Box 8: Key bound for transformal specialisations (preliminary version)

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Abstract

The aim of this talk is to give a proof of the key proposition 8.9 in [1].

Comments / connections to other talks etc.

A list of things which are needed from previous talks / from [1]

Lemma 1 ([1, Lemma 2.8]). *Let R be a well-mixed difference ring. Call an ideal P of R cofinally minimal if for any finite $F \subset R$ there exists a Noetherian subring S of R with $P \cap S$ a minimal prime ideal of S. Then any cofinally minimal ideal of R is a difference ideal, which is also algebraically prime.*

Remark 2 ([1, Remark 2.9]). Let A be a Noetherian commutative ring, R a countably generated A-algebra. Then cofinally minimal ideals exist. If S is a finitely generated subalgebra of R, and P a minimal prime ideal of S, then P extends to a cofinally minimal ideal of R.

Corollary 3. *Let R be a well-mixed finitely generated difference algebra over the difference field K. Then a minimal prime ideal is an algebraically prime difference ideal.*

The following lemma was needed in first proof of Lemma 8, and is not needed anymore. Its proof uses Remark 2.

Lemma 4 ([1, Lemma 2.10]). *Let R be a difference ring, and I a well-mixed radical ideal. Then I is the intersection of algebraically prime difference ideals.*

Apart from these, the following results are used in the talk:

1. [1, Lemma 2.1] is used in the proof of the key proposition. (This is an elementary fact.)
2. [1, Lemma 4.2] is used in the proof of the key proposition.
3. [1, Corollary 6.7] is used in the proof of Lemma 5.
4. [1, Proposition 7.16] is used in the proof of the key proposition. This in turn uses the notion of inertial dimension.

Comments / things to be noted

1. In many places, it is not explicitly stated that he is working with an ω-increasing valued difference field. Sometimes, what is written suggests that the results hold in a more general context. But one uses constantly results from the ω-increasing world, and so we added this in many places. In particular, we added it to the definition of $X_{\to 0}$.

2. There are two definitions of $\mathrm{rk}_{\mathrm{val}}$ in [1] (p. 52, (4), just before Not. 6.3; after Cor 6.33) which only coincide in the case of ω-increasing valued difference fields of transformal dimension 1 (finitely generated up to algebraic closure and completion).

1 Introduction

Context and notation. k is an inversive difference field, t is σ-transcendent over k.

$k(t)_\sigma = k(t, \sigma(t), \ldots)$ is the ω-increasing valued difference field, with v trivial on k and $0 < v(t)$.

$k[\breve{t}]_\sigma$ denotes the valuation ring of $k(t)_\sigma$, and $\breve{A}_k = \mathrm{Spec}^\sigma(k[\breve{t}]_\sigma)$.

Lemma 5 ([1, Corollary 7.22]). *Let K an ω-increasing valued difference field extending $k(t)_\sigma$, with valuation ring R. Let $K(0) \subseteq K(1) \subseteq \ldots$ be a sequence of subfields of K such that $k \subseteq K(0)$ and $\bigcup_{n\in\mathbb{N}} K(n) = K$. Set $R(n) = K(n) \cap R$.*

Assume that $t \in K(1)$, $\mathrm{tr.deg}\ (K(0)/k) = d < \infty$, and that for every $n \in \mathbb{N}$, one has $\mathrm{tr.deg}\ (K(n+1)/K(n)) \leq 1$ and $\sigma(K(n)) \subseteq K(n+1)$.

Then, the following properties hold.

1. *$P = \sqrt{tR}$ is an algebraically prime difference ideal of R. In particular, it is the unique minimal prime ideal containing tR.*

2. *$\sqrt[\sigma]{P}$ is equal to the maximal ideal $\mathfrak{m}$ of R.*

3. *$\mathrm{t.dim}\ (R/tR) \leq d$. If equality holds, then $P \cap R(0) = (0)$.*

4. *$\mathrm{rk}_{\mathrm{val}}\ (K/k(t)_\sigma) \leq \mathrm{t.dim}\ (R/tR)$. In particular, $\mathrm{rk}_{\mathrm{val}}\ (K/k(t)_\sigma) \leq d$, and if equality holds, then $\mathrm{t.dim}\ (R/tR) = d$ and $P \cap R(0) = (0)$.*

Proof. We denote the residue field of K by K_{res}, and its value group by Γ_K.

