrm{rig}}(r)^{\le \ell}\Big)\end{aligned}$$

where the injection on the left follows as usual from Definition 2.13.

Proposition 4.9. *For $1\le i\le \ell\le n$, we have an isomorphism of subspaces of $\mathrm{Ext}^1_{(\varphi,\Gamma_K)}(D_{\mathrm{rig}}(r)^{\le i},D_{\mathrm{rig}}(r)/D_{\mathrm{rig}}(r)^{\le \ell})$:*

$$\mathrm{Ext}^1_{(\varphi,\Gamma_K)}\Big(D_{\mathrm{rig}}(r)^{\le i}/t^{\Sigma_\ell(\mathbf{k},w_x)}D_{\mathrm{rig}}(r)^{\le i},D_{\mathrm{rig}}(r)/D_{\mathrm{rig}}(r)^{\le \ell}\Big)\cong \mathrm{Ext}^1_{\mathrm{cris}}\Big(D_{\mathrm{rig}}(r)^{\le i},D_{\mathrm{rig}}(r)/D_{\mathrm{rig}}(r)^{\le \ell}\Big).$$

Proof. To lighten notation, we write D_{rig} instead of $D_{\mathrm{rig}}(r)$ and drop the subscript (φ,Γ_K). By the exact sequence (4.16) and a dévissage on $D_{\mathrm{rig}}^{\le i}$ and $D_{\mathrm{rig}}/D_{\mathrm{rig}}^{\le \ell}$ (recall from Definition 2.13 and the discussion preceding Lemma 4.4 that $\mathrm{Ext}^1_{\mathrm{cris}}$ respects short exact sequences here), it is enough to prove (i) that the composition:

$$\mathrm{Ext}^1_{\mathrm{cris}}\Big(D_{\mathrm{rig}}^{\le i},D_{\mathrm{rig}}/D_{\mathrm{rig}}^{\le \ell}\Big)\subseteq \mathrm{Ext}^1\Big(D_{\mathrm{rig}}^{\le i},D_{\mathrm{rig}}/D_{\mathrm{rig}}^{\le \ell}\Big)\longrightarrow \mathrm{Ext}^1\Big(t^{\Sigma_\ell(\mathbf{k},w_x)}D_{\mathrm{rig}}^{\le i},D_{\mathrm{rig}}/D_{\mathrm{rig}}^{\le \ell}\Big)$$

is zero and (ii) that:

$$\mathrm{Ext}^1\Big(\mathrm{gr}_{\ell'}D_{\mathrm{rig}}/(t^{\Sigma_\ell(\mathbf{k},w_x)}),\mathrm{gr}_{\ell''}D_{\mathrm{rig}}\Big)\cong \mathrm{Ext}^1_{\mathrm{cris}}\Big(\mathrm{gr}_{\ell'}D_{\mathrm{rig}},\mathrm{gr}_{\ell''}D_{\mathrm{rig}}\Big)$$

(inside $\mathrm{Ext}^1(\mathrm{gr}_{\ell'}D_{\mathrm{rig}},\mathrm{gr}_{\ell''}D_{\mathrm{rig}})$) for all ℓ',ℓ'' such that $\ell'\le\ell$ and $\ell''\ge \ell+1$.

We prove (i). The map clearly factors through:

$$\mathrm{Ext}^1_{\mathrm{cris}}\Big(t^{\Sigma_\ell(\mathbf{k},w_x)}D_{\mathrm{rig}}^{\le i},D_{\mathrm{rig}}/D_{\mathrm{rig}}^{\le \ell}\Big),$$

let us prove that the latter vector space is zero. By dévissage again, it is enough to prove that:

$$\mathrm{Ext}^1_{\mathrm{cris}}\Big(t^{\Sigma_\ell(\mathbf{k},w_x)}\mathrm{gr}_{\ell'}D_{\mathrm{rig}}, \mathrm{gr}_{\ell''}D_{\mathrm{rig}}\Big) = 0$$

for ℓ', ℓ'' such that $\ell' \leq \ell$ and $\ell'' \geq \ell + 1$. It is enough to prove that, for all $\tau : K \hookrightarrow L$, we have $\mathrm{wt}_\tau(t^{\Sigma_\ell(\mathbf{k},w_x)}\mathrm{gr}_{\ell'}D_{\mathrm{rig}}) \geq \mathrm{wt}_\tau(\mathrm{gr}_{\ell''}D_{\mathrm{rig}})$ (using Definition 2.13 when these two weights are equal). This is equivalent to:

$$\sum_{j=1}^{\ell}(k_{\tau,j} - k_{\tau,w_{x,\tau}^{-1}(j)}) + k_{\tau,w_{x,\tau}^{-1}(\ell')} \geq k_{\tau,w_{x,\tau}^{-1}(\ell'')} \tag{4.17}$$

which indeed holds for ℓ', ℓ'' as above because $k_{\tau,1} > k_{\tau,2} > \cdots > k_{\tau,n}$.

We prove (ii). From (i) we have in particular an inclusion:

$$\mathrm{Ext}^1_{\mathrm{cris}}\Big(\mathrm{gr}_{\ell'}D_{\mathrm{rig}}, \mathrm{gr}_{\ell''}D_{\mathrm{rig}}\Big) \subseteq \mathrm{Ext}^1\Big(\mathrm{gr}_{\ell'}D_{\mathrm{rig}}/(t^{\Sigma_\ell(\mathbf{k},w_x)}), \mathrm{gr}_{\ell''}D_{\mathrm{rig}}\Big). \tag{4.18}$$

It is an easy (and well-known) exercise that we leave to the reader to check that:

$$\dim_{k(x)}\mathrm{Ext}^1_{\mathrm{cris}}\Big(\mathrm{gr}_{\ell'}D_{\mathrm{rig}}, \mathrm{gr}_{\ell''}D_{\mathrm{rig}}\Big) = \Big|\{\tau : K \hookrightarrow L, w_{x,\tau}^{-1}(\ell'') < w_{x,\tau}^{-1}(\ell')\}\Big|. \tag{4.19}$$

On the other hand, from (ii) and (iii) of Lemma 4.8, using (4.17) and $\mathcal{R}_{L,K}(\delta)/(t_\tau^{k_\tau}t_\sigma^{k_\sigma}) \cong \mathcal{R}_{L,K}(\delta)/(t_\tau^{k_\tau}) \times \mathcal{R}_{L,K}(\delta)/(t_\sigma^{k_\sigma})$ if $\tau \neq \sigma$, we deduce:

$$\dim_{k(x)}\mathrm{Ext}^1\Big(\mathrm{gr}_{\ell'}D_{\mathrm{rig}}/(t^{\Sigma_\ell(\mathbf{k},w_x)}), \mathrm{gr}_{\ell''}D_{\mathrm{rig}}\Big) = \Big|\{\tau : K \hookrightarrow L, w_{x,\tau}^{-1}(\ell'') < w_{x,\tau}^{-1}(\ell')\}\Big|. \tag{4.20}$$

(4.18), (4.19) and (4.20) imply $\mathrm{Ext}^1_{\mathrm{cris}}(\mathrm{gr}_{\ell'}D_{\mathrm{rig}}, \mathrm{gr}_{\ell''}D_{\mathrm{rig}}) \cong \mathrm{Ext}^1(\mathrm{gr}_{\ell'}D_{\mathrm{rig}}/(t^{\Sigma_\ell(\mathbf{k},w_x)}), \mathrm{gr}_{\ell''}D_{\mathrm{rig}})$ which finishes the proof. □

Corollary 4.10. *Let $i \in \{1, \ldots, n\}$, $\mathcal{E} \in \mathrm{Ext}^1_{(\varphi,\Gamma_K)}(D_{\mathrm{rig}}(r), D_{\mathrm{rig}}(r))$ and assume that the image of $\mathcal{E}$ (by pullback and pushforward) in:*

$$\mathrm{Ext}^1_{(\varphi,\Gamma_K)}\Big(t^{\Sigma_{i-1}(\mathbf{k},w_x)}D_{\mathrm{rig}}(r)^{\leq i-1}, D_{\mathrm{rig}}(r)/D_{\mathrm{rig}}(r)^{\leq i-1}\Big)$$

is zero. Then the image of $\mathcal{E}$ in:

$$\mathrm{Ext}^1_{(\varphi,\Gamma_K)}\Big(t^{\Sigma_i(\mathbf{k},w_x)}D_{\mathrm{rig}}(r)^{\leq i-1}, D_{\mathrm{rig}}(r)/D_{\mathrm{rig}}(r)^{\leq i}\Big)$$

is also zero.

Proof. By Proposition 4.9 applied with $(i,\ell) = (i-1, i-1)$, the image of $\mathcal{E}$ in $\mathrm{Ext}^1_{(\varphi,\Gamma_K)}(D_{\mathrm{rig}}(r)^{\leq i-1}, D_{\mathrm{rig}}(r)/D_{\mathrm{rig}}(r)^{\leq i-1})$ sits in:

$$\mathrm{Ext}^1_{\mathrm{cris}}\Big(D_{\mathrm{rig}}(r)^{\leq i-1}, D_{\mathrm{rig}}(r)/D_{\mathrm{rig}}(r)^{\leq i-1}\Big).$$

Hence its image in $\mathrm{Ext}^1_{(\varphi,\Gamma_K)}(D_{\mathrm{rig}}(r)^{\leq i-1}, D_{\mathrm{rig}}(r)/D_{\mathrm{rig}}(r)^{\leq i})$ sits in:

$$\mathrm{Ext}^1_{\mathrm{cris}}\Big(D_{\mathrm{rig}}(r)^{\leq i-1}, D_{\mathrm{rig}}(r)/D_{\mathrm{rig}}(r)^{\leq i}\Big).$$

It follows from Proposition 4.9 again applied with $(i,\ell) = (i-1, i)$ that it maps to zero in $\mathrm{Ext}^1_{(\varphi,\Gamma_K)}(t^{\Sigma_i(\mathbf{k},w_x)}D_{\mathrm{rig}}(r)^{\leq i-1}, D_{\mathrm{rig}}(r)/D_{\mathrm{rig}}(r)^{\leq i})$. □

Corollary 4.11. *For $2 \leq i \leq n$ we have a surjection:*

$$\mathrm{Ext}^1_{(\varphi,\Gamma_K)}\Big(D_{\mathrm{rig}}(r)^{\leq i}/t^{\Sigma_i(\mathbf{k},w_x)}D_{\mathrm{rig}}(r)^{\leq i}, D_{\mathrm{rig}}(r)/D_{\mathrm{rig}}(r)^{\leq i}\Big) \twoheadrightarrow \mathrm{Ext}^1_{(\varphi,\Gamma_K)}\Big(D_{\mathrm{rig}}(r)^{\leq i-1}/t^{\Sigma_i(\mathbf{k},w_x)}D_{\mathrm{rig}}(r)^{\leq i-1}, D_{\mathrm{rig}}(r)/D_{\mathrm{rig}}(r)^{\leq i}\Big)$$

where the map is the pullback along:

$$D_{\mathrm{rig}}(r)^{\leq i-1}/t^{\Sigma_i(\mathbf{k},w_x)}D_{\mathrm{rig}}(r)^{\leq i-1} \hookrightarrow D_{\mathrm{rig}}(r)^{\leq i}/t^{\Sigma_i(\mathbf{k},w_x)}D_{\mathrm{rig}}(r)^{\leq i}.$$

Proof. This follows from Proposition 4.9 (applied with $(i,\ell) = (i,i)$ and $(i,\ell) = (i-1,i)$) and the fact that the map:

$$\mathrm{Ext}^1_{\mathrm{cris}}\Big(D_{\mathrm{rig}}(r)^{\leq i}, D_{\mathrm{rig}}(r)/D_{\mathrm{rig}}(r)^{\leq i}\Big) \longrightarrow \mathrm{Ext}^1_{\mathrm{cris}}\Big(D_{\mathrm{rig}}(r)^{\leq i-1}, D_{\mathrm{rig}}(r)/D_{\mathrm{rig}}(r)^{\leq i}\Big)$$

is surjective. □

4.4. **End of proof of Theorem 2.15.** We use the results of §4.1 and §4.2 to prove Theorem 2.15.

We keep the previous notation.

Corollary 4.12. *The image of $T_{X,x}$ in $\mathrm{Ext}^1_{\mathcal{G}_K}(r,r)$ has dimension smaller or equal than:*

$$\dim_{k(x)} \mathrm{Ext}^1_{\mathcal{G}_K}(r,r) - d_x - \Big([K:\mathbb{Q}_p]\frac{n(n-1)}{2} - \lg(w_x)\Big).$$

Proof. It follows from Theorem 4.1 that the image of any $\vec{v} \in T_{X,x}$ in $\mathrm{Ext}^1_{\mathcal{G}_K}(r,r)$ is in V. It follows from Corollary 4.3 that the image of any $\vec{v} \in T_{X,x}$ in $\mathrm{Ext}^1_{\mathcal{G}_K}(r,r)$ is also in $V_1 \cap \cdots \cap V_{n-1}$. One concludes with Proposition 4.7. □

Lemma 4.13. *Let $x' = (r',\delta') \in X^{\square}_{\mathrm{tri}}(\overline{r})$ be any point, then there is an exact sequence of $k(x')$-vector spaces $0 \to K(r') \to T_{\mathfrak{X}^{\square}_{\overline{r}},r'} \to \mathrm{Ext}^1_{\mathcal{G}_K}(r',r') \to 0$ where $K(r')$ is a $k(x')$-subvector space of $T_{\mathfrak{X}^{\square}_{\overline{r}},r'}$ of dimension $\dim_{k(x')} \mathrm{End}_{k(x')}(r') - \dim_{k(x')} \mathrm{End}_{\mathcal{G}_K}(r') = n^2 - \dim_{k(x')} \mathrm{End}_{\mathcal{G}_K}(r')$.*

Proof. It easily follows from [37, Lem.2.3.3 & Prop.2.3.5] that there is a topological isomorphism $\widehat{\mathcal{O}}_{\mathfrak{X}^{\square}_{\overline{r}},r'} \cong R^{\square}_{r'}$ where the former is the completed local ring at r' to the rigid analytic variety $\mathfrak{X}^{\square}_{\overline{r}}$ and the latter is the framed local deformation ring of r' in equal characteristic 0. In particular from (2.11) we have $T_{\mathfrak{X}^{\square}_{\overline{r}},r'} \cong \mathrm{Hom}_{k(x')}\Big(R^{\square}_{r'}, k(x')[\varepsilon]/(\varepsilon^2)\Big)$. Then the result follows by the same argument as in [37, §2.3.4], seeing an element of $\mathrm{Ext}^1_{\mathcal{G}_K}(r',r')$ as a deformation of r' with values in $k(x')[\varepsilon]/(\varepsilon^2)$. □

Lemma 4.14. *Let $x' = (r',\delta') \in X^{\square}_{\mathrm{tri}}(\overline{r})$ be any point such that $H^2(\mathcal{G}_K, \mathrm{End}_{k(x')}(r')) = 0$, then $\dim_{k(x')} \mathrm{Ext}^1_{\mathcal{G}_K}(r',r') = \dim_{k(x')} \mathrm{End}_{\mathcal{G}_K}(r') + n^2[K:\mathbb{Q}_p]$.*

Proof. This follows by the usual argument computing $\dim_{k(x')} H^1(\mathcal{G}_K, \mathrm{End}_{k(x')}(r'))$ from the Euler characteristic formula of Galois cohomology using $\dim_{k(x')} H^2(\mathcal{G}_K, \mathrm{End}_{k(x')}(r')) = 0$ and $\dim_{k(x')} H^0(\mathcal{G}_K, \mathrm{End}_{k(x')}(r')) = \dim_{k(x')} \mathrm{End}_{\mathcal{G}_K}(r')$. □

Remark 4.15. Lemma 4.14 for instance holds if x' is crystalline and the Frobenius eigenvalues $(\varphi'_i)_{1\leq i\leq n}$ (see Lemma 2.1) satisfy $\varphi'_i{\varphi'_j}^{-1}\neq q$ for $1\leq i,j\leq n$. In particular it holds for $x'=x$.

