Preface

Introduction

Model theorists have often joked in recent years that the part of mathematical logic known as "pure model theory" (or stability theory), as opposed to the older and more traditional "model theory applied to algebra", turns out to have more and more to do with other subjects of mathematics and to yield genuine applications to combinatorial geometry, differential algebra and algebraic geometry.

We illustrate this by presenting the very striking application to diophantine geometry due to Ehud Hrushovski: using model theory, he has given the first proof valid in all characteristics of the "Mordell-Lang conjecture for function fields" (*The Mordell-Lang conjecture for function fields*, Journal AMS 9 (1996), 667-690). More recently he has also given a new (model theoretic) proof of the Manin-Mumford conjecture for semi-abelian varieties over a number field. His proof yields the first effective bound for the cardinality of the finite sets involved (*The Manin-Mumford conjecture*, preprint).

There have been previous instances of applications of model theory to algebra or number theory, but these applications had in common the feature that their proofs used a lot of algebra (or number theory) but only very basic tools and results from the model theory side: compactness, first-order definability, elementary equivalence...

Hrushovski's results are not only interesting as such but also due to the nature of their proofs which use in an essential way most of the beautiful and sophisticated recent developments of model theory. In fact he shows that these questions of diophantine geometry can be naturally integrated into the abstract framework which has been developed in model theory these last years.

Let us go back a few years to recall a bit of informal history, without any attempt at exhaustive coverage or precise attribution. Model theory during its first years of existence, from the 1930's to the 60's, was traditionaly related to universal algebra. Indeed, in model theory one considers classes of "abstract structures", of which the classical mathematical structures will be particular instances. These structures come equipped with a distinguished class of subsets, the definable sets, which as their name indicates, are "defined" from the basic operations of the structure.

Then in the sixties, first with M. Morley and then with the colossal work of S. Shelah, a new perspective arose around two main lines: the idea was to classify structures according to the type of combinatorial objects one could define in them (infinite orderings, infinite trees...) and also to use this classification to assign dimensions to the definable sets in certain cases. This was the birth of "classification theory" or "stability theory" , considered as "pure model theory", in contrast with the Robinson type of applied model theory. The inspiration here was primarily of a set theoretic and infinite combinatorial nature.

A second change of perspective took place in the 80's with B. Zilber in particular and then since 1986 with the work of Hrushovski. This was the birth of "geometric stability" or "geometric model theory", where the inspiration comes from combinatorial geometries and algebraic geometry. While integrating stability theory in the style of Shelah, one focuses here on the study of definable sets with finite dimension and their classification according to which type of algebraic objects (groups, fields) or geometries can be defined in them.

The following trichotomy turns out to be very relevant: structures where no group is definable, groups of linear type (which behave like vector spaces) and field-like structures. One context in which this trichotomy is particularly meaningful comes out of the beautiful work of Hrushovski and Zilber on "Zariski geometries", where they characterize abstractly amongst noetherian topologies the ones arising from the Zariski topology of an algebraic curve (over an algebraically closed field).

As we will see in this volume, the Mordell-Lang conjecture really says that certain subsets of a semi-abelian variety are of linear type. Hrushovski first sets the question in the adequate framework, differentially closed fields in characteristic zero and (non algebraically closed) separably closed fields in characteristic p. He proceeds to apply the powerful tools that have been developed around this trichotomy in the past years. One of the interesting features of this proof is the fact that it is uniform for both characteristic zero and characteristic p, only the basic settings are different. It should also be noted that in fact the characteristic zero case can be deduced from the characteristic p case, as is shown by Hrushovski in the last chapter of this volume.

Suggestions for further reading

In this volume we focus on one result, the Mordell-Lang conjecture for function fields and present only those parts of geometric model theory which are relevant and useful for this proof. In particular we say nothing of the more recent work of Hrushovski on the Manin-Mumford conjecture which also fits in this general model-theoretic framework (the adequate setting in this case is algebraically closed fields with an automorphism, ACFA). The reader can find a partial exposition of these results in *ACFA and the Manin-Mumford Conjecture*, A. Pillay, in Algebraic Model Theory, B. Hart, A. Lachlan and M. Valeriote ed., NATO ASI Series C 496, Kluwer 1997, pp: 195-205.