The ring $R(n)$ induces a valuation on $K(n)$, with residue field $K(n)_{\mathrm{res}}$ and value group $\Gamma_{K(n)} \subseteq \Gamma_K$. Denote by Γ_n the divisible hull of $\Gamma_{K(n)}$ in Γ_K.

Claim 1. *For every $n \in \mathbb{N}$, Γ_{n+1} contains an element which is greater than every element of Γ_n.*

Indeed, it follows from the assumptions that $\mathrm{tr.deg}\ (K(n)/k) < \infty$. Thus, Γ_n is of finite $\mathbb{Q}$-rank. Moreover, we have $\sigma(K(n)) \subseteq K(n+1)$, so in particular $\sigma(\Gamma_n) \subseteq \Gamma_{n+1}$. Let $0 < \gamma_n$ be maximal in Γ_n w.r.t. $\ll$. Then $\gamma_{n+1} = \sigma(\gamma_n) \gg \gamma_n$, and so $\gamma_{n+1} > \Gamma_n$, proving Claim 1.

Claim 2. $K_{\mathrm{res}} \subseteq (K(0)_{\mathrm{res}})^{alg}$.

As tr.deg $(K(n+1)/K(n)) \leq 1$ and $\mathrm{rk}_{\mathbb{Q}}\ (\Gamma_{n+1}/\Gamma_n) \geq 1$ by Claim 1, we infer from the fundamental inequality that both quantities are equal to 1 and that $K(n+1)_{\mathrm{res}} \subseteq (K(n)_{\mathrm{res}})^{alg}$ for all $n \in \mathbb{N}$. This proves Claim 2.

Claim 3. *Let* $\Delta = \{\gamma \in \Gamma_K \mid -\mathrm{val}(t) \ll \gamma \ll \mathrm{val}(t)\}$. *Then* $\Delta \subseteq \Gamma_0$.

Since $\mathrm{rk}_{\mathbb{Q}}\ (\Gamma_{n+1}/\Gamma_n) = 1$ and $\mathrm{val}(t) \in \Gamma_1$, one may deduce from Claim 1 that $\Delta \cap \Gamma_n = \Delta \cap \Gamma_{n+1}$ for all n. But $\Delta \subseteq \Gamma_K = \bigcup_{n \in \mathbb{N}} \Gamma_n$, and so Claim 3 follows.

It follows from the definition of Δ that $\sqrt{tR} = \{a \in R \mid \mathrm{val}(a) \notin \Delta\}$, and so $P := \sqrt{tR}$ is easily seen to be an algebraically prime difference ideal of R, proving (1).

In (2), $\sqrt[\sigma]{P} \subseteq \mathfrak{m}$ is clear. To prove the reverse inclusion, note that Δ has finite $\mathbb{Q}$-rank by Claim 3. In particular, for every $\delta > 0$, as $\delta \ll \sigma(\delta) \ll \ldots$, necessarily $\sigma^n(\delta) \notin \Delta$ for n large enough. Thus, for $a \in \mathfrak{m}$, there is $n \in \mathbb{N}$ such that $\mathrm{val}(\sigma^n(a)) \notin \Delta$. This proves $\sqrt[\sigma]{P} \supseteq \mathfrak{m}$.

(3) Consider the coarsening of val on K given by $\mathrm{val}' = \pi \circ \mathrm{val} : K \to \Gamma/\Delta$. The corresponding valuation ring is $R' = R_P$, with maximal ideal PR'. The divisible hull (inside Γ/Δ) of the value group of the valuation on $K(n)$ given by the valuation ring $R'(n) := K(n) \cap R'$ is equal to $\Gamma_n/\Delta =: \widetilde{\Gamma_n}$. Denote by $K'(n)_{\mathrm{res}}$ its residue field. It follows from Claims 1 and 3 that $\mathrm{rk}_{\mathbb{Q}}\ (\widetilde{\Gamma_{n+1}}/\widetilde{\Gamma_n}) \geq 1$ for all n. The proof of Claim 2 then shows that $K'_{\mathrm{res}} := \mathrm{Frac}(R/P) = R'/PR' \subseteq (K'(0)_{\mathrm{res}})^{alg}$. Thus,