Lemma 4.16. *There is an injection of $k(x)$-vector spaces $T_{X,x}\hookrightarrow T_{\mathfrak{X}_{\overline{r}}^{\square},r}$.*

Proof. The embedding $X\hookrightarrow X_{\rm tri}^{\square}(\overline{r})\hookrightarrow \mathfrak{X}_{\overline{r}}^{\square}\times\mathcal{T}_L^n$ induces an injection on tangent spaces (with obvious notation):

$$T_{X,x}\hookrightarrow T_{\mathfrak{X}_{\overline{r}}^{\square},r}\oplus T_{\mathcal{T}_L^n,\delta}.$$

We thus have to show that the composition with the projection $T_{\mathfrak{X}_{\overline{r}}^{\square},r}\oplus T_{\mathcal{T}_L^n,\delta}\twoheadrightarrow T_{\mathfrak{X}_{\overline{r}}^{\square},r}$ remains injective. Let $\vec{v}\in T_{X,x}$ which maps to $0\in T_{\mathfrak{X}_{\overline{r}}^{\square},r}$, and thus *a fortiori* to 0 in $\mathrm{Ext}^1_{\mathcal{G}_K}(r,r)$ via the surjection in Lemma 4.13. We have to show that the image of $\vec{v}$ in $T_{\mathcal{T}_L^n,\delta}$ is also 0. We know that the image of $\vec{v}$ in $T_{\mathcal{W}_L^n,\omega(x)}$ is zero since the Hodge-Tate weights don't vary (that is, the $d_{\tau,i,\vec{v}}$ are all zero). To conclude that the image in $T_{\mathcal{T}_L^n,\delta}$ is also 0, we can for instance use Theorem 4.2 (which uses the accumulation property of X at x) together with an obvious induction on i. ☐

Corollary 4.17. *Theorem 2.15 is true.*

Proof. From Lemma 4.16 and Lemma 4.13 we obtain a short exact sequence:

$$0\longrightarrow K(r)\cap T_{X,x}\longrightarrow T_{X,x}\longrightarrow \mathrm{Ext}^1_{\mathcal{G}_K}(r,r) \tag{4.21}$$

which by Corollary 4.12 gives the bound:

$$\dim_{k(x)}T_{X,x}\leq \dim_{k(x)}K(r)+\lg(w_x)-d_x+\dim_{k(x)}\mathrm{Ext}^1_{\mathcal{G}_K}(r,r)-[K:\mathbb{Q}_p]\frac{n(n-1)}{2}.$$

But from Lemma 4.13 and Lemma 4.14 + Remark 4.15 we have:

$$\dim_{k(x)}K(r)+\lg(w_x)-d_x+\dim_{k(x)}\mathrm{Ext}^1_{\mathcal{G}_K}(r,r)-[K:\mathbb{Q}_p]\frac{n(n-1)}{2}=$$
$$\lg(w_x)-d_x+n^2+[K:\mathbb{Q}_p]\frac{n(n+1)}{2}$$

which gives Theorem 2.15. ☐

5. Modularity and local geometry of the trianguline variety

We prove that the main conjecture of [13] (see [13, Conj.3.23]), and thus the classical modularity conjectures by [13, Prop.3.27], imply Conjecture 2.8 when $\overline{r}$ "globalizes" and x is very regular.

5.1. **Companion points on the patched eigenvariety.** We prove that the existence of certain points (e.g. companion points) on the patched eigenvariety $X_p(\overline{\rho})$ implies the existence of others (which are "less companion"), see Theorem 5.5. This result, which can be seen as an "un-companioning" process, is used in the proof of Proposition 5.9 below.

We use the notation of §3. We denote by $\mathfrak{g}$ (resp. $\mathfrak{b}$, resp. $\mathfrak{t}$) the $\mathbb{Q}_p$-Lie algebra of G_p (resp. B_p, resp. T_p). We also denote by $\mathfrak{n}$ (resp. $\overline{\mathfrak{n}}$) the $\mathbb{Q}_p$-Lie algebra of the inverse image N_p in B_p (resp. $\overline{N}_p$ in $\overline{B}_p$) of the subgroup of upper (resp. lower) unipotent matrices of

$\prod_{v\in S_p}\mathrm{GL}_n(F_{\tilde v})$. We add an index L for the L-Lie algebras obtained by scalar extension $\cdot\otimes_{\mathbb{Q}_p}L$ (e.g. $\mathfrak{g}_L$, etc.) and we denote by $U(\cdot)$ the corresponding enveloping algebras.

For $v\in S_p$ we denote by $\mathfrak{t}_v$ the $\mathbb{Q}_p$-Lie algebra of the torus T_v, so that $\mathfrak{t}=\prod_{v\in S_p}\mathfrak{t}_v$. Recall that $\mathfrak{t}_v$ is an $F_{\tilde v}$-vector space, and thus $\mathfrak{t}_{v,L}=\mathfrak{t}_v\otimes_{\mathbb{Q}_p}L\cong\prod_{\tau\colon F_{\tilde v}\hookrightarrow L}\mathfrak{t}_v\otimes_{F_{\tilde v},\tau}L$. We can see any $\eta=(\eta_v)_{v\in S_p}=(\eta_{v,1},\dots,\eta_{v,n})_{v\in S_p}\in\widehat{T}_{p,L}$ as an L-valued additive character of $\mathfrak{t}$, and thus of $\mathfrak{t}_L$ by L-linearity, via the usual derivative action $(\mathfrak{z}_{v,1},\dots,\mathfrak{z}_{v,n})_{v\in S_p}\mapsto\sum_{v\in S_p}\sum_{i=1}^n\frac{d}{dt}\eta_{v,i}(\exp(t\mathfrak{z}_{v,i}))|_{t=0}$. Recall that the character $\mathfrak{z}_{v,i}\in F_{\tilde v}\mapsto\frac{d}{dt}\eta_{v,i}(\exp(t\mathfrak{z}_{v,i}))|_{t=0}$ is nothing else than $\sum_{\tau\colon F_{\tilde v}\hookrightarrow L}\tau(\mathfrak{z}_{v,i})\mathrm{wt}_\tau(\eta_{v,i})\in L$.

In what follows we use notation and definitions from [46] concerning L-Banach representations of p-adic Lie groups and their locally $\mathbb{Q}_p$-analytic vectors. If Π is an admissible continuous representation of G_p on a L-Banach space we denote by $\Pi^{\mathrm{an}}\subseteq\Pi$ its invariant subspace of locally $\mathbb{Q}_p$-analytic vectors.

Lemma 5.1. *Let Π be an admissible continuous representation of G_p on a L-Banach space and assume that the continuous dual Π' is a finite projective $\mathcal{O}_L[\![K_p]\!][1/p]$-module. Let λ, μ be L-valued characters of $\mathfrak{t}_L$ that we see as L-valued characters of $\mathfrak{b}_L$ by sending $\mathfrak{n}_L$ to 0. If $U(\mathfrak{g}_L)\otimes_{U(\mathfrak{b}_L)}\mu\hookrightarrow U(\mathfrak{g}_L)\otimes_{U(\mathfrak{b}_L)}\lambda$ is an injection of $U(\mathfrak{g}_L)$-modules, then the map:*

$$\mathrm{Hom}_{U(\mathfrak{g}_L)}\Big(U(\mathfrak{g}_L)\otimes_{U(\mathfrak{b}_L)}\lambda,\Pi^{\mathrm{an}}\Big)\longrightarrow\mathrm{Hom}_{U(\mathfrak{g}_L)}\Big(U(\mathfrak{g}_L)\otimes_{U(\mathfrak{b}_L)}\mu,\Pi^{\mathrm{an}}\Big)$$

induced by functoriality is surjective.

Proof. We have as in [46, Prop.6.5] a K_p-equivariant isomorphism:

$$\Pi^{\mathrm{an}}\cong\lim_{\substack{r\to 1\\ r<1}}\Pi_r \tag{5.1}$$

where each $\Pi_r\subseteq\Pi^{\mathrm{an}}$ is a Banach space over L endowed with an admissible locally $\mathbb{Q}_p$-analytic action of K_p. In particular each Π_r is stable under $U(\mathfrak{g}_L)$ in Π^{an}. If $f:U(\mathfrak{g}_L)\otimes_{U(\mathfrak{b}_L)}\lambda\to\Pi^{\mathrm{an}}$ is a $U(\mathfrak{g}_L)$-equivariant morphism, the source, being of finite type over $U(\mathfrak{g}_L)$, factors through some Π_r by (5.1). Moreover the action of $U(\mathfrak{g}_L)$ on Π_r extends to an action of the L-Banach algebra $U_r(\mathfrak{g}_L)$ which is the topological closure of $U(\mathfrak{g}_L)$ in the completed distribution algebra $D_r(K_p,L)$ (see [46, §5]). Consequently f extends to a $U_r(\mathfrak{g}_L)$-equivariant morphism:

$$f_r:U_r(\mathfrak{g}_L)\otimes_{U(\mathfrak{g}_L)}(U(\mathfrak{g}_L)\otimes_{U(\mathfrak{b}_L)}\lambda)\cong U_r(\mathfrak{g}_L)\otimes_{U_r(\mathfrak{b}_L)}\lambda\longrightarrow\Pi_r$$

where $U_r(\mathfrak{b}_L)$ is the closure of $U(\mathfrak{b}_L)$ in $D_r(K_p,L)$ and the first isomorphism follows from the isomorphism $\lambda\xrightarrow{\sim}U_r(\mathfrak{b}_L)\otimes_{U(\mathfrak{b}_L)}\lambda$ (which holds since its image is dense and λ is finite dimensional). We deduce from [44, Prop.3.4.8] (applied with $w=1$) that the injection $U(\mathfrak{g}_L)\otimes_{U(\mathfrak{b}_L)}\lambda\hookrightarrow U(\mathfrak{g}_L)\otimes_{U(\mathfrak{b}_L)}\mu$ extends to an injection of $U_r(\mathfrak{g}_L)$-modules $U_r(\mathfrak{g}_L)\otimes_{U_r(\mathfrak{b}_L)}\lambda\hookrightarrow U_r(\mathfrak{g}_L)\otimes_{U_r(\mathfrak{b}_L)}\mu$. Moreover, as $U_r(\mathfrak{g}_L)\otimes_{U_r(\mathfrak{b}_L)}\lambda$ and $U_r(\mathfrak{g}_L)\otimes_{U_r(\mathfrak{b}_L)}\mu$ are $U_r(\mathfrak{g}_L)$-modules of finite type, they have a unique topology of Banach module over $U_r(\mathfrak{g}_L)$ and every $U_r(\mathfrak{g}_L)$-linear map of one of them into Π_r is automatically continuous (see [46, Prop.2.1]). We deduce from all this isomorphisms:

$$\begin{aligned}\mathrm{Hom}_{U(\mathfrak{g}_L)}\Big(U(\mathfrak{g}_L)\otimes_{U(\mathfrak{b}_L)}\lambda,\Pi^{\mathrm{an}}\Big)&\xrightarrow{\sim}\varinjlim_r\mathrm{Hom}_{U(\mathfrak{g}_L)}\Big(U(\mathfrak{g}_L)\otimes_{U(\mathfrak{b}_L)}\lambda,\Pi_r\Big)\\&\xrightarrow{\sim}\varinjlim_r\mathrm{Hom}_{U_r(\mathfrak{g}_L)-\mathrm{cont}}\Big(U_r(\mathfrak{g}_L)\otimes_{U_r(\mathfrak{b}_L)}\lambda,\Pi_r\Big)\end{aligned}$$

where $\mathrm{Hom}_{U_r(\mathfrak{g}_L)-\mathrm{cont}}$ means continuous homomorphisms of $U_r(\mathfrak{g}_L)$-Banach modules, and likewise with μ instead of λ. By exactitude of $\varinjlim_r$, we see that it is enough to prove that Π_r is an injective object (with respect to injections which have closed image) in the category of $U_r(\mathfrak{g}_L)$-Banach modules with continuous maps.

By assumption the dual Π' is a projective module of finite type over $\mathcal{O}_L[\![K_p]\!][1/p]$, hence a direct summand of $\mathcal{O}_L[\![K_p]\!][1/p]^{\oplus s}$ for some $s > 0$. From the proof of [46, Prop.6.5] together with [46, Th.7.1(iii)], we also know that Π_r is the continuous dual of the $D_r(K_p, L)$-Banach module:

$$\Pi'_r := D_r(K_p, L) \otimes_{\mathcal{O}_L[\![K_p]\!][1/p]} \Pi'.$$

We get that the $D_r(K_p, L)$-module Π'_r is a direct summand of $D_r(K_p, L)^{\oplus s}$. Now it easily follows from the results in [38, §1.4] that $D_r(K_p, L)$ is itself a free $U_r(\mathfrak{g}_L)$-module of finite rank. Dualizing, we finally obtain that there is a finite dimensional L-vector space W such that the left $U_r(\mathfrak{g}_L)$-Banach module Π_r is a direct factor of the left $U_r(\mathfrak{g}_L)$-Banach module $\mathrm{Hom}_{\mathrm{cont}}(U_r(\mathfrak{g}_L) \otimes_L W, L)$ (which is seen as a left $U_r(\mathfrak{g}_L)$-module via the automorphism on $U_r(\mathfrak{g}_L)$ extending the multiplication by -1 on $\mathfrak{g}_L$). Since direct summands and finite sums of injective modules are still injective, it is enough to prove the injectivity of $\mathrm{Hom}_{\mathrm{cont}}(U_r(\mathfrak{g}_L), L)$ in the category of $U_r(\mathfrak{g}_L)$-Banach modules with continuous maps.