For a recent survey on model theory and diophantine geometry with some sketches of proof (in particular for the characteristic p case of Mordell-Lang), see *Model theory and diophantine geometry*, A. Pillay, Bull. Am. Math. Soc. 34 (1997), pp: 405-422.

There is another related but different line of work which has seen substantial development recently, namely the relationship between differential algebra (with work in particular of P. Cassidy and of A. Buium) and the model theoretic point of view on differentially closed fields. This subject is not treated in this volume except when it relates to our own purpose (in the chapters of C. Wood and D. Marker). For more on this subject, see for example :

- *Model Theory of Fields*, D. Marker, M. Messmer and A. Pillay, Lecture Notes in Logic 5, Springer, 1996.

- *Model Theory, Differential Algebra and Number Theory*, A. Pillay, in Proceedings of ICM 94 Zurich, Birkhauser 1996.

Presentation of the volume

A few months after Hrushovski announced his model theoretic proof of the Mordell-Lang conjecture, it became apparent that there were some natural obstacles to the understanding of these new exciting results. In order to understand one needs a minimal knowledge of the basics of algebraic geometry but, more importantly, a good knowledge of the recent developments of model theory, or more precisely, of geometric stability theory.

Thus the idea arose naturally to present a more or less self-contained exposition of this proof. This was concretized first as a series of coordinated lectures in a summer school organized in September 1994 in Manchester (UK) in the framework of "RESMOD" and devoted exclusively to this subject (RESMOD is the acronym for the European Human Capital and Mobility Network on "Model Theory and its applications" coordinated by the Équipe de Logique Mathématique CNRS-Université Paris 7 (1994-1997)). This workshop was intended mainly for young researchers in model theory, and only a very basic knowledge of classical model theory was assumed. After the success of this workshop and the interest shown in the community, the need became apparent for something more elaborate than the notes that were distributed at the time of the workshop. Each of the speakers at the workshop agreed to write a chapter of this book on the subject of his/her lectures, and to collaborate with the others in order to obtain a progressive and coherent presentation. Hrushovski himself, who had not participated to the Summer School, contributed a final chapter, where he shows that "characteristic p implies characteristic 0".

The aim of this volume is to take a mathematician with a very basic knowledge of both model theory and algebraic geometry and to introduce her/him to the relationship between geometric stability theory and algebraic geometry, finishing with the detailed exposition of Hrushovski's proof of the Mordell-Lang conjecture for function fields.

In the hope that this might be used also by mathematicians with no previous knowledge whatsoever of model theory, a first chapter gives an informal presentation of the main basic definitions and results of "classical" model theory.

In order to be really self-contained and to avoid certain technical difficulties that might hinder the reader's understanding, we have chosen in this volume to make two restrictions. First, we restrict ourselves to the case of abelian varieties, but Hrushovski's proof really works in the same way for semi-abelian varieties with just a little extra effort needed on the model theoretic side in the characteristic p case. Secondly, we have chosen to present exhaustively the characteristic zero case, which is a little easier to describe from the model theoretic point of view. The characteristic p case goes through the same steps. We give here the details of the characteristic p setting (separably closed fields of finite invariant) and explain at the end where the main differences lie. We believe that a reader who understands both the characteristic zero case and the characteristic p setting should be able to see the characteristic p case quite well.

Some of the chapters are totally self-contained, with complete proofs. Others are surveys, either because the subject is too vast or because there already exist some good (and accessible) references. In the case of surveys, special effort has been made so that all results needed further along in the book are clearly stated and some comments and examples are given in order to give the reader some intuitive understanding of the subject. Each chapter has its own reference list.

Description of the chapters

- *Introduction to model theory:* an informal introduction to the very basic concepts of model theory, to help the reader with no previous knowledge of the subject.