$$\mathrm{tr.deg}\ (K'_{\mathrm{res}}/k) = \mathrm{tr.deg}\ (K'(0)_{\mathrm{res}}) \leq \mathrm{tr.deg}\ (K(0)/k) = d,$$

where the inequality is strict unless $R'(0) = K(0)$, in which case $P \cap R(0) = (0)$. This proves (3), since t.dim $(R/tR) =$ t.dim (R/P) by (1), and the latter is bounded by (in fact even equal to) tr.deg (K'_{res}/k).

(4) Note that K is of transformal dimension 1 over k, and finitely generated over k (up to algebraic closure), so it follows from [1, Corollary 6.7] that $\mathrm{rk}_{\mathbb{Q}}\ (\Gamma_K/\Gamma_{k(t)_\sigma}) = \mathrm{rk}_{\mathbb{Q}}\ (\Delta)$. Thus, by the fundamental inequality and the fact that K'_{res} is a valued field with value group Δ and residue field K_{res}, we get

$$\mathrm{rk}_{\mathrm{val}}\ (K/k(t)_\sigma) = \mathrm{tr.deg}\ (K_{\mathrm{res}}/k) + \mathrm{rk}_{\mathbb{Q}}\ (\Delta) \leq \mathrm{tr.deg}\ (K'_{\mathrm{res}}/k) = \mathrm{t.dim}\ (R/tR).$$

□

2 Lemma 8.2

We keep the notation and conventions from the previous section, to which we add the following.

Context and notation.
V is an irreducible, reduced affine algebraic variety over k.

$X \subseteq [\sigma]_k V_{\breve{A}_k} := [\sigma]_k V \times_{\mathrm{Spec}^\sigma(k)} \breve{A}_k$ is a difference subscheme (we may and will assume X affine).

$X_t := X \times_{\breve{A}_k} \mathrm{Spec}^\sigma(k(t)_\sigma)$ denotes the *generic fiber* of X.

$X_0 := X \times_{\breve{A}_k} \mathrm{Spec}^\sigma(k)$ denotes the *special fiber* of X.

Definition 6. We say that an affine difference scheme of finite type over $\breve{A}_k$ is *flat* if for every algebraically prime difference ideal P, multiplication by $\sigma^n(t)$ defines an injective endomorphism of the localization at P for any $n \geq 0$.

Remark 7. Assume that $X = \mathrm{Spec}^\sigma A$, let P be an algebraically prime difference ideal of A. Then the flatness of the difference scheme X implies the flatness of the $k[\breve{t}]_\sigma$-module A_P. Indeed, by [2, 4.12], the flatness of A_P is equivalent to: for any finitely generated ideal I of $k[\breve{t}]_\sigma$, the natural morphism $I \otimes_{k[\breve{t}]_\sigma} A_P \to IA_P$ is injective. As any non-zero element of $I \otimes A_P$ can be written $t^\alpha \otimes a$ with $0 \neq a \in A_P$, our definition of flat scheme implies that $t^\alpha a \neq 0$, and therefore the flatness of A_P.

More generally, the same proof gives that if M is an $k[\breve{t}]_\sigma$-module such that multiplication by $\sigma^n(t)$ defines an injective endomorphism of M for any $n \geq 0$, then M is a flat $k[\breve{t}]_\sigma$-module.

Lemma 8 ([1, Lemma 8.2])**.** *Let X be a flat difference scheme of finite type over $\breve{A}_k$. If* $\mathrm{t.dim}\ X_t = d$ *(over K), then* $\mathrm{t.dim}\ X_0 \leq d$ *(over k).*

Proof. Let $X = \mathrm{Spec}^\sigma A$ for some finitely generated $k[\breve{t}]_\sigma$-difference algebra A. Then $X_0 = \mathrm{Spec}^\sigma A/tA$ is a closed subscheme of X, and every algebraically prime difference ideal of A/tA lifts to an algebraically prime difference ideal of A. We may therefore assume that A is (reduced and) well-mixed.