If V is any $U_r(\mathfrak{g}_L)$-Banach module, it is not difficult to see that there is a canonical isomorphism of Banach spaces over L:

$$\mathrm{Hom}_{U_r(\mathfrak{g}_L)-\mathrm{cont}}\Big(V, \mathrm{Hom}_{\mathrm{cont}}(U_r(\mathfrak{g}_L), L)\Big) \xrightarrow{\sim} \mathrm{Hom}_{\mathrm{cont}}(V, L) \tag{5.2}$$

so that the required injectivity property is a consequence of the Hahn-Banach Theorem (see for example [45, Prop.9.2]). □

We go on with two technical lemmas which require more notation. Fix a compact open uniform normal pro-p subgroup H_p of K_p such that $H_p = (\overline{N}_p \cap H_p)(T_p \cap H_p)(N_p \cap H_p)$. For example H_p can be chosen of the form $\prod_{v \in S_p} H_v$ where H_v is the inverse image in K_v of matrices of $\mathrm{GL}_n(\mathcal{O}_{F_{\tilde{v}}})$ congruent to 1 mod p^m for m big enough. Let $N_0 := N_p \cap H_p$, $T_{p,0} := T_p \cap H_p$, $\overline{N}_0 := \overline{N}_p \cap H_p$ (which are still uniform pro-p-groups) and $T_p^+ := \{t \in T_p$ such that $tN_0t^{-1} \subseteq N_0\}$ (which is a multiplicative monoid in T_p). We also fix $z \in T_p^+$ such that $zN_0z^{-1} \subseteq N_0^p$ and we assume moreover $z^{-1}H_pz \subseteq K_p$ so that the elements of $z^{-1}H_pz$ normalize H_p (as H_p is normal in K_p). Note that such a z always exists, for instance take z such that $zN_0z^{-1} \subseteq N_0^p$, choose r such that $H_p^{p^r} \subseteq zK_pz^{-1}$ and replace H_p by $H_p^{p^r}$: with this new choice we still have $zN_0z^{-1} \subseteq N_0^p$.

For any uniform pro-p-group H we denote by $\mathcal{C}(H, L)$ the Banach space of continuous L-valued functions on H and, if $h \geq 1$, by $\mathcal{C}^{(h)}(H, L)$ the Banach space of h-analytic L-valued functions on H defined in [20, §0.3]. We have $\mathcal{C}(H_p, L) \cong \mathcal{C}(\overline{N}_0, L)\widehat{\otimes}_L\mathcal{C}(T_{p,0}, L)\widehat{\otimes}_L\mathcal{C}(N_0, L)$ and likewise with $\mathcal{C}^{(h)}(\cdot, L)$.

Lemma 5.2. *Let $f \in \mathcal{C}(H_p, L)$ such that for each left coset $(zH_pz^{-1} \cap H_p)n \subset H_p$, there exists $f_n \in \mathcal{C}^{(h)}(H_p, L)$ such that $f(gn) = f_n(z^{-1}gz)$ for $g \in zH_pz^{-1} \cap H_p$. Then we have:*

$$f \in \mathcal{C}^{(h-1)}(\overline{N}_0, L)\widehat{\otimes}_L\mathcal{C}^{(h)}(T_{p,0}, L)\widehat{\otimes}_L\mathcal{C}(N_0, L).$$

Proof. Representatives of the quotient $(zH_pz^{-1} \cap H_p)\backslash H_p$ can be chosen in N_0, whence the above notation n (do not confuse with the n of GL_n!). Restricting f to the left coset

$(zH_pz^{-1}\cap H_p)n$ for some $n\in N_0$ and translating on the right by n we can assume that the support of f is contained in $zH_pz^{-1}\cap H_p$. Then if $g\in zH_pz^{-1}\cap H_p$, we have by assumption $f(g)=F(z^{-1}gz)$ for some $F\in\mathcal{C}^{(h)}(H_p,L)$. Consequently $f|_{zH_pz^{-1}\cap H_p}$ can be extended to an h-analytic function on zH_pz^{-1} and f can be extended (by 0) on $zH_pz^{-1}N_0=z\overline{N}_0z^{-1}T_{p,0}N_0$ as an element of:

$$\mathcal{C}^{(h)}(z\overline{N}_0z^{-1},L)\widehat{\otimes}_L\mathcal{C}^{(h)}(T_{p,0},L)\widehat{\otimes}_L\mathcal{C}(N_0,L).$$

We deduce that f is in the image of the restriction map (note that $zN_0z^{-1}\subseteq N_0$ implies $\overline{N}_0\subseteq z\overline{N_0}z^{-1}$):

$$\mathcal{C}^{(h)}(z\overline{N}_0z^{-1},L)\widehat{\otimes}_L\mathcal{C}^{(h)}(T_{p,0},L)\widehat{\otimes}_L\mathcal{C}(N_0,L)\longrightarrow\mathcal{C}^{(h)}(\overline{N}_0,L)\widehat{\otimes}_L\mathcal{C}^{(h)}(T_{p,0},L)\widehat{\otimes}_L\mathcal{C}(N_0,L).$$

Now the stronger condition $zN_0z^{-1}\subseteq N_0^p$ implies $\overline{N}_0\subseteq z\overline{N_0}^pz^{-1}=(z\overline{N}_0z^{-1})^p$. But by [20, Rem.IV.12] the restriction to $(z\overline{N}_0z^{-1})^p$ (and *a fortiori* to $\overline{N}_0$) of an h-analytic function on $z\overline{N}_0z^{-1}$ is $(h-1)$-analytic and we can conclude. □

If Π is an admissible continuous representation of G_p on a L-Banach space and if $h\geq 1$, we denote by $\Pi_{H_p}^{(h)}$ the H_p-invariant Banach subspace of Π^{an} defined in [20, §0.3]. If V is any (left) $U(\mathfrak{t}_L)$-module over L and $\lambda:\mathfrak{t}_L\to L$ is a character, we let V_λ be the L-subvector space of V on which $\mathfrak{t}_L$ acts via the multiplication by λ. Recall that if V is any $L[G_p]$-module, the monoid T_p^+ acts on V^{N_0} via $v\longmapsto t\cdot v:=\delta_{B_p}(t)\sum_{n_0\in N_0/tN_0t^{-1}}n_0tv$ ($v\in V^{N_0}$, $t\in T_p^+$, see §3.1 for δ_{B_p}). This T_p^+-action respects the subspace $(\Pi_\lambda^{\mathrm{an}})^{N_0}$ of $(\Pi^{\mathrm{an}})^{N_0}$ (use that $tN_0t^{-1}=N_0$ for $t\in T_p^0$).

We don't claim any originality on the following lemma which is a variant of classical results (see e.g. [22]), however we couldn't find its exact statement in the literature.

Lemma 5.3. *Let Π be an admissible continuous representation of G_p on a L-Banach space, λ an L-valued character of $\mathfrak{t}_L$ and $h\geq 1$. Then the action of z on $(\Pi^{\mathrm{an}})^{N_0}$ preserves the subspace $(\Pi_{H_p}^{(h)})_\lambda^{N_0}=(\Pi_{H_p}^{(h)})^{N_0}\cap\Pi_\lambda^{\mathrm{an}}$ and is a* compact *operator on this subspace.*

Proof. Let $l_1,\dots,l_s$ be a system of generators of the continous dual Π' as a module over the algebra $\mathcal{O}_L[\![H_p]\!][1/p]$. Define a closed embedding of Π into $\mathcal{C}(H_p,L)^{\oplus s}$ via the map $v\mapsto(g\mapsto l_i(gv))_{1\leq i\leq s}$. This embedding is H_p-equivariant for the left action of H_p on $\mathcal{C}(H_p,L)$ by right translation on functions. By [20, Prop.IV.5], we have $\Pi_{H_p}^{(h)}=\Pi\cap\mathcal{C}^{(h)}(H_p,L)^{\oplus s}$. If $v\in\Pi_{H_p}^{(h)}$, $n\in N_0$ and $g\in H_p\cap zH_pz^{-1}$, we have $l_i(gnzv)=l_i(z(z^{-1}gz)(z^{-1}nz)v)$. Let $v\in\Pi_{H_p}^{(h)}$ and $n\in N_0$. As $z^{-1}N_0z$ normalizes H_p (by the choice of z) we have $w:=(z^{-1}nz)v\in\Pi_{H_p}^{(h)}$ (see [20, Prop.IV.16]). As $l_i(z\cdot)$ is a continuous linear form on Π, using [20, Thm.IV.6(i)] the function $f_n:H_p\to L$, $g\mapsto f_n(g):=l_i(zgw)$ is in $\mathcal{C}^{(h)}(H_p,L)$ and $l_i(gnzv)=f_n(z^{-1}gz)$ for $g\in zH_pz^{-1}\cap H_p$. We deduce from Lemma 5.2 applied to the functions $f:H_p\to L$, $g\mapsto l_i(gzv)$ for $1\leq i\leq s$ that:

$$z\Pi_{H_p}^{(h)}\subset\Big(\mathcal{C}^{(h-1)}(\overline{N}_0,L)\widehat{\otimes}_L\mathcal{C}^{(h)}(T_{p,0},L)\widehat{\otimes}_L\mathcal{C}(N_0,L)\Big)^{\oplus s}. \tag{5.3}$$

Let $v\in(\Pi_{H_p}^{(h)})^{N_0}$, the space on the right hand side of (5.3) being stable under N_0 (acting by right translation on functions), it still contains $z\cdot v=\sum_{n_0\in N_0/tN_0t^{-1}}n_0zv$. Since $z\cdot v$ is fixed under N_0, we deduce:

$$z\cdot v\ \in\ \Big(\mathcal{C}^{(h-1)}(\overline{N}_0,L)\widehat{\otimes}_L\mathcal{C}^{(h)}(T_{p,0},L)\Big)^{\oplus s}\subseteq\ \mathcal{C}^{(h)}(H_p,L)^{\oplus s}.$$

In particular $z\cdot(\Pi_{H_p}^{(h)})_\lambda^{N_0}\subseteq(\Pi_\lambda^{\rm an})^{N_0}\cap\mathcal{C}^{(h)}(H_p,L)^{\oplus s}=(\Pi_{H_p}^{(h)})_\lambda^{N_0}$ which shows the first statement. We also deduce:

$$z\cdot(\Pi_{H_p}^{(h)})_\lambda^{N_0}\subseteq\Big(\mathcal{C}^{(h-1)}(\overline{N}_0,L)\widehat{\otimes}_L\mathcal{C}^{(h)}(T_{p,0},L)\Big)_\lambda^{\oplus s}\cong\mathcal{C}^{(h-1)}(\overline{N}_0,L)\widehat{\otimes}_L\Big(\mathcal{C}^{(h)}(T_{p,0},L)_\lambda^{\oplus s}\Big).$$

But by [20, Prop. IV.13.(i)], we have $\mathcal{C}^{(h)}(T_{p,0},L)'\simeq D_r(T_{p,0},L)$ for $r=p^{-1/p^h}$ where $D_r(T_{p,0},L)$ is as in [46, §4]. Let $U_r(\mathfrak{t}_L)$ be the closure of $U(\mathfrak{t}_L)$ in $D_r(T_{p,0},L)$, then (as in the proof of Lemma 5.1) $D_r(T_{p,0},L)$ is a finite free $U_r(\mathfrak{t}_L)$-module ([38, §1.4]). Using $\lambda\xrightarrow{\sim}U_r(\mathfrak{t}_L)\otimes_{U(\mathfrak{t}_L)}\lambda$, it follows that $(\mathcal{C}^{(h)}(T_{p,0},L)_\lambda)'$, and hence $\mathcal{C}^{(h)}(T_{p,0},L)_\lambda^{\oplus s}$, are finite dimensional L-vector spaces. We denote the latter by W_λ.

We thus have $z\cdot(\Pi_{H_p}^{(h)})_\lambda^{N_0}\subseteq\mathcal{C}^{(h-1)}(\overline{N}_0,L)\otimes_L W_\lambda$: the endomorphism induced by z on $(\Pi_{H_p}^{(h)})_\lambda^{N_0}$ factors through the subspace $(\Pi_{H_p}^{(h)})_\lambda^{N_0}\cap\Big(\mathcal{C}^{(h-1)}(\overline{N}_0,L)\otimes_L W_\lambda\Big)$. As the inclusion of $\mathcal{C}^{(h-1)}(\overline{N}_0,L)$ into $\mathcal{C}^{(h)}(\overline{N}_0,L)$ is compact and W_λ is finite dimensional over L, the inclusion of $(\Pi_{H_p}^{(h)})_\lambda^{N_0}\cap\Big(\mathcal{C}^{(h-1)}(\overline{N}_0,L)\otimes_L W\Big)$ into $(\Pi_{H_p}^{(h)})_\lambda^{N_0}$ is compact, which proves the result. □

If $\delta_v,\epsilon_v\in\widehat{T}_{v,L}$, we write $\epsilon_v\uparrow_{\mathfrak{t}_v}\delta_v$ if, seeing δ_v,ϵ_v as $U(\mathfrak{t}_{v,L})$-modules, we have $\epsilon_v\uparrow\delta_v$ in the sense of [32, §5.1] with respect to the roots of the upper triangular matrices in $(\mathrm{Res}_{F_{\tilde v}/\mathbb{Q}_p}\mathrm{GL}_{n,F_{\tilde v}})_L$. Likewise if $\delta,\epsilon\in\widehat{T}_{p,L}$, we write $\epsilon\uparrow_{\mathfrak{t}}\delta$ if, seeing δ,ϵ as $U(\mathfrak{t}_L)$-modules, we have $\epsilon\uparrow\delta$ in the sense of [32, §5.1] with respect to the roots of the upper triangular matrices in $\prod_{v\in S_p}(\mathrm{Res}_{F_{\tilde v}/\mathbb{Q}_p}\mathrm{GL}_{n,F_{\tilde v}})_L$. Thus writing $\delta=(\delta_v)_{v\in S_p},\epsilon=(\epsilon_v)_{v\in S_p}$, we have $\epsilon\uparrow_{\mathfrak{t}}\delta$ if and only if $\epsilon_v\uparrow_{\mathfrak{t}_v}\delta_v$ for all $v\in S_p$.

Definition 5.4. *Let $\delta,\epsilon\in\widehat{T}_{p,L}$, we write $\epsilon\uparrow\delta$ if $\epsilon\uparrow_{\mathfrak{t}}\delta$ and if $\epsilon\delta^{-1}$ is an algebraic character of T_p, i.e. $\epsilon\delta^{-1}=\delta_\lambda$ for some $\lambda=(\lambda_v)_{v\in S_p}\in\prod_{v\in S_p}(\mathbb{Z}^n)^{\mathrm{Hom}(F_{\tilde v},L)}$.*

We can now prove the main theorem of this section.