- *Introduction to stability theory and Morley rank:* a self-contained detailed presentation, with all proofs included, of the necessary classical material from stability. The presentation and the choice of material are adapted to the context of this book.

- *Omega-stable groups:* classical results on groups of finite Morley rank as well as more recent results on one-based groups. Again this is self-contained, referring only to the previous chapter. Complete proofs are given.

- *Model theory of algebraically closed fields:* a survey, with proofs included, of the model theoretic approach to algebraically closed fields, algebraic varieties and algebraic groups. This is intended both as an introduction to the model theoretic point of view on basic algebraic geometry for the geometers and as an introduction to what is needed of algebraic geometry for the model theorists.

- *Introduction to abelian varieties and the Mordell-Lang conjecture:* an introduction to the conjecture and related questions, aimed at non specialists of algebraic geometry, with a survey of the main classical properties of abelian varieties which are needed in this volume. This chapter presents no proofs but includes a detailed bibliography of the subject, with comments.

- *The model-theoretic content of Lang's conjecture:* this short chapter explains (with proof) what the model theoretic equivalent of the Mordell-Lang conjecture is.

- *Zariski geometries:* a survey of the important paper by Hrushovski and Zilber. It would be impossible to give proofs here so the stress is on giving examples and comments that can shed some light on the results. At the end, the proof that strongly minimal sets in differentially closed fields are Zariski structures is given.

- *Differentially closed fields of characteristic zero:* a survey of basic results as well as more recent and sophisticated ones in the form they are needed in the other chapters. This contains mainly only indications of proofs as good references exist and are given.

- *Separably closed fields:* this chapter puts in place the setting for the characteristic p case of the conjecture. The first part consists of a short survey of the basic classical results on separably closed fields, for which again good references exist. The second part presents in detail, with all proofs given, the adequate topology for which dimension one subsets are Zariski geometries. The presentation and the proof given here by F. Delon are different from the ones given by Hrushovski in his paper. In particular the proof given here applies to all minimal types, removing the hypothesis of "thin" appearing in Hrushovski's account.

- *Proof of the Mordell-Lang conjecture:* a complete exposition of Hrushovski's proof in characteristic zero is given, the setting for the characteristic p case is set up in details and the main technical difficulties are pointed out. This chapter refers only to notions and results introduced in the previous chapters, except when mentioning the aspects of the characteristic p case which is not given in full detail here.

- *Proof of Manin's theorem by reduction to positive characteristic:* a proof is given of how to deduce the characteristic zero case from the positive characteristic case.

Prerequisites and notation

It will be assumed in all except the first chapter that the reader is familar with the very basic notions of classical model theory : models, formulas, definable sets, compactness theorem, saturation, types. All these notions are defined informally in the first chapter.

From the algebraic point of view, only a very basic knowledge of commutative algebra is assumed. All notions of algebraic geometry are defined, even the most basic ones.

Since these notes were written by different authors, they may in some instances use different terminology or notation, more adapted to their specific subject matter, but we have tried to signal these variations.

Otherwise we all use the standard notation of model theory, most of which is recalled in the first two chapters.

Acknowledgments

I would like first of all to thank all the contributors to this book. They showed great good will when asked again and again to make changes in order to converge

towards a coherent whole. Some of them agreed to act as "help referees" for the others' chapters. Finally they all showed infinite patience over the years since this project was first initiated, years made so numerous only by my own procrastination.

Over these numerous years many others were called upon to contribute by comments and suggestions on parts of the book. I cannot thank them all but let me mention L. Bélair, B. Herwig, and give particular thanks to Z. Chatzidakis who without authoring directly any chapter, made valuable contributions to several, going well beyond the work any referee would do.

Finally, this project would not have been contemplated without the first step, the Manchester RESMOD workshop. This was masterfully organised at the Mathematics Department of the University of Manchester under the supervision of M. Prest.

Elisabeth Bouscaren