Let P be an algebraically prime difference ideal of A/tA, and Q the corresponding algebraically prime difference ideal of A. By Remark 7, A_Q is a flat $k[\breve{t}]_\sigma$-module. Hence the natural map (of rings) $A_Q \to A_Q \otimes_{k[\breve{t}]_\sigma} k(t)_\sigma$ is injective, so in particular we may find a prime ideal P_0 in the ring $A_Q \otimes_{k[\breve{t}]_\sigma} k(t)_\sigma$. Now, let P_1 be a minimal prime ideal of $A \otimes_{k[\breve{t}]_\sigma} k(t)_\sigma$ contained in $P_0 \cap A \otimes_{k[\breve{t}]_\sigma} k(t)_\sigma$. Then P_1 is an algebraically prime difference ideal by Corollary 3 (since $A \otimes_{k[\breve{t}]_\sigma} k(t)_\sigma$ is well-mixed, being the localisation of a well-mixed ring) and we compute

$$\dim(A/tA)/P = \dim A/Q \leq \dim A_Q \otimes_{k[\breve{t}]_\sigma} k(t)_\sigma / P_0 \leq \dim A \otimes_{k[\breve{t}]_\sigma} k(t)_\sigma / P_1 \leq \mathrm{t.dim}\ X_t.$$

Only the inequality $\dim A/Q \leq \dim A_Q \otimes_{k[\breve{t}]_\sigma} k(t)_\sigma / P_0$ deserves an argument. Let $a = (a_1, \ldots, a_n)$ be a tuple from A satisfying a polynomial equation $F(a) = 0$ in $A_Q \otimes_{k[\breve{t}]_\sigma} k(t)_\sigma / P_0$, where F has coefficients from $k(t)_\sigma$. Multiplying F by a suitable element of K, we may assume that F has coeeficients from $k[\breve{t}]_\sigma$, and that at least one coefficient is of valuation 0. This means that the image of a in $A_Q \otimes_{k[\breve{t}]_\sigma} k$ satisfies a nontrivial equation with coefficients from k.

As Q contains tA, the same is true in A_Q/Q, and so in A/P. So the result follows.

□

3 Lemma 8.8

We keep the context and notation from the previous section.

Definition 9.

- An *ω-increasing valuative difference scheme over $\breve{A}_k$* is a difference scheme $\mathrm{Spec}^\sigma(R)$, where R is an ω-increasing transformal valuation domain extending $k[\breve{t}]_\sigma$, i.e., if $K = \mathrm{Frac}(R)$, then $(K, R) \supseteq (k(t)_\sigma, k[\breve{t}]_\sigma)$ is an extension of ω-increasing valued difference fields.

- In the above context, the *pathwise specialization* $X_{\to 0}$ of X is the smallest well-mixed difference subscheme Y of X_0 such that for any ω-increasing valuative difference scheme T over $\breve{A}_k$

and every morphism $f : T \to X$ over $\breve{A}_k$, the restriction to the special fibers $f_0 : T_0 \to X_0$ factors through Y.

Suppose that $X = \mathrm{Spec}^\sigma(A)$. A map $f : T \to X$ as in the definition of $X_{\to 0}$ then corresponds to a map of difference rings $h : A \to R$ (over $k[\breve{t}]_\sigma$), where $T = \mathrm{Spec}^\sigma(R)$.

Lemma 10. *Let $X = \mathrm{Spec}^\sigma(A)$ be as above. Then $X_{\to 0}$ exists.*

More precisely, suppose that $\{h_j : A \to R_j\}_{j \in J}$ is a family of maps of difference rings over $k[\breve{t}]_\sigma$, with R_j an ω-increasing transformal valued difference domain for every $j \in J$, and such that $\bigcap_{j \in J} h_j^{-1}(tR_j)$ is minimal with respect to inclusion. Then $\bigcap_{j \in J} h_j^{-1}(tR_j)$ equals the well-mixed difference ideal $I_{\to 0}$ defining $X_{\to 0}$.