Theorem 5.5. *Let $\mathfrak{m}\subseteq R_\infty[1/p]$ be a maximal ideal, $\delta,\epsilon\in\widehat{T}_{p,L}$ such that $\epsilon\uparrow\delta$ and L' a finite extension of L containing the residue fields $k(\delta)=k(\epsilon)$ and $k(\mathfrak{m})$. Then we have:*

$$\mathrm{Hom}_{T_p}\Big(\epsilon,J_{B_p}(\Pi_\infty^{R_\infty-\mathrm{an}}[\mathfrak{m}]\otimes_{k(\mathfrak{m})}L')\Big)\neq 0\Longrightarrow\mathrm{Hom}_{T_p}\Big(\delta,J_{B_p}(\Pi_\infty^{R_\infty-\mathrm{an}}[\mathfrak{m}]\otimes_{k(\mathfrak{m})}L')\Big)\neq 0.$$

Proof. We assume first $k(\delta)=k(\epsilon)=L$ and $L'=k(\mathfrak{m})$, so that we can forget about L'. Let Π be a locally $\mathbb{Q}_p$-analytic representation of B_p over L. The subspace $\Pi^{\mathfrak{n}_L}$ of vectors killed by $\mathfrak{n}_L$ is a smooth representation of the group N_0 and we denote by $\pi_{N_0}:\Pi^{\mathfrak{n}_L}\twoheadrightarrow\Pi^{N_0}\subseteq\Pi^{\mathfrak{n}_L}$ the unique N_0-equivariant projection on its subspace Π^{N_0}. It is preserved by the action of T_p inside Π, hence also by the action of $\mathfrak{t}_L$ and one easily checks that:

$$\pi_{N_0}\circ\mathfrak{x}=\mathfrak{x}\circ\pi_{N_0}\qquad(\mathfrak{x}\in\mathfrak{t}_L)\tag{5.4}$$

(use $tN_0t^{-1}=N_0$ for $t\in T_p^0$). The subspace $\Pi_\lambda^{\mathfrak{n}_L}:=\Pi_\lambda\cap\Pi^{\mathfrak{n}_L}\subseteq\Pi^{\mathfrak{n}_L}$ is still preserved by T_p and by (5.4) the projection π_{N_0} sends $\Pi_\lambda^{\mathfrak{n}_L}$ onto $\Pi_\lambda^{N_0}:=\Pi^{N_0}\cap\Pi_\lambda^{\mathfrak{n}_L}\subseteq\Pi_\lambda^{\mathfrak{n}_L}$. We have $t\cdot v=\pi_{N_0}(tv)$ for $t\in T_p^+$, $v\in\Pi_\lambda^{N_0}$ and in the rest of the proof we view $\Pi_\lambda^{N_0}$ as an $L[T_p^+]$-module via this monoid action.

The locally $\mathbb{Q}_p$-analytic character $\delta:T_p\to L^\times$ determines a surjection of L-algebras $L[T_p]\twoheadrightarrow L$ and we denote its kernel by $\mathfrak{m}_\delta$ (a maximal ideal of the L-algebra $L[T_p]$). We still write $\mathfrak{m}_\delta$ for its intersection with $L[T_p^+]$, which is then a maximal ideal of $L[T_p^+]$. Let $\lambda : \mathfrak{t}_L \to L$ be the derivative of δ, arguing as in [22, Prop.3.2.12] we get for $s \geq 1$:

$$J_{B_p}(\Pi)[\mathfrak{m}_\delta^s] \cong \Pi^{N_0}[\mathfrak{m}_\delta^s] \cong \Pi_\lambda^{N_0}[\mathfrak{m}_\delta^s], \tag{5.5}$$

(in particular $\mathrm{Hom}_{T_p}(\delta, J_{B_p}(\Pi)) \cong \Pi^{N_0}[\mathfrak{m}_\delta] \cong \Pi_\lambda^{N_0}[\mathfrak{m}_\delta]$). Likewise we have $J_{B_p}(\Pi)[\mathfrak{m}_\epsilon^s] \cong \Pi^{N_0}[\mathfrak{m}_\epsilon^s] \cong \Pi_\mu^{N_0}[\mathfrak{m}_\epsilon^s]$ if $\mu : \mathfrak{t}_L \to L$ is the derivative of ϵ.

Let $\mathfrak{J} \subset S_\infty[1/p]$ be an ideal such that $\dim_L(S_\infty[1/p]/\mathfrak{J}) < \infty$ and define $\Pi_\mathfrak{J} := \Pi_\infty[\mathfrak{J}]$. As the continuous dual Π'_∞ is a finite projective $S_\infty[\![K_p]\!][1/p]$-module (property (ii) in §3.2), the continuous dual $\Pi_\infty[\mathfrak{J}]'$ of the G_p-representation $\Pi_\infty[\mathfrak{J}]$, which is isomorphic to $\Pi'_\infty/\mathfrak{J}$ by the discussion at the end of [13, §3.1], is a finite projective $S_\infty[\![K_p]\!][1/p]/\mathfrak{J}S_\infty[\![K_p]\!][1/p] \cong \mathcal{O}_L[\![K_p]\!][1/p]$-module (in particular it is an admissible continuous representation of G_p over L). Moreover it is immediate to check that $\Pi_\infty^{R_\infty-\mathrm{an}}[\mathfrak{J}]$ is isomorphic to the subspace $\Pi_\mathfrak{J}^{\mathrm{an}}$ of locally $\mathbb{Q}_p$-analytic vectors of $\Pi_\mathfrak{J}$.

Taking the image of a vector in (the underlying L-vector space of) λ or μ gives natural isomorphisms:

$$\mathrm{Hom}_{U(\mathfrak{g}_L)}\Big(U(\mathfrak{g}_L) \otimes_{U(\mathfrak{b}_L)} \lambda, \Pi_\mathfrak{J}^{\mathrm{an}}\Big) \xrightarrow{\sim} (\Pi_\mathfrak{J}^{\mathrm{an}})_\lambda^{\mathfrak{n}_L} \text{ and } \mathrm{Hom}_{U(\mathfrak{g}_L)}\Big(U(\mathfrak{g}_L) \otimes_{U(\mathfrak{b}_L)} \mu, \Pi_\mathfrak{J}^{\mathrm{an}}\Big) \xrightarrow{\sim} (\Pi_\mathfrak{J}^{\mathrm{an}})_\mu^{\mathfrak{n}_L}.$$

As by assumption we have $\mu \uparrow \lambda$ in the sense of [32, §5.1] (for the algebraic group $(\mathrm{Res}_{F_{\tilde{v}}/\mathbb{Q}_p}\mathrm{GL}_{n,F_{\tilde{v}}})_L$ with respect to the roots of the upper triangular matrices), [32, Th.5.1] implies the existence of a unique (up to L-homothety) $U(\mathfrak{g}_L)$-equivariant injection:

$$\iota_{\mu,\lambda} : U(\mathfrak{g}_L) \otimes_{U(\mathfrak{b}_L)} \mu \hookrightarrow U(\mathfrak{g}_L) \otimes_{U(\mathfrak{b}_L)} \lambda \tag{5.6}$$

which induces an L-linear map:

$$\iota_{\mu,\lambda}^* : (\Pi_\mathfrak{J}^{\mathrm{an}})_\lambda^{\mathfrak{n}_L} \longrightarrow (\Pi_\mathfrak{J}^{\mathrm{an}})_\mu^{\mathfrak{n}_L}. \tag{5.7}$$

We claim that (5.7) maps the subspace $(\Pi_\mathfrak{J}^{\mathrm{an}})_\lambda^{N_0}$ to the subspace $(\Pi_\mathfrak{J}^{\mathrm{an}})_\mu^{N_0}$. It is enough to prove:

$$\iota_{\mu,\lambda}^* \circ \pi_{N_0} = \pi_{N_0} \circ \iota_{\mu,\lambda}^*. \tag{5.8}$$

Let $\mathfrak{v}$ be the image by (5.6) of a nonzero vector v in the underlying L-vector space of μ. Writing $U(\mathfrak{g}_L) \otimes_{U(\mathfrak{b}_L)} \lambda \cong U(\mathfrak{n}_L^-)$ we see that $\mathfrak{v} \in U(\mathfrak{n}_L^-)_{\mu-\lambda}$ (with obvious notation), that $\mathfrak{v}$ is killed by $\mathfrak{n}_L$ in $U(\mathfrak{g}_L) \otimes_{U(\mathfrak{b}_L)} \lambda$ and that the morphism $\iota_{\mu,\lambda}^*$ is given by the action (on the left) by $\mathfrak{v}$. To get (5.8) it is enough to prove $\pi_{N_0} \circ \mathfrak{v} = \mathfrak{v} \circ \pi_{N_0}$ on $(\Pi_\mathfrak{J}^{\mathrm{an}})_\lambda^{\mathfrak{n}_L}$, which itself follows from $n \circ \mathfrak{v} = \mathfrak{v} \circ n$ for $n \in N_0$ (n acting via the underlying G_p-action on $\Pi_\mathfrak{J}^{\mathrm{an}}$), or equivalently $\mathrm{Ad}(n)(\mathfrak{v}) = \mathfrak{v}$ on $(\Pi_\mathfrak{J}^{\mathrm{an}})_\lambda^{\mathfrak{n}_L}$. Writing $n = \exp(\mathfrak{m})$ with $\mathfrak{m} \in \mathfrak{n}_L$ (do not confuse here with the maximal ideal $\mathfrak{m}$!), recall we have $\mathrm{Ad}(n)(\mathfrak{v}) = \exp(\mathrm{ad}(\mathfrak{m}))(\mathfrak{v})$ (using standard notation). Since $\mathfrak{v}$ is killed by left multiplication by $\mathfrak{n}_L$ in $U(\mathfrak{g}_L) \otimes_{U(\mathfrak{b}_L)} \lambda$, we have:

$$\exp(\mathrm{ad}(\mathfrak{m}))(\mathfrak{v}) \in \mathfrak{v} + U(\mathfrak{g}_L)(\mathfrak{n}_L + \ker(\lambda))$$

where $\ker(\lambda) := \ker(U(\mathfrak{g}_L) \to U(\mathfrak{g}_L) \otimes_{U(\mathfrak{b}_L)} \lambda,\ \mathfrak{x} \mapsto \mathfrak{x} \otimes v)$. The action of $\mathrm{Ad}(n)(\mathfrak{v})$ on $(\Pi_\mathfrak{J}^{\mathrm{an}})_\lambda^{\mathfrak{n}_L}$ is thus the same as that of $\mathfrak{v}$.

We still write:

$$\iota_{\mu,\lambda}^* : (\Pi_\mathfrak{J}^{\mathrm{an}})_\lambda^{N_0} \longrightarrow (\Pi_\mathfrak{J}^{\mathrm{an}})_\mu^{N_0} \tag{5.9}$$

for the map induced by (5.7). Using $\mathfrak{v} \in U(\mathfrak{n}_L^-)_{\mu-\lambda}$ together with (5.8), it is easy to check that $\iota_{\mu,\lambda}^* \circ t = (\delta\epsilon^{-1})(t)(t \circ \iota_{\mu,\lambda}^*)$ for $t \in T_p^+$ (for the previous $L[T_p^+]$-module structure). Moreover, it follows from Lemma 5.1 that (5.7) is surjective, hence the top horizontal map and the two vertical maps are surjective in the commutative diagram:

$$\begin{array}{ccc} (\Pi_{\mathfrak{J}}^{\mathrm{an}})_\lambda^{\mathfrak{n}_L} & \xrightarrow{(5.7)} & (\Pi_{\mathfrak{J}}^{\mathrm{an}})_\mu^{\mathfrak{n}_L} \\ \downarrow{\scriptstyle \pi_{N_0}} & & \downarrow{\scriptstyle \pi_{N_0}} \\ (\Pi_{\mathfrak{J}}^{\mathrm{an}})_\lambda^{N_0} & \xrightarrow{(5.9)} & (\Pi_{\mathfrak{J}}^{\mathrm{an}})_\mu^{N_0} \end{array}$$

which implies that (5.9) is also surjective. Note also that both (5.7) and (5.9) trivially commute with the action of R_∞ (which factors through $R_\infty/\mathfrak{J}R_\infty$).

From [20, §0.3] we have $\Pi_{\mathfrak{J}}^{\mathrm{an}} \cong \lim_{h\to+\infty} \Pi_{\mathfrak{J},H_p}^{(h)}$ and thus:

$$(\Pi_{\mathfrak{J}}^{\mathrm{an}})_\lambda^{N_0} \cong \lim_{h\to+\infty} (\Pi_{\mathfrak{J},H_p}^{(h)})_\lambda^{N_0} \quad \text{and} \quad (\Pi_{\mathfrak{J}}^{\mathrm{an}})_\mu^{N_0} \cong \lim_{h\to+\infty} (\Pi_{\mathfrak{J},H_p}^{(h)})_\mu^{N_0}.$$

By Lemma 5.3 there is $z \in T_p^+$ which acts compactly on $(\Pi_{\mathfrak{J},H_p}^{(h)})_\lambda^{N_0}$ and $(\Pi_{\mathfrak{J},H_p}^{(h)})_\mu^{N_0}$. We deduce from this fact together with [47, Prop.9] and [47, Prop.12] that the map $\iota_{\mu,\lambda}^*$ in (5.9) remains surjective at the level of *generalized eigenspaces* for the action of T_p^+ (twisting this action by the character $\delta\epsilon^{-1}$ on the right hand side). Consequently $\iota_{\mu,\lambda}^*$ induces a surjective map:

$$\bigcup_{s\geq 1} (\Pi_{\mathfrak{J}}^{\mathrm{an}})_\lambda^{N_0}[\mathfrak{m}_\delta^s] \twoheadrightarrow \bigcup_{s\geq 1} (\Pi_{\mathfrak{J}}^{\mathrm{an}})_\mu^{N_0}[\mathfrak{m}_\epsilon^s].$$

As both the source and target of this map are unions of finite dimensional L-vector spaces (as follows from the admissibility of $\Pi_{\mathfrak{J}}^{\mathrm{an}}$, [22, Th.4.3.2] and (5.5)) which are stable under R_∞ and as $\iota_{\mu,\lambda}^*$ is R_∞-equivariant, the following map induced by $\iota_{\mu,\lambda}^*$ remains surjective:

$$\bigcup_{s,t\geq 1} (\Pi_{\mathfrak{J}}^{\mathrm{an}})_\lambda^{N_0}[\mathfrak{m}_\delta^s, \mathfrak{m}^t] \twoheadrightarrow \bigcup_{s,t\geq 1} (\Pi_{\mathfrak{J}}^{\mathrm{an}})_\mu^{N_0}[\mathfrak{m}_\epsilon^s, \mathfrak{m}^t]. \tag{5.10}$$

Since $\mathfrak{m}^t$ is an ideal of cofinite dimension in $R_\infty[1/p]$, the inverse image $\mathfrak{J}$ of $\mathfrak{m}^t$ in $S_\infty[1/p]$ is *a fortiori* of cofinite dimension in $S_\infty[1/p]$ and we can apply (5.10) with such an $\mathfrak{J}$. But we have for this $\mathfrak{J}$:

$$(\Pi_{\mathfrak{J}}^{\mathrm{an}})_\lambda^{N_0}[\mathfrak{m}_\delta^s, \mathfrak{m}^t] = (\Pi_\infty^{R_\infty-\mathrm{an}})_\lambda^{N_0}[\mathfrak{m}_\delta^s, \mathfrak{m}^t, \mathfrak{J}] = (\Pi_\infty^{R_\infty-\mathrm{an}})_\lambda^{N_0}[\mathfrak{m}_\delta^s, \mathfrak{m}^t]$$

and likewise with $\mathfrak{m}_\epsilon$, so that (5.10) is a surjection:

$$\bigcup_{s,t\geq 1} (\Pi_\infty^{R_\infty-\mathrm{an}})_\lambda^{N_0}[\mathfrak{m}_\delta^s, \mathfrak{m}^t] \twoheadrightarrow \bigcup_{s,t\geq 1} (\Pi_\infty^{R_\infty-\mathrm{an}})_\mu^{N_0}[\mathfrak{m}_\epsilon^s, \mathfrak{m}^t].$$

Looking at the eigenspaces on both sides, we obtain $\mathrm{Hom}_{T_p}(\delta, J_{B_p}(\Pi_\infty^{R_\infty-\mathrm{an}}[\mathfrak{m}])) \neq 0$ if $\mathrm{Hom}_{T_p}(\epsilon, J_{B_p}(\Pi_\infty^{R_\infty-\mathrm{an}}[\mathfrak{m}])) \neq 0$.
Finally, when $k(\delta) = k(\epsilon)$ is larger than L, we replace Π_∞ by $\Pi'_\infty := \Pi_\infty \otimes_L L'$, $S_\infty[1/p]$ by $S_\infty[1/p] \otimes_L L'$, $\mathfrak{m}$ by $\mathfrak{m}' := \ker(R_\infty[1/p] \otimes_L L' \twoheadrightarrow k(\mathfrak{m}) \otimes_L L' \twoheadrightarrow L')$ (the last surjection coming from the inclusion $k(\mathfrak{m}) \subseteq L'$) and the reader can check that all the arguments of the previous proof go through *mutatis mutandis.* □

5.2. **A closed embedding.** Assuming the main conjecture of [13] and using Theorem 5.5 we construct a certain closed embedding in the trianguline variety (Proposition 5.9).

