Some contributions to spatial statistic: non-stationarity and deformation

A model for non-stationarity

- $Z = \{Z(x), x \in G \subseteq \mathbb{R}^2\}$ (centered and standardized) with correlation r
- $\epsilon = \{\epsilon(u), u \in D \subseteq \mathbb{R}^2\}$ stationary or isotropic random field with correlation ρ
- f bijective and bi-continuous transformation (deformation) from G to D

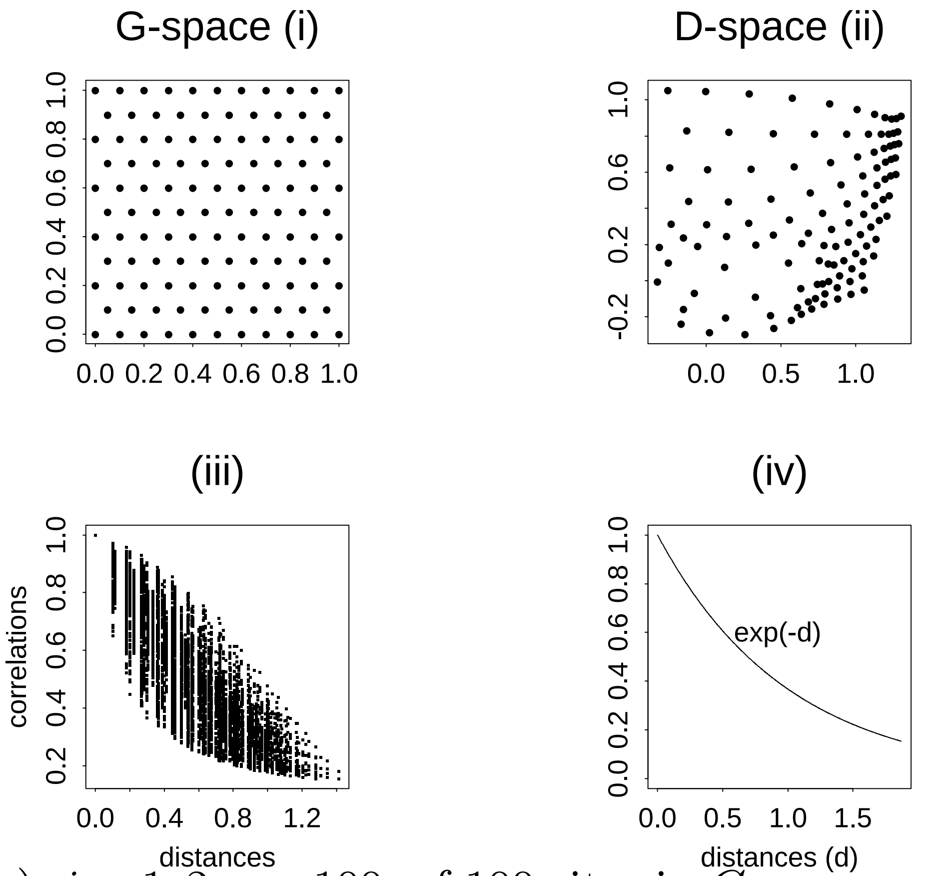
$$Z(x) = \epsilon(f(x)) \Longleftrightarrow Z(f^{-1}(u)) = \epsilon(u)$$

$$\Longrightarrow$$

$$r(x, y) = \rho(f(x) - f(y)) \quad \text{or} \quad r(x, y) = \rho(\|f(x) - f(y)\|)$$

(Guttorp, Sampson (1986, 1992, 1994) and Stock (1988))

Illustration for ρ isotropic



(i) positions (x_i, y_i), $i = 1, 2, \ldots, 100$, of 100 sites in G

(ii) their deformed positions $f(x_i, y_i)$ in D

(iii) non-stationary correlations $r(x_i, y_i, x_j, y_j)$ with respect to the distances $\|(x_i, y_i) - (x_j, y_j)\|$ in G

(iv) isotropic correlations $\exp(-\|f(x_i, y_i) - f(x_j, y_j)\|)$ with respect to the distances $\|f(x_i, y_i) - f(x_j, y_j)\|$ in D

Why such a model ?

- to "generalize" non-stationarity: stationarity when f = identity
- to model non-stationarity:

$$Y(x) = \mu(x) + \sigma(x)Z(x)$$

- to come down to a known framework: Haslett & Raftery (1989) followed with a discussion from Guttorp & Sampson. Finding a "well-behaved" covariance function where estimation is feasible.

Developments of this model (in terms of estimation):

Sampson (1986), Sampson & Guttorp (1992), Mardia & Goodall (1993), Guttorp & Sampson (1994), Meiring (1995), Sampson *et al.* (2000), Damian *et al.* (2000), Schmidt & O'Hagan (2003), ...

Headlines

- characterization
- estimation
- generalizations

Answer to this J.L. Krivine's question (Assouad (1980)):

How to distinguish, among hermitian kernels r of positive type (satisfying $r(x,x) =$ constant) those with the form $r(x,y) = \rho(f(x) - f(y))$, where ρ is a function of positive type on a locally compact Abelian group D and f is an application from G to D?

(Perrin & Senoussi (1999))

$$r(x,y) = \rho\left(f(x) - f(y)\right)$$

if and only if almost everywhere for $x \neq y$:

$$\partial_1 r(x,y)\frac{\partial_1 r(y,x_0)}{\partial_2 r(y,x_0)} + \partial_2 r(x,y)\frac{\partial_1 r(x,x_0)}{\partial_2 r(x,x_0)} = 0$$

(f,ρ) is given by:

$$f(x) = -\int_{x_0}^{x} \frac{\partial_1 r(u,x_0)}{\partial_2 r(u,x_0)} f^{(1)}(x_0)du$$

$$\rho(u) = r\left(x_0, f^{-1}(u)\right)$$

Example

Correlation of a fractional Brownian motion $Z(x), x > 0$:

$$r(x,y) = \frac{\left(x^{2H} + y^{2H} - |x-y|^{2H}\right)}{2x^H y^H}$$

where $H \in]0,1[$

$$r(x,y) = \rho(f(x) - f(y))$$

with

$$f(x) = \ln(x)$$

and

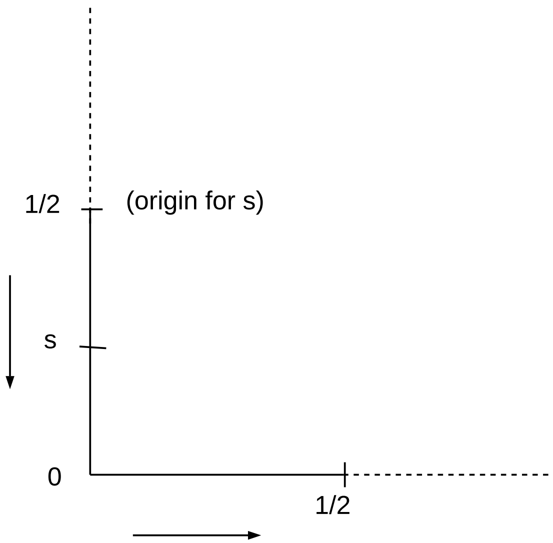
$$\rho(u) = \cosh(Hu) - 2^{(2H-1)} \left(\sinh(|u|/2)\right)^{2H}$$

Counter-example

- $P = \{P(x), x \in \mathbb{R}^2\}$ random field with correlation:

$$r(x,y) = \exp\left(-\|x-y\|^2\right)$$

- $Z(s) = P(x(s)),\ s \in [0,1]$



$$Z(s) \neq Y(f(s))$$

Generalization

- Let $Y = \{Y(u,v), (u,v) \in \mathbb{R}^2\}$ be a second-order stationary random field indexed by $\mathbb{R}^2$ with the covariance function R defined by

$$R(u,v) = Cov\left[Y(y,z), Y(y+u, z+v)\right] = \exp(-|u| - |v|)$$

for all $(u,v),\ (y,z) \in \mathbb{R}^2$.