Proof. Given a map $h : A \to R$ over $k[\breve{t}]_\sigma$, the ideal $h^{-1}(tR)$ is well-mixed. Indeed, tR is a well-mixed ideal, as R is a 1-increasing valued difference domain. Given a well-mixed ideal $I \subseteq A$, the map $h_0 : A/t \to R/t$ then factors through A/I iff $h^{-1}(tR) \supseteq I$.

The second statement of the lemma follows, since an arbitrary intersection of well-mixed ideals is well-mixed. □

Lemma 11 ([1, Lemma 8.8]). *Let Z be an algebraic component of $X_{\to 0}$. Then there is an ω-increasing valuative difference scheme T over $\breve{A}_k$ and a morphism $f : T \to X$ over $\breve{A}_k$ such that Z is contained in $f_0(T_0)$.*

Proof. Let $\mathfrak{p} \supseteq I_{\to 0}$ be the algebraically prime difference ideal corresponding to Z. If we reformulate the lemma on the level of difference rings, it states that there is an ω-increasing transformal valued difference domain R containing $k[\breve{t}]_\sigma$ and a map of difference rings $h : A \to R$ over $k[\breve{t}]_\sigma$ such that $h^{-1}(tR) \subseteq \mathfrak{p}$.

To prove this, consider first finitely many elements $a_1, \ldots, a_n$ from $A \setminus \mathfrak{p}$. Since $\mathfrak{p}$ is a prime ideal, $a = a_1 \cdots a_n \in A \setminus \mathfrak{p}$. As $\mathfrak{p} \supseteq I_{\to 0} = \bigcap_{j \in J} h_j^{-1}(tR_j)$ by Lemma 10 (for some family of $h_j : A \to R_j$), there is R' and $h' : A \to R'$ such that $a \notin h'^{-1}(tR')$ and thus $a_1, \ldots, a_n \notin h'^{-1}(tR')$.

By compactness, we may find $h : A \to R$ such that $h(a) \notin tR$ for any $a \in A \setminus \mathfrak{p}$. But this means that $h^{-1}(tR) \subseteq \mathfrak{p}$. □

4 The key proposition

We keep the hypotheses and notation from the previous sections.

Definition 12. Let Y be a difference subscheme of $[\sigma]_k V$. We say that Y is weakly Zariski dense (in V) if $Y[0] = V$.

Definition 13. The difference subscheme $Z \subseteq [\sigma]_k V$ is said to be *evenly spread out along* V if for any proper closed subvariety U of V, the difference scheme $Z \cap [\sigma]_k U$ has total dimension $< \dim(V)$.

For the definition of the *inertial dimension* which appears in the following proposition, we refer to talk 7 (see [1, Example 7.9 and Definition 7.10]).

Proposition 14 ([1, Proposition 8.9]). *In the above context, (1) implies (3), and (1&2) implies (3&4).*

1. *Consider ω-increasing transformal valued fields L generated over $k(t)_\sigma$ by $c \in X_t(L) \subseteq V(L)$. Whenever* $\mathrm{tr.deg}_{k(t)_\sigma}(L) \geq d$*, equality holds, and $\sigma(c) \in k(t, c)^{alg}$.*

2. *Every weakly Zariski dense (in V) algebraically integral difference subscheme of X_0 is transformally reduced.*

3. *$X_{\to 0}$ is evenly spread out along V, i.e., every algebraic component Y of $X_{\to 0}$ of total dimension $\geq d$ is weakly Zariski dense in V.*

4. *For W a subvariety of V, consider $^*W := \{x \in X_t(\mathcal{O}) \mid \mathrm{res}_X(x) \in W\}$, a quantifier free formula in the theory of ω-increasing valued difference fields containing $k(t)_\sigma$, where $\mathcal{O}$ denotes the valuation ring and $\mathrm{res}_X : X_t(\mathcal{O}^L) \to X_0(L_{\mathrm{res}})$ the reduction map induced by* res.

   If W is a proper closed subvariety of V, then inert.dim $(^*W) < d = \dim(V)$.