We fix a continuous representation $\overline{r}:\mathcal{G}_K\to \mathrm{GL}_n(k_L)$ as in §2.1 and keep the local notation of §2 and §4. We also assume that there exist number fields F/F^+, a unitary group G/F^+, a tame level U^p, a set of finite places S and an irreducible representation $\overline{\rho}$ as in §3.1 such that all the assumptions in §3.1 and §3.2 are satisfied, and such that for each place $v\in S_p$ there is a place $\tilde{v}$ of F dividing v satisfying $F_{\tilde{v}}\cong K$ and $\overline{\rho}_{\tilde{v}}\cong \overline{r}$. Note that this implies in particular $(2n,p)=1$ (as $p>2$ and as $(n,p)=1$ by the proof of [27, Th.9]). Assuming $(2n,p)=1$, it follows from [18, Lem.2.2] and [18, §2.3] that such $(F/F^+,G,U^p,S,\overline{\rho})$ always exist if $n=2$ or if $\overline{r}$ is (absolutely) semi-simple (increasing L if necessary).

We recall the statement of [13, Conj.3.23] (see §2.3 for $\widetilde{X}^{\square}_{\mathrm{tri}}(\overline{r})$).

Conjecture 5.6. *The rigid subvariety $X^{\mathfrak{X}^p-\mathrm{aut}}_{\mathrm{tri}}(\overline{\rho}_p)$ of $X^{\square}_{\mathrm{tri}}(\overline{\rho}_p)$ doesn't depend on $\mathfrak{X}^p$ and is isomorphic to $\widetilde{X}^{\square}_{\mathrm{tri}}(\overline{\rho}_p):=\prod_{v\in S_p}\widetilde{X}^{\square}_{\mathrm{tri}}(\overline{\rho}_{\tilde{v}})$.*

Remark 5.7. (i) By (3.12), Conjecture 5.6 is thus equivalent to $X_p(\overline{\rho})\xrightarrow{\sim}\mathfrak{X}_{\overline{\rho}^p}\times\widetilde{X}^{\square}_{\mathrm{tri}}(\overline{\rho}_p)\times\mathbb{U}^g$.
(ii) The authors do not know if $\widetilde{X}^{\square}_{\mathrm{tri}}(\overline{\rho}_p)$ is really strictly smaller than $X^{\square}_{\mathrm{tri}}(\overline{\rho}_p)$.
(iii) Finally, recall that Conjecture 5.6 is *equivalent* to the classical modularity lifting conjectures for $\overline{\rho}$ (in all weights with trivial inertial type), see [13, Prop.3.27 & Prop.3.28].

Let $\mathbf{k}:=(\mathbf{k}_i)_{1\le i\le n}$ where $\mathbf{k}_i:=(k_{\tau,i})_{\tau:K\hookrightarrow L}\in\mathbb{Z}^{\mathrm{Hom}(K,L)}$ is such that $k_{\tau,i}>k_{\tau,i+1}$ for all i and τ. For $w=(w_\tau)_{\tau:K\hookrightarrow L}\in W=\prod_{\tau:K\hookrightarrow L}\mathcal{S}_n$, denote by $\mathcal{W}^n_{w,\mathbf{k},L}\subset\mathcal{W}^n_L$ the Zariski-closed (reduced) subset of characters $(\eta_1,\dots,\eta_n)$ defined by the equations:

$$\mathrm{wt}_\tau(\eta_{w_\tau(i)}\eta_i^{-1})=k_{\tau,i}-k_{\tau,w_\tau^{-1}(i)},\quad 1\le i\le n,\quad \tau:K\hookrightarrow L. \tag{5.11}$$

For instance one always has:

$$(z^{\mathbf{k}_{w^{-1}(1)}}\chi_1,\dots,z^{\mathbf{k}_{w^{-1}(n)}}\chi_n)\in\mathcal{W}^n_{w,\mathbf{k},L} \tag{5.12}$$

where $\chi_i\in\mathcal{W}_L$ are finite order characters. Note that $\mathcal{W}^n_{1,\mathbf{k},L}=\mathcal{W}^n_L$. We define an automorphism $\jmath_{w,\mathbf{k}}:\mathcal{T}^n_L\xrightarrow{\sim}\mathcal{T}^n_L$, $\eta=(\eta_1,\dots,\eta_n)\mapsto\jmath_{w,\mathbf{k}}(\eta)=\jmath_{w,\mathbf{k}}(\eta_1,\dots,\eta_n)$ by:

$$\jmath_{w,\mathbf{k}}(\eta_1,\dots,\eta_n):=(z^{\mathbf{k}_1-\mathbf{k}_{w^{-1}(1)}}\eta_1,\dots,z^{\mathbf{k}_n-\mathbf{k}_{w^{-1}(n)}}\eta_n)$$

which we extend to an automorphism $\jmath_{w,\mathbf{k}}:\mathfrak{X}^{\square}_{\overline{r}}\times\mathcal{T}^n_L\xrightarrow{\sim}\mathfrak{X}^{\square}_{\overline{r}}\times\mathcal{T}^n_L$, $(r,\eta)\mapsto(r,\jmath_{w,\mathbf{k}}(\eta))$. We will be particularly interested in applying $\jmath_{w,\mathbf{k}}$ to points whose image in $\mathcal{W}^n_L$ lies in $\mathcal{W}^n_{w,\mathbf{k},L}$.

Example 5.8. Consider the case $[K:\mathbb{Q}_p]=2$ (so $\mathrm{Hom}(K,L)=\{\tau,\tau'\}$), $n=3$ and $w=(w_\tau,w_{\tau'})$ with $w_\tau=s_1s_2s_1$, $w_{\tau'}=s_2s_1$ (s_1,s_2 being the simple reflections in $\mathcal{S}_3$). Then $\mathcal{W}^3_{w,\mathbf{k},L}$ is the set of characters of the form:

$$\eta=(\eta_1,\eta_2,\eta_3)=\Big(\tau(z)^{k_{\tau,3}}\tau'(z)^{k_{\tau',2}}\chi_1,\ \tau(z)^{k_{\tau,2}}\tau'(z)^{k_{\tau',3}}\chi_2,\ \tau(z)^{k_{\tau,1}}\tau'(z)^{k_{\tau',1}}\chi_3\Big)$$

where $\mathrm{wt}_\tau(\chi_1)=\mathrm{wt}_\tau(\chi_3)$ and $\mathrm{wt}_{\tau'}(\chi_1)=\mathrm{wt}_{\tau'}(\chi_2)=\mathrm{wt}_{\tau'}(\chi_3)$. Note that there is no condition on $\mathrm{wt}_\tau(\chi_2)$ (so one could as well rewrite the middle character as just $\tau'(z)^{k_{\tau',3}}\chi_2$). One has (when the η_i, or equivalently the χ_i, come from characters in $\mathcal{T}_L$):

$$\jmath_{w,\mathbf{k}}(\eta)=\Big(\tau(z)^{k_{\tau,1}}\tau'(z)^{k_{\tau',1}}\chi_1,\ \tau(z)^{k_{\tau,2}}\tau'(z)^{k_{\tau',2}}\chi_2,\ \tau(z)^{k_{\tau,3}}\tau'(z)^{k_{\tau',3}}\chi_3\Big).$$

Let $\widetilde{U}_{\mathrm{tri}}^{\square}(\overline{r}) := U_{\mathrm{tri}}^{\square}(\overline{r}) \cap \widetilde{X}_{\mathrm{tri}}^{\square}(\overline{r})$ (a union of connected components of $U_{\mathrm{tri}}^{\square}(\overline{r})$), then $\widetilde{U}_{\mathrm{tri}}^{\square}(\overline{r}) \times_{\mathcal{W}_L^n} \mathcal{W}_{w,\mathbf{k},L}^n$ is reduced (since it is smooth over $\mathcal{W}_{w,\mathbf{k},L}^n$) and Zariski-open (but not necessarily Zariski-dense) in $(\widetilde{X}_{\mathrm{tri}}^{\square}(\overline{r}) \times_{\mathcal{W}_L^n} \mathcal{W}_{w,\mathbf{k},L}^n)^{\mathrm{red}}$ where $(-)^{\mathrm{red}}$ means the associated reduced closed analytic subvariety. We denote by $\overline{\widetilde{U}_{\mathrm{tri}}^{\square}(\overline{r}) \times_{\mathcal{W}_L^n} \mathcal{W}_{w,\mathbf{k},L}^n}$ its Zariski-closure, so that we have a chain of Zariski-closed embeddings:

$$\overline{\widetilde{U}_{\mathrm{tri}}^{\square}(\overline{r}) \times_{\mathcal{W}_L^n} \mathcal{W}_{w,\mathbf{k},L}^n} \subseteq (\widetilde{X}_{\mathrm{tri}}^{\square}(\overline{r}) \times_{\mathcal{W}_L^n} \mathcal{W}_{w,\mathbf{k},L}^n)^{\mathrm{red}} \subseteq \widetilde{X}_{\mathrm{tri}}^{\square}(\overline{r}) \times_{\mathcal{W}_L^n} \mathcal{W}_{w,\mathbf{k},L}^n \subseteq \widetilde{X}_{\mathrm{tri}}^{\square}(\overline{r}) \subseteq X_{\mathrm{tri}}^{\square}(\overline{r}) \subseteq \mathfrak{X}_{\overline{r}}^{\square} \times \mathcal{T}_L^n.$$

Proposition 5.9. *Assume Conjecture 5.6, then for $w \in W$ the automorphism $\jmath_{w,\mathbf{k}} : \mathfrak{X}_{\overline{r}}^{\square} \times \mathcal{T}_L^n \xrightarrow{\sim} \mathfrak{X}_{\overline{r}}^{\square} \times \mathcal{T}_L^n$ induces a closed embedding of reduced rigid analytic spaces over L:*

$$\jmath_{w,\mathbf{k}} : \overline{\widetilde{U}_{\mathrm{tri}}^{\square}(\overline{r}) \times_{\mathcal{W}_L^n} \mathcal{W}_{w,\mathbf{k},L}^n} \hookrightarrow \widetilde{X}_{\mathrm{tri}}^{\square}(\overline{r}) \subseteq X_{\mathrm{tri}}^{\square}(\overline{r}).$$

Proof. Since $\widetilde{U}_{\mathrm{tri}}^{\square}(\overline{r}) \times_{\mathcal{W}_L^n} \mathcal{W}_{w,\mathbf{k},L}^n$ is Zariski-dense in $\overline{\widetilde{U}_{\mathrm{tri}}^{\square}(\overline{r}) \times_{\mathcal{W}_L^n} \mathcal{W}_{w,\mathbf{k},L}^n}$, it is enough to prove $\jmath_{w,\mathbf{k}}(\widetilde{U}_{\mathrm{tri}}^{\square}(\overline{r}) \times_{\mathcal{W}_L^n} \mathcal{W}_{w,\mathbf{k},L}^n) \subseteq \widetilde{X}_{\mathrm{tri}}^{\square}(\overline{r})$, i.e. that any point $x' = (r', \delta')$ in $\widetilde{U}_{\mathrm{tri}}^{\square}(\overline{r})$ with $\omega(x') \in \mathcal{W}_{w,\mathbf{k},L}^n$ is such that $\jmath_{w,\mathbf{k}}(x')$ is still in $\widetilde{X}_{\mathrm{tri}}^{\square}(\overline{r})$.