- Then consider the restriction of Y to the curve $(x, x^2) \subset \mathbb{R}^2$ and define $Z(x) = Y(x, x^2)$. The process Z is indexed by $\mathbb{R}$ and its covariance function r is defined by

$$r(x,x') = cov(Z(x), Z(x')) = \exp(-|x - x'|)\exp(1 + |x + x'|).$$

- The covariance function r is nonstationary in $\mathbb{R}$. However,

$$cov(Z(x), Z(w)) = cov(Y(x, x^2)), Y(w, w^2))),$$

where Y is a second-order stationary process in $\mathbb{R}^2$.

(Perrin and Meiring (2003))

This counter-example motivates the general question: given any random field $Z(x)$ indexed by $\mathbb{R}^n$ with moments at least of order 2 and any function $\Psi : \mathbb{R}^n \to \mathbb{R}^n$, is there a Y indexed by $\mathbb{R}^{2n}$ such that

$$cov(Z(x), Z(x')) = cov(Y(x, \Psi(x)), Y(x', \Psi(x'))),$$

where the process Y is second-order stationary in $\mathbb{R}^{2n}$?

(Question from Pierre Jacob (University Montpellier II))

Let $Z = \{Z(x), x \in G \subseteq \mathbb{R}^n\}$, $n \geq 1$, be a centered and standardized (*a priori* nonstationary) random field indexed by a subset G of $\mathbb{R}^n$.

We denote by $r(x, x') = cov(Z(x), Z(x'))$ the covariance function of Z and by Δ the diagonal set of G: $\Delta = \{(x, x), x \in G\}$. We also let $\mathbf{0}$ denote the origin in $\mathbb{R}^{2n}$.

Let Φ be a function defined on $G \subseteq \mathbb{R}^n$ by $\Phi(x) = (x, \Psi(x))$, where $\Psi = (\psi_1, \ldots, \psi_n)$ is a vectorial function of dimension n such that the transformation

$$\begin{aligned} h : G \times G & \longrightarrow D - D \\ (x, x') & \longmapsto \Phi(x) - \Phi(x') = (x - x', \Psi(x) - \Psi(x')) \end{aligned}$$

is bijective from $(G \times G) \setminus \Delta$ onto $\{D - D\} \setminus \{\mathbf{0}\}$, where $D = \Phi(G) = \{(x, \Psi(x)), x \in G\}$, $D - D = \{u - u', (u, u') \in D \times D\}$. Note that D is the graph of Ψ. Then there exists a centered and standardized Gaussian stationary random field $Y = \{Y(u), u \in D \subseteq \mathbb{R}^{2n}\}$ indexed by D with covariance function R defined on $D - D$ such that, for all (x, x') in $G \times G$,

$$\begin{aligned} r(x, x') &= cov(Z(x), Z(x')) = cov(Y(\Phi(x)), Y(\Phi(x'))) \\ &= R(h(x, x')) = R(x - x', \Psi(x) - \Psi(x')). \end{aligned}$$

Example of transformation

The transformation Ψ is a functional parameter under our control.
We can take, for instance:

$$\psi_i(x_1, \ldots, x_n) = x_i^2, \quad i = 1, 2, \ldots, n.$$

The inverse transformation
$h^{-1} : w = (u_1, \ldots, u_n, v_1, \ldots, v_n) \longmapsto (x, x')$ is defined by:

$$\left\{ \begin{array}{rcl} x_i & = & \dfrac{1}{2}\left(\dfrac{v_i}{u_i} + u_i\right) \\ \\ x'_i & = & \dfrac{1}{2}\left(\dfrac{v_i}{u_i} - u_i\right). \end{array} \right.$$

(Perrin & Schlater (2005))

An $n \times n$ matrix $C = (C_{ij})_{i,j=1,\ldots,n}$ is real-valued, symmetric and positive definite and has identical values on the diagonal if and only if a real-valued positive definite function c on a graph of $\mathbb{R}^2$ and points $x_1, \ldots, x_n \in \mathbb{R}^2$ exist, so that

$$C = (c(x_i - x_j))_{i,j=1,\ldots,n}. \tag{1}$$

(Perrin & Senoussi (2000))

Correlations reducible to a stationary one

(f, ρ) is unique up to an affine transformation for f and up to a scaling for ρ.

Correlations reducible to an isotropic one

(f, ρ) is unique up to a homothetic Euclidean motion for f and up to a scaling for ρ.

Uniqueness of the solution (f, ρ) when ρ is monotonic (no need for differentiability assumptions anymore) is given in Perrin and Meiring (1999) (isometric embedding theorem from Schœnberg (1938)).

Example and counter-example ($n = 2$)

Correlation of the Lévy fractional Brownian field $Z(x)$, $x \neq 0$:

$$r(x, y) = \frac{\|x\| + \|y\| - \|y - x\|}{2\|x\|\|y\|}$$

Thus

$$r(x, y) = R(f(x) - f(y))$$

with

$$f(x) = (\ln(\|x\|), \arctan(x_2/x_1)), \quad \text{where } x = (x_1, x_2)$$

and

$$\begin{aligned} & R(u_1, u_2) \\ = & \frac{1}{2}\left(\exp(u_1/2) + \exp(-u_1/2) - \sqrt{\exp(u_1/2) + \exp(-u_1/2) - 2\cos(u_2)}\right) \end{aligned}$$

Model estimation

$Z = \{Z(x), x \in G \subseteq \mathbb{R}^2\}$ non-stationary random field with correlation

$$r(x,y) = \rho_\beta(f(x) - f(y)) \quad \text{or} \quad r(x,y) = \rho_\beta(\|f(x) - f(y)\|)$$

with $\beta \in \mathbb{R}^q$

Estimation with repetitions

T independent and identically distributed realizations of Z at n fixed monitoring sites $x_1, x_2, \ldots, x_n$

$$Z_t(x_i), \quad t = 1, \ldots, T, i = 1, \ldots, n$$

$\hat{f}$ and $\hat{\beta}$ minimize the objective function

$$U(f, \beta) = \sum_{i<j} \left[\hat{r}(x_i, x_j) - \rho_\beta(\|f(x_i) - f(x_j)\|)\right]^2$$

where

$$\hat{r}(x_i, x_j) = \frac{1}{T} \sum_{t=1}^{T} Z_t(x_i) Z_t(x_j)$$

Model estimation: parametric approach

(Perrin & Monestiez (1998)

Elementary radial basis deformations from $\mathbb{R}^2$ onto $\mathbb{R}^2$

$$f(\mathbf{x}) = \mathbf{c} + (\mathbf{x} - \mathbf{c})\Phi(u)$$

where $\mathbf{c}$ is the center of the deformation, Φ is a function from $\mathbb{R}^+$ to $\mathbb{R}$ and $u = \|\mathbf{x} - \mathbf{c}\|$. Φ can be

- cosine: $\Phi(u) = 1 + b\cos(au \wedge \pi)$
- exponential: $\Phi(u) = 1 + b\exp(-au)$
- Gaussian: $\Phi(u) = 1 + b\exp(-au^2)$

with $a > 0$.

Model estimation: non-parametric approach

(Perrin & Iovleff (2004))

Continuous state version of the simulated annealing algorithm for a Metropolis-Hastings dynamic subject to some non-folding constraints

Non-parametric deformation f:

$$f(\mathbf{x}_i) = \mathbf{y}_i,\ i = 1, \ldots, n$$

are the coordinates of the sites in the D-space.