Proof. (1)⇒(3): Let A be the difference algebra over $k[\check{t}]_\sigma$ such that $X = \mathrm{Spec}^\sigma\ (A)$, and let Y be an algebraic component of $X_{\to 0}$, with $\mathfrak{p}$ the corresponding algebraically prime difference ideal of A. By Lemma 11, there exists an ω-increasing transformal valued field (K, R) extending $(k(t)_\sigma, k[\check{t}]_\sigma)$ and a difference ring morphism $h : A \to R$ over $k[\check{t}]_\sigma$ such that $\mathfrak{p} \supseteq h^{-1}(tR)$. In other words, $A/\mathfrak{p}$ embeds into a quotient of R/tR.

Replacing K by a difference subfield if necessary, we may assume that $K = \mathrm{Frac}(h(A))$. In particular, by (1), we then have $\mathrm{tr.deg}\ _{k(t)_\sigma}(K) \leq d$.

Observe that for every $n \geq 0$, $\sigma^n(t)$ is not a 0-divisor in $A/\ker(h)$. Indeed, if $\sigma^n(t) \cdot \overline{a} = 0$ in $A/\ker(h)$, then $h(\sigma^n(t) \cdot a) = \sigma^n(t) \cdot h(a) = 0$ in R, and so $h(a) = 0$. It follows in particular that no $\sigma^n(t)$ is a 0-divisor in a localisation of $A/\ker(h)$ at some algebraically prime difference ideal, and so $\mathrm{Spec}^\sigma\ (A/\ker(h))$ is a flat difference scheme over $\breve{A}_k$. Thus, by Lemma 8, $\mathrm{t.dim}\ (\mathrm{Spec}^\sigma\ (A/\ker(h), t) \leq d$, hence also $\mathrm{t.dim}\ \big(\mathrm{Spec}^\sigma\ (A/h^{-1}(tR)\big) \leq d$.

If $\mathrm{tr.deg}\ _{k(t)_\sigma}(K) < d$, clearly $\mathrm{t.dim}\ \big(\mathrm{Spec}^\sigma\ (A/h^{-1}(tR))\big) < d$.

Note that $A/h^{-1}(tR)$ is well-mixed and finitely generated over k. Thus, by [1, Lemma 4.2], $\mathrm{t.dim}\ (A/h^{-1}(tR))$ is equal to the maximum over all $\mathrm{tr.deg}\ (\mathrm{Frac}(A/\mathfrak{q}))$, where $\mathfrak{q}$ is any prime ideal of A containing $h^{-1}(tR)$. It follows that if $\mathfrak{p}$ is not a minimal prime ideal (among the ones containing $h^{-1}(tR)$) or if $\mathrm{tr.deg}\ _{k(t)_\sigma}(K) < d$, then $\mathrm{t.dim}\ (Y = \mathrm{Spec}^\sigma\ (A/\mathfrak{p})) < d$, and there is nothing to show in these cases.

We may thus assume that $\mathfrak{p}$ is minimal among the prime ideals containing $h^{-1}(tR)$, and that $\mathrm{tr.deg}\ _{k(t)_\sigma}(K) = d$. The latter implies (by (1)) that h induces an embedding of $k(V)$ into K. In what follows, we will consider $k(V)$ as a subfield of K (via h).

Let $K[0] := k(V)^{alg} \cap K$, and $K[n] := K[0](t, \sigma(t), \ldots, \sigma^{n-1}(t))^{alg} \cap K$ for $n \geq 1$. Then $\sigma(K[0]) \subseteq K[1]^{alg}$ by (1), and so $\sigma(K[n]) \subseteq K[n+1]^{alg}$ for every $n \geq 0$. By (1), $\mathrm{tr.deg}\ _{K[n]}(K[n+1]) \leq 1$ for all $n \geq 0$, so the hypotheses of Lemma 5 are satisfied by $K = \bigcup_n K[n]$. In particular, $P = \sqrt{tR}$ is the unique minimal prime ideal containing tR, and it is a difference ideal.