Recall that by assumption:

$$X_p(\overline{\rho}) \overset{(3.12)}{\xrightarrow{\sim}} \mathfrak{X}_{\overline{\rho}^p} \times \widetilde{X}_{\mathrm{tri}}^{\square}(\overline{\rho}_p) \times \mathbb{U}^g \subseteq \mathfrak{X}_{\overline{\rho}^p} \times (\mathfrak{X}_{\overline{\rho}_p} \times \widehat{T}_{p,L}) \times \mathbb{U}^g. \tag{5.13}$$

Let $y' \in X_p(\overline{\rho})$ be any point such that its image in $\widetilde{X}_{\mathrm{tri}}^{\square}(\overline{\rho}_p)$ by (5.13) is $(x')_{v \in S_p}$. Write again w for the element $(w)_{v \in S_p} \in \prod_{v \in S_p}(\prod_{F_{\tilde{v}} \hookrightarrow L} \mathcal{S}_n)$ (that is, for each v we have the same element $w = (w_\tau)_{\tau: K \hookrightarrow L} \in \prod_{\tau: K \hookrightarrow L} \mathcal{S}_n$), $\mathbf{k}$ for $(\mathbf{k})_{v \in S_p} \in \prod_{v \in S_p} \mathbb{Z}^{\mathrm{Hom}(K,L)}$ (ibid.), $\jmath_{w,\mathbf{k}}$ for the automorphism $(\jmath_{w,\mathbf{k}})_{v \in S_p}$ of $\widehat{T}_{p,L} \cong \prod_{v \in S_p} \widehat{T}_{v,L} \cong \prod_{v \in S_p} \mathcal{T}_L^n$ and (again) $\jmath_{w,\mathbf{k}}$ for the automorphism $\mathrm{id} \times (\mathrm{id} \times \jmath_{w,\mathbf{k}}) \times \mathrm{id}$ of $\mathfrak{X}_{\overline{\rho}^p} \times (\mathfrak{X}_{\overline{\rho}_p} \times \widehat{T}_{p,L}) \times \mathbb{U}^g$. Then it is enough to prove that $\jmath_{w,\mathbf{k}}(y') \in X_p(\overline{\rho})$ (via (5.13)). Writing $y' = (\mathfrak{m}', \epsilon') \in X_p(\overline{\rho}) \subseteq \mathfrak{X}_\infty \times \widehat{T}_{p,L}$ where $\mathfrak{m}' \subset R_\infty[1/p]$ is the maximal ideal corresponding to the projection of y' in $\mathfrak{X}_\infty$ and $\epsilon' = (\imath_v(\delta'))_{v \in S_p}$, we have $\mathrm{Hom}_{T_p}(\epsilon', J_{B_p}(\Pi_\infty^{R_\infty-\mathrm{an}}[\mathfrak{m}'] \otimes_{k(\mathfrak{m}')} k(y'))) \neq 0$ (see (3.10)) and we have to prove (note that $\jmath_{w,\mathbf{k}} \circ \imath_v^{-1} = \imath_v^{-1} \circ \jmath_{w,\mathbf{k}}$ on $\widehat{T}_{v,L}$ and that $k(y') = k(\jmath_{w,\mathbf{k}}(y'))$):

$$\mathrm{Hom}_{T_p}\Big(\jmath_{w,\mathbf{k}}(\epsilon'), J_{B_p}(\Pi_\infty^{R_\infty-\mathrm{an}}[\mathfrak{m}'] \otimes_{k(\mathfrak{m}')} k(y'))\Big) \neq 0. \tag{5.14}$$

From Theorem 5.5, it is enough to prove $\epsilon' \uparrow \jmath_{w,\mathbf{k}}(\epsilon')$ in the sense of Definition 5.4. Since $\epsilon' \jmath_{w,\mathbf{k}}(\epsilon')^{-1}$ is clearly an algebraic character of T_p by definition of $\jmath_{w,\mathbf{k}}$, it is enough to prove $\imath_v(\delta') \uparrow_{\mathfrak{t}_v} \jmath_{w,\mathbf{k}}(\imath_v(\delta'))$ (with the notation of §5.1) for one, or equivalently all here, $v \in S_p$. From (5.11), we see that we can write:

$$\delta' = (z^{\mathbf{k}_{w^{-1}(1)}}\chi_1, \ldots, z^{\mathbf{k}_{w^{-1}(n)}}\chi_n) \quad \text{and} \quad \jmath_{w,\mathbf{k}}(\imath_v(\delta')) = (z^{\mathbf{k}_1}\chi_1, \ldots, z^{\mathbf{k}_n}\chi_n)$$

where $\mathrm{wt}_\tau(\chi_i) = \mathrm{wt}_\tau(\chi_{w_\tau(i)})$ for $1 \leq i \leq n$ and $\tau : K = F_{\tilde{v}} \hookrightarrow L$ (compare Example 5.8). As we only care about the $\mathfrak{t}_{v,L}$-action, setting $s_{\tau,i} := \mathrm{wt}_\tau(\chi_i) \in L$ and using usual additive notation, we can write $\imath_v(\delta')|_{\mathfrak{t}_{v,L}} = (\imath_v(\delta')_\tau)_{\tau: F_{\tilde{v}} \hookrightarrow L}$ and $\jmath_{w,\mathbf{k}}(\imath_v(\delta'))|_{\mathfrak{t}_{v,L}} = (\jmath_{w,\mathbf{k}}(\imath_v(\delta'))_\tau)_{\tau: F_{\tilde{v}} \hookrightarrow L}$

with:

$$\begin{aligned}\imath_v(\delta')_\tau &= (k_{\tau,w_\tau^{-1}(1)} + s_{\tau,1}, k_{\tau,w_\tau^{-1}(2)} + s_{\tau,2}+1, \dots, k_{\tau,w_\tau^{-1}(n)} + s_{\tau,n}+n-1)\\ \jmath_{w,\mathbf{k}}(\imath_v(\delta'))_\tau &= (k_{\tau,1}+s_{\tau,1}, k_{\tau,2}+s_{\tau,2}+1, \dots, k_{\tau,n}+s_{\tau,n}+n-1)\end{aligned}$$

(see the beginning of §5.1). Since $s_{\tau,i} = s_{\tau,w_\tau^{-1}(i)}$ for all i, τ, we can rewrite:

$$\imath_v(\delta')_\tau = (k_{\tau,w_\tau^{-1}(1)} + s_{\tau,w_\tau^{-1}(1)}, k_{\tau,w_\tau^{-1}(2)} + s_{\tau,w_\tau^{-1}(2)}+1, \dots, k_{\tau,w_\tau^{-1}(n)} + s_{\tau,w_\tau^{-1}(n)}+n-1)$$

hence we have $\imath_v(\delta')_\tau = w_\tau \cdot \jmath_{w,\mathbf{k}}(\imath_v(\delta'))_\tau$ for the "dot action" $\cdot$ with respect to the upper triangular matrices in $\mathrm{GL}_{n,F_{\tilde v}} \times_{F_{\tilde v},\tau} L$ (see [32, §1.8]). Let us write the permutation w_τ on $\{1,\dots,n\}$ as a product of commuting cycles $c_1 \circ \cdots \circ c_m$ with pairwise disjoint support $\mathrm{supp}(c_i) \subseteq \{1,\dots,n\}$. Let us denote by $\mathcal{S}_{n,i} \subseteq \mathcal{S}_n$ the subgroup of permutations which fixes the elements in $\{1,\dots,n\}$ *not* in $\mathrm{supp}(c_i)$ and set $\mathcal{S}_{n,w_\tau} := \prod_{i=1}^m \mathcal{S}_{n,i} \subseteq \mathcal{S}_n$. Then, arguing in each $\mathrm{supp}(c_i)$, it is not difficult to see that one can write w_τ as a product:

$$w_\tau = s_{\alpha_d} s_{\alpha_{d-1}} \cdots s_{\alpha_1}$$

where the α_i are (not necessarily simple) roots of the upper triangular matrices in $\mathrm{GL}_{n,F_{\tilde v}} \times_{F_{\tilde v},\tau} L$, the associated reflections s_{α_i} are in $\mathcal{S}_{n,w_\tau}$ and where $s_{\alpha_i+1}s_{\alpha_i}\cdots s_{\alpha_1} > s_{\alpha_i}\cdots s_{\alpha_1}$ for the Bruhat order in $\mathcal{S}_n$ ($1 \le i \le n-1$). By an argument analogous *mutatis mutandis* to the one in [32, §5.2], it then follows from the above assumptions (in particular $s_{\tau,i} = s_{\tau,w_\tau^{-1}(i)}$ for all i) that we have for $1 \le i \le n-1$ with obvious notation:

$$(s_{\alpha_i+1}\cdots s_{\alpha_1}) \cdot \jmath_{w,\mathbf{k}}(\imath_v(\delta'))_\tau \le (s_{\alpha_i}\cdots s_{\alpha_1}) \cdot \jmath_{w,\mathbf{k}}(\imath_v(\delta'))_\tau.$$

By definition this implies that $w_\tau \cdot \jmath_{w,\mathbf{k}}(\imath_v(\delta'))_\tau$ is *strongly linked* to $\jmath_{w,\mathbf{k}}(\imath_v(\delta'))_\tau$ ([32, §5.1]). As this holds for all τ, we have $\imath_v(\delta') \uparrow_{\mathfrak{t}_v} \jmath_{w,\mathbf{k}}(\imath_v(\delta'))$. □

Remark 5.10. It would be interesting to find a purely local proof of the local statement of Proposition 5.9 without assuming Conjecture 5.6.

5.3. **Tangent spaces on the trianguline variety.** We prove that Conjecture 5.6 implies Conjecture 2.8 (when $\bar r$ "globalizes" and x is very regular) and give one (conjectural) application.

We keep the notation and assumptions of §5.2. We fix $x = (r,\delta) \in \widetilde{X}_{\mathrm{tri}}^{\square}(\bar r) \subseteq X_{\mathrm{tri}}^{\square}(\bar r)$ which is crystalline strictly dominant very regular. Recall from Lemma 2.1 that $\delta = (\delta_1 \dots, \delta_n)$ where $\delta_i = z^{\mathbf{k}_i}\mathrm{unr}(\varphi_i)$ with $\mathbf{k}_i = (k_{\tau,i})_{\tau: K \hookrightarrow L} \in \mathbb{Z}^{\mathrm{Hom}(K,L)}$ and $\varphi_i \in k(x)^\times$. The following result immediately follows from Proposition 3.10 and Theorem 2.15 applied to $X = \widetilde{X}_{\mathrm{tri}}^{\square}(\bar r)$.

Corollary 5.11. *Assume Conjecture 5.6, then we have:*

$$\dim_{k(x)} T_{\widetilde{X}_{\mathrm{tri}}^{\square}(\bar r),x} \le \lg(w_x) - d_x + \dim X_{\mathrm{tri}}^{\square}(\bar r) = \lg(w_x) - d_x + n^2 + [K:\mathbb{Q}_p]\frac{n(n+1)}{2}.$$

The rest of this section is devoted to the proof of the converse inequality (still assuming Conjecture 5.6).

As in the proof of Proposition 4.7, we consider for $1 \le i \le n$ the cartesian diagram which defines W_i (with the notation of §4.1):

$$\begin{array}{ccc} \mathrm{Ext}^1_{(\varphi,\Gamma_K)}\Big(D_{\mathrm{rig}}(r), D_{\mathrm{rig}}(r)\Big) & \longrightarrow & \mathrm{Ext}^1_{(\varphi,\Gamma_K)}\Big(D_{\mathrm{rig}}(r)^{\le i}, D_{\mathrm{rig}}(r)\Big) \\ \uparrow & & \uparrow \\ W_i & \longrightarrow & \mathrm{Ext}^1_{(\varphi,\Gamma_K)}\Big(D_{\mathrm{rig}}(r)^{\le i}, D_{\mathrm{rig}}(r)^{\le i}\Big). \end{array}$$

We define $W_{\mathrm{cris},i} \subseteq \mathrm{Ext}^1_{\mathrm{cris}}(D_{\mathrm{rig}}(r), D_{\mathrm{rig}}(r))$ as W_i but replacing everywhere $\mathrm{Ext}^1_{(\varphi,\Gamma_K)}$ by its subspace $\mathrm{Ext}^1_{\mathrm{cris}}$. Note that $W_{\mathrm{cris},i} \subseteq W_i$ for $1 \le i \le n$.

Proposition 5.12. *For $1 \le i \le n$, we have isomorphisms of $k(x)$-vector spaces:*

$$W_1 \cap \cdots \cap W_{i-1}/W_1 \cap \cdots \cap W_i \ \xrightarrow{\sim}\ \mathrm{Ext}^1_{(\varphi,\Gamma_K)}\Big(\mathrm{gr}_i D_{\mathrm{rig}}(r), D_{\mathrm{rig}}(r)/D_{\mathrm{rig}}(r)^{\le i}\Big) \tag{5.15}$$

$$W_{\mathrm{cris},1} \cap \cdots \cap W_{\mathrm{cris},i-1}/W_{\mathrm{cris},1} \cap \cdots \cap W_{\mathrm{cris},i} \ \xrightarrow{\sim}\ \mathrm{Ext}^1_{\mathrm{cris}}\Big(\mathrm{gr}_i D_{\mathrm{rig}}(r), D_{\mathrm{rig}}(r)/D_{\mathrm{rig}}(r)^{\le i}\Big)$$

where $W_1 \cap \cdots \cap W_{i-1} := \mathrm{Ext}^1_{(\varphi,\Gamma_K)}(D_{\mathrm{rig}}(r), D_{\mathrm{rig}}(r))$ *(resp.* $W_{\mathrm{cris},1} \cap \cdots \cap W_{\mathrm{cris},i-1} := \mathrm{Ext}^1_{\mathrm{cris}}(D_{\mathrm{rig}}(r), D_{\mathrm{rig}}(r))$*) if $i = 1$.*

Proof. We write D_{rig} instead of $D_{\mathrm{rig}}(r)$ and drop the subscript (φ, Γ_K) in this proof. We start with the first isomorphism, the proof of which is analogous to (though simpler than) the proof of (4.10) in §4.2. We have the exact sequence (using Definition 2.13):

$$0 \to \mathrm{Ext}^1\Big(\mathrm{gr}_i D_{\mathrm{rig}}, D_{\mathrm{rig}}/D_{\mathrm{rig}}^{\le i}\Big) \to \mathrm{Ext}^1\Big(D_{\mathrm{rig}}^{\le i}, D_{\mathrm{rig}}/D_{\mathrm{rig}}^{\le i}\Big) \to \mathrm{Ext}^1\Big(D_{\mathrm{rig}}^{\le i-1}, D_{\mathrm{rig}}/D_{\mathrm{rig}}^{\le i}\Big) \to 0. \tag{5.16}$$

The composition:

$$W_1 \cap \cdots \cap W_{i-1} \hookrightarrow \mathrm{Ext}^1\Big(D_{\mathrm{rig}}, D_{\mathrm{rig}}\Big) \twoheadrightarrow \mathrm{Ext}^1\Big(D_{\mathrm{rig}}^{\le i}, D_{\mathrm{rig}}/D_{\mathrm{rig}}^{\le i}\Big)$$

lands in $\mathrm{Ext}^1(\mathrm{gr}_i D_{\mathrm{rig}}, D_{\mathrm{rig}}/D_{\mathrm{rig}}^{\le i})$ by (5.16). If $v \in W_1 \cap \cdots \cap W_{i-1}$ is also in W_i, then its image in $\mathrm{Ext}^1(D_{\mathrm{rig}}^{\le i}, D_{\mathrm{rig}}/D_{\mathrm{rig}}^{\le i})$ is 0. We thus deduce a canonical induced map:

$$W_1 \cap \cdots \cap W_{i-1}/W_1 \cap \cdots \cap W_i \to \mathrm{Ext}^1\Big(\mathrm{gr}_i D_{\mathrm{rig}}, D_{\mathrm{rig}}/D_{\mathrm{rig}}^{\le i}\Big). \tag{5.17}$$

Let us prove that (5.17) is surjective. One easily checks that $\mathrm{Ext}^1(D_{\mathrm{rig}}/D_{\mathrm{rig}}^{\le i-1}, D_{\mathrm{rig}}) \subseteq W_1 \cap \cdots \cap W_{i-1}$ and that the natural map $\mathrm{Ext}^1(D_{\mathrm{rig}}/D_{\mathrm{rig}}^{\le i-1}, D_{\mathrm{rig}}) \to \mathrm{Ext}^1(\mathrm{gr}_i D_{\mathrm{rig}}, D_{\mathrm{rig}}/D_{\mathrm{rig}}^{\le i})$ is surjective (again by Definition 2.13). This implies that *a fortiori* (5.17) must also be surjective. Let us prove that (5.17) is injective. If $w \in W_1 \cap \cdots \cap W_{i-1}$ maps to zero, then the image of w in $\mathrm{Ext}^1(D_{\mathrm{rig}}^{\le i}, D_{\mathrm{rig}}/D_{\mathrm{rig}}^{\le i})$ is also zero, i.e. $w \in W_i$ hence $w \in W_1 \cap \cdots \cap W_i$.