$\hat{\mathbf{y}} = (\hat{\mathbf{y}}_1, \hat{\mathbf{y}}_2, \ldots, \hat{\mathbf{y}}_n)$ and $\hat{\beta}$ minimize:

$$U(\mathbf{y}, \beta) = \sum_{i<j} [\hat{r}(\mathbf{x}_i, \mathbf{x}_j) - \rho_\beta(\|\mathbf{y}_i - \mathbf{y}_j\|)]^2, \qquad (2)$$

with respect to $\mathbf{y} = (\mathbf{y}_1, \mathbf{y}_2, \ldots, \mathbf{y}_n)$ and β, and subject to some non-folding constraints described latter.

Advantages of the simulated annealing

- it explores the whole objective function's surface and tries to optimize the function while moving both uphill and downhill. Thus, it is largely independent of the starting values, often a critical input in conventional algorithms;
- it can escape from local minima and go on to find the global minimum by the uphill and downhill moves;
- it makes less stringent regularity assumptions regarding the function than do conventional algorithms (it need not even be continuous);
- it is well suited for minimizing strongly non-convex functions of several variables ($2n + q$ variables in our problem) having plenty of local minima;
- it can take intricate constraints into account.

starting step: set $\mathbf{y}(0) = \{\mathbf{y}_i(0), i = 1, 2, \ldots, n\}$ and $\beta(0)$ to arbitrary values by running $2n + q$ changes of the parameters with the transition q: choose a candidate j uniformly in the set $\{1, \ldots, n + q\}$; if $j \leq n$ then it corresponds to one of the n sites, and move the corresponding site locally and uniformly at a position $\mathbf{y}$ with natural non-folding constraints we specify below; otherwise, do the change $\beta \longrightarrow \beta'$ where β' is chosen uniformly in a neighborhood of β.

Take a sequence of temperatures $(c_0, c_1, \ldots, c_k, \ldots)$ decreasing to 0 by step of length $n + q$:

$$c_k = \theta^{\lfloor k/(n+q) \rfloor} c_0, \quad \theta \in]0, 1[, \; k \in \mathbb{N};$$

step k: start from the configuration $(\mathbf{y}(k), \beta(k))$ of the sites and generate a candidate $((\tilde{\mathbf{y}}(k), \tilde{\beta}(k))$ according to the rule q_k. Then:

- If $\Delta_k U = U(\tilde{\mathbf{y}}(k), \tilde{\beta}(k)) - U(\mathbf{y}(k), \beta(k)) \leq 0$ then take $(\mathbf{y}(k+1), \beta(k+1)) = (\tilde{\mathbf{y}}(k), \tilde{\beta}(k))$;
- otherwise sample an uniform law V_k in $[0,1]$: if $V_k \leq \exp(-\Delta_k U/c_k)$ take $(\mathbf{y}(k+1), \beta(k+1)) = (\tilde{\mathbf{y}}(k), \tilde{\beta}(k))$; otherwise keep $(\mathbf{y}(k), \beta(k))$;

stopping criterion: if $|U(\mathbf{y}(k(n+q)), \beta(k(n+q))) - U(\mathbf{y}((k+1)(n+q)), \beta((k+1)(n+q)))| < 10^{-8}$ for two consecutive values of the integer k we stop the algorithm.

This algorithm is written in C language.

Non-folding constraints

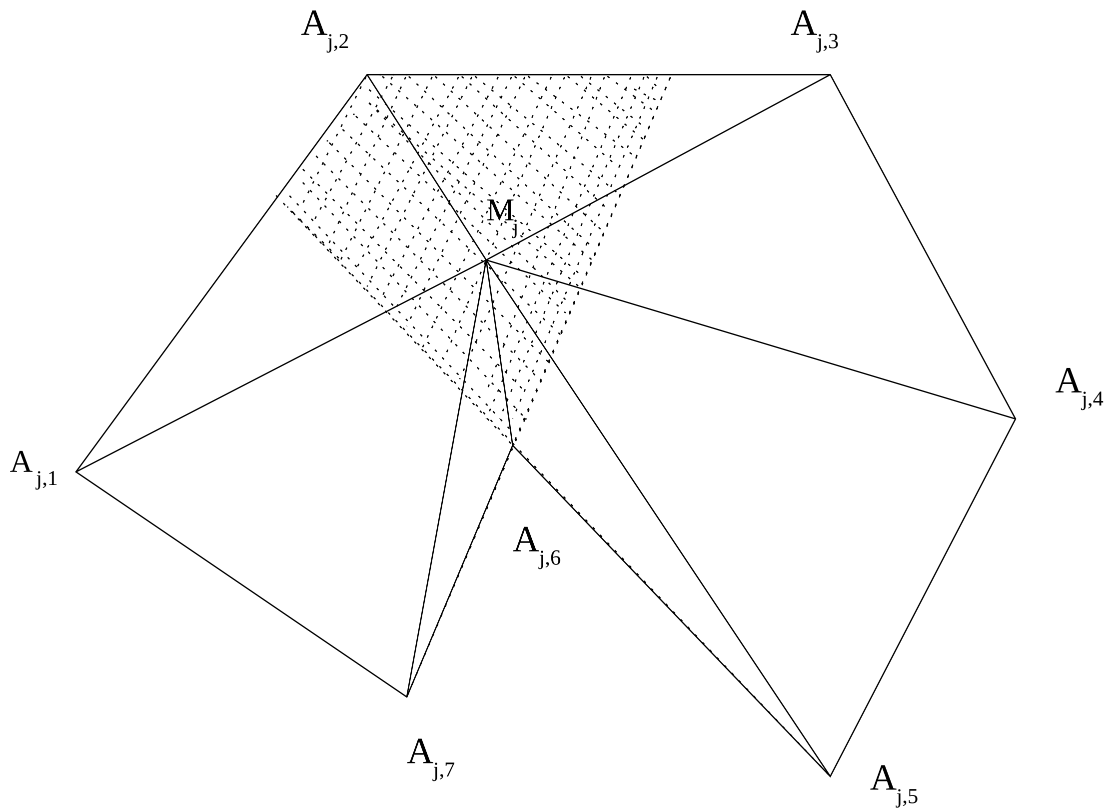


Figure 1: The marked area corresponds to the acceptable move for M_j.

These constraints mean that we impose moves that preserve the topological structure of the Delaunay triangulation the same.

Estimation on the whole G-space

- simulated annealing gives an estimation of the "discrete" mapping $\mathbf{x}_i \longmapsto \hat{\mathbf{y}}_i, \quad i = 1, \ldots, n$
- estimation $\hat{f}$ of f: a piecewise affine interpolation of $(\mathbf{x}_i, \hat{\mathbf{y}}_i)$

$$\hat{f}(\mathbf{x}) = a_{\mathbf{x},i}\hat{\mathbf{y}}_i + a_{\mathbf{x},j}\hat{\mathbf{y}}_j + a_{\mathbf{x},k}\hat{\mathbf{y}}_k$$

Precipitation data

- 10−days aggregated precipitation data
- $n = 20$ sites in the Languedoc-Roussillon, region of France, with similar altitudes
- $T = 108$: 6 records during November and December each year from 1975 through 1992
- very few missing values
- sample correlations calculated on the log scale (means and variances positively related)

We use the model:

$$\rho_\beta(u) = \epsilon \exp\left(-\beta u^\eta\right), \quad \epsilon \in]0,1], \beta > 0, \eta \in]0,2], \tag{3}$$

where β is the 3-dimension parameter (ϵ, α, η), so that the objective function (2) is re-written as follows:

$$U(\mathbf{y}_1, \mathbf{y}_2, \ldots, \mathbf{y}_n, \beta) = \sum_{i<j} [\hat{r}(\mathbf{x}_i, \mathbf{x}_j) - \epsilon \exp\left(-\alpha \|\mathbf{y}_j - \mathbf{y}_i\|^\eta\right)]^2. \tag{4}$$

In the cooling schedule we take $c_0 = 1000$ and $\theta = 0.9999$.