If $\mathrm{t.dim}\ (R/tR) < d$, then $\mathrm{t.dim}\ (Y) < d$ as well. We may thus assume that $\mathrm{t.dim}\ (R/tR) \geq d$. By Lemma 5, it follows that $\mathrm{t.dim}\ (R/tR) = d$ and $P \cap K[0] = (0)$. If $h(x) \in P$, then $h(x^n) \in tR$ for some $n \in \mathbb{N}$, so $x^n \in \mathfrak{p}$ and finally $x \in \mathfrak{p}$. This shows that $h^{-1}(P) \subseteq \mathfrak{p}$, and thus $h^{-1}(P) = \mathfrak{p}$ by minimality of $\mathfrak{p}$. But then $\mathfrak{p} \cap k(V) = (0)$ and so in particular $\mathfrak{p} \cap k[V] = (0)$, showing that Y is weakly Zariski dense in V.

(1&2)⇒(4): By one of the main results of Talk 7 (Proposition 7.16), it is enough to show that whenever $K = k(t, c)_\sigma$ is an ω-increasing valued difference field such that $c \in {}^*W$, then $\mathrm{rk}_{\mathrm{val}}\ (K/k(t)_\sigma) < d = \dim(V)$.

Recall that $\mathrm{rk}_{\mathrm{val}}\ (K/k(t)_\sigma) = \mathrm{tr.deg}\ _k(K_{\mathrm{res}}) + \mathrm{rk}_{\mathbb{Q}}\ (\Gamma_K/\Gamma_{k(t)_\sigma})$. By the fundamental inequality, the result is then clear if $\mathrm{tr.deg}\ _{k(t)_\sigma}(K) < d$, so we may assume that $\mathrm{tr.deg}\ _{k(t)_\sigma}(K) = d$ (by (1)).

Let $K(0) = k(c)$, and $K(n) = k(c, \sigma(c), \dots, \sigma^n(c), t, \dots, \sigma^{n-1}(t))$ for $n \geq 1$. By (1), $\sigma(c) \in k(t, c)^{alg}$, and so $\sigma^{n+1}(c) \in k(\sigma^n(c), \sigma^n(t))^{alg}$ for all n. In particular, the hypotheses of Lemma 5 hold for the increasing sequence $(K(n))_{n \in \mathbb{N}}$. Let R be the valuation ring of K. By Lemma 5(4) we have $\mathrm{rk}_{\mathrm{val}}\ (K/k(t)_\sigma) < d$, unless $\mathrm{rk}_{\mathrm{val}}\ (K/k(t)_\sigma) = \mathrm{t.dim}\ (R/tR) = d$ and $P \cap K(0) = (0)$.

So let us now assume that $\mathrm{rk}_{\mathrm{val}}\ (K/k(t)_\sigma) = \mathrm{t.dim}\ (R/tR) = d$ and $P \cap K(0) = (0)$. Let $A = k[\check{t}]_\sigma [c]_\sigma \subseteq R$, and let $A(0) = k[c] \subseteq K(0) \cap R$. We then have $A(0) \cap P = (0)$. Note that $t \in P$, and so $Z = \mathrm{Spec}^\sigma\ (A/A \cap P)$ is a closed difference subscheme of X_0 which is weakly Zariski dense in V. (It follows from (1) that $A(0) = k[V]$.) Hypothesis (2) then implies that Z is a component of X_0, i.e., $P \cap A$ is a transformally prime ideal of A. Moreover, since no $\sigma^n(t)$ is a 0-divisor in A, we have $\mathrm{t.dim}\ (Z) \leq d$ by Lemma 8, and so $\mathrm{t.dim}\ (Z) = d$.

We have $k \subseteq K_A = \mathrm{Frac}(A/A \cap P) \subseteq K' = \mathrm{Frac}(R/P)$, with $\mathrm{tr.deg}\ (K_A/k) = \mathrm{tr.deg}\ (K'/k) = d$. So K'/K_A is an algebraic extension, and as $P \cap A$ is transformally prime, P is transformally prime as well by [1, Lemma 2.1]. Lemma 5(2) then implies that $P = \mathfrak{m}$, where $\mathfrak{m}$ is the maximal ideal of R. As $P \cap K(0) = (0)$, the valuation on $K(0)$ is trivial and $\mathrm{res} : K(0) \to K(0)_{\mathrm{res}}$ is an isomorphism (over k). But $c \notin W$ and $\mathrm{res}(c) \in W$. Contradiction. $\square$

References

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