The proof for the second isomorphism is exactly the same replacing everywhere W_j by $W_{\mathrm{cris},j}$ and $\mathrm{Ext}^1_{(\varphi,\Gamma_K)}$ by $\mathrm{Ext}^1_{\mathrm{cris}}$. □

Corollary 5.13. *We have:*

$$\dim_{k(x)}\Big(W_1 \cap \cdots \cap W_{n-1}\Big) = \dim_{k(x)} \mathrm{Ext}^1_{(\varphi,\Gamma_K)}\Big(D_{\mathrm{rig}}(r), D_{\mathrm{rig}}(r)\Big) - [K:\mathbb{Q}_p]\frac{n(n-1)}{2}$$

$$\dim_{k(x)}\Big(W_{\mathrm{cris},1} \cap \cdots \cap W_{\mathrm{cris},n-1}\Big) = \dim_{k(x)} \mathrm{Ext}^1_{\mathrm{cris}}\Big(D_{\mathrm{rig}}(r), D_{\mathrm{rig}}(r)\Big) - \lg(w_x).$$

Proof. This follows from Proposition 5.12 together with (4.9) and Lemma 4.4 (both for $\ell = i$) by the same argument as at the end of the proof of Proposition 4.5. □

Remark 5.14. Note that $W_1 \cap \cdots \cap W_{n-1} \cap \mathrm{Ext}^1_{\mathrm{cris}}(D_{\mathrm{rig}}(r), D_{\mathrm{rig}}(r)) = W_{\mathrm{cris},1} \cap \cdots \cap W_{\mathrm{cris},n-1}$.

Now consider $x' := (r, \delta') = (r, \delta'_1, \dots, \delta'_n)$ with $\delta'_i := z^{\mathbf{k}_{w_x^{-1}(i)}} \mathrm{unr}(\varphi_i)$, then $x' \in \widetilde{U}^{\square}_{\mathrm{tri}}(\overline{r})$ by (4.1). We also have $\omega(x') \in \mathcal{W}^n_{w_x,\mathbf{k},L}$ by (5.12), thus $x' \in \widetilde{U}^{\square}_{\mathrm{tri}}(\overline{r}) \times_{\mathcal{W}^n_L} \mathcal{W}^n_{w_x,\mathbf{k},L} \subseteq \widetilde{U}^{\square}_{\mathrm{tri}}(\overline{r})$ and $\jmath_{w,\mathbf{k}}(x') = x$. Recall from §4.1 and the smoothness of $U^{\square}_{\mathrm{tri}}(\overline{r})$ over $\mathcal{W}^n_L$ that the weight map ω induces a $k(x)$-linear surjection on tangent spaces (note that $k(x') = k(x)$):

$$d\omega : T_{\widetilde{X}^{\square}_{\mathrm{tri}}(\overline{r}),x'} \cong T_{X^{\square}_{\mathrm{tri}}(\overline{r}),x'} \twoheadrightarrow T_{\mathcal{W}^n_L,\omega(x')} \cong k(x)^{[K:\mathbb{Q}_p]n}, \quad \vec{v} \longmapsto (d_{\tau,i,\vec{v}})_{1\leq i\leq n, \tau: K\hookrightarrow L}. \tag{5.18}$$

Proposition 5.15. *We have an isomorphism of $k(x)$-subvector spaces of $T_{\widetilde{X}^{\square}_{\mathrm{tri}}(\overline{r}),x'}$:*

$$T_{(\widetilde{X}^{\square}_{\mathrm{tri}}(\overline{r})\times_{\mathcal{W}^n_L}\mathcal{W}^n_{w_x,\mathbf{k},L})^{\mathrm{red}},x'} \xrightarrow{\sim} \Big\{\vec{v} \in T_{\widetilde{X}^{\square}_{\mathrm{tri}}(\overline{r}),x'} \text{ such that } d_{\tau,i,\vec{v}} = d_{\tau,w_{x,\tau}^{-1}(i),\vec{v}},\ 1 \leq i \leq n,\ \tau : K \hookrightarrow L\Big\}.$$

In particular $\dim_{k(x)} T_{(\widetilde{X}^{\square}_{\mathrm{tri}}(\overline{r})\times_{\mathcal{W}^n_L}\mathcal{W}^n_{w_x,\mathbf{k},L})^{\mathrm{red}},x'} = \dim X^{\square}_{\mathrm{tri}}(\overline{r}) - d_x$.

Proof. We write Hom instead of $\mathrm{Hom}_{k(x)-\mathrm{alg}}$ in this proof. Let $\widetilde{U}^{\square}_{\mathrm{tri},w_x,\mathbf{k}}(\overline{r}) := \widetilde{U}^{\square}_{\mathrm{tri}}(\overline{r}) \times_{\mathcal{W}^n_L} \mathcal{W}^n_{w_x,\mathbf{k},L}$, we have:

$$\mathcal{O}_{\widetilde{U}^{\square}_{\mathrm{tri},w_x,\mathbf{k}}(\overline{r}),x'} \cong \mathcal{O}_{\widetilde{U}^{\square}_{\mathrm{tri}}(\overline{r}),x'} \otimes_{\mathcal{O}_{\mathcal{W}^n_L,\omega(x')}} \mathcal{O}_{\mathcal{W}^n_{w_x,\mathbf{k},L},\omega(x')} \tag{5.19}$$

and note that $T_{(\widetilde{X}^{\square}_{\mathrm{tri}}(\overline{r})\times_{\mathcal{W}^n_L}\mathcal{W}^n_{w_x,\mathbf{k},L})^{\mathrm{red}},x'} = T_{\widetilde{U}^{\square}_{\mathrm{tri},w,\mathbf{k}}(\overline{r}),x'}$. Recall that, if A, B, C, D are commutative $k(x)$-algebras with B, C being A-algebras, we have:

$$\mathrm{Hom}(B \otimes_A C, D) \xrightarrow{\sim} \mathrm{Hom}(B, D) \times_{\mathrm{Hom}(A,D)} \mathrm{Hom}(C, D). \tag{5.20}$$

From (5.19) and (5.20) we deduce:

$$T_{\widetilde{U}^{\square}_{\mathrm{tri},w,\mathbf{k}}(\overline{r}),x'} = \mathrm{Hom}\Big(\mathcal{O}_{\widetilde{U}^{\square}_{\mathrm{tri},w_x,\mathbf{k}}(\overline{r}),x'}, k(x)[\varepsilon]/(\varepsilon^2)\Big) \cong T_{\widetilde{X}^{\square}_{\mathrm{tri}}(\overline{r}),x'} \times_{T_{\mathcal{W}^n_L,\omega(x')}} T_{\mathcal{W}^n_{w_x,\mathbf{k},L},\omega(x')}. \tag{5.21}$$

But from (5.11) we have:

$$T_{\mathcal{W}^n_{w_x,\mathbf{k},L},\omega(x')} = \Big\{(d_{\tau,i})_{1\leq i\leq n,\tau: K\hookrightarrow L} \in T_{\mathcal{W}^n_L,\omega(x')} \text{ such that } d_{\tau,i} = d_{\tau,w_{x,\tau}^{-1}(i)},\ \forall\ i,\ \forall\ \tau\Big\}$$

whence the first statement. The last statement comes from $\dim_{k(x)} T_{\widetilde{X}^{\square}_{\mathrm{tri}}(\overline{r}),x'} = \dim \widetilde{X}^{\square}_{\mathrm{tri}}(\overline{r}) = \dim X^{\square}_{\mathrm{tri}}(\overline{r})$ (since x' is smooth on $X^{\square}_{\mathrm{tri}}(\overline{r})$ as $x' \in U^{\square}_{\mathrm{tri}}(\overline{r})$), the surjectivity of $T_{\widetilde{X}^{\square}_{\mathrm{tri}}(\overline{r}),x'} \to T_{\mathcal{W}^n_L,\omega(x')}$ (since the morphism $\widetilde{U}^{\square}_{\mathrm{tri}}(\overline{r}) \to \mathcal{W}^n_L$ is smooth by [13, Th.2.6(iii)]) and the same argument as in the proof of Proposition 4.6. □

Recall from the discussion just before Conjecture 2.8 that we have a closed embedding $\iota_{\mathbf{k}} : \widetilde{\mathfrak{X}}^{\square,\mathbf{k}-\mathrm{cr}}_{\overline{r}} \hookrightarrow \widetilde{X}^{\square}_{\mathrm{tri}}(\overline{r})$ with $x \in \iota_{\mathbf{k}}(\widetilde{\mathfrak{X}}^{\square,\mathbf{k}-\mathrm{cr}}_{\overline{r}})$. We deduce an injection of $k(x)$-vector spaces:

$$T_{\iota_{\mathbf{k}}(\widetilde{\mathfrak{X}}^{\square,\mathbf{k}-\mathrm{cr}}_{\overline{r}}),x} \hookrightarrow T_{\widetilde{X}^{\square}_{\mathrm{tri}}(\overline{r}),x}.$$

Likewise we deduce from Proposition 5.9 (assuming Conjecture 5.6) another injection of $k(x)$-vector spaces:

$$T_{\jmath_{w_x,\mathbf{k}}\left(\overline{\widetilde{U}^{\square}_{\mathrm{tri}}(\overline{r})\times_{\mathcal{W}^n_L}\mathcal{W}^n_{w,\mathbf{k},L}}\right),x} \hookrightarrow T_{\widetilde{X}^{\square}_{\mathrm{tri}}(\overline{r}),x}.$$

Taking the sum in $T_{\widetilde{X}^{\square}_{\mathrm{tri}}(\overline{r}),x}$ of these two subspaces of $T_{\widetilde{X}^{\square}_{\mathrm{tri}}(\overline{r}),x}$, we have an injection of $k(x)$-vector spaces:

$$T_{\jmath_{w_x,\mathbf{k}}\left(\overline{\widetilde{U}^{\square}_{\mathrm{tri}}(\overline{r})\times_{\mathcal{W}^n_L}\mathcal{W}^n_{w,\mathbf{k},L}}\right),x}+T_{\iota_{\mathbf{k}}(\widetilde{\mathfrak{X}}^{\square,\mathbf{k}-\mathrm{cr}}_{\overline{r}}),x}\hookrightarrow T_{\widetilde{X}^{\square}_{\mathrm{tri}}(\overline{r}),x}. \tag{5.22}$$

Proposition 5.16. *Assume Conjecture 5.6, then we have:*

$$\dim_{k(x)}\left(T_{\jmath_{w_x,\mathbf{k}}\left(\overline{\widetilde{U}^{\square}_{\mathrm{tri}}(\overline{r})\times_{\mathcal{W}^n_L}\mathcal{W}^n_{w,\mathbf{k},L}}\right),x}+T_{\iota_{\mathbf{k}}(\widetilde{\mathfrak{X}}^{\square,\mathbf{k}-\mathrm{cr}}_{\overline{r}}),x}\right)=\lg(w_x)-d_x+\dim X^{\square}_{\mathrm{tri}}(\overline{r}).$$

Proof. The composition $\widetilde{X}^{\square}_{\mathrm{tri}}(\overline{r})\hookrightarrow\mathfrak{X}^{\square}_{\overline{r}}\times\mathcal{T}^n_L\twoheadrightarrow\mathfrak{X}^{\square}_{\overline{r}}$ induces a $k(x)$-linear morphism $T_{\widetilde{X}^{\square}_{\mathrm{tri}}(\overline{r}),x'}\to T_{\mathfrak{X}^{\square}_{\overline{r}},r}$. Since $x'\in\widetilde{U}^{\square}_{\mathrm{tri}}(\overline{r})\subseteq U^{\square}_{\mathrm{tri}}(\overline{r})$, it follows from [34, Th.Cor.6.3.10] (arguing e.g. as in the proof of [13, Lem.2.11]) that the triangulation $(D^{\leq i}_{\mathrm{rig}})_{1\leq i\leq n}$ "globalizes" in a small neighbourhood of x' in $U^{\square}_{\mathrm{tri}}(\overline{r})$, or equivalenty in $\widetilde{U}^{\square}_{\mathrm{tri}}(\overline{r})$. In particular, for any $\vec{v}\in T_{\widetilde{U}^{\square}_{\mathrm{tri}}(\overline{r}),x'}\cong T_{\widetilde{X}^{\square}_{\mathrm{tri}}(\overline{r}),x'}$ we have a triangulation of $D_{\mathrm{rig}}(r_{\vec{v}})$ by free (φ,Γ_K)-submodules over $\mathcal{R}_{k(x)[\varepsilon]/(\varepsilon^2),K}$ such that the associated parameter is $(\delta_{1,\vec{v}},\dots,\delta_{n,\vec{v}})$ (see §4.1 for the notation). This has two consequences: (1) the proof of Lemma 4.16 goes through and the above map $T_{\widetilde{X}^{\square}_{\mathrm{tri}}(\overline{r}),x'}\to T_{\mathfrak{X}^{\square}_{\overline{r}},r}$ is an injection of $k(x)$-vector spaces and (2) the image of the composition $T_{\widetilde{X}^{\square}_{\mathrm{tri}}(\overline{r}),x'}\hookrightarrow T_{\mathfrak{X}^{\square}_{\overline{r}},r}\twoheadrightarrow\mathrm{Ext}^1_{(\varphi,\Gamma_K)}(D_{\mathrm{rig}}(r),D_{\mathrm{rig}}(r))$ (see Lemma 4.13) lies in $W_1\cap\cdots\cap W_{n-1}\subseteq\mathrm{Ext}^1_{(\varphi,\Gamma_K)}(D_{\mathrm{rig}}(r),D_{\mathrm{rig}}(r))$. From Lemma 4.13 we thus obtain an exact sequence:

$$0\to K(r)\cap T_{\widetilde{X}^{\square}_{\mathrm{tri}}(\overline{r}),x'}\to T_{\widetilde{X}^{\square}_{\mathrm{tri}}(\overline{r}),x'}\to W_1\cap\cdots\cap W_{n-1}.$$

But $\dim_{k(x)}T_{\widetilde{X}^{\square}_{\mathrm{tri}}(\overline{r}),x'}=\dim X^{\square}_{\mathrm{tri}}(\overline{r})$ since $\widetilde{X}^{\square}_{\mathrm{tri}}(\overline{r})$ is smooth at x', and from Lemma 4.13, Lemma 4.14 and Corollary 5.13, we have:

$$\dim_{k(x)}K(r)+\dim_{k(x)}\Big(W_1\cap\cdots\cap W_{n-1}\Big)=n^2+[K:\mathbb{Q}_p]\frac{n(n+1)}{2}=\dim X^{\square}_{\mathrm{tri}}(\overline{r})$$

which forces a short exact sequence $0\to K(r)\to T_{\widetilde{X}^{\square}_{\mathrm{tri}}(\overline{r}),x'}\to W_1\cap\cdots\cap W_{n-1}\to 0$. It then follows from Proposition 5.15 that we have a short exact sequence of $k(x)$-vector spaces:

$$0\to K(r)\to T_{(\widetilde{X}^{\square}_{\mathrm{tri}}(\overline{r})\times_{\mathcal{W}^n_L}\mathcal{W}^n_{w_x,\mathbf{k},L})^{\mathrm{red}},x'}\to W_1\cap\cdots\cap W_{n-1}\cap V\to 0 \tag{5.23}$$

where $V\subseteq\mathrm{Ext}^1_{(\varphi,\Gamma_K)}(D_{\mathrm{rig}}(r),D_{\mathrm{rig}}(r))$ is as in the end of §4.2 (the intersection on the right hand side being in $\mathrm{Ext}^1_{(\varphi,\Gamma_K)}(D_{\mathrm{rig}}(r),D_{\mathrm{rig}}(r))$).