Form of the deformation

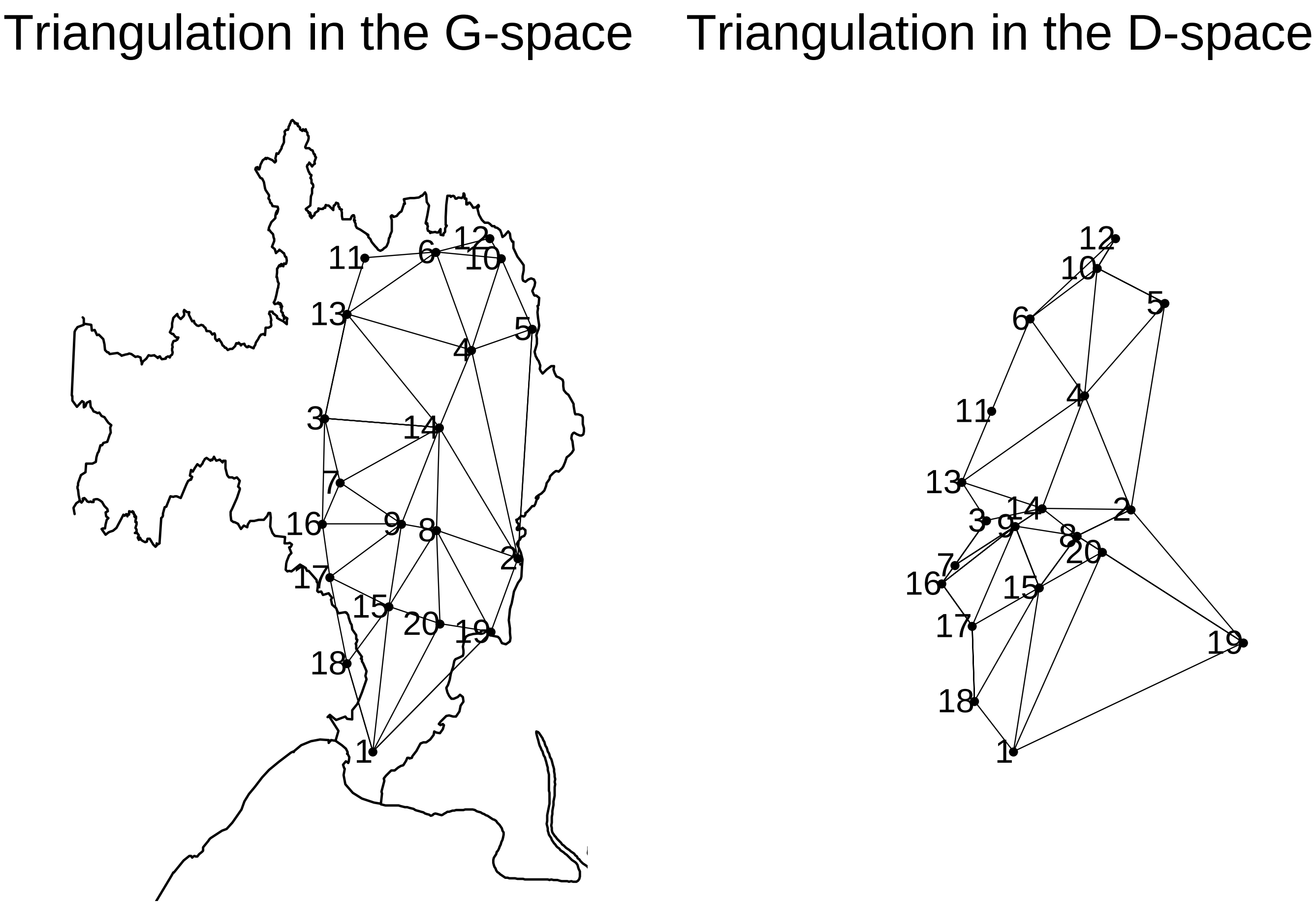
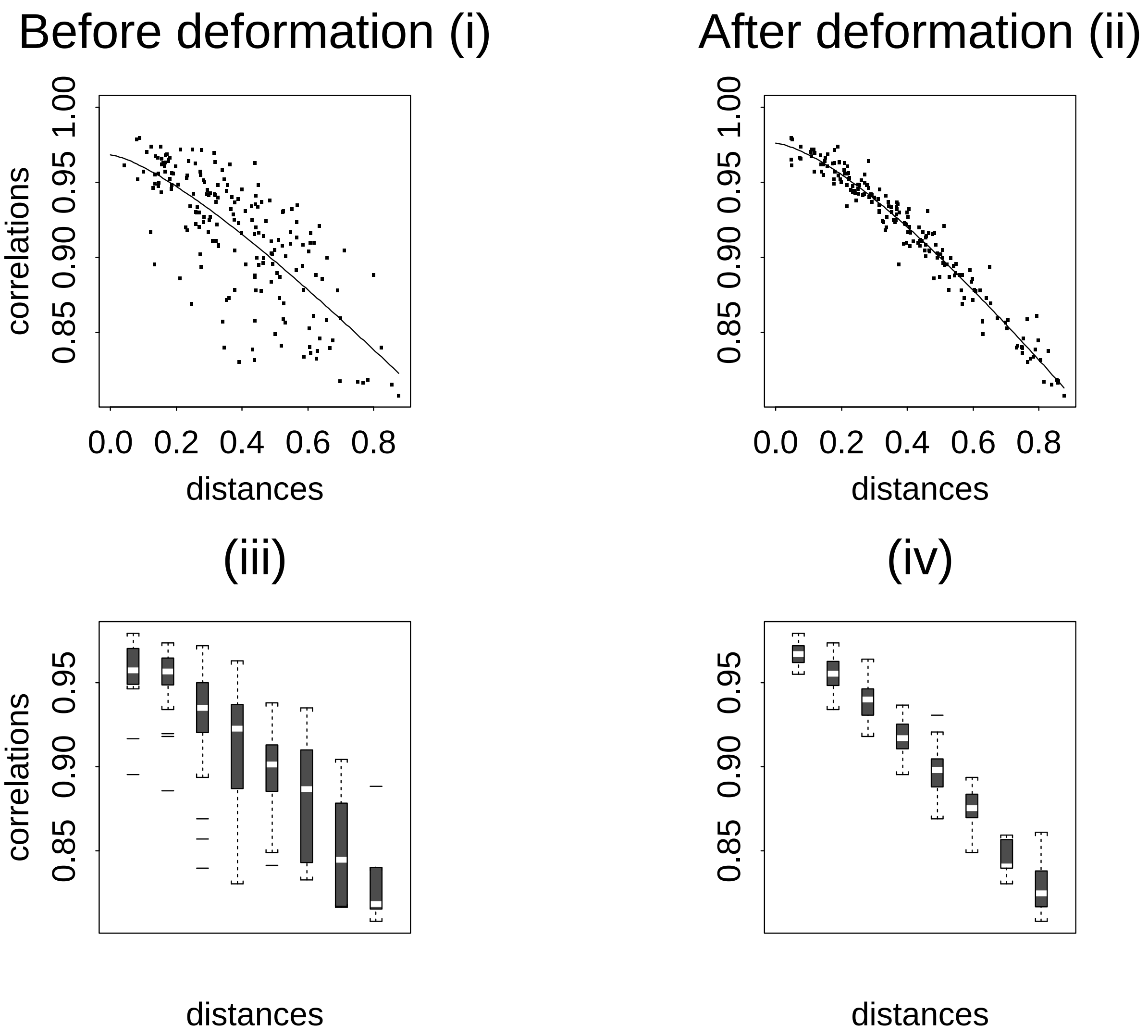


Figure 2: On the left: site locations and the corresponding Delaunay triangulation without the rectangle (the outlines indicate the French department of Gard and the coast). On the right: deformation of the triangulation.