Arguing as in [37, §2.3.5] we also have a short exact sequence (see [36, (3.3.5)]):

$$0\to K(r)\to T_{\mathfrak{X}^{\square,\mathbf{k}-\mathrm{cr}}_{\overline{r}},r}\to\mathrm{Ext}^1_{\mathrm{cris}}(D_{\mathrm{rig}}(r),D_{\mathrm{rig}}(r))\to 0. \tag{5.24}$$

Using $T_{\iota_{\mathbf{k}}(\widetilde{\mathfrak{X}}^{\square,\mathbf{k}-\mathrm{cr}}_{\overline{r}}),x}\cong T_{\mathfrak{X}^{\square,\mathbf{k}-\mathrm{cr}}_{\overline{r}},r}$ (which easily follows from the fact that the Frobenius eigenvalues $(\varphi_1,\dots,\varphi_n)$ are pairwise distinct) and:

$$T_{(\widetilde{X}^{\square}_{\mathrm{tri}}(\overline{r})\times_{\mathcal{W}^n_L}\mathcal{W}^n_{w,\mathbf{k},L})^{\mathrm{red}},x'}\cong T_{\widetilde{U}^{\square}_{\mathrm{tri}}(\overline{r})\times_{\mathcal{W}^n_L}\mathcal{W}^n_{w,\mathbf{k},L},x'}\cong T_{\overline{\widetilde{U}^{\square}_{\mathrm{tri}}(\overline{r})\times_{\mathcal{W}^n_L}\mathcal{W}^n_{w,\mathbf{k},L}},x'}\xrightarrow{\sim}T_{\jmath_{w_x,\mathbf{k}}(\overline{\widetilde{U}^{\square}_{\mathrm{tri}}(\overline{r})\times_{\mathcal{W}^n_L}\mathcal{W}^n_{w,\mathbf{k},L}}),x},$$

we deduce from (5.23) and (5.24) a short exact sequence of $k(x)$-vector spaces:

$$0 \to K(r) \to T_{\jmath_{w_x,\mathbf{k}}\left(\overline{\widetilde{U}^{\square}_{\mathrm{tri}}(\overline{r})\times_{\mathcal{W}^n_L}\mathcal{W}^n_{w,\mathbf{k},L}}\right),x} \cap T_{\iota_{\mathbf{k}}(\widetilde{\mathfrak{X}}^{\square,\mathbf{k}-\mathrm{cr}}_{\overline{r}}),x} \to W_1\cap\cdots\cap W_{n-1}\cap V\cap \mathrm{Ext}^1_{\mathrm{cris}}(D_{\mathrm{rig}}(r),D_{\mathrm{rig}}(r)) \to 0,$$

the intersection in the middle being in $T_{\mathfrak{X}^{\square}_{\overline{r}},r}$. But we have:

$$W_1\cap\cdots\cap W_{n-1}\cap V\cap \mathrm{Ext}^1_{\mathrm{cris}}(D_{\mathrm{rig}}(r),D_{\mathrm{rig}}(r)) \xrightarrow{\sim} W_{\mathrm{cris},1}\cap\cdots\cap W_{\mathrm{cris},n-1}\cap V \xrightarrow{\sim} W_{\mathrm{cris},1}\cap\cdots\cap W_{\mathrm{cris},n-1}$$

where the first isomorphism is Remark 5.14 and the second follows from $\mathrm{Ext}^1_{\mathrm{cris}}(D_{\mathrm{rig}}(r),D_{\mathrm{rig}}(r))\subseteq V$ (since the Hodge-Tate weights don't vary at all in $\mathrm{Ext}^1_{\mathrm{cris}}(D_{\mathrm{rig}}(r),D_{\mathrm{rig}}(r))$). From Corollary 5.13 we thus get:

$$\dim_{k(x)}\left(T_{\jmath_{w_x,\mathbf{k}}\left(\overline{\widetilde{U}^{\square}_{\mathrm{tri}}(\overline{r})\times_{\mathcal{W}^n_L}\mathcal{W}^n_{w,\mathbf{k},L}}\right),x} \cap T_{\iota_{\mathbf{k}}(\widetilde{\mathfrak{X}}^{\square,\mathbf{k}-\mathrm{cr}}_{\overline{r}}),x}\right) = \dim_{k(x)} K(r) + \dim_{k(x)}\mathrm{Ext}^1_{\mathrm{cris}}\Big(D_{\mathrm{rig}}(r),D_{\mathrm{rig}}(r)\Big) - \lg(w_x). \tag{5.25}$$

We now compute using Proposition 5.15, (5.24) and (5.25):

$$\begin{aligned}\dim_{k(x)}\left(T_{\jmath_{w_x,\mathbf{k}}\left(\overline{\widetilde{U}^{\square}_{\mathrm{tri}}(\overline{r})\times_{\mathcal{W}^n_L}\mathcal{W}^n_{w,\mathbf{k},L}}\right),x} + T_{\iota_{\mathbf{k}}(\widetilde{\mathfrak{X}}^{\square,\mathbf{k}-\mathrm{cr}}_{\overline{r}}),x}\right) &= \Big(\dim X^{\square}_{\mathrm{tri}}(\overline{r}) - d_x\Big) + \\ &\Big(\dim_{k(x)} K(r) + \dim_{k(x)}\mathrm{Ext}^1_{\mathrm{cris}}\Big(D_{\mathrm{rig}}(r),D_{\mathrm{rig}}(r)\Big)\Big) - \\ &\Big(\dim_{k(x)} K(r) + \dim_{k(x)}\mathrm{Ext}^1_{\mathrm{cris}}\Big(D_{\mathrm{rig}}(r),D_{\mathrm{rig}}(r)\Big) - \lg(w_x)\Big) = \\ &\dim X^{\square}_{\mathrm{tri}}(\overline{r}) - d_x + \lg(w_x).\end{aligned}$$

□

Corollary 5.17. *Conjecture 5.6 implies Conjecture 2.8 for $\overline{r}=\overline{\rho}_{\tilde{v}}$ ($v\in S_p$), i.e.:*

$$\dim_{k(x)} T_{\widetilde{X}^{\square}_{\mathrm{tri}}(\overline{r}),x} = \lg(w_x) - d_x + \dim X^{\square}_{\mathrm{tri}}(\overline{r}).$$

In particular x is smooth on $\widetilde{X}^{\square}_{\mathrm{tri}}(\overline{r})$ if and only if w_x is a product of distinct simple reflections.

Proof. It follows from (5.22) and Proposition 5.16 that we have $\lg(w_x) - d_x + \dim X^{\square}_{\mathrm{tri}}(\overline{r}) \leq \dim_{k(x)} T_{\widetilde{X}^{\square}_{\mathrm{tri}}(\overline{r}),x}$. The equality follows from Corollary 5.11 which gives the converse inequality. Note that we also deduce $T_{\jmath_{w_x,\mathbf{k}}\left(\overline{\widetilde{U}^{\square}_{\mathrm{tri}}(\overline{r})\times_{\mathcal{W}^n_L}\mathcal{W}^n_{w,\mathbf{k},L}}\right),x} + T_{\iota_{\mathbf{k}}(\widetilde{\mathfrak{X}}^{\square,\mathbf{k}-\mathrm{cr}}_{\overline{r}}),x} \xrightarrow{\sim} T_{\widetilde{X}^{\square}_{\mathrm{tri}}(\overline{r}),x}$. Finally, as we have already seen, the last statement follows from Lemma 2.7. □

We end up with an application of Corollary 5.17 (thus assuming Conjecture 5.6) to the classical eigenvariety $Y(U^p,\overline{\rho})$ of §3.1. We keep the notation and assumptions of §3.1 and §3.2 and we consider a point $x\in Y(U^p,\overline{\rho})$ which is crystalline strictly dominant very regular. In a recent preprint ([5]), Bergdall, inspired by the upper bound in Theorem 2.15, proved an analogous upper bound for $\dim_{k(x)} T_{Y(U^p,\overline{\rho}),x}$, and obtained in particular that $Y(U^p,\overline{\rho})$ is smooth at x when the Weyl group element w_x in (3.13) is a product of distinct simple reflections and when some Selmer group (which is always conjectured to be zero) vanishes.

As a consequence of Corollary 5.17 we prove that this should *not* remain so when w_x is *not* a product of distinct simple reflections.

Corollary 5.18. *Assume Conjecture 5.6 and assume that w_x is* not *a product of distinct simple reflections. Then the eigenvariety $Y(U^p,\overline{\rho})$ is singular at x.*

Proof. For $v\in S_p$ denote by x_v the image of x in $X_{\mathrm{tri}}^{\square}(\overline{\rho}_{\tilde{v}})$ via (3.3). Since $Y(U^p,\overline{\rho})\hookrightarrow X_p(\overline{\rho})$, we have $x_v\in \widetilde{X}_{\mathrm{tri}}^{\square}(\overline{\rho}_{\tilde{v}})$. It follows from Corollary 5.17 that it is enough to prove the following: if $Y(U^p,\overline{\rho})$ is smooth at x then $\widetilde{X}_{\mathrm{tri}}^{\square}(\overline{\rho}_{\tilde{v}})$ is smooth at x_v for all $v\in S_p$, or equivalently $\widetilde{X}_{\mathrm{tri}}^{\square}(\overline{\rho}_p)\cong\prod_{v\in S_p}\widetilde{X}_{\mathrm{tri}}^{\square}(\overline{\rho}_{\tilde{v}})$ is smooth at $(x_v)_{v\in S_p}$. The point x is very regular, consequently it follows from [13, Thm. 4.8] that we have:

$$Y(U^p,\overline{\rho})\cong X_p(\overline{\rho})\times_{(\mathrm{Spf}\,S_\infty)^{\mathrm{rig}}}\mathrm{Sp}\,L \tag{5.26}$$

at the neighborhood of x where the map $S_\infty\twoheadrightarrow L$ is $S_\infty\twoheadrightarrow (S_\infty/\mathfrak{a})[1/p]$ and where $X_p(\overline{\rho})\to\mathfrak{X}_\infty\to(\mathrm{Spf}\,S_\infty)^{\mathrm{rig}}$ is induced by the morphism $S_\infty\to R_\infty$. Let $\omega_\infty(x)$ be the image of x in $(\mathrm{Spf}\,S_\infty)^{\mathrm{rig}}$, by an argument similar to the one in the proof of Proposition 5.15 we deduce from (5.26):

$$T_{Y(U^p,\overline{\rho}),x}\cong\left\{\vec{v}\in T_{X_p(\overline{\rho}),x}\text{ mapping to 0 in }T_{(\mathrm{Spf}\,S_\infty)^{\mathrm{rig}},\omega_\infty(x)}\otimes_{k(\omega_\infty(x))}k(x)\right\}.$$

This obviously implies:

$$\dim_{k(x)}T_{Y(U^p,\overline{\rho}),x}\geq\dim_{k(x)}T_{X_p(\overline{\rho}),x}-\dim_{k(\omega_\infty(x))}T_{(\mathrm{Spf}\,S_\infty)^{\mathrm{rig}},\omega_\infty(x)}. \tag{5.27}$$

But $\dim_{k(\omega_\infty(x))}T_{(\mathrm{Spf}\,S_\infty)^{\mathrm{rig}},\omega_\infty(x)}=g+[F^+:\mathbb{Q}]\frac{n(n-1)}{2}+|S|n^2$ (see beginning of §3.2) and $\dim_{k(x)}T_{Y(U^p,\overline{\rho}),x}=\dim Y(U^p,\overline{\rho})=n[F^+:\mathbb{Q}]$ since x is assumed to be smooth on $Y(U^p,\overline{\rho})$, hence we deduce from (5.27):

$$\dim_{k(x)}T_{X_p(\overline{\rho}),x}\leq g+[F^+:\mathbb{Q}]\frac{n(n+1)}{2}+|S|n^2=\dim X_p(\overline{\rho})$$

where the last equality follows from [13, Cor.3.12]. We thus have $\dim_{k(x)}T_{X_p(\overline{\rho}),x}=\dim X_p(\overline{\rho})$ which implies that x is smooth on $X_p(\overline{\rho})$, and thus by (i) of Remark 5.7 that $(x_v)_{v\in S_p}$ is smooth on $\widetilde{X}_{\mathrm{tri}}^{\square}(\overline{\rho}_p)$. □

Remark 5.19. (i) Singular crystalline strictly dominant points on eigenvarieties are already known to exist by [1, §6]. However, the singular points of *loc. cit.* are different from the points x of Corollary 5.18 since they have reducible associated global Galois representations.
(ii) Following (i) of Remark 3.13 and since the image of x in $\mathfrak{X}_{\overline{\rho}^p}$ should be a smooth point (use that x should be classical by (iii) of Remark 5.7 and argue as in the proof of Corollary 3.12), we thus think that the rigid analytic variety $X_p(\overline{\rho})$ should be irreducible in a neighbourhood of x. Though this is not a statement about the variety $Y(U^p,\overline{\rho})$, it seems reasonable to us to also expect that $Y(U^p,\overline{\rho})$ should be irreducible at x (and singular if w_x is not a product of distinct simple reflections).

6. Erratum to [13]

It was pointed to our attention by Toby Gee that the isomorphism $R_\infty/\mathfrak{a}R_\infty\simeq R_{\overline{\rho},\mathcal{S}}$ claimed to exist in [13, Thm. 3.5.(ii)] is not a consequence of [18]. Consequently the correct statement of [13, Thm. 3.5.(ii)] should be that there exists a surjective morphism $R_\infty/\mathfrak{a}R_\infty\twoheadrightarrow R_{\overline{\rho},\mathcal{S}}$ and a compatible isomorphism $\Pi_\infty[\mathfrak{a}]\simeq\widehat{S}(U^p,L)_{\mathfrak{m}}$.

However this mistake does not affect the other results of [13]. Here is the point of proof which should be modified.

In the proof of [13, Thm. 4.8], the equality in the displayed formula should be a priori replaced by an inclusion $\subset$ of closed rigid analytic subspaces. However it follows from [13, Thm. 4.2] that the corresponding reduced analytic subspaces are equal and it follows from the remaining part of the proof of [13, Thm. 4.8] (which does not used that equality) that the right hand side is reduced. This justifies a posteriori the equality of rigid analytic spaces in the displayed formula.

We thank Toby Gee for drawing this fact to our attention.

References

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