Fitting of the correlation model



minima of the objective function: 0.147 (before) and 0.013 (after)

Application to prediction

$$\hat{Z}(x) = \sum_{i=1}^{n} \lambda_i Z(x_i).$$

Results of a cross validation study: for the previous model: 40.5% of improvement for the MSEP,

	before deformation	after deformation	% of improvement
$\exp(-\beta_1 u)$	0.224	0.129	42.4
$\beta_2 \exp(-\beta_1 u)$	0.203	0.136	33.1
$\exp(-\beta_1 u^2)$	0.271	0.132	51.3
$\beta_2 \exp(-\beta_1 u^2)$	0.180	0.115	36.1

Application to the prediction

Z random field with covariance r. Prediction at x:

$$\hat{Z}(x) = \sum_{i=1}^{n} \lambda_i Z(x_i),$$

with $\lambda_1, \lambda_2, \ldots, \lambda_n$ (kriging coefficients) solutions of $\min E[\hat{Z}(x) - Z(x)]^2$, *i.e.*

$$\begin{pmatrix} \lambda_1 \\ \vdots \\ \lambda_i \\ \vdots \\ \lambda_n \end{pmatrix} = \begin{pmatrix} r(x_1,x_1) & \cdots & r(x_1,x_j) & \cdots & r(x_1,x_n) \\ \vdots & \vdots & \vdots & \vdots & \vdots \\ r(x_i,x_1) & \cdots & r(x_i,x_j) & \cdots & r(x_i,x_n) \\ \vdots & \vdots & \vdots & \vdots & \vdots \\ r(x_n,x_1) & \cdots & r(x_n,x_j) & \cdots & r(x_n,x_n) \end{pmatrix}^{-1} \begin{pmatrix} r(x,x_1) \\ \vdots \\ r(x,x_i) \\ \vdots \\ r(x,x_n) \end{pmatrix}$$

(Perrin (1999))

$Z = \{Z(x), x \in [0,1]\}$ centered **Gaussian process** with covariance function $r(x,y)$ satisfying:

(A1) r is continuous in $[0,1]^2$ and has second derivatives which are uniformly bounded for $x \neq y$.

Singularity function α of Z: $\forall x \in [0,1]$:

$$\alpha(x) = \lim_{y \nearrow x} r^{(0,1)}(x,y) - \lim_{y \searrow x} r^{(0,1)}(x,y)$$

(A2) α has a bounded first derivative in $[0,1]$.

The process of the quadratic variations

$\forall n \in \mathbb{N}^\star$, $\forall k = 1, 2, \ldots, n$, we set:

$$\Delta Z_k = Z(k/n) - Z((k-1)/n).$$

For all $x \in [0,1]$, we define the **quadratic variations** $V_n(x)$ of Z along $\left\{0, \frac{1}{n}, \frac{2}{n}, \ldots, \frac{[nx]}{n}\right\}$ as follows:

$$V_n(x) = \sum_{k=1}^{[nx]} \left(\Delta Z_k\right)^2 .$$

Definition. The process of the quadratic variations of Z, $v_n = \{v_n(x), x \in [0,1]\}$, is defined as:

$$\begin{cases} v_n(x) &= V_n(x) + (nx - [nx]) \left(\Delta Z_{[nx]+1}\right)^2, \quad x \in [0,1[, \\ v_n(1) &= V_n(1). \end{cases}$$

Functional convergence in distribution

Theorem. **The process :** $\left\{\sqrt{\mathbf{n}}(\mathbf{v_n}(\mathbf{x}) - \int_{\mathbf{0}}^{\mathbf{x}} \alpha(\mathbf{u})\mathbf{du}), \mathbf{x} \in [\mathbf{0}, \mathbf{1}]\right\}$ **converges in distribution in** $\mathbf{C}([\mathbf{0}, \mathbf{1}])$ **to the Gaussian process** $\left\{\int_{\mathbf{0}}^{\mathbf{x}} \sqrt{\mathbf{2}}\alpha(\mathbf{u})\mathbf{dW}(\mathbf{u}), \mathbf{x} \in [\mathbf{0}, \mathbf{1}]\right\}$ **as** $\mathbf{n} \to \infty$**.**

Let Z be a centered, standardized Gaussian process with correlation function r satisfying **(A1)**. Consider the problem of estimating the function $f : [0,1] \longmapsto \mathbb{R}$ from one realization of Z in the model

$$Z(x) = \epsilon(f(x)), \quad x \in [0,1], \tag{5}$$

where ϵ is a stationary random process with known correlation R.

(B) f is bijective and has uniformly bounded second derivatives in [0,1], as well as its inverse.

Model (5) is equivalent to $r(x,y) = R(f(y) - f(x))$. Note that for any $b > 0$ and $c \in \mathbb{R}$, $(\tilde{f}, \tilde{R})$ with $\tilde{f}(x) = bf(x) + c$ and $\tilde{R}(u) = R(u/b)$ is a solution of the model as well. Thus, without loss of generality we may impose that $f(0) = 0$ and $f(1) = 1$.

Deformation and singularity function

From **(A1)** and **(B)** we deduce:

$$\begin{aligned} R^{(1)}(0^-) &= D^-(x)/f^{(1)}(x), \\ R^{(1)}(0^+) &= D^+(x)/f^{(1)}(x). \end{aligned}$$

Then:

$$\alpha(\mathbf{x}) = \mathbf{2R^{(1)}(0^-)f^{(1)}(x)}.$$

Finally, we get for all $x \in [0,1]$:

$$\mathbf{f}(\mathbf{x}) = \frac{\int_{\mathbf{0}}^{\mathbf{x}} \alpha(\mathbf{u})\mathbf{du}}{\int_{\mathbf{0}}^{\mathbf{1}} \alpha(\mathbf{u})\mathbf{du}}.$$

Conclusion: the **estimation** of f **requires an estimation of the primitive** $x \longmapsto \int_0^x \alpha(u)du$ of α

Estimation of the space deformation f

$$\hat{f}_n(x) = \frac{v_n(x)}{v_n(1)}.$$

Theorem. Almost surely:

$$\lim_{n\to\infty} \sup_{x\in[0,1]} |\hat{f}_n(x) - f(x)| = 0.$$

Corollary. The process $\left\{ \sqrt{n}(\hat{f}_n(x) - f(x)), x \in [0,1] \right\}$ converges in distribution in $C([0,1])$ to the Gaussian process :

$$\left\{ \frac{\sqrt{2}\int_0^x \alpha(u)dW(u)}{\int_0^1 \alpha(u)du} - f(x)\frac{\sqrt{2}\int_0^1 \alpha(u)dW(u)}{\int_0^1 \alpha(u)du}, x \in [0,1] \right\}$$

as $n \to \infty$.

Test for stationarity

$$f(x) = x \text{ against } f(x) \neq x.$$

Under the null hypothesis, $\left\{\sqrt{n}(\hat{f}_n(x) - x), x \in [0,1]\right\}$ converges in distribution $C([0,1])$ to the Brownian bridge $\left\{\sqrt{2}(W(x) - xW(1)), x \in [0,1]\right\}$ as $n \to \infty$. Thus, $\sqrt{n} \sup_{x \in [0,1]} |\hat{f}_n(x) - x|$ converges to the Kolmogorov distribution $\sqrt{2} \sup_{x \in [0,1]} |W(x) - xW(1)|$.

Estimating a non-stationarity spatial structure

(Guyon & Perrin (2000))

Problem to be solved: estimate the deformation f from observations of Z at the nodes of a rectangular partition $\{0, 1/n, 2/n, \ldots, 1\} \times \{0, 1/m, 2/m, \ldots, 1\}$ of $G = [0, 1]^2$ finer and finer ($n \to \infty$ and $m \to \infty$). The *geometry of the partition* $\lambda = \frac{m}{n}$ is a parameter under our control.

Identification of spatial deformations using linear and superficial quadratic variations

Assumptions for the correlation R and the deformation f

(A1) $R(u,v) = 1 - \alpha|u| - \beta|v| + O(uv), \quad \alpha > 0$ and $\beta > 0$.

(A2) $R^{(2,0)}(u,v), R^{(1,1)}(u,v), R^{(0,2)}(u,v)$ are uniformly bounded outside axis.

For instance $R(u,v) = \exp(-\alpha|u| - \beta|v|)$ satisfies **(A1)** and **(A2)**.

(B1) $f = (f_1, f_2)$ has uniformly bounded second order derivatives in $[0,1]^2$.

(B2) First partial derivatives of f satisfy:

$f_1^{(1,0)}(x,y) > 0, f_2^{(0,1)}(x,y) > 0, f_1^{(0,1)}(x,y) \geq 0, f_2^{(1,0)}(x,y) \geq 0.$

(B3) $a = \sup_{(x,y)\in[0,1]^2} \dfrac{f_1^{(0,1)}(x,y)}{f_1^{(1,0)}(x,y)} < \inf_{(x,y)\in[0,1]^2} \dfrac{f_2^{(0,1)}(x,y)}{f_2^{(1,0)}(x,y)} = b.$

The assumption **(B3)** strengthen the condition:

"The Jacobian determinant of f is strictly positive in $[0,1]^2$."

Consider the points $A = (x, y)$, $B = (x + \frac{1}{n}, y)$, $C = (x, y + \frac{1}{m})$ and $D = (x + \frac{1}{n}, y + \frac{1}{m})$; we deduce from assumptions **(B1)-(B3)** that, for all $\lambda = \frac{m}{n} \in]a, b[\cap \mathbb{Q}^+$, the slopes of the straight lines $f(A)f(B)$ and $f(A)f(C)$ are positive, and the slope of the straight line $f(B)f(C)$ is negative, when $n \to \infty$ and $m \to \infty$.

(B2) and **(B3)** can be weakened like this:

(B2') The first partial derivatives of f have a constant sign in $[0,1]^2$.

(B3') $$a = \sup_{(x,y)\in[0,1]^2} \left| \frac{f_1^{(0,1)}(x,y)}{f_1^{(1,0)}(x,y)} \right| < \inf_{(x,y)\in[0,1]^2} \left| \frac{f_2^{(0,1)}(x,y)}{f_2^{(1,0)}(x,y)} \right| = b.$$

Examples of bijections

- Bijections $f = (f_1, f_2)$ such that $f_1(x, y) = F(x)$ and $f_2(x, y) = G(y)$ where F and G are two increasing diffeomorphisms in $[0, 1]$ ($a = 0$ and $b = \infty$).

- Bilinear bijections:
$$\begin{pmatrix} x \\ y \end{pmatrix} \longmapsto \begin{pmatrix} c_1 x + c_2 y + c_3 xy \\ d_1 x + d_2 y + d_3 xy \end{pmatrix},$$
where $c_1 > 0$, $c_2 \geq 0$, $c_3 \geq 0$, $d_1 \geq 0$, $d_2 > 0$, $d_3 \geq 0$ and $(c_2 + c_3)(d_1 + d_3) < c_1 d_2$.

Superficial quadratic variations

Rectangular increase: $Z(\Delta) = Z(x', y') - Z(x', y) - Z(x, y') + Z(x, y)$
with:

$$\Delta_{k,y} = \left[\left(\frac{k}{n}, \frac{\lfloor my \rfloor}{m}\right), \left(\frac{k+1}{n}, \frac{\lfloor my \rfloor + 1}{m}\right)\right[, \; k = 0, \ldots, n-1, y \in [0,1],$$

$$\Delta_{x,l} = \left[\left(\frac{\lfloor nx \rfloor}{n}, \frac{l}{m}\right), \left(\frac{\lfloor nx \rfloor + 1}{n}, \frac{l+1}{m}\right)\right[, \; l = 0, \ldots, m-1, x \in [0,1].$$

Definition of two superficial quadratic variations:

$$H_{n,\lambda}(x,y) = \sum_{k=0}^{\lfloor nx \rfloor - 1} \left(Z(\Delta_{k,y})\right)^2,$$

$$V_{m,\lambda}(x,y) = \sum_{l=0}^{\lfloor my \rfloor - 1} \left(Z(\Delta_{x,l})\right)^2,$$

$\lambda = \frac{m}{n}$ geometry of the partition, parameter under our control

Definition of two linear quadratic variations:

$$h_n(x) = \sum_{k=0}^{\lfloor nx \rfloor - 1} \left(Z\left(\frac{k+1}{n}, 0\right) - Z\left(\frac{k}{n}, 0\right)\right)^2,$$

$$v_m(y) = \sum_{l=0}^{\lfloor my \rfloor - 1} \left(Z\left(0, \frac{l+1}{m}\right) - Z\left(0, \frac{l}{m}\right)\right)^2.$$

Mean square convergence

Define for all $\lambda > 0$ and for all $(x, y) \in [0,1]^2$:

$$H_\lambda(x,y) = 4\left(\beta(f_2(x,y) - f_2(0,y)) + \frac{\alpha}{\lambda}\int_0^x f_1^{(0,1)}(u,y)du\right),$$

$$V_\lambda(x,y) = 4\left(\beta\lambda\int_0^y f_2^{(1,0)}(x,v)dv + \alpha(f_1(x,y) - f_1(x,0))dy\right),$$

$$h(x) = 2\left(\alpha f_1(x,0) + \beta f_2(x,0)\right) \text{ and } v(y) = 2\left(\alpha f_1(0,y) + \beta f_2(0,y)\right).$$

Then for all $\lambda \in]a,b[\cap\mathbb{Q}^+$:

$$\lim_{n\to\infty} H_{n,\lambda}(x,y) \overset{L_2}{=} H_\lambda(x,y) \text{ and } \lim_{m\to\infty} V_{m,\lambda}(x,y) \overset{L_2}{=} V_\lambda(x,y).$$

Moreover we have:

$$\lim_{n\to\infty} h_n(x) \overset{L_2}{=} h(x) \text{ and } \lim_{m\to\infty} v_m(y) \overset{L_2}{=} v(y).$$

Remark: under the same assumptions we have the uniform almost sure convergence.

Estimation of f

For all $(x,y) \in [0,1]^2$ and two distinct values of λ (λ_1 and λ_2) in $]a,b[\cap\mathbb{Q}^+$ we obtain one estimator for $f = (f_1, f_2)$:

$$\hat{\alpha f}_{1,n}(x,y) = \frac{\lambda_1 V_{\lambda_2 n,\lambda_2}(x,y) - \lambda_2 V_{\lambda_1 n,\lambda_1}(x,y) + 2(\lambda_1-\lambda_2)h_n(x)}{4(\lambda_1-\lambda_2)} - \frac{(\lambda_1 H_{n,\lambda_1}(x,0) - \lambda_2 H_{n,\lambda_2}(x,0))}{4(\lambda_1-\lambda_2)}$$

$$\hat{\beta f}_{2,n}(x,y) = \frac{\lambda_1 H_{n,\lambda_1}(x,y) - \lambda_2 H_{n,\lambda_2}(x,y) + 2(\lambda_1-\lambda_2)v_n(y)}{4(\lambda_1-\lambda_2)}, - \frac{(\lambda_1 V_{\lambda_2 n,\lambda_2}(0,y) - \lambda_2 V_{\lambda_1 n,\lambda_1}(0,y))}{4(\lambda_1-\lambda_2)}.$$

RESULT: $\hat{f}_n = (\hat{f}_{1,n}, \hat{f}_{2,n})$ converge in L_2 to $f = (f_1, f_2)$, $n \to \infty$.

Estimation of f : idea of the proof

1. With the superficial quadratic variation $H_{n,\lambda}(x,y)$ we identify $H_\lambda(x,y) = 4\left(\beta(f_2(x,y) - f_2(0,y)) + \frac{\alpha}{\lambda}\int_0^x f_1^{(0,1)}(u,y)du\right)$. Thus, with *two distinct values* of λ we identify $f_2(x,y) - f_2(0,y)$ and $f_2(x,0)$ (we set $y = 0$ and without loss of generality we assume $f(0,0) = (0,0)$, the correlation deformation model being translation invariant).

2. Moreover, with the linear quadratic variation $h_n(x)$ we identify $2\left(\alpha f_1(x,0) + \beta f_2(x,0)\right)$. Thus we get the identification of $f_1(x,0)$.

3. Similarly, with the superficial quadratic variation $V_{m,\lambda}(x,y)$ we identify $f_1(x,y) - f_1(x,0)$ with two distinct values of λ.

4. Finally, we get the identification of $f_1(x,y)$.

A similar treatment leads to the identification of $f_2(x,y)$.

CONCLUSION and PERSPECTIVES

CONCLUSION: for a particular structure of the stationary correlation R, one single realization of the non-stationary Gaussian random field Z in $[0,1]^2$ (or on a dense grid in $[0,1]^2$) is enough to identify the deformation f that makes this field stationary.

PERSPECTIVES: (i) Study of other correlation structures for R;

(ii) Simultaneous identification of R and f.

Classical definition:

Definition 1 *A random field* $\{X(\mathbf{t}), \mathbf{t} \in \mathbb{R}^n\}$ *is self-similar with index* $H > 0$ *(H-ss) if for all* $a > 0$, *the finite-dimensional distributions of* $\{X(a\mathbf{t}), \mathbf{t} \in \mathbb{R}^n\}$ *are identical to the finite-dimensional distributions of* $\{a^H X(\mathbf{t}), \mathbf{t} \in \mathbb{R}^n\}$.

Let $X = \{X(\mathbf{t}), \mathbf{t} \in \mathbb{R}^2\}$ be a mean zero standardized fractional Brownian sheet with correlation $\mathrm{E}[X(\mathbf{t})X(\mathbf{u})]$

$$= \frac{1}{4}\left(|t_1|^{2H_1} + |u_1|^{2H_1} - |t_1 - u_1|^{2H_1}\right)\left(|t_2|^{2H_2} + |u_2|^{2H_2} - |t_2 - u_2|^{2H_2}\right),$$

where $\mathbf{t} = (t_1, t_2)^T$, $\mathbf{u} = (u_1, u_2)^T$, and $0 < H_1 \leq 1$, $0 < H_2 \leq 1$.

X is H-ss with $H = H_1 + H_2$, so that this global index H does not reflect the self-similarity component-wise.

(Genton, Perrin & Taqqu (2005))

New definition:

Definition 2 *A random field* $\{X(\mathbf{t}), \mathbf{t} = (t_1, \dots, t_n)^T \in \mathbb{R}^n\}$ *is multi-self-similar with index* $\mathbf{H} = (H_1, \dots, H_n)^T \in \mathbb{R}_+^n$ *(*$\mathbf{H}$*-mss) if*

$$\{X(a_1 t_1, \dots, a_n t_n)\} \stackrel{d}{=} \{a_1^{H_1} \cdots a_n^{H_n} X(t_1, \dots, t_n)\},$$

for all $a_1 > 0, \dots, a_n > 0$*, where, as usual,* $\stackrel{d}{=}$ *denotes equality of the finite-dimensional distributions.*

If $a_1 = \cdots = a_n = a > 0$ and $H_1 + \cdots + H_n = H > 0$, then our definition reduces to the classical one, for which the self-similarity index is the same in all dimensions.

Proposition 1 *If* $\{X(\mathbf{t}), \mathbf{t} = (t_1, \ldots, t_n)^T \in \mathbb{R}_+^n\}$ *is* $\mathbf{H}$*-mss, then*

$$Y(\mathbf{t}) = e^{-\mathbf{t}^T \mathbf{H}} X(e^{t_1}, \ldots, e^{t_n}), \quad \mathbf{t} = (t_1, \ldots, t_n)^T \in \mathbb{R}^n, \tag{6}$$

is stationary. Conversely, if $\{Y(\mathbf{t}), \mathbf{t} = (t_1, \ldots, t_n)^T \in \mathbb{R}^n\}$ *is stationary, then*

$$X(\mathbf{t}) = t_1^{H_1} \cdots t_n^{H_n} Y(\ln(t_1), \ldots, \ln(t_n)), \quad \mathbf{t} = (t_1, \ldots, t_n)^T \in \mathbb{R}_+^n \text{ is } \mathbf{H}\text{-mss.}$$

The covariance of X (when it has finite second moments) can be written as:

$$c(\mathbf{t}, \mathbf{u}) = R_1 \left(\frac{\mathbf{g}(\mathbf{t}) + \mathbf{g}(\mathbf{u})}{2} \right) R\left(\mathbf{g}(\mathbf{t}) - \mathbf{g}(\mathbf{u})\right),$$

where

$$R_1(\mathbf{w}) = e^{2\mathbf{H}^T \mathbf{w}}, \; \mathbf{g}(\mathbf{t}) = (\ln(t_1), \ldots, \ln(t_n))^T \text{ and } R \text{ is a stationary covariance.} \tag{7}$$

(“Generalization” of Genton & Perrin (2004))

Definition 3 *A random field* $\{X(\mathbf{t}), \mathbf{t} = (t_1, \ldots, t_n)^T \in \mathbb{R}^n\}$ *with finite second moments is locally stationary reducible (LSR) if its covariance function* c *can be written in the form:*

$$c(\mathbf{t}, \mathbf{u}) = R_1\left(\frac{\mathbf{g}(\mathbf{t}) + \mathbf{g}(\mathbf{u})}{2}\right) R\left(\mathbf{g}(\mathbf{t}) - \mathbf{g}(\mathbf{u})\right), \quad \mathbf{t}, \mathbf{u} \in \mathbb{R}^n, \tag{8}$$

where R_1 *is a nonnegative function,* R *is a stationary covariance and* $\mathbf{g}$ *is a bijective deformation of the index space* $\mathbb{R}^n$*. If* X *is Gaussian, then* $X(\mathbf{t}) \stackrel{d}{=} Y(\mathbf{g}(\mathbf{t}))$*, where* Y *is an LS random field. We call* Y *the reduced random field.*

Therefore, multi-self-similar random fields with finite second moments are a subclass of LSR random fields. In this particular case, the deformation $\mathbf{g}$ does not depend on the index $\mathbf{H}$.

Let $X = \{X(\mathbf{t}), \mathbf{t} \in \mathbb{R}_+^n\}$ be a mean zero standard fractional Brownian sheet with covariance

$$\mathrm{E}[X(\mathbf{t})X(\mathbf{u})] = \frac{1}{2^n}\prod_{i=1}^{n}\left(t_i^{2H_i} + u_i^{2H_i} - |t_i - u_i|^{2H_i}\right), \qquad (9)$$

where $\mathbf{t} = (t_1, \ldots, t_n)^T$, $\mathbf{u} = (u_1, \ldots, u_n)^T$, and $0 < H_i \leq 1$. Then it follows from Definition 2 that X is $\mathbf{H}$-mss with $\mathbf{H} = (H_1, \ldots, H_n)^T$. From Proposition 1, we obtain that:

$$X(\mathbf{t}) = t_1^{H_1} \cdots t_n^{H_n} Y(\ln(t_1), \ldots, \ln(t_n)),$$

where $Y(\mathbf{t})$ is a mean zero Gaussian stationary process with covariance $R(\mathbf{v}) = \prod_{i=1}^{n}\left(\cosh(H_i v_i) - 2^{(2H_i-1)}(\sinh(|v_i|/2))^{2H_i}\right)$. It follows from Definition 3 and Relation (7) that fractional Brownian sheets are LSR random fields with $R_1(\mathbf{w}) = e^{2\mathbf{H}^T\mathbf{w}}$, $\mathbf{g}(\mathbf{t}) = (\ln(t_1), \ldots, \ln(t_n))^T$ and R given above.

Theorem 1 *Let $X = \{X(\mathbf{t}), \mathbf{t} = (t_1, t_2)^T \in \mathbb{R}^2\}$ be a mean zero Lévy fractional Brownian random field with covariance*

$$E[X(\mathbf{t})X(\mathbf{u})] = \frac{1}{2}(\|\mathbf{t}\|^{2H} + \|\mathbf{u}\|^{2H} - \|\mathbf{t} - \mathbf{u}\|^{2H}), \tag{10}$$

where $0 < H \leq 1$ and $\| \cdot \|$ denotes the usual Euclidean norm in $\mathbb{R}^2$. Then

$$X(\mathbf{t}) \stackrel{d}{=} \rho_{\mathbf{t}}^{H} Y(\ln(\rho_{\mathbf{t}}), \theta_{\mathbf{t}}), \tag{11}$$

with $\rho_{\mathbf{t}} = \sqrt{t_1^2 + t_2^2}$, $\theta_{\mathbf{t}} = \arctan(t_2/t_1) + k\pi$, $k \in \mathbb{Z}$, and where $Y(\mathbf{t})$ is a mean zero Gaussian stationary process with correlation

$$R(\mathbf{v}) = \frac{1}{2}\left(e^{v_1 H} + e^{-v_1 H} - (e^{v_1} + e^{-v_1} - 2\cos(v_2))^H\right). \tag{12}$$

Conversely, if $Y(\mathbf{t})$, $\mathbf{t} = (t_1, t_2)^T \in \mathbb{R}^2$, is a mean zero Gaussian stationary process with correlation $R(\mathbf{v})$ given by (12), then $Y(\mathbf{t})$ can be expressed as: $Y(\mathbf{t}) \stackrel{d}{=} e^{-t_1 H} X(e^{t_1}\cos(t_2), e^{t_1}\sin(t_2))$, where X is a mean zero Lévy fractional Brownian random field.

According to Definition 1, X defined by (10) and (11) is H-ss with $0 < H \leq 1$. A natural question is whether X is also $\mathbf{H}$-mss? The answer is no in Cartesian coordinates, but yes in polar coordinates. Indeed, rewriting (11) as

$$Z(\rho_{\mathbf{t}}, \theta_{\mathbf{t}}) = X(\mathbf{t}) \stackrel{d}{=} \rho_{\mathbf{t}}^{H_1} (e^{\theta_{\mathbf{t}}})^{H_2} Y(\ln(\rho_{\mathbf{t}}), \ln(e^{\theta_{\mathbf{t}}})), \tag{13}$$

with $\mathbf{H} = (H_1, H_2)^T = (H, 0)^T$, we conclude from Proposition 1 that X is $\mathbf{H}$-mss with respect to the polar coordinates $(\rho_{\mathbf{t}}, \theta_{\mathbf{t}})$. Thus, from Definition 3, Lévy fractional Brownian random fields indexed by $\mathbb{R}^2$ are LSR random fields with

$$R_1(\mathbf{w}) = e^{2\mathbf{H}^T \mathbf{w}}, \ \mathbf{H} = (H_1, H_2)^T = (H, 0)^T, \ \mathbf{g}(\mathbf{t}) = (\ln(t_1), t_2)^T,$$

and $R(\mathbf{v})$ given by (12).

Properties of the stationary correlation function

Consider the stationary correlation function $R(\mathbf{v})$, $\mathbf{v} = (v_1, v_2) \in \mathbb{R}^2$, given by:

$$R(\mathbf{v}) = \frac{1}{2}\left(e^{v_1 H} + e^{-v_1 H} - (e^{v_1} + e^{-v_1} - 2\cos(v_2))^H\right). \tag{14}$$

The asymptotic behavior of $R(\mathbf{v})$ as $v_1 \to +\infty$ is given by

$$R(\mathbf{v}) \sim \begin{cases} \frac{1}{2} e^{-v_1 H} & \text{for } 0 < H \leq \frac{1}{2}, \\ H e^{-v_1(1-H)} \cos(v_2) & \text{for } \frac{1}{2} < H \leq 1. \end{cases} \tag{15}$$

It is interesting to note that, unlike the Lévy fractional Brownian random field X, the corresponding reduced process Y has a short-range dependence structure for $0 < H \leq 1$.

Modeling a latent dimension

Work in progress with Wendy Meiring (University of California, Santa Barbara)

Do not find a "good" deformation in the model:

$$r(x,y) = \rho(\|f(x) - f(y)\|_2)$$

may mean that such a deformation do not exist or that the underlying phenomenon (indexed by $\mathbb{R}^2$) depends on other (latent) dimension(s) (Sampson & Guttorp (1992)).

New model:

$$r(x,y) = \rho\left(\|(x, \psi(x)) - (y, \psi(y))\|_3\right), \tag{16}$$

where $\|.\|_3$ represents the canonical Euclidean norm in $\mathbb{R}^3$ and where ψ is an application from $\mathbb{R}^2$ to $\mathbb{R}$, modeling the latent dimension.

Latent dimension: characterization

For random processes indexed by $\mathbb{R}$:

$$r(x,y) = \rho\left(\sqrt{(x-y)^2 + (\psi(x)-\psi(y))^2}\right) \tag{17}$$

if and only if, for all $x, y \in \mathbb{R}$

$$(\, x - y \,)\left(r_1^{(1,0)}(x,y) \; + \; r_1^{(0,1)}(x,y) \right)$$

$$= -(\, \psi(x) - \psi(y) \,)\left(\psi^{(1)}(y)\, r_1^{(1,0)}(x,y) \; + \; \psi^{(1)}(x)\, r_1^{(0,1)}(x,y) \right).$$

where ψ from $\mathbb{R}$ to $\mathbb{R}$ satisfies:

$$\psi^{(1)}(x) \neq \psi^{(1)}(y), \; \forall \; x \neq y.$$

Comparison stationarity/non-stationarity

Work in progress with Maureen Clerc (INRIA, Sophia Antipolis) and Marc Genton (Texas A& M University)

- for estimating the deformation: comparison of the quadratic variations method (Guyon & Perrin (2000)) with the scalogram method (Clerc & Mallat (2003)
- application of the quadratic variations method to kriging, estimators of the kriging coefficients (their behavior?)
- influence of the deformation (comparison stationarity/non-stationarity)